# The junction of an open and a closed waveguide for periodic media

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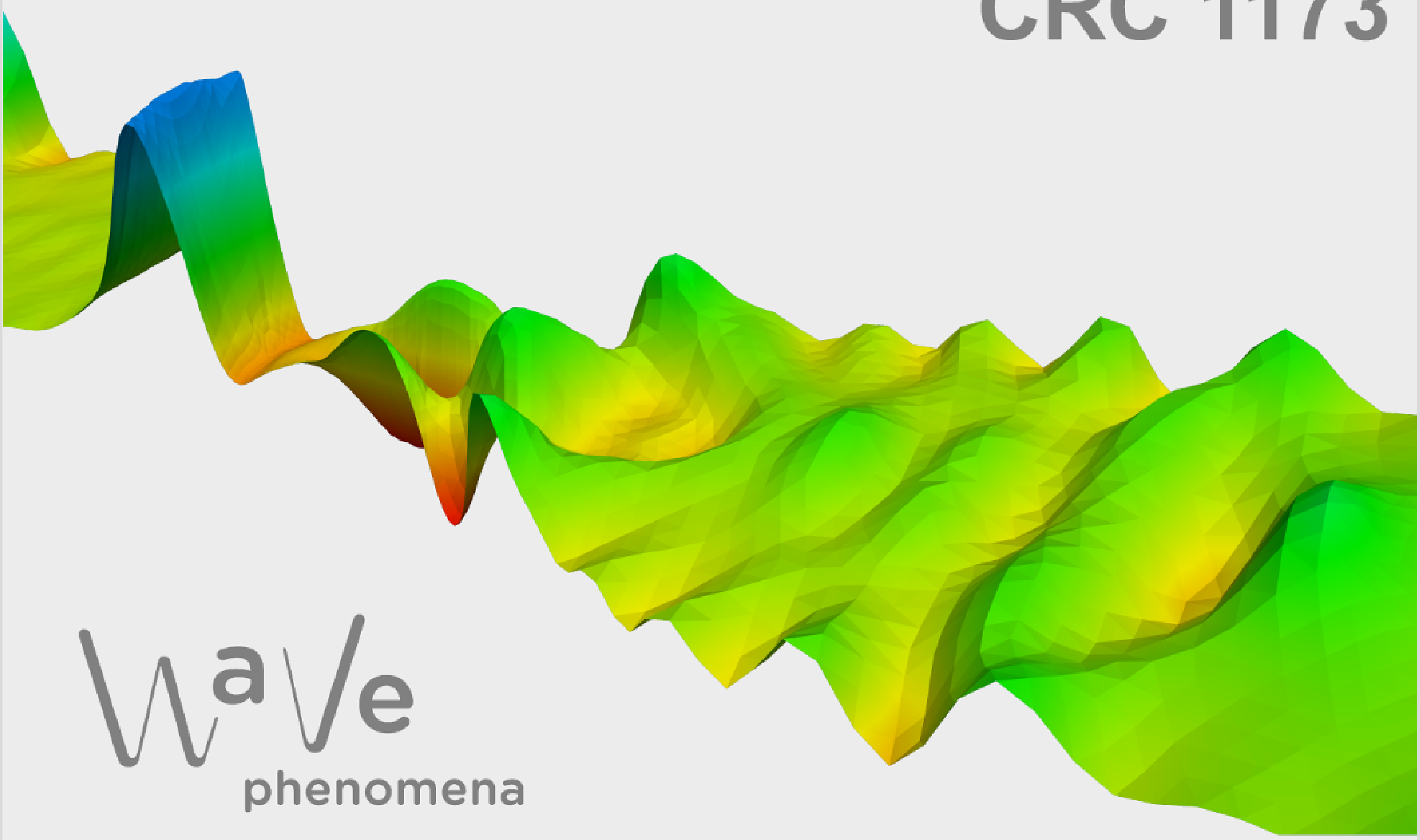

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# THE JUNCTION OF AN OPEN AND A CLOSED WAVEGUIDE FOR PERIODIC MEDIA.

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ABSTRACT. In this paper we investigate the junction of a closed waveguide with an open waveguide where the refractive indices of both waveguides are periodic with respect to the axis of the waveguides. We allow also that the refractive index is locally perturbed. We formulate a proper radiation condition for this problem which follows from a limiting absorption principle and decribes the bahavior of the solution along the axis of the waveguides (which we take to be the $x_1-$axis) and also, for the open waveguide, normal to it. Away from the junction the solution consists of linear combinations of propagating modes travelling to the left or right, respectively, and a radiating parts which decays (exponentially fast in the closed waveguide and of order $\mathcal{O}(1/x_1^{3/2})$ in the open waveguide) along the $x_1-$axis. We show well-posedeness of the problem by introducing Dirichlet-to-Neumann operators and reducing the problem to a bounded region containing the junction of the waveguides. The Fredholm property of the problem is shown. By intruducing the fluxes of this problem along the axis we show (partial) uniqueness.

**MSC:** 35J05

**Key words:** Helmholtz equation, open waveguide, closed waveguide, radiation condition, Dirichlet-to-Neumann operator, variational formulation,

## 1. INTRODUCTION

We first introduce some notations. We set $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$. The waveguide $W$ is given by $W = W^- \cup \hat{\gamma} \cup W^0 \cup \gamma \cup W^+$ where $W^- = (-\infty, \hat{a}) \times (0, h_0)$ denotes the closed semi-waveguide on the left of $\hat{\gamma} = \{\hat{a}\} \times (0, h_0)$ for some $h_0 > 0$ and $\hat{a} < 0$, and $W^+ = (0, \infty) \times (0, \infty)$ denotes the open semi-waveguide on the right hand side of $\gamma = \{0\} \times (0, h_0)$ and $W^0 = (\hat{a}, 0) \times (0, h_0)$ the bounded recatangle in between, see Figure 1. Let $\omega > 0$ be the frequency, $n_\pm \in L^\infty(W^\pm)$ be the refractive indices which are assumed to be the restrictions to $W^\pm$ of functions which are $2\pi-$periodic with respect to $x_1$. Furthermore, let $n_+(x) = 1$ for $x_2 > h_1$ for some $h_1 \geq h_0$. Let $n \in L^\infty(W)$ such that $n(x) \geq n_0$ for some $n_0 > 0$ and

$$n(x) \;=\; \begin{cases} n_-(x) & \text{for } x \in W^- \,, \\ n_+(x) & \text{for } x \in W^+ \,. \end{cases} \tag{1}$$

Without loss of generality we assume that $\hat{a} = -2\pi\hat{\ell}$ for some $\hat{\ell} \in \mathbb{Z}_{\geq 1}$. We also define the bounded rectangle $\Omega^+ = (0, 2\pi) \times (0, h_1)$ as part of $W^+$.

For any open set $U \subset \mathbb{R}^2$ and $m \in \{0, 1, \ldots\}$ the spaces $H^m_{loc}(\overline{U})$ are defined as

$$H^m_{loc}(\overline{U}) := \left\{ u|_U : U \to \mathbb{C} : u \in H^m_{loc}(\mathbb{R}^2) \right\}.$$

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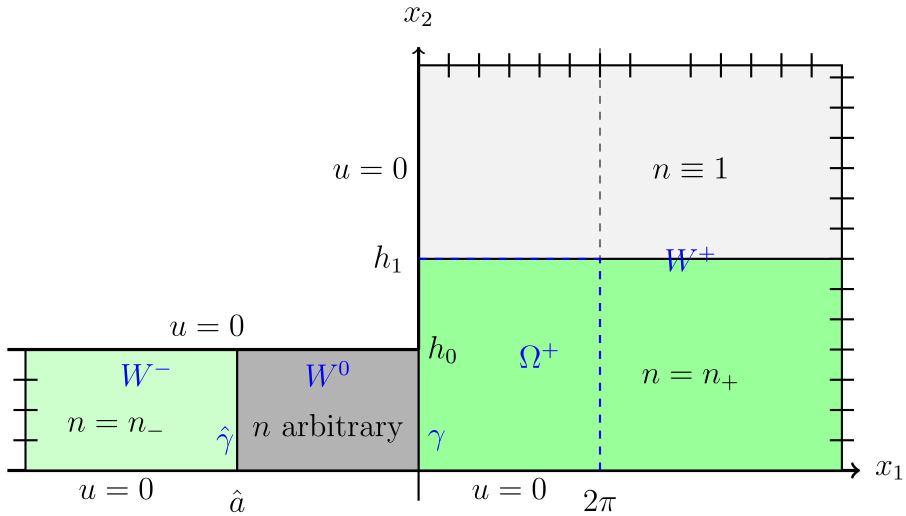


FIGURE 1. The geometry of the waveguide $W = W^- \cup \hat{\gamma} \cup W^0 \cup \gamma \cup W^+$

In particular, the spaces $H^m_{loc}(\overline{W})$ and $H^m_{loc}(\overline{W^\pm})$ are defined.

For some $f^- \in L^2(W^0)$ and $f^+ \in L^2(\Omega^+)$ we set $f = f^-$ in $W^0$ and $f = f^+$ in $\Omega^+$ and $f = 0$ in $W \setminus (W^0 \cup \Omega^+)$ and consider the source problem to determine $u \in H^1_{loc}(\overline{W})$ with

$$\Delta u + \omega^2 n\, u \;=\; -f \text{ in } W, \quad u = 0 \text{ for } x_2 \in \partial W, \tag{2}$$

and a radiation condition described in the following section.

We finish this section by some remarks concerning the literature on this topic. The coupling of closed waveguides of constant or layered (with respect to $x_2$) media has a long history, and it is impossible to list all of the relevant literature. The basic idea is to reduce the problem to a boundary value problem in a bounded domain (the "junction" of the waveguides) with the use of the Dirichlet-to-Neumann maps for the branches with are closed semi-waveguides. We just mention the classical references [6, 8] and the more recent works by [4, 3, 5] and the references therein. There exists much less work for the coupling with open waveguides. The difficulty here is the correct choice of the trace space on the vertical part $\{0\} \times (0, \infty)$ for solutions of the open waveguides. For a layered medium an elegant approach has been suggested in [1]. The authors use a generalized Fourier transform with respect to $x_2$, based on the spectral decomposition of the vertical differential operator $\frac{d^2}{dx_2^2} + n(x_2)$, to construct the Dirichlet-to-Neumann map. We were not able to treat the same problem for periodic media. Instead, our work was inspired by the reference [2] where the same geometry as in Figure 1 was considered, but for local perturbations of constant refractive indices.

The plan of this paper is as follows. In Section 2 we first formulate the correct radiation condition corresponding to solutions of (2). This radiation condition consists of two parts. With respect to $x_1 \to \pm\infty$ it combines the conditions for the closed semi-waveguide $W^-$ and the open semi-waveguide $W^+$. Furthermore, it contains a radiation condition as $x_2$ tends to infinity in the part $W^+$. This radiation condition has been derived (separately for both semi-waveguides) by the limiting absorption principle, see, e.g., [13, 14, 9]. Then, in Theorem 2.7, we state the well-posedness of the problem.

Section 3 is devoted to the proof of Theorem 2.7. In Subsection 3.1 we recall the results for the closed semi-waveguide $W^-$ (Theorem 3.1) and prove the corresponding result for the open semi-waveguide $W^+$ in Theorem 3.3 under the additional assumption that $n^+$ is an even function with respect to $x_1$. We consider the fluxes (which are used for the proof of uniqueness) and show that are constant outside the supports of the sources. In Subsection 3.2 we finish the proof of Theorem 2.7.
Finally, in Section 4 we add some remarks concerning possible extensions.

## 2. Formulation of the Radiation Condition

First we recall the definitions of cut-off values and propagative wave numbers for periodic open or closed waveguides. We recall that, for some $\alpha \in \mathbb{R}$, a function $\psi : \mathbb{R} \to \mathbb{C}$ is called $\alpha-$quasi-periodic if $\psi(x+2\pi) = e^{i2\pi\alpha}\psi(x)$ for all $x \in \mathbb{R}$. Obviously, $\psi$ is $\alpha-$quasi-periodic if and only if the function $x \mapsto e^{-i\alpha x}\psi(x)$ is $2\pi-$periodic.

**Definition 2.1.** *(a) $\alpha \in \mathbb{R}$ is called a cut-off value if there exists $\ell \in \mathbb{Z}$ with $|\alpha+\ell| = \omega$.*
*(b) $\alpha \in \mathbb{R}$ is called a propagative wave number for the refractive index $n_+$ and frequency $\omega$ if there exists a non-trivial solution $\phi \in H^1_{loc}(\overline{\mathbb{R}^2_+})$ of $\Delta\phi+\omega^2 n_+\phi = 0$ in $\mathbb{R}^2_+$ with $\phi = 0$ for $x_2 = 0$ which is $\alpha-$quasi-periodic with respect to $x_1$ and satisfies the Rayleigh expansion*

$$\phi(x) \;=\; \sum_{\ell\in\mathbb{Z}} \phi_\ell \, e^{i\sqrt{\omega^2-(\ell+\alpha)^2}(x_2-h_0)+i(\ell+\alpha)x_1}\,, \quad x_2 > h_0\,, \tag{3}$$

*for some $\phi_\ell \in \mathbb{C}$ where the convergence is uniform for $x_2 \geq h$ for all $h > h_0$.*
*(c) $\alpha \in \mathbb{R}$ is called a propagative wave number for the refractive index $n_-$ and frequency $\omega$ if there exists a non-trivial solution $\phi \in H^1_{loc}(\mathbb{R}\times[0,h_0])$ of $\Delta\phi+\omega^2 n_-\phi = 0$ in $\mathbb{R}\times(0,h_0)$ with $\phi = 0$ for $x_2 = 0$ and $x_2 = h_0$ which is $\alpha-$quasi-periodic with respect to $x_1$.*
*The sets of propagative wave numbers $\alpha \in \mathbb{R}$ for $n_\pm$ are denoted by $\mathcal{A}^\pm$. For $\alpha \in \mathcal{A}^\pm$ we denote the space of (Floquet or Bloch) modes $\phi$ by $\mathcal{M}^\pm(\alpha)$.*

The sets $\mathcal{A}^\pm$ depend on the frequency $\omega$ and on the refractive indices $n_\pm$, of course, but we do not indicate this dependence.
We note that $\alpha$ is a propagative wave number or a cut-off value if, and only if, $\alpha+\ell$ is a propagative wave number or a cut-off value, respectively, for every $\ell \in \mathbb{Z}$.
From the definition we observe directly that the set of all cut-off values is given by $\{+\omega,-\omega\}+\mathbb{Z} = \{\sigma\omega+\ell : \sigma \in \{+1,-1\},\ \ell \in \mathbb{Z}\}$. It is obvious that $\alpha$ is a propagative wave number with mode $\phi$ if, and only if, $-\alpha$ is a propagative wave number with mode $\bar\phi$. Finally, we note that $\mathcal{M}^\pm(\alpha) = \mathcal{M}^\pm(\alpha+\ell)$ for all $\ell \in \mathbb{Z}$.

The following hermetian sesqui-linear forms play an important role.

$$\mathcal{E}^\pm(u,v) \;:=\; i\int_{Q^\pm}\bigl[u\,\partial_{x_1}\bar v - \bar v\,\partial_{x_1}u\bigr]\,dx \quad \text{for } u,v \in H^1(Q^\pm) \tag{4}$$

where $Q^+ = (0,2\pi)\times(0,\infty)$ and $Q^- = (0,2\pi)\times(0,h_0)$. We make the following assumptions.

**Assumption 2.2.** *(a) Let $\pm\omega$ not be propagative wave numbers for index $n_+$, i.e. $\pm\omega \notin \mathcal{A}^+$.*

*(b) For every $\alpha \in \mathcal{A}^\pm$ and every $v \in \mathcal{M}^\pm(\alpha)$ with $v \neq 0$ let the linear form $\mathcal{E}^\pm(\cdot, v)$ be not trivial on $\mathcal{M}^\pm(\alpha)$.*

Frequencies $\omega$ for which assumption (b) does not hold are sometimes called thresholds (or cut-off values, see [5]), but we do not use these notations in the following. Then we have

**Lemma 2.3.** *Let Assumption 2.2 hold.*

(a) *Let $\alpha \in \mathcal{A}^+$ and $\phi \in \mathcal{M}^+(\alpha)$. Then the coefficients $\phi_\ell$ in the Rayleigh expansion (3) vanish for all $\ell \in \mathbb{Z}$ with $|\ell + \alpha| < \omega$. The corresponding modes $\phi$ are evanescent, i.e. for every $h > h_0$ there exist $c, \sigma > 0$ with $|\phi(x)| \le c\, e^{-\sigma x_2}$ for $x_2 \ge h$.*

(b) *The mode spaces $\mathcal{M}^\pm(\alpha)$ are finite dimensional.*

(c) *Let $\Gamma_b^+ := \{b\} \times (0, \infty)$ for $b \in \mathbb{R}$ and let $u \in \mathcal{M}^+(\alpha)$ and $v \in \mathcal{M}^+(\beta)$ for $\alpha, \beta \in \mathcal{A}^+$ and at least one of them is evanescent. Then $\mathcal{E}^+(u,v)$ exists and*

$$\mathcal{E}^+(u,v) \;=\; 2\pi i \int_{\Gamma_b^+} \left( u\, \partial_{x_1} \bar{v} - \bar{v}\, \partial_{x_1} u \right) ds \quad \text{for any } b \in \mathbb{R}\,. \tag{5}$$

*Furthermore, $\mathcal{E}^+(u,v) = 0$ if $\alpha - \beta \notin \mathbb{Z}$.*

(d) *Let $\Gamma_b^- = \{b\} \times (0, h_0)$ for $b \in \mathbb{R}$ and let $u \in \mathcal{M}^-(\alpha)$ and $v \in \mathcal{M}^-(\beta)$ for $\alpha, \beta \in \mathcal{A}^-$. Then*

$$\mathcal{E}^-(u,v) \;=\; 2\pi i \int_{\Gamma_b^-} \left( u\, \partial_{x_1} \bar{v} - \bar{v}\, \partial_{x_1} u \right) ds \quad \text{for any } b \in \mathbb{R}\,. \tag{6}$$

*Furthermore, $\mathcal{E}^-(u,v) = 0$ if $\alpha - \beta \notin \mathbb{Z}$.*

(e) *The sets $\mathcal{A}^\pm \cap (-1/2, 1/2]$ are finite.*

Because of (e) we can number the propagative wave numbers in $(-1/2, 1/2]$ by $\hat{\alpha}_j^\pm$, $j \in J^\pm$, where $J^\pm \subset \mathbb{Z}$ are finite sets, i.e.

$$\mathcal{A}^\pm \cap (-1/2, 1/2] \;=\; \{\hat{\alpha}_j^\pm : j \in J^\pm\}\,.$$

With the forms $\mathcal{E}^\pm$ we construct a basis of the mode spaces $\mathcal{M}^\pm(\hat{\alpha}_j^\pm)$ for every $j \in J^\pm$. Let $j \in J^\pm$ and let $\mathcal{M}^\pm(\hat{\alpha}_j^\pm) \neq \{0\}$. With the hermetian sesqui-linear form $\mathcal{E}^\pm$ from (4) and the inner product $\langle u, v\rangle_\pm = \omega^2 \int_{Q^\pm} n_\pm\, u\, \bar{v}\, dx$ in $\mathcal{M}^\pm(\hat{\alpha}_j^\pm)$ we consider the self-adjoint eigenvalue problem to determine $\lambda_{\ell,j}^\pm \in \mathbb{R}$ and non-trivial $\phi_{\ell,j}^\pm \in \mathcal{M}(\hat{\alpha}_j^\pm)$, $\ell = 1, \ldots, m_j^\pm := \dim \mathcal{M}^\pm(\hat{\alpha}_j^\pm)$, with

$$\mathcal{E}^\pm(\phi_{\ell,j}^\pm, \psi) \;=\; \lambda_{\ell,j}^\pm\, \omega^2 \int_{Q^\pm} n_\pm\, \phi_{\ell,j}^\pm\, \bar{\psi}\, dx \quad \text{for all } \psi \in \mathcal{M}^\pm(\hat{\alpha}_j^\pm)\,. \tag{7}$$

The eigenfunctions $\phi_{\ell,j}^\pm$ are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_\pm$. We normalize the eigenfunctions such that $\langle \phi_{\ell,j}^\pm, \phi_{\ell',j}^\pm \rangle_\pm = \delta_{\ell,\ell'}$. Then $\{\phi_{\ell,j}^\pm : \ell = 1, \ldots, m_j^\pm\}$ is an orthonormal basis of $\mathcal{M}(\hat{\alpha}_j)$ for every $j \in J$. Furthermore, $\lambda_{\ell,j}^\pm \neq 0$ because of Assumption 2.2 and, by part (d) of Lemma 2.3,

$$\mathcal{E}^\pm(\phi_{\ell,j}^\pm, \phi_{\ell',j'}^\pm) \;=\; \delta_{\ell,\ell'}\, \delta_{j,j'}\, \lambda_{\ell,j}^\pm \quad \text{for all } \ell = 1, \ldots, m_j^\pm \text{ and } \ell' = 1, \ldots, m_{j'}^\pm\,.$$

We have shown in [10] that the Limiting Absorption Principle leads to this basis of $\mathcal{M}^\pm(\hat{\alpha}_j)$.

Now we can formulate the waveguide radiation condition.

**Definition 2.4.** *Let $\psi_+ \in C^\infty(\mathbb{R})$ satisfy $\psi_+(x_1) = 1$ for $x_1 \geq R$ and $\psi_+(x_1) = 0$ for $x_1 \leq R-1$ for some $R > 1$. Define $\psi_-$ by $\psi_-(x_1) = \psi_+(-x_1)$. Then $\psi_-(x_1) = 1$ for $x_1 \leq -R$ and $\psi_-(x_1) = 0$ for $x_1 \geq -R+1$.*
*Let $W_h := W \cap \big(\mathbb{R} \times (0,h)\big)$ for any $h \geq h_0$ and let again $f \in L^2(W)$ with support in $W^0 \cup \Omega^+$.*
*A solution $u \in H^1_{loc}(\overline{W})$ of $\Delta u + \omega^2 n u = -f$ in $W$ with $u = 0$ on $\partial W$ satisfies the waveguide radiation condition if $u$ has a decomposition in the form $u = u_{rad} + u_{prop}$ such that*

(a) *$u_{rad} \in H^1(W_h)$ for all $h > h_0$ and*

$$u_{prop}(x) \;=\; \psi_+(x_1) \sum_{j \in J^+} \sum_{\ell : \lambda^+_{\ell,j} > 0} a^+_{\ell,j}\, \phi^+_{\ell,j}(x) \;+\; \psi_-(x_1) \sum_{j \in J^-} \sum_{\ell : \lambda^-_{\ell,j} < 0} a^-_{\ell,j}\, \phi^-_{\ell,j}(x) \tag{8}$$

*for $x \in W$ and some $a^\pm_{\ell,j} \in \mathbb{C}$.*

(b) *Let*

$$(\mathcal{F} u_{rad})(\xi, x_2) \;:=\; -i \sqrt{\frac{2}{\pi}} \int_0^\infty u_{rad}(x_1, x_2)\, \sin(x_1 \xi)\, dx_1\,, \qquad \xi \in \mathbb{R}\,,\; x_2 > h_0\,,$$

*be the Fourier transform of the odd extension of $u_{rad}(x)$ (or the sine transform of $u_{rad}$ if one drops the factor $-i$, see (19) below) for $x_2 > h_0$.*[1] *Then $u_{rad}$ satisfies the generalized angular spectrum radiation condition*

$$\lim_{x_2 \to \infty} \int_0^\infty \left| \partial_{x_2} (\mathcal{F} u_{rad})(\xi, x_2) - i \sqrt{\omega^2 - \xi^2}\, (\mathcal{F} u_{rad})(\xi, x_2) \right|^2 d\xi \;=\; 0\,. \tag{9}$$

**Remark 2.5.**
- *We note that the choice of $\psi_\pm$ has no influence of the form of $a^\pm_{\ell,j}$ but only on $u_{rad}$.*
- *We observe that*

$$u_{prop}(x) \;=\; \sum_{j \in J^+} \sum_{\lambda^+_{\ell,j} > 0} a^+_{\ell,j}\, \phi^+_{\ell,j}(x) \quad \text{for } x_1 \geq R,\; x_2 > 0 \quad \text{and}$$

$$u_{prop}(x) \;=\; \sum_{j \in J^-} \sum_{\lambda^-_{\ell,j} < 0} a^-_{\ell,j}\, \phi^-_{\ell,j}(x) \quad \text{for } x_1 \leq -R,\; 0 < x_2 < h_0\,.$$

It is the aim to prove well-posedness of the problem under the following additional assumptions.

**Assumption 2.6.** *(a) Let $n_+(\cdot, x_2)$ be an even function (i.e. $n_+(-x_1, x_2) = n_+(x_1, x_2)$ for almost all $x \in \mathbb{R}^2_+ := \mathbb{R} \times (0,\infty)$).*
*(b) Let $\omega^2$ not be an eigenvalue of $-\frac{1}{n}\Delta$ in $H^1_0(W)$.*
*(c) Let $\omega^2$ not be an eigenvalue of $-\frac{1}{n_-}\Delta$ for both closed semi-waveguides $(-\infty, 0) \times (0, h_0)$ and for $(0,\infty) \times (0, h_0)$ with respect to homogeneous Dirichlet boundary condition.*

[1] It exists as a $L^2$-Function because $u_{rad} \in H^1(W_h)$.

Assumption (a) is a technical assumption needed for our proof. We believe that it is not necessary for the result to hold. Part (b) is a standard assumption and excludes resonances $\omega$ for the problem. Part (c) of Assumption 2.6 is also a technical assumption and is always satisfied if $n_-$ is an even function with respect to $x_1$. (This follows from part (b) of Assumption 2.2 and the finiteness of $\mathcal{A}_-$ by Lemma 2.3.) This assumption implies that $\omega$ is not a resonance for both closed semi-waveguides $W^- = (-\infty, \hat{a}) \times (0, h_0)$ and $(\hat{a}, \infty) \times (0, h_0)$ with respect to $n_-$. Indeed if $u \in H_0^1(W^-)$ satisfies $\Delta u + \omega^2 n_- u = 0$ in $W^-$ then $v(x_1, x_2) = u(x_1 - 2\pi\hat{\ell}, x_2)$ satisfies $v \in H_0^1\big((-\infty, 0) \times (0, h_0)\big)$ and $\Delta v + \omega^2 n_- v = 0$ in $(-\infty, 0) \times (0, h_0)$ because $\hat{a} = -2\pi\hat{\ell}$. The same argument holds for solutions $u \in H_0^1\big((\hat{a}, \infty) \times (0, h_0)\big)$.

It is the aim to prove the following main result of this paper.

**Theorem 2.7.** *Let Assumptions 2.2 and 2.6 hold. Then, for any $f \in L^2(W)$ with support in $W^0 \cup \Omega^+$ there exists a unique solution $u \in H^1_{loc}(\overline{W})$ with $\Delta u + \omega^2 n u = -f$ in $W$, the boundary condition $u = 0$ on $\partial W$, and the waveguide radiation condition of Definition 2.4 in $W$. Furthermore, the mapping $f \mapsto u|_V$ is bounded from $L^2(W^0 \cup \Omega^+)$ into $H^2(V)$ for all bounded domains $V \subset W$.*

We will prove this theorem in Section 4.

## 3. The Semi–Waveguide Problems and the Flux

We define the Sobolev space

$$H_0^{1/2}(0, h_0) \;:=\; \big\{\phi|_{(0,h_0)} : \phi \in H^{1/2}(\mathbb{R})\,,\ \phi(t) = 0 \text{ for } t \notin (0, h_0)\big\}$$

and identify $H_0^{1/2}(\gamma)$ and $H_0^{1/2}(\hat{\gamma})$ with $H_0^{1/2}(0, h_0)$ where $\hat{\gamma} = \{\hat{a}\} \times (0, h_0)$ and again $\gamma = \{0\} \times (0, h_0)$.

We consider first the closed semi–waveguide $W^- = (-\infty, \hat{a}) \times (0, h_0)$. The following theorem has been proven in, e.g. [7, 14, 5].

**Theorem 3.1.** *Let Assumption 2.2 and part (c) of Assumption 2.6 hold. For any $\phi \in H_0^{1/2}(\hat{\gamma})$ there exists a unique solution $u^- \in H^1_{loc}(\overline{W^-})$ of $\Delta u^- + \omega^2 n_- u^- = 0$ in $W^-$ and $u^- = 0$ on $\partial W \cap \overline{W^-}$ and $u^- = \phi$ on $\hat{\gamma}$ which has a decomposition into $u^- = u^-_{rad} + u^-_{prop}$ where $u^-_{rad} \in H^1(W^-)$ and $u^-_{prop}$ has the form*

$$u^-_{prop}(x) \;=\; \sum_{j \in J^-} \sum_{\lambda^-_{\ell,j} < 0} a^-_{\ell,j}\, \phi^-_{\ell,j}(x)\,,\ x_1 < \hat{a}\,, \tag{10}$$

*for some $a^-_{\ell,j} \in \mathbb{C}$. Furthermore, for every bounded domain $V \subset W^-$ the mapping $\phi \mapsto u^-|_V$ is bounded from $H_0^{1/2}(\hat{\gamma})$ into $H^1(V)$.*

We turn to the open semi-waveguide $W^+$ and want to prove an analogous theorem. First we prove an auxiliary result.

**Lemma 3.2.** *For any $\phi \in H_0^{1/2}(\gamma)$ there exists a unique solution $w^+ \in H^1(W^+)$ of $\Delta w^+ - w^+ = 0$ in $W^+$ and $w^+ = 0$ on $\partial W \cap \overline{W^+}$ and $w^+ = \phi$ on $\gamma$. Furthermore, the mapping $\phi \mapsto w^+$ is bounded from $H_0^{1/2}(\gamma)$ into $H^1(W^+)$.*

*Proof.* This follows easily from the coercivity of $(u, \psi) \mapsto \int_{W^+} \big[\nabla u \cdot \nabla \overline{\psi} + u\, \overline{\psi}\big]\, dx$. □

Under Assumption 2.6 the function $x_1 \mapsto n_+(x_1,x_2)$ is even. Then we note that, if $\phi^+_{\ell,j}$ is right going, i.e. $\lambda^+_{\ell,j} = \mathcal{E}^+(\phi^+_{\ell,j}.\phi^+_{\ell,j}) > 0$, then $\tilde{\phi}^+_{\ell,j}(x) = \phi^+_{\ell,j}(-x_1,x_2)$ is also a solution of the differential equation which is left going because $\mathcal{E}^+(\tilde{\phi}^+_{\ell,j},\tilde{\phi}^+_{\ell,j}) = -\mathcal{E}^+(\phi^+_{\ell,j},\phi^+_{\ell,j}) < 0$. Furthermore, $\phi^+_{\ell,j}$ and $\tilde{\phi}^+_{\ell,j}$ do not coincide because otherwise $\lambda^+_{\ell,j} = \mathcal{E}^+(\phi^+_{\ell,j}.\phi^+_{\ell,j}) = 0$ which contradicts Assumption 2.2. Therefore, there are exactly the same number of right going modes as there are left going modes (corresponding to the same $\hat{\alpha}^+_j$), i.e. $m^+_j$ is an even integer. Therefore, it is convenient to renumber the modes such that $\{\phi^+_{\ell,j} : \ell = 1,\ldots,m^+_j/2\}$ are right going and $\{\phi^+_{\ell,j} : \ell = -m^+_j/2,\ldots,-1\}$ are left going and $\phi^+_{\ell,j}(-x_1,x_2) = \phi^+_{-\ell,j}(x_1,x_2)$ and $\lambda^+_{-\ell,j} = -\lambda^+_{\ell,j}$ for all $\ell$ and $j$.

**Theorem 3.3.** *Let Assumption 2.2 hold and let $n_+$ be an even function with respect to $x_1$. For any $\phi \in H_0^{1/2}(\hat{\gamma})$ and $f^+ \in L^2(\Omega^+)$ there exists a unique solution $u^+ \in H^1_{loc}(\overline{W^+})$ of $\Delta u^+ + \omega^2 n_+ u^+ = -f^+$ in $W^+$ and $u^+ = 0$ on $\partial W \cap \overline{W^+}$ and $u^+ = \phi$ on $\gamma$ which has a decomposition into $u^+ = u^+_{rad} + u^+_{prop}$ where $u^+_{rad} \in H^1(W_h \cap W^+)$ for all $h > h_0$*[2] *and $u^+_{prop}$ has the form*

$$u^+_{prop}(x) \;=\; \sum_{j\in J^+}\sum_{\ell=1}^{m_j^+/2} a^+_{\ell,j}\,\phi^+_{\ell,j}(x)\,,\ \ x_1>0,\ \ x_2>0\,, \tag{11}$$

*for some $a^+_{\ell,j} \in \mathbb{C}$.*
*Furthermore, for $x_2 > h_0$ the radiating part $u^+_{rad}$ satisfies the generalized angular spectrum radiation condition* (9)*, the Sommerfeld radiation condition in the form*

$$\sup_{x\in A_\tau}\sqrt{|x|}\,|u^+_{rad}(x)| \;+\; \sup_{x\in A_\tau}\sqrt{|x|}\,\big|\partial_r u^+_{rad}(x)\big| \;<\; \infty\,, \tag{12}$$

*for all $\tau > 0$ where $A_\tau = (0,\infty)\times(h_0+\tau,\infty)$, and*

$$\sqrt{r}\,\sup_{x\in C^r_\tau}\big|\partial_r u^+_{rad}(x) - ik\,u^+_{rad}(x)\big| \;\longrightarrow\; 0\,,\quad r\to\infty\,, \tag{13}$$

*for all $\tau > 0$ where $C^r_\tau = \{x\in\mathbb{R}^2 : x_1>0,\ x_2>h_0+\tau,\ |x|=r\}$.*
*Furthermore, $u^+_{rad}$ has the following asymptoptic behavior as $x_1\to\infty$ or $x_2\to\infty$:*

- *Let $Q_{\ell,h} = (2\ell\pi-\pi),2\ell\pi+\pi))\times(0,h)$ for $\ell\in\mathbb{Z}$. For every $h>h_0$ there exists $c(h)>0$ with*

$$\|u^+_{rad}\|_{H^1(Q_{\ell,h})} \;\le\; \frac{c}{|\ell|^{3/2}}\,\|f^+\|_{L^2(\Omega^+)} \quad \text{for all } \ell\in\mathbb{N},\ \ell\ge 1\,. \tag{14}$$

- *There exists $\rho\in L^1(\mathbb{R})$ with*

$$\big|u^+_{rad}(x)\big| \;+\; \big|\nabla u_{rad}(x)\big| \;\le\; x_2\,\rho(x_1) \tag{15}$$

  *for all $x\in W^+$ with $x_2\ge h_0+1$.*
- *For every $b>0$ there exists $c(b)>0$ with*

$$\big|u^+_{rad}(x)\big| \le \frac{c(b)}{\sqrt{x_2}} \quad \text{and} \quad \big|\partial_{x_1}u^+_{rad}(x)\big| \le \frac{c(b)}{x_2} \quad \text{for all } x_1\in(0,b) \text{ and } x_2\ge h_0+1\,. \tag{16}$$

*Finally, for every bounded domain $V\subset W^+$ the mapping $(\phi,f^+)\mapsto u^+|_V$ is bounded from $H_0^{1/2}(\gamma)\times L^2(\Omega^+)$ into $H^1(V)$.*

[2]Recall that $W_h\cap W^+ = (0,\infty)\times(0,h)$

*Proof.* For given $\phi \in H_0^{1/2}(\gamma)$ let $w^+ \in H^1(W^+)$ be the solution of $\Delta w^+ - w^+ = 0$ in $W^+$ and $w^+ = \phi$ on $\gamma$ as in Lemma 3.2. Let $\eta \in C^\infty(\mathbb{R}^2)$ such that $\eta(x) = 1$ on $(-1,1) \times (-1, h_0+1)$ and $\eta(x) = 0$ for $x \notin (-2,2) \times (-2, h_0+2)$. We define the function $g \in L^2\big(\mathbb{R} \times (0,\infty)\big)$ by

$$g \;:= (\Delta + \omega^2 n_+)\big(\eta\, w^+\big) \;=\; \eta\, w^+(1+\omega^2 n_+) + 2\,\nabla\eta \cdot \nabla w^+ + w^+\,\Delta\eta \quad \text{for } x_1 > 0$$

and $g(x_1,x_2) = -g(-x_1,x_2)$ for $x_1 < 0$. Then $g(\cdot,x_2)$ is an odd function. Analogously, we define $\tilde f$ by $\tilde f(x) = f^+(x)$ for $x_1 > 0$ and $\tilde f(x_1,x_2) = -f^+(-x_1,x_2)$ for $x_1 < 0$. Then also $\tilde f(\cdot,x_2)$ is odd.

Now we consider the open waveguide problem in $\mathbb{R}^2_+ = \mathbb{R} \times (0,\infty)$ with refractive index $n_+$ and source $\tilde f + g$. Since $\tilde f$ and $g$ have compact supports there exists a unique $v \in H^1_{loc}(\overline{\mathbb{R}^2_+})$ with $\Delta v + \omega^2 n_+ v = -\tilde f - g$ in $\mathbb{R}^2_+$ and $v = 0$ for $x_2 = 0$, satisfying the open waveguide radiation condition (see, e.g, [9]). That is, $v = v_{rad} + v_{prop}$ with $v_{rad} \in H^1\big(\mathbb{R} \times (0,h)\big)$ for all $h > h_0$ and

$$v_{prop}(x) \;=\; \sum_{j\in J^+} \sum_{\ell=1}^{m_j^+/2} a_{\ell,j}\, \phi^+_{\ell,j}(x) \quad \text{for } x_1 \geq 1 \quad \text{and}$$

$$v_{prop}(x) \;=\; \sum_{j\in J^+} \sum_{\ell=-m_j^+/2}^{-1} a_{\ell,j}\, \phi^+_{\ell,j}(x) \quad \text{for } x_1 \leq -1\,,$$

where $a_{\ell,j} \in \mathbb{C}$ are given by

$$a_{\ell,j} \;=\; \frac{2\pi i}{|\lambda^+_{\ell,j}|} \int\limits_{\mathbb{R}^2_+} \big[\tilde f(x) + g(x)\big]\, \overline{\phi^+_{\ell,j}(x)}\, dx\,, \quad \ell = -m_j^+/2, \ldots, m_j^+/2\,,\; j \in J^+\,. \tag{17}$$

Furthermore, $v_{rad}$ satisfies the generalized angular spectrum radiation condition

$$\lim_{x_2\to\infty} \int\limits_{-\infty}^{\infty} \big|\partial_{x_2}(\mathcal{F}v_{rad})(\xi,x_2) - i\sqrt{\omega^2-\xi^2}\,(\mathcal{F}v_{rad})(\xi,x_2)\big|^2 d\xi \;=\; 0 \tag{18}$$

where

$$(\mathcal{F}v_{rad})(\xi,x_2) \;:=\; \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} v_{rad}(x_1,x_2)\, e^{-ix_1\xi}\, dx_1\,, \quad \xi \in \mathbb{R}\,,\; x_2 > h_0\,, \tag{19}$$

denotes the Fourier transform with respect to $x_1$. Finally, $v_{rad}$ satisfies the Sommerfeld radiation condition analogously to (12), (13) and has the asymptotic behavior (14), (15), and (16). (For (12), (13), and (14) we refer to Theorem 6.1 of [9], for (15) to Lemma 2.7 of [9] or Theorem 2.10 of [11]. The first estimate of (16) has been shown in Theorem 6.2 of [9] or in Theorem 2.10 of [11], the second estimate in Theorem 3.2 of [12].)

Now we observe that $v$, $v_{rad}$, and $v_{prop}$ are odd functions with respect to $x_1$ because $n_+$ is even and $g + \tilde f$ is odd with respect to $x_1$. Indeed, the form (17) and the facts that $\tilde f + g$ is odd and $\phi^+_{\ell,j}(-x_1,x_2) = \phi^+_{-\ell,j}(x_1,x_2)$ implies $a_{\ell,j} = -a_{-\ell,j}$ which shows that $v_{prop}(\cdot,x_2)$ is odd. Since $v_{rad}$ solves a source problem with the even index $n^+$ and an odd right hand side also $v_{rad}$ has to be odd because of the uniqueness result.

Therefore, (18) is equivalent to (9) and $v$ vanishes for $x_1 = 0$. The function $u^+ \in H^1_{loc}(\overline{W^+})$, defined as $u^+ := \eta\, w^+ + v$, solves $\Delta u^+ + \omega^2 n_+ u^+ = g + \Delta v + \omega^2 n_+ v = -f^+$ in $W^+$ and $u^+ = 0$ on $\partial W \cap \overline{W^+}$ and $u^+ = \phi$ on $\gamma$, and has the asymptotic behavior (14), (15), and (16). Furthermore, $u^+$ has the decomposition $u^+ = u^+_{prop} + u^+_{rad}$ with $u^+_{prop}$ from (11) and $u^+_{rad} = u^+ - u^+_{prop} = v_{rad} + \eta w^+ + v_{prop} - u^+_{prop}$ in $W^+$. Since $\eta w^+ + v_{prop} - u^+_{prop}$ vanishes for $x_1 > 1$ and decays exponentially to zero as $x_2 \to \infty$ we conclude that also $u^+_{rad} \in H^1(W_h \cap W^+)$ for all $h > h_0$ and satisfies (12)–(16).

Finally, the mapping $\phi \mapsto w^+$ is bounded, thus also $\phi \mapsto g$ from $H^{1/2}_0(\gamma)$ into $L^2\big((-2,2)\times(-2,h_0+2)\big)$. Since, for given bounded domain $V \subset W^+$, the mapping $(g,\tilde f) \mapsto v|_V$ is also bounded we conclude that the mapping $(\phi, f^+) \mapsto u^+|_V$ is bounded from $H^{1/2}_0(\gamma)\times L^2(\Omega^+)$ into $H^1(V)$. □

The following theorem shows that the fluxes are constant and negative on $W^-$ and constant and positive on $W^+ \setminus \Omega^+$.

**Theorem 3.4.** *Let Assumptions 2.2 and 2.6 hold. Let again* $\Gamma^+_b := \{b\}\times(0,\infty)$ *for* $b > 0$ *and* $\Gamma^-_b = \{b\}\times(0,h_0)$ *for* $b < 0$.

(a) *Let* $u^+ \in H^1_{loc}(\overline{W^+})$ *and* $u^- \in H^1_{loc}(\overline{W^-})$ *be the solutions of the boundary value problems described in Theorem 3.3 and Theorem 3.1, respectively, for some* $(\phi, f^+) \in H^{1/2}_0(\gamma)\times L^2(\Omega^+)$ *and* $\phi \in H^{1/2}_0(\hat\gamma)$*, respectively. Then*

$$\mathcal{E}^\pm_b(u^\pm) \;:=\; 2\,\mathrm{Im}\int\limits_{\Gamma^\pm_b} \overline{u^\pm}\,\partial_{x_1} u^\pm\, ds$$

*are well defined for every* $b > 0$ *or* $b < \hat a$*, respectively.*

(b) *The fluxes* $\mathcal{E}^+_b(u^+)$ *and* $\mathcal{E}^-_b(u^-)$ *are constant with respect to* $b > 2\pi$ *and* $b < \hat a$*, respectively, and*

$$(20)\quad \begin{aligned} \mathcal{E}^+_b(u^+) \;&=\; \frac{1}{2\pi}\,\mathcal{E}^+(u^+,u^+) = \frac{1}{2\pi}\,\mathcal{E}^+(u^+_{rad},u^+_{rad}) + \frac{1}{4\pi}\sum_{j\in J^+}\sum_{\ell=1}^{m^+_j/2}\lambda^+_{\ell,j}|a^+_{\ell,j}|^2,\ b > 2\pi\,,\\ \mathcal{E}^-_b(u^-) \;&=\; \frac{1}{2\pi}\,\mathcal{E}^-(u^-,u^-) = \frac{1}{2\pi}\,\mathcal{E}^-(u^-_{rad},u^-_{rad}) + \frac{1}{4\pi}\sum_{j\in J^-}\sum_{\ell=-m^+_j/2}^{-1}\lambda^-_{\ell,j}|a^-_{\ell,j}|^2,\ b < \hat a\,, \end{aligned}$$

*with* $\mathcal{E}^\pm$ *from (4). Furthermore,* $\pm\mathcal{E}^\pm(u^\pm_{rad},u^\pm_{rad}) \geq 0$ *and thus* $\pm\mathcal{E}^\pm(u^\pm,u^\pm) \geq 0$.

We observe from (20) that $u^+_{rad}$, $u^+_{prop}$ and $u^+$ are right-going (their fluxes are non-negative) and $u^-_{rad}$, $u^-_{prop}$ and $u^-$ are left-going (their fluxes are non-positive). This is exactly what one expects.

*Proof.* (a) The existence of $\mathcal{E}^-_b(u^-_{rad})$ and $\mathcal{E}_b(u^-)$ for $b < \hat a$ are obvious by the trace theorem and the boundedness of $\Gamma^-_b$. The existence of $\mathcal{E}^+_b(u^+_{rad})$ for $b > 0$ follows since $\overline{u^+_{rad}(b,x_2)}\,\partial_{x_1}u^+_{rad}(b,x_2) = \mathcal{O}(1/x_2^{3/2})$ by (16). The existence of $\mathcal{E}^+_b(u^+)$ is obvious by the exponential decay of $u^+_{prop}$.

We continue with $\mathcal{E}^+_b$. The case of $\mathcal{E}^-_b$ is treated analogously (replace $H$ by $h_0$ in the application of Green's theorem below). Let $2\pi < a < b$ and $v = u^+$ or $v = u^+_{rad}$ or

$v = u^+_{prop}$. Then $\Delta v + \omega^2 n v = 0$ in $(a,b)\times(0,\infty)$ because $f^+$ vanishes in this region. Application of Green's theorem in the rectangle $(a,b)\times(0,H)$ for some $H > h_0$ yields

$$\int_0^H \big[\bar v\,\partial_{x_1} v - v\,\partial_{x_1}\bar v\big]_{x_1=a}\,dx_2 \;-\; \int_0^H \big[\bar v\,\partial_{x_1} v - v\,\partial_{x_1}\bar v\big]_{x_1=b}\,dx_2 \;=\; \int_a^b \big[\bar v\,\partial_{x_2} v - v\,\partial_{x_2}\bar v\big]_{x_2=H}\,dx_1\,.$$

Therefore $\mathcal{E}^+_a(v) = \mathcal{E}^+_b(v)$ follows for all choices of $v$ since the integral on the segment $(a,b)\times\{H\}$ tends to zero as $H$ tends to infinity. From (5) we conclude that $\mathcal{E}^+_b(v) = \frac{1}{2\pi}\mathcal{E}^+(v,v)$.

(b) We restrict ourselves again to $\mathcal{E}^+_b(u^+)$ for $b > 2\pi$. By the decomposition $u^+ = u^+_{rad} + u^+_{prop}$ we obtain $\mathcal{E}^+_b(u) = \mathcal{E}^+_b(u_{rad}) + \mathcal{E}^+_b(u_{prop}) + 2\,\mathrm{Im}\int_{\Gamma^+_b}[\overline{u^+_{rad}}\,\partial_{x_1}u^+_{prop} + \overline{u^+_{prop}}\,\partial_{x_1}u^+_{rad}]ds$. We show that the integral vanishes. For $R > b$ and $H > h_0$ we apply Green's theorem in $(b,R)\times(0,H)$ and obtain

$$\mathrm{Im}\int_0^H [\overline{u^+_{rad}}\,\partial_{x_1}u^+_{prop} + \overline{u^+_{prop}}\,\partial_{x_1}u^+_{rad}]_{x_1=b}\,dx_2 = \mathrm{Im}\int_0^H [\overline{u^+_{rad}}\,\partial_{x_1}u^+_{prop} - u^+_{prop}\,\partial_{x_1}\overline{u^+_{rad}}]_{x_1=b}\,dx_2$$

$$= \mathrm{Im}\int_0^H [\overline{u^+_{rad}}\,\partial_{x_1}u^+_{prop} - u^+_{prop}\,\partial_{x_1}\overline{u^+_{rad}}]_{x_1=R}\,dx_2 + \mathrm{Im}\int_b^R [\overline{u^+_{rad}}\,\partial_{x_2}u^+_{prop} - u^+_{prop}\,\partial_{x_2}\overline{u^+_{rad}}]_{x_2=H}\,dx_1$$

We keep $H$ fixed and let $R$ tend to infinity. Then the first integral on the right hand side tends to zero and thus

$$(21)\qquad \begin{aligned} &\left|\mathrm{Im}\int_0^H [\overline{u^+_{rad}}\,\partial_{x_1}u^+_{prop} + \overline{u^+_{prop}}\,\partial_{x_1}u^+_{rad}]_{x_1=b}\,dx_2\right| \\ =\;& \left|\mathrm{Im}\int_b^\infty [\overline{u^+_{rad}}\,\partial_{x_2}u^+_{prop} - u^+_{prop}\,\partial_{x_2}\overline{u^+_{rad}}]_{x_2=H}\,dx_1\right| \\ \le\;& c\left[\|u^+_{rad}(\cdot,H)\|_{L^1(\mathbb{R})} + \|\partial_{x_2}u^+_{rad}(\cdot,H)\|_{L^1(\mathbb{R})}\right] e^{-\sigma H}\end{aligned}$$

where we used the estimates $|u^+_{prop}(x_1,H)| + |\partial_{x_2}u^+_{prop}(x_1,H)| \le c\,e^{-\sigma H}$. From (15) we obtain the estimate $\|u^+_{rad}(\cdot,H)\|_{L^1(\mathbb{R})} + \|\partial_{x_2}u^+_{rad}(\cdot,H)\|_{L^1(\mathbb{R})} \le c\,H$. Therefore, the integral (21) is estimated by $cHe^{-\sigma H}$ which tends to zero as $H$ tends to infinity. This shows $\mathcal{E}^+_b(u^+) = \mathcal{E}^+_b(u^+_{rad}) + \mathcal{E}^+_b(u^+_{prop})$.

Next we compute $\mathcal{E}^+_b(u^+_{prop})$ for $b > 2\pi$. From the form of $u^+_{prop}$ for $x_1 > 2\pi$, and the orthogonaliy of $\{\phi^+_{\ell,j}\}$ with respect to $\mathcal{E}^+$ we obtain

$$\begin{aligned}\mathcal{E}^+_b(u^+_{prop}) \;&=\; \frac{1}{2\pi}\,\mathcal{E}^+(u^+_{prop},u^+_{prop}) \;=\; \frac{1}{2\pi}\sum_{j\in J^+}\sum_{\ell=1}^{m^+_j/2} |a^+_{\ell,j}|^2\,\mathcal{E}^+(\phi^+_{\ell,j},\phi^+_{\ell,j}) \\ &=\; \frac{1}{2\pi}\sum_{j\in J^+}\sum_{\ell=1}^{m^+_j/2} |a^+_{\ell,j}|^2\,\lambda^+_{\ell,j} \;\ge\; 0\,.\end{aligned}$$

Finally, we show that $\mathcal{E}_b^+(u_{rad}^+) \geq 0$ for $b > 2\pi$. For $H > h_0 + 1$ we apply Green's theorem in the region

$$D_H \;:=\; \left\{x \in \mathbb{R}^2_+ : |x| < H,\; b < x_1 < R(H)\right\} \quad \text{where} \quad R(H) \;:=\; \sqrt{H^2 - (h_0+1)^2}$$

(see Figure 2) and obtain

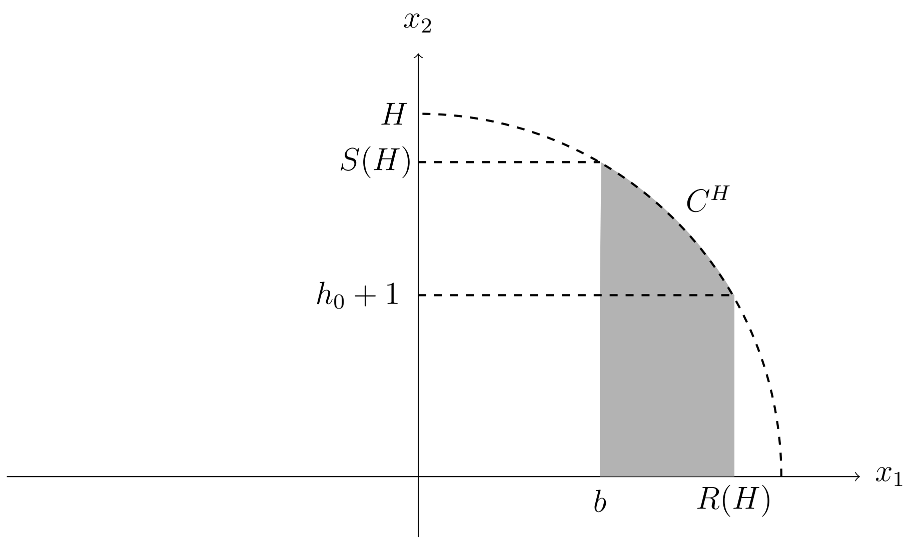


FIGURE 2. The region $D_H$

$$\operatorname{Im} \int_0^{S(H)} \left[\overline{u_{rad}^+}\, \partial_{x_1} u_{rad}^+\right]_{x_1=b} dx_2 \;=\; \operatorname{Im} \int_{C^H} \overline{u_{rad}^+}\, \partial_\nu u_{rad}^+\, ds \;+\; \operatorname{Im} \int_0^{h_0+1} \left[\overline{u_{rad}^+}\, \partial_{x_1} u_{rad}^+\right]_{x_1=R(H)} dx_2$$

where $S(H) = \sqrt{H^2 - b^2}$ and $C^H := \{x \in \mathbb{R}^2_+ : |x| = H,\; b < x_1 < R(H)\}$. The last integral tends to zero as $H \to \infty$ by the decay of $u_{rad}^+$ and $\partial_{x_1} u_{rad}^+$ as $x_1 \to \infty$. We write the first integral on the right hand side as

$$\operatorname{Im} \int_{C^H} \overline{u_{rad}^+}\, \partial_\nu u_{rad}^+\, ds \;=\; \operatorname{Im} \int_{C^H} \overline{u_{rad}^+}\, [\partial_\nu u_{rad}^+ - iku_{rad}^+]\, ds \;+\; k \int_{C^H} |u_{rad}^+|^2\, ds\,.$$

The first term on the right hand side converges to zero by the Sommerfeld radiation conditions (12), (13). The second term is non-negative.

The case of $\mathcal{E}_b^-(u^-)$ is treated analogously. The proof is simpler because $x_2$ is restricted to the bounded interval $(0, h_0)$. $\square$

## 4. Reduction to a Bounded Domain and Proof of Theorem 2.7

By Theorems 3.1 and 3.3 we can define the Dirichlet-to-Neumann operators $\Lambda^- : H_0^{1/2}(\hat\gamma) \to H^{-1/2}(\hat\gamma)$ and $\Lambda^+ : H_0^{1/2}(\gamma) \times L^2(\Omega^+) \to H^{-1/2}(\gamma)$ by $\phi \mapsto \partial_{x_1} u^-|_{\hat\gamma}$ and $(\phi, f^+) \mapsto \partial_{x_1} u^+|_\gamma$, respectively. We consider the following variational equation in the region $W^0 = (\hat a, 0) \times$

$(0,h_0)$ to determine $u \in X$ with

$$
(22)\quad
\begin{aligned}
\int\limits_{W^0} \big[\nabla u\cdot\nabla\bar\psi - \omega^2 n\,u\,\bar\psi\big]\,dx \;&=\int\limits_{\gamma}\Lambda^+(u,f^+)\,\bar\psi\,ds \;-\int\limits_{\hat\gamma}\Lambda^- u\,\bar\psi\,ds \;+\int\limits_{W^0} f^-\,\bar\psi\,dx\\
&=\int\limits_{\gamma}\Lambda^+(u,0)\,\bar\psi\,ds \;-\int\limits_{\hat\gamma}\Lambda^- u\,\bar\psi\,ds \;+\;\int\limits_{\gamma}\Lambda^+(0,f^+)\,\bar\psi\,ds \;+\int\limits_{W^0} f^-\,\bar\psi\,dx
\end{aligned}
$$

for all $\psi \in X$ where $X = \big\{u \in H^1(W^0) : u = 0 \text{ on } \partial W \cap \overline{W^0}\big\} = \big\{u \in H^1(W^0) : u = 0 \text{ for } x_2 \in \{0,h_0\}\big\}$.

**Lemma 4.1.** *Let Assumptions 2.2 and 2.6 hold. Then $\Lambda^- : H_0^{1/2}(\hat\gamma) \to H^{-1/2}(\hat\gamma)$ and $\Lambda^+(\cdot,0) : H_0^{1/2}(\gamma) \to H^{-1/2}(\gamma)$ have decompositions into $\Lambda^- = \hat\Lambda^- + C^-$ and $\Lambda^+(\cdot,0) = \hat\Lambda^+ + C^+$ where $C^\pm$ are compact and $\mp\hat\Lambda^\pm$ are coercive, i.e. $\int_{\hat\gamma}(\hat\Lambda^-\,\phi)\,\bar\phi\,ds \geq c\|\phi\|^2_{H^{1/2}(\hat\gamma)}$ for all $\phi \in H_0^{1/2}(\hat\gamma)$ and $-\int_\gamma(\hat\Lambda^+\,\phi)\,\bar\phi\,ds \geq c\|\phi\|^2_{H^{1/2}(\gamma)}$ for all $\phi \in H_0^{1/2}(\gamma)$.*

*Proof.* We begin with $\Lambda^-$, set $Q^- := (-\infty,\hat a)\times(0,h_0)$ for the moment, and define $\hat\Lambda^-$ by $\hat\Lambda^-\phi = \partial_{x_1}w^-\big|_{\hat\gamma}$ where $w^- \in H^1(Q^-)$ solves $\Delta w^- - w^- = 0$ in $Q^-$, $w^- = 0$ for $x_2 \in \{0,h_0\}$ and $w^- = \phi$ on $\hat\gamma$. Existence and uniqueness is assured because $(u,\psi) \mapsto \int_{Q^-}\big[\nabla u\cdot\nabla\bar\psi + u\,\bar\psi\big]\,dx$ is coercive. We conclude that $\int_{\hat\gamma}(\hat\Lambda^-\,\phi)\,\bar\phi\,ds = \int_{\hat\gamma}\partial_{x_1}w^-\,\overline{w^-}\,ds = \int_{Q^-}[|\nabla w^-|^2 + |w^-|^2]dx = \|w^-\|^2_{H^1(Q^-)} \geq c\|\phi\|^2_{H^{1/2}(\hat\gamma)}$ by the trace theorem.
We now choose $\rho \in C^\infty(\mathbb{R})$ with $\rho(x_1) = 1$ for $x_1 \geq \hat a - 1$ and $\rho(x_1) = 0$ for $x_1 \leq \hat a - 2$. Then $\hat\Lambda^-\phi = \partial_{x_1}(\rho w^-)\big|_{\hat\gamma}$. We define the operator $C^-$ by $C^- = \Lambda^- - \hat\Lambda^-$, thus $C^-\phi = \partial_{x_1}(u - \rho w^-)\big|_{\hat\gamma} = \partial_{x_1}v\big|_{\hat\gamma}$ where $v := u^- - \rho w^- \in H^1(Q^-)$ solves

$$
\Delta v + \omega^2 n_- v \;=\; -(\Delta + \omega^2 n_-)(\rho w^-) \;=\; -\big[(\omega^2 n_- + 1)\,\rho\,w^- + 2\rho'\,\partial_{x_1}w^- + \rho'' w^-\big]
$$

in $Q^-$ and $v = 0$ on $\partial Q^-$. Since the right hand side is compactly supported and in $L^2(Q^-)$ standard regularity results imply that $v \in H^2_{loc}(\overline{Q^-})$, which shows compactness of $C^-$.

Now we turn to $\Lambda^+$ and proceed similarly. The operator $\hat\Lambda^+$ is defined as $\hat\Lambda^+\phi = \partial_{x_1}w^+\big|_\gamma$ where $w^+ \in H^1(W^+)$ is given as the solution of $\Delta w^+ - w^+ = 0$ in $W^+$, $w^+ = 0$ on $\partial W \cap \overline{W^+}$, and $w^+ = \phi$ on $\gamma$, see Lemma 3.2. Then $-\hat\Lambda^+$ is coercive by the same arguments as above. Now we choose $\rho = \rho(x_1,x_2)$ as in the proof of Lemma 3.2 and observe that $C^+\phi := \Lambda^+\phi - \hat\Lambda^+\phi = \partial_{x_1}v\big|_\gamma$ where $v := u^+ + \rho w^+ \in H^1(W^+)$ solves

$$
\Delta v + \omega^2 n_+ v \;=\; -(\Delta + \omega^2 n_+)(\rho w^+) \;=\; -\big[(\omega^2 n_+ + 1)\,\rho\,w^+ + 2\nabla\rho\cdot\nabla w^+ + \Delta\rho\,w^+\big]
$$

in $W^+$ and $v = 0$ on $\partial W^+$. □

The variational equation (22) is written as

$$
a_1(u,\psi) + a_2(u,\psi) \;=\; \int\limits_{\gamma}\Lambda^+(0,f^+)\,\bar\psi\,ds \;+\; \int\limits_{W^0} f^-\,\bar\psi\,dx \quad \text{for all } \psi \in X
$$

where

$$
\begin{aligned}
a_1(v,\psi) &= \int\limits_{W^0} \big[\nabla v\cdot\nabla\bar\psi + v\,\bar\psi\big]\,dx \;-\int\limits_{\gamma} \hat\Lambda^+ v\,\bar\psi\,ds \;+\int\limits_{\hat\gamma} \hat\Lambda^- v\,\bar\psi\,ds \\
a_2(v,\psi) &= -\int\limits_{W^0} (\omega^2 n+1)\,v\,\bar\psi\,dx \;-\int\limits_{\gamma} C^+ v\,\bar\psi\,ds \;+\int\limits_{\hat\gamma} C^- v\,\bar\psi\,ds
\end{aligned}
$$

for $v,\psi\in X$. Using the theorem of Riesz, the compact embedding of $X$ into $L^2(W^0)$, and the previous lemma we can write this as

$$A_1 u + A_2 u = r$$

with a coercive operator $A_1$ and a compact operator $A_2$ from $X$ into itself.

Therefore, $A_1+A_2$ is a Fredholm operator, and it remains to show injectivity of $A_1+A_2$. This is equivalent to showing that $u=0$ is the only solution in $X$ of (22) for $f^+=f^-=0$ which in turn is equivalent to showing that $u=0$ is the only solution in $H^1_{loc}(\overline{W})$ of (2) for $f=0$ satisfying the radiation condition of Definition 2.4.
Therefore, let $u\in H^1_{loc}(\overline{W})$ be a solution of (2) for $f=0$ satisfying the radiation condition of Definition 2.4. Then the restrictions $u^+:=u|_{W^+}$ and $u^-:=u|_{W^-}$ are solutions of the semi-waveguide problems in $W^+$ and in $W^-$, respectively, for $\phi=u|_\gamma$ and $\phi=u|_{\hat\gamma}$, respectively.
Let $a<\hat a$ and $b>2\pi$ and $H>h_0$ and define the part $W_{a,b,H}$ of the waveguide by $W_{a,b,H}=\{x\in W: a<x_1<b,\ x_2<H\}$. Application of Green's theorem in $W_{a,b,H}$ yields

$$
\begin{aligned}
\int\limits_{W_{a,b,H}} \big[|\nabla u|^2-\omega^2 n\,|u|^2\big]\,dx &= -\int\limits_0^{h_0} [\bar u\,\partial_{x_1}u]_{x_1=a}\,dx_2 \;+\int\limits_0^{H} [\bar u\,\partial_{x_1}u]_{x_1=b}\,dx_2 \\
&\quad +\int\limits_0^{b} [\bar u^+\,\partial_{x_2}u^+]_{x_2=H}\,dx_1\,.
\end{aligned}
$$

In this equation we can let $H$ tend to infinity. Since $\overline{u(x_1,H)}\,\partial_{x_2}u(x_1,H)$ converges to zero as $H\to\infty$ uniformly with respect to $x_1\in(0,b)$ (use (16) for $u^+_{rad}$ and the exponential decay of $u^+_{prop}$) we observe that the third term on the right hand side tends to zero and thus (taking the imaginary part) $\mathcal{E}^-_a(u)=\mathcal{E}^+_b(u)$. Since $a\mapsto\mathcal{E}^-_a(u)$ is constant on $(-\infty,\hat a)$ and non-positive and $a\mapsto\mathcal{E}^+_a(u)$ is constant on $(2\pi,\infty)$ and non-negative by Theorem 3.4 we conclude that $\mathcal{E}^-(u,u)=\mathcal{E}^+(u,u)=0$ which implies from (20) that $a^\pm_{\ell,j}$ vanish for all $\ell$ and $j$. Therefore, $u=u_{rad}\in H^1(W_h)$ for all $h\geq h_0$ where again $W_h:=W\cap\big(\mathbb{R}\times(0,h)\big)$.
Now we proceed as in the proof of Theorem 3.3. of [9]. For fixed $h>h_0$ we apply Green's Theorem to $u$ in $W_h$ which yields

$$\operatorname{Im}\int\limits_0^\infty \overline{u^+(x_1,h)}\,\partial_{x_2}u^+(x_1,h)\,dx_1 \;=\; 0$$

because $u$ vanishes on $\partial W$. Now we extend $u^+$ to an odd function $u^{odd}$ (with respect to $x_1$) into the half plane $\mathbb{R}\times(h,\infty)$. Then $u^{odd}\in H^1\big(\mathbb{R}\times(h,H)\big)$ for all $H>h$ (since $u^+$ vanishes on $\{0\}\times(h,\infty)$) and $\operatorname{Im}\int_{-\infty}^\infty \overline{u^{odd}(x_1,h)}\,\partial_{x_2}u^{odd}(x_1,h)\,dx_1=0$. We denote by $\hat{u} = \mathcal{F}u^{odd}$ the Fourier transform of $u^{odd}$ with $\mathcal{F}$ from (19). Since $u^{odd}$ satisfies the Helmholtz equation $\Delta u^{odd} + \omega^2 u^{odd} = 0$ for $x_2 > h$ we conclude that $\hat{u}(\xi, x_2)$ has the form

$$\hat{u}(\xi, x_2) \;=\; \begin{cases} \hat{u}(\xi,h)\, e^{i\sqrt{\omega^2-\xi^2}(x_2-h)}\,, & |\xi| < \omega\,, \\ \hat{u}(\xi,h)\, e^{-\sqrt{\xi^2-\omega^2}(x_2-h)}\,, & |\xi| > \omega\,, \end{cases} \qquad \text{and} \quad x_2 > h\,,$$

and

$$\partial_{x_2}\hat{u}(\xi, x_2) \;=\; \begin{cases} i\sqrt{\omega^2-\xi^2}\,\hat{u}(\xi,h)\, e^{i\sqrt{\omega^2-\xi^2}(x_2-h)}\,, & |\xi| < \omega\,, \\ -\sqrt{\xi^2-\omega^2}\,\hat{u}(\xi,h)\, e^{-\sqrt{\xi^2-\omega^2}(x_2-h)}\,, & |\xi| > \omega\,, \end{cases} \qquad \text{and} \quad x_2 > h\,.$$

Plancherel's Theorem yields

$$0 \;=\; \operatorname{Im}\int_{-\infty}^{\infty} \overline{\hat{u}(\xi,h)}\,\partial_{x_2}\hat{u}(\xi,h)\,d\xi \;=\; \int_{-\omega}^{\omega} \sqrt{\omega^2-\xi^2}\,|\hat{u}(\xi,h)|^2\,d\xi$$

which implies that $\hat{u}(\xi,h) = 0$ for $|\xi| < \omega$ and thus also $\hat{u}(\xi,x_2) = 0$ for $|\xi| < \omega$ and $x_2 > h$. For $|\xi| > \omega$ we compute

$$\int_h^\infty |\hat{u}(\xi,x_2)|^2\,dx_2 \;= |\hat{u}(\xi,h)|^2 \int_h^\infty e^{-2\sqrt{\xi^2-\omega^2}(x_2-h)}\,dx_2 \;=\; \frac{|\hat{u}(\xi,h)|^2}{2\sqrt{\xi^2-\omega^2}}\,.$$

Since $u^+$ is a solution of the semi-waveguide problem in $W^+$ with $\phi = u|_\gamma \in H_0^{1/2}(\gamma)$ Theorem 3.3 (estimate (14)) yields that $x \mapsto u^{odd}(x)$ is in $L^1\big(\mathbb{R}\times(h,h+1)\big)$. Therefore the Fourier transform $(\xi,x_2) \mapsto \hat{u}(\xi,x_2)$ is continuous and thus also $\xi \mapsto \hat{u}(\xi,h)$. This implies that $\xi \mapsto \frac{|\hat{u}(\xi,h)|^2}{\sqrt{\xi^2-\omega^2}}$ is in $L^1(\mathbb{R})$ and proves that (note that $\hat{u}(\cdot,h)$ is odd)

$$\int_h^\infty\int_{-\infty}^\infty |u^{odd}(x_1,x_2)|^2\,dx_1\,dx_2 \;=\; 2\int_h^\infty\int_\omega^\infty |\hat{u}(\xi,x_2)|^2\,d\xi\,dx_2 \;=\; \int_\omega^\infty \frac{|\hat{u}(\xi,h)|^2}{\sqrt{\xi^2-\omega^2}}\,d\xi \;<\; \infty\,.$$

The arguments for the derivatives with respect to $x_1$ and $x_2$ follow the same arguments. Therefore, $u \in H^1(W)$ and thus has to vanish by Assumption 2.6. This ends the proof of uniqueness and thus also of the proof of Theorem 2.7.

## 5. Extensions and Remarks

5.1. **A scattering problem.** For given incident field $u^{inc}$ the scattering problem is to determine the total field $u^{tot}$ as a solution of $\Delta u^{tot} + \omega^2 n u^{tot} = 0$ in $W$ and $u^{tot} = 0$ on $\partial W$ such that the scattered field $u^s = u^{tot} - u^{inc}$ satisfies the radiation condition. We make the Assumptions 2.2 and 2.6. Let $u^{inc}$ be a right-going mode with respect to $n_-$, i.e. $u^{inc} = \phi^-_{\ell_0,j_0}$ for some $j_0 \in J^-$ and $\ell_0 \in \{1,\ldots,m^-_{j_0}\}$ with $\lambda_{\ell_0,j_0} > 0$. We choose $\rho \in C^\infty(\mathbb{R})$ with $\rho(x_1) = 1$ for $x_1 \le \frac{2}{3}\hat{a}$ and $\rho(x_1) = 0$ for $x_1 \ge \frac{1}{3}\hat{a}$ and make the ansatz $u^{tot}(x) = \rho(x_1)u^{inc}(x) + u(x)$. Then $u$ solves (2) with $f(x) = 2\rho'(x_1)\partial_{x_1}u^{inc}(x) + \rho''(x_1)\,u^{inc}(x)$ which is supported in $\big(\frac{2}{3}\hat{a},\frac{1}{3}\hat{a}\big)\times(0,h_0)$. By Theorem 2.7 there exists a unique solution $u = u_{rad} + u_{prop}$ which satisfies the properties of Theorem 2.7. In particular $u$ is right-going for $x_1 > 0$ and left-going for $x_1 < 0$.
For $x \in W^-$ with $x_1 \le \frac{2}{3}\hat{a}$ we have $u^{tot}(x) = u^{inc}(x) + u(x)$, i.e. $u^s(x) = u(x) =$

$u_{rad}(x) + u_{prop}(x)$ which is left-going. For $x \in W^+$ we have $u^{tot}(x) = u(x)$, i.e. $u^s(x) = u(x) - \phi^-_{\ell_0,j_0}(x) = u_{rad}(x) + u_{prop}(x) - \phi^-_{\ell_0,j_0}(x)$ which is right-going.

5.2. **Different periods of $n_-$ and $n_+$.** In this manuscript the periods of $n_+$ and $n_-$ are both chosen to be $2\pi$. This is done by keeping the notations as simple as possible. However, we can replace them by $L^+$ and $L^-$, respectively. The terms $\ell + \alpha$ have to be replaced by $\ell\frac{2\pi}{L^+}$ in $W^+$ and $\ell\frac{2\pi}{L^-}$ in $W^-$, respectively, which has implications in the definitions of, e.g., quasi-periodicity, cut-off values, and the Rayleigh expansion. The analysis, however, is the same, only the notations are more technical.

5.3. **Changing the shape of the upper part of the boundary of $W^-$.** We can replace the part $(-\infty, 0) \times \{h_0\}$ of $\partial W^-$ by the graph $\{(x_1, h(x_1) : x_1 < 0)\}$ of a periodic (with the same period as $n_-$) Lipschitz continuous function $h : \mathbb{R} \to \mathbb{R}_{>0}$ such that $h$ and $1/h$ are bounded on $\mathbb{R}$. Then we set $h_0 := h(0)$ in the coupling. This would not change anything of the analysis – besides a bit more technical description including more notations.

5.4. **Junction of several closed semi-waveguides.** The axes of the closed and open waveguides don't have to coincide. Also, the coupling of more than one closed semi-waveguides with an open waveguide can be treated, such as, e.g., in the configuration of Figure 3.

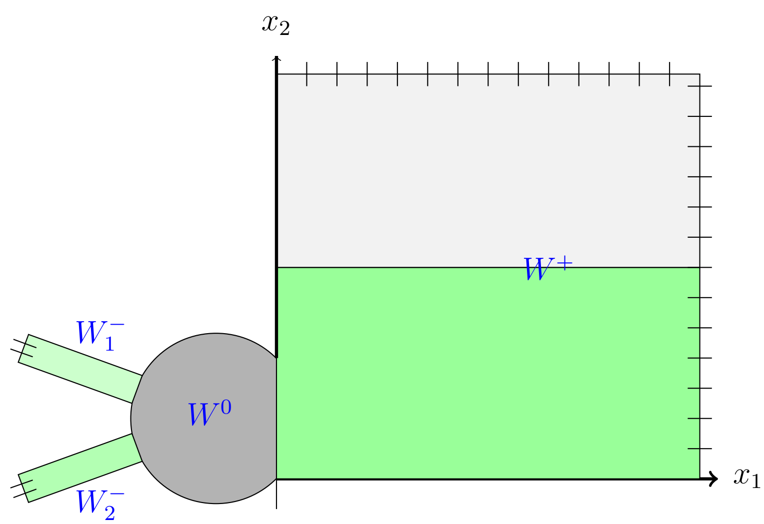


Figure 3. A more general configeration

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