

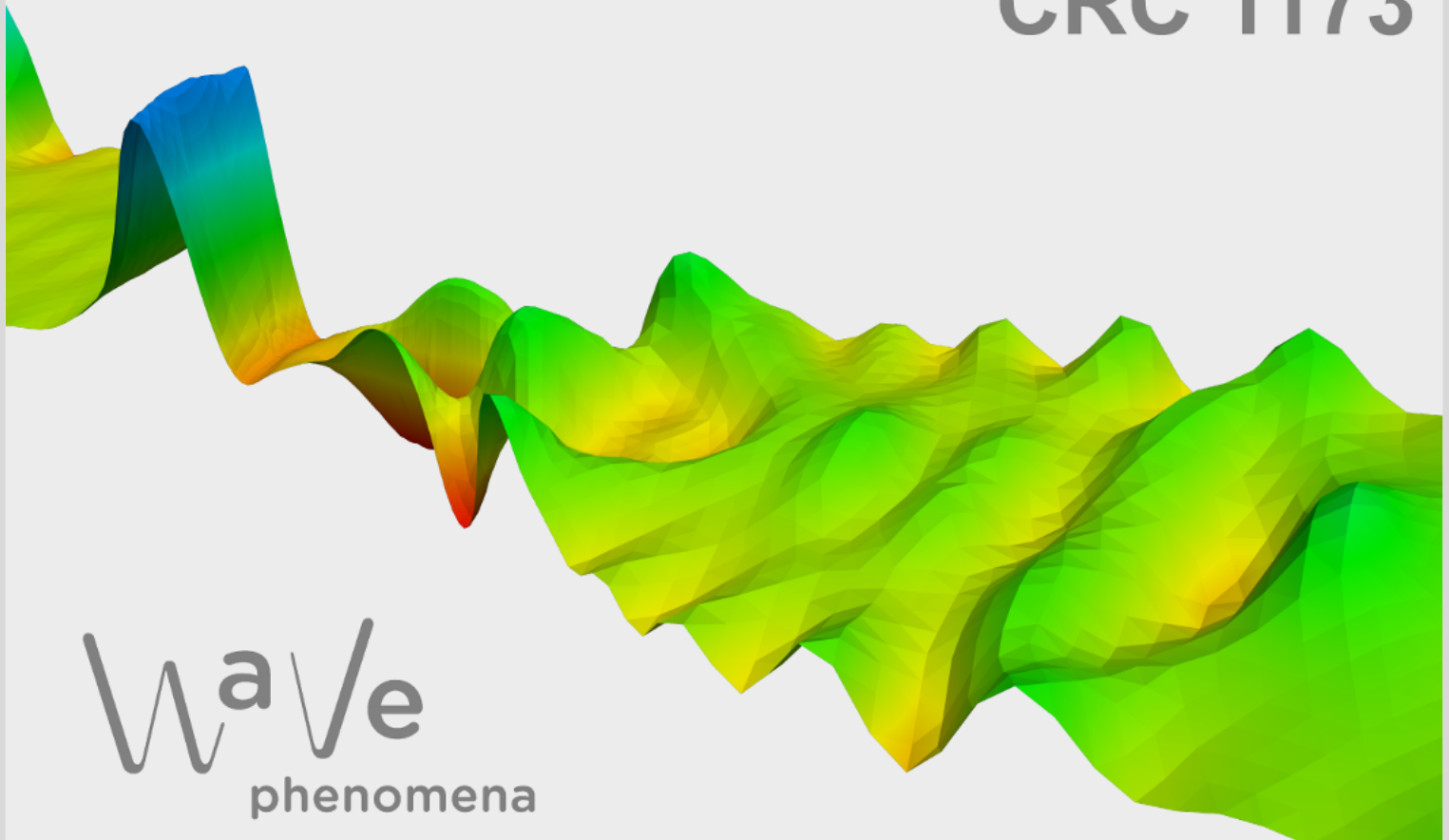
Scattering from thin-coated objects: A high-order finite element method

Théophile Chaumont-Frelet, Julian Dörner

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SCATTERING FROM THIN-COATED OBJECTS: A HIGH-ORDER FINITE ELEMENT METHOD

THÉOPHILE CHAUMONT-FRELET AND JULIAN DÖRNER

ABSTRACT. We study the scattering of a time-harmonic acoustic wave by a thinly coated object. The coating is replaced by an effective boundary condition of second order, which involves the Laplace–Beltrami operator on the surface of the object. Conditions of this type are also known as Ventcel boundary conditions. For a coating made from a metamaterial with negative effective density, the usual “coercive + compact” framework degenerates as the absorption of the coating tends to zero, and it fails for a lossless coating. In this work, we introduce new arguments to show that the problem is well-posed even in the degenerate case. We then analyze its finite element discretization by Lagrange elements of degree p on a curved mesh and show that a quasi-optimal solution under an optimal resolution condition already known in the absence of Ventcel boundary conditions. The second-order boundary condition in the metamaterial regime therefore degrades neither the resolution condition nor the achievable order of convergence.

1. INTRODUCTION

1.1. Motivation. Scattering objects can be coated by a thin reactive layer in order to tailor their response to acoustic waves. From a computational perspective, resolving a thin layer by a mesh is prohibitively expensive. The layer is therefore often replaced by an effective boundary condition on the surface of the coating. Such a condition is obtained from an asymptotic expansion in the layer thickness δ , going back at least to [21]. We briefly sketch this expansion for time-harmonic

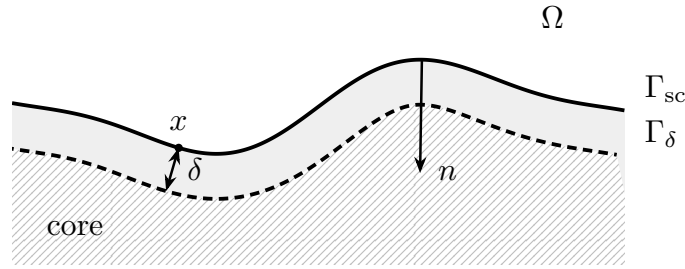


FIGURE 1. Close-up of the coating. The boundary Γ_{sc} of the scattering object carries the layer $\Gamma_{\delta} = \{x + sn(x) \mid x \in \Gamma_{sc}, s \in (0, \delta)\}$ of thickness δ . The hatched region is the rigid core.

acoustic scattering. The acoustic wave is assumed to be time-harmonic with the convention $e^{-i\omega t}$, where $\omega > 0$ is the frequency. The material is assumed to be homogeneous throughout, described by the density ρ_0 and compressibility κ_0 in

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Ω , and the relative density ρ_f and relative compressibility κ_f in the thin film Γ_δ . Then, the wave pressure u satisfies the following Helmholtz equations in Ω and Γ_δ :

$$\Delta u + k^2 u = 0, \quad \text{in } \Omega, \quad \Delta u + k_f^2 u = 0, \quad \text{in } \Gamma_\delta,$$

with the wavenumbers $k^2 = \omega^2 \rho_0 \kappa_0$ and $k_f^2 = \rho_f \kappa_f k^2$. At the material interface, the pressure satisfies the transmission conditions

$$(u|_\Omega)|_{\Gamma_{\text{sc}}} = (u|_{\Gamma_\delta})|_{\Gamma_{\text{sc}}}, \quad (\rho_0^{-1} \partial_n u|_\Omega)|_{\Gamma_{\text{sc}}} = ((\rho_0 \rho_f)^{-1} \partial_n u|_{\Gamma_\delta})|_{\Gamma_{\text{sc}}}, \quad \text{on } \Gamma_{\text{sc}}.$$

To expand u around Γ_{sc} in the direction of the normal n , we assume that the layer is thin compared to the wavelength inside the film, i.e. $k_f \delta \ll 1$, and thin compared to the radii of curvature of Γ_{sc} . This allows for a parametrization of the thin layer relative to the scattering objects boundary Γ_{sc} :

$$\Gamma_\delta = \{x + sn(x) \mid x \in \Gamma_{\text{sc}}, s \in (0, \delta)\}.$$

Thus, we employ the ansatz

$$u = u_0 + su_1 + \frac{s^2}{2} u_2 + O(s^3).$$

It remains to match the transmission condition at $s = 0$, the film equation inside Γ_δ , and the sound-hard boundary condition of the core at $s = \delta$. The latter, i.e. $\partial_s u = 0$ at $s = \delta$, implies that $u_1 = -\delta u_2 + O(\delta^2)$. For the film equation inside the layer, we write $\Delta = \partial_s^2 + H(x, s) \partial_s + \Delta_{\Gamma_{\text{sc}}}$, where H denotes the sum of the principal curvatures. Evaluation at $s = 0$ gives $u_2 = -(\Delta_{\Gamma_{\text{sc}}} + k_f^2) u_0 + O(\delta)$. Finally, continuity of the normal velocity at $s = 0$ gives $\rho_f^{-1} u_1 = \partial_n u$. Chaining the three relations and using $k_f^2 / \rho_f = \kappa_f k^2$ provides

$$\partial_n u = \frac{1}{\rho_f} u_1 = -\frac{\delta}{\rho_f} u_2 + O(\delta^2) = \frac{\delta}{\rho_f} (\Delta_{\Gamma_{\text{sc}}} + k_f^2) u + O(\delta^2).$$

The coating is replaced, to leading order in δ , by the effective boundary condition

$$\partial_n u - \alpha k u + \frac{\beta}{k} \Delta_{\Gamma_{\text{sc}}} u = 0, \quad \text{on } \Gamma_{\text{sc}}, \quad \alpha = k \delta \kappa_f, \quad \beta = -\frac{k \delta}{\rho_f}. \quad (1)$$

The same condition, with ρ_f and κ_f replaced by the relative permittivity and permeability, governs the electromagnetic problem of a TM-polarized wave scattered by a coated perfect conductor.

Both coefficients in (1) are dimensionless and carry the small thickness $k \delta \ll 1$ of the layer, so that they are of order one only for a so-called metamaterial. Indeed, $\beta = O(1)$ forces $|\rho_f| \ll 1$. Metamaterials exhibit properties beyond those found in nature and are typically built from fine structures that can be modeled by effective, continuous material parameters. For example, materials with near-zero or even negative effective density ρ_f have been realized, and we refer to [16] for a comprehensive introduction to sonic metamaterials.

Mechanically, $1/\rho$ is a measure of mobility. The tangential term is therefore large precisely when the layer is easy to move along the surface, so that it can transport energy tangentially. This non-locality allows for evanescent surface waves on the scattering object, see, e.g., [29, 35]. In contrast, an ordinary lining has $\rho_f = O(1)$, hence $\beta = O(k \delta)$, and one recovers the classical impedance condition, in which every point of the surface responds independently of its neighbors. The surface term is thus exactly what makes the coating non-locally reacting.

In this manuscript, we are interested in materials that exhibit a negative effective mass density, i.e. $\text{Re } \rho_f < 0$. Such a negative effective density is realized by locally resonant [29] and by membrane-type [34] metamaterials.

Likewise, $\alpha = O(1)$ forces $|\kappa_f^{-1}| = |K_f| = O(k \delta)$, a near-zero effective bulk modulus. Such a metamaterial is realized in [22]. Materials that exhibit both a negative density and a negative modulus are described in [16, 30].

Assuming that the layer is passive, i.e. $\text{Im } \rho_f \geq 0$ and $\text{Im } \kappa_f \geq 0$, we arrive at the following parameter regime, which we cover in this manuscript,

$$\alpha \in \mathbb{C}, \quad \text{Im } \alpha \geq 0, \quad \beta \in \mathbb{C}, \quad \text{Re } \beta > 0, \text{Im } \beta \geq 0. \quad (2)$$

Generalized impedance boundary conditions like (1) have been studied thoroughly in the literature. See, e.g., [2, 6, 7] for the Helmholtz case, or [11, 27] for the Maxwell case. To the best of our knowledge, a parameter regime like (2) has not been covered yet. It breaks or degenerates the ellipticity assumption on the principal part of the equation as $\text{Im } \beta \rightarrow 0$, so the problem no longer fits into the “coercive + compact” framework.

Our model problem (3) below truncates the domain Ω at an exterior boundary Γ_{ex} with an absorbing boundary condition and is reformulated as a scattering problem. Furthermore, we are interested in the high-frequency regime $k \gg 1$. We show that the problem is still well-posed in the sense of Fredholm, and that a unique solution exists under the passivity assumption in (2). The remainder of the manuscript is then concerned with the finite element discretization of the problem. On a curved mesh of mesh size h , we show that Lagrange elements of degree p are quasi-optimal subject to the common resolution condition. Precisely, if

$$kh \ll 1 \quad \text{and} \quad k(kh)^p C_{\text{sol}} \ll 1,$$

then the finite element discretization u_h satisfies

$$\|u - u_h\| \lesssim \min_{v_h} \|u - v_h\|,$$

with a constant independent of k and h . Here, C_{sol} denotes the normalized norm of the solution operator. In particular, the second-order boundary condition with the non-standard parameter regime (2) degrades neither the resolution condition nor the achievable order of convergence.

1.2. Setting. We consider for $d \in \{2, 3\}$ a bounded domain $\Omega \subset \mathbb{R}^d$ with boundary $\Gamma = \partial\Omega \in C^{\gamma,1}$, where $\gamma \in \mathbb{N}_+$. The boundary is further divided into two disjoint parts $\Gamma = \Gamma_{\text{ex}} \dot{\cup} \Gamma_{\text{sc}}$. Here, Γ_{ex} is the connected exterior boundary of Ω , and Γ_{sc} is the connected interior boundary of the scattering object. We assume that both boundaries are well separated, i.e. $\text{dist}(\Gamma_{\text{ex}}, \Gamma_{\text{sc}}) \gtrsim 1$. Note that we restrict ourselves to the case of one scattering object to keep the presentation tight.

We seek a solution $u : \Omega \rightarrow \mathbb{C}$ of the scattering problem

$$-k^2 u - \Delta u = f, \quad \text{in } \Omega, \quad (3a)$$

$$\partial_n u - \alpha k u + \frac{\beta}{k} \Delta_{\Gamma_{\text{sc}}} u = g_{\text{sc}}, \quad \text{on } \Gamma_{\text{sc}}, \quad (3b)$$

$$\partial_n u - i k u = g_{\text{ex}}, \quad \text{on } \Gamma_{\text{ex}}, \quad (3c)$$

where $k \gg 1$ denotes the wavenumber and n the outer unit normal. The functions $f : \Omega \rightarrow \mathbb{C}$, $g_{\text{ex}} : \Gamma_{\text{ex}} \rightarrow \mathbb{C}$, and $g_{\text{sc}} : \Gamma_{\text{sc}} \rightarrow \mathbb{C}$ are given data. The two material parameters $\alpha, \beta \in \mathbb{C}$ on the scattering boundary Γ_{sc} satisfy (2). Note that we are only interested in the high-frequency regime and assume that the wavenumber is bounded from below, i.e. $k \geq k_0 > 0$. Hidden constants might depend on k_0 .

1.3. Notation. Throughout the manuscript, we consider the Lebesgue space $L^2(\Omega)$ of square integrable, complex valued functions $u : \Omega \rightarrow \mathbb{C}$. The inner product is denoted by $(\cdot | \cdot)_{\Omega}$, which is assumed to be linear in the first component, and anti-linear in the second component.

Similarly, we denote by $L^2(\Gamma)$ the space induced by the surface measure with inner product $(\cdot | \cdot)_{\Gamma}$, following the same convention with respect to linearity.

For real $s \geq 0$ we denote by $H^s(\Omega)$ the fractional Sobolev spaces and follow the convention that $H^0(\Omega) = L^2(\Omega)$. For $s \in (1/2, 1]$, we denote by $H_0^s(\Omega)$ the

completion of compactly supported test functions $C_c^\infty(\Omega)$ with respect to the $H^s(\Omega)$ norm, and by $H^{-s}(\Omega)$ its topological dual.

The dual pairing is always denoted with angled brackets $\langle \cdot | \cdot \rangle$ and an additional index if necessary.

We use the notation $K \Subset \Omega$ for a compact inclusion, and write $A \lesssim B$ for $A, B \in \mathbb{R}$ if there exists a k -independent constant $C > 0$ such that $A \leq CB$. We write $A \approx B$ if the inequality holds in both directions.

Wavenumber scaled norms. Throughout the manuscript, we make use of Sobolev norms that scale appropriately with respect to the wavenumber k . For $m \in \mathbb{N}$ and $u \in H^m(\Omega)$ we define

$$\|u\|_{H_k^m(\Omega)}^2 = \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq m} k^{-2|\alpha|+2} \|\partial^\alpha u\|_{L^2(\Omega)}^2. \quad (4a)$$

This definition is extended to $H_k^s(\Omega)$ for real $s > 0$ via interpolation.

Let $\Upsilon \in \{\Gamma_{\text{ex}}, \Gamma_{\text{sc}}\}$. We introduce k -scaled Sobolev spaces on the boundary Υ in the fashion of [31, Sec. 3]. Since Υ is a compact and boundary-less hypersurface, there exists a finite $C^{\gamma,1}$ -atlas $\mathcal{A}_\Upsilon = \{(U_i, \chi_i)\}_{i=1}^N$, where $\bigcup_{i=1}^N U_i = \Upsilon$ and $\chi_i : U_i \rightarrow V_i \subset \mathbb{R}^{d-1}$ is a homeomorphism, called chart. The inverse $\theta_i = \chi_i^{-1} : V_i \rightarrow U_i$ of the chart is called the parametrization. Given a subordinate partition of unity $\{\eta_i\}_{i=1}^N$, i.e. $\eta_i \in C^{\gamma,1}(\Upsilon)$, $\text{supp } \eta_i \subset U_i$ and $\sum_{i=1}^N \eta_i \equiv 1$, we say that a measurable function $\phi : \Upsilon \rightarrow \mathbb{C}$ is an element of $H^\ell(\Upsilon)$, where $\ell \in \{0, \dots, \gamma+1\}$, if the following norm is finite,

$$\|\phi\|_{H_k^\ell(\Upsilon)}^2 = \sum_{i=1}^N \sum_{\lambda \in \mathbb{N}^{d-1}, |\lambda| \leq \ell} k^{-2|\lambda|+1} \|\partial^\lambda ((\eta_i \phi) \circ \theta_i)\|_{L^2(V_i)}^2. \quad (4b)$$

We emphasize the different k -scaling in the latter definition compared to (4a), which accounts for the loss of differentiability while taking traces. This definition does not depend on the specific choice of atlas, and we refer to [31, Sec. 3] for details. We extend this definition via interpolation to $H_k^s(\Upsilon)$, where $s \in [0, \gamma+1]$. Furthermore, $H_k^{-s}(\Upsilon)$ are defined as the topological dual spaces, realized through the $L^2(\Upsilon)$ inner product.

Surface differential operators. Similar to the Sobolev norms, the tangential gradient $\nabla_\Upsilon : H^1(\Upsilon) \rightarrow L^2(\Upsilon)$ and the Laplace–Beltrami operator $\Delta_\Upsilon : H^1(\Upsilon) \rightarrow H^{-1}(\Upsilon)$ can be defined in terms of charts and a subordinate partition of unity. Alternatively, they are defined in terms of projections from a tubular neighborhood. We refer to the review in [19, Sec. 2] for details, or to Appendix B where we use the chart-based definition. Note that $k \|\phi\|_{L^2(\Upsilon)}^2 + k^{-1} \|\nabla_\Upsilon \phi\|_{L^2(\Upsilon)}^2$ is equivalent to the $H_k^1(\Upsilon)$ norm as defined in (4b).

2. VARIATIONAL SETTING AND WELL-POSEDNESS

To derive a variational formulation of (3), we have to accommodate the Laplace operator in the interior (3a) as well as the Laplace–Beltrami operator on the scattering boundary (3b). This leads naturally to the energy space

$$V = \{u \in H^1(\Omega) \mid u|_{\Gamma_{\text{sc}}} \in H^1(\Gamma_{\text{sc}})\} \quad (5a)$$

with associated norm

$$\|u\|^2 = k^2 \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + k \|u|_{\Gamma_{\text{sc}}}\|_{L^2(\Gamma_{\text{sc}})}^2 + k^{-1} \|\nabla_{\Gamma_{\text{sc}}} u\|_{L^2(\Gamma_{\text{sc}})}^2. \quad (5b)$$

Note that $\|u\|^2$ is equivalent to the sum norm $\|u\|_{H_k^1(\Omega)}^2 + \|u\|_{H_k^1(\Gamma_{\text{sc}})}^2$. We obtain a weak formulation of (3) by multiplying (3a) with a test function and integrating

by parts, taking the boundary conditions (3b) and (3c) into account. A second integration by parts, now for the Laplace–Beltrami operator in (3b), leads to the following variational problem:

$$\text{Seek } u \in V \text{ such that } b(u|\phi) = \ell(\phi), \quad \text{for all } \phi \in V, \quad (6a)$$

with the sesquilinear form $b : V \times V \rightarrow \mathbb{C}$ defined by

$$b(u, \phi) = (\nabla u | \nabla \phi)_{\Omega} - k^2 (u | \phi)_{\Omega} - \alpha k (u | \phi)_{\Gamma_{\text{sc}}} - i k (u | \phi)_{\Gamma_{\text{ex}}} - \frac{\beta}{k} (\nabla_{\Gamma} u | \nabla_{\Gamma} \phi)_{\Gamma_{\text{sc}}}, \quad (6b)$$

and the antilinear form $\ell : V \rightarrow \mathbb{C}$ defined by

$$\ell(\phi) = (f | \phi)_{\Omega} + (g_{\text{sc}} | \phi)_{\Gamma_{\text{sc}}} + (g_{\text{ex}} | \phi)_{\Gamma_{\text{ex}}}. \quad (6c)$$

Lack of coercivity. We briefly review why the usual “coercive + compact” framework does not work in our situation. Let $\phi = u$ in (6b). Taking the real part gives

$$\text{Re } b(u|u) = \|\nabla u\|_{L^2(\Omega)}^2 - k^2 \|u\|_{L^2(\Omega)}^2 - \text{Re } \alpha k \|u\|_{L^2(\Gamma_{\text{sc}})}^2 - \text{Re } \beta k^{-1} \|\nabla_{\Gamma} u\|_{L^2(\Gamma_{\text{sc}})}^2.$$

The second and the third term are compact perturbations, since both embeddings $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and $H^1(\Gamma_{\text{sc}}) \hookrightarrow L^2(\Gamma_{\text{sc}})$ are compact. The fourth term $\text{Re } \beta k^{-1} \|\nabla_{\Gamma} u\|_{L^2(\Gamma_{\text{sc}})}^2$ is not coercive, because $\text{Re } \beta > 0$ by assumption (2). It is not a compact perturbation either, since no residual regularity is left on the boundary. Hence, $\text{Re } b(u|u)$ is not coercive up to a compact perturbation.

One usually deals with this by taking the imaginary part of (6b) as well, which reads

$$\text{Im } b(u|u) = -\text{Im } \alpha k \|u\|_{L^2(\Gamma_{\text{sc}})}^2 - k \|u\|_{L^2(\Gamma_{\text{ex}})}^2 - \text{Im } \beta k^{-1} \|\nabla_{\Gamma} u\|_{L^2(\Gamma_{\text{sc}})}^2.$$

This gives control over the indefinite boundary term. If $\text{Im } \beta > 0$, we can choose an angle $\theta \in [0, 2\pi)$ such that

$$\text{Re}(e^{i\theta} b(u|u)) \geq c \|u\|^2 - C \|u\|_{L^2(\Omega)}^2, \quad u \in V, \quad (7)$$

with constants $C > c > 0$, i.e. the problem is coercive up to a compact perturbation. This argument does not cover the case $\text{Im } \beta = 0$, however. Moreover, the ratio Cc^{-1} grows like $(\text{Im } \beta)^{-1}$, which is undesirable if $\text{Im } \beta$ is positive but small.

In the remainder of this section, we show that problem (6a) is well-posed in the Fredholm sense, cf. [4, Def. 4.2.21]. To cover the parameter range in (2), we use ideas from the so-called T-coercivity, cf. [10], which was developed precisely for sign-changing material coefficients. Together with a unique continuation principle, this allows us to prove the following well-posedness result at the end of the section.

Proposition 1. *Let $f \in L^2(\Omega)$, $g_{\text{sc}} \in L^2(\Gamma_{\text{sc}})$, and $g_{\text{ex}} \in L^2(\Gamma_{\text{ex}})$. Then (6a) admits a unique solution $u \in V$. Moreover, the solution depends continuously on the data, i.e.*

$$\|u\| \leq k C_{\text{sol}} \left(\|k^{-2} f\|_{H_k^0(\Omega)} + \|k^{-1} g_{\text{sc}}\|_{H_k^0(\Gamma_{\text{sc}})} + \|k^{-1} g_{\text{ex}}\|_{H_k^0(\Gamma_{\text{ex}})} \right), \quad (8)$$

where $C_{\text{sol}} = k^{-1} \|B^{-1}\|_{V \leftarrow V}$, and B is the operator associated with (6b).

Remark 1. *The constant C_{sol} depends on the wavenumber and can grow at most exponentially in k . The factor k^{-1} in the definition normalizes C_{sol} to the best possible behavior of the solution operator: for non-trapping configurations one expects $C_{\text{sol}} \approx 1$.*

2.1. T-Gårding inequality. In this subsection, we restore a Gårding inequality like (7). We insert an isomorphism into the second argument of the sesquilinear form b . It flips the sign of the boundary term that is neither coercive nor compact. The constants we obtain do not depend on $\text{Im } \beta$, so the argument also covers the case $\text{Im } \beta = 0$.

We first introduce an extension operator for boundary data on Γ_{sc} that additionally minimizes the energy norm $\|\cdot\|$ of (5).

Definition 1. For $s \in [0, 1/2]$ let

$$E_k : H^{1/2+s}(\Gamma_{\text{sc}}) \rightarrow H^{1+s}(\Omega), \quad h \mapsto u,$$

where u is the unique solution of $k^2 u - \Delta u = 0$ in Ω with $u|_{\Gamma_{\text{sc}}} = h$ and $u|_{\Gamma_{\text{ex}}} = 0$.

Note that this definition implicitly demands elliptic regularity of the Laplace operator. Such regularity already holds on Lipschitz domains, and in particular on domains with $C^{\gamma,1}$ -regular boundary for $\gamma \geq 1$, cf. [31, Thm. 4.16, 4.18]. The bounds derived in [1, Thm. 2.17] are k -explicit and read for $h \in H^1(\Gamma_{\text{sc}})$

$$\|E_k h\|_{H_k^{3/2}(\Omega)} + \|k^{-1} \partial_n E_k h\|_{H_k^0(\Gamma)} \lesssim \|h\|_{H_k^1(\Gamma_{\text{sc}})}, \quad (9)$$

with a hidden constant that depends only on Ω .

Composing E_k with the trace on Γ_{sc} isolates the boundary part of a function $u \in V$. We use this to define an operator that flips the sign of the trace on the scattering boundary.

Definition 2. Let $T_k : V \rightarrow V$ be defined by $u \mapsto u - 2E_k u|_{\Gamma_{\text{sc}}}$.

Like E_k , the operator T_k depends on the wavenumber k . We collect the properties of T_k that we need in the following lemma.

Lemma 1. For all $u \in V$ the operator T_k satisfies

- (i) $T_k T_k u = u$, in particular T_k is an isomorphism with $T_k^{-1} = T_k$,
- (ii) $\|T_k u\| \leq C_{T_k} \|u\|$, where $C_{T_k} > 0$ depends only on Ω ,
- (iii) $T_k u|_{\Gamma_{\text{sc}}} = -u|_{\Gamma_{\text{sc}}}$,
- (iv) $T_k u|_{\Gamma_{\text{ex}}} = u|_{\Gamma_{\text{ex}}}$.

Proof. Properties (iii) and (iv) follow directly from Definition 1. For (i), we use (iii) and compute

$$T_k T_k u = T_k u - 2E_k(T_k u)|_{\Gamma_{\text{sc}}} = T_k u + 2E_k u|_{\Gamma_{\text{sc}}} = u.$$

Hence, T_k is bijective and its inverse is T_k itself, so the bound (ii) also applies to the inverse. It remains to prove this bound. Using (iii) once more, we obtain

$$\|T_k u\|^2 \lesssim \|u\|^2 + k^2 \|E_k u|_{\Gamma_{\text{sc}}}\|_{L^2(\Omega)}^2 + \|\nabla E_k u|_{\Gamma_{\text{sc}}}\|_{L^2(\Omega)}^2.$$

The estimates (9) prove the claim since

$$\|E_k u|_{\Gamma_{\text{sc}}}\|_{H_k^1(\Omega)} \lesssim \|E_k u|_{\Gamma_{\text{sc}}}\|_{H_k^{3/2}(\Omega)} \lesssim \|u|_{\Gamma_{\text{sc}}}\|_{H_k^1(\Gamma_{\text{sc}})} \lesssim \|u\|.$$

□

With Lemma 1(iii) at hand, we can prove the following Gårding inequality.

Lemma 2. For all $u \in V$ it holds

$$\text{Re } b(u|T_k u) \geq c_G \|u\|^2 - C_G k^2 \|u\|_{L^2(\Omega)}^2, \quad (10)$$

with constants $C_G > c_G > 0$ that depend on Ω , $|\alpha|$, and $\text{Re } \beta$, but not on k and $\text{Im } \beta$.

Proof. Definition 2 together with Lemma 1(iii) and (iv) gives

$$\begin{aligned} b(u|T_k u) &= (\nabla u | \nabla T_k u)_\Omega - k^2 (u | T_k u)_\Omega \\ &\quad - \frac{\beta}{k} (\nabla_\Gamma u | \nabla_\Gamma T_k u)_{\Gamma_{\text{sc}}} - \alpha k (u | T_k u)_{\Gamma_{\text{sc}}} - ik (u | T_k u)_{\Gamma_{\text{ex}}} \\ &= \|\nabla u\|_{L^2(\Omega)}^2 - k^2 \|u\|_{L^2(\Omega)}^2 - 2(\nabla u | \nabla E_k u|_{\Gamma_{\text{sc}}})_\Omega + 2k^2 (u | E_k u|_{\Gamma_{\text{sc}}})_\Omega \\ &\quad + \frac{\beta}{k} \|\nabla_\Gamma u\|_{L^2(\Gamma_{\text{sc}})}^2 + \alpha k \|u\|_{L^2(\Gamma_{\text{sc}})}^2 - ik \|u\|_{L^2(\Gamma_{\text{ex}})}^2. \end{aligned}$$

Taking the real part and estimating from below, we obtain

$$\begin{aligned} \operatorname{Re} b(u|T_k u) &\gtrsim \min\{1, \operatorname{Re} \beta\} \|u\|^2 - 2k^2 \|u\|_{L^2(\Omega)}^2 \\ &\quad - |\alpha k \|u\|_{L^2(\Gamma_{\text{sc}})}^2 - |(\nabla u | \nabla E_k u|_{\Gamma_{\text{sc}}})_\Omega - k^2 (u | E_k u|_{\Gamma_{\text{sc}}})_\Omega|, \end{aligned}$$

where the contribution on Γ_{ex} drops out since it is purely imaginary.

It remains to estimate the last three terms against the energy norm and the $L^2(\Omega)$ -term. The trace inequality in product form, see, e.g., [26, Thm. 1.5.1.10], gives for every $\epsilon_1 > 0$

$$|\alpha k \|u\|_{L^2(\Gamma_{\text{sc}})}^2 \lesssim \epsilon_1 |\alpha| \|\nabla u\|_{L^2(\Omega)}^2 + \epsilon_1^{-1} |\alpha k^2| \|u\|_{L^2(\Omega)}^2.$$

We choose ϵ_1 such that $\epsilon_1 |\alpha| \lesssim \min\{1, \operatorname{Re} \beta\}$ and absorb the first term on the right-hand side into the energy norm.

Next, we treat the L^2 -term. The Cauchy–Schwarz and the Young inequality give for every $\epsilon_2 > 0$

$$k^2 |(u | E_k u|_{\Gamma_{\text{sc}}})_\Omega| \lesssim \epsilon_2 k^2 \|E_k u|_{\Gamma_{\text{sc}}}\|_{L^2(\Omega)}^2 + \epsilon_2^{-1} k^2 \|u\|_{L^2(\Omega)}^2.$$

We bound the first term with the estimate (9), i.e.

$$\epsilon_2 k^2 \|E_k u|_{\Gamma_{\text{sc}}}\|_{L^2(\Omega)}^2 \lesssim \epsilon_2 \|u|_{\Gamma_{\text{sc}}}\|_{H_k^1(\Gamma_{\text{sc}})}^2 \lesssim \epsilon_2 \|u\|^2,$$

and absorb it into the energy norm by choosing $\epsilon_2 \lesssim \min\{1, \operatorname{Re} \beta\}$.

The last term cannot be estimated directly. Instead, we use the additional regularity $\partial_n E_k u|_{\Gamma_{\text{sc}}} \in L^2(\Gamma)$. Testing the equation in Definition 1 with $\bar{u}$ and integrating by parts gives

$$(\nabla u | \nabla E_k u|_{\Gamma_{\text{sc}}})_\Omega = -k^2 (u | E_k u|_{\Gamma_{\text{sc}}})_\Omega + (u | \partial_n E_k u|_{\Gamma_{\text{sc}}})_\Gamma.$$

We have already treated the first term on the right-hand side, so only the second one is left. Again, Cauchy–Schwarz, Young, and (9) give for every $\epsilon_3 > 0$

$$\begin{aligned} k |(u | k^{-1} \partial_n E_k u|_{\Gamma_{\text{sc}}})_\Gamma| &\lesssim \epsilon_3 \|k^{-1} \partial_n E_k u|_{\Gamma_{\text{sc}}}\|_{H_k^0(\Gamma)}^2 + \epsilon_3^{-1} \|u\|_{H_k^0(\Gamma)}^2 \\ &\lesssim \epsilon_3 \|u\|^2 + \epsilon_3^{-1} \|u\|_{H_k^0(\Gamma)}^2. \end{aligned}$$

We absorb the first term into the energy norm by choosing $\epsilon_3 \lesssim \min\{1, \operatorname{Re} \beta\}$. The second term is handled as in the first estimate above with the trace inequality in product form. This proves the claim. $\square$

2.2. Well-posedness. We now combine the Gårding inequality (10) with the isomorphism property of T_k from Lemma 1(i). This shows that the problem is well-posed in the Fredholm sense.

Lemma 3. *The operator $B : V \rightarrow V'$ defined by*

$$\langle Bu | \phi \rangle_{V' \times V} = b(u | \phi), \quad \phi \in V$$

is Fredholm with index $\operatorname{ind}(B) = 0$.

Proof. Let $J : V \rightarrow V'$ be defined by $\langle Ju|\phi \rangle_{V' \times V} = (u|\phi)_\Omega$ for all $\phi \in V$. The operator J is compact. Indeed, the embedding $V \hookrightarrow H^1(\Omega)$ is bounded, and $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact by the Rellich-Kondrachov theorem.

We denote the adjoint of T_k by $T_k^* : V' \rightarrow V'$, which is again an isomorphism. Hence, the operator $A = T_k^* B + C_G k^2 J$ is bounded. Moreover, Lemma 2 gives

$$\begin{aligned} \operatorname{Re} \langle Au|u \rangle_{V' \times V} &= \operatorname{Re} \langle T_k^* B u|u \rangle_{V' \times V} + C_G k^2 \langle Ju|u \rangle_{V' \times V} \\ &= \operatorname{Re} b(u|T_k u) + C_G k^2 (u|u)_\Omega \\ &\geq c_G \|u\|^2. \end{aligned}$$

Thus, A is bounded and coercive, hence invertible. Since J is compact, $T_k^* B = A - C_G k^2 J$ is therefore Fredholm with index $\operatorname{ind}(T_k^* B) = 0$. Finally, T_k^* is an isomorphism, and we conclude that B is Fredholm with index $\operatorname{ind}(B) = 0$. $\square$

It remains to show that B is injective, since Lemma 3 then makes it invertible. For such problems, injectivity is commonly shown with a *unique continuation principle* (UCP). The passivity condition (2) allows us to apply a UCP here.

Lemma 4. *If $u \in V$ satisfies $b(u|\phi) = 0$ for all $\phi \in V$, then $u = 0$.*

Proof. We follow the lines of [32, Sec. 8] or [25, Thm. 2.1], where first-order impedance conditions are treated.

First, the passivity condition (2) on the parameters gives control over the exterior trace. Indeed, taking the imaginary part of $b(u|u) = 0$ yields

$$0 = \frac{\operatorname{Im} \beta}{k} \|\nabla_\Gamma u\|_{L^2(\Gamma_{\text{sc}})}^2 + \operatorname{Im} \alpha k \|u\|_{L^2(\Gamma_{\text{sc}})}^2 + k \|u\|_{L^2(\Gamma_{\text{ex}})}^2 \geq k \|u\|_{L^2(\Gamma_{\text{ex}})}^2,$$

since $\operatorname{Im} \beta, \operatorname{Im} \alpha \geq 0$. Thus, the solution has homogeneous trace on the exterior boundary, i.e. $u|_{\Gamma_{\text{ex}}} = 0$.

Next, we extend u by zero across Γ_{ex} . Let D be the scattering object, i.e. the bounded domain with $\partial D = \Gamma_{\text{sc}}$, and fix a ball B_R with $\overline{\Omega} \subset B_R$. We consider the bounded open set $\Omega^* = B_R \setminus \overline{D}$. It decomposes into $\Omega^* = \Omega \dot{\cup} \Gamma_{\text{ex}} \dot{\cup} \Omega_+$ with the open shell $\Omega_+ = B_R \setminus (\overline{D} \cup \overline{\Omega})$. Since the hypersurface Γ_{sc} is connected, the set Ω^* is connected as well.

Let $\tilde{u}$ denote the extension of u by zero. Then $\tilde{u} \in H^1(\Omega^*)$, and we show that $\tilde{u}$ satisfies the homogeneous Helmholtz equation in Ω^* . Let $v \in H_0^1(\Omega^*)$. The trace of v vanishes on $\partial\Omega^* \supset \Gamma_{\text{sc}}$, so that $v|_\Omega \in V$ with $v|_{\Gamma_{\text{sc}}} = 0$. Together with $u|_{\Gamma_{\text{ex}}} = 0$, all boundary terms in $b(u|v|_\Omega)$ vanish, and we conclude

$$\int_{\Omega^*} \nabla \tilde{u} \cdot \overline{\nabla v} - k^2 \tilde{u} \bar{v} \, dx = b(u|v|_\Omega) = 0, \quad \text{for all } v \in H_0^1(\Omega^*).$$

Interior elliptic regularity then shows that $\tilde{u}|_K \in H^2(K)$ for all $K \Subset \Omega^*$, see, e.g., [31, Thm. 4.16], and therefore

$$|\Delta \tilde{u}(x)| = k^2 |\tilde{u}(x)|, \quad \text{a.e. in } \Omega^*.$$

Hence, the unique continuation principle of [3] applies. The extension $\tilde{u}$ vanishes on the non-empty shell Ω_+ , and therefore on all of Ω^* . In particular, $u = 0$. $\square$

Both lemmas together give unique solvability for every right-hand side in V' .

Theorem 1. *The operator B is an isomorphism. In particular, for every $\ell \in V'$ there is a unique $u \in V$ with $b(u|\phi) = \ell(\phi)$ for all $\phi \in V$, and this solution satisfies*

$$\|u\| \leq k C_{\text{sol}} \|\ell\|_{V'}, \quad C_{\text{sol}} = k^{-1} \|B^{-1}\|_{V \leftarrow V'}. \quad (11)$$

The same holds for the adjoint operator B^ , with the same constant C_{sol} .*

Proof. By Lemma 3, B is Fredholm with index $\text{ind}(B) = 0$. Hence, B is an isomorphism if and only if it is injective, which is Lemma 4. The solution of $Bu = \ell$ then satisfies $\|u\| \leq \|B^{-1}\|_{V \leftarrow V'} \|\ell\|_{V'} = kC_{\text{sol}} \|\ell\|_{V'}$. Finally, B^* is an isomorphism as well and $\|(B^*)^{-1}\|_{V \leftarrow V'} = \|B^{-1}\|_{V \leftarrow V'}$, which covers the adjoint problem. $\square$

Proposition 1 is the special case of Theorem 1 where the right-hand side is given by (6c).

Proof of Proposition 1. It remains to estimate ℓ from (6c) in V' . By (5) we have $\|\phi\|_{H_k^0(\Omega)} \leq \|\phi\|$ and $\|\phi\|_{H_k^0(\Gamma_{\text{sc}})} \leq \|\phi\|$, while the trace inequality in product form, see, e.g., [26, Thm. 1.5.1.10], gives $\|\phi\|_{H_k^0(\Gamma_{\text{ex}})} \lesssim \|\phi\|$. With the Cauchy–Schwarz inequality, (6c) therefore yields

$$|\ell(\phi)| \lesssim (\|k^{-2}f\|_{H_k^0(\Omega)} + \|k^{-1}g_{\text{sc}}\|_{H_k^0(\Gamma_{\text{sc}})} + \|k^{-1}g_{\text{ex}}\|_{H_k^0(\Gamma_{\text{ex}})}) \|\phi\|.$$

Hence, $\ell \in V'$, and (11) proves (8). $\square$

Following the idea in [32, Prop. 8.2.7], we combine Theorem 1 with the operator T_k from Definition 2 to obtain an inf-sup condition.

Corollary 1. *For all $k > 0$ it holds*

$$\inf_{u \in V \setminus \{0\}} \sup_{\phi \in V \setminus \{0\}} \frac{\text{Re } b(u|\phi)}{\|u\| \|\phi\|} \gtrsim \frac{1}{kC_{\text{sol}}},$$

where the hidden constant depends on c_G, C_G , and C_{T_k} , but not on $\text{Im } \beta$.

Proof. Let $u \in V \setminus \{0\}$ and let $\zeta_u \in V$ be the solution of the adjoint problem

$$b(\phi|\zeta_u) = k^2(\phi|u)_\Omega, \quad \phi \in V.$$

Theorem 1 provides the stability bound

$$\|\zeta_u\| \leq kC_{\text{sol}} \|ku\|_{L^2(\Omega)} \leq kC_{\text{sol}} \|u\|.$$

We set $u^* = T_k u + C_G \zeta_u$. Lemma 1(ii) and the latter estimate then give

$$\|u^*\| \leq \|T_k u\| + C_G \|\zeta_u\| \leq (C_{T_k} + C_G kC_{\text{sol}}) \|u\|. \quad (12)$$

Finally, we bound the supremum from below with the candidate u^* , i.e.

$$\sup_{\phi \in V \setminus \{0\}} \frac{\text{Re } b(u|\phi)}{\|u\| \|\phi\|} \geq \frac{\text{Re } b(u|T_k u) + C_G k^2 \|u\|_{L^2(\Omega)}^2}{\|u\| \|u^*\|}.$$

The inequality in Lemma 2 and (12) prove the claim. $\square$

3. FINITE ELEMENT METHOD

3.1. Discrete space. For spatial discretization, we consider a curved mesh $\mathcal{T}_h$ of Ω . The mesh $\mathcal{T}_h$ is a finite collection of closed curved elements K . Every element is equipped with a bi-Lipschitz reference map $\mathcal{F}_K : \hat{K}_d \rightarrow K$, so that the elements do not overlap and cover $\bar{\Omega}$. We further assume that the mesh is matching. Let $K_\pm \in \mathcal{T}_h$ be distinct, and let $F = K_- \cap K_+$ be a $(d-1)$ -manifold. Then $\mathcal{F}_{K_\pm}^{-1}(F)$ are faces of $\hat{K}_d$, and the two maps coincide up to an affine reparametrization. We define the mesh size h as the maximal diameter $h_K = \text{diam}(K)$ of the individual elements. Finally, we define the shape-regularity of an individual element K as

$$M_K = \max\{h_K \|(J\mathcal{F}_K)^{-1}\|_{L^\infty(\hat{K}_d)}, h_K^{-1} \|J\mathcal{F}_K\|_{L^\infty(\hat{K}_d)}\}, \quad (13)$$

where $J\mathcal{F}_K$ is the Jacobian of $\mathcal{F}_K$, and we use the 2-norm for matrices. The mesh $\mathcal{T}_h$ is shape-regular if $M_{\mathcal{T}_h} = \max_K M_K$ is bounded independently of h . Note that the reference maps $\mathcal{F}_K$ are not assumed to be polynomial. We refer to the construction in [13, Def. A.1] or to Appendix A.

For fixed $p \in \mathbb{N}_+$, we define the Lagrange finite element space associated with the mesh $\mathcal{T}_h$ as

$$V_h = \{v_h \in C(\overline{\Omega}) \mid v_h|_K \circ \mathcal{F}_K \in \mathbb{P}_p(\widehat{K}_d), \quad K \in \mathcal{T}_h\}, \quad (14)$$

where $\mathbb{P}_p(\widehat{K}_d)$ denotes the space of polynomials in d variables of total degree at most p .

Mesh regularity. The shape-regularity (13) controls only the first derivatives of the reference maps. High-order approximation on a curved mesh needs control over the higher derivatives as well. We therefore define for $r \geq 1$ the mesh quantities

$$M_K(r) = \max_{1 \leq |\alpha| \leq r} h_K^{-|\alpha|} \|\partial^\alpha \mathcal{F}_K\|_{L^\infty(\widehat{K}_d)}, \quad M_{\mathcal{T}_h}(r) = \max_K M_K(r). \quad (15)$$

A mesh is called r -regular if $\mathcal{F}_K \in C^r(\widehat{K}_d, K)$ and $M_{\mathcal{T}_h}(r)$ is bounded independently of h .

Remark 2. (a) *The abstract description of curved finite elements in Section 3.1 is in line with the construction of Bernardi in [9]. The reference map $F_K = \widetilde{F}_K + \Phi_K : \widehat{K}_d \rightarrow K$ with affine $\widetilde{F}_K$ and bi-Lipschitz Φ_K in (2.1) of [9] is recovered by Taylor. For any $\widehat{\mathbf{a}} \in \widehat{K}_d$ we set $\widetilde{F}_K(\widehat{\mathbf{x}}) = \mathcal{F}_K(\widehat{\mathbf{a}}) + J\mathcal{F}_K(\widehat{\mathbf{a}})(\widehat{\mathbf{x}} - \widehat{\mathbf{a}})$ and $\Phi_K = \mathcal{F}_K - \widetilde{F}_K$. The mesh constants c_K of (2.2) and c_r of (2.7) in [9] are equivalent to (13) and (15) respectively.*

(b) *Furthermore, in Section 6 of [9], different algorithms for the construction of regular curved elements for a smooth domain are discussed, with the procedure due to Lenoir in Section 6.2 guaranteeing the existence of a $(\gamma+1)$ -regular curved mesh for a $C^{1,\gamma}$ -smooth domain.*

3.2. Discrete problem. The mesh and the discrete space are now fixed. We discretize (6a) by replacing V with V_h and obtain:

$$\text{Seek } u_h \in V_h \text{ such that } b(u_h|\phi_h) = \ell(\phi_h), \quad \text{for all } \phi_h \in V_h. \quad (16)$$

The remainder of this subsection proves a discrete counterpart of the inf-sup stability in Corollary 1. The argument is standard once the Gårding inequality in Lemma 2 is available. Section 3.3 then proves high-order quasi-optimality. Two auxiliary results are stated along the way. We prove them in Section 3.4 and in Section 3.5.

Approximability. We introduce two quantities that measure how well the finite element space resolves the problem. The first one concerns the adjoint problem. We define the adjoint approximability as

$$\gamma_{\text{app}} = k \sup_{g \in L^2(\Omega) \setminus \{0\}} \frac{\min_{v_h \in V_h} \|\sigma_g - v_h\|}{\|g\|_{L^2(\Omega)}}, \quad (17)$$

where $\sigma_g \in V$ solves $b(\phi|\sigma_g) = (\phi|g)_\Omega$ for all $\phi \in V$. The embedding $V \hookrightarrow L^2(\Omega)$ is compact. This ensures that $\gamma_{\text{app}} \rightarrow 0$ as $h \rightarrow 0^+$.

The second quantity concerns the operator T_k . We only have a Gårding inequality with respect to T_k . We therefore measure how far T_k moves a discrete function out of V_h . Hence, we define

$$\gamma_{T_k} = \max_{u_h \in V_h \setminus \{0\}} \frac{\min_{v_h \in V_h} \|T_k u_h - v_h\|}{\|u_h\|}. \quad (18)$$

We prove the following approximation result in Section 3.4.

Theorem 2. *Let $\mathcal{T}_h$ be 2-regular. Then $\gamma_{T_k} \lesssim \sqrt{kh}$, where the hidden constant depends on Ω , d , p , and on $M_{\mathcal{T}_h}(2)$, but not on h , k , and C_{sol} .*

The assumption that the mesh is 2-regular is not a restriction. We only deal with $C^{\gamma,1}$ -regular domains Ω , where $\gamma \in \mathbb{N}_+$. In that case, a 2-regular mesh can be constructed, cf. Remark 2 (b).

Discrete inf-sup condition and well-posedness. The analysis repeatedly compares a function in V with its best approximation in V_h . We therefore introduce the best-approximation mapping

$$\Pi_h : V \rightarrow V_h, \quad \Pi_h v = \arg \min_{v_h \in V_h} \|v - v_h\|. \quad (19)$$

The space V_h is a closed subspace of V . Moreover, V is complete with respect to the inner product induced by the energy norm (5). Hence, Π_h is the orthogonal projection onto V_h . In particular, it is linear with $\|\Pi_h\|_{V \rightarrow V} \leq 1$.

We are now able to prove the discrete inf-sup stability. The argument follows along the lines of Corollary 1. The candidate u^* constructed there is not discrete, so we replace it by its best approximation. The two quantities (17) and (18) measure the error of this replacement.

Lemma 5. *Let $\mathcal{T}_h$ be 2-regular. If $kh \ll 1$ and $\gamma_{\text{app}} \ll 1$ sufficiently small, then*

$$\min_{u_h \in V_h \setminus \{0\}} \max_{v_h \in V_h \setminus \{0\}} \frac{\operatorname{Re} b(u_h|v_h)}{\|u_h\| \|v_h\|} \gtrsim \frac{1}{kC_{\text{sol}}}, \quad (20)$$

where the hidden constant depends on c_G , C_G , C_{T_k} , and on the continuity constant of b . In particular, it remains bounded as $\operatorname{Im} \beta \rightarrow 0$, with α and $\operatorname{Re} \beta > 0$ fixed.

Proof. Let $u_h \in V_h$. We define $u^* = T_k u_h + C_G \zeta_{u_h} \in V$, where $\zeta_{u_h} \in V$ solves

$$b(\phi|\zeta_{u_h}) = k^2(\phi|u_h)_\Omega, \quad \phi \in V.$$

Theorem 1 gives $\|\zeta_{u_h}\| \lesssim kC_{\text{sol}} \|u_h\|_{H_k^0(\Omega)}$. We approximate u^* in V_h by its best approximation, i.e.

$$u_h^* = \Pi_h u^* = \Pi_h T_k u_h + C_G \Pi_h \zeta_{u_h}.$$

Then $\|T_k u_h - \Pi_h T_k u_h\| \leq \gamma_{T_k} \|u_h\|$ by (18) and

$$\frac{\|\zeta_{u_h} - \Pi_h \zeta_{u_h}\|}{\|u_h\|} \lesssim \frac{\|\zeta_{u_h} - \Pi_h \zeta_{u_h}\|}{k \|u_h\|_{L^2(\Omega)}} = k \frac{\|\zeta_{u_h} - \Pi_h \zeta_{u_h}\|}{\|k^2 u_h\|_{L^2(\Omega)}} \lesssim \gamma_{\text{app}}$$

by (17). Hence, $\|u^* - u_h^*\| \lesssim (\gamma_{T_k} + \gamma_{\text{app}}) \|u_h\|$. With Lemma 2 it holds

$$\operatorname{Re} b(u_h|u_h^*) = \operatorname{Re} b(u_h|u^*) - \operatorname{Re} b(u_h|u^* - u_h^*) \gtrsim (C - \gamma_{T_k} - \gamma_{\text{app}}) \|u_h\|^2,$$

with a constant $C > 0$ that depends on the continuity constant of $\operatorname{Re} b$, on C_G , and on c_G . We further obtain with $\|\Pi_h\|_{V \rightarrow V} \leq 1$ that

$$\|u_h^*\| \lesssim \|T_k u_h\| + \|\zeta_{u_h}\| \lesssim C_{T_k} \|u_h\| + kC_{\text{sol}} \|u_h\|_{H_k^0(\Omega)} \leq (C_{T_k} + kC_{\text{sol}}) \|u_h\|.$$

It remains to bound the maximum in (20) from below by the candidate u_h^* , that is

$$\max_{v_h \in V_h \setminus \{0\}} \frac{\operatorname{Re} b(u_h|v_h)}{\|u_h\| \|v_h\|} \geq \frac{\operatorname{Re} b(u_h|u_h^*)}{\|u_h\| \|u_h^*\|} \gtrsim \frac{C - \gamma_{T_k} - \gamma_{\text{app}}}{C_{T_k} + kC_{\text{sol}}} \gtrsim \frac{1}{kC_{\text{sol}}},$$

where we used the smallness assumption and $1 \lesssim kC_{\text{sol}}$. $\square$

Trial and test space in (16) coincide, and V_h is finite dimensional. The discrete inf-sup condition (20) therefore already gives discrete well-posedness.

Corollary 2. *Let $\mathcal{T}_h$ be 2-regular, and let $kh \ll 1$ and $\gamma_{\text{app}} \ll 1$ be sufficiently small. For any $\ell \in V_h'$, the discrete problem (16) has a unique solution $u_h \in V_h$, and this solution satisfies*

$$\|u_h\| \lesssim kC_{\text{sol}} \|\ell\|_{V_h'},$$

with the hidden constant of Lemma 5.

3.3. High-order quasi-optimality. We prove high-order quasi-optimality of the discretization in this section. The argument uses ideas from [12, 14, 23]. Its starting point is a spectral low-pass filter that keeps only the frequencies below the wavenumber k . The filter improves the Gårding inequality of Lemma 2, since it replaces the compact perturbation. It also maps into a space of smooth functions. This is what makes the duality step of the usual Schatz argument pay off.

The main result of this section reads as follows.

Theorem 3. *Let $p \leq \gamma$ and $\mathcal{T}_h$ be $(p+1)$ -regular. Let $u \in V$ and $u_h \in V_h$ denote the solutions of (6a) and (16), respectively. If $kh \ll 1$ and $k(kh)^p C_{\text{sol}} \ll 1$ sufficiently small, then*

$$\|u - u_h\| \lesssim \min_{v_h \in V_h} \|u - v_h\|,$$

where the hidden constant and the two implicit thresholds are independent of k , h , u , u_h , and of the data $(f, g_{\text{sc}}, g_{\text{ex}})$.

The proof needs three ingredients. The first one is the low-pass filter. We use it only through the following two properties. Section 3.5 constructs an operator with these properties.

Theorem 4. *Let $p \leq \gamma$. We denote by $\Pi_\Lambda : V \rightarrow \Lambda_k$ the L^2 -orthogonal projection onto a subspace $\Lambda_k \subset V$. The subspace is finite dimensional and has the following two properties.*

- (i) *For all $u \in V$ it holds*

$$\text{Re } b(u|T_k u) \geq \tilde{c}_G \|u\|^2 - \tilde{C}_G k^2 \|\Pi_\Lambda u\|_{L^2(\Omega)}^2,$$

with constants $\tilde{C}_G > \tilde{c}_G > 0$ that depend on Ω , $|\alpha|$, and $\text{Re } \beta$, but not on k and $\text{Im } \beta$.

- (ii) *The subspace Λ_k is contained in $H^{p+1}(\Omega)$, and for all $u \in V$ it holds*

$$\|\Pi_\Lambda u\|_{H_k^{p+1}(\Omega)} \lesssim \|\Pi_\Lambda u\|_{H_k^0(\Omega)},$$

where the hidden constant depends on Ω , p , and on the constants in the T -Gårding inequality (10), but not on k or $\text{Im } \beta$.

Property (i) improves Lemma 2, since it replaces the full $L^2(\Omega)$ -norm by the norm of the filtered function. Property (ii) is what makes the improvement pay off. It states that the filter maps into a space of smooth functions. On this space, all k -scaled Sobolev norms up to order $p+1$ are equivalent.

The second ingredient is elliptic regularity of the adjoint problem. We use it in a duality argument where the right-hand side is in the range of Π_Λ , and therefore smooth. Theorem 6 and Corollary 3 in Section 4 give the following result. Indeed, their proofs also apply to the adjoint boundary conditions $\partial_n \zeta + ik\zeta = 0$ on Γ_{ex} and $\partial_n \zeta + \bar{\beta}k^{-1}\Delta_{\Gamma_{\text{sc}}}\zeta - \bar{\alpha}k\zeta = 0$ on Γ_{sc} , since the signs and phases of the coefficients do not enter the regularity estimates.

Proposition 2. *Let $f \in H^{p-1}(\Omega)$ and let $\zeta \in V$ solve the adjoint problem*

$$b(\psi|\zeta) = (\psi|f)_\Omega, \quad \psi \in V. \quad (21)$$

Then $\zeta \in H^{p+1}(\Omega)$ and $\zeta|_{\Gamma_{\text{sc}}} \in H^{p+1}(\Gamma_{\text{sc}})$ with

$$\|\zeta\|_{H_k^{p+1}(\Omega)} + \|\zeta\|_{H_k^{p+1}(\Gamma_{\text{sc}})} \lesssim k C_{\text{sol}} \|k^{-2}f\|_{H_k^{p-1}(\Omega)}, \quad (22)$$

where the hidden constant is independent of k , f , and ζ .

The third ingredient is a pair of bulk and surface interpolation operators.

Proposition 3. *Let $p \leq \gamma$ and $\mathcal{T}_h$ be $(p+1)$ -regular. There exists a pair of interpolation operators*

$$(\mathcal{I}_h, \mathcal{I}_h^{\Gamma_{\text{sc}}}) : H^{p+1}(\Omega) \times H^{p+1}(\Gamma_{\text{sc}}) \rightarrow V_h \times V_h|_{\Gamma_{\text{sc}}},$$

such that for $m \in \{2, \dots, p+1\}$ it holds

$$\|v - \mathcal{I}_h v\|_{H_k^1(\Omega)} + \|v|_{\Gamma_{\text{sc}}} - \mathcal{I}_h^{\Gamma_{\text{sc}}} v|_{\Gamma_{\text{sc}}}\|_{H_k^1(\Gamma_{\text{sc}})} \lesssim (kh)^{m-1} (\|v\|_{H_k^m(\Omega)} + \|v\|_{H_k^m(\Gamma_{\text{sc}})}).$$

The hidden constant is independent of v and h .

An Oswald-type interpolator that realizes $\mathcal{I}_h$ in our mesh setting is constructed in [13, App. A]. Remark 2 (b) places our description in the setting of Bernardi. The Cl  ment-type quasi-interpolator of [9, Thm. 4.1] is therefore a second realization of $\mathcal{I}_h$. A Lagrange-type interpolator that realizes $\mathcal{I}_h^{\Gamma_{\text{sc}}}$ is constructed in [17, Prop. 2.7], choosing $I = I_h$ in the notation of that paper. This construction is compatible with [9], and hence with our setting. We also refer to [20, Prop. 5.4] for a bulk surface interpolator in the fashion of Proposition 3.

All three ingredients are in place, and we can prove the main result.

Proof of Theorem 3. All hidden constants in this proof are independent of k , h , and the data. Theorem 4(i) gives

$$\|u - u_h\|^2 \lesssim \text{Re } b(u - u_h | T_k(u - u_h)) + k^2 \|\Pi_\Lambda(u - u_h)\|_{L^2(\Omega)}^2. \quad (23)$$

We treat the first term on the right-hand side. We add and subtract the best approximation (19) and obtain

$$\text{Re } b(u - u_h | T_k(u - u_h)) = \text{Re } b(u - u_h | T_k(1 - \Pi_h)(u - u_h) + T_k \Pi_h(u - u_h)).$$

For the first term, the uniform bound in Lemma 1(ii) gives

$$\begin{aligned} \text{Re } b(u - u_h | T_k(1 - \Pi_h)(u - u_h)) &\lesssim \|u - u_h\| \|T_k(1 - \Pi_h)(u - u_h)\| \\ &\lesssim \|u - u_h\| \|(1 - \Pi_h)(u - u_h)\| \\ &\lesssim \|u - u_h\| \|(1 - \Pi_h)u\|, \end{aligned}$$

where we used that $(1 - \Pi_h)u_h = 0$. For the second term, Galerkin orthogonality and the definition (18) of γ_{T_k} give

$$\begin{aligned} |b(u - u_h | T_k \Pi_h(u - u_h))| &= \min_{v_h \in V_h} |b(u - u_h | T_k \Pi_h(u - u_h) - v_h)| \\ &\lesssim \gamma_{T_k} \|u - u_h\| \|\Pi_h(u - u_h)\| \\ &\lesssim \gamma_{T_k} \|u - u_h\|^2 + \gamma_{T_k} \|u - u_h\| \|(1 - \Pi_h)u\|, \end{aligned}$$

where we used that $\Pi_h(u - u_h) = u - u_h - (1 - \Pi_h)u$.

We now divide the first term in (23) by $\|u - u_h\|$. The previous two estimates then give

$$\frac{\text{Re } b(u - u_h | T_k(u - u_h))}{\|u - u_h\|} \lesssim \gamma_{T_k} \|u - u_h\| + (1 + \gamma_{T_k}) \min_{v_h \in V_h} \|u - v_h\|. \quad (24)$$

Theorem 2 gives $\gamma_{T_k} \lesssim \sqrt{kh}$. The condition $kh \ll 1$ therefore lets us absorb the first term on the right-hand side into the left-hand side. It remains to treat the second term of (23) with a duality argument.

Let $\zeta \in V$ denote the solution of

$$b(\psi | \zeta) = k(\psi | \Pi_\Lambda(u - u_h))_\Omega, \quad \psi \in V. \quad (25)$$

Theorem 4(ii) gives $k\Pi_\Lambda(u - u_h) \in H^{p+1}(\Omega) \subset H^{p-1}(\Omega)$. Proposition 2 therefore provides $\zeta \in H^{p+1}(\Omega)$ and $\zeta|_{\Gamma_{\text{sc}}} \in H^{p+1}(\Gamma_{\text{sc}})$, together with the estimate

$$\begin{aligned} \|\zeta\|_{H_k^{p+1}(\Omega)} + \|\zeta\|_{H_k^{p+1}(\Gamma_{\text{sc}})} &\lesssim kC_{\text{sol}} \|k^{-2}k\Pi_\Lambda(u - u_h)\|_{H_k^{p-1}(\Omega)} \\ &= C_{\text{sol}} \|\Pi_\Lambda(u - u_h)\|_{H_k^{p-1}(\Omega)} \\ &\lesssim C_{\text{sol}} \|\Pi_\Lambda(u - u_h)\|_{H_k^0(\Omega)}, \end{aligned} \quad (26)$$

where we used Theorem 4(ii) for the latter estimate.

Next, we insert $\psi = u - u_h$ into (25). Galerkin orthogonality then shows

$$k \|\Pi_\Lambda(u - u_h)\|_{L^2(\Omega)}^2 = \min_{v_h \in V_h} |b(u - u_h, \zeta - v_h)| \lesssim \|u - u_h\| \min_{v_h \in V_h} \|\zeta - v_h\|. \quad (27)$$

Since the Scott–Zhang operator of Proposition 4 is uniformly stable and is a projection both onto V_h and onto its discrete trace space, the stable discrete-lifting argument of [18, Thm. 2.1 and Rem. 4] allows the bulk and surface approximations to be chosen independently. The interpolation results in Proposition 3 therefore provide

$$\begin{aligned} \min_{v_h \in V_h} \|\zeta - v_h\| &\lesssim \|\zeta - \mathcal{I}_h \zeta\|_{H_k^1(\Omega)} + \|\zeta|_{\Gamma_{\text{sc}}} - \mathcal{I}_h^{\Gamma_{\text{sc}}} \zeta|_{\Gamma_{\text{sc}}}\|_{H_k^1(\Gamma_{\text{sc}})} \\ &\lesssim (hk)^p (\|\zeta\|_{H_k^{p+1}(\Omega)} + \|\zeta\|_{H_k^{p+1}(\Gamma_{\text{sc}})}). \end{aligned}$$

With (26) we conclude that

$$\begin{aligned} \min_{v_h \in V_h} \|\zeta - v_h\| &\lesssim (hk)^p C_{\text{sol}} \|\Pi_\Lambda(u - u_h)\|_{H_k^0(\Omega)} \\ &= k(hk)^p C_{\text{sol}} \|\Pi_\Lambda(u - u_h)\|_{L^2(\Omega)}. \end{aligned}$$

We insert this into (27) and divide by $\|\Pi_\Lambda(u - u_h)\|_{L^2(\Omega)}$. This gives

$$k \|\Pi_\Lambda(u - u_h)\|_{L^2(\Omega)} \lesssim k(hk)^p C_{\text{sol}} \|u - u_h\|.$$

The condition $k(hk)^p C_{\text{sol}} \ll 1$ therefore lets us absorb the second term of (23) into the left-hand side. Together with (24), this proves the claim. $\square$

3.4. Approximation of the operator T_k . It remains to prove Theorem 2 and Theorem 4. We start with the approximation of T_k . The main tool is the Scott–Zhang quasi-interpolator of [33]. It is a projection onto V_h , and it is also a projection on the discrete trace space. To the best of our knowledge, such an operator has not been constructed on curved meshes yet. We provide a construction in Appendix A and collect the properties that we need below.

Proposition 4 (Scott–Zhang interpolator). *Let $kh \lesssim 1$. There is a linear operator $\Pi_{\text{SZ}} : H^1(\Omega) \rightarrow V_h$ with the following properties.*

- (i) *It is a projection onto V_h , that is $\Pi_{\text{SZ}} v_h = v_h$ for all $v_h \in V_h$.*
- (ii) *It is a projection on the discrete trace space, that is $(\Pi_{\text{SZ}} v)|_\Gamma = v|_\Gamma$ whenever $v \in H^1(\Omega)$ and $v_h \in V_h$ satisfy $v|_\Gamma = v_h|_\Gamma$.*
- (iii) *It is stable in $H^1(\Omega)$, that is $\|\Pi_{\text{SZ}} v\|_{H_k^1(\Omega)} \lesssim \|v\|_{H_k^1(\Omega)}$ for all $v \in H^1(\Omega)$.*

The hidden constant in (iii) depends on d , p , and on the shape-regularity of $\mathcal{T}_h$, but not on h and v .

Proof of Theorem 2. Let $u_h \in V_h \setminus \{0\}$. The image $T_k u_h = u_h - 2E_k u_h|_{\Gamma_{\text{sc}}}$ is a sum whose first summand is discrete. Hence, γ_{T_k} reduces to

$$\gamma_{T_k} = 2 \max_{u_h \in V_h \setminus \{0\}} \frac{\min_{w_h \in V_h} \|E_k u_h|_{\Gamma_{\text{sc}}} - w_h\|}{\|u_h\|}. \quad (28)$$

Set $w = E_k u_h|_{\Gamma_{\text{sc}}}$. Definition 1 together with Lemma 1(iii) shows that E_k reproduces discrete traces. Indeed, $w|_\Gamma = (E_k u_h|_{\Gamma_{\text{sc}}})|_\Gamma \in V_h|_\Gamma$. Let Π_{SZ} denote the

Scott–Zhang operator from Proposition 4. With Proposition 4(ii) and (iii), we obtain for any $v_h \in V_h$

$$\|(1 - \Pi_{SZ})w\|^2 = \|(1 - \Pi_{SZ})w\|_{H_k^1(\Omega)}^2 \lesssim \|w - v_h\|_{H_k^1(\Omega)}^2,$$

since $(1 - \Pi_{SZ})v_h = 0$ by Proposition 4 (i).

Definition 1 gives $w \in H^{3/2}(\Omega)$. We make use of this fractional regularity by choosing v_h as the image of w under a suitable interpolation operator. For example, we can interpolate the Cl  ment-type quasi-interpolator $\Pi_{CL} : H^1(\Omega) \rightarrow V_h$ of [9, Thm. 4.1] between $H^1(\Omega)$ and $H^2(\Omega)$. With $v_h = \Pi_{CL}w$ we obtain

$$\begin{aligned} \|w - \Pi_{CL}w\|_{H_k^1(\Omega)}^2 &= k^2 \|w - \Pi_{CL}w\|_{L^2(\Omega)}^2 + |w - \Pi_{CL}w|_{H^1(\Omega)}^2 \\ &\lesssim kh(k^2h^2 + 1)k^{-1} |w|_{H^{3/2}(\Omega)}^2 \leq kh(k^2h^2 + 1) \|w\|_{H_k^{3/2}(\Omega)}^2. \end{aligned}$$

We do not interpolate here between Sobolev spaces with prescribed boundary conditions. The half-integer case therefore causes no problem.

It remains to estimate the last factor. The bounds in (9) give

$$\|w\|_{H_k^{3/2}(\Omega)} \lesssim \|u_h|_{\Gamma_{sc}}\|_{H_k^1(\Gamma_{sc})} \leq \|u_h\|.$$

With (28) this proves the claim. $\square$

3.5. Construction of the low-pass filter. It remains to prove Theorem 4. We build the filter from the eigenfunctions of the Neumann Laplacian. These functions are smooth, and this is what property (ii) of Theorem 4 rests on.

Let $\{(\psi_j, \lambda_j^2)\}_{j \geq 0}$ denote the eigenpairs of

$$-\Delta \psi_j = \lambda_j^2 \psi_j, \quad \text{in } \Omega, \quad \partial_n \psi_j = 0, \quad \text{on } \Gamma, \quad j \geq 0, \quad (29)$$

where the eigenvalues are sorted as $0 = \lambda_0 < \lambda_1 \leq \dots$ and repeated according to their multiplicity. The eigenfunctions corresponding to $\lambda_0 = 0$ are the constant functions. In particular, $\int_{\Omega} \psi_j \, dx = 0$ for $j \geq 1$. We choose the eigenfunctions so that they are orthonormal with respect to the L^2 -product, i.e.

$$(\psi_j | \psi_\ell)_\Omega = \delta_{j\ell}, \quad (\nabla \psi_j | \nabla \psi_\ell)_\Omega = \lambda_j^2 \delta_{j\ell}, \quad j, \ell \geq 0. \quad (30)$$

We obtain the second identity in (30) by multiplying the first expression of (29) with $\overline{\psi_\ell}$ and integrating by parts. Consequently, every $u \in H^1(\Omega)$ has the expansion $u = \sum_{j \geq 0} (u | \psi_j)_\Omega \psi_j$. With (30) this gives the identities

$$\|u\|_{L^2(\Omega)}^2 = \sum_{j \geq 0} |(u | \psi_j)_\Omega|^2, \quad \|\nabla u\|_{L^2(\Omega)}^2 = \sum_{j \geq 0} \lambda_j^2 |(u | \psi_j)_\Omega|^2. \quad (31)$$

Let $C^* = c_G^{-1}(C_G - c_G) > 0$, where $C_G > c_G > 0$ denote the constants from Lemma 2. We define the low-pass filter as

$$\Pi_\Lambda : H^1(\Omega) \rightarrow \Lambda_k = \text{span}\{\psi_j \mid \lambda_j^2 \leq 2C^*k^2\}, \quad u \mapsto \sum_{\lambda_j^2 \leq 2C^*k^2} (u | \psi_j)_\Omega \psi_j. \quad (32)$$

We choose the cut-off $2C^*k^2$ since the frequencies above it already carry enough coercivity. This is the content of Lemma 6 below.

Lemma 6. *For all $u \in V$ it holds*

$$\text{Re } b(u | T_k u) \geq \tilde{c}_G \|u\|^2 - \tilde{C}_G k^2 \|\Pi_\Lambda u\|_{L^2(\Omega)}^2,$$

with constants $\tilde{C}_G > \tilde{c}_G > 0$ that depend on Ω , $|\alpha|$, and $\text{Re } \beta$, but not on k and $\text{Im } \beta$.

Proof. Let $u \in V$. Lemma 2 gives

$$\begin{aligned} & c_G^{-1} \operatorname{Re} b(u|T_k u) + 2k^2 C^* \|\Pi_\Lambda u\|_{L^2(\Omega)}^2 \\ & \geq \|u\|^2 - c_G^{-1} C_G k^2 \|u\|_{L^2(\Omega)}^2 + 2k^2 C^* \|\Pi_\Lambda u\|_{L^2(\Omega)}^2 \\ & \geq k^{-1} \|\nabla_{\Gamma_{\text{sc}}} u\|_{L^2(\Gamma_{\text{sc}})}^2 + k \|u\|_{L^2(\Gamma_{\text{sc}})}^2 \\ & \quad + \left(\|\nabla u\|_{L^2(\Omega)}^2 - C^* k^2 \|u\|_{L^2(\Omega)}^2 + 2k^2 C^* \|\Pi_\Lambda u\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

It remains to bound the last term from below. The identities (31) and the definition (32) give

$$\begin{aligned} & \|\nabla u\|_{L^2(\Omega)}^2 - C^* k^2 \|u\|_{L^2(\Omega)}^2 + 2k^2 C^* \|\Pi_\Lambda u\|_{L^2(\Omega)}^2 \\ & = \sum_{j \geq 0} (\lambda_j^2 - C^* k^2) |(u|\psi_j)_\Omega|^2 + 2k^2 C^* \sum_{\lambda_j^2 \leq 2C^* k^2} |(u|\psi_j)_\Omega|^2 \\ & = \sum_{\lambda_j^2 \leq 2C^* k^2} (\lambda_j^2 + C^* k^2) |(u|\psi_j)_\Omega|^2 + \sum_{\lambda_j^2 > 2C^* k^2} (\lambda_j^2 - C^* k^2) |(u|\psi_j)_\Omega|^2 \\ & \geq \frac{1}{4} \sum_{j \geq 0} (\lambda_j^2 + C^* k^2) |(u|\psi_j)_\Omega|^2, \end{aligned}$$

where we used in the last inequality that

$$\lambda_j^2 > 2C^* k^2 \implies \lambda_j^2 - C^* k^2 \geq \frac{1}{4} (\lambda_j^2 + C^* k^2).$$

This proves the claim. $\square$

The second property concerns the smoothness of the filtered functions. It rests on the following regularity of the eigenfunctions.

Lemma 7. *For any non-constant eigenfunction ψ_j in (29) it holds*

$$\psi_j \in H^{\gamma+1}(\Omega), \quad \|\psi_j\|_{H^m(\Omega)} \lesssim |\lambda_j|^m, \quad m \leq \gamma + 1,$$

where the hidden constant depends on Ω and γ , but not on j .

Proof. Let $j \geq 1$. The claim for $m = 0$ and $m = 1$ holds by (30). Assume that it is true for some $0 \leq m \leq \gamma - 1$. Elliptic regularity, see, e.g., [31, Thm. 4.16, 4.18], gives

$$\|\psi_j\|_{H^{m+2}(\Omega)} \lesssim \|\Delta \psi_j\|_{H^m(\Omega)} + \|\psi_j\|_{H^1(\Omega)} = \lambda_j^2 \|\psi_j\|_{H^m(\Omega)} + \|\psi_j\|_{H^1(\Omega)}.$$

The induction hypothesis and the case $m = 1$ then show that

$$\|\psi_j\|_{H^{m+2}(\Omega)} \lesssim \lambda_j^{2+m} + \lambda_j \lesssim \lambda_j^{2+m} (1 + \lambda_1^{-m-1}),$$

where we used that $1 \leq \lambda_1^{-1} \lambda_j$. Interpolation then provides the intermediate estimate

$$\|\psi_j\|_{H^{m+1}(\Omega)} \leq \|\psi_j\|_{H^m(\Omega)}^{1/2} \|\psi_j\|_{H^{m+2}(\Omega)}^{1/2} \lesssim \lambda_j^{m/2} \lambda_j^{m/2+1}.$$

This proves the claim. $\square$

Lemma 8. *For all $f \in \Lambda_k$ it holds*

$$\|f\|_{H_k^{\gamma+1}(\Omega)} \lesssim \|f\|_{H_k^0(\Omega)},$$

where the hidden constant depends on Ω , γ , and on the constants in the T -Gårding inequality (10), but not on k , $\operatorname{Im} \beta$, or the dimension of Λ_k .

Proof. Let $f = \sum_{\lambda_j^2 \leq 2C^*k^2} c_j \psi_j \in \Lambda_k$ with $c_j = (f|_{\psi_j})_\Omega$. By the definition (4a) of the k -scaled norms, it suffices to show

$$\|f\|_{H^m(\Omega)} \lesssim k^m \|f\|_{L^2(\Omega)}, \quad 0 \leq m \leq \gamma + 1, \quad (33)$$

with a constant independent of k and of $\dim \Lambda_k$. For $\ell \in \mathbb{N}$ we set

$$g_\ell := (-\Delta)^\ell f = \sum_{\lambda_j^2 \leq 2C^*k^2} \lambda_j^{2\ell} c_j \psi_j \in \Lambda_k,$$

so that

$$\Delta g_\ell = -g_{\ell+1}, \quad \text{in } \Omega, \quad \partial_n g_\ell = 0, \quad \text{on } \Gamma.$$

We have $\lambda_j^2 \leq 2C^*k^2$, so the identities (31) give

$$\begin{aligned} \|g_\ell\|_{L^2(\Omega)}^2 &= \sum_{\lambda_j^2 \leq 2C^*k^2} \lambda_j^{4\ell} |c_j|^2 \leq (2C^*k^2)^{2\ell} \|f\|_{L^2(\Omega)}^2, \\ \|\nabla g_\ell\|_{L^2(\Omega)}^2 &= \sum_{\lambda_j^2 \leq 2C^*k^2} \lambda_j^{4\ell+2} |c_j|^2 \leq (2C^*k^2)^{2\ell+1} \|f\|_{L^2(\Omega)}^2, \end{aligned} \quad (34)$$

i.e. $\|g_\ell\|_{H^m(\Omega)} \lesssim k^{m+2\ell} \|f\|_{L^2(\Omega)}$ for $m \in \{0, 1\}$ and all $\ell \in \mathbb{N}$. The orthonormality (30) gives a constant that does not deteriorate with $\dim \Lambda_k$. A triangle inequality would give a constant that does.

We show $\|g_\ell\|_{H^m(\Omega)} \lesssim k^{m+2\ell} \|f\|_{L^2(\Omega)}$ for all $\ell \in \mathbb{N}$ and $m + 2\ell \leq \gamma + 1$ by induction over m . The cases $m \in \{0, 1\}$ are (34). Assume that the assertion holds for some $m \geq 0$ and all admissible ℓ . We apply elliptic regularity for the Neumann problem, see, e.g., [31, Thm. 4.16, 4.18], to g_ℓ . With the induction hypothesis for $g_{\ell+1}$ this gives

$$\begin{aligned} \|g_\ell\|_{H^{m+2}(\Omega)} &\lesssim \|\Delta g_\ell\|_{H^m(\Omega)} + \|g_\ell\|_{H^1(\Omega)} \\ &= \|g_{\ell+1}\|_{H^m(\Omega)} + \|g_\ell\|_{H^1(\Omega)} \\ &\lesssim (k^{m+2\ell+2} + k^{2\ell+1}) \|f\|_{L^2(\Omega)} \\ &\lesssim k^{m+2\ell+2} \|f\|_{L^2(\Omega)}, \end{aligned}$$

where we used $k \gg 1$ in the last step. The choice $\ell = 0$ gives (33) and thus

$$\|f\|_{H_k^{\gamma+1}(\Omega)}^2 \lesssim \sum_{m=0}^{\gamma+1} k^{-2m+2} \|f\|_{H^m(\Omega)}^2 \lesssim k^2 \|f\|_{L^2(\Omega)}^2 = \|f\|_{H_k^0(\Omega)}^2. \quad \square$$

The three lemmas together prove Theorem 4.

Proof of Theorem 4. We take the operator Π_Λ of (32), which is linear and maps $V \subset H^1(\Omega)$ into $\Lambda_k \subset V$. Property (i) is Lemma 6. For (ii) we note that Λ_k is spanned by finitely many eigenfunctions. Lemma 7 therefore gives $\Lambda_k \subset H^{\gamma+1}(\Omega) \subset H^{p+1}(\Omega)$. The norm bound is Lemma 8 applied to $f = \Pi_\Lambda u$. $\square$

4. WAVENUMBER-EXPLICIT ELLIPTIC REGULARITY

We first introduce norms that are localized to a relatively open subset of the boundary. The definition (4b) of Section 1.3 extends to relatively open subsets $\omega \subset \Upsilon$. For integer orders, we restrict the integrals in (4b) to $\chi_i(U_i \cap \omega)$. Fractional orders follow again by interpolation. For negative orders, we let

$$\tilde{H}_k^s(\omega) = \text{closure of } \{ \phi \in H_k^s(\Upsilon) \mid \text{supp } \phi \Subset \omega \} \text{ in } H_k^s(\Upsilon), \quad (35)$$

and define $H_k^{-s}(\omega)$ as the topological dual of $\tilde{H}_k^s(\omega)$, realized through the pivot space $L^2(\omega)$ and equipped with the scaled norm

$$\|f\|_{H_k^{-s}(\omega)} = k \sup_{\phi \in \tilde{H}_k^s(\omega) \setminus \{0\}} \frac{|\langle f | \phi \rangle_\omega|}{\|\phi\|_{H_k^s(\Upsilon)}}, \quad f \in H_k^{-s}(\omega), \quad (36)$$

which matches the k -scaling of (4b). We emphasize that the test functions in (36) are measured in the ambient norm of $H_k^s(\Upsilon)$. For $s < 1/2$, this is equivalent to the intrinsic norm of $H_k^s(\omega)$, since the extension by zero is bounded in this range. For $\omega = \Upsilon$, the definition reduces to the one of Section 1.3.

The regularity close to Γ_{sc} rests on two local ingredients: a shift theorem for the Laplace–Beltrami operator, and a k -explicit estimate for the normal trace. We state both here and prove them for completeness in Appendix B and Appendix C, respectively.

Theorem 5. *Let $G_1, G_2 \subset \mathbb{R}^d$ be open with $G_1 \Subset G_2$, $G_1 \cap \Gamma \neq \emptyset$ and $G_2 \cap \Gamma \subset \Gamma_{\text{sc}}$, and let $s \in [0, \gamma]$. If $\psi \in H_k^s(G_2 \cap \Gamma)$ with $\Delta_{\Gamma_{\text{sc}}} \psi \in H_k^{s-1}(G_2 \cap \Gamma)$, then $\psi \in H_k^{s+1}(G_1 \cap \Gamma)$ and*

$$\|\psi\|_{H_k^{s+1}(G_1 \cap \Gamma)} \lesssim \|\psi\|_{H_k^s(G_2 \cap \Gamma)} + \|k^{-2} \Delta_{\Gamma_{\text{sc}}} \psi\|_{H_k^{s-1}(G_2 \cap \Gamma)}. \quad (37)$$

The hidden constant depends on d, s, γ, k_0 and $\text{dist}(G_1, \partial G_2)$, but not on k or ψ .

The proof rests on a calculus of the scaled norms in the charts of Γ_{sc} , which we develop in Appendix B.

Lemma 9. *Let G_1, G_2 be open subsets of $\mathbb{R}^d$ with $G_1 \Subset G_2$ and $G_1 \cap \Gamma_{\text{sc}} \neq \emptyset$. Define*

$$\Omega_j = \Omega \cap G_j, \quad \Gamma_j = \Gamma_{\text{sc}} \cap G_j, \quad j = 1, 2.$$

(i) *If $u \in H^1(\Omega_2)$ with $\Delta u \in L^2(\Omega_2)$, then $\partial_n u \in H^{-1/2}(\Gamma_1)$ with*

$$\|k^{-1} \partial_n u\|_{H_k^{-1/2}(\Gamma_1)} \lesssim \|u\|_{H_k^1(\Omega_2)} + \|k^{-2} \Delta u\|_{H_k^0(\Omega_2)}. \quad (38)$$

(ii) *If $u \in H^{m+2}(\Omega_2)$, for $m \in \{0, \dots, \gamma - 1\}$, then $\partial_n u \in H^{m+1/2}(\Gamma_1)$ with*

$$\|k^{-1} \partial_n u\|_{H_k^{m+1/2}(\Gamma_1)} \lesssim \|u\|_{H_k^{m+2}(\Omega_2)}. \quad (39)$$

In both cases, the hidden constant depends on $\Omega, m, \gamma, \text{dist}(G_1, \partial G_2)$ and k_0 , but not on k or u .

The proof is given in Appendix C.

A bootstrapping argument combines Theorem 5 and Lemma 9 into the following local regularity result near the scattering boundary Γ_{sc} .

Lemma 10. *Let G_1, G_2 be open subsets of $\mathbb{R}^d$ with $G_1 \Subset G_2$ and $G_1 \cap \Gamma_{\text{sc}} \neq \emptyset$. Let $m \in \{0, \dots, \gamma - 1\}$ and define*

$$\Omega_j = \Omega \cap G_j, \quad \Gamma_j = \Gamma_{\text{sc}} \cap G_j, \quad j = 1, 2.$$

Assume $u \in H^1(\Omega_2)$ with $\Delta u \in H^m(\Omega_2)$, and $u|_{\Gamma_2} \in H^1(\Gamma_2)$. If $\partial_n u + \frac{\beta}{k} \Delta_{\Gamma_{\text{sc}}} u - \alpha k u|_{\Gamma_2} = 0$ on Γ_2 , then $u \in H^{m+2}(\Omega_1)$ and $u|_{\Gamma_1} \in H^{m+2}(\Gamma_1)$ with

$$\|u\|_{H_k^{m+2}(\Omega_1)} + \|u\|_{H_k^{m+2}(\Gamma_1)} \lesssim \|u\|_{H_k^1(\Omega_2)} + \|u\|_{H_k^1(\Gamma_2)} + \|k^{-2} \Delta u\|_{H_k^m(\Omega_2)},$$

with the hidden constant only depending on d, m, γ, Ω , on $|\beta^{-1}|$ and $|\alpha\beta^{-1}|$, on $\text{dist}(G_1, \partial G_2)$, and on k_0 , but not on k or u .

The proof is given at the end of this section. Our main result now reads as follows.

Theorem 6. Let $m \in \{0, \dots, \gamma-1\}$, $u \in H^1(\Omega)$ with $\Delta u \in H^m(\Omega)$ and $\partial_n u - iku = 0$ on Γ_{ex} . Furthermore, let $u|_{\Gamma_{\text{sc}}} \in H^1(\Gamma_{\text{sc}})$ with $\partial_n u + \frac{\beta}{k} \Delta_{\Gamma_{\text{sc}}} u - \alpha k u = 0$ on Γ_{sc} . Then $u \in H^{m+2}(\Omega)$ and $u|_{\Gamma_{\text{sc}}} \in H^{m+2}(\Gamma_{\text{sc}})$ with

$$\|u\|_{H_k^{m+2}(\Omega)} + \|u\|_{H_k^{m+2}(\Gamma_{\text{sc}})} \lesssim \|u\|_{H_k^1(\Omega)} + \|u\|_{H_k^1(\Gamma_{\text{sc}})} + \|k^{-2} \Delta u\|_{H_k^m(\Omega)}, \quad (40)$$

where the hidden constant depends on d, m, γ, Ω , on the material parameters α and β , and on k_0 , but not on k or u .

Proof. We abbreviate the right-hand side of (40) by

$$\Lambda = \|u\|_{H_k^1(\Omega)} + \|u\|_{H_k^1(\Gamma_{\text{sc}})} + \|k^{-2} \Delta u\|_{H_k^m(\Omega)} < \infty.$$

We cover $\bar{\Omega}$ by finitely many pairs of balls and apply a local regularity result on each pair. Set $c = \text{dist}(\Gamma_{\text{ex}}, \Gamma_{\text{sc}}) > 0$ and define, for $x \in \bar{\Omega}$,

$$r_x = \begin{cases} \frac{1}{4} \text{dist}(x, \Gamma), & \text{dist}(x, \Gamma) \geq c/4, \\ c/8, & \text{otherwise.} \end{cases}$$

In the first case $B(x, 2r_x) \subset \Omega$. In the second case there is a $z \in \Gamma$ with $|x-z| < c/4$, and every $y \in B(x, 2r_x) \cap \Gamma = B(x, c/4) \cap \Gamma$ satisfies $|y-z| < c/2 < c$, so that y and z lie in the same connected component of Γ . Hence, $B(x, 2r_x) \cap \Gamma$ is contained in Γ_{ex} or in Γ_{sc} , but never in both.

The balls $\{B(x, r_x)\}_{x \in \bar{\Omega}}$ cover the compact set $\bar{\Omega}$, so that finitely many of them,

$$B_i = B(x_i, r_i), \quad B'_i = B(x_i, 2r_i), \quad r_i = r_{x_i}, \quad i = 1, \dots, N,$$

already do. Since $r_x \geq c/16$ for every $x \in \bar{\Omega}$, the number N and the distances $\text{dist}(B_i, \partial B'_i) = r_i$ are bounded in terms of Ω alone. In particular, they do not depend on k or on u . By construction, each pair $B_i \Subset B'_i$ is of one of the three types

- (A) $B'_i \subset \Omega$,
- (B) $B'_i \cap \Gamma \neq \emptyset$ and $B'_i \cap \Gamma \subset \Gamma_{\text{ex}}$,
- (C) $B'_i \cap \Gamma \neq \emptyset$ and $B'_i \cap \Gamma \subset \Gamma_{\text{sc}}$,

and we abbreviate $\Omega_i = \Omega \cap B_i$, $\Omega'_i = \Omega \cap B'_i$, $\Gamma_i = \Gamma_{\text{sc}} \cap B_i$, $\Gamma'_i = \Gamma_{\text{sc}} \cap B'_i$.

Next, we apply local estimates for the different cases. Let i be of type (A). Since $\Delta u \in H^m(B'_i)$, the interior shift theorem [31, Thm. 4.7], extended to k -dependent norms, gives $u \in H^{m+2}(B_i)$ with

$$\|u\|_{H_k^{m+2}(\Omega_i)} \lesssim \|u\|_{H_k^1(\Omega'_i)} + \|k^{-2} \Delta u\|_{H_k^m(\Omega'_i)} \leq \Lambda. \quad (41)$$

Lemma B.3 below proves a k -explicit estimate that can be used instead of [31, Thm. 4.7]. The norm (65) needs a different scaling there, but the proof carries over.

Let i be of type (B). Then u satisfies the homogeneous impedance condition $\partial_n u - iku = 0$ on the whole of $B'_i \cap \Gamma$. The corresponding local regularity result near an impedance boundary of class $C^{\gamma,1}$, see [23, Thm. 3.4], gives $u \in H^{m+2}(\Omega_i)$ with

$$\|u\|_{H_k^{m+2}(\Omega_i)} \lesssim \|u\|_{H_k^1(\Omega'_i)} + \|k^{-2} \Delta u\|_{H_k^m(\Omega'_i)} \leq \Lambda. \quad (42)$$

Finally, let i be of type (C). Then $B'_i \cap \Gamma \subset \Gamma_{\text{sc}}$, so that Lemma 10, applied with $G_1 = B_i \Subset B'_i = G_2$, provides $u \in H^{m+2}(\Omega_i)$ and $u|_{\Gamma_i} \in H^{m+2}(\Gamma_i)$ with

$$\|u\|_{H_k^{m+2}(\Omega_i)} + \|u\|_{H_k^{m+2}(\Gamma_i)} \lesssim \|u\|_{H_k^1(\Omega'_i)} + \|u\|_{H_k^1(\Gamma'_i)} + \|k^{-2} \Delta u\|_{H_k^m(\Omega'_i)} \leq \Lambda. \quad (43)$$

In all three cases the hidden constant depends on $\text{dist}(B_i, \partial B'_i) = r_i$, on α, β , and on k_0 , but not on $k \geq k_0$ or u .

Note that $m \leq \gamma - 1$ keeps m in the admissible range of Lemma 10 and of the local results near Γ_{ex} .

It remains to sum over the cover. The definition (4a) of the scaled norm gives

$$\begin{aligned} \|u\|_{H_k^{m+2}(\Omega)}^2 &= \sum_{|\lambda| \leq m+2} k^{-2|\lambda|+2} \|\partial^\lambda u\|_{L^2(\Omega)}^2 \\ &\leq \sum_{i=1}^N \sum_{|\lambda| \leq m+2} k^{-2|\lambda|+2} \|\partial^\lambda u\|_{L^2(\Omega_i)}^2 = \sum_{i=1}^N \|u\|_{H_k^{m+2}(\Omega_i)}^2, \end{aligned}$$

that is, the scaled norms of integer order are subadditive with respect to a covering of the domain. Together with (41), (42), and (43) this shows $u \in H^{m+2}(\Omega)$ with $\|u\|_{H_k^{m+2}(\Omega)} \lesssim \sqrt{N}\Lambda$.

For the boundary norm we first note that every $x \in \Gamma_{\text{sc}}$ belongs to some B_i , and that this i is necessarily of type (C). Hence, the sets Γ_i with i of type (C) cover Γ_{sc} , and consequently $\chi_j(U_j \cap \Gamma_{\text{sc}}) \subset \bigcup_i \chi_j(U_j \cap \Gamma_i)$ for every chart index j . Since the localized norm of integer order is given by (4b) with the integrals restricted to $\chi_j(U_j \cap \Gamma_i)$, the same argument as above yields

$$\|u\|_{H_k^{m+2}(\Gamma_{\text{sc}})}^2 \leq \sum_{i \text{ of type (C)}} \|u\|_{H_k^{m+2}(\Gamma_i)}^2 \lesssim N\Lambda^2,$$

where we used $m+2 \leq \gamma+1$, so that the left-hand side is covered by (4b). This proves $u|_{\Gamma_{\text{sc}}} \in H^{m+2}(\Gamma_{\text{sc}})$. $\square$

Corollary 3. *Let $m \in \{0, \dots, \gamma-1\}$ and let $u \in V$ be the solution of (6a) for the data $f \in H^m(\Omega)$ and $g_{\text{sc}} = g_{\text{ex}} = 0$. Then $u \in H^{m+2}(\Omega)$ and $u|_{\Gamma_{\text{sc}}} \in H^{m+2}(\Gamma_{\text{sc}})$ with*

$$\|u\|_{H_k^{m+2}(\Omega)} + \|u\|_{H_k^{m+2}(\Gamma_{\text{sc}})} \lesssim \|u\| + \|k^{-2}f\|_{H_k^m(\Omega)}. \quad (44)$$

In particular, the stability estimate (8) gives

$$\|u\|_{H_k^{m+2}(\Omega)} + \|u\|_{H_k^{m+2}(\Gamma_{\text{sc}})} \lesssim (1 + kC_{\text{sol}}) \|k^{-2}f\|_{H_k^m(\Omega)}. \quad (45)$$

The hidden constants are those of Theorem 6.

It remains to prove Lemma 10.

Proof of Lemma 10. We prove the claim for $m = 0$. Let $G_{1,1}, G_{1,2}, G_{1,3}, G_{1,4} \subset \mathbb{R}^d$ be open sets that form a cascade $G_1 \Subset G_{1,4} \Subset G_{1,3} \Subset G_{1,2} \Subset G_{1,1} \Subset G_2$. Define

$$\Omega_{1,i} = \Omega \cap G_{1,i}, \quad \Gamma_{1,i} = \Gamma_{\text{sc}} \cap G_{1,i}, \quad i = 1, 2, 3, 4.$$

We bootstrap along the cascade to obtain the final regularity. Throughout, we abbreviate the right-hand side of the lemma by

$$\Lambda = \|u\|_{H_k^1(\Omega_2)} + \|u\|_{H_k^1(\Gamma_2)} + \|k^{-2}\Delta u\|_{H_k^0(\Omega_2)}.$$

Note that the number of intermediate sets $G_{1,i}$ is fixed and therefore, there exists a fixed constant $C \in [1, \infty)$ such that

$$\text{dist}(G_1, \partial G_2) \leq C \text{dist}(G_{1,i}, \partial G_2), \quad i = 1, 2, 3, 4.$$

Hence, the hidden constants below do not degenerate.

First, Lemma 9 (i) gives $\partial_n u \in H^{-1/2}(\Gamma_{1,1})$ with

$$\|k^{-1}\partial_n u\|_{H_k^{-1/2}(\Gamma_{1,1})} \lesssim \|u\|_{H_k^1(\Omega_2)} + \|k^{-2}\Delta u\|_{H_k^0(\Omega_2)} \lesssim \Lambda. \quad (46)$$

Hence, the boundary condition provides

$$\Delta_{\Gamma_{\text{sc}}} u = -k\beta^{-1}\partial_n u + k^2\beta^{-1}\alpha u|_{\Gamma_2} \in H^{-1/2}(\Gamma_{1,1}),$$

Theorem 5 then gives $u|_{\Gamma_{1,2}} \in H^{3/2}(\Gamma_{1,2})$ with

$$\begin{aligned} \|u\|_{H_k^{3/2}(\Gamma_{1,2})} &\lesssim \|u\|_{H_k^1(\Gamma_{1,1})} + \|k^{-2}\Delta_{\Gamma_{\text{sc}}}u\|_{H_k^{-1/2}(\Gamma_{1,1})} \\ &\lesssim \|u\|_{H_k^1(\Gamma_{1,1})} + \|k^{-1}\partial_n u\|_{H_k^{-1/2}(\Gamma_{1,1})} \\ &\lesssim \Lambda, \end{aligned} \quad (47)$$

where we use (46) for the last estimate.

Second, we use a Dirichlet boundary result to gain regularity in the domain. It states that $u \in H^1(\Omega_2)$ with $\Delta u \in L^2(\Omega_2)$ and $u|_{\Gamma_{1,2}} \in H^{3/2}(\Gamma_{1,2})$ implies $u \in H^2(\Omega_{1,3})$ with

$$\|u\|_{H_k^2(\Omega_{1,3})} \lesssim \|u\|_{H_k^1(\Omega_{1,2})} + \|k^{-2}\Delta u\|_{H_k^0(\Omega_{1,2})} + \|u\|_{H_k^{3/2}(\Gamma_{1,2})} \lesssim \Lambda,$$

where we use (47) for the last estimate. See [23, Thm. 3.2] for the homogeneous k -explicit regularity estimate. The inhomogeneous case is treated similarly with a k -explicit boundary lifting, cf. (75) in Appendix C.

Third, since $u \in H^2(\Omega_{1,3})$, Lemma 9 (ii) gives $\partial_n u \in H^{1/2}(\Gamma_{1,4})$ with

$$\|k^{-1}\partial_n u\|_{H_k^0(\Gamma_{1,4})} \lesssim \|k^{-1}\partial_n u\|_{H_k^{1/2}(\Gamma_{1,4})} \lesssim \|u\|_{H_k^2(\Omega_{1,3})} \lesssim \Lambda.$$

Thus, Theorem 5 provides $u|_{\Gamma_1} \in H^2(\Gamma_1)$ with

$$\|u\|_{H_k^2(\Gamma_1)} \lesssim \|u\|_{H_k^1(\Gamma_{1,4})} + \|k^{-2}\Delta_{\Gamma_{\text{sc}}}u\|_{H_k^0(\Gamma_{1,4})} \lesssim \Lambda.$$

This proves the claim for $m = 0$.

For $m > 0$ we argue by induction. The first step is then replaced by Lemma 9 (ii). $\square$

5. NUMERICAL EXPERIMENTS

We complement the analysis with a numerical study of the resolution condition of Theorem 3. For a non-trapping configuration we expect $C_{\text{sol}} \approx 1$, cf. Remark 1, so that the condition $k(kh)^p C_{\text{sol}} \ll 1$ reduces to $k(kh)^p \ll 1$. The coarsest mesh on which the discretization is still quasi-optimal should therefore behave like

$$kh_{\text{crit}} \approx k^{-1/p}, \quad \text{i.e.} \quad h_{\text{crit}} \approx k^{-(p+1)/p}. \quad (48)$$

We measure h_{crit} for $p \in \{1, 2, 3\}$ and compare the observed decay with the exponents -2 , $-3/2$, and $-4/3$ predicted by (48). These are the exponents known for the Helmholtz equation without a second-order boundary condition, cf. [23].

Following the Guidelines for Safeguarding Good Research Practice of the Deutsche Forschungsgemeinschaft,¹ we make the software underlying this section publicly available. The experiments are implemented in FEniCSx [5], and the meshes are generated with Gmsh [24]. The source code, a version-pinned environment, and the parameter sets that reproduce the results reported below are available at

https://github.com/juliandoerner/thin_scattering_code.

5.1. Setting. We consider a scattering object of circular shape in two dimensions. Let B denote the unit disk centered at the origin and let $\Omega = (-L, L)^2 \setminus \overline{B}$, so that Γ_{sc} is the unit circle and Γ_{ex} is the boundary of the square. We choose $L = 2$ for $p \in \{1, 2\}$ and $L = 5/2$ for $p = 3$. The material parameters on Γ_{sc} are $\alpha = 1$ and $\beta = 1$. In particular $\text{Im } \beta = 0$, which is the degenerate case of (2) that the classical “coercive + compact” framework does not reach.

The data of (3) are chosen such that u is the field scattered by an incident plane wave $u_{\text{in}}(x) = e^{ikd \cdot x}$ with direction $d = (1, 0)$. For a circular obstacle, this scattered field is known in closed form. The incident wave is expanded into cylindrical

¹Deutsche Forschungsgemeinschaft, *Guidelines for Safeguarding Good Research Practice. Code of Conduct*, <https://doi.org/10.5281/zenodo.3923601>.

harmonics by the Jacobi–Anger expansion, and the scattered field is the Mie series whose coefficients are fixed by the boundary condition (3b), cf. [15]. Accordingly, we set $f = 0$ and let g_{sc} be the negative of the boundary operator in (3b) applied to u_{in} , so that the total field $u + u_{\text{in}}$ satisfies the homogeneous condition on Γ_{sc} . We prescribe g_{ex} by evaluating (3c) for the Mie series. The exact solution of (3) is thus known, and the error we report below contains no contribution from the truncation of the domain at Γ_{ex} . The discrete problem (16) is solved with Lagrange elements of degree p on triangular meshes whose elements along Γ_{sc} are curved of geometric order p .

5.2. Measured quantity. For a wavenumber k and a mesh of size h , we compute the quasi-optimality ratio

$$R(k, h) = \frac{\|u - u_h\|}{\|u - \Pi_h u\|}, \quad (49)$$

where Π_h is the best-approximation mapping (19). We evaluate the denominator by computing the orthogonal projection of u onto V_h with respect to the inner product induced by (5). We call a mesh resolved if $R(k, h) \leq 1.2$ and define $h_{\text{crit}}(k)$ as the largest h with this property.

For every wavenumber, we sweep 20 geometrically distributed values of kh within a fixed window, and obtain h_{crit} by log-linear interpolation between the two neighboring values of kh at which the threshold is crossed. The window is $kh \in [0.25, 3]$ for $p = 1$, $kh \in [0.5, 3]$ for $p = 2$, and $kh \in [1, 3.5]$ for $p = 3$. For $p = 1$ we use nine equidistant wavenumbers in $[6, 14]$, for $p = 2$ nine in $[10, 26]$, and for $p = 3$ eleven in $[60, 110]$.

5.3. Results. Figure 2 shows h_{crit} against k on a doubly logarithmic scale, one panel per polynomial degree, together with the decay predicted by (48). A least-

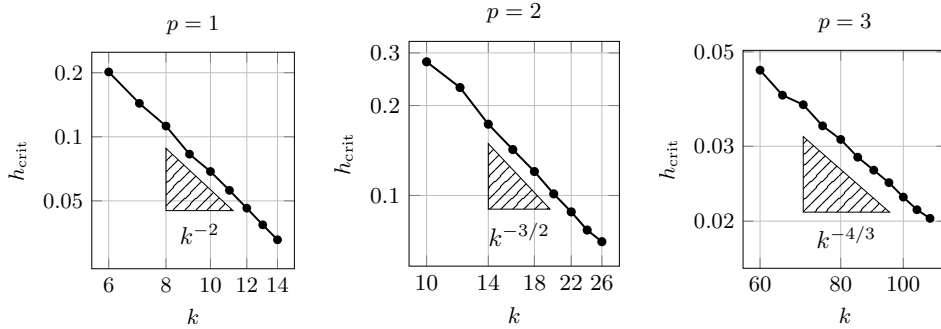


FIGURE 2. Critical mesh size h_{crit} against the wavenumber k for $p = 1, 2, 3$. The hatched triangle indicates the decay predicted by (48).

squares fit of the exponent gives -2.14 for $p = 1$, -1.50 for $p = 2$, and -1.33 for $p = 3$. The two high-order cases match the predicted exponents $-3/2$ and $-4/3$ to two digits. For $p = 1$ the measured decay is slightly steeper than predicted.

The experiment confirms the prediction of Theorem 3. The second-order boundary condition in the parameter regime (2), and in particular in the critical case $\text{Im } \beta = 0$, does not degrade the resolution condition.

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APPENDIX A. A SCOTT–ZHANG OPERATOR ON CURVED MESHES

We construct a Scott–Zhang operator on a curved mesh. The construction follows the original one in [33]. Our curved finite element framework is the one of [13, App. A]. We prove that this operator is stable in integer Sobolev norms. Theorem A.1 gives a local estimate. Proposition A.1 gives the global counterpart that we use in the manuscript. We close with the proof of Proposition 4.

A.1. Setting and notation. We restate the setting, so that this appendix can be read on its own. Throughout, $\Omega \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ is a domain of diameter $L = 1$. We fix a reference d -simplex $\hat{K}_d$ with $|\hat{K}_d| \leq 1$ and a reference $(d-1)$ -simplex $\hat{F}_{d-1} \subset \mathbb{R}^{d-1}$.

Definition A.1 ([13, App. A]). *A curved matching mesh $\mathcal{T}_h$ of Ω is a finite collection of elements K . Every element is equipped with a bi-Lipschitz reference map $\mathcal{F}_K : \hat{K}_d \rightarrow K$. The elements do not overlap and cover $\bar{\Omega}$. We further assume that the mesh is matching. Let $K_{\pm} \in \mathcal{T}_h$ be distinct, and let $F = K_- \cap K_+$ be a $(d-1)$ -manifold. Then $\mathcal{F}_{K_{\pm}}^{-1}(F)$ are faces of $\hat{K}_d$, and the two reference maps coincide on F up to an affine reparametrization of the reference face.*

For $K \in \mathcal{T}_h$ we write $h_K = \text{diam}(K)$. The mesh size is $h = \max_{K \in \mathcal{T}_h} h_K$, and in particular $h_K \leq h \leq L = 1$. We quantify the shape-regularity of an element K by

$$M_K = \max\{h_K \|(J\mathcal{F}_K)^{-1}\|_{L^\infty(\hat{K}_d)}, h_K^{-1} \|J\mathcal{F}_K\|_{L^\infty(\hat{K}_d)}\}, \quad (50)$$

where $J\mathcal{F}_K$ is the Jacobian of $\mathcal{F}_K$ and we use the 2-norm for matrices. The mesh $\mathcal{T}_h$ is *shape-regular* if $M_{\mathcal{T}_h} = \max_{K \in \mathcal{T}_h} M_K$ is bounded independently of h . Higher order norms need control over the higher derivatives of the reference maps as well. For $r \geq 1$ we therefore define

$$M_K(r) = \max_{1 \leq |\alpha| \leq r} h_K^{-|\alpha|} \|\partial^\alpha \mathcal{F}_K\|_{L^\infty(\hat{K}_d)}, \quad M_{\mathcal{T}_h}(r) = \max_{K \in \mathcal{T}_h} M_K(r). \quad (51)$$

A mesh is called *r-regular* if $\mathcal{F}_K \in C^r(\hat{K}_d, K)$ for all $K \in \mathcal{T}_h$ and $M_{\mathcal{T}_h}(r)$ is bounded independently of h . Throughout this appendix, the mesh is assumed to be shape-regular. All hidden constants depend only on d , p , $M_{\mathcal{T}_h}$, and on the order of the norms involved. Wherever higher order norms appear, they also depend on $M_{\mathcal{T}_h}(r)$. They never depend on h or on the function at hand.

The *patch* of $K \in \mathcal{T}_h$ is $\mathcal{T}_h^K := \{K' \in \mathcal{T}_h \mid K' \cap K \neq \emptyset\}$ with associated patch domain $\omega_K := \text{int} \bigcup_{K' \in \mathcal{T}_h^K} K'$.

Lemma A.1 ([13, Lem. A.4]). *The cardinality of $\mathcal{T}_h^K$ is bounded, and $h_{K'} \approx h_K$ for all $K' \in \mathcal{T}_h^K$. Both constants depend only on $M_{\mathcal{T}_h}$. In particular, the patch domains have finite overlap, i.e. $\#\{K \in \mathcal{T}_h \mid x \in \omega_K\} \lesssim 1$ for a.e. $x \in \Omega$.*

For fixed $p \in \mathbb{N}_+$, the Lagrange finite element space associated with $\mathcal{T}_h$ is

$$V_h := \{v_h \in C(\bar{\Omega}) \mid v_h|_K \circ \mathcal{F}_K \in \mathbb{P}_p(\hat{K}_d) \text{ for all } K \in \mathcal{T}_h\} \subset H^1(\Omega).$$

It admits a nodal basis $\{\phi_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{D}_h\}$ associated with a finite set of nodes $\mathcal{D}_h \subset \bar{\Omega}$. On the reference element we fix a $\mathbb{P}_p(\hat{K}_d)$ -unisolvant set of reference nodes $\hat{\mathcal{D}} \subset \hat{K}_d$, together with the associated reference shape functions $\hat{\Phi} = \{\hat{\phi}_{\hat{\mathbf{a}}} \mid \hat{\mathbf{a}} \in \hat{\mathcal{D}}\} \subset \mathbb{P}_p(\hat{K}_d)$. The nodes contained in an element are the images of the reference nodes,

$$\mathcal{D}_h(K) := \mathcal{D}_h \cap \bar{K} = \mathcal{F}_K(\hat{\mathcal{D}}), \quad K \in \mathcal{T}_h, \quad (52)$$

and the basis functions are the transported reference shape functions,

$$\phi_{\mathbf{a}}|_K = \hat{\phi}_{\hat{\mathbf{a}}} \circ \mathcal{F}_K^{-1} \quad \text{with} \quad \hat{\mathbf{a}} = \mathcal{F}_K^{-1}(\mathbf{a}), \quad \mathbf{a} \in \mathcal{D}_h(K), \quad K \in \mathcal{T}_h. \quad (53)$$

Finally, $\phi_{\mathbf{a}}$ vanishes identically on every element K with $\mathbf{a} \notin \bar{K}$.

A.2. Scaling of volume and surface measures. Let $K \in \mathcal{T}_h$ be an element, and let F be a $(d-1)$ -dimensional face of K . There is a face $\widehat{F}$ of the reference element $\widehat{K}_d$ and an affine map $A : \widehat{F}_{d-1} \rightarrow \widehat{F}$ with

$$\mathcal{F}_{K,F} = \mathcal{F}_K \circ A : \widehat{F}_{d-1} \rightarrow F.$$

We transform integrals over faces to the reference face. This requires the square root of the determinant of the first fundamental form,

$$\sqrt{g_{K,F}(\widehat{x}')} = \sqrt{\det J\mathcal{F}_{K,F}^T(\widehat{x}') \cdot J\mathcal{F}_{K,F}(\widehat{x}')}, \quad \widehat{x}' \in \widehat{F}_{d-1}.$$

As in the affine case, we expect $\sqrt{g_{K,F}(\widehat{x}')} \approx h_K^{d-1}$. Singular values and an interlacing inequality prove this.

Lemma A.2. *There exists $\mu_K \geq 1$ depending only on M_K such that*

$$\mu_K^{-1} h_K^d \leq |\det J\mathcal{F}_K(\widehat{x})| \leq \mu_K h_K^d, \quad \text{for a.e. } \widehat{x} \in \widehat{K}_d. \quad (54)$$

Moreover, with the same constant it holds

$$\mu_K^{-1} h_K^{d-1} \leq \sqrt{g_{K,F}(\widehat{x}')} \leq \mu_K h_K^{d-1}, \quad \text{for a.e. } \widehat{x}' \in \widehat{F}_{d-1}. \quad (55)$$

Proof. Let $\sigma_1(\widehat{x}) \geq \sigma_2(\widehat{x}) \geq \dots \geq \sigma_d(\widehat{x})$ denote the singular values of $J\mathcal{F}_K(\widehat{x})$ for $\widehat{x} \in \widehat{K}_d$. The largest singular value is the spectral norm, so that

$$\sigma_1(\widehat{x}) = \|J\mathcal{F}_K(\widehat{x})\|_2 \leq \|J\mathcal{F}_K\|_{L^\infty(\widehat{K}_d)} \leq M_K h_K.$$

For the smallest one, we use the spectral norm of the inverse, so that

$$\sigma_d(\widehat{x}) = \frac{1}{\|(J\mathcal{F}_K)^{-1}(\widehat{x})\|_2} \geq \frac{1}{\|(J\mathcal{F}_K)^{-1}\|_{L^\infty(\widehat{K}_d)}} \geq M_K^{-1} h_K.$$

Every singular value therefore satisfies

$$M_K^{-1} h_K \leq \sigma_d(\widehat{x}) \leq \sigma_i(\widehat{x}) \leq \sigma_1(\widehat{x}) \leq M_K h_K, \quad i = 1, \dots, d. \quad (56)$$

In particular $M_K \geq 1$, and hence $\mu_K \geq 1$. The well-known formula

$$\prod_{i=1}^d \sigma_i(\widehat{x}) = |\det J\mathcal{F}_K(\widehat{x})|$$

therefore proves (54).

Let $\nu_1(\widehat{x}') \geq \nu_2(\widehat{x}') \geq \dots \geq \nu_{d-1}(\widehat{x}')$ denote the singular values of $J\mathcal{F}_{K,F}(\widehat{x}')$ for $\widehat{x}' \in \widehat{F}_{d-1}$. Without restriction, we can assume that

$$\mathcal{F}_{K,F}(\widehat{x}') = \mathcal{F}_K((\widehat{x}', 0)), \quad \widehat{x}' \in \widehat{F}_{d-1}.$$

Hence, $J\mathcal{F}_{K,F}$ arises from $J\mathcal{F}_K$ by deletion of a column. The Cauchy interlace theorem, see, e.g., [28, Thm. 4.3.17], shows that

$$\sigma_{i+1}((\widehat{x}', 0)) \leq \nu_i(\widehat{x}') \leq \sigma_i((\widehat{x}', 0)), \quad i = 1, \dots, d-1, \quad \widehat{x}' \in \widehat{F}_{d-1}.$$

With (56), every $\nu_i(\widehat{x}')$ lies between $M_K^{-1} h_K$ and $M_K h_K$ as well. The identity

$$\prod_{i=1}^{d-1} \nu_i(\widehat{x}') = \sqrt{\det J\mathcal{F}_{K,F}^T(\widehat{x}') \cdot J\mathcal{F}_{K,F}(\widehat{x}')}, \quad \widehat{x}' \in \widehat{F}_{d-1},$$

therefore gives (55) with the constant $M_K^{d-1} \leq \mu_K$. $\square$

A.3. Change of variables. We collect the transformation properties of Sobolev norms under the reference maps. In contrast to affine maps, a single power of h_K per differentiation order is not available for curved maps. Instead, each estimate couples all lower order seminorms.

Lemma A.3 ([13, Lem. A.5,A.6]). *Let $\mathcal{T}_h$ be r -regular, $K \in \mathcal{T}_h$ and $\ell \in \{1, \dots, r\}$. For $v \in L^2(K)$ it holds*

$$\mu_K^{-1/2} h_K^{-d/2} \|v\|_{L^2(K)} \leq \|v \circ \mathcal{F}_K\|_{L^2(\widehat{K}_d)} \leq \mu_K^{1/2} h_K^{-d/2} \|v\|_{L^2(K)}, \quad (57)$$

and for $v \in H^\ell(K)$ it holds

$$|v \circ \mathcal{F}_K|_{H^\ell(\widehat{K}_d)} \lesssim h_K^{-d/2} \sum_{t=1}^{\ell} h_K^t |v|_{H^t(K)}, \quad (58)$$

$$|v|_{H^\ell(K)} \lesssim h_K^{d/2} \sum_{t=1}^{\ell} h_K^{-t} |v \circ \mathcal{F}_K|_{H^t(\widehat{K}_d)}, \quad (59)$$

where the hidden constant depends on M_K and $M_K(\ell)$.

A.4. Construction. We follow the construction of [33]. The set of mesh faces is $\mathcal{F}_{\mathcal{T}_h}$. Every $F \in \mathcal{F}_{\mathcal{T}_h}$ is a face of at least one element $K \in \mathcal{T}_h$. For every node $\mathbf{a} \in \mathcal{D}_h$ we choose an associated control volume $\sigma_{\mathbf{a}}$, which is either an element or a face. The choice follows the rules below.

Rule 1: If there exists an element $K \in \mathcal{T}_h$ such that $\mathbf{a} \in \text{int}(K)$, then $\sigma_{\mathbf{a}} = K$.

Rule 2: If there exists $F \in \mathcal{F}_{\mathcal{T}_h}$ such that $\mathbf{a} \in \text{int}(F)$, then $\sigma_{\mathbf{a}} = F$.

Else: There exists at least one $F \in \mathcal{F}_{\mathcal{T}_h}$ such that $\mathbf{a} \in \overline{F}$.

Rule 3a: If $\mathbf{a} \in \partial\Omega$, then choose $F \subset \partial\Omega$ and define $\sigma_{\mathbf{a}} = F$.

Rule 3b: Else, choose $\sigma_{\mathbf{a}} = F$.

In all cases $\mathbf{a} \in \overline{\sigma_{\mathbf{a}}}$. We fix an element $K_{\mathbf{a}} \in \mathcal{T}_h$ with $\sigma_{\mathbf{a}} \subset \overline{K_{\mathbf{a}}}$.

For every node $\mathbf{a} \in \mathcal{D}_h$ we denote by $\mathcal{D}_h(\sigma_{\mathbf{a}}) = \mathcal{D}_h \cap \overline{\sigma_{\mathbf{a}}}$ the set of nodes within the control volume. The restrictions of the associated basis functions are collected in

$$\Phi_{\sigma_{\mathbf{a}}} = \{ \phi_{\mathbf{b}}|_{\sigma_{\mathbf{a}}} \mid \mathbf{b} \in \mathcal{D}_h(\sigma_{\mathbf{a}}) \}.$$

These restrictions are linearly independent. Composed with the reference map, they coincide by (53) with the reference shape functions, respectively with their restrictions to the reference face. Linear independence is therefore the unisolvence assumption of the Lagrange element on $\widehat{K}_d$, respectively on $\widehat{F}_{d-1}$. Hence, $V_{\sigma_{\mathbf{a}}} := \text{span } \Phi_{\sigma_{\mathbf{a}}}$ has dimension $n_{\sigma_{\mathbf{a}}} = \#\mathcal{D}_h(\sigma_{\mathbf{a}})$. There exists a unique $L^2(\sigma_{\mathbf{a}})$ -dual basis

$$\Psi_{\sigma_{\mathbf{a}}} = \{ \psi_{\mathbf{a},\mathbf{b}} \in V_{\sigma_{\mathbf{a}}} \mid \mathbf{b} \in \mathcal{D}_h(\sigma_{\mathbf{a}}) \} \subset V_{\sigma_{\mathbf{a}}},$$

characterized by

$$\int_{\sigma_{\mathbf{a}}} \psi_{\mathbf{a},\mathbf{b}}(t) \phi_{\mathbf{c}}(t) d\eta(t) = \delta_{\mathbf{b},\mathbf{c}}, \quad \mathbf{b}, \mathbf{c} \in \mathcal{D}_h(\sigma_{\mathbf{a}}), \quad (60)$$

where the measure η is either the d -dimensional Lebesgue measure or the induced $(d-1)$ -dimensional surface measure. We abbreviate $\psi_{\mathbf{a}} = \psi_{\mathbf{a},\mathbf{a}}$ for all $\mathbf{a} \in \mathcal{D}_h$.

The Scott–Zhang operator is now defined for $s > 1/2$ as

$$\Pi_{\text{SZ}} : H^s(\Omega) \rightarrow V_h, \quad \Pi_{\text{SZ}} v(x) = \sum_{\mathbf{a} \in \mathcal{D}_h} \phi_{\mathbf{a}}(x) \int_{\sigma_{\mathbf{a}}} \psi_{\mathbf{a}}(t) v(t) d\eta(t). \quad (61)$$

The integrals are well-defined. For a control volume of dimension d , this follows from $\psi_{\mathbf{a}} \in L^\infty(\sigma_{\mathbf{a}})$ and $v \in L^2(\sigma_{\mathbf{a}})$. On a face $\sigma_{\mathbf{a}} \subset \partial K_{\mathbf{a}}$, the trace of $v|_{K_{\mathbf{a}}} \in H^s(K_{\mathbf{a}})$ with $s > 1/2$ belongs to $L^2(\sigma_{\mathbf{a}})$.

A.5. Stability. We collect three lemmas from which the stability of Π_{SZ} follows.

Lemma A.4. *For every $\mathbf{a} \in \mathcal{D}_h$ it holds $\|\psi_{\mathbf{a}}\|_{L^\infty(\sigma_{\mathbf{a}})} \lesssim h_{K_{\mathbf{a}}}^{-\dim \sigma_{\mathbf{a}}}$.*

Proof. Write $K = K_{\mathbf{a}}$ and assume first that $\sigma_{\mathbf{a}}$ is a face, i.e. $\dim \sigma_{\mathbf{a}} = d - 1$. We transform (60) to the reference face. With $\hat{\phi}_{\mathbf{b}} := \phi_{\mathbf{b}} \circ \mathcal{F}_{K, \sigma_{\mathbf{a}}}$ and $\hat{\psi}_{\mathbf{b}} := \psi_{\mathbf{a}, \mathbf{b}} \circ \mathcal{F}_{K, \sigma_{\mathbf{a}}}$, the orthogonality reads

$$\int_{\hat{F}_{d-1}} \hat{\psi}_{\mathbf{b}}(\hat{x}') \hat{\phi}_{\mathbf{c}}(\hat{x}') \sqrt{g_{K, \sigma_{\mathbf{a}}}(\hat{x}')} d\hat{x}' = \delta_{\mathbf{b}, \mathbf{c}}, \quad \mathbf{b}, \mathbf{c} \in \mathcal{D}_h(\sigma_{\mathbf{a}}).$$

Since $\psi_{\mathbf{a}} \in V_{\sigma_{\mathbf{a}}}$, we can expand $\hat{\psi}_{\mathbf{a}} = \sum_{\mathbf{c}} \beta_{\mathbf{c}} \hat{\phi}_{\mathbf{c}}$. The coefficient vector β solves $G\beta = e_{\mathbf{a}}$ with the weighted Gram matrix

$$G_{\mathbf{bc}} = \int_{\hat{F}_{d-1}} \hat{\phi}_{\mathbf{b}} \hat{\phi}_{\mathbf{c}} \sqrt{g_{K, \sigma_{\mathbf{a}}}} d\hat{x}'.$$

With the weight bounds of Lemma A.2 this gives, for every coefficient vector ξ ,

$$\xi^T G \xi = \int_{\hat{F}_{d-1}} \left| \sum_{\mathbf{c}} \xi_{\mathbf{c}} \hat{\phi}_{\mathbf{c}} \right|^2 \sqrt{g_{K, \sigma_{\mathbf{a}}}} d\hat{x}' \geq \mu_K^{-1} h_K^{d-1} \left\| \sum_{\mathbf{c}} \xi_{\mathbf{c}} \hat{\phi}_{\mathbf{c}} \right\|_{L^2(\hat{F}_{d-1})}^2 \gtrsim h_K^{d-1} |\xi|^2.$$

Hence $|\beta| \leq \|G^{-1}\|_2 \lesssim h_K^{-(d-1)}$. With the uniform bound $\|\hat{\phi}_{\mathbf{c}}\|_{L^\infty(\hat{F}_{d-1})} \lesssim 1$ we conclude

$$\|\psi_{\mathbf{a}}\|_{L^\infty(\sigma_{\mathbf{a}})} = \|\hat{\psi}_{\mathbf{a}}\|_{L^\infty(\hat{F}_{d-1})} \leq |\beta| \max_{\mathbf{c}} \|\hat{\phi}_{\mathbf{c}}\|_{L^\infty(\hat{F}_{d-1})} \sqrt{n_{\sigma_{\mathbf{a}}}} \lesssim h_K^{-(d-1)}.$$

If $\sigma_{\mathbf{a}} = K$ is an element, the same argument applies with $\mathcal{F}_{K, \sigma_{\mathbf{a}}}$ replaced by $\mathcal{F}_K$ and the weight $\sqrt{g_{K, \sigma_{\mathbf{a}}}}$ replaced by $|\det J\mathcal{F}_K| \approx h_K^d$, cf. (54). $\square$

Lemma A.5. *Let $\mathcal{T}_h$ be r -regular, $\mathbf{a} \in \mathcal{D}_h$, and $\ell \in \{1, \dots, r\}$. For $v \in H^\ell(K_{\mathbf{a}})$ it holds*

$$\|v\|_{L^1(\sigma_{\mathbf{a}})} \lesssim h_{K_{\mathbf{a}}}^{\dim \sigma_{\mathbf{a}} - d/2} \sum_{j=0}^{\ell} h_{K_{\mathbf{a}}}^j |v|_{H^j(K_{\mathbf{a}})}.$$

Proof. Write $K = K_{\mathbf{a}}$ and assume first that $\sigma_{\mathbf{a}}$ is a face. With the reference map $\mathcal{F}_{K, \sigma_{\mathbf{a}}} : \hat{F}_{d-1} \rightarrow \sigma_{\mathbf{a}}$ and Lemma A.2 we obtain

$$\|v\|_{L^1(\sigma_{\mathbf{a}})} = \int_{\hat{F}_{d-1}} |v \circ \mathcal{F}_{K, \sigma_{\mathbf{a}}}(\hat{x}')| \sqrt{g_{K, \sigma_{\mathbf{a}}}(\hat{x}')} d\hat{x}' \lesssim h_K^{d-1} \|v \circ \mathcal{F}_{K, \sigma_{\mathbf{a}}}\|_{L^1(\hat{F}_{d-1})}.$$

We have $\mathcal{F}_{K, \sigma_{\mathbf{a}}} = \mathcal{F}_K \circ A$ with an affine map $A : \hat{F}_{d-1} \rightarrow \hat{F}$ onto a face $\hat{F}$ of $\hat{K}_d$, and there are only finitely many such reparametrizations. This gives

$$\|v \circ \mathcal{F}_{K, \sigma_{\mathbf{a}}}\|_{L^1(\hat{F}_{d-1})} \lesssim \|v \circ \mathcal{F}_K\|_{L^1(\hat{F})} \lesssim \|v \circ \mathcal{F}_K\|_{L^2(\hat{F})} \lesssim \|v \circ \mathcal{F}_K\|_{H^\ell(\hat{K}_d)},$$

where the last step is the trace theorem on the reference element, see, e.g., [31, Thm. 3.38]. It is applicable in $L^2(\partial \hat{K}_d)$ since $\ell \geq 1 > 1/2$. If $\sigma_{\mathbf{a}} = K$ is an element, the same argument applies with $\mathcal{F}_{K, \sigma_{\mathbf{a}}}$ replaced by $\mathcal{F}_K$ and the weight $\sqrt{g_{K, \sigma_{\mathbf{a}}}}$ replaced by $|\det J\mathcal{F}_K| \approx h_K^d$, cf. (54). In both cases the claim follows with Lemma A.3. $\square$

Lemma A.6. *Let $\mathcal{T}_h$ be r -regular, $K \in \mathcal{T}_h$, $m \in \{0, \dots, r\}$ and let $v \in H^s(\Omega)$ for $s > 1/2$. Then*

$$\|\Pi_{\text{SZ}} v\|_{H^m(K)} \lesssim h_K^{d/2-m} \sum_{\mathbf{a} \in \mathcal{D}_h(K)} \left| \int_{\sigma_{\mathbf{a}}} \psi_{\mathbf{a}} v d\eta \right|.$$

Proof. On K , only the basis functions of the nodes $\mathcal{D}_h(K)$ do not vanish, so that

$$(\Pi_{\text{SZ}}v)|_K = \sum_{\mathbf{a} \in \mathcal{D}_h(K)} \left(\int_{\sigma_{\mathbf{a}}} \psi_{\mathbf{a}} v \, d\eta \right) \phi_{\mathbf{a}}|_K.$$

By (53), (57), and (59), the basis functions satisfy

$$\|\phi_{\mathbf{a}}\|_{H^m(K)} \lesssim h_K^{d/2-m} \|\phi_{\mathbf{a}} \circ \mathcal{F}_K\|_{H^m(\hat{K}_d)} \lesssim h_K^{d/2-m},$$

where we used $h_K \leq 1$. This proves the claim. $\square$

It remains to bound the nodal functionals.

Theorem A.1. *Let $\mathcal{T}_h$ be r -regular, $K \in \mathcal{T}_h$, $m \in \{0, \dots, r\}$ and $v \in H^\ell(\omega_K)$ for $\ell \in \{\max\{1, m\}, \dots, r\}$. Then*

$$\|\Pi_{\text{SZ}}v\|_{H^m(K)} \lesssim h_K^{-m} \sum_{j=0}^{\ell} h_K^j |v|_{H^j(\omega_K)}. \quad (62)$$

Proof. Hölder's inequality together with Lemma A.4 and Lemma A.5 gives

$$\begin{aligned} \left| \int_{\sigma_{\mathbf{a}}} \psi_{\mathbf{a}} v \, d\eta \right| &\leq \|\psi_{\mathbf{a}}\|_{L^\infty(\sigma_{\mathbf{a}})} \|v\|_{L^1(\sigma_{\mathbf{a}})} \\ &\lesssim h_{K_{\mathbf{a}}}^{-\dim \sigma_{\mathbf{a}}} h_{K_{\mathbf{a}}}^{\dim \sigma_{\mathbf{a}}-d/2} \sum_{j=0}^{\ell} h_{K_{\mathbf{a}}}^j |v|_{H^j(K_{\mathbf{a}})}. \end{aligned}$$

Since $\mathbf{a} \in \overline{K} \cap \overline{K_{\mathbf{a}}}$ implies $K_{\mathbf{a}} \in \mathcal{T}_h^K$, Lemma A.1 gives $h_{K_{\mathbf{a}}} \approx h_K$. Finite overlap of the patches ω_K and Lemma A.6 prove the claim. $\square$

Summation over the elements gives the global counterpart. We only need it for the first order wavenumber scaled norm.

Proposition A.1. *Let $\mathcal{T}_h$ be 1-regular and $kh \lesssim 1$. For all $v \in H^1(\Omega)$ it holds*

$$\|\Pi_{\text{SZ}}v\|_{H_k^1(\Omega)} \lesssim \|v\|_{H_k^1(\Omega)},$$

where the hidden constant depends on d , p , and $M_{\mathcal{T}_h}(1)$, but not on h , k , and v .

Proof. We first treat the $L^2(\Omega)$ -part. Here we square (62) with $m = 0$ and sum over $K \in \mathcal{T}_h$. The patch domains have finite overlap by Lemma A.1, and $h_K \leq h$, so that

$$\|\Pi_{\text{SZ}}v\|_{L^2(\Omega)}^2 \lesssim \|v\|_{L^2(\Omega)}^2 + h^2 |v|_{H^1(\Omega)}^2.$$

With $kh \lesssim 1$ this gives $k^2 \|\Pi_{\text{SZ}}v\|_{L^2(\Omega)}^2 \lesssim \|v\|_{H_k^1(\Omega)}^2$.

It remains to estimate the H^1 -part. Let $K \in \mathcal{T}_h$ and $c \in \mathbb{C}$. Constants belong to V_h , so that $\Pi_{\text{SZ}}c = c$ by the projection property. With (62) and $m = 1$, we obtain

$$|\Pi_{\text{SZ}}v|_{H^1(K)}^2 \leq \|\Pi_{\text{SZ}}(v - c)\|_{H^1(K)}^2 \lesssim h_K^{-2} \|v - c\|_{L^2(\omega_K)}^2 + |v|_{H^1(\omega_K)}^2.$$

We choose $c = |\omega_K|^{-1} \int_{\omega_K} v \, dx$ as the mean value of v on the patch. The Poincaré inequality then provides

$$\|v - c\|_{L^2(\omega_K)}^2 \leq \text{diam}(\omega_K)^2 |v|_{H^1(\omega_K)}^2 \lesssim h_K^2 |v|_{H^1(\omega_K)}^2,$$

where the hidden constant in the last inequality depends only on $M_{\mathcal{T}_h}$, cf. Lemma A.1. Summation over $K \in \mathcal{T}_h$ and the finite overlap of the patch domains provides $|\Pi_{\text{SZ}}v|_{H^1(\Omega)} \lesssim |v|_{H^1(\Omega)}$. This proves the claim. $\square$

Proof of Proposition 4. Part (iii) is Proposition A.1.

It remains to prove the projection properties (i) and (ii). Let $v_h \in V_h$. Then, $v_h = \sum_{\mathbf{b} \in \mathcal{D}_h} c_{\mathbf{b}} \phi_{\mathbf{b}}$ with coefficients $c_{\mathbf{b}} \in \mathbb{C}$. Hence, by (61), we obtain for $\mathbf{c} \in \mathcal{D}_h$

$$\Pi_{\text{SZ}} v_h(\mathbf{c}) = \int_{\sigma_{\mathbf{c}}} \psi_{\mathbf{c}}(t) v_h(t) d\eta(t) = \sum_{\mathbf{b} \in \mathcal{D}_h} c_{\mathbf{b}} \int_{\sigma_{\mathbf{c}}} \psi_{\mathbf{c}}(t) \phi_{\mathbf{b}}(t) d\eta(t) = c_{\mathbf{c}}.$$

Thus, v_h and its interpolation agree at every node. Hence $v_h = \Pi_{\text{SZ}} v_h$. Part (ii) follows in the same way. Let $v \in H^1(\Omega)$ and $v_h \in V_h$ with $v|_{\Gamma} = v_h|_{\Gamma}$. Rule 2 and Rule 3a give $\sigma_{\mathbf{a}} \subset \partial\Omega$ for every boundary node $\mathbf{a} \in \mathcal{D}_h \cap \partial\Omega$. The value $\Pi_{\text{SZ}} v(\mathbf{a})$ therefore depends on v only through $v|_{\Gamma}$, so that $\Pi_{\text{SZ}} v(\mathbf{a}) = \Pi_{\text{SZ}} v_h(\mathbf{a}) = v_h(\mathbf{a})$. A discrete trace is determined by the nodal values on $\partial\Omega$, and hence $(\Pi_{\text{SZ}} v)|_{\Gamma} = v_h|_{\Gamma}$. $\square$

APPENDIX B. THE LAPLACE–BELTRAMI OPERATOR

In this appendix we establish the local shift theorem for the Laplace–Beltrami operator on Γ_{sc} with constants that are explicit in the wavenumber.

We work in the setting of Section 1.2 and fix open sets $G_1, G_2 \subset \mathbb{R}^d$ with $G_1 \Subset G_2$, $G_1 \cap \Gamma \neq \emptyset$, and $G_2 \cap \Gamma \subset \Gamma_{\text{sc}}$. Throughout, $\{(U_i, \chi_i)\}_{i=1}^N$ denotes the atlas of Γ_{sc} from Section 1.3, with subordinate partition of unity $\{\eta_i\}_{i=1}^N$. Recall that the transition maps $\tau_{ji} = \chi_j \circ \theta_i$ are $C^{\gamma,1}$, cf. [31, Sec. 3]. All hidden constants in this appendix are independent of the wavenumber k .

We follow the ideas of [19, Thm. 3.3], but keep track of the wavenumber. Our argument flattens every boundary patch, localizes, and then uses a global k -explicit regularity result.

Surface differential operators. As for the Sobolev norms, we define the surface differential operators in terms of charts and a subordinate partition of unity. We recall the definition for a single chart $(U_i, \chi_i) = (U, \chi)$. Let $x \in V$ and define

$$g_{m,n} = \partial_{x_m} \theta(x) \cdot \partial_{x_n} \theta(x), \quad m, n = 1, \dots, d-1.$$

Then, $G = (g_{m,n})_{m,n=1}^{d-1}$ is the first fundamental form. We denote by $G^{-1} = (g^{m,n})_{m,n=1}^{d-1}$ the inverse and by $g = \det G$ the determinant. Note that the regularity assumptions give $\theta \in C^{\gamma,1}(V)$ and therefore $g_{m,n} \in C^{\gamma-1,1}(V)$. For a smooth $f : \Upsilon \rightarrow \mathbb{C}$ we define the *tangential gradient* and the *Laplace–Beltrami operator* as

$$(\nabla_{\Gamma} f) \circ \theta = \sum_{m,n=1}^{d-1} g^{m,n} (\partial_{x_n} (f \circ \theta)) \partial_{x_m} \theta, \quad (63a)$$

$$(\Delta_{\Gamma} f) \circ \theta = \frac{1}{\sqrt{g}} \sum_{m,n=1}^{d-1} \partial_{x_n} \left(g^{m,n} \sqrt{g} \partial_{x_m} (f \circ \theta) \right). \quad (63b)$$

Scaled norms on $\mathbb{R}^{d-1}$. The proof of Theorem 5 rests on flattening every boundary patch and applying elliptic regularity of the full-space problem on $\mathbb{R}^{d-1}$. To this end, we introduce k -scaled norms of real order on $\mathbb{R}^{d-1}$. We then collect the calculus rules and the regularity result that we need.

Let $\mathcal{S}(\mathbb{R}^{d-1})$ be the Schwartz space and $\mathcal{S}'(\mathbb{R}^{d-1})$ its dual, the space of tempered distributions. Following the convention of Section 1.3, we write $\langle v | \phi \rangle_{\mathbb{R}^{d-1}} = v(\bar{\phi})$ for $v \in \mathcal{S}'(\mathbb{R}^{d-1})$ and $\phi \in \mathcal{S}(\mathbb{R}^{d-1})$. The pairing reduces to the $L^2(\mathbb{R}^{d-1})$ -inner product whenever $v \in L^2(\mathbb{R}^{d-1})$. The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^{d-1}) \rightarrow \mathcal{S}(\mathbb{R}^{d-1})$

is an isomorphism with inverse $\mathcal{F}^*$, defined by

$$\widehat{v}(\xi) = (\mathcal{F}v)(\xi) = \int_{\mathbb{R}^{d-1}} v(x) e^{-2\pi i x \cdot \xi} dx, \quad \xi \in \mathbb{R}^{d-1}, \quad (64a)$$

$$v(x) = (\mathcal{F}^*v)(x) = \int_{\mathbb{R}^{d-1}} \widehat{v}(\xi) e^{2\pi i x \cdot \xi} d\xi, \quad x \in \mathbb{R}^{d-1}. \quad (64b)$$

We use the standard facts about the Fourier transform, like Plancherel's theorem, without further notice. For details, we refer to the subsection ‘‘Fourier Transform’’ of Section 3 in [31].

For $\sigma \in \mathbb{R}$ we set

$$\|v\|_{H_k^\sigma(\mathbb{R}^{d-1})}^2 = k \int_{\mathbb{R}^{d-1}} (1 + k^{-2}|\xi|^2)^\sigma |\widehat{v}(\xi)|^2 d\xi, \quad (65)$$

and define the Sobolev space of order σ as

$$H_k^\sigma(\mathbb{R}^{d-1}) = \{v \in \mathcal{S}'(\mathbb{R}^{d-1}) \mid \widehat{v} \text{ measurable, } \|v\|_{H_k^\sigma(\mathbb{R}^{d-1})} < \infty\}. \quad (66)$$

By Plancherel's theorem this is a Hilbert space, and $\mathcal{S}(\mathbb{R}^{d-1})$ is dense in it. Note that the k -scaling matches (4b), i.e. it accounts for the loss of differentiability when taking traces.

The following lemma gathers all the k -scaled auxiliary results we need.

Lemma B.1. *Let $\sigma \in \mathbb{R}$ and $v \in H_k^\sigma(\mathbb{R}^{d-1})$.*

(i) *For $\sigma_0, \sigma_1 \in \mathbb{R}$ and $\vartheta \in (0, 1)$ it holds*

$$[H_k^{\sigma_0}(\mathbb{R}^{d-1}), H_k^{\sigma_1}(\mathbb{R}^{d-1})]_{\vartheta} = H_k^{(1-\vartheta)\sigma_0 + \vartheta\sigma_1}(\mathbb{R}^{d-1})$$

with equal norms, where $[\cdot, \cdot]_{\vartheta}$ denotes the complex interpolation functor. In particular, the interpolation theorem for linear bounded maps applies to this scale.

(ii) *For $m \in \{1, \dots, d-1\}$ it holds $\partial_m v \in H_k^{\sigma-1}(\mathbb{R}^{d-1})$ with*

$$\|\partial_m v\|_{H_k^{\sigma-1}(\mathbb{R}^{d-1})} \lesssim k \|v\|_{H_k^\sigma(\mathbb{R}^{d-1})}.$$

(iii) *The pairing $\langle \cdot | \cdot \rangle_{\mathbb{R}^{d-1}}$ extends uniquely to a sesquilinear form on $H_k^\sigma(\mathbb{R}^{d-1}) \times H_k^{-\sigma}(\mathbb{R}^{d-1})$, and $w \mapsto k \langle w | \cdot \rangle_{\mathbb{R}^{d-1}}$ identifies $H_k^\sigma(\mathbb{R}^{d-1})$ isometrically with the antidual of $H_k^{-\sigma}(\mathbb{R}^{d-1})$. In particular,*

$$\|v\|_{H_k^\sigma(\mathbb{R}^{d-1})} = k \sup_{\phi \in H_k^{-\sigma}(\mathbb{R}^{d-1}) \setminus \{0\}} \frac{|\langle v | \phi \rangle_{\mathbb{R}^{d-1}}|}{\|\phi\|_{H_k^{-\sigma}(\mathbb{R}^{d-1})}}. \quad (67)$$

(iv) *Let $r \in \mathbb{N}$, $b \in W^{r,\infty}(\mathbb{R}^{d-1})$ and $|\sigma| \leq r$. For $\sigma \geq 0$, let bv be the pointwise product. For $\sigma < 0$, define $bv \in H_k^\sigma(\mathbb{R}^{d-1})$ by duality,*

$$\langle bv | \phi \rangle_{\mathbb{R}^{d-1}} = \langle v | \bar{b}\phi \rangle_{\mathbb{R}^{d-1}}, \quad \text{for all } \phi \in H_k^{-\sigma}(\mathbb{R}^{d-1}). \quad (68)$$

It holds

$$\|bv\|_{H_k^\sigma(\mathbb{R}^{d-1})} \lesssim \|b\|_{W^{r,\infty}(\mathbb{R}^{d-1})} \|v\|_{H_k^\sigma(\mathbb{R}^{d-1})},$$

uniformly in k .

Proof. Assertion (i) follows from complex interpolation of weighted L^2 -spaces, see, e.g., [8, Thm. 5.4.1]. The same theorem contains the interpolation theorem.

Assertion (ii) follows from the identity $\mathcal{F}(\partial_m v)(\xi) = 2\pi i \xi_m \widehat{v}(\xi)$ and the estimate

$$|\xi_m| (1 + k^{-2}|\xi|^2)^{(\sigma-1)/2} \leq k (1 + k^{-2}|\xi|^2)^{\sigma/2}.$$

For (iii), Plancherel's theorem and the Cauchy–Schwarz inequality give

$$|\langle w | \phi \rangle_{\mathbb{R}^{d-1}}| \leq k^{-1} \|w\|_{H_k^\sigma(\mathbb{R}^{d-1})} \|\phi\|_{H_k^{-\sigma}(\mathbb{R}^{d-1})}, \quad (69)$$

for $w \in H_k^\sigma(\mathbb{R}^{d-1})$ and $\phi \in \mathcal{S}(\mathbb{R}^{d-1})$. Since $\mathcal{S}(\mathbb{R}^{d-1})$ is dense in $H_k^{-\sigma}(\mathbb{R}^{d-1})$, the pairing extends uniquely to $H_k^\sigma(\mathbb{R}^{d-1}) \times H_k^{-\sigma}(\mathbb{R}^{d-1})$ with the same bound, so $w \mapsto k \langle w | \cdot \rangle_{\mathbb{R}^{d-1}}$ maps into the antidual of $H_k^{-\sigma}(\mathbb{R}^{d-1})$ with norm at most one. This shows

$$k \sup_{\phi \in H_k^{-\sigma}(\mathbb{R}^{d-1}) \setminus \{0\}} \frac{|\langle v | \phi \rangle_{\mathbb{R}^{d-1}}|}{\|\phi\|_{H_k^{-\sigma}(\mathbb{R}^{d-1})}} \leq \|v\|_{H_k^\sigma(\mathbb{R}^{d-1})}.$$

If $v \neq 0$, the function ϕ with $\hat{\phi} = (1 + k^{-2}|\xi|^2)^\sigma \hat{v}$ belongs to $H_k^{-\sigma}(\mathbb{R}^{d-1}) \setminus \{0\}$ and satisfies

$$\|\phi\|_{H_k^{-\sigma}(\mathbb{R}^{d-1})} = \|v\|_{H_k^\sigma(\mathbb{R}^{d-1})}, \quad k \frac{|\langle v | \phi \rangle_{\mathbb{R}^{d-1}}|}{\|\phi\|_{H_k^{-\sigma}(\mathbb{R}^{d-1})}} = \|v\|_{H_k^\sigma(\mathbb{R}^{d-1})}.$$

This proves the reverse inequality. Hence, (67) follows.

For (iv) and integer $\sigma \in [0, r]$, the uniform equivalence with the derivative-based norms, the Leibniz rule, and $k^{-(|\lambda|-|\mu|)} \leq 1$ for $\mu \leq \lambda$ give the claim. For integer $\sigma \in [-r, 0)$, the previous case applied with the order $-\sigma \in (0, r]$ gives $\bar{b}\phi \in H_k^{-\sigma}(\mathbb{R}^{d-1})$ and

$$\begin{aligned} |\langle v | \bar{b}\phi \rangle_{\mathbb{R}^{d-1}}| &\leq k^{-1} \|v\|_{H_k^\sigma(\mathbb{R}^{d-1})} \|\bar{b}\phi\|_{H_k^{-\sigma}(\mathbb{R}^{d-1})} \\ &\lesssim k^{-1} \|b\|_{W^{r,\infty}(\mathbb{R}^{d-1})} \|v\|_{H_k^\sigma(\mathbb{R}^{d-1})} \|\phi\|_{H_k^{-\sigma}(\mathbb{R}^{d-1})} \end{aligned}$$

for $\phi \in \mathcal{S}(\mathbb{R}^{d-1})$. The right-hand side of (68) is therefore an antilinear functional on $\mathcal{S}(\mathbb{R}^{d-1})$ that is bounded with respect to $\|\cdot\|_{H_k^{-\sigma}(\mathbb{R}^{d-1})}$, and (iii) provides a unique $bv \in H_k^\sigma(\mathbb{R}^{d-1})$ representing it, with the asserted bound. The remaining cases follow by interpolation with (i). $\square$

Localization. The next lemma transports functions on a boundary patch to $\mathbb{R}^{d-1}$ and back, with bounds that are uniform in k .

Lemma B.2. *Let $\omega \subset \Gamma_{\text{sc}}$ be relatively open and $i \in \{1, \dots, N\}$. Recall that the atlas consists of N charts (U_i, χ_i) with parametrization $\theta_i = \chi_i^{-1} : V_i \rightarrow U_i$, cf. Section 1.3.*

- (i) *Let $r \in \{0, \dots, \gamma + 1\}$, $b \in W^{r,\infty}(\mathbb{R}^{d-1})$ with $\text{supp } b \Subset \chi_i(U_i \cap \omega)$. For $\sigma \in [0, r]$ and $\phi \in H_k^\sigma(\omega)$ it holds*

$$\|b(\phi \circ \theta_i)\|_{H_k^\sigma(\mathbb{R}^{d-1})} \lesssim \|\phi\|_{H_k^\sigma(\omega)}.$$

- (ii) *Let $r \in \{0, \dots, \gamma + 1\}$, $c \in W^{r,\infty}(\mathbb{R}^{d-1})$ with $\text{supp } c \Subset \chi_i(U_i \cap \omega)$. For $\sigma \in [0, r]$ and $v \in H_k^\sigma(\mathbb{R}^{d-1})$ the zero extension of $(cv) \circ \chi_i$ satisfies*

$$\|(cv) \circ \chi_i\|_{H_k^\sigma(\omega)} \lesssim \|v\|_{H_k^\sigma(\mathbb{R}^{d-1})}.$$

- (iii) *For real-valued $b \in W^{1,\infty}(\mathbb{R}^{d-1})$ with $\text{supp } b \Subset \chi_i(U_i \cap \omega)$, $\sigma \in [-1, 0)$, and $\phi \in H_k^\sigma(\omega)$, the tempered distribution defined by*

$$\langle \sqrt{g}b(\phi \circ \theta_i) | v \rangle_{\mathbb{R}^{d-1}} = \langle \phi | (bv) \circ \chi_i \rangle_\omega, \quad v \in \mathcal{S}(\mathbb{R}^{d-1}),$$

satisfies $\|\sqrt{g}b(\phi \circ \theta_i)\|_{H_k^\sigma(\mathbb{R}^{d-1})} \lesssim \|\phi\|_{H_k^\sigma(\omega)}$ and coincides with the point-wise product of (i) if $\phi \in L^2(\omega)$.

The hidden constants depend on b and c , but not on k .

Proof. For (i) and integer $\sigma = \ell \leq r$, set

$$D_j = \chi_j(U_j \cap \omega), \quad w_j = (\eta_j \phi) \circ \theta_j \quad \text{on } D_j.$$

We work with these functions locally. The product $b(\phi \circ \theta_i)$ is extended by zero outside $\chi_i(U_i \cap \omega)$. This preserves its regularity because $\text{supp } b \Subset \chi_i(U_i \cap \omega)$. On $\chi_i(U_i \cap \omega)$ we have the pointwise identity

$$\phi \circ \theta_i = \sum_{j=1}^N (\eta_j \phi) \circ \theta_i, \quad (70)$$

because $\sum_j \eta_j \equiv 1$ on Γ_{sc} . The transition map τ_{ji} is defined on $\chi_i(U_i \cap U_j)$ only, so we localize before changing charts.

Fix i and j and set $A_{ij} = \chi_i(\theta_i(\text{supp } b) \cap \text{supp } \eta_j)$. Since $\text{supp } b \Subset \chi_i(U_i \cap \omega)$, the set $\theta_i(\text{supp } b)$ is a compact subset of $U_i \cap \omega$, so A_{ij} is compact, and $\text{supp } \eta_j \Subset U_j$ gives $A_{ij} \Subset \chi_i(U_i \cap U_j)$. Hence, A_{ij} is a compact subset of the open domain of τ_{ji} . Since $\tau_{ji} \in C^{\gamma,1}$, this gives $\tau_{ji} \in W^{\gamma+1,\infty}(A_{ij})$. The hidden constant depends on $\text{supp } b$, but not on k or ϕ .

The j -th summand of (70), multiplied by b , is supported in A_{ij} and satisfies

$$b((\eta_j \phi) \circ \theta_i) = b((\eta_j \phi) \circ \theta_j \circ \tau_{ji}) = b(w_j \circ \tau_{ji}) \quad \text{on } \chi_i(U_i \cap U_j \cap \omega).$$

By the Leibniz and chain rule, $\partial^\lambda(b(w_j \circ \tau_{ji}))$, $|\lambda| \leq \ell$, is a finite sum of products of derivatives of b and of τ_{ji} up to order ℓ , which are bounded on A_{ij} by the above, and of terms $(\partial^\mu w_j) \circ \tau_{ji}$ with $|\mu| \leq |\lambda|$, which are evaluated on $\tau_{ji}(A_{ij}) \subset \chi_j(U_j \cap \omega)$. Hence, using $k^{-(|\lambda|-|\mu|)} \leq 1$ yields

$$\begin{aligned} k^{-|\lambda|+1/2} \|\partial^\lambda(b(\phi \circ \theta_i))\|_{L^2(\mathbb{R}^{d-1})} &\lesssim \sum_{j=1}^N \sum_{|\mu| \leq \ell} k^{-|\mu|+1/2} \|\partial^\mu w_j\|_{L^2(\chi_j(U_j \cap \omega))} \\ &\lesssim \|\phi\|_{H_k^\ell(\omega)}. \end{aligned}$$

Since $\phi \mapsto b(\phi \circ \theta_i)$ is linear, fractional orders follow with Lemma B.1 (i), because the patch spaces are defined via interpolation too.

The proof of (ii) is analogous.

For (iii), the test function $(bv) \circ \chi_i$ has compact support in ω . The definition of the negative order norms, Lemma B.1 (iii), and Part (ii) with $r = 1$ therefore give

$$\begin{aligned} \|\sqrt{g}b(\phi \circ \theta_i)\|_{H_k^\sigma(\mathbb{R}^{d-1})} &= k \sup_{v \in \mathcal{S}(\mathbb{R}^{d-1}) \setminus \{0\}} \frac{|\langle \phi | (bv) \circ \chi_i \rangle_\omega|}{\|v\|_{H_k^{-\sigma}(\mathbb{R}^{d-1})}} \\ &\leq \|\phi\|_{H_k^\sigma(\omega)} \sup_{v \in \mathcal{S}(\mathbb{R}^{d-1}) \setminus \{0\}} \frac{\|(bv) \circ \chi_i\|_{H_k^{-\sigma}(\Gamma_{\text{sc}})}}{\|v\|_{H_k^{-\sigma}(\mathbb{R}^{d-1})}} \\ &\lesssim \|\phi\|_{H_k^\sigma(\omega)}. \end{aligned}$$

For $\phi \in L^2(\omega)$, transforming the surface integral to the chart produces exactly the factor $\sqrt{g}$, which shows the consistency. $\square$

A shift theorem on $\mathbb{R}^{d-1}$. Flattening leads to a full-space elliptic equation on $\mathbb{R}^{d-1}$. The next lemma provides the k -explicit shift theorem that we need for it.

Lemma B.3. *Let $a \in W^{\gamma,\infty}(\mathbb{R}^{d-1}; \mathbb{R}^{(d-1) \times (d-1)})$ be symmetric-valued with $c_a |\zeta|^2 \leq a(x) \zeta \cdot \zeta \leq C_a |\zeta|^2$ for all $x \in \mathbb{R}^{d-1}$, $\zeta \in \mathbb{R}^{d-1}$, and some $0 < c_a \leq C_a$. Define*

$$A_k v = v - k^{-2} \text{div}(a \nabla v).$$

For every $\sigma \in [0, \gamma]$, if $v \in H_k^\sigma(\mathbb{R}^{d-1})$ and $A_k v \in H_k^{\sigma-1}(\mathbb{R}^{d-1})$, then $v \in H_k^{\sigma+1}(\mathbb{R}^{d-1})$ and

$$\|v\|_{H_k^{\sigma+1}(\mathbb{R}^{d-1})} \lesssim \|A_k v\|_{H_k^{\sigma-1}(\mathbb{R}^{d-1})}.$$

The hidden constant depends only on d , γ , σ , c_a , C_a , k_0 , and $\|a\|_{W^{\gamma,\infty}(\mathbb{R}^{d-1})}$, but not on k or v .

Proof. For every $\sigma \in [0, \gamma]$, Lemma B.1 (ii),(iv) componentwise give

$$\begin{aligned} \|k^{-2} \operatorname{div}(a \nabla v)\|_{H_k^{\sigma-1}(\mathbb{R}^{d-1})} &\lesssim k^{-1} \|a \nabla v\|_{H_k^{\sigma}(\mathbb{R}^{d-1})} \\ &\lesssim k^{-1} \|\nabla v\|_{H_k^{\sigma}(\mathbb{R}^{d-1})} \leq \|v\|_{H_k^{\sigma+1}(\mathbb{R}^{d-1})}, \end{aligned}$$

which proves the a-priori bound.

Let $\sigma = 0$. By (65) and Plancherel's theorem, the sesquilinear form

$$\langle A_k v | w \rangle_{\mathbb{R}^{d-1}} = B(v, w) = (v | w)_{\mathbb{R}^{d-1}} + k^{-2} (a \nabla v | \nabla w)_{\mathbb{R}^{d-1}},$$

defined on $H_k^1(\mathbb{R}^{d-1})$ is bounded and coercive. Hence, the Lax–Milgram lemma shows invertibility of $A_k : H_k^1(\mathbb{R}^{d-1}) \rightarrow H_k^{-1}(\mathbb{R}^{d-1})$. The bounded inverse theorem then provides that A_k is an isomorphism with the k -uniform lower bound.

Next, we apply elliptic regularity in the case that $\sigma = m \in \{1, \dots, \gamma\}$. Let $f \in H_k^{m-1}(\mathbb{R}^{d-1}) \subset H_k^{-1}(\mathbb{R}^{d-1})$ and let $v \in H_k^1(\mathbb{R}^{d-1})$ solve $A_k v = f$, so that $\operatorname{div}(a \nabla v) = k^2(v - f)$. The interior regularity result [31, Thm. 4.16], applied on the balls $B_1(z) \subset B_2(z)$ with $z \in \mathbb{Z}^{d-1}$, provides

$$\|v\|_{H^{j+1}(\mathbb{R}^{d-1})} \lesssim \|\operatorname{div}(a \nabla v)\|_{H^{j-1}(\mathbb{R}^{d-1})} + \|v\|_{H^1(\mathbb{R}^{d-1})}, \quad j \in \{1, \dots, m\}, \quad (71)$$

with constants that do not depend on k , since the operator $\operatorname{div}(a \nabla \cdot)$ contains no k .

For $w \in H^{j-1}(\mathbb{R}^{d-1})$, the elementary inequality $k^{-j+3/2} \leq k^{-|\lambda|+1/2}$ for $|\lambda| \leq j-1$ gives $k^{-j+3/2} \|w\|_{H^{j-1}(\mathbb{R}^{d-1})} \lesssim \|w\|_{H_k^{j-1}(\mathbb{R}^{d-1})}$, by the uniform equivalence of the integer order norms. Multiplying (71) by $k^{-j-1/2}$ and inserting $\operatorname{div}(a \nabla v) = k^2(v - f)$ therefore yields

$$\begin{aligned} k^{-j-1/2} \|v\|_{H^{j+1}(\mathbb{R}^{d-1})} &\lesssim \|v\|_{H_k^{j-1}(\mathbb{R}^{d-1})} + \|f\|_{H_k^{j-1}(\mathbb{R}^{d-1})} + k^{-j} \|v\|_{H_k^1(\mathbb{R}^{d-1})}. \end{aligned}$$

Summing over $j = 1, \dots, m$ gives

$$\|v\|_{H_k^{m+1}(\mathbb{R}^{d-1})} \lesssim \|f\|_{H_k^{m-1}(\mathbb{R}^{d-1})} + \|v\|_{H_k^m(\mathbb{R}^{d-1})}.$$

Iterating this estimate down to the case $\sigma = 0$ proves the claim.

For $\sigma \in (m, m+1)$, interpolating A_k^{-1} between the integer orders m and $m+1$ with Lemma B.1 (i) shows that $A_k^{-1} : H_k^{\sigma-1}(\mathbb{R}^{d-1}) \rightarrow H_k^{\sigma+1}(\mathbb{R}^{d-1})$ is bounded, uniformly in k . The identity $A_k A_k^{-1} = I = A_k^{-1} A_k$ holds on dense subspaces, and all involved operators are continuous. Hence, A_k is an isomorphism for every $\sigma \in [0, \gamma]$.

It remains to prove the shift. Let $v \in H_k^{\sigma}(\mathbb{R}^{d-1})$ with $f = A_k v \in H_k^{\sigma-1}(\mathbb{R}^{d-1})$, and let $\tilde{v} = A_k^{-1} f \in H_k^{\sigma+1}(\mathbb{R}^{d-1})$. Then $z = v - \tilde{v} \in L^2(\mathbb{R}^{d-1})$ and $A_k z = 0$ in the sense of distributions. We show that this implies $z = 0$. Let $g \in C_c^{\infty}(\mathbb{R}^{d-1})$, and use the already established case $\sigma = 1$ to define $q = A_k^{-1} g \in H^2(\mathbb{R}^{d-1})$. Choose $q_n \in C_c^{\infty}(\mathbb{R}^{d-1})$ with $q_n \rightarrow q$ in $H^2(\mathbb{R}^{d-1})$. Since $a \in W^{1,\infty}(\mathbb{R}^{d-1})$, we have $A_k q_n \rightarrow A_k q = g$ in $L^2(\mathbb{R}^{d-1})$. The symmetry of a therefore gives

$$(z | g)_{\mathbb{R}^{d-1}} = \lim_{n \rightarrow \infty} (z | A_k q_n)_{\mathbb{R}^{d-1}} = \lim_{n \rightarrow \infty} \langle A_k z | q_n \rangle_{\mathbb{R}^{d-1}} = 0.$$

Since $C_c^{\infty}(\mathbb{R}^{d-1})$ is dense in $L^2(\mathbb{R}^{d-1})$, it follows that $z = 0$. Thus $v = \tilde{v} \in H_k^{\sigma+1}(\mathbb{R}^{d-1})$, and the asserted estimate follows from the bound for A_k^{-1} . $\square$

The next lemma extends a coefficient from the parameter domain V_i to all of $\mathbb{R}^{d-1}$.

Lemma B.4. *Let $\gamma \in \mathbb{N}_+$ and let $V \subset \mathbb{R}^{d-1}$ be bounded and open. Furthermore, let $a \in W^{\gamma,\infty}(V; \mathbb{R}_{\text{sym}}^{(d-1) \times (d-1)})$ be pointwise positive definite on V . For any $K \Subset V$*

there exist $\kappa \in [1, \infty)$ and $\tilde{a} \in W^{\gamma, \infty}(\mathbb{R}^{d-1}; \mathbb{R}_{\text{sym}}^{(d-1) \times (d-1)})$ such that

$$\tilde{a} \equiv a \quad \text{on } K, \quad \kappa^{-1}|\zeta|^2 \leq \zeta \cdot \tilde{a}(x)\zeta \leq \kappa|\zeta|^2, \quad x \in \mathbb{R}^{d-1}, \zeta \in \mathbb{R}^{d-1}.$$

The constant κ depends on a , V and K .

Proof. Let $\delta = \text{dist}(K, \partial V) > 0$ and define $K_s = \{x \in \mathbb{R}^{d-1} \mid \text{dist}(x, K) \leq s\}$. We choose a smooth cut-off $\rho \in C_c^\infty(\mathbb{R}^{d-1})$ with

$$0 \leq \rho \leq 1, \quad \text{on } \mathbb{R}^{d-1}, \quad \rho \equiv 1, \quad \text{on } K_{\delta/4}, \quad \text{supp } \rho \subset K_{\delta/2}.$$

The extended coefficient is defined as

$$\tilde{a} = I + b, \quad b = \rho(a - I) \quad \text{on } V, \quad b = 0 \quad \text{on } \mathbb{R}^{d-1} \setminus \text{supp } \rho.$$

The sets V and $\mathbb{R}^{d-1} \setminus \text{supp } \rho$ cover $\mathbb{R}^{d-1}$, and the two formulas agree on the overlap. Hence, $\tilde{a}$ is well-defined and symmetric, with $\tilde{a} \equiv a$ on $K_{\delta/4} \supset K$.

Furthermore, a is continuous and positive definite on $\text{supp } \rho \Subset V$. Hence, it attains a minimum $\lambda > 0$ and a maximum $\Lambda > 0$ such that

$$\lambda|\zeta|^2 \leq \zeta \cdot a(x)\zeta \leq \Lambda|\zeta|^2, \quad x \in \text{supp } \rho, \zeta \in \mathbb{R}^{d-1}.$$

Set $\kappa = \max\{\Lambda, \lambda^{-1}, 1\} \in [1, \infty)$. Let $x \in \mathbb{R}^{d-1}$ and write $t = \rho(x) \in [0, 1]$. If $t = 0$, then $\zeta \cdot \tilde{a}(x)\zeta = |\zeta|^2$. Otherwise, $x \in \text{supp } \rho$ and we obtain

$$\begin{aligned} \zeta \cdot \tilde{a}(x)\zeta &= t\zeta \cdot a(x)\zeta + (1-t)|\zeta|^2 \\ &\in [\min\{\lambda, 1\}|\zeta|^2, \max\{\Lambda, 1\}|\zeta|^2] \subset [\kappa^{-1}|\zeta|^2, \kappa|\zeta|^2]. \end{aligned}$$

Since $\rho \in C_c^\infty(\mathbb{R}^{d-1})$ and $a \in W^{\gamma, \infty}(V)$, the product $b = \rho(a - I)$ belongs to $W^{\gamma, \infty}(\mathbb{R}^{d-1})$. This proves the claim. $\square$

Proof of Theorem 5. Fix $\eta \in C_c^\infty(G_2)$ with $\eta \equiv 1$ on a neighborhood of $\overline{G_1}$ and set $\zeta_i = (\eta_i \eta)|_{\Gamma_{\text{sc}}}$, so that $\text{supp } \zeta_i \Subset U_i \cap G_2 \cap \Gamma$ and $\sum_{i=1}^N \zeta_i = \eta|_{\Gamma_{\text{sc}}} \equiv 1$ on a neighborhood of $\overline{G_1} \cap \Gamma$. Since η is smooth on $\mathbb{R}^d$ and $\eta_i \in C^{\gamma, 1}(\Gamma_{\text{sc}})$, the zero extensions of $\tilde{\zeta}_i = \zeta_i \circ \theta_i$ belong to $W^{\gamma+1, \infty}(\mathbb{R}^{d-1})$.

Fix i , write $\tilde{\zeta} = \tilde{\zeta}_i$, $\tilde{\psi} = \psi \circ \theta_i$, and let $w_i = \tilde{\zeta}\tilde{\psi}$, extended by zero to $\mathbb{R}^{d-1}$. Lemma B.2 (i) with $b = \tilde{\zeta}$ gives

$$w_i \in H_k^s(\mathbb{R}^{d-1}), \quad \text{with } \|w_i\|_{H_k^s(\mathbb{R}^{d-1})} \lesssim \|\psi\|_{H_k^s(G_2 \cap \Gamma)}. \quad (72)$$

The matrix $a = \sqrt{g}G^{-1} \in W^{\gamma, \infty}(V_i, \mathbb{R}_{\text{sym}}^{(d-1) \times (d-1)})$ is symmetric and positive definite on V_i . Lemma B.4 applied with $K = \text{supp } \tilde{\zeta} \Subset V_i$ provides a uniformly elliptic extension $\tilde{a} \in W^{\gamma, \infty}(\mathbb{R}^{d-1}, \mathbb{R}^{(d-1) \times (d-1)})$ that agrees with a on $\text{supp } \tilde{\zeta}$. The coefficient $\tilde{a}$ satisfies the assumptions of Lemma B.3.

The formal commutator identity

$$\Delta_{\Gamma_{\text{sc}}}(\zeta_i \psi) = \zeta_i \Delta_{\Gamma_{\text{sc}}} \psi + 2 \text{div}_{\Gamma_{\text{sc}}}(\psi \nabla_{\Gamma_{\text{sc}}} \zeta_i) - \psi \Delta_{\Gamma_{\text{sc}}} \zeta_i,$$

reads in the chart

$$\text{div}(\tilde{a} \nabla w_i) = \sqrt{g} \tilde{\zeta} (\Delta_{\Gamma_{\text{sc}}} \psi) \circ \theta_i + 2 \text{div}(\tilde{\psi} \tilde{a} \nabla \tilde{\zeta}) - \tilde{\psi} \text{div}(\tilde{a} \nabla \tilde{\zeta}),$$

where for $s < 1$ the first term on the right is understood as in Lemma B.2 (iii). Consequently, using the operator A_k from Lemma B.3 gives

$$A_k w_i = w_i - k^{-2} \sqrt{g} \tilde{\zeta} (\Delta_{\Gamma_{\text{sc}}} \psi) \circ \theta_i - 2k^{-2} \text{div}(\tilde{\psi} \tilde{a} \nabla \tilde{\zeta}) + k^{-2} \tilde{\psi} \text{div}(\tilde{a} \nabla \tilde{\zeta}). \quad (73)$$

We estimate the four terms on the right-hand side of (73) in $H_k^{s-1}(\mathbb{R}^{d-1})$ individually. Throughout, we abbreviate $\Lambda = \|\psi\|_{H_k^s(G_2 \cap \Gamma)} + \|k^{-2} \Delta_{\Gamma_{\text{sc}}} \psi\|_{H_k^{s-1}(G_2 \cap \Gamma)}$.

For the first term, we obtain with (72) that

$$\|w_i\|_{H_k^{s-1}(\mathbb{R}^{d-1})} \leq \|w_i\|_{H_k^s(\mathbb{R}^{d-1})} \lesssim \Lambda.$$

For the second term, we distinguish two cases. First, let $s < 1$. Then, we apply Lemma B.2 (iii) with $\sigma = s - 1$, $b = \tilde{\zeta}$ and $\phi = k^{-2}\Delta_{\Gamma_{\text{sc}}}\psi \in H^{s-1}(G_2 \cap \Gamma)$. This provides

$$\left\| k^{-2}\sqrt{g}\tilde{\zeta}(\Delta_{\Gamma_{\text{sc}}}\psi \circ \theta_i) \right\|_{H_k^{s-1}(\mathbb{R}^{d-1})} = \left\| \sqrt{g}b(\phi \circ \theta_i) \right\|_{H_k^{s-1}(\mathbb{R}^{d-1})} \lesssim \Lambda.$$

The case $s \geq 1$ follows by Lemma B.2 (i) with $b = \sqrt{g}\tilde{\zeta}$.

We treat the third term componentwise. Let $b = (\tilde{a}\nabla\tilde{\zeta})_m \in W^{\gamma,\infty}(\mathbb{R}^{d-1})$ for $m \in \{1, \dots, d-1\}$. Applying Lemma B.1 (ii) and Lemma B.2 (i) then gives

$$\left\| k^{-2}\partial_m(b\tilde{\psi}) \right\|_{H_k^{s-1}(\mathbb{R}^{d-1})} \lesssim \left\| k^{-1}b\tilde{\psi} \right\|_{H_k^s(\mathbb{R}^{d-1})} \lesssim \Lambda.$$

For the fourth term, we define $b = \text{div}(\tilde{a}\nabla\tilde{\zeta}) \in W^{\gamma-1,\infty}(\mathbb{R}^{d-1})$. If $\sigma = s - 1 \geq 0$, we apply Lemma B.2 (i). Otherwise, $\sigma = s - 1 \in [-1, 0)$, and the monotonicity $\|\cdot\|_{H_k^{s-1}(\mathbb{R}^{d-1})} \leq \|\cdot\|_{H_k^0(\mathbb{R}^{d-1})}$ reduces this to the previous case. In both cases, we obtain

$$\left\| k^{-2}\text{div}(\tilde{a}\nabla\tilde{\zeta})\tilde{\psi} \right\|_{H_k^{s-1}(\mathbb{R}^{d-1})} = \left\| k^{-2}b\tilde{\psi} \right\|_{H_k^{s-1}(\mathbb{R}^{d-1})} \lesssim \Lambda.$$

Collecting the four bounds gives $\|A_k w_i\|_{H_k^{s-1}(\mathbb{R}^{d-1})} \lesssim \Lambda$. By the shift property in Lemma B.3, $w_i \in H_k^{s+1}(\mathbb{R}^{d-1})$ with

$$\|w_i\|_{H_k^{s+1}(\mathbb{R}^{d-1})} \lesssim \|A_k w_i\|_{H_k^{s-1}(\mathbb{R}^{d-1})} \lesssim \Lambda, \quad i \in \{1, \dots, N\}.$$

Finally, $w_i \circ \chi_i = \zeta_i \psi$ on U_i , and both sides vanish outside U_i . Since $\sum_{i=1}^N \zeta_i \equiv 1$ on $G_1 \cap \Gamma$, this gives $\psi = \sum_{i=1}^N w_i \circ \chi_i$ there. Choose $\beta_i \in C_c^\infty(\chi_i(U_i \cap G_2 \cap \Gamma))$ with $\beta_i \equiv 1$ on $\text{supp } \tilde{\zeta}_i$, so that $\beta_i w_i = w_i$. By the triangle inequality, monotonicity of the norm in the domain, and Lemma B.2 (ii) with $c = \beta_i$ and $\omega = G_2 \cap \Gamma$, we conclude

$$\|\psi\|_{H_k^{s+1}(G_1 \cap \Gamma)} \leq \sum_{i=1}^N \|w_i \circ \chi_i\|_{H_k^{s+1}(G_2 \cap \Gamma)} \lesssim \sum_{i=1}^N \|w_i\|_{H_k^{s+1}(\mathbb{R}^{d-1})} \lesssim \Lambda.$$

This proves the claim. $\square$

APPENDIX C. THE NORMAL TRACE

In this appendix we prove the local, k -explicit normal trace estimates of Lemma 9.

Classically, the normal trace of a function $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ is well-defined as an element of $H^{-1/2}(\Gamma)$. It satisfies $\|\partial_n u\|_{H^{-1/2}(\Gamma)} \lesssim \|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)}$, see, e.g., [31, Lem. 4.3]. We first replicate this result globally with k -explicit norms, and then localize.

Throughout, we use the k -explicit trace estimates

$$\|v|_\Gamma\|_{H_k^{\sigma-1/2}(\Gamma)} \lesssim \|v\|_{H_k^\sigma(\Omega)}, \quad 1/2 < \sigma \leq \gamma, \quad (74)$$

uniformly in $k \geq k_0$. They follow by flattening the boundary and applying the Fourier transform. We omit the details and refer to [31, Lem. 3.35, Thm. 3.37]. Finally, we recall that for $\sigma = 1$ the trace has a right inverse $\mathcal{L}$

$$\mathcal{L} : H_k^{1/2}(\Gamma) \rightarrow H_k^1(\Omega), \quad (\mathcal{L}\phi)|_\Gamma = \phi, \quad \|\mathcal{L}\phi\|_{H_k^1(\Omega)} \lesssim \|\phi\|_{H_k^{1/2}(\Gamma)}, \quad (75)$$

uniformly in $k \geq k_0$. We obtain this from the Fourier transform on the flattened boundary and the k -explicit norms. The version without explicit k is [31, Thm. 3.37]. Alternatively, one constructs $\mathcal{L}$ in the same way as the operator E_k in Definition 1, and uses k -explicit regularity estimates.

Proof of Lemma 9. We start with Part (i). By (35) and (36), the assertion means

$$|\langle \partial_n u | \phi \rangle_{\Gamma_1}| \lesssim \Lambda \|\phi\|_{H_k^{1/2}(\Gamma_{\text{sc}})}, \quad \Lambda = \|u\|_{H_k^1(\Omega_2)} + \|k^{-2} \Delta u\|_{H_k^0(\Omega_2)}, \quad (76)$$

for all $\phi \in H_k^{1/2}(\Gamma_{\text{sc}})$ with $\text{supp } \phi \Subset \Gamma_1$. Note that the factor k^{-1} in (38) cancels the factor k in the definition of the negative order norm (36).

First, we show the global k -explicit estimate. Let $w \in H^1(\Omega)$ with $\Delta w \in L^2(\Omega)$ and $\phi \in H_k^{1/2}(\Gamma)$. We define

$$\langle \partial_n w | \phi \rangle_{\Gamma} = (\nabla w | \nabla v)_{\Omega} + (\Delta w | v)_{\Omega}, \quad v \in H^1(\Omega) \text{ with } v|_{\Gamma} = \phi. \quad (77)$$

The right-hand side does not depend on the choice of v . Indeed, the difference of two liftings has vanishing trace, and by the definition of the distributional Laplacian the right-hand side vanishes for such functions. For smooth w , Green's formula shows that $\partial_n w$ defined by (77) is the classical normal derivative. Choosing $v = \mathcal{L}\phi$, the Cauchy–Schwarz inequality and (75) yield

$$\begin{aligned} |\langle \partial_n w | \phi \rangle_{\Gamma}| &\leq \|\nabla w\|_{L^2(\Omega)} \|\nabla \mathcal{L}\phi\|_{L^2(\Omega)} + \|\Delta w\|_{L^2(\Omega)} \|\mathcal{L}\phi\|_{L^2(\Omega)} \\ &\leq \|w\|_{H_k^1(\Omega)} \|\mathcal{L}\phi\|_{H_k^1(\Omega)} + k \|k^{-2} \Delta w\|_{H_k^0(\Omega)} k^{-1} \|\mathcal{L}\phi\|_{H_k^1(\Omega)} \\ &\lesssim \left(\|w\|_{H_k^1(\Omega)} + \|k^{-2} \Delta w\|_{H_k^0(\Omega)} \right) \|\phi\|_{H_k^{1/2}(\Gamma)}. \end{aligned}$$

Note that the powers of k match exactly.

Next, we localize. Fix $\eta \in C_c^\infty(G_2)$ with $\eta \equiv 1$ on a neighborhood of $\overline{G_1}$, and let w be the extension of ηu by zero to Ω . Then $w \in H^1(\Omega)$ and $\Delta w = \eta \Delta u + 2\nabla \eta \cdot \nabla u + u \Delta \eta \in L^2(\Omega)$. Hence,

$$\|w\|_{H_k^1(\Omega)} \lesssim (1 + k_0^{-1} \|\eta\|_{W^{1,\infty}}) \|u\|_{H_k^1(\Omega_2)}, \quad \|k^{-2} \Delta w\|_{H_k^0(\Omega)} \lesssim \Lambda,$$

where the second bound follows from

$$\begin{aligned} k^{-1} \|\nabla \eta \cdot \nabla u\|_{L^2(\Omega_2)} &\leq k_0^{-1} \|\eta\|_{W^{1,\infty}} \|u\|_{H_k^1(\Omega_2)}, \\ k^{-1} \|u \Delta \eta\|_{L^2(\Omega_2)} &\leq k_0^{-2} \|\eta\|_{W^{2,\infty}} \|u\|_{H_k^0(\Omega_2)}. \end{aligned}$$

Thus, the global result shows that $|\langle \partial_n w | \phi \rangle_{\Gamma}| \lesssim \Lambda \|\phi\|_{H_k^{1/2}(\Gamma)}$.

Finally, let $\phi \in H_k^{1/2}(\Gamma_{\text{sc}})$ with $\text{supp } \phi \Subset \Gamma_1$, extended by zero to Γ . In (77) we use the lifting $v = \chi \mathcal{L}\phi$, where $\chi \in C_c^\infty(G_1)$ with $\chi \equiv 1$ on a neighborhood of $\text{supp } \phi$. Then $v \in H^1(\Omega)$ with $v|_{\Gamma} = \chi \phi = \phi$. Since $\text{supp } v \Subset G_1$ and $\eta \equiv 1$ there, we have $\nabla w \cdot \nabla \bar{v} = \nabla u \cdot \nabla \bar{v}$ and $\Delta w \bar{v} = \Delta u \bar{v}$ a.e. in G_1 . Hence,

$$\langle \partial_n w | \phi \rangle_{\Gamma} = (\nabla u | \nabla \bar{v})_{\Omega_2} + (\Delta u | \bar{v})_{\Omega_2} = \langle \partial_n u | \phi \rangle_{\Gamma_1}.$$

This quantity depends on u only through $u|_{\Omega_2}$. As above, it is independent of the choice of χ and of the lifting. This proves the claim.

For Part (ii), we apply the k -explicit trace estimates (74) to ∇u and localize as above.

First, let η and w be as above. Multiplication by the fixed cut-off η is bounded on $H_k^{m+2}(\Omega)$ uniformly in $k \geq k_0$,

$$\|w\|_{H_k^{m+2}(\Omega)} \lesssim \|\eta\|_{W^{m+2,\infty}} \|u\|_{H_k^{m+2}(\Omega_2)},$$

since by (4a) every derivative falling on η in the Leibniz rule carries a factor $k^{-1} \leq k_0^{-1}$. This is the domain analog of Lemma B.1 (iv).

Second, (4a) gives

$$\|\nabla v\|_{H_k^\ell(\Omega)} \leq k \|v\|_{H_k^{\ell+1}(\Omega)}, \quad \ell \in \mathbb{N}.$$

Third, since $\Gamma \in C^{\gamma,1}$, the components of the normal vector n belong to $W^{\gamma,\infty}(\Gamma)$ in every chart. By Lemma B.1 (iv) and Lemma B.2, multiplication by n is therefore

bounded on $H_k^\sigma(\Gamma)$ for $0 \leq \sigma \leq \gamma$. Note that the restriction $m+1 \leq \gamma$ enters here. The trace estimate below is applied with $\sigma = m+1$, which lies in the range of (74). Since $\eta \equiv 1$ on a neighborhood of $\overline{G_1}$, we have $\partial_n u = n \cdot (\nabla w)|_\Gamma$ on Γ_1 .

Combining the three estimates in reverse order with (74) for $\sigma = m+1$ shows

$$\begin{aligned} \|\partial_n u\|_{H_k^{m+1/2}(\Gamma_1)} &= \|n \cdot (\nabla w)|_\Gamma\|_{H_k^{m+1/2}(\Gamma_1)} \lesssim \|(\nabla w)|_\Gamma\|_{H_k^{m+1/2}(\Gamma)} \\ &\lesssim \|\nabla w\|_{H_k^{m+1}(\Omega)} \lesssim k \|w\|_{H_k^{m+2}(\Omega)} \lesssim k \|u\|_{H_k^{m+2}(\Omega_2)}. \end{aligned}$$

Dividing by k proves the claim. $\square$

IRIA UNIV. LILLE AND LABORATOIRE PAUL PAINLEVÉ

Current address: 59655 Villeneuve-d'Ascq, France

Email address: `theophile.chaumont@inria.fr`

INSTITUTE FOR APPLIED AND NUMERICAL MATHEMATICS, KARLSRUHE INSTITUTE OF TECHNOLOGY, 76149 KARLSRUHE, GERMANY

Current address: Englerstr. 2, 76131 Karlsruhe

Email address: `julian.doerner@kit.edu`