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Wave phenomena
analysis and numerics

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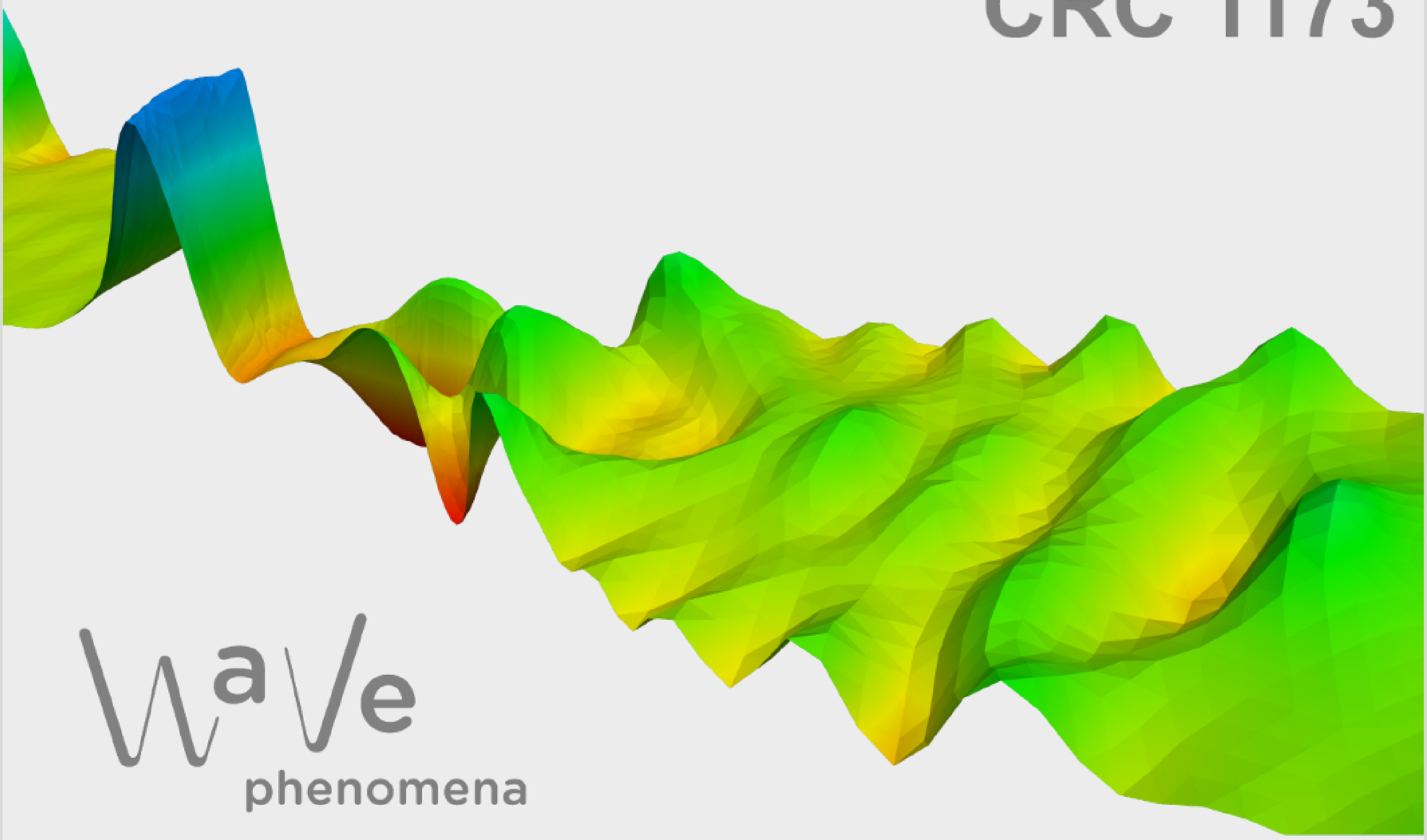

Wave phenomena

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SCATTERING BY A DOUBLE-PERIODIC LAYER IN $\mathbb{R}^3$

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ABSTRACT. We study the Helmholtz equation in a closed waveguide of the form $\Omega = \mathbb{R}^2 \times (0,h) \subset \mathbb{R}^3$ where the refractive index $n = n(x_1, x_2, x_3)$ is periodic with respect to x_1 and x_2. A functional analytic approach is used to prove the limiting absorption principle under an assumption on the set of propagative wave vectors. This assumption is formulated without the use of the band functions. Finally, we use the stationary phase method to obtain the asymptotic behavior of the limiting absorption solution as $|x|$ tends to infinity.

Keywords Helmholtz equation, closed waveguide, propagating modes, limiting absorption principle
MSC: 35J05

1. Formulations of the Main Results

1.1. **Introduction and the Floquet-Bloch Transform.** Let $n = n(x_1, x_2, x_3) \in L^\infty(\Omega)$ be $2\pi-$periodic with respect to x_1 and x_2 where $\Omega = \mathbb{R}^2 \times (0,h) \subset \mathbb{R}^3$ and let $g \in L^2(\Omega)$ have compact support. It is the aim to determine $u \in H^1_{loc}(\bar{\Omega})$ such that

$$\Delta u + \omega^2 n u \;=\; -g \text{ in } \Omega\,, \quad u = 0 \text{ on } \partial\Omega\,, \tag{1}$$

satisfying an appropriate radiation condition as $|x| \to \infty$. We call this the three-dimensional waveguide problem because $\Omega \subset \mathbb{R}^3$ although the refractive index $n = n(x)$ is periodic with respect to the two variables x_1 and x_2.

In this contribution we will derive the radiation condition by a limiting absorption principle, i.e. for every $\varepsilon > 0$ we determine $u_\varepsilon \in H^1(\Omega)$ with

$$\Delta u_\varepsilon + (\omega^2 + i\varepsilon) n u_\varepsilon \;=\; -g \text{ in } \Omega\,, \quad u_\varepsilon = 0 \text{ on } \partial\Omega\,, \tag{2}$$

and let ε tend to zero. It is well known that there exists a unique solution of (2) for every $\varepsilon > 0$, see Remark 1.1 below.

The general idea for proving the convergence is to transform (by the Floquet-Bloch transform) this equation into a family of quasi-periodic problems in a bounded cell (called fundamental domain or Wigner-Seitz cell by some authors), express the solution of the transformed equation by the eigenfunctions of the unperturbed quasiperiodic problem, apply the inverse Floquet-Bloch transform and let ε tend to zero.

For two-dimensional waveguides $\mathbb{R} \times (0,h) \subset \mathbb{R}^2$ where $n = n(x_1, x_2)$ is periodic with respect to x_1 this approach is well understood and leads to optimal results. We refer to the well-written article [1] and the references therein. This approach needs strong smoothness properties of the spectral parameters of the eigenvalue problem with respect to the Floquet parameter (which is also called quasi-momentum by some authors and will be denoted by α in the following) which are known to hold for $\alpha \in \mathbb{R}$.

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For three dimensional problems of the type considered in this manuscript, however, these strong smoothness properties do not hold. Let us formulate the quasi-periodic eigenvalue problem. We denote by $Q = (0, 2\pi) \times (0, 2\pi) \times (0, h)$ the fundamental domain and define the space of periodic functions by

$$H^1_{per,0}(Q) = \left\{ u \in H^1(Q) : u = 0 \text{ on } \partial\Omega \cap \overline{Q},\ u(\cdot, x_3) \text{ is } 2\pi\text{−periodic with respect to } x_1, x_2 \right\}.$$

For every $\alpha \in \mathbb{R}^2$ we determine $\mu_j = \mu_j(\alpha) \in \mathbb{R}$ (called band functions) and non-trivial $\phi_j = \phi_j(\cdot, \alpha) \in H^1_{per,0}(Q)$ with

$$-\frac{1}{n(x)}\,\Delta\big(e^{i\alpha\cdot\tilde{x}}\phi_j(x,\alpha)\big) \;=\; \mu_j(\alpha)\,\big(e^{i\alpha\cdot\tilde{x}}\phi_j(x,\alpha)\big)\,, \quad x \in Q\,, \tag{3}$$

where $\tilde{x} = (x_1, x_2)$, i.e.

$$\Delta\phi_j(x,\alpha) \;+\; 2i\alpha\cdot\tilde{\nabla}\phi_j(x,\alpha) \;+\; \big[\mu_j(\alpha)\,n(x) - |\alpha|^2\big]\,\phi_j(x,\alpha) \;=\; 0\,, \quad x \in Q\,, \tag{4}$$

where $\tilde{\nabla} = (\partial_{x_1}, \partial_{x_2})^\top$. We normalize the eigenfunctions such that $\int_Q n(x)\,\phi_j(x,\alpha)\,\overline{\phi_\ell(x,\alpha)}\,dx = \delta_{j,\ell}$. The spectral system $\{\mu_j(\alpha), \phi_j(\cdot,\alpha) : j \in \mathbb{N}\}$ exists for every $\alpha \in \mathbb{R}^2$ because the left hand side of (3) is a selfadjoint operator with compact resolvent. The difficulty for the three dimensional case compared to the two dimensional case relies in the missing analyticity of $\mu_j(\alpha)$ and $\phi_j(x,\alpha)$ with respect to $\alpha \in \mathbb{R}^2$. As mentioned, e.g., in the survey article [7], the functions are only piecewise analytic and globally Lipschitz continuous. This latter property has been used in [9] to derive a limiting absorption result for a general class of periodic problems. However, the convergence has to be understood in a very weak sense (distributional sense with respect to ω^2). We refer also to [8] for an application to a nonlinear Helmholtz equation. In [10] the multi-dimensional transformed problem (for $\alpha \in \mathbb{R}^n$) is reduced to a family of one-dimensional problems where known techniques can be applied.

Motivated by open waveguide problems where the above approach does not work in [3, 4, 5] we derived an alternative approach for treating the limiting absorption principle for two-dimensional closed and open waveguide problems. It avoids completely the use of the spectral problem (4) and formulates the conditions for convergence in terms of the flux of the modes. It is the intention of this paper to extend this approach to the double-periodic refractive index.

A basic difference between the two-dimensional and three-dimensional problem lies in the topology of the set $\mathcal{A}$ of propagative wave numbers/vectors (see Definition 1.2 below). While in the two dimensional case this set $\mathcal{A}$ is a discrete set in $\mathbb{R}$ it consists of one-dimensional arcs in $\mathbb{R}^2$ in the three dimensional case (under suitable assumptions, see Assumption 1.3 below), or at least of a set of measure zero, see [7], Theorem 5.20.

We recall that the two-dimensional (periodic) Floquet Bloch transform is defined as the extension of

$$(F\phi)(\tilde{x},\alpha) \;:=\; \sum_{j\in\mathbb{Z}^2} \phi(\tilde{x} + 2\pi j)\, e^{i(\tilde{x}+2\pi j)\cdot\alpha}\,, \quad \tilde{x}, \alpha \in \mathbb{R}^2\,,$$

to an operator from $L^2(\mathbb{R}^2)$ into $L^2\big((0,2\pi)^2, I\big)$ where $I = (-1/2, 1/2)^2 \subset \mathbb{R}^2$ denotes the Brillouin zone. Furthermore, it is known that F has an extension to an isomorphism from $H^1_0(\Omega)$ onto $L^2\big(I, H^1_{per,0}(Q)\big)$ where again $Q = (0,2\pi) \times (0,2\pi) \times (0,h) \subset \Omega$ denotes the fundamental domain. The inverse transform is then given as

$$(F^{-1}v)(x) \;=\; \int_I v(x,\alpha)\, e^{i\alpha\cdot\tilde{x}}\, d\alpha\,, \quad x \in \Omega\,, \tag{5}$$

where $v(\cdot,\alpha)$ is extented to a $2\pi-$periodic function with respect to x_1 and x_2.

Application of the Floquet-Bloch transform to the variational form of (2) yields

$$\int_Q \Big(\nabla u_\varepsilon(\alpha)\cdot\overline{\nabla\psi} - 2i\,\alpha\cdot\tilde\nabla u_\varepsilon(\alpha)\,\overline{\psi} + \big[|\alpha|^2-(\omega^2+i\varepsilon)n\big]\,u_\varepsilon(\alpha)\,\overline{\psi}\Big)\,dx = \int_Q (Fg)(\alpha)\,e^{-i\alpha\cdot\tilde x}\overline{\psi}\,dx$$

for all $\psi\in H^1_{per,0}(Q)$ and every Bloch parameter $\alpha\in\mathbb{R}^2$. Again, $\tilde\nabla u=(\partial_{x_1}u,\partial_{x_2}u)^\top$ and $\tilde x=(x_1,x_2)$. With the usual inner product $\langle\cdot,\cdot\rangle$ in $H^1(Q)$ we write this as

$$\begin{aligned}\langle u_\varepsilon(\alpha),\psi\rangle\ &-\int_Q \big(2i\,\alpha\cdot\tilde\nabla u(\alpha)\,\overline{\psi} + \big[\omega^2 n+1-|\alpha|^2\big]\,u(\alpha)\,\overline{\psi}\big)\,dx\ -\ i\varepsilon\int_Q n\,u_\varepsilon\overline{\psi}\mathrm{d}x\\ =\ &\int_Q (Fg)(x,\alpha)\,e^{-i\alpha\cdot\tilde x}\,\overline{\psi(x)}\,dx\quad\text{for all }\psi\in H^1_{per,0}(Q)\end{aligned}$$

or

$$\big[I-K(\alpha)-i\varepsilon A\big]u_\varepsilon(\alpha)\ =\ \tilde g(\alpha)\text{ in }H_{per,0}(Q)\tag{6}$$

with compact and selfadjoint operators $K(\alpha)$ and A and $\tilde g(\alpha)\in H^1_{per,0}(Q)$, defined as

$$\langle K(\alpha)u,\psi\rangle\ =\ \int_Q\Big(2i\,\alpha\cdot\tilde\nabla u\,\overline{\psi} + \big[\omega^2 n+1-|\alpha|^2\big]\,u\,\overline{\psi}\Big)\,dx\,,\quad u,\psi\in H^1_{per,0}(Q)\,,\tag{7}$$

$$\langle Au,\psi\rangle\ =\ \int_Q n\,u\,\overline{\psi}\,dx\,,\quad u,\psi\in H^1_{per,0}(Q)\,,\tag{8}$$

$$\langle\tilde g(\alpha),\psi\rangle\ =\ \int_Q (Fg)(x,\alpha)\,e^{-i\alpha\cdot\tilde x}\,\overline{\psi(x)}\,dx\,,\quad \psi\in H^1_{per,0}(Q)\,,\tag{9}$$

respectively.

Remark 1.1. *For every $\varepsilon>0$ and every $\alpha\in I$ there exists a unique solution $u_\varepsilon(\alpha)\in H^1_{per,0}(Q)$ of (6). This follows directly from the Fredholm property of $I-K(\alpha)-i\varepsilon A$ and the fact that $K(\alpha)$ is selfadjoint and A is positive, i.e. $\langle A\psi,\psi\rangle>0$ for all $\psi\neq 0$.*

1.2. **The Structure of the Set of Propagative Wave Vectors (Fermi Surface).**

Definition 1.2. *$\alpha\in\mathbb{R}^2$ is called propagative wave vector if $L(\alpha):=I-K(\alpha)$ is not invertible. We denote by $\tilde{\mathcal{A}}\subset\mathbb{R}^2$ the set of all propagative wave vectors (also called Fermi surface) and by $\mathcal{A}=\tilde{\mathcal{A}}\cap[-1/2,1/2]^2$ the restriction to $[-1/2,1/2]^2=[-1/2,1/2]\times[-1/2,1/2]$. The functions $\phi\in\mathcal{N}(L(\alpha))$ satisfy*

$$\Delta\big(\phi(x)e^{i\alpha\cdot\tilde x}\big)+\omega^2 n(x)\left(\phi(x)e^{i\alpha\cdot\tilde x}\right)=0\ \text{in }\Omega\,,\quad \phi(x)e^{i\alpha\cdot\tilde x}=0\ \text{on }\partial\Omega\,,\tag{10}$$

and are called (Floquet-)modes.[1]

Furthermore, for every pair (ϕ,ψ) of modes corresponding to $\alpha\in\tilde{\mathcal{A}}$ we define the flux vector $\mathcal{F}(\phi,\psi)\in\mathbb{C}^2$ by

$$\mathcal{F}(\phi,\psi)\ :=\ -2i\int_Q\tilde\nabla\big(\phi(x)\,e^{i\alpha\cdot\tilde x}\big)\,\overline{\left(\psi(x)\,e^{i\alpha\cdot\tilde x}\right)}\,dx\,.\tag{11}$$

[1]More precisely, the corresponding $\alpha-$quasi-periodic functions $\phi(x)\,e^{i\alpha\cdot\tilde x}$ are called modes.

We note that for every unit vector $q \in \mathbb{R}^2$ the form $q \cdot \mathcal{F}$ defines a hermitean sesqui-linear form on $H^1_{per,0}(Q) \times H^1_{per,0}(Q)$. Actually, $q \cdot \mathcal{F}(\phi, \phi)$ is the flux of ϕ in the direction q. We set $\mathcal{F}(\phi) = \mathcal{F}(\phi, \phi)$ for abbreviation.

We observe that α is a propagative wave vector if, and only if, $\mu_j(\alpha) = \omega^2$ for some $j \in \mathbb{N}$ where $\mu_j(\alpha)$ are the eigenvalues of (4).

It is easily seen that $\tilde{\mathcal{A}}$ is 1−periodic with respect to α_1 and α_2, i.e. $\alpha \in \tilde{\mathcal{A}}$ implies $\alpha + m \in \tilde{\mathcal{A}}$ for all $m \in \mathbb{Z}^2$. Furthermore, $\mathcal{A}$ is a compact subset of $[-1/2, 1/2] \times [-1/2, 1/2]$. Figure 1 shows $\mathcal{A}$ for an example of a constant refractive index with three different values of ω. In the appendix we describe a simple spectral method for the approximation of $\mathcal{A}$ for a special class of non-constant indices. Three examples are shown in Figure 2 at the end of the paper.

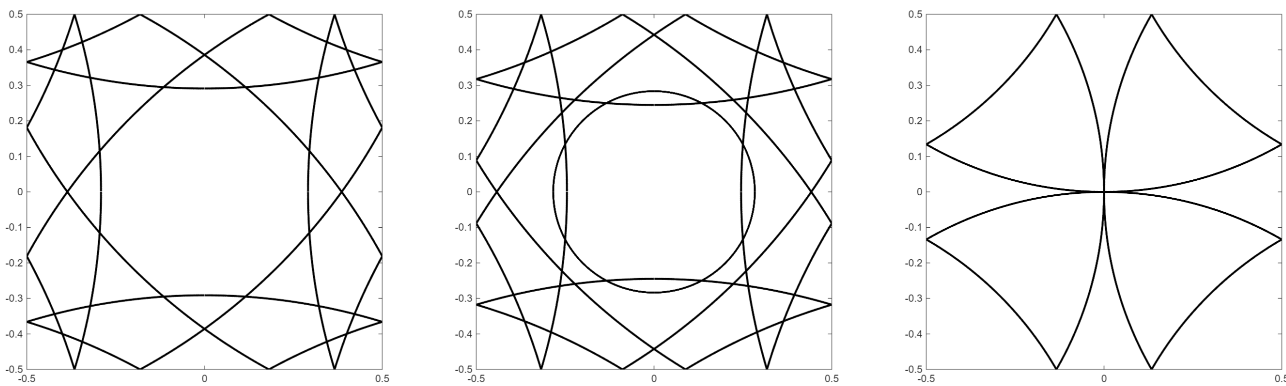

FIGURE 1. The set of propagative wave vectors for $h = \pi$ and $n \equiv 1$ and $\omega = 1.98$ (left), $\omega = 2.02$ (center), and $\omega = \sqrt{2}$ (right), respectively.

We make a strong assumption on the topology of the set $\tilde{\mathcal{A}}$. This assumption is formulated for every $\hat{\alpha} \in \tilde{\mathcal{A}}$.

Assumption 1.3. *For every propagative wave vector $\hat{\alpha} \in \tilde{\mathcal{A}}$ we assume that*

$$\mathcal{F}(\phi, \phi) \;=\; -2i \int_Q \tilde{\nabla}_x \big(\phi(x) e^{i\hat{\alpha}\cdot\tilde{x}}\big) \, \overline{\big(\phi(x) e^{i\hat{\alpha}\cdot\tilde{x}}\big)} \, dx \;\neq\; \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{12}$$

for all $\phi \in \mathcal{N}$ with $\phi \neq 0$ where $\mathcal{N} := \mathcal{N}(L(\hat{\alpha}))$ and $L(\hat{\alpha}) = I - K(\hat{\alpha})$. Furthermore, every propagative wave vector $\hat{\alpha} \in \tilde{\mathcal{A}}$ is of one of the following type (A) or type (B).

Type (A) $\dim \mathcal{N} = 1$ *or*

Type (B) $\dim \mathcal{N} = 2$, *and there exist two linearly independent unit vectors $q^{(m)} \in \mathbb{R}^2$, $m = 1, 2$, such that the hermitean sesqui-linear forms $(\phi, \psi) \mapsto q^{(m)} \cdot \mathcal{F}(\phi, \psi)$ are singular on $\mathcal{N} \times \mathcal{N}$ for $m = 1, 2$. Here we call a hermitean sesqui-linear form $a : \mathcal{N} \times \mathcal{N} \to \mathbb{C}$ singular if there exists a non-trivial $\phi \in \mathcal{N}$ such that $a(\phi, \psi) = 0$ for all $\psi \in \mathcal{N}$.*

We note that the cases $\omega = 2$ and $\omega = 2.01$ of Figure 1 seem to satisfy Assumption 1.3 with respect to the dimension. In the right example $\omega = \sqrt{2}$ it is violated because $\dim \mathcal{N}(L(0)) = 4$. In part (b) of the following remarks we relate the assumptions to the eigenvalues $\mu_j(\alpha)$ of (4).

Remarks 1.4. *(a) From the definition of $L(\hat\alpha) = I - K(\hat\alpha)$ we note that*

$$\begin{aligned}\big\langle \partial_{\alpha_m} L(\hat\alpha)\phi, \psi\big\rangle = &\int\limits_Q \big[2\hat\alpha_m\,\phi - 2i\,\partial_{x_m}\phi\big]\,\overline{\psi}\,dx \;=\; -2i\int\limits_Q \partial_{x_m}\big(\phi(x)\,e^{i\hat\alpha\cdot\tilde x}\big)\,\overline{\big(\psi(x)\,e^{i\hat\alpha\cdot\tilde x}\big)}\,dx \\ = &\;\mathcal{F}_m(\phi,\psi) \quad \text{for } m=1,2\,.\end{aligned} \tag{13}$$

Therefore, condition (12) *is equivalent to $\langle \nabla_\alpha L(\hat\alpha)\phi,\phi\rangle \neq \binom{0}{0}$ for all $\phi \in \mathcal{N}(L(\hat\alpha))$, $\phi \neq 0$, and the additional condition for $\hat\alpha$ of type (B) is equivalent to $q^{(m)} \cdot P\nabla L(\hat\alpha)|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ are singular for $m = 1,2$.*

(b) Let us assume – which is not always the case – that $\alpha \mapsto \mu_j(\alpha)$ and $\alpha \mapsto \phi_j(\cdot,\alpha)$ are differentiable for all j and α in a neighborhood of $\hat\alpha$ where (without loss of generality) $\mu_1(\hat\alpha) = \mu_2(\hat\alpha) = \omega^2$. Let us assume that $\mu_j(\hat\alpha) \neq \omega^2$ for all $j \geq 3$. Then $\dim\mathcal{N}(L(\hat\alpha)) = 2$. We differentiate equation (4) *(for $j=1$) with respect to α_m, $m = 1,2$, at $\hat\alpha$ and obtain*

$$\big(\Delta + 2i\hat\alpha\cdot\tilde\nabla + \big[\omega^2\,n - |\hat\alpha|^2\big]\big)\,\partial_{\alpha_m}\phi_1(\cdot,\hat\alpha) + 2i\big(\partial_{x_m} + i\hat\alpha_m\big)\phi_1(\cdot,\hat\alpha) \;=\; -\partial_{\alpha_m}\mu_1(\hat\alpha)\,n\,\phi_1(\cdot,\hat\alpha)\,.$$

Multiplication of this equation with $\phi_\ell(x,\hat\alpha)$ for $\ell \in \{1,2\}$, integration over Q, and application of Green's theorem yields

$$2i\int\limits_Q \big[\partial_{x_m}\phi_1(x,\hat\alpha) + i\hat\alpha_m\,\phi_1(x,\hat\alpha)\big]\,\overline{\phi_\ell(x,\hat\alpha)}\,dx \;=\; -\partial_{\alpha_m}\mu_1(\hat\alpha)\int\limits_Q n(x)\,\phi_1(x,\hat\alpha)\,\overline{\phi_\ell(x,\hat\alpha)}\,dx\,,$$

for $m,\ell = 1,2$, i.e.

$$\mathcal{F}\big(\phi_1(\cdot,\hat\alpha),\phi_\ell(\cdot,\hat\alpha)\big) \;=\; \nabla\mu_1(\hat\alpha)\,\delta_{1,\ell}\,,\quad \ell = 1,2\,.$$

For $\ell = 1$ we obtain $\nabla\mu_1(\hat\alpha) = \mathcal{F}\big(\phi_1(\cdot,\hat\alpha),\phi_1(\cdot,\hat\alpha)\big)$. The same holds for $\mu_2(\hat\alpha)$, and (12) *is equivalent to $\nabla\mu_j(\hat\alpha) \neq \binom{0}{0}$ for $j = 1,2$. This is a standard assumption for the limiting absorption principle to hold, see, e.g.,* [9].
Furthermore, with respect to the basis $\{\phi_1(\cdot,\hat\alpha),\phi_2(\cdot,\hat\alpha)\}$ the sesqui-linear form $(\phi,\psi) \mapsto q\cdot\mathcal{F}(\phi,\psi)$ is given by the matrix

$$\begin{bmatrix} q\cdot\mathcal{F}\big(\phi_1(\cdot,\hat\alpha),\phi_1(\cdot,\hat\alpha)\big) & 0 \\ 0 & q\cdot\mathcal{F}\big(\phi_2(\cdot,\hat\alpha)\phi_2(\cdot,\hat\alpha)\big)\end{bmatrix} \;=\; \begin{bmatrix} q\cdot\nabla\mu_1(\hat\alpha) & 0 \\ 0 & q\cdot\nabla\mu_2(\hat\alpha)\end{bmatrix}.$$

Therefore, the additional assumption for type (B) is satisfied if $\nabla\mu_1(\hat\alpha)$ and $\nabla\mu_2(\hat\alpha)$ are linearly independent.

For the formulation of the main result some preparations are needed. An open smooth arc Γ in $\mathbb{R}^2$ is a set of the form $\Gamma = \{\alpha = \gamma(s) : a < s < b\}$ where $\gamma \in C^\infty((a,b),\mathbb{R}^2) \cap C^1([a,b],\mathbb{R}^2)$ and $|\dot\gamma(s)| = 1$ for all s. We set $\overline{\Gamma} = \{\alpha = \gamma(s) : a \leq s \leq b\}$. If Γ is an open smooth arc, H a Hilbert space and $\phi(\alpha) \in H$ for $\alpha \in \Gamma$ then ϕ is said to depend smoothly on α if $s \mapsto \phi(\gamma(s))$ is in $C^\infty((a,b),H)$. For such an arc and any $\alpha \in \overline{\Gamma}$ let $\{\tau(\alpha),\nu(\alpha)\}$ denote an orthonormal system of tangential vector and normal vector, oriented in the mathematically positive way.

In the following theorem we consider the case of $\hat\alpha$ being of type (A).

Theorem 1.5. *Let $\hat\alpha \in \tilde{\mathcal{A}}$ with $\dim\mathcal{N}(L(\hat\alpha)) = 1$ and let assumption* (12) *hold. Then the exists a smooth open arc $\Gamma \subset \tilde{\mathcal{A}}$ with $\hat\alpha \in \Gamma$ and functions $\phi(\alpha) \in H^1_{per,0}(Q)$ which depend smoothly on $\alpha \in \Gamma$ such that $\mathcal{N}(L(\alpha)) = \operatorname{span}\{\phi(\alpha)\}$ for all $\alpha \in \Gamma$.*

Proof. Set $H := H^1_{per,0}(Q)$ and let $\mathcal{N} = \mathcal{N}(L(\hat\alpha)) = \operatorname{span}\{\hat\phi\}$ with $\|\hat\phi\| = 1$. Without loss of generality we assume that $\langle \partial_{\alpha_1} L(\hat\alpha)\hat\phi, \hat\phi\rangle \neq 0$. We define the mapping ψ from $H \times \mathbb{C} \times \mathbb{C}$ into $H \times \mathbb{C}$ by

$$\psi(\phi, \alpha_1, \alpha_2) \ := \ \begin{bmatrix} L(\alpha_1, \alpha_2)\phi \\ \langle \phi, \hat\phi\rangle - 1 \end{bmatrix}.$$

Then $\psi(\hat\phi, \hat\alpha) = \binom{0}{0}$, and the partial (Fréchet-)derivative of ψ at $(\hat\phi, \hat\alpha)$ with respect to (ϕ, α_1) is given by

$$\nabla_{(\phi,\alpha_1)}\psi(\hat\phi, \hat\alpha)(\phi, \alpha_1) \ = \ \begin{bmatrix} \alpha_1\, \partial_{\alpha_1} L(\hat\alpha)\hat\phi + L(\hat\alpha)\phi \\ \langle \phi, \hat\phi\rangle \end{bmatrix}$$

which is a linear mapping from $H \times \mathbb{C}$ into itself. We show that it is an isomorphism from $H \times \mathbb{C}$ onto itself. Indeed, let $\nabla_{(\phi,\alpha_1)}\psi(\hat\phi, \hat\alpha)(\phi, \alpha_1) = \binom{0}{0}$. Multiplication of the first equation by $\hat\phi$ and using the selfadjointness of $L(\hat\alpha)$ yields $\alpha_1 \langle \partial_{\alpha_1} L(\hat\alpha)\hat\phi, \hat\phi\rangle = 0$ and thus $\alpha_1 = 0$. Therefore, $\phi \in \mathcal{N}$. The second condition $\langle \phi, \hat\phi\rangle = 0$ yields $\phi = 0$ because $\mathcal{N} = \mathcal{N}(L(\hat\alpha))$ is one-dimensional.
Application of the implicit function theorem yields the existence of smooth functions $\eta = \eta(\alpha_2) \in \mathbb{C}$ and $\phi = \phi(\cdot, \alpha_2) \in H$ with $\psi(\phi(\cdot,\alpha_2), \eta(\alpha_2), \alpha_2) = 0$ for all $|\alpha_2 - \hat\alpha_2| < \delta$ and $\eta(\hat\alpha_2) = \hat\alpha_1$ and $\phi(\cdot, \hat\alpha_2) = \hat\phi$. Then $L(\gamma(\alpha_2))\phi(\cdot, \alpha_2) = 0$ and $\langle \phi(\cdot, \alpha_2), \hat\phi\rangle = 1$ where γ is given by $\gamma(s) = \binom{\eta(s)}{s}$. Also, $\phi(\cdot, \alpha_2)$ does not vanish for small $|\alpha_2 - \hat\alpha_2|$ because $\phi(\cdot, \hat\alpha_2) = \hat\phi$. □

Let now $\hat\alpha \in \tilde{\mathcal{A}}$ with $\dim \mathcal{N}(L(\hat\alpha)) = 2$, and let $P : H^1_{per,0}(Q) \to \mathcal{N}$ denote again the orthogonal projection onto $\mathcal{N} := \mathcal{N}(L(\hat\alpha))$. The following theorem is the analogue to Theorem 1.5.

Theorem 1.6. *Let $\hat\alpha \in \tilde{\mathcal{A}}$ with $\dim \mathcal{N}(L(\hat\alpha)) = 2$, and let Assumption 1.3 hold. Then there exist two smooth open arcs $\Gamma^{(1)}, \Gamma^{(2)} \subset \tilde{\mathcal{A}}$ and functions $\phi^{(m)}(\alpha) \in H^1_{per,0}(Q)$ which depend smoothly on $\alpha \in \Gamma^{(m)}$ for $m = 1, 2$ such that $\Gamma^{(1)} \cap \Gamma^{(2)} = \{\hat\alpha\}$ and $\mathcal{N}(L(\alpha)) = \operatorname{span}\{\phi^{(m)}(\alpha)\}$ for all $\alpha \in \Gamma^{(m)}$, $\alpha \neq \hat\alpha$, and $\mathcal{N}(L(\hat\alpha)) = \operatorname{span}\{\phi^{(1)}(\hat\alpha), \phi^{(2)}(\hat\alpha)\}$. Finally, the intersection is non-tangential (i.e. the tangential vectors of $\Gamma^{(1)}$ and $\Gamma^{(2)}$ at $\hat\alpha \in \Gamma^{(1)} \cap \Gamma^{(2)}$ are linearly independent).*
Without loss of generality, we can parametrize the curves $\Gamma^{(1)}$ and $\Gamma^{(2)}$ by $\alpha = \gamma^{(m)}(s)$, $s \in (-\delta, \delta)$ for $m = 1, 2$ such that $\gamma^{(m)}(0) = \hat\alpha_j$ and $|\dot\gamma^{(m)}(s)| = 1$ and $|\dot\gamma^{(m)}(s)| = 1$ for all s and $m = 1, 2$. Furthermore, we take $\delta > 0$ sufficiently small such that

$$\tau^{(1)}(\alpha) \cdot \nu^{(2)}(\beta) \neq 0\,, \quad \nu^{(1)}(\alpha) \cdot \tau^{(2)}(\beta) \neq 0 \quad \text{and} \quad \left| \int_Q n(x)\, \phi^{(1)}(x, \alpha)\, \overline{\phi^{(2)}(x, \beta)}\, dx \right| < 1 \tag{14}$$

for all $(\alpha, \beta) \in \Gamma^{(1)} \times \Gamma^{(2)}$ with $|\alpha - \hat\alpha| \leq \delta$ and $|\beta - \hat\alpha| \leq \delta$.

We note that the properties of (14) are possible because they hold for $\alpha = \beta = \hat\alpha$. Indeed, the first assertions follow directly from the linear independence of $\tau^{(1)}(\hat\alpha)$ and $\tau^{(2)}(\hat\alpha)$. The second assertion follows from the linear independence of $\phi^{(1)}(x, \hat\alpha)$ and $\phi^{(2)}(x, \hat\alpha)$ by the Cauchy-Schwarz inequality.

The proof of this theorem is more complicated and moved to Section 2.

We formulate the following corollary of Theorems 1.5 and 1.6.

Corollary 1.7. *Let Assumption 1.3 be satisfied. Then the set $\mathcal{A} := \tilde{\mathcal{A}} \cap [-1/2, 1/2]^2$ of propagative wave vectors in $[-1/2, 1/2] \times [-1/2, 1/2]$ can be covered by a finite number of smooth open arcs $\Gamma_\ell \subset \tilde{\mathcal{A}}$, $\ell \in \mathcal{L}$, (for some finite set $\mathcal{L} \subset \mathbb{N}$), i.e. $\mathcal{A} \subset \bigcup_{\ell\in\mathcal{L}} \Gamma_\ell$. There exist at most finitely many intersections $\mathcal{S} := \{\hat{\alpha}_j : j \in \mathcal{J}\}$ of these curves where $\mathcal{J} \subset \mathcal{L} \times \mathcal{L}$, and in each $\hat{\alpha}_j = \hat{\alpha}_{(j_1,j_2)}$ exactly two curves, which we denote by Γ_{j_1} and Γ_{j_2} for $(j_1, j_2) \in \mathcal{J}$, intersect. There are no other intersections. By $\mathcal{L}_{arc} := \mathcal{L} \setminus \{\ell \in \mathcal{L} : \exists j = (j_1, j_2) \in \mathcal{J}$ with $\ell = j_1$ or $\ell = j_2\}$ we indicate the arcs without intersections.*

Furthermore, there exist nontrivial modes $\phi_\ell(\alpha) \in \mathcal{N}(L(\alpha))$ for $\alpha \in \Gamma_\ell$ and $\ell \in \mathcal{L}$ which depend smoothly on $\alpha \in \Gamma_\ell$, are normalized by $\int_Q n(x)|\phi_\ell(x, \alpha)|^2 dx = 1$, and span $\mathcal{N}(L(\alpha))$ for $\alpha \in \Gamma_\ell \setminus \mathcal{S}$. Furthermore, $\mathcal{N}(L(\hat{\alpha}_j)) = \operatorname{span}\{\phi_{j_1}(\hat{\alpha}_j), \phi_{j_2}(\hat{\alpha}_j)\}$ if $\Gamma_{j_1} \cap \Gamma_{j_2} = \{\hat{\alpha}_j\}$ for $j = (j_1, j_2) \in \mathcal{J}$. We can assume that (14) holds for $\Gamma^{(m)}$ and $\phi^{(m)}$ replaced by Γ_{j_m} and ϕ_{j_m}, respectively, for $m = 1, 2$.

Finally, $\inf_{\alpha\in\Gamma_\ell} \big|\nu_\ell(\alpha) \cdot \mathcal{F}(\phi_\ell(\alpha))\big| > 0$ for all $\ell \in \mathcal{L}$. We choose the (mathematically positive orientated) orthonormal system $\{\tau_\ell(\alpha), \nu_\ell(\alpha)\}$ for $\ell \in \mathcal{L}$ such that $f_\ell(\alpha) := \nu_\ell(\alpha \cdot \mathcal{F}(\phi_\ell(\alpha)) > 0$ for all $\alpha \in \overline{\Gamma_\ell}$, $\ell \in \mathcal{L}$.

Proof. For every $\hat{\alpha} \in \tilde{\mathcal{A}}$ of type (A) or (B) the assumptions of Theorem 1.5 and Theorem 1.6 are satisfied by Assumption 1.3. Application of Theorem 1.5 yields that for every $\hat{\alpha} \in \tilde{\mathcal{A}}$ of type (A) there exists a smooth arc $\Gamma(\hat{\alpha})$ and non-trivial smooth functions $\phi(\cdot, \alpha) \in H^1_{per,0}(Q)$ with $\mathcal{N}(L(\alpha)) = \operatorname{span}\{\phi(\cdot, \alpha)\}$ for $\alpha \in \Gamma(\hat{\alpha})$.
Furthermore, by Theorem 1.6, for $\hat{\alpha} \in \mathcal{A}$ of type (B) there exist two smooth arcs $\Gamma^{(1)}(\hat{\alpha})$ and $\Gamma^{(2)}(\hat{\alpha})$ which intersect in $\hat{\alpha}$ non-tangentially, and there exist two non-trivial smooth functions $\phi^{(m)}(\cdot, \alpha) \in H^1_{per,0}(Q)$ with $\mathcal{N}(L(\alpha)) = \operatorname{span}\{\phi^{(m)}(\cdot, \alpha)\}$ for $\alpha \in \Gamma^{(m)}(\hat{\alpha}) \setminus \{\hat{\alpha}\}$, $m = 1, 2$, and $\mathcal{N}(L(\hat{\alpha})) = \operatorname{span}\{\phi^{(1)}(\cdot, \hat{\alpha}), \phi^{(2)}(\cdot, \hat{\alpha})\}$.

The set $\tilde{\mathcal{A}} \subset \mathbb{R}^2$ is a metric space as a subset of $\mathbb{R}^2$. Also, $\mathcal{A}$ is a compact subset of $\tilde{\mathcal{A}}$ which is covered by the open arcs , i.e.

$$\mathcal{A} \subset \bigcup_{\hat{\alpha} \text{ of type (A)}} \Gamma(\hat{\alpha}) \cup \bigcup_{\hat{\alpha} \text{ of type (B)}} \left[\Gamma^{(1)}(\hat{\alpha}) \cup \Gamma^{(2)}(\hat{\alpha})\right].$$

Since $\mathcal{A}$ is compact finitely many arcs suffice to cover $\mathcal{A}$. We number the arcs into the set $\{\Gamma_\ell : \ell \in \mathcal{L}\}$ for some finite set $\mathcal{L} \subset \mathbb{N}$ and denote the corresponding modes on Γ_ℓ by ϕ_ℓ. As described in the formulation of the corollary we denote the intersections $\Gamma_{j_1} \cap \Gamma_{j_2}$ by $\hat{\alpha}_j$ where $j = (j_1, j_2) \in \mathcal{J} \subset \mathcal{L} \times \mathcal{L}$ and collect them into the set $\mathcal{S}$ of intersections.

Finally, we show that $\nu_\ell(\alpha) \cdot \mathcal{F}(\phi_\ell(\alpha)) \neq 0$ for all $\alpha \in \overline{\Gamma_\ell}$ and $\ell \in \mathcal{L}$. Let the curve Γ_ℓ be parametrized by $\alpha = \gamma_\ell(s)$, $s \in (a_\ell, b_\ell)$. Differentiating $L(\gamma_\ell(s))\phi_\ell(s) = 0$ yields $\dot{\gamma}_\ell(s) \cdot \nabla L(\gamma_\ell(s))\phi_\ell(s) + L(\gamma_\ell(s))\dot{\phi}_\ell(s) = 0$ and thus $\big\langle \dot{\gamma}_\ell(s) \cdot \nabla L(\gamma_\ell(s))\phi_\ell(s), \phi_\ell(s)\big\rangle = 0$ for all $s \in [a_\ell, b_\ell]$. If also $\big\langle \nu_\ell(\hat{s}) \cdot \nabla L(\gamma_\ell(\hat{s}))\phi_\ell(\hat{s}), \phi_\ell(\hat{s})\big\rangle = 0$ for some $\hat{s} \in [a_\ell, b_\ell]$ then $\big\langle \nabla L(\gamma_\ell(\hat{s}))\phi_\ell(\hat{s}), \phi_\ell(\hat{s})\big\rangle = \binom{0}{0}$, a contradiction to (12) (by part (a) of Remarks 1.4). ☐

At the end of the previous proof we have shown the following corollary.

Corollary 1.8. *Let the assumptions of the previous corollary hold.*
Then $\big\langle \dot{\gamma}_\ell(s) \cdot \nabla L(\gamma_\ell(s))\phi_\ell(s), \phi_\ell(s)\big\rangle = 0$ for all $s \in [a_\ell, b_\ell]$ and $\ell \in \mathcal{L}$, i.e. $\tau_\ell(\alpha) \cdot \mathcal{F}(\phi_\ell(\alpha)) = 0$ for all $\alpha \in \Gamma_\ell$ and $\ell \in \mathcal{L}$.

1.3. **The Limiting Absorption Principle.** As mentioned already in Corollary 1.7 we decompose the curves $\{\Gamma_\ell : \ell \in \mathcal{L}\}$ into two classes. Curves which intersect at $\hat{\alpha}_j$ for $j = (j_1, j_2) \in \mathcal{J}$ are denoted by Γ_{j_1} and Γ_{j_2}. The remaining curves (which contain no intersections) are denoted by Γ_ℓ, $\ell \in \mathcal{L}_{arc}$.

Under the assumptions of Corollary 1.7 we define a neighborhood of $\mathcal{A}$ in the following way. We define the neighborhoods

$$U_\ell \;:=\; \big\{\alpha = \gamma_\ell(t_1) + t_2\nu_\ell(t_1) : t_1 \in (a_\ell, b_\ell),\ \gamma_\ell(t_1) \notin \mathcal{S},\ |t_2| < \delta\big\}\,,\ \ell \in \mathcal{L}_{arc}\,, \tag{15}$$

$$V_j \;:=\; \big\{\alpha = \gamma_{j_1}(t_1) + \gamma_{j_2}(t_2) - \hat{\alpha}_j : |t_1| < \delta,\ |t_2| < \delta\big\}\,,\ j = (j_1, j_2) \in \mathcal{J}\,, \tag{16}$$

of Γ_ℓ and $\hat{\alpha}_j$, respectively, With $U_0 := (-1/2 - \delta, 1/2 + \delta)^2 \setminus \mathcal{A}$ we have a covering of the compact set $I = [-1/2, 1/2]^2$ by the open sets U_ℓ ($\ell \in \mathcal{L}_{arc}$), and V_j ($j \in \mathcal{J}$). Let ψ_ℓ ($\ell \in \mathcal{L}_{arc}$), and ψ_j ($j \in \mathcal{J}$) be a corresponding partition of unity, i.e. $\operatorname{supp}\psi_\ell \subset U_\ell$ and $\operatorname{supp}\psi_j \subset V_j$ and $\sum_{\ell\in\mathcal{L}_{arc}} \psi_\ell(\alpha) + \sum_{j\in\mathcal{J}} \psi_j(\alpha) = 1$ for all $\alpha \in I$.[2] The inverse Floquet-Bloch transform is then decomposed as

$$\begin{aligned} u_\varepsilon(x) \;&=\; \int_I u_\varepsilon(x,\alpha)\, e^{i\alpha\cdot\tilde{x}}\, d\alpha \;=\; \int_{U_0} \psi_0(\alpha)\, u_\varepsilon(x,\alpha)\, e^{i\alpha\cdot\tilde{x}}\, d\alpha \\ &\quad + \sum_{\ell\in\mathcal{L}_{arc}} \int_{U_\ell} \psi_\ell(\alpha)\, u_\varepsilon(x,\alpha)\, e^{i\alpha\cdot\tilde{x}}\, d\alpha \;+\; \sum_{j\in\mathcal{J}} \int_{V_j} \psi_j(\alpha)\, u_\varepsilon(x,\alpha)\, e^{i\alpha\cdot\tilde{x}}\, d\alpha\,. \end{aligned} \tag{17}$$

We introduce the following functions which will play an essential role.

$$\Psi^{(\pm)}(s) \;:=\; \frac{1}{2} \;\pm\; \frac{1}{\pi}\int_0^s \frac{\sin\tau}{\tau}\, d\tau\,, \quad s \in \mathbb{R}\,. \tag{18}$$

We note that $|\Psi^{(\pm)}(s) - 1| \le \frac{c}{|s|+1}$ for $\pm s \ge 0$ and $|\Psi^{(\pm)}(s)| \le \frac{c}{|s|+1}$ for $\pm s \le 0$.

Theorem 1.9. *Let Assumption 1.3 be satisfied. We recall the modes $\phi_\ell(\cdot,\alpha)$ for $\alpha \in \Gamma_\ell$, $\ell \in \mathcal{L}$, and define the fluxes $f_\ell(\alpha)$ and coefficients $g_\ell(\alpha)$ corresponding to the arcs Γ_ℓ by*

$$f_\ell(\alpha) \;:=\; \nu_\ell(\alpha)\cdot\mathcal{F}(\phi_\ell(\alpha)) \quad \text{for } \alpha \in \Gamma_\ell\,,\ \ell \in \mathcal{L}\,, \tag{19}$$

$$g_\ell(\alpha) \;:=\; \int_\Omega g(y)\, \overline{e^{i\alpha\cdot\tilde{y}}\, \phi_\ell(y,\alpha)}\, dy \quad \text{for } \alpha \in \Gamma_\ell\,,\ \ell \in \mathcal{L}\,, \tag{20}$$

respectively, and signs $\sigma_j^{(1)}, \sigma_j^{(2)} \in \{+,-\}$ by

$$\begin{aligned} \sigma_j^{(1)} \;&:=\; \operatorname{sign}\big[\tau_{j_2}(\hat{\alpha}_j)\cdot\nu_{j_1}(\alpha)\big]\,,\ \alpha \in \Gamma_{j_1}\,,\ |\alpha - \hat{\alpha}_j| < \delta\,, \quad j \in \mathcal{J}\,, \\ \sigma_j^{(2)} \;&:=\; \operatorname{sign}\big[\tau_{j_1}(\hat{\alpha}_j)\cdot\nu_{j_2}(\alpha)\big]\,,\ \alpha \in \Gamma_{j_2}\,, |\alpha - \hat{\alpha}_j| < \delta\,, \quad j \in \mathcal{J}\,. \end{aligned} \tag{21}$$

Note that $\sigma_j^{(m)}$ does not depend on α because of (14). *Furthermore, $\sigma_j^{(1)} = -\sigma_j^{(2)}$.*

Then u_ε converges locally to some $u_0 \in H^1_{loc}(\Omega)$, i.e. $u_\varepsilon|_{\Omega_R} \to u_0|_{\Omega_R}$ in $H^1(\Omega_R)$ for all $R > 0$ where $\Omega_R = \{x \in \Omega : |\tilde{x}| < R\}$. The function u_0 is given by $u_0 = u_{arc} + u_{sec} + v$ where $v \in H^1(\Omega)$

[2] Note that we used the index $\ell \in \{1, \ldots, L\}$ for the functions ψ_ℓ with support in U_ℓ and the index $j = (j_1, j_2) \in \mathcal{J}$ for the functions $\psi_j = \psi_{(j_1,j_2)}$ with support in V_j.

and

$$\begin{aligned} u_{arc}(x) &= 2\pi i \sum_{\ell\in\mathcal{L}_{arc}} \int\limits_{\Gamma_\ell} \frac{\psi_\ell(\alpha)}{f_\ell(\alpha)}\, g_\ell(\alpha)\, \Psi^{(+)}\big(\nu_\ell(\alpha)\cdot\tilde{x}\big)\, \phi_\ell(x,\alpha)\, e^{i\alpha\cdot\tilde{x}}\, ds(\alpha)\,, \\ u_{sec}(x) &= 2\pi i \sum_{j\in\mathcal{J}} \Bigg[\Psi^{(\sigma_j^{(1)})}\big(\tau_{j_2}(\hat{\alpha}_j)\cdot\tilde{x}\big) \int\limits_{\Gamma_{j_1}} \frac{\psi_j(\alpha)}{f_{j_1}(\alpha)}\, g_{j_1}(\alpha)\, \phi_{j_1}(x,\alpha)\, e^{i\alpha\cdot\tilde{x}} \mathrm{d}s(\alpha) \\ &\qquad + \Psi^{(\sigma_j^{(2)})}\big(\tau_{j_1}(\hat{\alpha}_j)\cdot\tilde{x}\big) \int\limits_{\Gamma_{j_2}} \frac{\psi_j(\alpha)}{f_{j_2}(\alpha)}\, g_{j_2}(\alpha)\, \phi_{j_2}(x,\alpha)\, e^{i\alpha\cdot\tilde{x}}\, ds(\alpha) \Bigg]\,. \end{aligned}$$

We note that $f_\ell(\alpha) > 0$ and $f_{j_m}(\alpha) > 0$ by Corollary 1.7.

We will prove this theorem in Section 4 but look at an example first.

Example 1.10. *Let $n = n(x_3)$ be constant with respect to x_1 and x_2. Let $\{\mu_p^2 : p = 1,\ldots,p_0\}$ be the positive eigenvalues of the problem $\varphi''(x_3) + \omega^2 n(x_3)\,\varphi(x_3) = \mu^2\,\varphi(x_3)$ for $0 < x_3 < h$ and $\varphi(0) = \varphi(h) = 0$. (There are at most finitely many of them.) Let $\{\varphi_p\}$ be the corresponding eigenfunctions, normalized as $\int_0^h n(x_3)\,\varphi_p(x_3)\,\varphi_q(x_3)\,dx_3 = \delta_{p,q}$. (If $h = \pi$ and $n \equiv 1$ then $\mu_p = \sqrt{\omega^2 - p^2}$ and $\varphi_p(x_3) = \sqrt{\frac{2}{\pi}}\sin(px_3)$.) Then the (quasi-periodic) modes are given by*

$$\phi(x)\, e^{i\alpha\cdot\tilde{x}} = e^{i\beta\cdot\tilde{x}}\,\varphi_p(x_3)\,, \quad x\in\Omega\,,$$

for $p \in \{1,\ldots,p_0\}$ and $\beta \in \mathbb{R}^2$ with $|\beta| = \mu_p$ and $\alpha = \alpha(\beta) \in I$ such that $\alpha - \beta \in \mathbb{Z}^2$. Therefore, the propagative wave vectors, restricted to $I = [-1/2, 1/2]^2$ are given by

$$\mathcal{A} = \big\{\beta - m \in \mathbb{R}^2 : |\beta| = \mu_p \text{ for some } p \text{ and } m \in \mathbb{Z}^2\big\} \cap I\,.$$

Therefore, $\mathcal{A}$ consists of finitely many arcs which are subsets of circles. In Figure 1 this set $\mathcal{A}$ is shown for $h = \pi$, $n \equiv 1$ and $\omega = 2$, $\omega = 2.01$, and $\omega = \sqrt{2}$, respectively.
If $\hat{\alpha} \in \mathcal{A}$ is of type (A) then the arc $\Gamma_j \subset I$ containing $\hat{\alpha}$ is parametrized by $\gamma(s) = \mu_p\binom{\cos(s/\mu_p)}{\sin(s/\mu_p)} + m$ for $s \in (a,b)$ where $0 \le a < b < 2\pi\,\mu_p$ and $m \in \mathbb{Z}^2$ such that $\gamma(s) \in I$ for $s \in [a,b]$. Assumption (12) *is satisfied because $\mathcal{F}(\phi,\phi) = 2(2\pi)^2\,\beta$.*
If $\hat{\alpha} \in \mathcal{A}$ is of type (B) then

$$\hat{\alpha} = \mu_p\begin{pmatrix}\cos(\hat{s}/\mu_p)\\ \sin(\hat{s}/\mu_p)\end{pmatrix} + m = \mu_q\begin{pmatrix}\cos(\hat{t}/\mu_q)\\ \sin(\hat{t}/\mu_q)\end{pmatrix} + n$$

with $p, q \in \{1,\ldots,p_0\}$ and $m, n \in \mathbb{Z}^2$ and some $\hat{s} \in (0, 2\pi\mu_p)$ and $\hat{t} \in (0, 2\pi\mu_q)$. The modes at $\hat{\alpha}$ are given by

$$\phi^{(1)}(x)\, e^{i\hat{\alpha}\cdot\tilde{x}} = e^{i\beta^{(1)}\cdot\tilde{x}}\,\varphi_p(x_3)\,, \quad \phi^{(2)}(x)\, e^{i\hat{\alpha}\cdot\tilde{x}} = e^{i\beta^{(2)}\cdot\tilde{x}}\,\varphi_q(x_3)\,, \quad x\in\Omega\,,$$

where $\beta^{(1)} = \mu_p\binom{\cos(\hat{s}/\mu_p)}{\sin(\hat{s}/\mu_p)}$ *and* $\beta^{(2)} = \mu_q\binom{\cos(\hat{t}/\mu_p)}{\sin(\hat{t}/\mu_p)}$. *The flux-matrix is computed as* $\mathcal{F}(\phi^{(\ell)}) = 2(2\pi)^2\,\beta^{(\ell)}$ *for* $\ell = 1, 2$ *and*

$$\begin{aligned}\mathcal{F}(\phi^{(1)}, \phi^{(2)}) &= 2\,\beta^{(1)} \int_0^{2\pi}\int_0^{2\pi} e^{i(\beta^{(1)}-\beta^{(2)})\cdot\tilde{x}}\,dx_1\,dx_2 \int_0^h \varphi_p(x_3)\,\varphi_q(x_3)\,dx_3 \\ &= 2\,\beta^{(1)} \int_0^{2\pi}\int_0^{2\pi} e^{i(m-n)\cdot\tilde{x}}\,dx_1\,dx_2\,\delta_{p,q} \;=\; 0\end{aligned}$$

since $m \neq n$. *The matrix with entries* $v \cdot \mathcal{F}(\phi^{(\ell)}, \phi^{(j)})$ *is thus given as* $2(2\pi)^2 \begin{bmatrix} v\cdot\beta^{(1)} & 0 \\ 0 & v\cdot\beta^{(2)} \end{bmatrix}$. *We choose* $q^{(m)}$ *to be orthogonal to* $\beta^{(m)}$. *Then also the additional assumption for* $\hat{\alpha}$ *of type (B) of Assumption 1.3 is satisfied.*

1.4. **The Asymptotic Behavior as** $|x| \to \infty$**.** In this subsection we make the additional assumption that the curvature of the arcs Γ_ℓ for $\ell \in \mathcal{L}$ do not vanish, i.e.

Assumption 1.11. *We assume that* $\kappa_\ell(\alpha) \neq 0$ *for all* $\alpha \in \Gamma_\ell$ *and* $\ell \in \mathcal{L}$, *i.e. if* $\alpha = \gamma_\ell(s)$ *is the parametrization of* Γ_ℓ *by arc length then* $\kappa_\ell(s) = \kappa_\ell(\gamma_\ell(s)) := \ddot{\gamma}_\ell(s) \neq 0$ *for all* s *and* $\ell \in \mathcal{L}$.

We will determine the asymptotic behavior of the limiting absorption solution u_0 along fixed directions $\hat{\theta} \in \mathbb{R}^2$ with $|\hat{\theta}| = 1$. Therefore, we consider $x = (r\hat{\theta}_1, r\hat{\theta}_2, x_3)$ with $x_3 \in (0, h)$ and $r > 0$. Then $\tilde{x} = r\hat{\theta}$ and $x = (r\hat{\theta}, x_3)$.

We consider the integrals of Theorem 1.9 which are of the form (with the form x of above)

$$(22) \quad \int_{\Gamma_\ell} \tilde{g}(\alpha)\,\Psi^{(+)}\big(r\,\nu_\ell(\alpha)\cdot\hat{\theta}\big)\,\phi_\ell(r\hat{\theta}, x_3, \alpha)\,e^{ir\alpha\cdot\hat{\theta}}\,ds(\alpha) \quad \text{and} \int_{\Gamma_{j_m}} \tilde{g}(\alpha)\,\phi_{j_m}(r\hat{\theta}, x_3, \alpha)\,e^{ir\alpha\cdot\hat{\theta}}\,ds(\alpha)$$

with $\tilde{g}(\alpha) = \frac{\psi_\ell(\alpha)}{f_\ell(\alpha)}\,g_\ell(\alpha)$ or $\tilde{g}(\alpha) = \frac{\psi_j(\alpha)}{f_{j_m}(\alpha)}\,g_{j_m}(\alpha)$ for $m = 1, 2$, respectively. For the application of the stationary phase method the sets

$$(23) \qquad \mathcal{A}_\pm(\hat{\theta}) \;:=\; \{\alpha \in \mathcal{A} : \hat{\theta} = \pm\nu(\alpha)\}$$

will play the essential role. Note that for points $\hat{\alpha}_j \in \mathcal{S}$ of intersections only one of the normals $\nu_{j_1}(\hat{\alpha}_j)$ or $\nu_{j_2}(\hat{\alpha}_j)$ can coincide with $\pm\hat{\theta}$. Furthermore, we note that $\ddot{\gamma}_\ell(\hat{s})\cdot\hat{\theta} \neq 0$ if $\dot{\gamma}_\ell(\hat{s})\cdot\hat{\theta} = 0$ because $\ddot{\gamma}_\ell(\hat{s}) \neq 0$ by Assumption 1.11 and $\{\ddot{\gamma}_\ell(\hat{s}), \dot{\gamma}_\ell(\hat{s})\}$ are orthogonal to each other. From this fact and a compactness argument we conclude also that the set $\mathcal{A}_+(\hat{\theta})\cup\mathcal{A}_-(\hat{\theta})$ of stationary points is finite.

Theorem 1.12. *Let Assumptions 1.3 and 1.11 hold. Then*

$$(24) \quad u_0(r\hat{\theta}, x_3) = v(r\hat{\theta}, x_3) + 2\pi i\sqrt{\frac{2\pi}{r}} \sum_{\hat{\alpha}\in\mathcal{A}_+(\hat{\theta})} \frac{g(\hat{\alpha})}{f(\hat{\alpha})}\,\big[\phi(r\hat{\theta}, x_3, \hat{\alpha})\,e^{ir\hat{\alpha}\cdot\hat{\theta}}\big]e^{\sigma i\pi/4}\sqrt{\frac{1}{|\kappa(\hat{\alpha})|}} + \mathcal{O}\left(\frac{1}{r}\right)$$

as $r \to \infty$ *uniformly with respect to* $x_3 \in (0, h)$. *Here,* $\sigma = \sigma(\hat{\alpha}, \hat{\theta}) = \operatorname{sign}[\kappa(\hat{\alpha})\cdot\hat{\theta}]$ *and* $v \in H^1(\Omega)$. *Recall that* $f(\hat{\alpha})$ *and* $g(\hat{\alpha})$ *and* $\kappa(\hat{\alpha})$ *are given by* (19) *and* (20) *and* $\kappa(\hat{\alpha}) = \kappa_\ell(\hat{\alpha})$, *respectively, if* $\hat{\alpha} \in \Gamma_\ell$.

2. Proof of Theorem 1.6

First we recall from (12) and part (a) of Remarks 1.4 that $\langle \nabla_\alpha L(\hat\alpha)\phi, \phi\rangle \neq \binom{0}{0}$ for all $\phi \in \mathcal{N}$ with $\phi \neq 0$.

Let $\phi_m \in H^1_{per,0}(Q)$, $\phi_m \neq 0$, with $q^{(m)} \cdot P\nabla L(\hat\alpha)\phi_m = 0$ for $m = 1, 2$. We note that $\{\phi_1, \phi_2\}$ are linearly independent. Indeed, otherwise we would have $q^{(m)} \cdot P\nabla L(\hat\alpha)\phi_1 = 0$ for $m = 1, 2$ and thus $q^{(m)} \cdot \langle \nabla L(\hat\alpha)\phi_1, \phi_1\rangle = 0$ for $m = 1, 2$, i.e. $\langle \nabla L(\hat\alpha)\phi_1, \phi_1\rangle = \binom{0}{0}$ because $\{q^{(1)}, q^{(2)}\}$ span $\mathbb{R}^2$. This would contradict the assumption.

The further proof consists of three steps.

(i) Let $C \subset \mathbb{C}$ be a circle in the resolvent set of $K(\hat\alpha)$ with center 1 and radius so small such that 1 is the only eigenvalue of $K(\hat\alpha)$ inside of C. Then the orthogonal projection P onto the two-dimensional eigenspace $\mathcal{N}$ is given by

$$P \;=\; \frac{1}{2\pi i}\int_C [\lambda I - K(\hat\alpha)]^{-1} d\lambda\,.$$

Define

$$\hat P(\alpha) \;=\; \frac{1}{2\pi i}\int_C [\lambda I - K(\alpha)]^{-1} d\lambda$$

for $|\alpha - \hat\alpha| \leq \delta$ and $\delta > 0$ so small such that C is also in the resolvent set of $K(\alpha)$. Then $\hat P(\alpha)$ is the projection onto

$$\hat{\mathcal{N}}(\alpha) \;:=\; \bigoplus_{\lambda \in \Lambda(\alpha)} \mathcal{N}\big(\lambda I - K(\alpha)\big) \tag{25}$$

where $\Lambda(\alpha)$ is the set of the two eigenvalues of $K(\alpha)$ inside of C. By Kato's monograph [2], Chapter II, §4 (see also [6]), there exist unitary transforms[3] $U(\alpha)$ which depend smoothly on α such that $U(\hat\alpha) = I$ and $\hat P(\alpha) = U(\alpha)PU(\alpha)^*$. We set $\tilde\phi^{(m)}(\alpha) := U(\alpha)\phi_m$ for $m = 1, 2$. Then $\tilde\phi^{(m)}(\alpha) = U(\alpha)P\phi_m = \hat P(\alpha)U(\alpha)\phi_m \in \hat{\mathcal{N}}(\alpha)$, $m = 1, 2$, form a basis of $\hat{\mathcal{N}}(\alpha)$.

(ii) With respect to this basis $\{\tilde\phi^{(1)}(\alpha), \tilde\phi^{(2)}(\alpha)\}$ the operator $\hat P(\alpha)L(\alpha)|_{\hat{\mathcal{N}}(\alpha)}$ is given by the matrix $\hat L(\alpha) \in \mathbb{C}^{2\times 2}$ where $\hat L(\alpha)_{j,\ell} = \big\langle L(\alpha)\tilde\phi^{(j)}(\alpha), \tilde\phi^{(\ell)}(\alpha)\big\rangle$ depends smoothly on α. We note that $\hat L(\hat\alpha) = 0$. We search for a curve $\alpha = \gamma(s)$ with $\gamma(0) = \hat\alpha$ and $\det \hat L(\gamma(s)) = 0$. From the definition of $\hat L(\alpha)$ we compute

$$\begin{aligned}\nabla \hat L_{j,\ell}(\hat\alpha) \;&=\; \big\langle \nabla L(\hat\alpha)\tilde\phi^{(j)}(\hat\alpha), \tilde\phi^{(\ell)}(\hat\alpha)\big\rangle + \big\langle L(\hat\alpha)\nabla\tilde\phi^{(j)}(\hat\alpha), \tilde\phi^{(\ell)}(\hat\alpha)\big\rangle + \big\langle L(\hat\alpha)\tilde\phi^{(j)}(\hat\alpha), \nabla\tilde\phi^{(\ell)}(\hat\alpha)\big\rangle \\ &=\; \big\langle \nabla L(\hat\alpha)\phi_j, \phi_\ell\big\rangle\end{aligned}$$

because $\big\langle L(\hat\alpha)\nabla\tilde\phi^{(j)}(\hat\alpha), \tilde\phi^{(\ell)}(\hat\alpha)\big\rangle + \big\langle L(\hat\alpha)\tilde\phi^{(j)}(\hat\alpha), \nabla\tilde\phi^{(\ell)}(\hat\alpha)\big\rangle = \big\langle \nabla\tilde\phi^{(j)}(\hat\alpha), L(\hat\alpha)\phi_\ell\big\rangle + \big\langle L(\hat\alpha)\phi_j, \nabla\tilde\phi^{(\ell)}(\hat\alpha)\big\rangle = 0$. From the properties of $q^{(1)}$ and $q^{(2)}$ we observe that $q^{(1)} \cdot \nabla\hat L_{1,\ell}(\hat\alpha) = \big\langle q^{(1)} \cdot \nabla L(\hat\alpha)\phi_1, \phi_\ell\big\rangle = 0$ for $\ell = 1, 2$, and $q^{(2)} \cdot \nabla\hat L_{2,\ell}(\hat\alpha) = \big\langle q^{(2)} \cdot \nabla L(\hat\alpha)\phi_2, \phi_\ell\big\rangle = 0$ for $\ell = 1, 2$. Furthermore, $q^{(2)} \cdot \nabla\hat L_{1,1}(\hat\alpha) = \big\langle q^{(2)} \cdot \nabla L(\hat\alpha)\phi_1, \phi_1\big\rangle \neq 0$ because otherwise $\big\langle q^{(2)} \cdot \nabla L(\hat\alpha)\phi_1, \phi_1\big\rangle = 0$ and $\big\langle q^{(1)} \cdot \nabla L(\hat\alpha)\phi_1, \phi_1\big\rangle = 0$, i.e. $\big\langle \nabla L(\hat\alpha)\phi_1, \phi_1\big\rangle = \binom{0}{0}$, a contradiction. Analogously, $q^{(1)} \cdot \nabla\hat L_{2,2}(\hat\alpha) = \big\langle q^{(1)} \cdot \nabla L(\hat\alpha)\phi_2, \phi_2\big\rangle \neq 0$.

[3] The assertions in [2, 6] are for $\alpha \in \mathbb{C}$. However, as shown in [3], the assertion holds also for $\alpha \in \mathbb{C}^2$.

We define the matrices $M(t_1,t_2)\in\mathbb{C}^{2\times 2}$ by

$$M(t_1,t_2) \;:=\; \hat L(\hat\alpha + t_1\, q^{(1)} + t_2\, q^{(2)})\,, \quad |t_j|<\delta_0\,. \tag{26}$$

Then $M(0)=0$ and $\partial_{t_1}M_{1,\ell}(0)=0$ and $\partial_{t_2}M_{2,\ell}(0)=0$ for $\ell=1,2$ and $\partial_{t_2}M_{1,1}(0)\neq 0$ and $\partial_{t_1}M_{2,2}(0)\neq 0$. Therefore, all assumptions of Lemma B.2 of the appendix are satisfied, and there exist two curves $t=\tilde\gamma^{(m)}(s)$ and vectors $z^{(m)}(s)\in\mathbb{C}^2$, $m=1,2$, depending smoothly on $|s|<\delta$, with $\tilde\gamma^{(m)}(0)=0$ and $\frac{d}{ds}\tilde\gamma^{(1)}(0)=\binom{1}{0}=z^{(1)}(0)$ and $\frac{d}{ds}\tilde\gamma^{(2)}(0)=\binom{0}{1}=z^{(2)}(0)$ and $M(\tilde\gamma^{(m)}(s))z^{(m)}(s)=0$. We set $\gamma^{(m)}(s)=\hat\alpha+\tilde\gamma_1^{(m)}(s)\,q^{(1)}+\tilde\gamma_2^{(m)}(s)\,q^{(2)}$. Then $\gamma^{(m)}(0)=\hat\alpha$ and $\frac{d}{ds}\gamma^{(m)}(0)=q^{(m)}$ and $\hat L(\gamma^{(m)}(s))z^{(m)}(s)=0$ for $|s|<\delta$.

(iii) Now we set $\phi^{(m)}(s):=\sum_{\ell=1}^2 z_\ell^{(m)}(s)\,\tilde\phi^{(\ell)}(\gamma^{(m)}(s))$. Then $\phi^{(m)}(s)\in\hat{\mathcal{N}}(\gamma^{(m)}(s))$ and

$$\begin{aligned}\big\langle L(\gamma^{(m)}(s))\phi^{(m)}(s),\tilde\phi^{(j)}(\gamma^{(m)}(s))\big\rangle \;&=\; \sum_{\ell=1}^2 z_\ell^{(m)}(s)\,\big\langle L(\gamma^{(m)}(s))\tilde\phi^{(\ell)}(\gamma^{(m)}(s)),\tilde\phi^{(j)}(\gamma^{(m)}(s))\big\rangle\\ &=\; \sum_{\ell=1}^2 z_\ell^{(m)}(s)\,\hat L_{\ell,j}(\gamma^{(m)}(s)) \;=\; 0\end{aligned}$$

for $j=1,2$. Therefore, $\hat P(\gamma^{(m)}(s))L(\gamma^{(m)}(s))\phi^{(m)}(s)=0$ and thus $L(\gamma^{(m)}(s))\phi^{(m)}(s)=0$ since $\hat P(\alpha)L(\alpha)=L(\alpha)$. This ends the proof of Theorem 1.6.

3. Representation of the Solution u_ε

We use the notations of Corollary 1.7 and the neighborhoods U_ℓ and V_j of Γ_ℓ and $\hat\alpha_j$ from (15) and (16), respectively, and consider the solution u_ε of equation (6), i.e. of $[L(\alpha)-i\varepsilon A]u_\varepsilon(\alpha)=\tilde g(\alpha)$.

Bevor we begin with the proof of Theorem 1.9 (in Section 4) we represent the solution u_ε in the neighborhoods U_ℓ of Γ_ℓ, $\ell\in\mathcal{L}_{arc}$, and V_j of $\hat\alpha_j$, $j\in\mathcal{J}$, respectively, to determine the precise singular behavior with respect to α and ε. We begin with the neighborhoods U_ℓ.

3.1. **A representation of u_ε in the neighborhood U_ℓ of Γ_ℓ.** Let now $\ell\in\mathcal{L}_{arc}$, and let $\alpha=\gamma_\ell(t_1)$ for $t_1\in(a_\ell,b_\ell)$ be the parametrization of Γ_ℓ. We recall the unit tangential vector $\tau_\ell(t_1)=\pm\dot\gamma_\ell(t_1)$ and the corresponding unit normal vector $\nu_\ell(t_1)$ at $\gamma_\ell(t_1)$, where the sign is chosen such that $f_\ell(t_1)=\nu_\ell(t_1)\cdot\mathcal{F}(\phi_\ell(t_1))>0$ for all t_1, see Corollary 1.7 and (19). Here, $\phi_\ell(s)=\phi_\ell(\cdot,\gamma_\ell(s))\in\mathcal{N}(L(\gamma_\ell(s)))$ are the corresponding modes, normalized as $\int_Q n(x)\,|\phi_\ell(x,s)|^2dx=1$. We recall $g_\ell(\alpha)$ from (20), i.e. $g_\ell(t_1)=g_\ell(\gamma_\ell(t_1))=\int_\Omega g(y)\,\overline{e^{i\gamma_\ell(t_1)\cdot\tilde y}\phi_\ell(y,t_1)}\,dy$ for $t_1\in(a_\ell,b_\ell)$.
In the neighborhood $U_\ell=\{\gamma_\ell(t_1)+t_2\nu_\ell(t_1)\,:\,t_1\in(a_\ell,b_\ell),\ |t_2|<\delta\}$ of Γ_ℓ (where $\delta>0$ is sufficiently small) we introduce the local coordinate system

$$\alpha \;=\; \hat\gamma_\ell(t_1,t_2) \;:=\; \gamma_\ell(t_1)+t_2\,\nu_\ell(t_1) \tag{27}$$

for $t_1\in(a_\ell,b_\ell)$ and $|t_2|<\delta$.

Theorem 3.1. *Let Assumption 1.3 hold. Then the solution u_ε of (6) is represented in the neighborhood U_ℓ for $\ell\in\mathcal{L}_{arc}$ as*

$$u_\varepsilon\big(x,\hat\gamma_\ell(t_1,t_2)\big) \;=\; \frac{g_\ell(t_1)}{f_\ell(t_1)t_2-i\varepsilon}\,\phi_\ell(x,t_1) \;+\; w_\ell(x,t,\varepsilon) \tag{28}$$

for $(t_2,\varepsilon)\neq(0,0)$. Furthermore, $\|w_\ell(\cdot,t,\varepsilon)\|$ is uniformly bounded with respect to $(t_1,t_2,\varepsilon)\in(a_\ell,b_\ell)\times(-\delta,\delta)\times(0,\varepsilon_0)$.

Proof. We fix $\ell \in \mathcal{L}_{arc}$ and set $\tilde{L}(t_1,t_2) = L\big(\hat{\gamma}_\ell(t_1,t_2)\big)$ and $y(t) = \tilde{g}\big(\hat{\gamma}_\ell(t)\big)$ and $\tilde{u}(t_1,t_2,\varepsilon) = u_\varepsilon\big(\hat{\gamma}_\ell(t_1,t_2)\big)$ for abbreviation. Equation (6) is then equivalent to

$$[\tilde{L}(t) - i\varepsilon A]\tilde{u}(t,\varepsilon) \;=\; y(t)\,, \quad t \in (a_\ell, b_\ell) \times (-\delta,\delta)\,. \tag{29}$$

Note that $\tilde{L}(t_1,t_2)$ is invertible for $t_2 \neq 0$ and $\tilde{L}(t_1,0) = L\big(\gamma_\ell(t_1)\big)$ is not invertible with the one-dimensional kernel $\mathcal{N}(t_1) := \mathcal{N}(\tilde{L}(t_1,0)) = \operatorname{span}\{\phi_\ell(t_1)\}$ for $t_1 \in [a_\ell, b_\ell]$. First we observe from part (a) of Remarks 1.4 that

$$\begin{aligned} f_\ell(t_1) &= \nu_\ell(t_1)\cdot\mathcal{F}(\phi_\ell(t_1)) \;=\; \big\langle \nu_\ell(t_1)\cdot\nabla_\alpha L(\gamma_\ell(t_1))\phi_\ell(t_1), \phi_\ell(t_1)\big\rangle \\ &= \big\langle \partial_{t_2}\tilde{L}(t_1,0)\phi_\ell(t_1), \phi_\ell(t_1)\big\rangle\,. \end{aligned} \tag{30}$$

From now on we drop the index ℓ, i.e. write $f(t_1)$ and $g(t_1)$ and $\phi(t_1)$ for $f_\ell(t_1)$ and $g_\ell(t_1)$ and $\phi_\ell(t_1)$, respectively. We make the ansatz

$$u(t,\varepsilon) \;=\; \frac{\rho(t,\varepsilon)}{f(t_1)t_2 - i\varepsilon\mu}\phi(t_1) + v(t,\varepsilon) \tag{31}$$

with $\rho(t,\varepsilon) \in \mathbb{C}$ and $v(t,\varepsilon) \in \mathcal{R}(t_1) := \mathcal{R}(\tilde{L}(t_1,0)) = \mathcal{N}(t_1)^\perp$. Let $P(t_1)\psi = \frac{1}{\|\phi(t_1)\|^2}\langle\psi,\phi(t_1)\rangle\phi(t_1)$ and $Q(t_1) = I - P(t_1)$ be the projections onto $\mathcal{N}(t_1)$ and $\mathcal{R}(t_1)$, respectively, Multiplying the equation $[\tilde{L}(t) - i\varepsilon A]u(t,\varepsilon) = y(t)$ by $\phi(t_1)$ and applying $Q(t_1)$ yields the equivalent system

$$\frac{\rho(t,\varepsilon)}{f(t_1)t_2 - i\varepsilon}\big[\big\langle \tilde{L}(t)\phi(t_1),\phi(t_1)\big\rangle - i\varepsilon\big] \;+\; \big\langle [\tilde{L}(t) - i\varepsilon A]v(t,\varepsilon),\phi(t_1)\big\rangle \;=\; \langle y(t),\phi(t_1)\rangle\,,$$

$$\frac{\rho(t,\varepsilon)}{f(t_1)t_2 - i\varepsilon}Q(t_1)[\tilde{L}(t) - i\varepsilon A]\phi(t_1) \;+\; Q(t_1)[\tilde{L}(t) - i\varepsilon A]v(t,\varepsilon) \;=\; Q(t_1)y(t)\,,$$

which we write in matrix form as $M(t,\varepsilon)\begin{bmatrix}\rho\\ v\end{bmatrix} = \begin{bmatrix}\langle y(t),\phi(t_1)\rangle\\ Q(t_1)y(t)\end{bmatrix}$ where

$$M(t,\varepsilon) \;=\; \begin{bmatrix} \frac{\langle\tilde{L}(t)\phi(t_1),\phi(t_1)\rangle - i\varepsilon}{f(t_1)t_2 - i\varepsilon} & \langle[\tilde{L}(t) - i\varepsilon A]\cdot,\phi(t_1)\rangle \\ \frac{Q(t_1)[\tilde{L}(t) - i\varepsilon A]\phi(t_1)}{f(t_1)t_2 - i\varepsilon} & Q(t_1)[\tilde{L}(t) - i\varepsilon A]|_{\mathcal{R}(t_1)} \end{bmatrix}.$$

Now we observe that $\tilde{L}(t_1,0)\phi(t_1) = 0$ and also $Q(t_1)\tilde{L}(t_1,0) = 0$. Therefore, $\langle\tilde{L}(t)\phi(t_1),\phi(t_1)\rangle = t_2 f(t_1) + o(t_2)$ uniformly with respect to $t_1 \in (a,b)$ and $b(t,\varepsilon) := \frac{Q(t_1)[\tilde{L}(t) - i\varepsilon A]\phi(t_1)}{f(t_1)t_2 - i\varepsilon}$ is uniformly bounded and thus $M(t,\varepsilon) = M_0(t,\varepsilon) + M_1(t,\varepsilon)$ where

$$M_0(t,\varepsilon) \;=\; \begin{bmatrix} 1 & 0 \\ b(t,\varepsilon) & Q(t_1)\tilde{L}(t_1,0|_{\mathcal{R}(t_1)} \end{bmatrix} \quad \text{and} \quad M_1(t,\varepsilon) \to 0 \text{ as } (t_2,\varepsilon) \to (0,0)$$

uniformly with respect to $t_1 \in (a,b)$. The inverse $M_0(t,\varepsilon)$ is explicitly given by

$$M_0(t,\varepsilon)^{-1} \;=\; \begin{bmatrix} 1 & 0 \\ -V(t_1)b(t,\varepsilon) & V(t_1) \end{bmatrix} \quad \text{with} \quad V(t_1) := \big[Q(t_1)\tilde{L}(t_1,0|_{\mathcal{R}(t_1)}\big]^{-1} : \mathcal{R}(t_1) \to \mathcal{R}(t_1)\,,$$

and is uniformly bounded with respect to $(t_1,t_2,\varepsilon) \in (a,b) \times (-\delta,\delta) \times (0,\delta)$. We rewrite $M(t,\varepsilon)\begin{bmatrix}\rho\\ v\end{bmatrix} = \begin{bmatrix}\langle y(t),\phi(t_1)\rangle\\ Q(t_1)y(t)\end{bmatrix}$ in the equivalent form

$$\big[I + M_0(t,\varepsilon)^{-1}M_1(t,\varepsilon)\big]\begin{bmatrix}\rho\\ v\end{bmatrix} = M_0(t,\varepsilon)^{-1}\begin{bmatrix}\langle y(t),\phi(t_1)\rangle\\ Q(t_1)y(t)\end{bmatrix}$$

and observe that $M_0(t,\varepsilon)^{-1}M_1(t,\varepsilon) \to 0$ as $t_2,\varepsilon) \to (0,0)$ uniformly with respect to t_1. A perturbation argument yields

$$\begin{aligned}\rho(t,\varepsilon) &= \langle y(t),\phi(t_1)\rangle + \mathcal{O}\bigl(\sqrt{t_2^2+\varepsilon^2}\bigr) = \langle y(t_1,0),\phi(t_1)\rangle + \mathcal{O}\bigl(\sqrt{t_2^2+\varepsilon^2}\bigr),\\ v(t,\varepsilon) &= -\langle y(t),\phi(t_1)\rangle\, V(t_1)b(t,\varepsilon) + V(t_1)Q(t_1)y(t) + \mathcal{O}\bigl(\sqrt{t_2^2+\varepsilon^2}\bigr)\end{aligned}$$

uniformly with respect to $t_1 \in (a,b)$. Finally, we compute

$$\begin{aligned}\langle y(t_1,0),\phi_\ell(t_1)\rangle &= \int_Q \hat{g}(y,\gamma_\ell(t_1))\,\overline{e^{i\gamma_\ell(t_1)\cdot\tilde{y}}\phi_\ell(y,t_1)}\,dy\\ &= \int_\Omega g(y)\,\overline{e^{i\gamma_\ell(t_1)\cdot\tilde{y}}\phi_\ell(y,t_1)}\,dy = g_\ell(t_1)\end{aligned} \tag{32}$$

where we used the definition of the Floquet-Bloch transform in the last step. Substituting this into the ansatz (31) yields the assertion. □

3.2. **A representation of u_ε in the neighborhood V_j of $\mathcal{S}$.** Let now $\hat{\alpha} \in \mathcal{S}$, i.e. $\hat{\alpha} = \hat{\alpha}_j \in \Gamma_{j_1} \cap \Gamma_{j_2}$ for some $j = (j_1,j_2) \in \mathcal{J}$. We denote the parametrizations Γ_{j_1} and Γ_{j_2} by γ_{j_1} and γ_{j_2}, respectively, and choose the local coordinate system

$$\alpha = \hat{\gamma}_j(t_1,t_2) := \gamma_{j_1}(t_1) + \gamma_{j_2}(t_2) - \hat{\alpha}_j \tag{33}$$

for $|t_m| \le \delta$, $m = 1,2$, in the neighborhood V_j of $\hat{\alpha}_j$. Note that $\hat{\gamma}_j(0,0) = \hat{\alpha}_j$ and $\hat{\gamma}_j(t_1,0) = \gamma_{j_1}(t_1)$ and $\hat{\gamma}_j(0,t_2) = \gamma_{j_2}(t_2)$. We recall the definitions of $f_{j_m}(t_m)$ and $g_{j_m}(t_m)$ for $m = 1,2$ from (19) and (20), respectively, i.e., $f_{j_m}(t_m) = \nu_{j_m}(t_m) \cdot \mathcal{F}(\phi_{j_m}(t_m)) > 0$ and $g_{j_m}(t_m) = \int_\Omega g(y)\,\overline{e^{i\gamma_{j_m}(t_m)\cdot\tilde{y}}\,\phi_{j_m}(y,t_m)}\,dy$ for $t_m \in (-\delta,\delta)$ and $m = 1,2$ where we wrote $\phi_{j_m}(t_m)$ for $\phi_{j_m}(\gamma_{j_m}(t_m))$.

Theorem 3.2. *Let Assumption 1.3 hold. Then the solution u_ε is represented in the neighborhood V_j as*

$$\begin{aligned}u_\varepsilon(x,\hat{\gamma}_j(t_1,t_2)) = {}& \frac{g_{j_1}(t_1)}{D_j^{(1)}(t_1)\,f_{j_1}(t_1)t_2 - i\varepsilon}\,\phi_{j_1}(x,t_1) + \frac{g_{j_2}(t_2)}{D_j^{(2)}(t_2)f_{j_2}(t_2)t_1 - i\varepsilon}\,\phi_{j_2}(x,t_2)\\ &+ u_j(x,t,\varepsilon) + w_j(x,t,\varepsilon)\end{aligned} \tag{34}$$

for $(t_1,\varepsilon) \neq (0,0)$ and $(t_2,\varepsilon) \neq (0,0)$. Furthermore, $\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\|u_j(\cdot,t,\varepsilon)\|\,dt$ converges to zero in $H^1_{per,0}(Q)$ as $\varepsilon \to 0$, and $\|w_j(\cdot,t,\varepsilon)\|$ is uniformly bounded with respect to $(t_1,t_2,\varepsilon) \in (-\delta,\delta) \times (-\delta,\delta) \times (0,\varepsilon_0)$. The quantities $D_j^{(m)}(t_m)$ for $m = 1,2$ are given by

$$D_j^{(1)}(t_1) = \dot{\gamma}_{j_2}(0) \cdot \nu_{j_1}(t_1) \quad \textit{and} \quad D_j^{(2)}(t_2) = \dot{\gamma}_{j_1}(0) \cdot \nu_{j_2}(t_2)\,. \tag{35}$$

Proof. We set again $\tilde{L}(t_1,t_2) = L\bigl(\hat{\gamma}_j(t_1,t_2)\bigr)$ and $y(t_1,t_2) = \tilde{g}\bigl(\hat{\gamma}_j(t_1,t_2)\bigr)$ and $\tilde{u}(t_1,t_2,\varepsilon) = u_\varepsilon\bigl(\hat{\gamma}_j(t_1,t_2)\bigr)$ for abbreviation. Furthermore, we define $f^{(m)}(t_m)$ by

$$f^{(m)}(t_m) := D_j^{(m)}(t_m)\,f_{j_m}(t_m)\,, \quad m = 1,2\,.$$

As in the previous section the equation (6) is then equivalent to (29), i.e. $[\tilde{L}(t) - i\varepsilon A]\tilde{u}(t,\varepsilon) = y(t)$. We note that now

$$\tilde{L}(t_1,0) = L\bigl(\gamma_{j_1}(t_1)\bigr) \quad \text{and} \quad \tilde{L}(0,t_2) = L\bigl(\gamma_{j_2}(t_2)\bigr)$$

are not invertible and $\mathcal{N}(\tilde{L}(t_1,0)) = \mathcal{N}(L(\gamma_{j_1}(t_1))) = \text{span}\{\phi_{j_1}(t_1)\}$ for $t_1 \neq 0$ and $\mathcal{N}(\tilde{L}(0,t_2)) = \mathcal{N}(L(\gamma_{j_2}(t_2))) = \text{span}\{\phi_{j_2}(t_2)\}$ for $t_2 \neq 0$. Furthermore, the kernel of $\tilde{L}(0,0)$ is two-dimensional and $\mathcal{N}(\tilde{L}(0,0)) = \mathcal{N}(L(\hat{\alpha})) = \text{span}\{\phi_{j_1}(0), \phi_{j_2}(0)\}$. For (t_1,t_2) with $t_1 \neq 0$ and $t_2 \neq 0$ the operators $L(t_1,t_2)$ are invertible. Furthermore, from part (a) of Remarks 1.4 we recall that

$$\begin{aligned} f^{(1)}(t_1) &= D_j^{(1)}(t_1)\, f_{j_1}(t_1) \;=\; [\dot{\gamma}_{j_2}(0)\cdot\nu_{j_1}(t_1)]\,\nu_{j_1}(t_1)\cdot\mathcal{F}(\phi_{j_1}(t_1)) \;=\; \dot{\gamma}_{j_2}(0)\cdot\mathcal{F}(\phi_{j_1}(t_1)) \\ &= \left\langle \partial_{t_2} L(t_1,0)\phi_{j_1}(t_1)\,,\phi_{j_1}(t_1)\right\rangle, \\ f^{(2}(t_2) &= D_j^{(2)}(t_2)\, f_{j_2}(t_2) \;=\; [\dot{\gamma}_{j_1}(0)\cdot\nu_{j_2}(t_2)]\,\nu_{j_2}(t_2)\cdot\mathcal{F}(\phi_{j_2}(t_2)) \;=\; \dot{\gamma}_{j_1}(0)\cdot\mathcal{F}(\phi_{j_2}(t_2)) \\ &= \left\langle \partial_{t_1} L(0,t_2)\phi_{j_2}(t_2)\,,\phi_{j_2}(t_2)\right\rangle, \end{aligned}$$

because $\dot{\gamma}_{j_1}(t_1)\cdot\mathcal{F}(\phi_{j_1}(t_1)) = 0$ and $\dot{\gamma}_{j_2}(t_2)\cdot\mathcal{F}(\phi_{j_2}(t_2)) = 0$ by Corollary 1.8. Set $\mathcal{N}(t) = \mathcal{N}(t_1,t_2) := \text{span}\{\phi_{j_1}(t_1), \phi_{j_2}(t_2)\}$, and let $P(t)$ and $Q(t) = I - P(t)$ be the orthogonal projections onto $\mathcal{N}(t)$ and $\mathcal{N}(t)^\perp$, respectively. We note that the operators depend smoothly on t. We make the ansatz

$$u(t,\varepsilon) \;=\; \frac{\rho_1(t,\varepsilon)}{f^{(1)}(t_1)t_2 - i\varepsilon}\,\phi_{j_1}(t_1) \;+\; \frac{\rho_2(t,\varepsilon)}{f^{(2)}(t_2)t_1 - i\varepsilon}\,\phi_{j_2}(t_2) \;+\; v(t,\varepsilon)$$

with $\rho_j(t,\varepsilon) \in \mathbb{C}$ and $v \in \mathcal{N}(t)^\perp$. We set $\tilde{L}(t,\varepsilon) := \tilde{L}(t) - i\varepsilon A$ for abbreviation and multiply the equation $\tilde{L}(t,\varepsilon)u(t,\varepsilon) = y(t)$ by $\phi_{j_1}(t_1)$ and $\phi_{j_2}(t_2)$ (i.e. apply the projection $P(t)$) and apply the projection $Q(t)$. This yields the system of equations

$$\begin{aligned} &\frac{\rho_1(t,\varepsilon)}{f^{(1)}(t_1)t_2 - i\varepsilon}\left\langle \tilde{L}(t,\varepsilon)\phi_{j_1}(t_1), \phi_{j_1}(t_1)\right\rangle + \frac{\rho_2(t,\varepsilon)}{f^{(2)}(t_2)t_1 - i\varepsilon}\left\langle \tilde{L}(t,\varepsilon)\phi_{j_2}(t_2), \phi_{j_1}(t_1)\right\rangle \\ &+ \left\langle \tilde{L}(t,\varepsilon)v(t,\varepsilon), \phi_{j_1}(t_1)\right\rangle \;=\; \langle y(t), \phi_{j_1}(t_1)\rangle\,, \\ &\frac{\rho_1(t,\varepsilon)}{f^{(1)}(t_1)t_2 - i\varepsilon}\left\langle \tilde{L}(t,\varepsilon)\phi_{j_1}(t_1), \phi_{j_2}(t_2)\right\rangle + \frac{\rho_2(t,\varepsilon)}{f^{(2)}(t_2)t_1 - i\varepsilon}\left\langle \tilde{L}(t,\varepsilon)\phi_{j_2}(t_2), \phi_{j_2}(t_2)\right\rangle \\ &+ \left\langle \tilde{L}(t,\varepsilon)v(t,\varepsilon), \phi_{j_2}(t_2)\right\rangle \;=\; \langle y(t), \phi_{j_2}(t_2)\rangle\,, \\ &\frac{\rho_1(t,\varepsilon)}{f^{(1)}(t_1)t_2 - i\varepsilon}Q(t)\tilde{L}(t,\varepsilon)\phi_{j_1}(t_1) \;+\; \frac{\rho_2(t,\varepsilon)}{f^{(2)}(t_2)t_1 - i\varepsilon}Q(t)\tilde{L}(t,\varepsilon)\phi_{j_2}(t_2) \\ &+ Q(t)\tilde{L}(t,\varepsilon)v(t,\varepsilon) \;=\; Q(t)y(t)\,. \end{aligned}$$

We can replace the last equation by

$$\begin{aligned} &\frac{\rho_1(t,\varepsilon)}{f^{(1)}(t_1)t_2 - i\varepsilon}Q(t)\tilde{L}(t,\varepsilon)\phi_{j_1}(t_1) \;+\; \frac{\rho_2(t,\varepsilon)}{f^{(2)}(t_2)t_1 - i\varepsilon}Q(t)\tilde{L}(t,\varepsilon)\phi_{j_2}(t_2) \\ &+ [Q(t)\tilde{L}(t,\varepsilon) + P(t)]v(t,\varepsilon) \;=\; Q(t)y(t)\,. \end{aligned}$$

and require only $v(t,\varepsilon) \in H$. Indeed, if $(\rho_1, \rho_2, v) \in \mathbb{C}\times\mathbb{C}\times H$ solves this last equation then, applying $P(t)$, we observe that $P(t)v(t,\varepsilon) = 0$ which yields $v(t,\varepsilon) \in \mathcal{N}(t)^\perp$.

We write this system of equations in matrix form as

$$M(t,\varepsilon)\begin{bmatrix}\rho_1\\ \rho_2\\ v\end{bmatrix} = \begin{bmatrix}\langle y(t), \phi_{j_1}(t_1)\rangle\\ \langle y(t), \phi_{j_2}(t_2)\rangle\\ Q(t)y(t)\end{bmatrix}$$

where

$$M(t,\varepsilon) \;=\; \begin{bmatrix} \frac{\langle \tilde{L}(t)\phi_{j_1}(t_1),\phi_{j_1}(t_1)\rangle - i\varepsilon}{f^{(1)}(t_1)t_2 - i\varepsilon} & \frac{\langle [\tilde{L}(t)-i\varepsilon A]\phi_{j_2}(t_2),\phi_{j_1}(t_1)\rangle}{f^{(2)}(t_2)t_1 - i\varepsilon} & \langle [\tilde{L}(t)-i\varepsilon A]\cdot,\phi_{j_1}(t_1)\rangle \\ \frac{\langle [\tilde{L}(t)-i\varepsilon A]\phi_{j_1}(t_1),\phi_{j_2}(t_2)\rangle}{f^{(1)}(t_1)t_2 - i\varepsilon} & \frac{\langle \tilde{L}(t)\phi_{j_2}(t_2),\phi_{j_2}(t_2)\rangle - i\varepsilon}{f^{(2)}(t_2)t_1 - i\varepsilon} & \langle [\tilde{L}(t)-i\varepsilon A]\cdot,\phi_{j_2}(t_2)\rangle \\ \frac{Q(t)[\tilde{L}(t)-i\varepsilon A]\phi_{j_1}(t_1)}{f^{(1)}(t_1)t_2 - i\varepsilon} & \frac{Q(t)[\tilde{L}(t)-i\varepsilon A]\phi_{j_2}(t_2)}{f^{(2)}(t_2)t_1 - i\varepsilon} & Q(t)[\tilde{L}(t)-i\varepsilon A] + P(t) \end{bmatrix},$$

and consider it as an operator from $\mathbb{C}\times\mathbb{C}\times H$ into itself. Here we used the normalizations of $\phi_{j_1}(t_1)$ and $\phi_{j_2}(t_2)$. We discuss the terms of the matrix with respect to t and ε. From $\tilde{L}(t_1,0)\phi_{j_1}(t_1)=0$ and $\tilde{L}(0,t_2)\phi_{j_2}(t_2)=0$ and the selfadjointness of $\tilde{L}(t)$ and $\partial_{t_j}\tilde{L}(t)$ we obtain $\langle\partial_{t_1}\tilde{L}(0)\phi_{j_1}(0),\phi^{(j)}(0)\rangle=0$ and $\langle\partial_{t_2}\tilde{L}(0)\phi_{j_2}(0),\phi^{(j)}(0)\rangle=0$ for $j=1,2$. Therefore,

$$\begin{aligned}\langle\tilde{L}(t_1,t_2)\phi_{j_1}(t_1),\phi_{j_1}(t_1)\rangle \;&=\; \langle\tilde{L}(t_1,0)\phi_{j_1}(t_1),\phi_{j_1}(t_1)\rangle \;+\; t_2\langle\partial_{t_2}\tilde{L}(t_1,0)\phi_{j_1}(t_1),\phi_{j_1}(t_1)\rangle \\ &\quad + o(t_2) \\ &=\; t_2 f^{(1)}(t_1) \;+\; o(t_2) \quad \text{and thus}\end{aligned}$$

$$\lim_{t\to 0}\frac{\langle\tilde{L}(t_1,t_2)\phi_{j_1}(t_1),\phi_{j_1}(t_1)\rangle - i\varepsilon}{t_2 f^{(1)}(t_1)-i\varepsilon} \;=\; 1$$

because $\frac{|t_2|}{|t_2 f^{(1)}(t_1)-i\varepsilon|}$ is bounded. Analogously,

$$\lim_{t\to 0}\frac{\langle\tilde{L}(t_1,t_2)\phi_{j_2}(t_2),\phi_{j_2}(t_2)\rangle - i\varepsilon}{t_1 f^{(2)}(t_2)-i\varepsilon} \;=\; 1$$

and

$$\begin{aligned}&\langle\tilde{L}(t_1,t_2)\phi_{j_2}(t_2),\phi_{j_1}(t_1)\rangle - i\varepsilon\langle A\phi_{j_2}(t_2),\phi_{j_1}(t_1)\rangle \\ =\;& \langle\tilde{L}(0,t_2)\phi_{j_2}(t_2),\phi_{j_1}(t_1)\rangle \;+\; t_1\langle\partial_{t_1}\tilde{L}(0,t_2)\phi_{j_2}(t_2),\phi_{j_1}(t_1)\rangle \;+\; o(t_1) -\; i\varepsilon\overline{\mu(t)} \\ =\;& -\, i\varepsilon\overline{\mu(t)} \;+\; \mathcal{O}(t_1|t|) \;+\; o(t_1)\end{aligned}$$

with $\mu(t) := \langle A\phi^{(1)}(t_1),\phi^{(2)}(t_2)\rangle$. Therefore,

$$\frac{\langle\tilde{L}(t_1,t_2)\phi_{j_2}(t_2),\phi_{j_1}(t_1)\rangle - i\varepsilon\langle A\phi_{j_2}(t_2),\phi_{j_1}(t_1)\rangle}{t_1 f^{(2)}(t_2)-i\varepsilon} \;=\; \frac{-i\varepsilon\overline{\mu(t)}}{t_1 f^{(2)}(t_2)-i\varepsilon} \;+\; R_1(t,\varepsilon)$$

with $R_1(t,\varepsilon)\to 0$ as $(t,\varepsilon)\to(0,0)$. Analogously,

$$\frac{\langle\tilde{L}(t_1,t_2)\phi_{j_1}(t_1),\phi_{j_2}(t_2)\rangle - i\varepsilon\langle A\phi_{j_1}(t_1),\phi_{j_2}(t_2)\rangle}{t_2 f^{(1)}(t_1)-i\varepsilon} \;=\; \frac{-i\varepsilon\mu(t)}{t_2 f^{(1)}(t_1)-i\varepsilon} \;+\; R_2(t,\varepsilon)$$

with $R_2(t,\varepsilon)\to 0$ as $(t,\varepsilon)\to(0,0)$. Furthermore, we note that $\langle[\tilde{L}(t)-i\varepsilon A]\cdot,\phi_j(t_j)\rangle = \mathcal{O}\big(\sqrt{|t|^2+\varepsilon^2}\big)$ for $j=1,2$. Therefore, $M(t,\varepsilon)$ has the form $M(t,\varepsilon)=M_0(t,\varepsilon)+M_1(t,\varepsilon)$ with

$$M_0(t,\varepsilon) \;=\; \begin{bmatrix} 1 & \beta_1(t,\varepsilon) & 0 \\ \beta_2(t,\varepsilon) & 1 & 0 \\ b_1(t,\varepsilon) & b_2(t,\varepsilon) & \tilde{L}(0)+P(0)\end{bmatrix} \quad\text{and}\quad \lim_{(t,\varepsilon)\to(0,0)} M_1(t,\varepsilon) \;=\; 0$$

(note that $Q(0)\tilde{L}(0)=\tilde{L}(0)$) where

$$\beta_1(t,\varepsilon) \;=\; -\frac{i\varepsilon\overline{\mu(t)}}{t_1 f^{(2)}(t_2)-i\varepsilon}, \quad \beta_2(t,\varepsilon) \;=\; -\frac{i\varepsilon\mu(t)}{t_2 f^{(1)}(t_1)-i\varepsilon}$$

and

$$b_1(t,\varepsilon) \;=\; \frac{Q(t)[\tilde{L}(t)-i\varepsilon A]\phi_{j_1}(t_1)}{t_2 f^{(1)}(t_1)-i\varepsilon}\,, \quad b_2(t,\varepsilon) \;=\; \frac{Q(t)[\tilde{L}(t)-i\varepsilon A]\phi_{j_2}(t_2)}{t_1 f^{(2)}(t_2)-i\varepsilon}$$

are uniformly bounded. We can compute the inverse of $M_0(t,\varepsilon)$ explicitly. With

$$d(t,\varepsilon) \;=\; 1-\beta_1(t,\varepsilon)\beta_2(t,\varepsilon) \;=\; 1 \;+\; \frac{\varepsilon^2|\mu(t)|^2}{[t_1 f^{(2)}(t_2)-i\varepsilon][t_2 f^{(1)}(t_1)-i\varepsilon]}\,, \tag{36}$$

and $V := [\tilde{L}(0)+P(0)]^{-1} : H \to H$ we have

$$M_0(t,\varepsilon)^{-1} \;=\; \frac{1}{d(t,\varepsilon)}\begin{bmatrix} 1 & -\beta_1 & 0 \\ -\beta_2 & 1 & 0 \\ V(\beta_2 b_2 - b_1) & V(\beta_1 b_1 - b_2) & dV \end{bmatrix}$$

where we dropped the argument (t,ε) in the matrix. To show bounds of $|d(t,\varepsilon)|$ from below and above we estimate

$$|d(t,\varepsilon)| \;\leq\; 1 \;+\; \frac{\varepsilon^2|\mu(t)|^2}{\sqrt{|f^{(2)}(t_2)|^2 t_1^2+\varepsilon^2}\sqrt{|f^{(1)}(t_1)|^2 t_2^2+\varepsilon^2}} \;\leq\; 1+|\mu(t)|^2\,,$$

$$|d(t,\varepsilon)| \;\geq\; 1 \;-\; \frac{\varepsilon^2|\mu(t)|^2}{\sqrt{|f^{(2)}(t_2)|^2 t_1^2+\varepsilon^2}\sqrt{|f^{(1)}(t_1)|^2 t_2^2+\varepsilon^2}} \;\geq\; 1-|\mu(t)|^2$$

for all t and ε. Furthermore, $|\mu(0)|^2 = |\langle A\phi_{j_1}(0),\phi_{j_2}(0)\rangle|^2 < \langle A\phi_{j_1}(0),\phi_{j_1}(0)\rangle\langle A\phi_{j_2}(0),\phi_{j_2}(0)\rangle = 1$ by the Cauchy-Schwarz inequality and the fact that $\phi_{j_1}(0)$ and $\phi_{j_2}(0)$ are linearly independent.

From this representation we conclude that $M_0(t,\varepsilon)^{-1}$ is uniformly bounded with respect to (t,ε).

We go back to the equation $M(t,\varepsilon)\begin{bmatrix}\rho_1\\ \rho_2\\ v\end{bmatrix} = \begin{bmatrix}\langle y(t),\phi_{j_1}(t_1)\rangle\\ \langle y(t),\phi_{j_2}(t_2)\rangle\\ Q(t)y(t)\end{bmatrix}$ and write it in the equivalent form

$$\big[I+M_0(t,\varepsilon)^{-1}M_1(t,\varepsilon)\big]\begin{bmatrix}\rho_1\\ \rho_2\\ v\end{bmatrix} \;=\; M_0(t,\varepsilon)^{-1}\begin{bmatrix}\langle y(t),\phi_{j_1}(t_1)\rangle\\ \langle y(t),\phi_{j_2}(t_2)\rangle\\ Q(t)y(t)\end{bmatrix}.$$

Since $M_0(t,\varepsilon)^{-1}M_1(t,\varepsilon)\to 0$ as $(t,\varepsilon)\to(0,0)$ we conclude by a perturbation argument that

$$\begin{bmatrix}\rho_1(t,\varepsilon)\\ \rho_2(t,\varepsilon)\\ v(t,\varepsilon)\end{bmatrix} = M_0(t,\varepsilon)^{-1}\begin{bmatrix}\langle y(t),\phi_{j_1}(t_1)\rangle\\ \langle y(t),\phi_{j_2}(t_2)\rangle\\ Q(0)y(0)\end{bmatrix} + \mathcal{O}\big(\sqrt{|t|^2+\varepsilon^2}\big)\,.$$

Substituting this into the ansatz $u(t,\varepsilon)=\frac{\rho_1(t,\varepsilon)}{f^{(1)}(t_1)t_2-i\varepsilon}\,\phi_{j_1}(t_1)+\frac{\rho_2(t,\varepsilon)}{f^{(2)}(t_2)t_1-i\varepsilon}\,\phi_{j_2}(t_2)+v(t,\varepsilon)$ yields

$$
\begin{aligned}
u(t,\varepsilon) &= \frac{1}{d(t,\varepsilon)[f^{(1)}(t_1)t_2-i\varepsilon]}\left[\langle y(t),\phi_{j_1}(t_1)\rangle-\beta_1(t,\varepsilon)\langle y(t),\phi_{j_2}(t_2)\rangle\right]\phi_{j_1}(t_1)\\
&\quad+\frac{1}{d(t,\varepsilon)[f^{(2)}(t_2)t_1-i\varepsilon]}\left[-\beta_2(t,\varepsilon)\langle y(t),\phi_{j_1}(t_1)\rangle+\langle y(t),\phi_{j_2}(t_2)\rangle\right]\phi_{j_2}(t_2)\\
&\quad+\frac{1}{d(t,\varepsilon)}\left[\langle y(t),\phi_{j_1}(t_1)\rangle\,V(\beta_2b_2-b_1)+\langle y(t),\phi_{j_2}(t_2)\rangle\,V(\beta_1b_1-b_2)+d\,VQ(0)y(0)\right]\\
&= \frac{1}{d(t,\varepsilon)}\Bigg[\frac{\langle y(t_1,0),\phi_{j_1}(t_1)\rangle}{f^{(1)}(t_1)t_2-i\varepsilon}\,\phi_{j_1}(t_1)+\frac{\langle y(0,t_2),\phi_{j_2}(t_2)\rangle}{f^{(2)}(t_2)t_1-i\varepsilon}\,\phi_{j_2}(t_2)\\
&\quad+\frac{i\varepsilon}{[f^{(1)}(t_1)t_2-i\varepsilon][f^{(2)}(t_2)t_1-i\varepsilon]}\Big(\overline{\mu(t)}\,\langle y(t),\phi_{j_2}(t_2)\rangle\phi_{j_1}(t_1)\\
&\quad+\mu(t)\,\langle y(t),\phi_{j_1}(t_1)\rangle\phi_{j_2}(t_2)\Big)\Bigg]\;+\;w(t,\varepsilon)
\end{aligned}
$$

where $w(t,\varepsilon)$ is uniformly bounded. Using $\frac{1}{d(t,\varepsilon)}=1-\frac{\varepsilon^2|\mu(t)|^2}{[f^{(2)}(t_2)t_1-i\varepsilon][f^{(1)}(t_1)t_2-i\varepsilon]+\varepsilon^2|\mu(t)|^2}$ yields the form (34) with $u_j(t,\varepsilon)=\tilde u_1(t,\varepsilon)+\tilde u_2(t,\varepsilon)$ where

$$
\begin{aligned}
\tilde u_1(t,\varepsilon) &= -\frac{\varepsilon^2|\mu(t)|^2}{[f^{(1)}(t_1)t_2-i\varepsilon][f^{(2)}(t_2)t_1-i\varepsilon]+\varepsilon^2|\mu|^2}\Bigg[\frac{\langle y(t_1,0),\phi^{(1)}(t_1)\rangle}{f^{(1)}(t_1)t_2-i\varepsilon}\,\phi^{(1)}(t_1)+\\
&\quad+\frac{\langle y(0,t_2),\phi^{(2)}(t_2)\rangle}{f^{(2)}(t_2)t_1-i\varepsilon}\,\phi^{(2)}(t_2)\Bigg]\\
\tilde u_2(t,\varepsilon) &= \frac{i\varepsilon}{d(t,\varepsilon)[f^{(1)}(t_1)t_2-i\varepsilon][f^{(2)}(t_2)t_1-i\varepsilon]}\left[\overline{\mu(t)}\,\left\langle y(t),\phi^{(2)}(t_2)\right\rangle\phi^{(1)}(t_1)\right.\\
&\quad\left.+\,\mu(t)\left\langle y(t),\phi^{(1)}(t_1)\right\rangle\phi^{(2)}(t_2)\right]
\end{aligned}
$$

Finally, we estimate $\tilde u_1(t,\varepsilon)$ and $\tilde u_2(t,\varepsilon)$, using $|\mu(t)|\le\mu_0<1$ for all $t\in V$,

$$
\begin{aligned}
\int_V\|\tilde u_1(t,\varepsilon)\|\,dt &\le c\,\varepsilon^2\sup_{t\in V}\|y(t)\|\int_V\Bigg[\frac{1}{\big|[f^{(1)}(t_1)t_2-i\varepsilon][f^{(2)}(t_2)t_1-i\varepsilon]+\varepsilon^2|\mu(t)|^2\big|\,|f^{(1)}(t_1)t_2-i\varepsilon|}\\
&\quad+\frac{1}{\big|[f^{(1)}(t_1)t_2-i\varepsilon][f^{(2)}(t_2)t_1-i\varepsilon]+\varepsilon^2|\mu(t)|^2\big|\,|f^{(2)}(t_2)t_1-i\varepsilon|}\Bigg]dt\,,\\
&\le c_1\,\varepsilon^2\sup_{t\in V}\|y(t)\|\int_0^\delta\!\!\int_0^\delta\frac{1}{\sqrt{t_2^2+\varepsilon^2}\sqrt{t_1^2+\varepsilon^2}-\varepsilon^2\mu_0}\left[\frac{1}{\sqrt{t_1^2+\varepsilon^2}}+\frac{1}{\sqrt{t_2^2+\varepsilon^2}}\right]dt_1\,dt_2\\
&= c_1\,\varepsilon\sup_{t\in V}\|y(t)\|\int_0^{\delta/\varepsilon}\!\!\int_0^{\delta/\varepsilon}\frac{1}{\sqrt{s_2^2+1}\sqrt{s_1^2+1}-\mu_0}\left[\frac{1}{\sqrt{s_1^2+1}}+\frac{1}{\sqrt{s_2^2+1}}\right]ds_1\,ds_2\\
&\le c_2\,\varepsilon\sup_{t\in V}\|y(t)\|\int_0^{\delta/\varepsilon}\frac{ds_2}{\sqrt{s_2^2+1}-\mu_0}\int_0^{\delta/\varepsilon}\frac{ds_1}{\sqrt{s_1^2+1}}\;\le\;c_3\sup_{t\in V}\|y(t)\|\,\varepsilon\big[\ln(1/\varepsilon)\big]^2\,,
\end{aligned}
$$

which tends to zero as $\varepsilon \to 0$. Analogously, one obtains

$$\int\limits_V \|\tilde{u}_2(t,\varepsilon)\|\,dt \;\le\; c_1\,\varepsilon \sup_{t\in V}\|y(t)\| \int\limits_0^{\delta/\varepsilon} \frac{ds_2}{\sqrt{s_2^2+1}} \int\limits_0^{\delta/\varepsilon} \frac{ds_1}{\sqrt{s_1^2+1}} \;\le\; c_2 \sup_{t\in V}\|y(t)\|\,\varepsilon\big[\ln(1/\varepsilon)\big]^2 .$$

$\square$

4. Proof of Theorem 1.9

We investigate the integrals of (17) with respect to convergence as $\varepsilon \to 0$.

For the first term in (17) we note that $\operatorname{supp}\psi_0 \cap \mathcal{A} = \emptyset$. Therefore, we conclude that $\psi_0(\alpha)\,u_\varepsilon(\cdot,\alpha)$ converges to $\psi_0(\alpha)\,u_0(\cdot,\alpha)$ uniformly with respect to $\alpha \in I$ and $\psi_0 u_0 \in L^2\big(I, H^1(Q)\big)$. Therefore, convergence of the first term in the form (17) follows from the boundedness of the inverse Floquet-Bloch transform.

4.1. The terms corresponding to U_ℓ.

Next, we consider the integrals in the second term of (17) corresponding to $\ell \in \mathcal{L}_{arc}$. We drop the index ℓ and recall the open set $U = U_\ell = \{\hat{\gamma}(t_1,t_2) : a < t_1 < b,\ |t_2| < \delta\}$ from (15), the local coordinate system $\hat{\gamma} = \hat{\gamma}_\ell$ from (27), and the function $\psi = \psi_\ell$ from the partition of unity.

We have to investigate the integral $\int_U \psi(\alpha)\,u_\varepsilon(x,\alpha)\,e^{i\alpha\cdot\tilde{x}}d\alpha$ and transform the integral into

$$\begin{aligned}
\int\limits_U \psi(\alpha)\,u_\varepsilon(x,\alpha)\,e^{i\alpha\cdot\tilde{x}}d\alpha \;&=\; \int\limits_{U'} \psi(\hat{\gamma}(t))\,u_\varepsilon(x,\hat{\gamma}(t))\,e^{i\gamma(t)\cdot\tilde{x}}\,D(t)\,dt \\
&=\; \int\limits_{U'} \psi(\hat{\gamma}(t))\,u_\varepsilon(x,\hat{\gamma}(t))\,e^{it_2\nu(t_1)\cdot\tilde{x}}\,e^{i\gamma(t_1)\cdot\tilde{x}}\,D(t)\,dt
\end{aligned}$$

where $U' = (a,b)\times(-\delta,\delta)$ and $D(t_1,t_2) = \big|\det[\dot{\gamma}(t_1) + t_2\dot{\nu}(t_1)|\nu(t_1)]\big| = \big|1 - t_2\det[\dot{\nu}(t_1)|\nu(t_1)]\big|$. We set $\tilde{\psi}(t) := \psi(\hat{\gamma}(t))D(t)$ and substitute the form (28) into the integral and obtain

$$\begin{aligned}
\int\limits_U \psi(\alpha)\,u_\varepsilon(x,\alpha)\,e^{i\alpha\cdot\tilde{x}}\,d\alpha \;&=\; \int\limits_a^b\int\limits_{-\delta}^{\delta} \tilde{\psi}(t_1,t_2)\,u_\varepsilon\big(x,\hat{\gamma}(t_1,t_2)\big)\,e^{it_2\nu(t_1)\cdot\tilde{x}}\,dt_2\,e^{i\gamma(t_1)\cdot\tilde{x}}\,dt_1 \\
&=\; \int\limits_a^b\Bigg[\int\limits_{-\delta}^{\delta} \frac{\tilde{\psi}(t_1,t_2)}{f(t_1)t_2 - i\varepsilon}\,e^{it_2\nu(t_1)\cdot\tilde{x}}dt_2\Bigg]\,g(t_1)\,\phi(x,t_1)\,e^{i\gamma(t_1)\cdot\tilde{x}}\,dt_1 \\
&\quad+ \int\limits_a^b\int\limits_{-\delta}^{\delta} \tilde{\psi}(t_1,t_2)\,w(x,t,\varepsilon)\,e^{it_2\nu(t_1)\cdot\tilde{x}}\,e^{i\gamma(t_1)\cdot\tilde{x}}\,dt_2\,dt_1 .
\end{aligned}$$

First we note that $t_1 \mapsto f(t_1)$ is continuous and strictly positive on $[a,b]$. We discuss the integral with respect to t_2 of the first part and obtain

$$\int_{-\delta}^{\delta} \frac{\tilde\psi(t_1,t_2)}{f(t_1)t_2 - i\varepsilon}\, e^{it_2\nu(t_1)\cdot\tilde x}\, dt_2 \;=\; \tilde\psi(t_1,0)\int_{-1}^{1} \frac{1}{f(t_1)t_2 - i\varepsilon}\, e^{it_2\nu(t_1)\cdot\tilde x}\, dt_2$$

$$-\,\tilde\psi(t_1,0)\int_{\delta<|t_2|<1} \frac{1}{f(t_1)t_2 - i\varepsilon}\, e^{it_2\nu(t_1)\cdot\tilde x}\, dt_2 \;+\; \int_{-\delta}^{\delta} \frac{\tilde\psi(t_1,t_2)-\tilde\psi(t_1,0)}{f(t_1)t_2 - i\varepsilon}\, e^{it_2\nu(t_1)\cdot\tilde x}\, dt_2\,.$$

The second and third integral on the right hand side are uniformly bounded with respect to (t_1,ε) and tend to $-\frac{\tilde\psi(t_1,0)}{f(t_1)}\int_{\delta<|t_2|<1}\frac{1}{t_2}\, e^{it_2\nu(t_1)\cdot\tilde x}\, dt_2 + \frac{1}{f(t_1)}\int_{-\delta}^{\delta}\frac{\tilde\psi(t_1,t_2)-\tilde\psi(t_1,0)}{t_2}\, e^{it_2\nu(t_1)\cdot\tilde x} dt_2$ as $\varepsilon \to 0$ uniformly with respect to $t_1 \in (a,b)$.

For the first integral we apply Lemma B.1 of the appendix which yields that for any $R>0$

$$\sup_{|\tilde x|\le R}\left|\int_{-1}^{1} \frac{e^{it_2\nu(t_1)\cdot\tilde x}}{f(t_1)t_2 - i\varepsilon}\, dt_2\right| \;\le\; c_R \quad\text{and}\quad \lim_{\varepsilon\to 0}\int_{-1}^{1} \frac{e^{it_2\nu(t_1)\cdot\tilde x}}{f(t_1)t_2 - i\varepsilon}\, dt_2 \;=\; \frac{2\pi i}{f(t_1)}\,\Psi^{(+)}\big(\nu(t_1)\cdot\tilde x\big)$$

(note that $f(t_1)>0$) and $\Psi^{(+)}$ given by (18). Therefore, by Lebesgue's theorem on dominated convergence,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_U \psi(\alpha)\, u_\varepsilon(x,\alpha)\, e^{i\alpha\cdot\tilde x}\, d\alpha \\
=\;& \int_a^b \tilde\psi(t_1,0)\frac{2\pi i}{f(t_1)}\;\Psi^{(+)}\big(\nu(t_1)\cdot\tilde x\big)\, g(t_1)\,\phi(x,t_1)\, e^{i\gamma(t_1)\cdot\tilde x}\, dt_1 \\
&+ \int_a^b\int_{-\delta}^{\delta} \frac{\tilde\psi(t_1,t_2)-\tilde\psi(t_1,0)}{f(t_1)\,t_2}\, e^{it_2\nu(t_1)\cdot\tilde x}\, g(t_1)\,\phi(x,t_1)\, e^{i\gamma(t_1)\cdot\tilde x}\, dt_2\, dt_1 \\
&- \int_a^b\int_{\delta<|t_2|<1} \frac{\tilde\psi(t_1,0)}{f(t_1)\,t_2}\, e^{it_2\nu(t_1)\cdot\tilde x}\, g(t_1)\,\phi(x,t_1)\, e^{i\gamma(t_1)\cdot\tilde x}\, dt_2\, dt_1 \\
&+ \int_a^b\int_{-\delta}^{\delta} \tilde\psi(t_1,t_2)\, w(x,t,0)\, e^{it_2\nu(t_1)\cdot\tilde x}\, e^{i\gamma(t_1)\cdot\tilde x}\, dt_2\, dt_1\,.
\end{aligned}$$

Note that integrands of the second, third, and forth integral are bounded functions, thus in $L^2\big((a,b)\times(-\delta,\delta), H^1(Q)\big)$. By the inverse Floquet-Bloch transform these represent functions in $L^2(W)$, i.e.

$$\lim_{\varepsilon\to 0}\int\limits_U \psi(\alpha)\, u_\varepsilon(x,\alpha)\, e^{i\alpha\cdot\tilde{x}}\, d\alpha$$

$$= 2\pi i\int\limits_a^b \frac{\psi(\gamma(t_1))}{f(t_1)}\ \Psi^{(+)}\big(\nu(t_1)\cdot\tilde{x}\big)\, g(t_1)\,\phi(x,t_1)\, e^{i\gamma(t_1)\cdot\tilde{x}}\, dt_1 \ +\ v(x)$$

with $v \in H^1(W)$. In the last integral we have used the definition of $\tilde{\psi}$ and the fact that $D(t_1,0)=1$, i.e. $\tilde{\psi}(t_1,0)=\psi(\gamma(t_1))$.

This is exactly the form of $\tilde{u}$ in Theorem 1.9 corresponding to ℓ in the parametrized form.

4.2. **The terms corresponding to V_j.** In this subsection we consider the integrals in the terms of u_{sec} corresponding to $j\in\mathcal{J}$. We recall the open neighborhood $V_j=\{\hat{\gamma}_j(t_1,t_2) : |t_m|<\delta,\ m=1,2\}$ of $\hat{\alpha}_j$ (see (16)), the local coordinate system $\hat{\gamma}_j$ from (33), and the function $\psi_j\in C^\infty(\mathbb{R}^2)$ from the partition of unity.

We investigate the integral $\int_{V_j}\psi_j(\alpha)\, u_\varepsilon(x,\alpha)\, e^{i\alpha\cdot\tilde{x}} d\alpha$ and transform the integral using the transformation (33) and obtain

$$\begin{aligned}\int\limits_{V_j}\psi_j(\alpha)\, u_\varepsilon(x,\alpha)\, e^{i\alpha\cdot\tilde{x}} d\alpha\ &=\ \int\limits_{V_j'}\psi_j(\hat{\gamma}(t_1,t_2))\, u_\varepsilon(x,\hat{\gamma}_j(t))\, e^{i\gamma(t)\cdot\tilde{x}}\big|\det[\dot{\gamma}_{j_1}(t_1)\,|\,\dot{\gamma}_{j_2}(t_2)]\big|\, dt\\ &=\ e^{-i\hat{\alpha}\cdot\tilde{x}}\int\limits_{V'}\tilde{\psi}(t)\, u_\varepsilon(x,\hat{\gamma}(t))\, e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}} e^{i\gamma_{j_2}(t_2)\cdot\tilde{x}}\, dt\end{aligned}$$

where $V'=(-\delta,\delta)\times(-\delta,\delta)$ and

$$\tilde{\psi}(t_1,t_2)\ =\ \psi_j(\hat{\gamma}(t_1,t_2))\,\big|\det[\dot{\gamma}_{j_1}(t_1)\,|\,\dot{\gamma}_{j_2}(t_2)]\big|\ =\ \psi_j(\hat{\gamma}(t_1,t_2))\,|\dot{\gamma}_{j_1}(t_1)\cdot\nu_{j_2}(t_2)|\,. \tag{37}$$

We recall (see (14)) that $\delta>0$ had been chosen such that $\dot{\gamma}_{j_1}(t_1)\cdot\nu_{j_2}(t_2)$ does not change its sign for $t\in V'$.

We substitute the first part of $u_\varepsilon(x,\hat{\gamma}(t))$ from (34) into $e^{-i\hat{\alpha}\cdot\tilde{x}}\tilde{\psi}(t)\, u_\varepsilon(x,\hat{\gamma}(t))\, e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}} e^{i\gamma_{j_2}(t_2)\cdot\tilde{x}}$ and split the result into the form

$$\begin{aligned}& e^{-i\hat{\alpha}\cdot\tilde{x}}\,\frac{\tilde{\psi}(t)\, g_{j_1}(t_1)}{D_j^{(1)}(t_1) f_{j_1}(t_1) t_2 - i\varepsilon}\, e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}}\, e^{i\gamma_{j_2}(t_2)\cdot\tilde{x}}\,\phi_{j_1}(x,t_1)\\ =\ & \frac{\tilde{\psi}(t_1,0)\, g_{j_1}(t_1)}{D_j^{(1)}(t_1) f_{j_1}(t_1) t_2 - i\varepsilon}\, e^{it_2\dot{\gamma}_{j_2}(0)\cdot\tilde{x}}\, e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}}\,\phi_{j_1}(x,t_1)\ +\ B_1(x,t,\varepsilon)\end{aligned} \tag{38}$$

where $B_1(x,t,\varepsilon)$ is uniformly bounded with respect to (t,ε) and x from bounded sets.

Analogously, the second term in (34) has the form

$$\begin{aligned}& e^{-i\hat{\alpha}\cdot\tilde{x}}\,\frac{\tilde{\psi}(t)\, g_{j_2}(t_2)}{D_j^{(2)}(t_2) f_{j_2}(t_2) t_1 - i\varepsilon}\, e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}}\, e^{i\gamma_{j_2}(t_2)\cdot\tilde{x}}\,\phi_{j_2}(x,t_2)\\ =\ & \frac{\tilde{\psi}(0,t_2)\, g_{j_2}(t_2)}{D_j^{(2)}(t_2) f_{j_2}(t_2) t_1 - i\varepsilon}\, e^{it_1\dot{\gamma}_{j_1}(0)\cdot\tilde{x}}\, e^{i\gamma_{j_2}(t_2)\cdot\tilde{x}}\,\phi_{j_2}(x,t_2)\ +\ B_2(x,t,\varepsilon)\end{aligned} \tag{39}$$

where $B_2(x,t,\varepsilon)$ is uniformly bounded with respect to (t,ε) and $\tilde{x}$ from bounded sets.

Now we integrate these contributions of (34) with respect to $(t_1,t_2) \in V' = (-\delta,\delta)\times(-\delta,\delta)$ and let ε tend to zero. The integrals of $u_j(\cdot,t,\varepsilon)$ tend to zero by the last statement of Theorem 1.6. Integration of (38) yields

$$
\begin{aligned}
& e^{-i\hat{\alpha}\cdot\tilde{x}} \int\limits_{V'} \frac{\tilde{\psi}(t)\, g_{j_1}(t_1)}{D_j^{(1)}(t_1) f_{j_1}(t_1) t_2 - i\varepsilon}\, e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}}\, e^{i\gamma_{j_2}(t_2)\cdot\tilde{x}}\, \phi_{j_1}(x,t_1)\, dt \\
= & \int\limits_{-\delta}^{\delta} \Big[\int\limits_{-1}^{1} \frac{1}{D_j^{(1)}(t_1) f_{j_1}(t_1) t_2 - i\varepsilon}\, e^{it_2\dot{\gamma}_{j_2}(0)\cdot\tilde{x}}\, dt_2 \Big]\, \tilde{\psi}(t_1,0)\, g_{j_1}(t_1)\, e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}}\, \phi_{j_1}(x,t_1)\, dt_1 \\
& - \int\limits_{-\delta}^{\delta} \Big[\int\limits_{\delta<|t_2|<1} \frac{1}{D_j^{(1)}(t_1) f_{j_1}(t_1) t_2 - i\varepsilon}\, e^{it_2\dot{\gamma}_{j_2}(0)\cdot\tilde{x}}\, dt_2 \Big]\, \tilde{\psi}(t_1,0)\, g_{j_1}(t_1)\, e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}}\, \phi_{j_1}(x,t_1)\, dt_1 \qquad (40) \\
& + \int\limits_{V'} B_1(x,t,\varepsilon)\, dt\,.
\end{aligned}
$$

The integrands of the second and third integral are uniformly bounded with respect (t_1,t_2,ε) and converge pointwise as $\varepsilon \to 0$. Therefore, by Lebesgue's theorem on dominated convergence these integrals converge to a function of the form $v_1(x) = \int_I \tilde{B}(x,t)\, dt$ which is in $H^1(W)$ by the inverse Floquet-Bloch transform.

Furthermore, by Lemma B.1 of the Appendix we have that

$$
\int\limits_{-1}^{1} \frac{1}{D_j^{(1)}(t_1) f_{j_1}(t_1) t_2 - i\varepsilon}\, e^{it_2\dot{\gamma}_{j_2}(0)\cdot\tilde{x}}\, dt_2
$$

is uniformly bounded with respect to (t_1,ε) and $\tilde{x}$ from bounded sets and

$$
\lim_{\varepsilon\to 0} \int\limits_{-1}^{1} \frac{1}{D_j^{(1)}(t_1) f_{j_1}(t_1) t_2 - i\varepsilon}\, e^{it_2\dot{\gamma}_{j_2}(0)\cdot\tilde{x}}\, dt_2 \;=\; \frac{2\pi i}{|D_j^{(1)}(t_1)| f_{j_1}(t_1)}\, \Psi^{(\sigma_j^{(1)})}\big(\dot{\gamma}_{j_2}(0)\cdot\tilde{x}\big)
$$

where $\sigma_j^{(1)} = \operatorname{sign}\big[D_j^{(1)}(t_1)\big] = \operatorname{sign}\big[\dot{\gamma}_{j_2}(0)\cdot\nu_{j_1}(t_1)\big]$ (see (35) and note that $f_{j_1}(t_1) > 0$ by Corollary 1.8 and the orientation of Γ_{j_1}) and $\Psi^{(\pm)}$ is given by (18). We observe that $\sigma_j^{(1)}$ is independent of t_1. The (now one-dimensional) Lebesgue's theorem the first integral of (40) converges to

$$
2\pi i\, \Psi^{(\sigma_j^{(1)})}\big(\dot{\gamma}_{j_2}(0)\cdot\tilde{x}\big) \int\limits_{-\delta}^{\delta} \frac{1}{|D_j^{(1)}(t_1)|\, f_{j_1}(t_1)}\, \tilde{\psi}(t_1,0)\, g_{j_1}(t_1)\, e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}}\, \phi_{j_1}(x,t_1)\, dt_1\,.
$$

From the definitions of $\tilde{\psi}$ and $D_j^{(1)}(t_1)$ we note that $\tilde{\psi}(t_1, 0) = \psi_j(\gamma_{j_1}(t_1))|D_j^{(1)}(t_1)|$ and thus

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big[e^{-i\hat{\alpha}\cdot\tilde{x}}\int\limits_{V'}\frac{\tilde{\psi}(t)\,g_{j_1}(t_1)}{D_j^{(1)}(t_1)f_{j_1}(t_1)t_2 - i\varepsilon}\,e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}}\,e^{i\gamma_{j_2}(t_2)\cdot\tilde{x}}\,\phi_{j_1}(x,t_1)\,dt\Big] \\
&= \; 2\pi i\,\Psi^{(\sigma_j^{(1)})}\big(\dot{\gamma}_{j_2}(0)\cdot\tilde{x}\big)\int\limits_{-\delta}^{\delta}\frac{\psi_j(\gamma_{j_1}(t_1)\,g_{j_1}(t_1)}{f_{j_1}(t_1)}\,e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}}\,\phi_{j_1}(x,t_1)\,dt_1 \;+\; v_1(x)\,.
\end{aligned}$$

Analogously, we obtain for the second part of (34)

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\Big[e^{-i\hat{\alpha}\cdot\tilde{x}}\int\limits_{V}\frac{\tilde{\psi}(t)\,g_{j_2}(t_2)}{D_j^{(2)}(t_2)f_{j_2}(t_2)t_1 - i\varepsilon}\,e^{i\gamma_{j_2}(t_2)\cdot\tilde{x}}\,e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}}\,\phi_{j_2}(x,t_2)\,dt\Big] \\
&= \; 2\pi i\,\Psi^{(\sigma_j^{(2)})}\big(\dot{\gamma}_{j_1}(0)\cdot\tilde{x}\big)\int\limits_{-\delta}^{\delta}\frac{\psi_j(\gamma_{j_2}(t_2))\,g_{j_2}(t_2)}{f_{j_2}(t_2)}\,e^{i\gamma_{j_2}(t_2)\cdot\tilde{x}}\,\phi_{j_2}(x,t_2)\,dt_2 \;+\; v_2(x)
\end{aligned}$$

with $\sigma_j^{(2)} = \sigma_j^{(2)}(\tilde{x}) = \operatorname{sign}\big[D_j^{(2)}(t_2)\big]\operatorname{sign}\big[\gamma_{j_1}(0)\cdot\tilde{x}\big]$ and $v_2 \in H^1(W)$.
Now we add all of these terms and obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int\limits_{V}\psi_j(\alpha)\,u_\varepsilon(x,\alpha)\,e^{i\alpha\cdot\tilde{x}}\,d\alpha \\
&= \; 2\pi i\,\Psi^{(\sigma_j^{(1)})}\big(\dot{\gamma}_{j_2}(0)\cdot\tilde{x}\big)\int\limits_{-\delta}^{\delta}\frac{\psi_j(\gamma_{j_1}(t_1))\,g_{j_1}(t_1)}{f_{j_1}(t_1)}\,\phi_{j_1}(x,t_1)\,e^{i\gamma_{j_1}(t_1)\cdot\tilde{x}}\,dt_1 \\
&\quad + 2\pi i\,\Psi^{(\sigma_j^{(2)})}\big(\dot{\gamma}_{j_1}(0)\cdot\tilde{x}\big)\int\limits_{-\delta}^{\delta}\frac{\psi_j(\gamma_{j_2}(t_2))\,g_{j_2}(t_2)}{f_{j_2}(t_2)}\,\phi_{j_2}(x,t_2)\,e^{i\gamma_{j_2}(t_2)\cdot\tilde{x}}\,dt_2 \;+\; v(x)
\end{aligned}$$

where $v \in H^1(W)$ which coincides the form of u_S in Theorem 1.9 in the parametrized form and ends the proof of Theorem 1.9.

5. Proof of Theorem 1.12

We apply the following stationary phase result to the integrals of (22).

Lemma 5.1. *Let $G : [a,b]\times(0,\infty)\times[0,h]\to\mathbb{C}$ and $\varphi : [a,b]\to\mathbb{R}$ be smooth and $\varphi'(\hat{s}) = 0$ and $\varphi''(\hat{s}) \neq 0$ for some $\hat{s}\in(a,b)$. Furthermore, let φ' have no other zeros in $[a,b]$ other than $\hat{s}$ and let $G(s,r,x_3)$ and $\partial_s G(s,r,x_3)$ be bounded on $[a,b]\times(0,\infty)\times[0,h]$. Then*

$$\int\limits_a^b G(s,r,x_3)\,e^{ir\varphi(s)}\,ds \;=\; e^{ir\varphi(\hat{s})}\,G(\hat{s},r,x_3)\,e^{\sigma i\pi/4}\sqrt{\frac{2\pi}{r\,|\varphi''(\hat{s})|}} \;+\; \mathcal{O}\left(\frac{1}{r}\right) \quad \text{as } r\to\infty\,,$$

uniformly with respect to $x_3\in[0,h]$. Here, $\sigma = \operatorname{sign}[\varphi''(\hat{s})]$.

With the parametrization $\alpha = \gamma_\ell(s)$ of Γ_ℓ we note that the stationary points are given by $\{\alpha = \gamma_\ell(\hat{s}) : \dot{\gamma}_\ell(\hat{s})\cdot\hat{\theta} = 0\} = \mathcal{A}_+(\hat{\theta})\cup\mathcal{A}_-(\hat{\theta})$ with $\mathcal{A}_\pm(\hat{\theta})$ from (23).
We consider now the two integrals of (22).

(A) For the first integral fix $\ell \in \mathcal{L}_{arc}$ and decompose Γ_ℓ into finitely many closed arcs Γ (which intersect only at the endpoints) such that each of them contains at most one stationary point. If such an arc Γ has the parametrization $\alpha = \gamma_\ell(s)$, $s \in [a,b]$, then the first integral of (22), reduced to Γ, has the form

$$I(r,x_3) \;:=\; \int\limits_\Gamma \tilde{g}(\alpha)\,\Psi^{(+)}\big(r\,\nu_\ell(\alpha)\cdot\hat{\theta}\big)\,\phi_\ell(r\hat{\theta},x_3,\alpha)\,e^{ir\,\alpha\cdot\hat{\theta}}\,ds(\alpha) \;=\; \int\limits_a^b G(s,r,x_3)\,e^{ir\varphi(s)}\,ds$$

with smooth functions H and $\varphi(s) = \gamma_\ell(s)\cdot\hat{\theta}$.
If Γ contains no stationary point then $\varphi'(s) = \dot{\gamma}_\ell(s)\cdot\hat{\theta} \neq 0$ for all $s \in [a,b]$, and we can make the substitution $t = \varphi(s)$ with inverse $s = \varphi^{-1}(t)$. The integral $I(r,x_3)$ is then written as

$$I(r,x_3) \;=\; \int\limits_{\varphi(a)}^{\varphi(b)} G(\varphi^{-1}(t),r,x_3)\,\big(\varphi^{-1}\big)'(t)\,e^{irt}\,dt$$

and partial integration yields $I(r,x_3) = \mathcal{O}(1/r)$.
If Γ contains the stationary point $\hat{\alpha} = \gamma_\ell(\hat{s}) \in \mathcal{A}_+(\hat{\theta}) \cup \mathcal{A}_-(\hat{\theta})$ then we can apply Lemma 5.1 and have

$$\begin{aligned} I(r,x_3) \;&=\; e^{ir\varphi(\hat{s})}\,G(\hat{s},r,x_3)\,e^{\sigma i\pi/4}\sqrt{\frac{2\pi}{r\,|\varphi''(\hat{s})|}} \;+\; \mathcal{O}\left(\frac{1}{r}\right) \\ &=\; e^{ir\hat{\alpha}\cdot\hat{\theta}}\tilde{g}(\hat{\alpha})\,\Psi^{(+)}\big(r\,\nu_\ell(\hat{\alpha})\cdot\hat{\theta}\big)\,\phi_\ell(r\hat{\theta},x_3,\hat{\alpha})\,e^{\sigma i\pi/4}\sqrt{\frac{2\pi}{r\,|\kappa_\ell(\hat{\alpha})\cdot\hat{\theta})|}} \;+\; \mathcal{O}\left(\frac{1}{r}\right) \end{aligned}$$

as $r \to \infty$ uniformly with respect to $x_3 \in (0,h)$. We recall the definitions of $\Psi^{(+)}$ from (18). Therefore, if $\nu_\ell(\hat{\alpha}) = -\hat{\theta}$, i.e. $\hat{\alpha} \in \mathcal{A}_-(\hat{\theta})$, then $\Psi^{(+)}\big(r\,\nu_\ell(\hat{\alpha})\cdot\hat{\theta}\big) = \mathcal{O}(1/r)$, i.e. $I(r,x_3) = \mathcal{O}(1/r)$. If $\nu_\ell(\hat{\alpha}) = \hat{\theta}$, i.e. $\hat{\alpha} \in \mathcal{A}_+(\hat{\theta})$, then $\Psi^{(+)}\big(r\,\nu_\ell(\hat{\alpha})\cdot\hat{\theta}\big) = 1 + \mathcal{O}(1/r)$. With $|\kappa_\ell(\hat{\alpha})\cdot\hat{\theta})| = |\kappa_\ell(\hat{\alpha})|$ and the substitution of $\tilde{g}(\alpha)$ we obtain

$$I(r,x_3) \;=\; e^{ir\hat{\alpha}\cdot\hat{\theta}}\,\frac{\psi_\ell(\hat{\alpha})}{f_\ell(\hat{\alpha})}\,g_\ell(\hat{\alpha})\,\phi_\ell(r\hat{\theta},x_3,\hat{\alpha})\,e^{\sigma_\ell i\pi/4}\sqrt{\frac{2\pi}{r\,|\kappa_\ell(\hat{\alpha})|}} \;+\; \mathcal{O}\left(\frac{1}{r}\right)$$

as $r \to \infty$ uniformly with respect to $x_3 \in (0,h)$. Here, $\sigma_\ell = \mathrm{sign}[\kappa_\ell(\hat{\alpha})\cdot\hat{\theta})]$.
(B) We consider now the second integral of (22) and proceed in exactly the same way. If $\Gamma \subset \Gamma_{j_m}$ contains the stationary point $\hat{\alpha} = \gamma_{j_m}(\hat{s}) \in \mathcal{A}_+(\hat{\theta}) \cup \mathcal{A}_-(\hat{\theta})$ then we can apply Lemma 5.1 and obtain

$$\begin{aligned} &\int\limits_\Gamma \tilde{g}(\alpha)\,\phi_{j_m}(r\hat{\theta},x_3,\alpha)\,e^{ir\alpha\cdot\hat{\theta}}\,ds(\alpha) \\ &=\; e^{ir\hat{\alpha}\cdot\hat{\theta}}\,\frac{\psi_j(\hat{\alpha})}{f_{j_m}(\hat{\alpha})}\,g_{j_m}(\hat{\alpha})\,\phi_{j_m}(r\hat{\theta},x_3,\hat{\alpha})\,e^{\sigma_{j_m} i\pi/4}\sqrt{\frac{2\pi}{r\,|\kappa_{j_m}(\hat{\alpha})|}} \;+\; \mathcal{O}\left(\frac{1}{r}\right) \end{aligned}$$

as $r \to \infty$ uniformly with respect to $x_3 \in (0,h)$. Here, $\sigma_{j_m} = \mathrm{sign}[\kappa_{j_m}(\hat{\alpha})\cdot\hat{\theta})]$.
With respect to u_S we have to consider also $\Psi^{(\sigma_j^{(1)})}\big(\tau_{j_2}(\hat{\alpha}_j)\cdot\tilde{x}\big) = \Psi^{(\sigma_j^{(1)})}\big(r\,\tau_{j_2}(\hat{\alpha}_j)\cdot\hat{\theta}\big)$ and $\Psi^{(\sigma_j^{(2)})}\big(\tau_{j_1}(\hat{\alpha}_j)\cdot\tilde{x}\big) = \Psi^{(\sigma_j^{(2)})}\big(r\,\tau_{j_1}(\hat{\alpha}_j)\cdot\hat{\theta}\big)$ with $\sigma_j^{(m)}$ from (21).

Let us assume that $\hat\alpha \in \Gamma_{j_1}$. Then $\nu_{j_1}(\hat\alpha) = \pm\hat\theta$. If $\nu_{j_1}(\hat\alpha) = +\hat\theta$ then $\sigma_j^{(1)} = \operatorname{sign}[\tau_{j_2}(\hat\alpha)\cdot\hat\theta)]$ and $\Psi^{\sigma_j^{(1)}}(r\,\tau_{j_2}(\hat\alpha)\cdot\hat\theta) = 1 + \mathcal{O}(1/r)$ as $r \to \infty$. If $\nu_{j_1}(\hat\alpha) = -\hat\theta$ then $\sigma_j^{(1)} = -\operatorname{sign}[\tau_{j_2}(\hat\alpha)\cdot\hat\theta)]$ and $\Psi^{\sigma_j^{(1)}}(r\,\tau_{j_2}(\hat\alpha)\cdot\hat\theta) = \mathcal{O}(1/r)$ as $r \to \infty$.

Finally we note that $\psi_\ell(\hat\alpha) = 1$ if $\hat\alpha \in \Gamma_\ell \setminus \Gamma_m$ for all $m \neq \ell$. If $\hat\alpha \in \Gamma_\ell \cap \Gamma_m$ for some $m \neq \ell$ then $\psi_\ell(\hat\alpha) + \psi_m(\hat\alpha) = 1$. This ends the proof of Theorem 1.12.

Appendix A. A Simple Numerical Method for the Computation of Propagative Wave Numbers for a Special class of Non-constant Indices

Let, without loss of generality, $h = \pi$. We consider the class of trigonometric polynominals of the form $n = 1 + q$ where

$$q(x_1,x_2,x_3) \;=\; \sum_{j_1=-J_1}^{J_1}\sum_{j_2=-J_2}^{J_2} q_j\, e^{ij\cdot\tilde x}\,, \quad x = (x_1,x_2,x_3) \in \Omega = \mathbb{R}^2 \times (0,\pi)\,,$$

where again $\tilde x = (x_1,x_2) \in \mathbb{R}^2$ and $j = (j_1,j_2) \in \mathbb{Z}^2$. In particular $q(x)$ is constant with respect to x_3. We assume that $\overline{q_j} = q_{-j}$ for all j to make $q(x)$ real valued. We make the ansatz for the $\alpha-$quasi-periodic mode as

$$\phi(x)\, e^{i\alpha\cdot\tilde x} \;=\; \sum_{m_3=1}^{\infty}\sum_{\tilde m\in\mathbb{Z}^2} \phi_m\, e^{i(\tilde m+\alpha)\cdot\tilde x}\, \sin(m_3x_3)\,, \quad x \in \Omega\,,$$

where $\tilde m = (m_1,m_2)$. Substituting this ansatz into (10) yields the infinite system of equations

$$\left[-|\tilde m+\alpha|^2 - m_3^2 + \omega^2\right]\phi_m \;+\omega^2 \sum_{j_1=-J_1}^{J_1}\sum_{j_2=-J_2}^{J_2} q_j\,\phi_{\tilde m-j,m_3} \;=\; 0\,, \quad m \in \mathbb{Z}^2\times\mathbb{N}\,.$$

Truncating this system to $m \in \{m\in\mathbb{Z}^3 : |m_1| \le M_1,\ |m_2| \le M_2,\ 1 \le m_3 \le M_3\}$ and setting $q_j = 0$ for $j \notin \{-J_1,\dots,J_1\}\times\{-J_2,\dots,J_2\}$ leads to a system of $(2M_1+1)(2M_2+1)M_3$ equations and unknowns. For every $\alpha \in [-1/2,1/2]^2$ we compute the eigenvalue $\mu(\alpha)$ of the corresponding matrix which is closest to zero. The contour lines with $|\mu(\alpha)| \approx 0$ should approximate the set $\mathcal{A}$ of propagative wave numbers.

The following plots correspond to the particular example $q(x) = q_0\cos x_1 \sin x_2$ with amplitude $q_0 \in (0,1)$ for certain parameters (ω, q_0).

Appendix B. Elementary Results for an Integral and a Perturbed System of Linear Equations.

Lemma B.1. *For $f\in\mathbb{R}$, $f\neq 0$, and $r\in\mathbb{R}$ the integrals*

$$\int_{-1}^{1} \frac{1}{ft - i\varepsilon}\, e^{itr}\, dt$$

are bounded by $\frac{c}{|f|}(1+\ln(1+|r|))$ for all $\varepsilon > 0$ and $r\in\mathbb{R}$ where c is independent of f, ε, and r. Furthermore,

$$\lim_{\varepsilon\to 0}\int_{-1}^{1} \frac{1}{ft-i\varepsilon}\, e^{itr}\, dt \;=\; \frac{2\pi i}{|f|}\,\Psi^{(\sigma)}(r) \quad \textit{where} \quad \sigma = \operatorname{sign}(f)$$

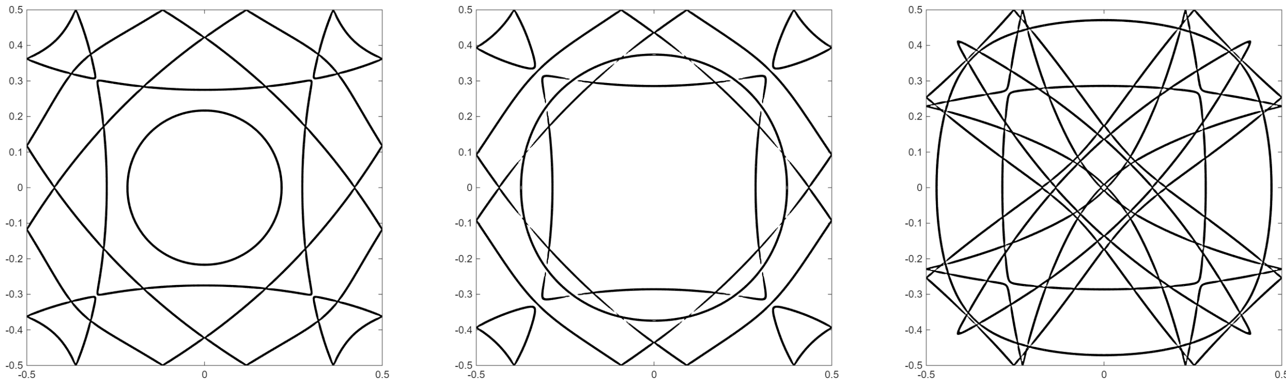

FIGURE 2. The set of propagative wave vectors for $h = \pi$ and $n = 1 + q$ and $M_1 = M_2 = M_3 = 3$ and $(\omega, q_0) = (2, 0.15)$ (left), $(\omega, q_0) = (2, 0.25)$ (center), and $(\omega, q_0) = (2.5, 0.15)$ (right), respectively.

and $\Psi^{(\pm)}(r)$ *is given by* (18), *i.e.* $\Psi^{(\pm)}(r) = \frac{1}{2} \pm \frac{1}{\pi}\int_0^r \frac{\sin\tau}{\tau}\,d\tau$ *for* $r \in \mathbb{R}$.

Proof. We compute

$$\begin{aligned}\int_{-1}^{1} \frac{1}{ft - i\varepsilon}\, e^{itr}\, dt \;&=\; \int_{-1}^{1} \frac{ft + i\varepsilon}{f^2t^2 + \varepsilon^2}\, e^{itr}\, dt \;=\; 2if\int_0^1 \frac{t\sin(tr)}{f^2t^2+\varepsilon^2}\, dt \;+\; 2i\varepsilon\int_0^1 \frac{\cos(tr)}{f^2t^2+\varepsilon^2}\, dt \\ &=\; 2if\int_0^1 \frac{t\sin(tr)}{f^2t^2+\varepsilon^2}\, dt \;+\; \frac{2i}{|f|}\int_0^{1|f|/\varepsilon} \frac{\cos(\varepsilon s/|f|)}{s^2+1}\, ds\end{aligned} \tag{41}$$

and thus

$$\begin{aligned}\left|\int_{-1}^{1} \frac{1}{ft - i\varepsilon}\, e^{itr}\, dt\right| \;&\le\; \frac{2}{|f|}\int_0^1 \frac{|\sin(tr)|}{t}\, dt \;+\; \frac{2}{|f|}\int_0^{\infty} \frac{1}{s^2+1}\, ds \;=\; \frac{2}{|f|}\int_0^{|r|} \frac{|\sin s|}{s}\, ds \;+\; \frac{\pi}{|f|} \\ &\le\; \frac{2}{|f|}\int_0^1 \frac{|\sin s|}{s}\, ds \;+\; \frac{2}{|f|}\int_1^{|r|} \frac{1}{s}\, ds \;+\; \frac{\pi}{|f|} \;\le\; \frac{c}{|f|}\,[1 + \ln|r|]\end{aligned}$$

for $|r| \ge 1$. To determine the limit as $\varepsilon \to 0$ we go back to (41) and obtain

$$\lim_{\varepsilon\to 0}\int_{-1}^{1} \frac{1}{ft - i\varepsilon}\, e^{itr}\, dt \;=\; \frac{2i}{f}\int_0^1 \frac{\sin(tr)}{t}\, dt \;+\; \frac{2i}{|f|}\int_0^{\infty} \frac{1}{s^2+1}\, ds\,.$$

In the first integral we substitute $\tau = t|r|$, the second integral is given by $\int_0^\infty \frac{1}{s^2+1}\, ds = \frac{\pi}{2}$. This yields the desired form. $\square$

Lemma B.2. *Let* $M(t_1, t_2) \in \mathbb{C}^{2\times 2}$ *be self-adjoint and smooth with respect to* $t_j \in (-\delta_0, \delta_0)$, $j = 1, 2$. *Furthermore, we assume that* $M(0) = 0 \in \mathbb{C}^{2\times 2}$ *and* $\partial_{t_j} M(0)$ *have the forms*

$$\partial_{t_1} M(0) = \begin{bmatrix} 0 & 0 \\ 0 & a_1 \end{bmatrix} \quad \text{and} \quad \partial_{t_2} M(0) = \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}$$

with $a_j \neq 0$. Then there exists $\delta \in (0,\delta_0)$ and smooth curves $\gamma^{(j)} : (-\delta,\delta) \to \mathbb{R}^2$ with $\gamma^{(j)}(0) = 0$ and $\dot\gamma^{(1)}(0) = \binom{1}{0}$ and $\dot\gamma^{(2)}(0) = \binom{0}{1}$ and $\det M(\gamma^{(j)}(s)) = 0$ for all $s \in (-\delta,\delta)$ and $j = 1,2$. Furthermore, for $j = 1,2$ the corresponding kernel of $M(\gamma^{(j)}(s))$ is one-dimensional for $s \neq 0$ and spanned by some $z^{(j)}(s) \in \mathbb{C}^2$ which depends smoothly on s with $z^{(1)}(0) = \binom{1}{0}$ and $z^{(2)}(0) = \binom{0}{1}$.

Proof. We use the asymptotic forms

$$\begin{aligned} M_{1,1}(t) &= t_2 a_2 + m_{1,1}(t)\,, \quad & M_{1,2}(t) &= m_{1,2}(t)\,, \\ M_{2,1}(t) &= m_{2,1}(t)\,, \quad & M_{2,2}(t) &= t_1 a_1 + m_{2,2}(t)\,, \end{aligned}$$

where $m_{j,\ell}(t)$ are smooth and $|m_{j,\ell}(t)| \leq c|t|^2$. Therefore, $\det M(t) = 0$ is equivalent to

$$[t_2 a_2 + m_{1,1}(t)]\,[t_1 a_1 + m_{2,2}(t)] - |m_{1,2}(t)|^2 = 0\,,$$

i.e.

$$t_1 t_2\, a_1 a_2 - m(t) = 0 \tag{42}$$

where $m(t)$ is smooth and $|m(t)| \leq |t|^3$. We search for a solution of the form $t_2 = t_1 \eta^{(1)}(t_1)$ for some function $\eta^{(1)} = \eta^{(1)}(t_1)$. Then (42) is equivalent to

$$a_1 a_2\, \eta^{(1)}(t_1) - \frac{1}{t_1^2}\, m\big(t_1, t_1\eta^{(1)}(t_1)\big) = 0 \tag{43}$$

for $t_1 \neq 0$. We define the function f by

$$f(t_1,\eta) := a_1 a_2\, \eta - \frac{1}{t_1^2}\, m(t_1, t_1\eta)\,, \ |\eta| < 1,\ |t_1| < 1\,.$$

We note that f smooth and $f(0,0) = 0$. Furthermore, $\partial_\eta f(t_1,\eta) = a_1 a_2 - \frac{1}{t_1}\,\partial_{t_2} m(t_1, t_1\eta)$ and thus $\partial_\eta f(0,0) = a_1 a_2 \neq 0$. The implicit function theorem implies the existence of a smooth function $\eta^{(1)} : (-\delta,\delta) \to \mathbb{R}$ with $\eta^{(1)}(0) = 0$ and $g(t_1, \eta^{(1)}(t_1)) = 0$, i.e. (43). Therefore, the vectors $t = \gamma^{(1)}(s) := \begin{pmatrix} s \\ s\,\eta^{(1)}(s) \end{pmatrix}$ satisfy $\gamma^{(1)}(0) = \binom{0}{0}$ and $\dot\gamma^{(1)}(0) = \binom{1}{0}$ and $\det M(\gamma^{(1)}(s)) = 0$ for $|s| \leq \delta$.

To compute the eigenvectors $z(s)$ we set $\tilde M(s) = M(\gamma^{(1)}(s))$ for simplicity and note that $\sqrt{\tilde M_{22}(s)^2 + |\tilde M_{21}(s)|^2} = |s|\sqrt{a_1^2 + \mathcal{O}(s^2)} \neq 0$ for small $|s| \neq 0$ because $a_1 \neq 0$. Therefore, the vector

$$z^{(1)}(s) := \frac{\operatorname{sign} s\ \operatorname{sign} a_1}{\sqrt{\tilde M_{22}(s)^2 + |\tilde M_{21}(s)|^2}} \begin{pmatrix} \tilde M_{22}(s) \\ -\tilde M_{21}(s) \end{pmatrix} = \frac{\operatorname{sign} a_1}{s\sqrt{\big(\frac{\tilde M_{22}(s)}{s}\big)^2 + \big|\frac{\tilde M_{21}(s)}{s}\big|^2}} \begin{pmatrix} \tilde M_{22}(s) \\ -\tilde M_{21}(s) \end{pmatrix}$$

is well defined and smooth for small $|s| \neq 0$, satisfies $\tilde M(s) z^{(1)}(s) = 0$, and it converges to $z^{(1)}(0) := \binom{1}{0}$ as $s \to 0$. This ends the construction of $\gamma^{(1)}(s)$ and $z^{(1)}(s)$.
For the construction of $\gamma^{(2)}(s)$ and $z^{(2)}(s)$ we proceed analogously by looking for solutions of (42) in the form $t_1 = t_2 \eta^{(2)}(t_2)$. ∎

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