

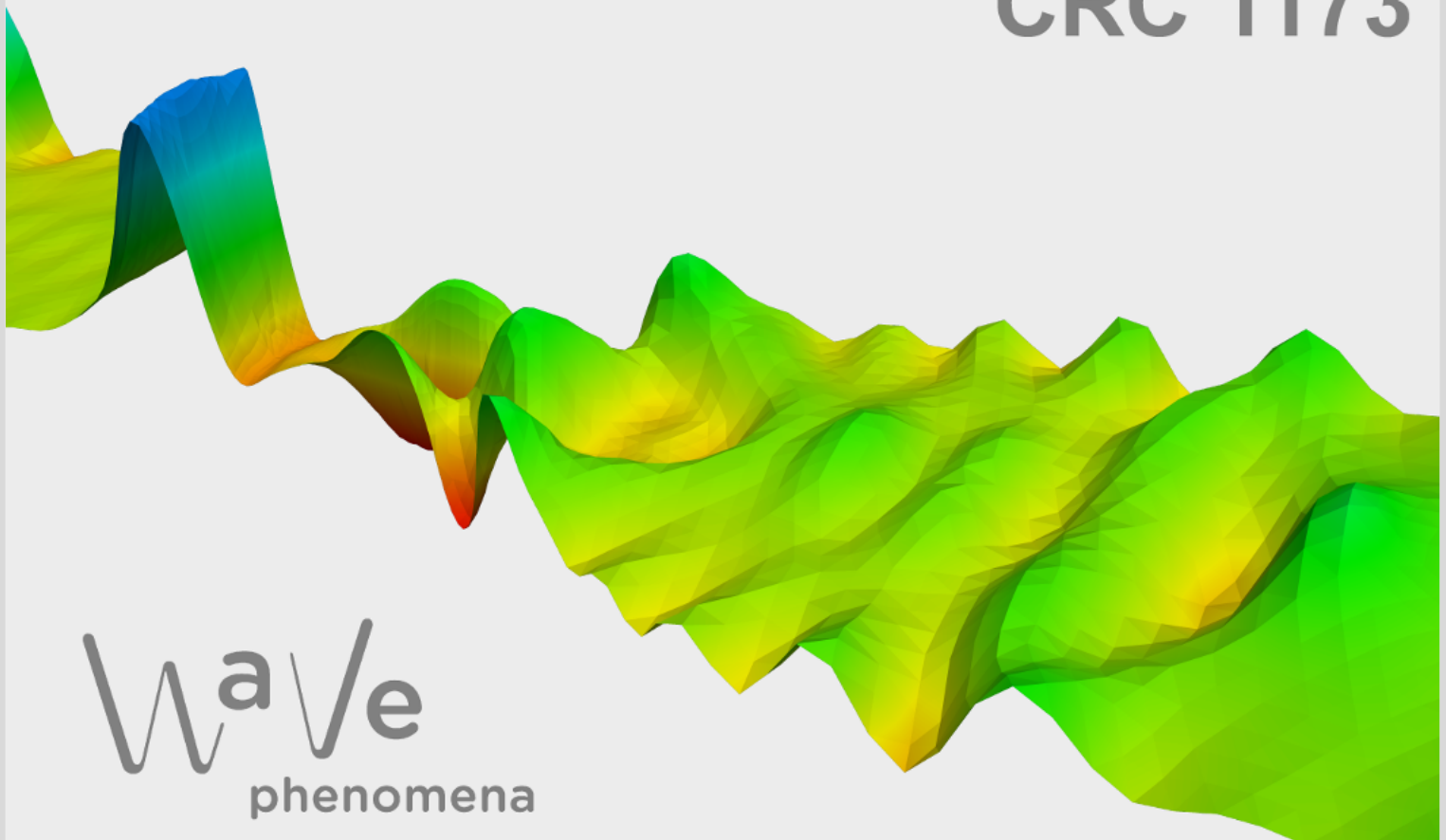
Asymptotics for wild and large data in the nonconservative NLS

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Abstract

We study the nonconservative nonlinear Schrödinger equation (NLS) $u_t = idu_{xx} + iu^n$, $d > 0$, and in contrast to most of the previous publications, we focus on the whole real line instead of periodic boundary conditions. Allowing for growth of the semigroup $e^{i\partial_x^2 t}$ and based on a scaling argument, we show that solutions exist globally and admit decay to zero in L^∞ when they start L^∞ -close to constants while the quantity $d^{\frac{1}{4}}\|\partial_x u(0)\|_{L^2}$ is large. The second main result of this paper removes this L^∞ -smallness at the cost of localization assumptions.

Keywords. Nonconservative NLS, nonlinear iteration, Zhidkov spaces, modulation spaces, dispersive smoothing, ODE dynamics

Mathematics Subject Classification (2020). 35Q55, 35B40, 35B44

1 Introduction

The nonconservative NLS

$$u_t = idu_{xx} + iu^n, \quad (x, t) \in \mathbb{R} \times [0, \infty), \quad (1.1)$$

where $n \in \mathbb{N}_{\geq 2}$ and $d > 0$, recently attracts an increasing amount of attention through the paper [11]. The named paper studies fundamental properties on periodic boundary conditions with crucial extensions by one of these authors in [9]. One basic observation in [11] is that the complex plane may be decomposed by $\mathbb{C} = \mathcal{M}_b^n \cup \mathcal{M}_g^n$ with

$$\mathcal{M}_g^n = \{z \in \mathbb{C} : \sin((n-1)\arg(z)) > -1\}, \quad \mathcal{M}_b^n = \{z \in \mathbb{C} : \sin((n-1)\arg(z)) = -1\}.$$

Considering the associated ODE to (1.1), i.e., $d = 0$, given by

$$\partial_t u = iu^n \quad (1.2)$$

the corresponding solution $z(t)$ to (1.2) with $z_0 := z(0) \in \mathcal{M}_b^n$ admits $|z(t)| \rightarrow \infty$ as $t \uparrow ((n-1)|z_0|^{n-1})^{-1}$ while

$$|z(t)| \lesssim_{z_0} (1+t)^{-\frac{1}{n-1}} \quad (1.3)$$

for all $t \geq 0$ with $z_0 \in \mathcal{M}_g^n$.¹ The works [11, 8] study in particular space-time dependent solutions which start close to points from $\mathcal{M}_g^n$ from an appropriate Banach space to show global existence and temporal decay of the solutions. The main contribution from [8] is that one may consider any Banach algebra X for

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¹Let S be a set, and let $A, B: S \rightarrow \mathbb{R}$. Throughout the paper, the expression ' $A(x) \lesssim B(x)$ for $x \in S$ ', means that there exists a constant $C > 0$, independent of x , such that $A(x) \leq CB(x)$ holds for all $x \in S$. An explicit dependence of the constant C by a variable a is denoted by $\lesssim_a$.

which $i\partial_x^2$ generates a C_0 -semigroup on X such that $\|e^{i\partial_x^2 t}\|_{X \rightarrow X} \lesssim 1$ which includes interesting initial data such as almost periodic functions or localized plus almost periodic functions. In this paper, we are interested in minimal conditions on the class of initial data X close to constants from $\mathcal{M}_g^n$ to guarantee global existence and temporal decay of the solutions measured in the scaling invariant L^∞ -norm. A crucial condition on the initial datum $u(0)$ with $u(0) - z_0 \in L^\infty$ for some $z_0 \in \mathcal{M}_g^n$ is that the L^∞ -distance to $\mathcal{M}_b^n$ is sufficiently small, otherwise, we may choose $u(0) \in \mathcal{M}_b^n$ and arrive at the ODE blow-up solution. Nevertheless, even the condition $\|u(0) - z_0\|_{C_b^1} \ll 1$ for some $z_0 \in \mathcal{M}_g^n$ seems not sufficient to guarantee global existence since, following [2, 1], dispersive blow-up is expected: choosing $u(0) - z_0 = \varepsilon \frac{e^{id^{-1}|x|^2}}{(1+d^{-1}x^2)^{\frac{1}{2}}}$ for any $\varepsilon > 0$, one arrives at $|e^{id\partial_x^2 t}(u(0) - z_0)| \gtrsim_{z_0, d, \varepsilon} |\log(\frac{1}{4} - t)|$ as $t \uparrow \frac{1}{4}$. Our main result of this paper is now that the condition $\|u(0) - z_0\|_{L^\infty} \ll 1$ is sufficient to conclude global existence and the same temporal decay rates as in (1.3) when we additionally assume that $\partial_x u(0) \in L^2$. Remarkably, the size of the initial L^∞ -perturbation only depends on the size of $d^{\frac{1}{4}}\|\partial_x u(0)\|_{L^2}$ and we make a distinction of the cases

$$d^{\frac{1}{4}}\|\partial_x u(0)\|_{L^2} \ll 1$$

and

$$d^{\frac{1}{4}}\|\partial_x u(0)\|_{L^2} \gg 1. \quad (1.4)$$

If (1.4) holds, we call the initial datum $u(0)$ wild. To be more precise, we show temporal decay for wild initial data where we need to adapt the L^∞ -distance to z_0 if the quantity $d^{\frac{1}{4}}\|\partial_x u(0)\|_{L^2}$ increases. We refer to Figure 1 for an illustration. In the following theorems, we suppress the explicit dependence of $|z_0|$ on the size of the initial deviation to z_0 .

Theorem 1.1 (Wild initial data close to constants). *There exists some constant $C > 0$ such that the following holds. Fix $n \in \mathbb{N}_{\geq 2}$. Let $z_0 \in \mathcal{M}_g^n$ and denote $z(t)$ the solution of (1.2) with $z(0) = z_0$. For all $K > 0$ there exists some $\varepsilon > 0$ such that for every $\|w_0\|_{L^\infty} < \varepsilon$ and $d^{\frac{1}{4}}\|\partial_x w_0\|_{L^2} \leq K$, there exists a unique (mild) solution $u \in C([0, \infty], X^0)$ of (1.1) with $u(0) = w_0 + z_0$ and²*

$$\|u(t) - z(t)\|_{L^\infty} \leq C(1 + K)\varepsilon^{\frac{1}{3}}(1 + t)^{-\frac{5}{6} - \frac{1}{n-1}}, \quad \|\partial_x u(t)\|_{L^2} \leq CKd^{-\frac{1}{4}}(1 + t)^{-1 - \frac{1}{n-1}}$$

for all $t \geq 0$. In particular, there exists some constant $\tilde{C} = \tilde{C}(|z_0|)$ such that

$$\|u(t)\|_{L^\infty} \leq C(\tilde{C} + (1 + K)\varepsilon^{\frac{1}{3}}(1 + t)^{-\frac{5}{6}})(1 + t)^{-\frac{1}{n-1}}$$

for all $t \geq 0$.

This result allows for interesting initial data:

Example 1.2. As a first example, we consider steep data which connect constant states at $\pm\infty$ such as $w_0(x) = \varepsilon \arctan(L\varepsilon^{-2}x)$ with $d^{\frac{1}{4}}L^{\frac{1}{2}} \leq K$ and sufficiently small $\varepsilon = \varepsilon(K) > 0$ as $K \rightarrow \infty$. In the similar spirit, we allow for fast oscillating data such as $w_0(x) = \varepsilon \sin\left(\frac{1}{(1+(L\varepsilon^{-2}x)^2)^\alpha}\right)$ with $\frac{1}{4} > \alpha > -\frac{1}{4}$, $d^{\frac{1}{4}}L^{\frac{1}{2}} \leq K$ and sufficiently small $\varepsilon = \varepsilon(K) > 0$ as $K \rightarrow \infty$.

It is well-known, see e.g. [6], that X^0 is a Banach algebra for which $i\partial_x^2$ generates a C_0 -semigroup and $\|e^{i\partial_x^2 t}\|_{X^0 \rightarrow X^0} \lesssim (1 + t)^{\frac{1}{6}}$, $t \geq 0$. Therefore, the first step in the next section is to show that we may allow such a growth following essentially the argument from [8] to conclude temporal decay of solutions to (1.1). We will see that we may even allow temporal growth up to order $(1 + t)^\alpha$ with $0 < \alpha < \frac{1}{2}$ instead of the boundedness from [8]. This further allows to consider the full localization range by applying our observation to modulation spaces $M_{p,1}$, $1 \leq p < \infty$, which serve as alternatives for L^p -spaces under linear dispersive evolutions and are defined in Appendix A. We also observe then that we may allow for Zhidkov data for

²See Appendix A for the definition of the Zhidkov space X^0 .

which $d^{\frac{1}{4}}\|\partial_x u(0)\|_{L^2}$ and $\|u(0)\|_{L^\infty}$ are both sufficiently small. We refer to Theorem 2.2, Corollaries 2.3, 2.4 and Remark 2.5. To show Theorem 1.1 and therefore allowing for (1.4), we introduce the rescaled solutions $u_\varepsilon(x, t) = u(\sqrt{d}\varepsilon x, \varepsilon^2 t)$ with a scaling parameter ε being the size of initial L^∞ -deviation to z_0 . Using the L^2 -scaling of the derivative, we observe that $\|\partial_x u_\varepsilon(t)\|_{L^2} \sim d^{\frac{1}{4}}\sqrt{\varepsilon}$ which allows to finely balance the non-critical growth $(1+t)^{\frac{1}{6}}$ of the L^∞ -norm with the temporal decay and growth in $\varepsilon^{-1} > 0$ through the contribution of $z(\varepsilon^2 t)^{n-1}$ given by $(1+\varepsilon^2 t)^{-2}$. For sufficiently small $\varepsilon > 0$, we then perform an absorption argument. We strongly expect that similar scaling arguments are also useful for other dispersive equations with different leading order linear operator and whose associated ODE admits an attracting set such as for the generalized Gross-Pitaevskii $\partial_t u = id\partial_x^2 u + \gamma u + \beta|u|^2 u$ with $\text{Re}(\gamma)$ and $\text{Re}(\beta)$ nonzero, studied in [1], or variants of the Lugiato-Lefever equation $\partial_t u = id\partial_x^2 u + (\mu i - 1)u - i|u|^2 u + F$. These examples in particular admit attracting sets of the associated ODEs which do not only consist of $\{0\}$. We also refer to Remark 3.4.

We note that the restriction on the temporal growth of the semigroup $e^{i\partial_x^2 t}$ does not include $M_{\infty,1}$ to conclude temporal decay of solutions to (1.1) because of the critical and optimal growth bound $(1+t)^{\frac{1}{2}}$ of the semigroup $e^{i\partial_x^2 t}$ from $M_{\infty,1}$ into itself. By the embedding $C_b^{1+\eta} \hookrightarrow_\eta M_{\infty,1} \hookrightarrow C_b$ for all $\eta > 0$, this would give a sharp threshold to the dispersive blow-up phenomenon in C_b^1 - similarly as in the result of the paper [1].

Conjecture. Fix $n \in \mathbb{N}_{\geq 2}$ and let $z_0 \in \mathcal{M}_g^n$. For all $\eta > 0$ there exists $\varepsilon > 0$ such that for $\|w\|_{C^{1+\eta}} < \varepsilon$ the corresponding solution $u(t)$ of (1.1) with $u(0) = w_0 + z_0$ exists globally and satisfies $\|u(t)\|_{L^\infty} \lesssim (1+t)^{-\frac{1}{n-1}}$ for all $t \geq 0$.

We aim to address the proof of this conjecture in future. As an interesting question for further research, we ask for detailed information on the dynamical picture for space-time dependent solutions and concretely formulate the following task in an extension to [11, Open questions 1, page 59] which restricts to real-valued initial data:

Find suitable conditions to prove statements on the long-time respectively blow-up behavior of solutions starting away from $\mathcal{M}_b^n$, i.e., $u_0(x) \in \mathcal{M}_g^n \setminus \mathcal{M}_b^n$ for all $x \in \mathbb{R}$.

Furthermore, we remark that the numerical study of [10] indicates that real, periodic data which are not close to some constant may admit blow-up in finite time. An analytical proof seems open to the author's current knowledge.

Our second main result allows now for L^∞ -large initial data at the cost of localization assumptions. By considering initially a localized deviation to a constant $z_0 \in \mathcal{M}_g^n$, this constant is the only one with localized distance to the prescribed initial datum. This heuristically suggests that the dynamics close to z_0 should be again described through $z(t)$. Notably, this allows initial data u_0 such that $u_0(x) \in \mathcal{M}_b^n$ for some $x \in \mathbb{R}$, see Example 1.5.

Theorem 1.3 (L^∞ -large initial data). *Fix $n \in \mathbb{N}_{\geq 2}$ and $d > 0$. Let $z_0 \in \mathcal{M}_g^n$ and denote $z(t)$ the solution of (1.2) with $z(0) = z_0$. For all $K > 0$ there exists some $C, \varepsilon > 0$ such that for every $E_0 = \|w_0\|_{L^1 \cap L^2} < \varepsilon$ and $\|\partial_x w_0\|_{L^2} \leq K$, there exists a unique (mild) solution $u \in C([0, \infty); H^1 + \mathbb{C})$ of (1.1) with $u(0) = w_0 + z_0$ and*

$$\|u(t) - z(t)\|_{L^\infty} \leq CE_0(1+t)^{-\frac{n}{n-1} - \frac{1}{2}} \quad (1.5)$$

and in particular

$$\|u(t)\|_{L^\infty} \leq C(\tilde{C} + E_0(1+t)^{-1-\frac{1}{2}})(1+t)^{-\frac{1}{n-1}},$$

for some $\tilde{C} = \tilde{C}(|z_0|)$ and all $t \geq 0$.

Remark 1.4. We clarify the different logic in how we formulate Theorems 1.1 and 1.3. Let $C > 0$ be the constant from Theorem 1.3 and let $\alpha > 0$ be small. If we choose $E_0^\alpha < \frac{1}{\max\{1, C\}}$ and $E_0 < \varepsilon$, then (1.5) becomes

$$\|u(t) - z(t)\|_{L^\infty} \leq \varepsilon^{1-\alpha} (1+t)^{-\frac{n}{n-1}-\frac{1}{2}}.$$

This shows that analogously to the formulation of Theorem 1.1, we may choose the constant in the estimate (1.5) independently of every parameter by making $\|w_0\|_{L^1 \cap L^2}$ smaller.

For the proof of Theorem 1.3 in Section 4, we use the dispersive smoothing estimate $\|e^{i\partial_x^2 t}\|_{L^1 \rightarrow L^\infty} \lesssim t^{-\frac{1}{2}}$, $t > 0$, and an energy estimate to control the derivative in L^2 .

Example 1.5. Let $n = 2$ and $z_0 > 0$. Choose $w_0(x) = L \exp(-(L^3 x)^2)$ with $L \gg 1$ as a straightforward example. We can modify this example as $w_0(x) = -z_0 L \exp(-(L^3 x)^2) + \frac{i}{L} \exp(-x^2)$ with $L \gg 1$ intersecting $\mathcal{M}_b^2$.

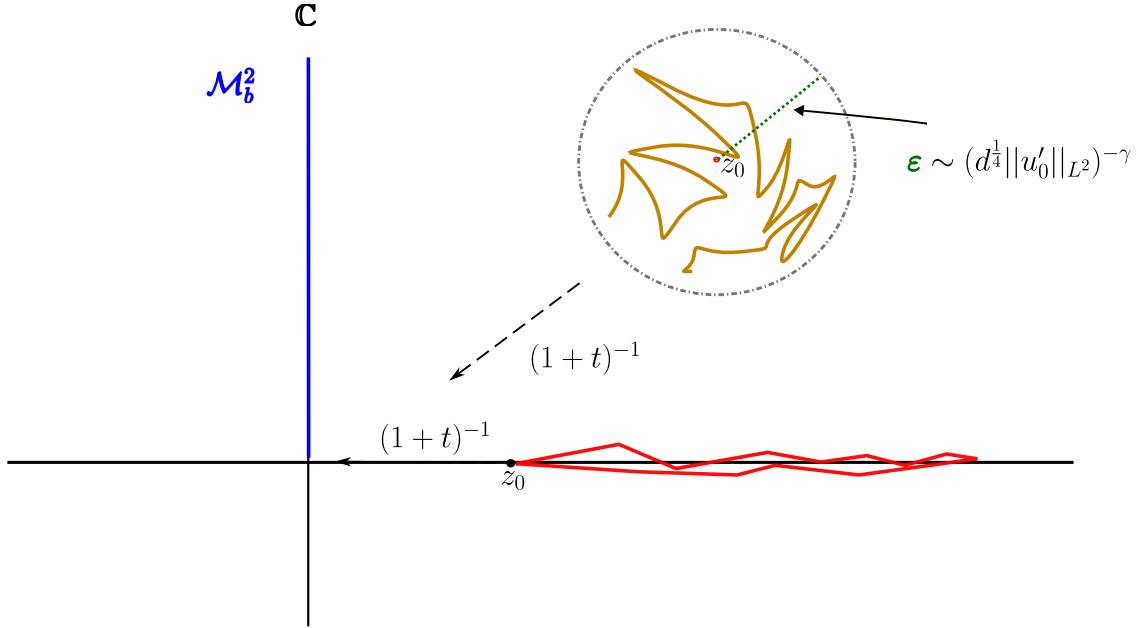


Figure 1: This figure serves as an illustration for Theorems 1.1 and 1.3. First, it displays an initial datum u_0 for $n = 2$ in orange with $\|u'_0\|_L^2$ large. The green dot line indicates the maximal L^∞ -distance to z_0 which is roughly of the scale $(d^{\frac{1}{4}} \|u'_0\|_{L^2})^{-\gamma}$ for some suitable $\gamma \gg 1$ which we did not specify in the theorem but which becomes apparent in its proof in Section 3; we refer to Remark 3.3. The red line is an example for an initial datum covered by Theorem 1.3.

Remark 1.6. We consider it as interesting extension to study space-dependent dispersion coefficients in the context of non-decaying data such as in Theorem 1.1. Furthermore, it would be also interesting to consider data admitting the weaker assumption $\partial_x w_0 \in H^s$ with $0 > s > -\frac{1}{2}$ instead of $\partial_x w_0 \in L^2$ in Theorem 1.1.

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2 Incorporating temporal growth of the semigroup

We first setup the local well-posedness. For this purpose, let $z_0 \in \mathbb{C}$ and $z(t)$ the solution to (1.2) with $z(0) = z_0$. Let $u(t)$ be a solution of (1.1) and we write it as

$$u(t) = z(t) + z(t)^n v(t) \quad (2.1)$$

following [11, 8], to measure the distance of $u(t)$ to $z(t)$. Again following [11, 8], this yields the equation for $v(t)$,

$$\partial_t v = i d \partial_x^2 v + i \sum_{m=2}^n \binom{n}{m} z^{m(n-1)} v^m. \quad (2.2)$$

Proposition 2.1. *Let $z_0 \in \mathbb{C} \setminus \{0\}$. Let X be a Banach algebra of functions on $\mathbb{R}$ for which $i \partial_x^2$ generates a C_0 -semigroup and $w_0 \in X$. There exists a unique solution $u \in C([0, T_{max}); X + \mathbb{C})$ of (1.1) with $u(0) = w_0 + z_0$ for some $T_{max} \in (0, \infty]$. If $z_0 \in \mathcal{M}_g^n$, then $T_{max} < \infty$ if and only if $\lim_{t \rightarrow T_{max}} \|v(t)\|_X = \infty$ with v given by (2.1).*

Proof. The Duhamel formula for v as chosen in (2.1), is given by

$$v(t) = e^{i d \partial_x^2 t} v_0 + i \int_0^t e^{i d \partial_x^2 (t-s)} \sum_{m=2}^n \binom{n}{m} z(s)^{m(n-1)} v(s)^m ds, \quad v(0) = \frac{w_0}{z_0^n} \quad (2.3)$$

The local-wellposedness of (2.3) in $C([0, T_{max}), X)$ is given by standard semigroup theory, see e.g. [3]. Proceeding as in [8, Proposition 2.4], we observe that $u(t)$ is a solution from $C([0, T_{max}); X + \mathbb{C})$ with $u(0) = z_0 + u_0$ if and only if v solves 2.3 in $C([0, T_{max}); X)$. The blow-up alternative for $z_0 \in \mathcal{M}_g^n$ then follows again from standard semigroup theory and the equivalence of the solutions. $\square$

We note that X does not necessarily contain constants in contrast to [8].

Theorem 2.2 (Generalization of [8]). *Let $z_0 \in \mathcal{M}_g^n$ and $0 < \alpha < \frac{1}{2}$. Let $X \hookrightarrow L^\infty$ be a Banach algebra for which $i \partial_x^2$ generates a C_0 -semigroup such that $\|e^{i \partial_x^2 t}\|_{X \rightarrow X} \lesssim (1+t)^\alpha$ for all $t \geq 0$. There exist $\varepsilon, C > 0$ such that for every $d > 0$ and $w_0 \in X$ with $\|w_0(\cdot \sqrt{d})\|_X \leq \varepsilon$, there exists a unique solution $u \in C([0, \infty); X + \mathbb{C})$ of (1.1) with $u(0) = w_0 + z_0$ such that*

$$\|u(\cdot \sqrt{d}, t) - z(t)\|_X \leq C \|w_0(\cdot \sqrt{d})\|_X (1+t)^{-\frac{n}{n-1} + \alpha}$$

for all $t \geq 0$. In particular, there exists a constant $\tilde{C} = \tilde{C}(|z_0|)$ such that

$$\|u(t)\|_{L^\infty} \leq C(\tilde{C} + \|w_0(\cdot \sqrt{d})\|_X (1+t)^{-1+\alpha}) (1+t)^{-\frac{1}{n-1}}$$

for $t \geq 0$.

We apply the theorem to Zhidkov and modulation spaces. We define these spaces in Appendix A in more generality and here we focus on a selection of special regimes. The proofs of the corollaries are immediate consequences from Propositions A.1- A.3.

Corollary 2.3 (Application to modulation spaces). *Fix $z_0 \in \mathcal{M}_g^n$ and $p \in (1, \infty)$. There exist $\varepsilon, C > 0$ such that for every $d > 0$ and*

$$\|w_0\|_{M_{p,1}} \leq \varepsilon \begin{cases} d^{\frac{1}{p} - \frac{1}{2}}, & 2 < p < \infty, d \geq 1 \\ 1, & 1 < p \leq 2, d \geq 1 \\ d^{\frac{1}{2p}}, & 0 < d \leq 1, \end{cases}$$

there exists a unique solution $u \in C([0, \infty); M_{p,1} + \mathbb{C})$ of (1.1) with $u(0) = w_0 + z_0$ such that

$$\|u(\cdot\sqrt{d}, t) - z(t)\|_{M_{p,1}} \leq C\varepsilon(1+t)^{-\frac{n}{n-1} + |\frac{1}{2} - \frac{1}{p}|}$$

and in particular

$$\|u(t)\|_{L^\infty} \leq C(\tilde{C} + \varepsilon(1+t)^{-1 + |\frac{1}{2} - \frac{1}{p}|})(1+t)^{-\frac{1}{n-1}}, \quad (2.4)$$

for some $\tilde{C} = \tilde{C}(|z_0|)$ and all $t \geq 0$.

We emphasize that due to the embedding $W^{2,p} \hookrightarrow M_{p,1}$, we cover every $W^{2,p}$ -space with $1 < p < \infty$, where we comment separately on the case $p = 1$ in Remark 2.5. We note that the stronger localized norms work well for large dispersion. More precisely, we mean: for $1 < p \leq 2$ and fixing some $d_0 > 0$, there exists an $\varepsilon > 0$ such that $\|w_0\|_{M_{p,1}} < \varepsilon$ implies (2.4) for all $d \geq d_0$. For small dispersions, this kind of uniformity gets lost.

Corollary 2.4 (Application to Zhidkov spaces). *Fix $z_0 \in \mathcal{M}_g^n$. There exist $\varepsilon, C > 0$ such that for every $d > 0$ and $\|w_0\|_{L^\infty} + d^{\frac{1}{4}}\|\partial_x w_0\|_{L^2} \leq \varepsilon$, there exists a unique solution $u \in C([0, \infty); X^0)$ of (1.1) with $u(0) = w_0 + z_0$ such that*

$$\|u(\cdot\sqrt{d}, t) - z(t)\|_{X^0} \leq C\varepsilon(1+t)^{-\frac{n}{n-1} + \frac{1}{6}}$$

and in particular

$$\|u(t)\|_{L^\infty} \leq C(\tilde{C} + \varepsilon(1+t)^{-1 + \frac{1}{6}})(1+t)^{-\frac{1}{n-1}},$$

for some $\tilde{C} = \tilde{C}(|z_0|)$ and all $t \geq 0$.

This corollary shows that X^0 works in particular well for small dispersions: in fact, it even includes the observation that $d > 0$ may be large when $\|\partial_x u(0)\|_{L^2}$ is sufficiently small, respectively, $\|\partial_x u(0)\|_{L^2}$ may be large when $d > 0$ is sufficiently small. In contrast, Theorem 1.1 allows now that both quantities may be large.

Proof of Theorem 2.2. Let $z_0 \in \mathcal{M}_g^n$. Let u given by Proposition 2.1 with $v \in C([0, T_{max}), X)$ fulfilling (2.3). We consider the spatially rescaled variable $v^d(x, t) = v(\sqrt{d}x, t)$ which solves now (2.2) with $d = 1$.³ We set $v_0^d = v^d(0)$. We set $\eta(t) = \sup_{0 \leq s \leq t} (1+s)^{-\alpha} \|v^d(s)\|_X$ which is continuous, by Proposition 2.1, and a monotone function. Let $0 \leq s \leq t < T_{max}$ with $\eta(t) \leq \frac{1}{2}$. We bound

$$\left\| \sum_{m=2}^n \binom{n}{m} z(s)^{m(n-1)} v^d(s)^m \right\|_X \lesssim \sum_{m=2}^n (1+s)^{-m} (1+s)^{\alpha m} \eta(t)^m \lesssim (1+s)^{2(-1+\alpha)} \eta(t)^2,$$

using the monotonicity of η and (1.3). Noting that $2(-1+\alpha) < -1$, we obtain

$$\eta(s) \lesssim \|v_0^d\|_X + \eta(t)^2 (1+s)^{-\alpha} \int_0^s (1+s-\tau)^\alpha (1+\tau)^{2(-1+\alpha)} d\tau \lesssim \|v_0^d\|_X + \eta(t)^2.$$

We conclude that there exists some $|z_0|$ -dependent constant $C \geq 1$ such that

$$\eta(t) \leq C(\|v_0^d\|_X + \eta(t)^2) \quad (2.5)$$

for all $0 \leq t < T_{max}$. We claim that $\|v_0^d\|_X < \frac{1}{4C^2}$ implies $T_{max} = \infty$ and $\eta(t) \leq 2C\|v_0^d\|_X$ for all $t \geq 0$. In fact, assume there exists some time in the interval of maximal existence for which η is larger than $2C\|v_0^d\|_X$,

³To justify this, one can start first with smooth data and finally do an approximation argument, we omit the details on this procedure and refer to [3]. The same applies to the proof of Theorems 1.1 and 1.3.

then by continuity, there has to be a $0 < t_0 < T_{max}$ such that $|\eta(t_0)| = 2C\|v_0^d\|_X \leq \frac{1}{2}$. Using (2.5), this implies

$$\eta(t_0) \leq C\|v_0^d\|_X + C(2C)^2(\|v_0^d\|_X)(\|v_0^d\|_X) < 2C\|v_0^d\|_X$$

which is a contradiction and the claim and therefore the result follow by the blow-up alternative in Proposition 2.1. $\square$

Remark 2.5 (The case $p = 1$). *The case $p = 1$ in Corollary 2.3 is omitted as straightforward consequence of Theorem 2.2 since for $M_{1,1}^s$, the semigroup also admits the critical growth $(1+t)^{\frac{1}{2}}$. Nevertheless, invoking [5, Lemma 2.8] gives*

$$M_{1,1} \hookrightarrow \mathcal{FL}^1 \hookrightarrow L^\infty,$$

where $\mathcal{FL}^1$ contains every L^∞ -function whose Fourier transform is an integrable function. Equipped with the L^1 -norm of the Fourier transform, $\mathcal{FL}^1$ is a Banach algebra for which $\|e^{i\partial_x^2 t}\|_{\mathcal{FL}^1 \rightarrow \mathcal{FL}^1} \lesssim 1$, as shown in [8]. Invoking Theorem 2.2 or [8, Theorem 2.6], we may conclude (2.4) for every solution of (1.1) with initial datum $w_0 + z_0$ for $z_0 \in \mathcal{M}_g^n$ and $\|w_0\|_{M_{1,1}} \ll \min\{1, d^{\frac{1}{2}}\}$.

3 Proof of Theorem 1.1

For our proof, we require a slightly refined semigroup estimate compared to Proposition A.1 which is a straightforward consequence of the analysis from [6].

Proposition 3.1. *There exists a constant $C_0 \geq 1$ such that for all $\kappa \geq 1$, $u_0 \in X^0$ and $t \geq 0$, we have the semigroup bound*

$$\|e^{i\partial_x^2 t} u_0\|_{L^\infty} \leq C_0(1+t)^{\frac{1}{6}}(\kappa\|u_0\|_{L^\infty} + \kappa^{-\frac{1}{2}}\|\partial_x u_0\|_{L^2}).$$

Proof. In [6], it is shown that

$$\|e^{i\partial_x^2 t} u_0\|_{L^\infty} \lesssim (\beta + \frac{1}{\beta})\|u_0\|_{L^\infty} + t^{\frac{1}{4}}\beta^{-\frac{1}{2}}\|u_0\|_{X^0}$$

for every $\beta > 0$. Choosing $\beta = \kappa t^{\frac{1}{6}}$ for large times and $\beta = \kappa$ for short times yields the result. $\square$

Now, let $z_0 \in \mathcal{M}_g^n$ and $z(t)$ be the corresponding ODE solution. Let u and v be as defined in (2.1) and Proposition 2.1. Let $\varepsilon > 0$ and $M \geq 1$. We suppose

$$\|v_0\|_{L^\infty} < \varepsilon^2 \text{ and } \|\partial_x v_0\|_{L^2} \leq M$$

and introduce the rescaled variable

$$v^\varepsilon(x, t) = v(\sqrt{d}\varepsilon x, \varepsilon^2 t)$$

which satisfies the equation

$$\partial_t v^\varepsilon(x, t) = i\partial_x^2 v^\varepsilon(x, t) + N(z, v^\varepsilon)(x, t) \text{ with } N(z, v^\varepsilon)(x, t) = i\varepsilon^2 \sum_{m=2}^n \binom{n}{m} z(\varepsilon^2 t)^{m(n-1)} v^\varepsilon(x, t)^m. \quad (3.1)$$

Note that given through Proposition 2.1, $v^\varepsilon \in C([0, \frac{T_{max}}{\varepsilon^2}], X^0)$ is a solution of (1.1) for some $T_{max} > 0$. Writing $v_0^\varepsilon = v^\varepsilon(0)$, we bound

$$\|v_0^\varepsilon\|_{L^\infty} = \|v_0\|_{L^\infty} < \varepsilon^2, \quad \|\partial_x v_0^\varepsilon\|_{L^2} \leq \sqrt{\sqrt{d}\varepsilon} \|\partial_x v_0\|_{L^2} \leq d^{\frac{1}{4}} M \sqrt{\varepsilon}. \quad (3.2)$$

Let C_0 be the constant from Proposition 3.1. Furthermore, we assume $\varepsilon \leq \frac{1}{2}$ and further specify its smallness in the following. We introduce the template functions

$$\eta_1(t) = \sup_{0 \leq s \leq t} \|v^\varepsilon(s)\|_{L^\infty} (1+s)^{-\frac{1}{6}}, \quad \eta_2(t) = \sup_{0 \leq s \leq t} \|\partial_x v^\varepsilon(s)\|_{L^2}.$$

We establish the following bounds on the nonlinearity.

Lemma 3.2. *Fix $K > 0$ and assume $\eta_1(t) + \eta_2(t) \leq K$. There exist a t -independent constant $C > 0$ such that we have the following nonlinear bounds*

$$\|\partial_x N(z, v^\varepsilon)(\cdot, s)\|_{L^2} \leq C\varepsilon^2 \sum_{m=2}^n (1+\varepsilon^2 s)^{-m} (1+s)^{\frac{m-1}{6}} \eta_2(t) \eta_1(t)^{m-1}$$

and

$$\|N(z, v^\varepsilon)(\cdot, s)\|_{X^0} \leq C\varepsilon^2 \sum_{m=2}^n (1+\varepsilon^2 s)^{-m} \left((1+s)^{\frac{m}{6}} \eta_1(t)^m + (1+s)^{\frac{m-1}{6}} \eta_2(t) \eta_1(t)^{m-1} \right)$$

for every $0 \leq s \leq t$.

Proof of Theorem 1.1. We make use of the Duhamel formula

$$v^\varepsilon(t) = e^{i\partial_x^2 t} v_0^\varepsilon + \int_0^t e^{i\partial_x^2(t-s)} N(z, v^\varepsilon)(s) ds \quad (3.3)$$

for $0 \leq t < \frac{T_{max}}{\varepsilon^2}$.

Approach. We fix $t \in (0, \frac{T_{max}}{\varepsilon^2})$ such that

$$\eta_1(t) \leq 2M_1\varepsilon, \quad \eta_2(t) \leq 2d^{\frac{1}{4}} M \sqrt{\varepsilon} \quad (3.4)$$

with $M_1 = C_0(1 + d^{\frac{1}{4}} M)$ where such times exist because of the continuity of η_1 and η_2 and the initial assumption on the size (3.2). We show that (3.4) implies that

$$\eta_1(t) < 2M_1\varepsilon, \quad \eta_2(t) < 2d^{\frac{1}{4}} M \sqrt{\varepsilon} \quad (3.5)$$

whenever $0 < \varepsilon \leq \varepsilon_0$ for some sufficiently small $\varepsilon_0 > 0$. This implication shows that there exists some constant $C > 0$ such that

$$\|v^\varepsilon(t)\|_{L^\infty} \leq 2M_1\varepsilon(1+t)^{\frac{1}{6}}, \quad \|\partial_x v^\varepsilon(t)\|_{L^2} \leq 2d^{\frac{1}{4}} M \sqrt{\varepsilon}$$

for all $0 \leq t < \frac{T_{max}}{\varepsilon^2}$. Scaling back now yields a universal constant $C > 0$ such that

$$\|v(t)\|_{L^\infty} \leq C(1 + d^{\frac{1}{4}} M) \varepsilon^{\frac{2}{3}} (1+t)^{\frac{1}{6}}, \quad \|\partial_x v(t)\|_{L^2} \leq C \quad (3.6)$$

for all $0 \leq t < T_{max}$ whenever $\|v_0\|_{L^\infty} < \varepsilon^2 \leq \varepsilon_0^2$. Finally, Proposition 2.1 implies that $T_{max} = \infty$ in (3.6) and therefore the result follows.

We now assume (3.4) and aim to show (3.5).

Estimate of $\|v^\varepsilon(t)\|_{L^\infty}$. Let $0 \leq s \leq t$. We take the L^∞ -norm in the Duhamel formula and use Proposition 3.1 for the initial contribution with $\kappa = \varepsilon^{-1}$ and for the nonlinear contributions with $\kappa = 1$ to find

$$(1+s)^{-\frac{1}{6}} \|v^\varepsilon(s)\|_{L^\infty} \leq M_1\varepsilon + C_0(1+s)^{-\frac{1}{6}} \int_0^s (1+s-\tau)^{\frac{1}{6}} \|N(z, v^\varepsilon)(\cdot, \tau)\|_{X^0} d\tau.$$

Noting that

$$(1 + \varepsilon^2 \tau)^{-m} \leq (1 + \varepsilon^2 \tau)^{-(1 + \frac{m}{6} + \frac{1}{18})} \leq (1 + \varepsilon^2 \tau)^{-(1 + \frac{m-1}{6} + \frac{1}{18})}$$

for every $m \geq 2$, we use (1.3) and Lemma 3.2 to estimate

$$\begin{aligned} & \|N(z, v^\varepsilon)(\cdot, \tau)\|_{X^0} \\ & \lesssim \sum_{m=2}^n \varepsilon^2 \left(\varepsilon^{m-1} (1 + \varepsilon^2 \tau)^{-(1 + \frac{m}{6} + \frac{1}{18})} (1 + \tau)^{\frac{m}{6}} + \varepsilon^{m-2+\frac{1}{2}} (1 + \varepsilon^2 \tau)^{-(1 + \frac{m-1}{6} + \frac{1}{18})} (1 + \tau)^{\frac{m-1}{6}} \right) \eta_1(t) \\ & \lesssim \varepsilon^2 \sum_{m=2}^n \left(\varepsilon^{m-1-2(1 + \frac{m}{6} + \frac{1}{18})} (\varepsilon^{-2} + \tau)^{-(1 + \frac{m}{6} + \frac{1}{18})} (1 + \tau)^{\frac{m}{6}} \right. \\ & \quad \left. + \varepsilon^{m-2+\frac{1}{2}-2(1 + \frac{m-1}{6} + \frac{1}{18})} (\varepsilon^{-2} + \tau)^{-(1 + \frac{m-1}{6} + \frac{1}{18})} (1 + \tau)^{\frac{m-1}{6}} \right) \eta_1(t) \\ & \lesssim \varepsilon^{\frac{1}{18}} (1 + \tau)^{-(1 + \frac{1}{18})} \eta_1(t) \end{aligned}$$

where we observed that

$$2 + (m-1) - 2(1 + \frac{m}{6} + \frac{1}{18}) \geq \frac{4}{3} - \frac{10}{9} = \frac{2}{9}, \quad 2 + m - 2 + \frac{1}{2} - 2(1 + \frac{m-1}{6} + \frac{1}{18}) \geq \frac{14}{9} - \frac{3}{2} = \frac{1}{18}$$

for every $m \geq 2$. We have shown that there exists an ε -independent constant $C_1 \geq 1$ such that

$$\eta_1(t) \leq M_1 \varepsilon + C_1 \varepsilon^{\frac{1}{18}} \eta_1(t).$$

Choosing $\varepsilon^{\frac{1}{18}} < \frac{1}{2C_1}$, we find

$$\eta_1(t) < 2M_1 \varepsilon.$$

Estimate of $\|\partial_x v^\varepsilon(t)\|_{L^2}$. Let $0 \leq s \leq t$. Using the Duhamel formula (3.3) and the fact that $e^{i\partial_x^2 t}$ is an isometry on L^2 and $\partial_x e^{i\partial_x^2 t} = e^{i\partial_x^2 t} \partial_x$, we have

$$\|\partial_x v^\varepsilon(s)\|_{L^2} \leq d^{\frac{1}{4}} M \sqrt{\varepsilon} + \int_0^s \|\partial_x N(z, v^\varepsilon)(\cdot, \tau)\|_{L^2} d\tau.$$

Invoking Lemma 3.2, we estimate

$$\begin{aligned} \|\partial_x N(z, v^\varepsilon)(\cdot, \tau)\|_{L^2} & \lesssim \varepsilon^2 \sum_{m=2}^n \varepsilon^{m-1-2(1 + \frac{m-1}{6} + \frac{1}{18})} (1 + \tau)^{-(1 + \frac{1}{18})} \eta_2(t) \\ & \lesssim \varepsilon^{\frac{5}{9}} (1 + \tau)^{-(1 + \frac{1}{18})} \eta_2(t) \end{aligned}$$

where we detect

$$2 + m - 1 - 2(1 + \frac{m-1}{6} + \frac{1}{18}) \geq \frac{5}{9}$$

for every $m \geq 2$. We have shown that there exists an ε -independent constant $C_2 \geq 1$ such that

$$\eta_2(t) \leq M_1 \sqrt{\varepsilon} + C_2 \varepsilon^{\frac{5}{9}} \eta_2(t).$$

Choosing $\varepsilon^{\frac{5}{9}} < \frac{1}{2C_2}$, we find

$$\eta_2(t) < 2M \sqrt{\varepsilon}.$$

Conclusion. Setting $\varepsilon_0 = \min \left\{ \frac{1}{2}, \left(\frac{1}{2C_1} \right)^{18}, \left(\frac{1}{2C_2} \right)^{\frac{9}{5}} \right\}$, our estimates yield (3.5) and therefore Theorem 1.1 is proved. □

Remark 3.3. We can now estimate the size of ε indicated in Figure 1. We write $K = d^{\frac{1}{4}} \|\partial_x u(0)\|_{L^2} \gg 1$. It is straightforward to check that there is a K -independent constant $C > 0$ such that $C_1, C_2 \leq CK$. This yields $\|v_0\|_{L^\infty} < C^{-36} K^{-36}$.

Remark 3.4 (Comment on the proof). The heart of our proof is Proposition 3.1. In fact, an absorption argument also works by a non-trivial application of this proposition on the nonlinearities: assuming $\|v_0\|_{L^\infty} \lesssim \varepsilon$ and $\|v'_0\|_{L^2} \leq K$, one establishes

$$\eta_1(t) \lesssim \varepsilon^{1-\alpha} + \varepsilon^{\frac{\alpha}{2}} + \eta_1(t) \max\{\varepsilon^{1-\alpha-\tilde{\alpha}}, \varepsilon^{\frac{\alpha}{2}-\tilde{\alpha}}, \varepsilon^{\frac{\alpha}{2}}\}, \quad \eta_2(t) \lesssim 1 + \eta_2(t) \max\{\varepsilon^{1-\alpha}, \varepsilon^{\frac{\alpha}{2}}\}$$

with the *a-priori* bound $\eta_1(t) \lesssim \max\{\varepsilon^{1-\alpha}, \varepsilon^{\frac{\alpha}{2}}\}$ and $\eta_2(t) \lesssim 1$. For the absorption argument, one wants $\tilde{\alpha} > 0$, $\frac{\alpha}{2} - \tilde{\alpha} > 0$ and $1 - \alpha - \tilde{\alpha} > 0$. Choosing $\alpha = \frac{2}{3}$ and $0 < \tilde{\alpha} < \frac{1}{3}$ fulfills these requirements with $\max\{\varepsilon^{1-\alpha}, \varepsilon^{\frac{\alpha}{2}}\} = \varepsilon^{\frac{1}{3}}$. For more general operators, this argument works whenever a trading property as in Proposition 3.1 is available while in absence of this property our scaling argument still allows a balance of the L^∞ -norm and the L^2 -norm of the derivative.

4 Proof of Theorem 1.3

Let $\varepsilon, K > 0$ and $v_0 \in L^1 \cap H^1$ with $E_0 = \|v_0\|_{L^1 \cap L^2} \leq \varepsilon$ and $\|v_0\|_{H^1} \leq K$. Let $v \in C([0, T_{max}), H^1)$ be given through Proposition 2.1. As predicted by the linear estimates, we choose the template functions

$$\eta_1(t) = \sup_{0 \leq s \leq t} \left(\|v(s)\|_{L^2} + t^{\frac{1}{2}} \|v(s)\|_{L^\infty} \right), \quad \eta_2(t) = \sup_{0 \leq s \leq t} \|\partial_x v(s)\|_{L^2},$$

for $t \in [0, T_{max})$. We note that $\eta_{1,2}$ are continuous and monotonically increasing.

Approach. Fix $t \in (0, T_{max})$. We show that $\eta_1(t) \leq \frac{1}{2}$ and $\eta_2(t) \leq 2K$ imply

$$\eta_1(t) \leq C(E_0 + \eta_1(t)^2), \quad \eta_2(t) \leq \|v_0\|_{H^1} e^{C\eta_1(t)}$$

for some t -independent constant $C \geq 1$. Therefore, choosing $E_0 < \frac{1}{4C^2}$ implies that $\eta_1(t) \leq 2CE_0$ and $\eta_2(t) \leq 2K$ for all $t \in [0, T_{max})$ and necessarily $T_{max} = \infty$ by Proposition 2.1.

Estimate on $\|\partial_x v(t)\|_{L^2}$. Let $0 \leq s \leq t$. Due to continuity, we choose some $t_0 > 0$ such that

$$\sup_{0 \leq s \leq t_0} \|v(s)\|_{H^1} \leq 2K.$$

Therefore, we bound

$$\begin{aligned} \sup_{0 \leq s \leq t} \|v(s)\|_{L^\infty} &\leq \sup_{0 \leq s \leq t_0} \|v(s)\|_{L^\infty} + \mathbf{1}_{(t_0, \infty)}(t) \sup_{t_0 < s \leq t} \|v(s)\|_{L^\infty} \\ &\leq 2C_1 K + \eta_1(t) t_0^{-\frac{1}{2}} \leq 2C_1 K + (4t_0)^{-\frac{1}{2}} \end{aligned} \tag{4.1}$$

for some constant $C_1 > 0$ arising by Sobolev embedding. Second, we recall the equation for v in (2.2) and set $\tilde{v} = \partial_x v$ which satisfies

$$\partial_t \tilde{v} = id \partial_x^2 \tilde{v} + i \sum_{m=2}^n m \binom{n}{m} z^{m(n-1)} v^{m-1} \tilde{v}.$$

Let $0 \leq \tau \leq s$. Invoking (4.1) and (1.3), we obtain the inequality

$$\partial_\tau \|\tilde{v}(\tau)\|_{L^2}^2 \lesssim \eta_1(t) (1 + \tau)^{-2} \|\tilde{v}(\tau)\|_{L^2}^2.$$

Integrating in $\tau \in [0, s)$ yields

$$\|\tilde{v}(s)\|_{L^2}^2 \lesssim \|v_0\|_{H^1}^2 e^{C\eta_1(t)}$$

for some t -independent constant $C > 0$.

Estimate on $\|v(t)\|_{L^\infty}$. Let $0 \leq s \leq t$. We first note that $\|v(s)^m\|_{L^1} \lesssim \eta_1(t)^2$ for all $2 \leq m \leq n$ due to the Cauchy-Schwarz inequality and (4.1). Using $\|e^{i\partial_x^2 t}\|_{L^1 \rightarrow L^\infty} \lesssim t^{-\frac{1}{2}}$, $t > 0$, and (1.3), we estimate

$$s^{\frac{1}{2}} \|v(s)\|_{L^\infty} \lesssim \|v_0\|_{L^1} + \eta_1(t)^2 s^{\frac{1}{2}} \int_0^s (s-\tau)^{-\frac{1}{2}} (1+\tau)^{-2} d\tau \lesssim \|v_0\|_{L^1} + \eta_1(t)^2.$$

Estimate on $\|v(t)\|_{L^2}$. Let $0 \leq s \leq t$. We first note that $\|v(s)^m\|_{L^2} \lesssim s^{\frac{1}{2}} \eta_1(t)^2$ for all $2 \leq m \leq n$. Using (1.3), we estimate

$$\|v(s)\|_{L^2} \lesssim \|v_0\|_{L^2} + \eta_1(t)^2 \int_0^s (1+\tau)^{-2} \tau^{-\frac{1}{2}} d\tau \lesssim \|v_0\|_{L^2} + \eta_1(t)^2.$$

Taking the supremum in $0 \leq s \leq t$ for every estimate, we have shown the claimed bounds and the theorem follows.

A Zhidkov and modulation spaces

We define Zhidkov and modulation spaces and collect crucial properties.

A.1 Zhidkov spaces

Let $s > -\frac{1}{2}$. We introduce the Zhidkov space

$$X^s = \{u \in L^\infty(\mathbb{R}) : u' \in H^s(\mathbb{R})\}, \quad \|u\|_{X^s} = \|u\|_{L^\infty} + \|u'\|_{H^s}.$$

Proposition A.1 ([6, 7]). *Let $s > -\frac{1}{2}$. Then, X^s is a Banach algebra on which $i\partial_x^2$ generates a C_0 -semigroup. There exists a constant $C > 0$ such that*

$$\|e^{i\partial_x^2 t}\|_{X^s \rightarrow X^s} \leq C(1+t)^{\frac{1}{6}}$$

for all $t \geq 0$.

A.2 Modulation spaces

Fix a non-zero Schwartz function g . Let $p, q \in [1, \infty]$ and $s \in \mathbb{R}$. We introduce the modulation space $M_{p,q}^s$ consisting of all $u \in \mathcal{S}'(\mathbb{R})$ such that

$$\|k \mapsto (1+k^2)^{\frac{s}{2}} \|V_g u(\cdot, k)\|_{L^p}\|_{L^q} < \infty, \quad V_g u(x, k) := \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} u(y) g(y-x) e^{-iky} dy,$$

we equip it with the norm

$$\|u\|_{M_{p,q}^s} := \|k \mapsto (1+k^2)^{\frac{s}{2}} \|V_g u(\cdot, k)\|_{L^p}\|_{L^q}.$$

Proposition A.2 ([5], Lemma 2.2. (iv)). *If $s > 1 - \frac{1}{q}$ or ($s \geq 0$ and $q = 1$), then $M_{p,q}^s$ is a Banach algebra and $M_{p,q}^s \hookrightarrow L^\infty$.*

Proposition A.3. [4, 12], [13, Theorem 1.1] *Let $p \in [1, \infty]$, $q \in [1, \infty)$ and $s \in \mathbb{R}$. We have the following assertions.*

- The operator $i\partial_x^2$ generates a C_0 -semigroup on $M_{p,q}^s$ and there exists some $C > 0$ such that the bound

$$\|e^{i\partial_x^2 t}\|_{M_{p,q}^s \rightarrow M_{p,q}^s} \leq C(1+t)^{|\frac{1}{2}-\frac{1}{p}|}, \quad t \geq 0$$

holds.

- There exists some $C > 0$ such that for all $\lambda > 0$, we have

$$\|f(\lambda \cdot)\|_{M_{p,q}} \leq C\|f\|_{M_{p,q}} \begin{cases} \lambda^{-\frac{2}{p}+1}, & p \geq 2, \lambda \geq 1 \\ 1, & 1 \leq p \leq 2, \lambda \geq 1 \\ \lambda^{-\frac{1}{p}}, & 0 < \lambda \leq 1. \end{cases}$$

(See [13] for analogous lower bounds.)

- The embeddings hold

$$W^{2,p} \hookrightarrow M_{p,1} \hookrightarrow L^p \cap C_b \text{ and } M_{p,1} \hookrightarrow M_{p',1},$$

for every $p' \geq p$.

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