

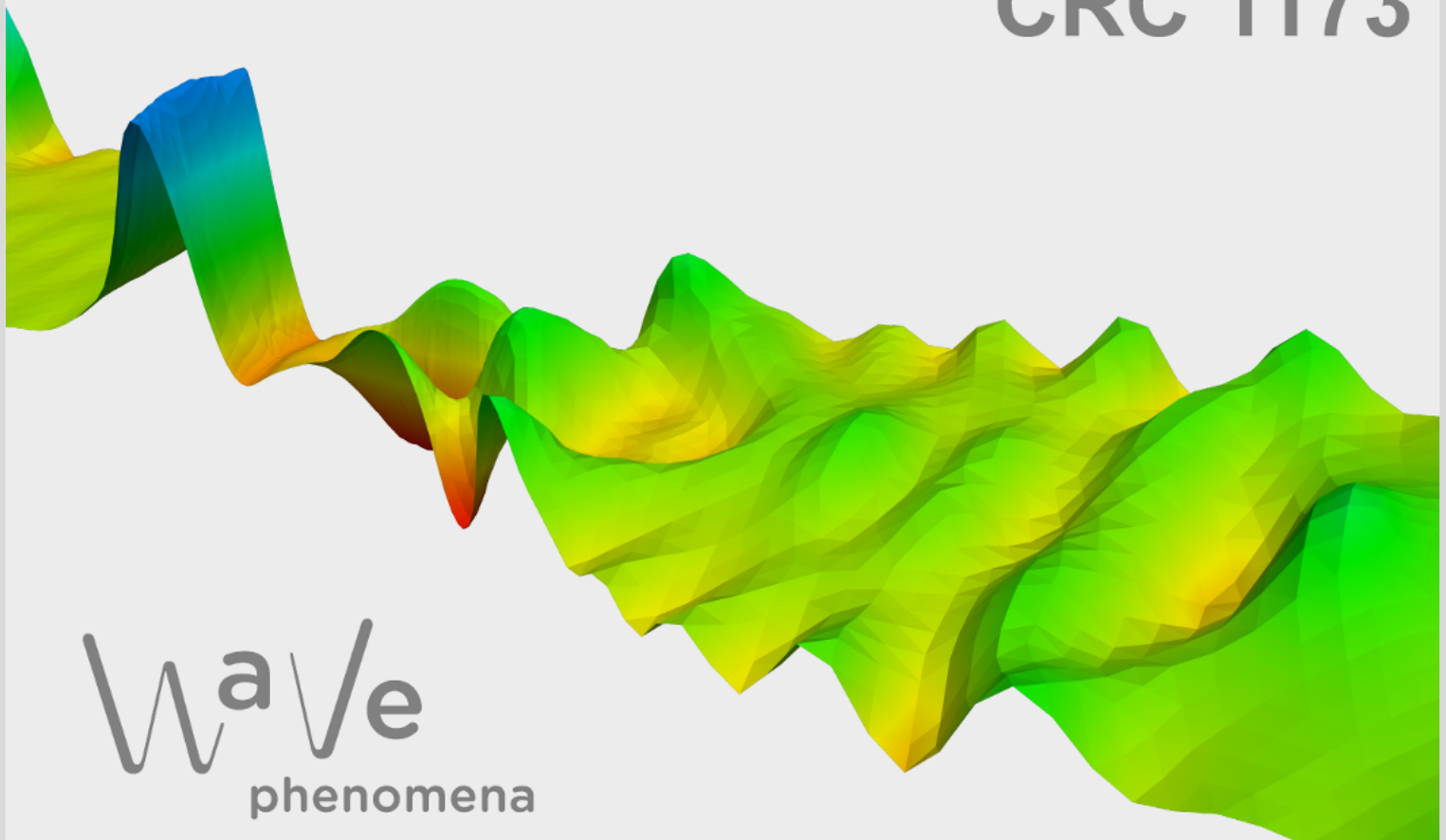
A blow-up approach for a priori bounds in semilinear planar elliptic systems: the Brezis–Merle critical case

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A blow-up approach for a priori bounds in semilinear planar elliptic systems: the Brezis-Merle critical case

Laura Baldelli*, Gabriele Mancini†, Giulio Romani‡

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Abstract

We establish uniform a priori estimates for solutions of semilinear planar Hamiltonian elliptic systems in a ball with Dirichlet boundary conditions. We consider a broad class of coupled nonlinearities with asymptotic critical behaviour in the sense of Brezis–Merle. The approach we follow is based on a blow-up analysis combined with Liouville–type theorems and integral estimates. Our results extend the scalar theory of uniform a priori bounds to the Hamiltonian case, and solve an open problem in [12]. We believe that this approach is new in this setting. As a consequence of our a priori estimates, we prove the existence of a positive solution by means of Fixed Point Index theory.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and consider the semilinear Hamiltonian elliptic system with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = f(v) & \text{in } \Omega, \\ -\Delta v = g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

It is well known that for $N \geq 3$ polynomial growth conditions at infinity have to be fulfilled in order to set these variational problems in Sobolev spaces and prove existence of solutions. If $N \geq 3$, for systems of the kind (1.1), the good notion of criticality is represented by a critical hyperbola: in the model case $f(v) = |v|^{p-1}v$ and $g(u) = |u|^{q-1}u$ with $p, q > 1$, then the critical hyperbola defined as

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N} \quad (1.2)$$

divides existence and nonexistence of classical positive solutions, see [19, 18, 27]. Note that if (1.1) is set in the entire $\mathbb{R}^N$ space, then it is proved that the Sobolev hyperbola (1.2) plays the same role for $N = 3, 4$, while it is still an open problem, under the name of Lane-Emden conjecture, for dimensions $N \geq 5$, see [28, 29, 39]. In contrast, much faster growths are admissible in dimension two: in particular, exponential nonlinearities can be treated, with the Trudinger-Moser inequality taking the role of the Sobolev embedding theorem valid for $N \geq 3$. For the system (1.1) we still can find a notion of critical

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hyperbola in the setting of Sobolev-Lorentz spaces: to give an idea, if $f(t) \sim e^{|t|^\alpha}$ and $g(t) \sim e^{|t|^\beta}$ with $0 < \alpha, \beta < +\infty$ (here the symbol $\sim$ means having the same asymptotic growth), then the maximal growth is obtained on the “conformal” critical hyperbola

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad (1.3)$$

see [36] and the more recent advance in [14].

When one treats equations or systems which are not necessarily variational (e.g. (1.1) when f and g depend on both variables u and v), a typical technique to prove the existence of solutions is the topological degree theory. An essential step in the proof is to provide a priori uniform bounds for solutions in L^∞ norm. In fact, from this one infers that a certain fixed point (or Leray-Schauder) index in a large ball of the functional space is zero; moreover, by imposing some natural assumptions on the nonlinearities at 0, the index in a neighbourhood of the origin is not zero, and therefore, by additivity of the index, one finds a nontrivial solution.

This strategy has been extensively employed in various contexts, and the study of L^∞ a priori bounds for solutions has become a topic of independent interest in the literature. For the scalar case, we refer to the classical papers by Gidas and Spruck [17] and de Figueiredo, Lions, and Nussbaum [11] for semilinear problems involving subcritical polynomial nonlinearities in dimension $N \geq 3$. Such results have been extended to the higher-order and quasilinear settings, and for wider classes of subcritical nonlinearities, see e.g. [32, 2, 35, 42, 3, 10]. In broad terms, these results indicate that a priori bounds for strong as well as distributional solutions can be established for (almost) all subcritical nonlinearities, up to the threshold of Sobolev critical growth. Concerning the system (1.1) with $N \geq 3$, the situation is comparable, in the sense that the critical hyperbola still represents the threshold for both existence and L^∞ a priori bounds, see among others [9, 41, 31, 29, 4].

The situation considerably changes in the case of the *conformal dimension*, where the critical threshold for the existence $t \mapsto e^{t^{\frac{N}{N-1}}}$ is given by the Trudinger-Moser inequality. Here, as the seminal work of Brezis-Merle [6] shows for the scalar second-order case in dimension $N = 2$, the limiting growth up to which one may expect to find a priori bounds within the class of *distributional* solutions of $-\Delta u = f(x, u)$ with Dirichlet boundary conditions is only $t \mapsto e^t$. Within this setting, in [6] the authors prove a priori bounds for nonlinearities bounded from below and above by e^t , under the assumption $\int_\Omega f(x, u) dx \leq \Lambda$; see also [8]. We note that such a condition follows from the analysis near the boundary already obtained in [11]. More in general, the respective quasilinear problem involving $-\Delta_N$ in $\Omega \subset \mathbb{R}^N$ with $N \geq 2$, was investigated in [24], where the nonlinearities behave either like e^{pt} for some $p > 0$, or grow less than e^{t^α} with $\alpha \in (0, 1)$. The authors use regularity estimates in Orlicz spaces to cover the latter case, while extend the arguments from [6] in the former. This diversity of the method causes a gap between the two classes of nonlinearities, and e.g. $f(t) = \frac{e^t}{t+1}$ could not be treated with such techniques. This gap was filled in [34], where instead a blow-up approach was used. The same technique has been also applied in [25] to the respective problem of higher-order, namely when the operator is $(-\Delta)^m$ in $\mathbb{R}^{2m}$, $m \geq 2$, in both cases of Dirichlet and Navier boundary conditions. In these works, the key assumption which allows to deal at once with both classes of nonlinearities in [24], is the existence of

$$\lim_{t \rightarrow +\infty} \frac{f'}{f}(t) = b \in [0, +\infty) \quad (1.4)$$

where the case $b > 0$ corresponds to the critical setting in the sense of Brezis and Merle, while $b = 0$ for all subcritical growths. We point out that with a blow-up approach one may further derive uniform bounds for *weak solutions* of Dirichlet problems with nonlinearity in the class $f(x, u) = u^{p-1}e^{u^p}$ even for $p \in (1, 2]$, that is, up to the critical Trudinger-Moser growth, under different integral bounds of energy type, see [16, Theorem 2.2] and [26, Theorem 2.1]; see also [33, 15] for related results.

When dealing with planar systems, a priori bounds have been investigated in [12] for systems of the kind (1.1) where f, g are positive nonlinearities (which may also depend on x), and then extended also

for nonvariational elliptic systems in [13]. First, by a moving plane techniques the authors find uniform bounds of the kind $\int_{\Omega} f(v) dx \leq \Lambda$ and $\int_{\Omega} g(u) dx \leq \Lambda$. Then, in order to apply either Orlicz spaces techniques or the argument inspired by [6], again as in [24], a dichotomy in the assumptions was needed: in fact, they impose that f and g grow either less than e^{t^α} and e^{t^β} , respectively, with $\alpha, \beta > 0$ and $\alpha + \beta < 2$, or both like e^t . In addition to the already recalled gap between the growths e^{t^α} and e^t , the limiting case

$$\alpha + \beta = 2 \quad \text{with} \quad \alpha \neq 1 \neq \beta \quad (1.5)$$

is stated as an open problem in [12, Remark 1.3].

The main purpose of this article is to progress with the study of a priori bounds for system (1.1) in the conformal setting $N = 2$, by covering this case of limiting nonlinearities (which from now on will be called *critical in the sense of Brezis-Merle*), by performing a blow-up analysis in the spirit of [34, 25]. For the model nonlinearities e^{t^α} and e^{t^β} , this will correspond to prove such bounds under (1.5). In the companion paper [5] we treat instead the subcritical case. We note indeed that such a case is not fully covered by [12, Theorem 1.3], since there are nonlinearities for which $b = 0$ but however grow more than any function of the kind e^{pt^α} irrespective of the choices of $p \geq 1$ and $\alpha \in (0, 1)$. For instance, it is sufficient to consider $g(t) = e^t$ and $f(t) = e^{\frac{t}{\log t}}$ so that $f'/f(t) = \frac{\log t - 1}{(\log t)^2} \sim \frac{1}{\log t} \rightarrow 0$, but slower than any function of the kind e^{pt^α} . Crucial steps in our methods will be first to identify the right assumption on f and g , which detects the “critical” nonlinearities and extends (1.4) for systems, and next to determine a suitable scaling, that allows to find a limit system in $\mathbb{R}^2$ to which one may apply a Liouville-type argument. During the blow-up analysis, several technical difficulties will arise due to the non-scalar nature of the problem. In fact, in the proof there will be a continuous exchange of information between the two components of the system. As a first contribution, here we restrict to consider the symmetric case of Ω being the ball $B_1(0)$: indeed, in this setting we can take advantage to the Schwarz symmetry of the solutions (u, v) , which implies that both components cannot have blow-up points except for the origin, and this simplifies to some extent our arguments. However, we believe that with a nontrivial effort one can also apply our strategy even in the case of a general bounded domain of $\mathbb{R}^2$, also in view of the L^1 -bounds already proved in [12].

We also mention that blow-up techniques for elliptic systems have been extensively employed in the framework of Toda-type systems, where, differently from (1.1), a strong algebraic structure and integrability properties allow a detailed analysis of bubbling solutions and mass quantization phenomena, see e.g. [20, 23, 22] and references therein. However, the techniques developed in that setting strongly exploit the specific Cartan matrix structure of the system and therefore do not appear to be directly transferable to the Hamiltonian systems studied here. To the best of our knowledge, we believe that the approach we follow here is new in the literature, and we hope it could serve for other kind of problems.

Let us now enter into the details of our results.

1.1 Assumptions and main results

Having in mind the model growths

$$f(t) \sim e^{t^\alpha}, \quad g(t) \sim e^{t^\beta} \quad \text{with} \quad \alpha > 0, \beta > 0 \quad (1.6)$$

with the critical condition

$$\alpha + \beta = 2,$$

we consider the following assumptions:

(H_1) $f, g \in C^1(\overline{\mathbb{R}^+})$, positive in $\mathbb{R}^+$, such that $\lim_{t \rightarrow +\infty} f(t) = +\infty$ and $\lim_{t \rightarrow +\infty} g(t) = +\infty$;

(H_2) there exist $t_0 > 0$ such that $f'(t) > 0$ and $g'(t) > 0$ for all $t \in (t_0, +\infty)$;

$$(H_3) \quad \left(\frac{f}{f'} \right)'(t) \rightarrow 0 \text{ and } \left(\frac{g'}{g} \right)'(t) \rightarrow 0 \text{ as } t \rightarrow +\infty;$$

$$(H_b) \quad \text{there exists } \lim_{t \rightarrow +\infty} \frac{f'}{f}(t) \frac{g'}{g}(t) := b \in (0, +\infty).$$

Note that f'/f and g'/g are the derivatives of the exponents of $f = e^{\log f}$ and $g = e^{\log g}$, respectively. The case $b = 0$ corresponds for the model nonlinearities in (1.6) to the case $\alpha + \beta < 2$, and therefore will be denoted as *Brezis-Merle subcritical*. On the other hand, $\alpha + \beta = 2$ implies $b = \alpha\beta > 0$, and we will call it *Brezis-Merle critical case*. In this paper we will only deal with the latter case, postponing the analysis of the subcritical one in the companion paper [5], since substantial differences in the respective blow-up arguments will occur.

In this critical setting, we will also need to distinguish between the case for which the limits of both ratios in (H_b) exist and are finite, from the remaining critical cases. We call the former scenario *Liouville*, which then occurs when there exist $p, q > 0$ such that

$$\lim_{t \rightarrow +\infty} \frac{f'}{f}(t) = p \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{g'}{g}(t) = q. \quad (1.7)$$

In the latter case, which we call *non-Liouville*, we may instead suppose

$$\lim_{t \rightarrow +\infty} \frac{f'}{f}(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{g'}{g}(t) = +\infty \quad (1.8)$$

without loss of generality, since the roles of f and g in (1.1) are symmetric. In this last context, we need some monotonicities and growth conditions on f and g , namely we assume that there exists $t_0 > 0$ such that

$$(H_4) \quad \left(\frac{f}{f'} \right)'(t) \geq 0 \text{ and } \left(\frac{g'}{g} \right)'(t) \geq 0 \text{ as } t > t_0;$$

$$(H_5) \quad \text{the map } t \mapsto t \frac{f'}{f}(t) \text{ is increasing and } t \mapsto \frac{1}{t} \frac{g'}{g}(t) \text{ is decreasing in } (t_0, +\infty).$$

Note that the existence of the two limits in (1.8) is a consequence of assumptions (H_b) and (H_4) . Finally, we suppose one of the following conditions on the growth of f : there exists $a > 1$ such that

$$(H_6) \quad t \frac{f'}{f}(t) \geq a(\log t)^2 \text{ for all } t > t_0;$$

$$(H'_6) \quad t \frac{f'}{f}(t) \geq a(\log \log F(t))^2 \text{ for all } t > t_0, \text{ where } F(t) := \int_0^t f(\tau) d\tau.$$

Before presenting the main result of the paper, let us briefly discuss the assumptions introduced above. First, observe that, except (H_1) , are all assumptions on the behaviour at ∞ . While (H_1) and (H_2) are standard conditions, (H_3) ensures subcriticality in the sense of Trudinger-Moser and that the growth of f and g are superpolynomial (see Lemma 2.7 below). The key assumption (H_b) , which characterises the problem as critical, can be regarded as the vectorial counterpart of (1.4), see (A3) in [34] or [25], where the scalar case is considered. In the non-Liouville setting, i.e., under condition (1.8), assumptions (H_4) and (H_5) provide suitable monotonicity properties, which are instrumental for our arguments. Note that the existence of the two limits in (1.8) is a consequence of the assumptions (H_b) and (H_4) . Finally, (H_6) and (H'_6) are technical assumptions which imply a mild lower bound on the growth of f , which has to be of the kind $e^{\log t (\log \log t)^p}$ (see Lemma 2.10 below). Note that, although (H_6) is of easier verification than (H'_6) , the latter permits to include a larger class of nonlinearities. A more detailed analysis of the consequences of our assumptions can be found in Section 2.

Since in the blow-up analysis we will need the radial symmetry of the solutions, as already mentioned in the Introduction, we will only deal with the case of Ω be a ball in $\mathbb{R}^2$. Our main result is the following:

Theorem 1.1. *Let $\Omega = B_1(0)$ be the unit ball in $\mathbb{R}^2$, assume conditions (H_1) , (H_2) , (H_3) , (H_b) , and alternatively*

- i) (1.7) holds;*
- ii) (1.8), (H_4) , (H_5) , and either (H_6) or (H'_6) hold.*

Then there exists a constant $C > 0$ such that

$$\|u\|_\infty \leq C \quad \text{and} \quad \|v\|_\infty \leq C$$

for all (possible) solutions (u, v) of system (1.1).

Note that, under such growth assumptions, one can easily show that f grows less than or comparable to e^{pt} for some $p > 0$ (see Lemma 2.4 below). Hence we may apply the regularity result [12, Theorem 1.1] and conclude that distributional solutions of (1.1) are indeed classical. Therefore, from now on we will simply talk about *solutions* of (1.1).

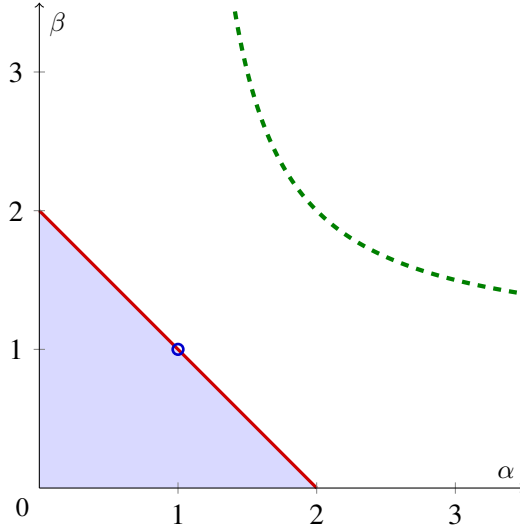


Figure 1: The state of the art on uniform a priori bounds for (1.1) for the model case (1.6).

In Figure 1 we can see the state of the art for problem (1.1) for the model nonlinearities in (1.6). In particular, a priori bounds in the blue area and in the point $(1, 1)$ were proved in [12], the red line is our contribution, while the dashed curve in green is the “conformal” critical hyperbola (1.3). This picture underlines that our case is largely subcritical in the sense of Trudinger-Moser. Note that we retrieve the scalar results in [24, 34] if we restrict on the diagonal of the graph. Therefore, in light of the counterexamples in [6, 25, 34], we expect that the region where now the a priori bounds for (1.1) applies is optimal for *distributional* solutions at least in the presence of singular potentials. However, it is still not clear – even in the scalar case – whether a priori bounds for *weak* solutions may be proved up to the critical growth in the sense of Trudinger-Moser.

Although reminiscent of some ideas developed in [25, 34] to deal with corresponding scalar problems, the method that we here present is new and completely different from the one in [12], from which we take only the a priori L^1 bounds in Proposition 2.3. Our blow-up technique is based on a suitable new scaling for system (1.1), and the main challenge is to precisely understand how to profit from the

exchange of information between the two components u and v in order to find contradictions with such L^1 -bounds. Let us try to briefly sketch the main line of our argument. We start in Section 3 by assuming by contradiction that there exists a sequence of solutions $(u_k, v_k)_k$ of (1.1) such that

$$\max\{\|u_k\|_\infty, \|v_k\|_\infty\} = +\infty.$$

First, we exclude the possibility that only one among $(\|u_k\|_\infty)_k$ and $(\|v_k\|_\infty)_k$ is unbounded and, exploiting the symmetric setting, we easily see that the origin is the unique blow-up point for both sequences.

The goal is to reach a contradiction either with the uniform L^1 -bounds on the nonlinear terms (see Proposition 2.3) or by applying a suitable Liouville-type theorem in the plane. In both cases, detecting a suitable scaling is essential: it produces $(\tilde{u}_k, \tilde{v}_k)_k$ defined on balls Ω_k with diverging radii, which formally locally converges to a solution of a Liouville's system on $\mathbb{R}^2$. In order to ensure this, we need that the scalings for both components are coupled by a compatibility condition, which we analyse in detail in Section 3.2.

Choosing to base the scaling on $\|u_k\|_\infty = u_k(0)$, so that $\tilde{u}_k(0) = 0$, our blow-up analysis in Section 4 then proceeds by distinguishing several cases according to the behavior of $\tilde{v}_k(0)$ as $k \rightarrow \infty$. If $\tilde{v}_k(0) \rightarrow -\infty$ as $k \rightarrow \infty$, then the Harnack inequality can be directly applied to $\tilde{u}_k$, which leads to a simple contradiction with the energy bounds. If $\tilde{v}_k(0) \rightarrow c \in \mathbb{R}$, then in the non-Liouville case (1.8) the boundedness of $\tilde{v}_k$ on compact sets yields a similar situation. In the Liouville case (1.7), we infer the contradiction from the analysis of the limit Liouville's system, relying on a characterisation result in [7]. Finally, the analysis becomes more technical when $\tilde{v}_k(0) \rightarrow +\infty$ and we need to find finer global estimates on $\tilde{u}_k$ and $\tilde{v}_k$, especially in the non-Liouville setting. Here the additional assumptions (H_6) and (H'_6) will permit to find the desired contradiction.

Under a standard superlinear growth condition on the nonlinearities in 0, in Section 5 we show that once the a priori bound is obtained, then the existence of a positive continuous solution (u, v) of (1.1) follows by means of Fixed Point index theory.

Theorem 1.2. *Besides the conditions under which Theorem 1.1 holds, assume moreover*

$$(H_9) \quad \limsup_{t \rightarrow 0^+} \frac{f(t)}{t} < \lambda_1 \quad \text{and} \quad \limsup_{t \rightarrow 0^+} \frac{g(t)}{t} < \lambda_1,$$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions. Then (1.1) admits a positive solution.

Notation. For $R > 0$ and $x_0 \in \mathbb{R}^N$ we denote by $B_R(x_0)$ the ball of radius R and center x_0 . The symbol $o_n(1)$ denotes a vanishing real sequence as $n \rightarrow +\infty$. Hereafter, the letter C will be used to denote positive constants which are independent of relevant quantities and whose value may change from line to line.

2 Preliminary results

In this section, we begin by recalling classical results, as well as some consequences of the assumptions, that will be used throughout the proof. Moreover, we collect some examples of families of nonlinearities, to which our results apply.

2.1 Elliptic regularity estimates

We begin by recalling classical local elliptic regularity estimates and a Harnack-type inequality, which will be frequently used in the sequel.

Lemma 2.1 ([37], Theorem 2). *Let u be a weak solution of $-\Delta u = h$ in $B_{2R}(0) \subset \Omega \subset \mathbb{R}^2$. Then,*

$$\|u\|_{C^{1,\alpha}(B_R(0))} \leq C(\|u\|_{L^2(B_{2R}(0))} + KR),$$

for a suitable $\alpha \in (0, 1)$ and positive constants $C(\alpha, R, \|h\|_{L^{\frac{2}{2-\varepsilon}}(\Omega)})$ and $K(R, \|h\|_{L^{\frac{2}{2-\varepsilon}}(\Omega)})$, for some $\varepsilon \in (0, 1)$.

Lemma 2.2 ([37], Theorem 6). *Let $u \geq 0$ be a solution of $-\Delta u = h$ in $B_{3R}(0) \subset \Omega \subset \mathbb{R}^2$. Then*

$$\max_{B_R(0)} u \leq C\left(\min_{B_R(0)} u + K\right),$$

where the constants C, K depending on R and on $\|h\|_{L^{\frac{2}{2-\varepsilon}}(\Omega)}$, $\varepsilon \in (0, 1)$.

Crucial for the blow-up analysis will be a uniform estimate on the L^1 norm of the nonlinear terms. This in fact holds for general domains since it follows from an application of the moving plane technique and the Kelvin transform.

Proposition 2.3 (Theorem 1.2, [12]). *Assume that there exists $c > 0$ and $p > 0$ such that $f(t) \leq ce^{pt}$ for all $t > 0$, or that the same holds for g . Then there exists a positive constant Λ , depending only on f , g , and Ω , such that*

$$\int_{\Omega} f(v) \leq \Lambda \quad \text{and} \quad \int_{\Omega} g(u) \leq \Lambda \quad (\Lambda)$$

for all solutions of (1.1).

Note that the assumption needed by Proposition 2.3 is satisfied under our hypothesis (1.7) or (1.8) since (H_b) implies that $f'/f(t) \leq \nu$ for $t > t_\nu$ for some $\nu > 0$ and $t_\nu > 0$, which in turn yields $f(t) \leq f(t_\nu)e^{\nu(t-t_\nu)}$ for $t > t_\nu$.

2.2 Consequences of the assumptions

First, we report examples of couples of nonlinearities which verify our assumptions.

Examples. 1. *The nonlinearities in (1.6), namely $f(t) = e^{t^\alpha}$, $g(t) = e^{t^\beta}$ with for $t > t_0 > 0$, $0 < \alpha \leq 1 \leq \beta < 2$ and the “critical tuning” $\alpha + \beta = 2$ are our model case. It is easy to verify assumptions (H_1) – (H_6) and (H_b) holds with $b = \alpha\beta$; it is also clear that mild modifications of such f, g in the spirit of [25, 34], namely of the kind $(\log t)^\tau t^p e^{t^\alpha}$ with $0 < \alpha < 2$, $p \geq 0$, and $\tau \in \mathbb{R}$, or $\frac{e^{\gamma t}}{t^q}$ with $\gamma > 0$ and $q \in \mathbb{R}$, also fulfill assumptions (H_1) – (H_6) and (H_b) .*

2. *A more general class of growths which is covered by our result is*

$$f(t) = e^{t^\alpha (\log t)^p} \quad \text{and} \quad g(t) = e^{\frac{t^\beta}{(\log t)^q}} \quad (2.1)$$

for $\alpha, \beta > 0$ and $\alpha + \beta = 2$, when one chooses $p = q \in \mathbb{R}$. In the case $p > 1$, this example also allows somehow to retrieve the borderline case $\alpha = 0$, $\beta = 2$. Indeed, $f(t) = e^{(\log t)^p}$ and $g(t) = e^{\frac{t^2}{(\log t)^q}}$ satisfy (H_b) if one modifies the “critical tuning” as $q = p + 1$ (in this case $b = 2p$). We note that (H_1) – (H_5) are verified for all $p > 1$. On the other hand, (H_6) is fulfilled in case $p > 3$ with $a > 1$, and if $p = 3$ with $a \in (1, 3]$; if, instead, $p \in (1, 3)$, then (H_6) does not hold anymore, but (H'_6) does.

3. *In the “double-borderline” case $\alpha = 0$ and $p = 1$ in (2.1), it is clear that the nonlinearity $f(t) = e^{\log t} = t$ is not allowed, but in principle we can keep going modifying it in order to obtain again admissible nonlinearities, e.g. $f(t) = e^{\log t (\log \log t)^p}$, which may be coupled with $g(t) = e^{\frac{t^2}{(\log \log t)^p}}$, so that (H_b) holds with $b = 2$. Here assumptions (H_1) – (H_5) are verified, while (H_6) never holds. However, the more relaxed assumption (H'_6) continues to hold, provided $p \geq 2$.*

Remark 1. We envision that the lower bound $p \geq 2$ in the last example is just a technical obstruction, and that our analysis can be pushed forward, at least up to $p > 1$, see the discussion in Remark 4 in Section 4.3.

From this point on, we present some properties which can be deduced by the conditions we impose on our nonlinearities. In what follows, (H_1) and (H_2) are always assumed.

Lemma 2.4. *Under (1.7) or (1.8), there exist $C_1, C_2 > 0$ and $\delta_1, \delta_2 > 0$ such that $f(s) < C_1 e^{\delta_1 s}$ and $g(s) > C_2 e^{\delta_2 s}$ for s large enough.*

Proof. By assumption, $\frac{g'}{g}(s) \rightarrow q \in \mathbb{R}^+ \cup \{+\infty\}$ as $s \rightarrow +\infty$. Hence, for $s > s_0$ large one has $\frac{g'}{g}(s) > \delta > 0$ for some $\delta \in (0, q)$. Integrating, one has $\log g(s) > \delta s + C_0$, from which the conclusion follows. The estimate for f follows analogously, since $\frac{f'}{f}(s) \rightarrow p \in \mathbb{R}^+ \cup \{0\}$ as $s \rightarrow +\infty$. $\square$

Lemma 2.5. *Under (1.8), there exists $s_1 > 0$ such that $f(s) < e^s < g(s)$ for all $s > s_1$.*

Proof. By de l'Hôpital's theorem and (1.8),

$$\frac{\log g(s)}{s} \rightarrow +\infty \quad \text{and} \quad \frac{\log f(s)}{s} \rightarrow 0$$

as $s \rightarrow +\infty$. Hence, letting $M > 0$, there exists $s_M > 0$ such that $\log g(s) > Ms$ and $\log f(s) < \frac{s}{M}$ for all $s > s_M$. Choosing $M = 1$ we therefore infer $\log f(s) < s < \log g(s)$ for all $s > s_1$, which proves the lemma. $\square$

Lemma 2.6. *Under (1.7) or (1.8), condition (H_3) implies*

$$i) \left(\frac{f'}{f} \right)'(t) \rightarrow 0 \quad \text{and} \quad \left(\frac{g'}{g} \right)'(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty;$$

$$ii) t \frac{f'}{f}(t) \rightarrow +\infty \quad \text{and} \quad t \frac{g'}{g}(t) \rightarrow +\infty \quad \text{as } t \rightarrow +\infty;$$

$$iii) \frac{1}{t} \frac{f'}{f}(t) \rightarrow 0 \quad \text{and} \quad \frac{1}{t} \frac{g'}{g}(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Proof. (i) Under (1.7) or (1.8), it is easy to see that

$$\left(\frac{f'}{f} \right)'(t) = - \left(\frac{f}{f'} \right)'(t) \left(\frac{f'}{f}(t) \right)^2 \rightarrow 0$$

and similarly

$$\left(\frac{g'}{g} \right)'(t) = - \left(\frac{g}{g'} \right)'(t) \left(\frac{g'}{g}(t) \right)^2 \rightarrow 0.$$

(ii)-(iii) If (1.8) is assumed, by de l'Hôpital's theorem, we compute

$$\lim_{t \rightarrow +\infty} \frac{f/f'(t)}{t} = \lim_{t \rightarrow +\infty} \left(\frac{f}{f'} \right)'(t) = 0.$$

Since $f/f' > 0$ by (H_1) and (H_2) , we get

$$\lim_{t \rightarrow +\infty} t \frac{f'}{f}(t) = \lim_{t \rightarrow +\infty} \frac{t}{f/f'(t)} = +\infty. \quad (2.2)$$

Similarly

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \frac{g'}{g}(t) = \lim_{t \rightarrow +\infty} \frac{g'/g(t)}{t} = \lim_{t \rightarrow +\infty} \left(\frac{g'}{g} \right)'(t) = 0.$$

All other conclusions are trivially true. $\square$

We next prove by (H_3) that both f and g are superpolynomial as well as subcritical in the sense of Trudinger-Moser. Note that the only assumption (H_b) would not have implied either of the two (e.g. $f(t) = t^p$ for $p > 1$ and $g(t) = e^{t^2}$ satisfy (H_b) with $b = 2p$).

Lemma 2.7. *Under (1.7) or (1.8), condition (H_3) implies that for all $p > 0$ one has*

$$\lim_{t \rightarrow +\infty} \frac{f(t)}{t^p} = +\infty = \lim_{t \rightarrow +\infty} \frac{g(t)}{t^p}$$

and, for all $\gamma > 0$,

$$\lim_{t \rightarrow +\infty} \frac{f(t)}{e^{\gamma t^2}} = 0 = \lim_{t \rightarrow +\infty} \frac{g(t)}{e^{\gamma t^2}}.$$

Proof. By Lemma 2.6 (iii), we know that

$$\lim_{t \rightarrow +\infty} \frac{\log f(t)}{t^2} = \lim_{t \rightarrow +\infty} \frac{f'(t)}{2tf(t)} = 0.$$

Therefore

$$\frac{f(t)}{e^{\gamma t^2}} = e^{\log(f(t)) - \gamma t^2} = e^{o(t^2) - \gamma t^2} \rightarrow 0$$

as $t \rightarrow +\infty$, for all $\gamma > 0$. Similarly, Lemma 2.6 (ii) yields

$$\lim_{t \rightarrow +\infty} \frac{\log f(t)}{\log t} = \lim_{t \rightarrow +\infty} \frac{tf'(t)}{f(t)} = +\infty,$$

so

$$\frac{f(t)}{t^p} = e^{\log f(t) - p \log t} = e^{\log f(t) + o(\log f(t))} \rightarrow +\infty$$

as $t \rightarrow +\infty$. Since (1.7) and Lemma 2.6(iii) are symmetric for f and g , the same proof holds also for g . $\square$

Lemma 2.8. *Under (1.7) or (1.8), (H_b) , and (H_3) , we have $\frac{f'(s)}{f(s)}g(s) \rightarrow +\infty$ as $s \rightarrow +\infty$.*

Proof. Under (1.7) the result is trivial. If instead (1.8) holds, using Lemmas 2.6(ii) and 2.7 we get

$$\frac{f'(s)}{f(s)}g(s) = s \frac{f'(s)}{f(s)} \cdot \frac{g(s)}{s} \rightarrow +\infty.$$

$\square$

Lemma 2.9. *Under (1.8) and (H_3) , the map $t \mapsto g(t)$ is invertible for t large, and its inverse function satisfies*

$$\lim_{s \rightarrow +\infty} \frac{g^{-1}(s)}{\sqrt{\log s}} = +\infty.$$

Proof. The invertibility of g follows by (H_2) . For $\alpha > 0$ by Lemma 2.7 there exists t_α such that $g(t) < e^{\alpha t^2}$ for all $t > t_\alpha$. This yields $t > \sqrt{\frac{1}{\alpha} \log g(t)}$, that is, for $s = g(t)$, $\frac{g^{-1}(s)}{\sqrt{\log s}} > \frac{1}{\sqrt{\alpha}}$. The conclusion follows by the arbitrariness of $\alpha > 0$. $\square$

To end this section, we show that from our technical conditions (H_6) and (H'_6) it is possible to deduce a bound from below on the growth of the function f . We will see that, although (H_6) is of easy verification, it excludes some nonlinearities that we would like to handle with our argument, and this is the reason why we introduced also the condition (H'_6) , which is a relaxed version of (H_6) and which gives us a much better bound from below of f . Henceforth, the symbol $\gtrsim$ is used when an inequality is true up to an omitted structural constant.

Lemma 2.10. *Suppose (1.8), then*

- i) *if (H_6) is assumed, then $f(t) \gtrsim e^{\frac{a}{3}(\log t)^3}$ for t large;*
- ii) *if (H'_6) is assumed, then $f(t) \gtrsim e^{\frac{a}{2} \log t (\log \log t)^2}$ for t large.*

Proof. i) Rewriting (H_6) as

$$\frac{f'}{f}(t) \geq a \frac{(\log t)^2}{t},$$

and integrating, one infers $\log f(t) \geq \frac{a}{3}(\log t)^3 + C(t_0)$, from which the estimate follows.

ii) One may rewrite (H'_6) as

$$\frac{f'}{f}(t) \geq \frac{a}{t} (\log \log F(t))^2.$$

Plainly it is $F(t) > t$ for large t , hence

$$(\log f)'(t) \geq \frac{a}{t} (\log \log t)^2,$$

which, integrated, gives

$$\begin{aligned} \log f(t) &\geq a \left((\log \log t)^2 - 2 \log \log t + 2 \right) \log t + C \\ &= a \log t (\log \log t)^2 (1 + o(1)) \geq \frac{a}{2} \log t (\log \log t)^2. \end{aligned}$$

□

Remark 2. This growth condition from below is somehow optimal for (H_6) -(H'_6) to hold. Indeed, e.g. if one considers nonlinearities of the kind $e^{a \log t (\log \log t)^p}$, then (H'_6) is satisfied if $p > 2$, or $p = 2$ and $a > 1$.

3 Scaling and compatibility condition

Since $\Omega = B_1(0)$, by [40, Theorem 1] we know that all solutions of (1.1) are radially symmetric and decreasing along the radial variable, fact that will be essential in our arguments.

We first show that, for a sequence $(u_k, v_k)_k$ of solutions of (1.1), either there exists a constant $C > 0$ such that $\|u_k\|_\infty, \|v_k\|_\infty \leq C$, or

$$\|u_k\|_\infty = u_k(0) \rightarrow +\infty \quad \text{and} \quad \|v_k\|_\infty = v_k(0) \rightarrow +\infty, \quad (3.1)$$

by excluding the possibility that only one between $\|u_k\|_\infty$ and $\|v_k\|_\infty$ is unbounded. Indeed, suppose there is a constant $C > 0$ such that $\|u_k\|_\infty \leq C$ and $\|v_k\|_\infty \rightarrow +\infty$ as $k \rightarrow +\infty$. By (H_1) , then $-\Delta v_k = g(u_k) \leq C$, so Harnack's inequality yields $\max v_k(0) \leq C$, a contradiction. The argument when we switch the roles of u_k and v_k , that is $\|u_k\|_\infty \rightarrow +\infty$ and $\|v_k\|_\infty \leq C$, is identical.

To prove Theorem 1.1, we then need to exclude the case (3.1). We will reason by contradiction, so we will henceforth assume that (3.1) holds.

3.1 The scaling

A key step in our argument is to identify a nice limit profile of a sequence of rescaled solutions, which formally satisfies a Liouville's equation in the plane. To this aim, crucial is to find a good scaling. Motivated by the usual one in the case of a pure exponential nonlinearity in the scalar case (i.e. $\tilde{u}_k(x) := u_k(x_k + \lambda_k x) - M_k$, where $u_k(x_k) = \|u_k\|_\infty =: M_k$ for some suitably defined $\lambda_k \rightarrow 0$ as $k \rightarrow +\infty$, see e.g. [21]), which can also be used to deal with nonlinearity with lower growth as shown in [25, 34], we look for an appropriate scaling of the form

$$\tilde{u}_k(x) := (u_k(x_k + \lambda_k x) - S_k) A_k \quad \text{and} \quad \tilde{v}_k(x) := (v_k(x_k + \lambda_k x) - T_k) B_k, \quad (3.2)$$

with $A_k, B_k, S_k, T_k, \lambda_k > 0$ and $x_k \in \Omega$. Let us then compute the system that $(\tilde{u}_k, \tilde{v}_k)$ solves. For $x \in \Omega_k := \lambda_k^{-1}(\Omega - x_k)$ one has

$$\begin{aligned} -\Delta \tilde{u}_k(x) &= A_k \lambda_k^2 f(v_k(x_k + \lambda_k x)) = A_k \lambda_k^2 f(T_k) e^{\log(f(v_k(x_k + \lambda_k x))) - \log(f(T_k))} \\ &= A_k \lambda_k^2 f(T_k) e^{f'/f(\eta_k(x))(v_k(x_k + \lambda_k x) - T_k)} = A_k \lambda_k^2 f(T_k) e^{\frac{1}{B_k} f'/f(\eta_k(x)) \tilde{v}_k(x)}. \end{aligned}$$

Note that in the second-to-last step we used Lagrange's theorem, and linearised $\log f$ around T_k as

$$\log f(v_k(x_k + \lambda_k x)) = \log f(T_k) + \frac{f'}{f}(\eta_k(x))(v_k(x_k + \lambda_k x) - T_k),$$

where

$$\eta_k(x) := T_k + \theta_k(x)(v_k(x_k + \lambda_k x) - T_k) = T_k \left(1 + \frac{\theta_k(x)}{T_k B_k} \tilde{v}_k(x) \right), \quad \theta_k(x) \in (0, 1). \quad (3.3)$$

Analogously for the second equation, one gets

$$-\Delta \tilde{v}_k(x) = B_k \lambda_k^2 g(S_k) e^{\frac{1}{A_k} g'/g(\zeta_k(x)) \tilde{u}_k(x)}, \quad x \in \Omega_k,$$

where

$$\zeta_k(x) := S_k + \bar{\theta}_k(x)(u_k(x_k + \lambda_k x) - S_k) = S_k \left(1 + \frac{\bar{\theta}_k(x)}{A_k S_k} \tilde{u}_k(x) \right), \quad \bar{\theta}_k(x) \in (0, 1). \quad (3.4)$$

We impose that the coefficients in front of the exponential terms are identically 1, namely

$$\lambda_k^2 = \frac{1}{A_k f(T_k)} = \frac{1}{B_k g(S_k)}. \quad (3.5)$$

We also notice that either f'/f or g'/g appears in the exponents. In order to exploit (H_b) , we also define

$$A_k := \frac{f(S_k)}{f'(S_k)} \quad \text{and} \quad B_k := \frac{g(T_k)}{g'(T_k)}. \quad (3.6)$$

With these choices one reaches the system

$$\begin{cases} -\Delta \tilde{u}_k = e^{f'/f(\eta_k(x))g'/g(T_k)\tilde{v}_k(x)} & \text{in } \Omega_k, \\ -\Delta \tilde{v}_k = e^{f'/f(S_k)g'/g(\zeta_k(x))\tilde{u}_k(x)} & \text{in } \Omega_k, \\ \tilde{u}_k = \tilde{v}_k = 0 & \text{on } \partial\Omega_k, \end{cases} \quad (3.7)$$

together with the *compatibility condition* which connects the parameters S_k and T_k

$$\frac{f'(S_k)}{f(S_k)} g(S_k) = \frac{g'(T_k)}{g(T_k)} f(T_k), \quad (3.8)$$

which follows by combining (3.5) and (3.6). Note that so far we still have the freedom to choose the points x_k and one of between S_k and T_k .

3.2 The compatibility condition

In order to better understand the relation (3.8) between S_k and T_k , let us denote by $J : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ the function given by

$$J(s, t) = \frac{f'}{f}(s) g(s) - \frac{g'}{g}(t) f(t) \quad (3.9)$$

with the aim of investigating its zero set. First, note that if $f \equiv g$, then the zero-set coincides with the bisector. In the model case (1.6), this happens when $\alpha = 1 = \beta$.

Next we verify the conditions of the implicit function theorem. We compute

$$\partial_t J(s, t) = - \left(\left(\frac{g'}{g} \right)'(t) f(t) + \frac{g'}{g}(t) f'(t) \right) = -f(t) \left(\left(\frac{g'}{g} \right)'(t) + \frac{g'}{g}(t) \frac{f'}{f}(t) \right) < 0 \quad (3.10)$$

independently of s , for t large by (H_1) , (H_3) and (H_b) . Then,

$$\lim_{t \rightarrow +\infty} J(s, t) = -\infty \quad \text{for all } s \text{ fixed,} \quad (3.11)$$

by (H_1) , and

$$\lim_{s \rightarrow +\infty} J(s, t) = +\infty \quad \text{for all } t \text{ fixed,} \quad (3.12)$$

by Lemma 2.8. Hence there exist $s_0, t_0 \in \mathbb{R}^+$ such that $\partial_t J(s, t) < 0$ for all $t > t_0$ and $s > 0$, and $J(s, t_0) > 0$ for all $s > s_0$. Therefore, one may apply the implicit function theorem in $(s_0, +\infty) \times (t_0, +\infty)$ and infer the existence of the implicit function $t = t(s)$ of class C^1 such that

$$J(s, t(s)) = 0 \quad \text{for all } s > s_0 \quad (3.13)$$

and that

$$t(s) \rightarrow +\infty \quad \text{as } s \rightarrow +\infty. \quad (3.14)$$

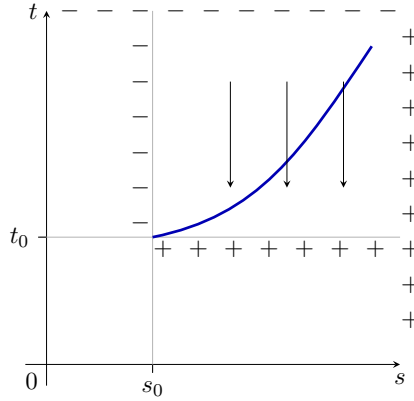


Figure 2: A qualitative sketch of the implicit function $t = t(s)$ in blue colour; the arrows indicate the growth of the function J from negative to positive.

Indeed, if by contradiction for a sequence $s_j \rightarrow +\infty$ there holds $t_j := t(s_j) \leq C$ for some positive constant C , then

$$0 = J(s_j, t(s_j)) = \frac{f'(s_j)}{f(s_j)} g(s_j) - \frac{g'(t_j)}{g(t_j)} f(t_j) \rightarrow +\infty$$

by Lemma 2.8 and (H_1) , a contradiction. Moreover, we know that

$$\begin{aligned} t'(s) &= -\frac{\partial_s J(s, t(s))}{\partial_t J(s, t(s))} = \frac{\left(\frac{f'}{f}\right)'(s)g(s) + \frac{f'}{f}(s)g'(s)}{\left(\frac{g'}{g}\right)'(t(s))f(t(s)) + \frac{g'}{g}(t(s))f'(t(s))} \\ &= \frac{g(s)}{f(t(s))} \cdot \frac{\left(\frac{f'}{f}\right)'(s) + \left(\frac{f'}{f} \frac{g'}{g}\right)(s)}{\left(\frac{g'}{g}\right)'(t(s)) + \left(\frac{f'}{f} \frac{g'}{g}\right)(t(s))}. \end{aligned}$$

Note that by (3.14), (H_3) , Lemma 2.6(i), and (H_b) , the second fraction converges to 1 as $s \rightarrow +\infty$ since $b > 0$. Hence,

$$f(t(s))t'(s) = g(s)(1 + o(1)), \quad (3.15)$$

which integrated implies that for any $\varepsilon > 0$ there exists $s_1(\varepsilon) > s_0$ so that for all $s > s_1(\varepsilon)$ then

$$F(t(s)) - F(t(s_1)) \begin{cases} < (1 + \varepsilon)(G(s) - G(s_1)) \\ > (1 - \varepsilon)(G(s) - G(s_1)) \end{cases}.$$

where $F(t) := \int_0^t f(\tau) d\tau$ and $G(t) := \int_0^t g(\tau) d\tau$. Noticing that F is invertible by (H_1) , we deduce an estimate on the behaviour of the implicit function at ∞ , namely, up to a different ε , that

$$F^{-1}((1 - \varepsilon)G(s)) < t(s) < F^{-1}((1 + \varepsilon)G(s)), \quad \text{for } s \text{ large.} \quad (3.16)$$

Actually, in case (1.8) holds, or (1.7) is satisfied with $p < q$, one can improve this by proving that

$$(1 - \varepsilon)F^{-1}(G(s)) < t(s) < (1 + \varepsilon)F^{-1}(G(s)) \quad (3.17)$$

for s large. To this end, define $\gamma(s) := F^{-1}(G(s))$, and we show that

$$\gamma'(s) = \frac{g(s)}{f(\gamma(s))} \quad \text{and} \quad \gamma(s) \geq s \quad \text{for } s \text{ large,} \quad (3.18)$$

so $\gamma(s) \rightarrow +\infty$ as $s \rightarrow +\infty$. The first is a direct computation, while the second follows by the fact that, if (1.8) holds or (1.7) is satisfied with $p < q$, then $g(s) > f(s)$ for large s by Lemma 2.5, which clearly implies $G(s) \geq F(s)$. Note, instead, that this is not anymore true when $p = q$, as the choice of $f(t) = e^{t+\varphi(t)}$ e $g(t) = e^{t-\varphi(t)}$ with $\varphi(t) = \frac{\sin t}{1+t}$ shows.

Letting $\varepsilon \in (0, 1)$ and using (H_b) , (H_3) , (H_2) and (3.18), for large s one has

$$\begin{aligned} \frac{d}{ds} J(s, (1 + \varepsilon)\gamma(s)) &= \partial_s J(s, (1 + \varepsilon)\gamma(s)) + (1 + \varepsilon)\gamma'(s)\partial_t J(s, (1 + \varepsilon)\gamma(s)) \\ &= g(s)(b + o(1)) - (1 + \varepsilon)\gamma'(s)f((1 + \varepsilon)\gamma(s))(b + o(1)) \\ &\leq g(s)(b + o(1)) - (1 + \varepsilon)\gamma'(s)f(\gamma(s))(b + o(1)) \\ &= g(s)(-\varepsilon b + o(1)) \rightarrow -\infty \end{aligned}$$

as $s \rightarrow +\infty$. This implies that $J(s, (1 + \varepsilon)\gamma(s)) < 0$ which, together with the fact that $t(s)$ is the unique zero of (3.9), then $t(s) < (1 + \varepsilon)\gamma(s)$ for s large. Analogously, one infers $t(s) > (1 - \varepsilon)\gamma(s)$, and (3.17) is proved. Combining (3.17) and (3.18) yields

$$t(s) \geq (1 - \varepsilon)s$$

for s large enough. Under (1.8) we can also improve this lower bound.

Lemma 3.1. Under (1.8), (H_1) - (H_3) and (H_b) , if $t = t(s)$ is given by (3.13), then

$$\lim_{s \rightarrow +\infty} \frac{t(s)}{s} = +\infty.$$

In particular, $t(s) \geq s$ for large s . Moreover,

$$\log F(t(s)) \cdot \frac{f'}{f}(s) \rightarrow +\infty \quad \text{as } s \rightarrow +\infty. \quad (3.19)$$

Proof. By de l'Hôpital's rule, (3.15), and (1.8) one computes

$$\begin{aligned} \lim_{s \rightarrow +\infty} \frac{t(s)}{s} &= \lim_{s \rightarrow +\infty} t'(s) = \lim_{s \rightarrow +\infty} \frac{g(s)}{f(t(s))} = \lim_{s \rightarrow +\infty} \frac{g'(s)}{f'(t(s))t'(s)} \\ &= \lim_{s \rightarrow +\infty} \frac{g'}{g}(s) \cdot \frac{f}{f'}(t(s))(1 + o(1)) = +\infty. \end{aligned}$$

The second conclusion is immediate from (3.16) and Lemma 2.6(ii). Indeed, fixing $\varepsilon > 0$, for s large, by (1.8), one has

$$\log F(t(s)) \cdot \frac{f'}{f}(s) \geq \log((1 - \varepsilon)G(s)) \cdot \frac{f'}{f}(s) \geq \log(1 - \varepsilon) \cdot \frac{f'}{f}(s) + s \frac{f'}{f}(s) \rightarrow +\infty,$$

having used Lemma 2.5 to have $\log G(s) > s$ for s large. $\square$

Lemma 3.2. In addition to the assumptions of Lemma 3.1, suppose that either (H_6) or (H'_6) hold. Then, there exists $\varepsilon > 0$ and $s_0 > 0$ such that

$$\frac{t(s)f'/f(t(s))}{(\log f/f'(s))^2} > \varepsilon > 0. \quad (3.20)$$

for all $s > s_0$.

Proof. By simple algebraic manipulations, (3.20) amounts to showing that

$$e^{\sqrt{\frac{1}{\varepsilon}t(s)f'/f(t(s))}} \frac{f'}{f}(s) > 1. \quad (3.21)$$

In case (H_6) is fulfilled, taking $\varepsilon = \frac{1}{a}$ and recalling Lemma 3.1, then

$$e^{\sqrt{\frac{1}{\varepsilon}t(s)f'/f(t(s))}} \frac{f'}{f}(s) > t(s) \frac{f'}{f}(s) > s \frac{f'}{f}(s) \rightarrow +\infty$$

by Lemma 2.6(ii), which proves (3.20). On the other hand, under (H'_6) , again taking $\varepsilon = \frac{1}{a}$,

$$e^{\sqrt{\frac{1}{\varepsilon}t(s)f'/f(t(s))}} \frac{f'}{f}(s) > \log F(t(s)) \cdot \frac{f'}{f}(s) \rightarrow +\infty$$

by Lemma 3.1. $\square$

Remark 3. Let us give some examples of behaviours at ∞ of the implicit function $t = t(s)$ found above, according to model choices of our nonlinearities.

- i) Let $f(t) = \alpha t^{\alpha-1} e^{t^\alpha}$ and $g(t) = \beta t^{\beta-1} e^{t^\beta}$, which satisfy our assumptions, and in particular (H_b) with $b = \alpha\beta$ when $\alpha, \beta \in (0, 2)$ and $\alpha + \beta = 2$. Then $F(t) = e^{t^\alpha}$ and $G(t) = e^{t^\beta}$, therefore $F^{-1}(t) = (\log t)^{\frac{1}{\alpha}}$. A simple computation gives

$$F^{-1}(G(s)) = \left(\log \left(e^{s^\beta} \right) \right)^{\frac{1}{\alpha}} = s^{\frac{\beta}{\alpha}},$$

therefore, (3.17) yields

$$(1 - \varepsilon) s^{\frac{\beta}{\alpha}} < t(s) < (1 + \varepsilon) s^{\frac{\beta}{\alpha}}.$$

Actually, arguing instead with (3.16), we can have a better asymptotic: indeed, by (3.16) one gets

$$(s^\beta - \varepsilon)^{\frac{1}{\alpha}} < t(s) < (s^\beta + \varepsilon)^{\frac{1}{\alpha}}.$$

and

$$(s^\beta - \varepsilon)^{\frac{1}{\alpha}} = s^{\frac{\beta}{\alpha}} \left(1 - \frac{\varepsilon}{s^\beta} \right)^{\frac{1}{\alpha}} = s^{\frac{\beta}{\alpha}} \left(1 - \frac{1}{\alpha} \frac{\varepsilon}{s^\beta} + o(\varepsilon) \right) = s^{\frac{\beta}{\alpha}} - \frac{\varepsilon}{\alpha} s^{\frac{\beta}{\alpha}(1-\alpha)} + o(\varepsilon),$$

and similarly for the right-hand side, so we also find the second term of the asymptotic expansion.

- ii) If f, g are such that $F(t) = e^{(\log t)^p}$ and $G(t) = e^{\frac{t^2}{(\log t)^q}}$ with $q = p + 1$ in order to satisfy (H_b) with $b = 2p$, with a similar argument one shows that the leading term of the map $s \mapsto t(s)$ is $e^{(s^2(\log s)^{-q})^{\frac{1}{p}}}$.

4 The blow-up argument: Proof of Theorem 1.1

By the radial symmetry, it is immediate to see that the only blow-up point for $(u_k)_k$ and $(v_k)_k$ is the origin. Indeed, if $\bar{x} \neq 0$, $\bar{x} \in B_1(0)$, such that $\exists \bar{x}_k \rightarrow \bar{x}$ with $u_k(\bar{x}_k) \rightarrow +\infty$ or $v_k(\bar{x}_k) \rightarrow +\infty$, since both u_k, v_k are radially decreasing about the origin, then $u_k \rightarrow +\infty$ or $v_k \rightarrow +\infty$, resp., uniformly on the whole ball $B_{|\bar{x}|}(0)$. This clearly contradicts the uniform L^1 bounds in (Λ) , since f and g are increasing by (H_2) . In light of this, the natural choice of the scaling parameters in (3.2) is $x_k = 0$ and

$$S_k := \|u_k\|_\infty = u_k(0) \rightarrow +\infty.$$

Then $T_k = t(S_k)$ where $t = t(s)$ is the implicit function defined by (3.13), and the parameters A_k, B_k , and λ_k are consequently fixed by (3.6) and (3.5). Note that

$$T_k \rightarrow +\infty \quad \text{and} \quad \lambda_k \rightarrow 0$$

by (3.14) and (3.5), respectively, under (H_1) in both settings (1.7) and (1.8). By the choice of S_k one immediately has

$$\tilde{u}_k \leq 0 \quad \text{in } \Omega_k \quad \text{and} \quad \tilde{u}_k(0) = 0.$$

Let us also define

$$\bar{v}_k(x) := \tilde{v}_k(x) - \tilde{v}_k(0) \leq 0 \tag{4.1}$$

since 0 is the maximum point of $\tilde{v}_k$, so $\bar{v}_k(0) = 0$ trivially. We have

$$-\Delta \bar{v}_k(x) = -\Delta \tilde{v}_k(x) = e^{f'/f(S_k)g'/g(\zeta_k(x))} \tilde{u}_k(x), \quad x \in \Omega_k. \tag{4.2}$$

Hence by (H_2) one infers $-\Delta \bar{v}_k \leq 1$ in Ω_k . Therefore one may apply the Harnack inequality in Lemma 2.2 with $\tilde{w}_k := -\bar{v}_k \geq 0$ in Ω_k and get

$$\max_{B_R(0)} |\bar{v}_k| = \max_{B_R(0)} \tilde{w}_k \leq C_R \left(\min_{B_R(0)} \tilde{w}_k + K(\|\Delta \tilde{w}_k\|_\infty, R) \right) \leq C_R.$$

By standard elliptic regularity estimates, this implies

$$\begin{aligned}\|\bar{v}_k\|_{H^2(B_{R/2}(0))} &\leq C_R (\|\bar{v}_k\|_{L^2(B_R(0))} + \|\Delta \tilde{v}_k\|_{L^2(B_R(0))}) \\ &\leq C_R (\|\bar{v}_k\|_{L^\infty(B_R(0))} + \|\Delta \bar{v}_k\|_{L^\infty(B_R(0))}) \leq C_R.\end{aligned}\quad (4.3)$$

Hence, by compact embedding, there exists $\bar{v} \in H_{loc}^2(\mathbb{R}^2)$ such that

$$\bar{v}_k \rightarrow \bar{v} \quad \text{uniformly on compact sets of } \mathbb{R}^2 \quad (4.4)$$

(which from now on will be shortened by the acronym *u.c.s.*) and in turn

$$\tilde{v}_k(x) = \tilde{v}_k(0) + \bar{v}_k(x) = \tilde{v}_k(0) + \bar{v}(x) + o_k(1) \quad \text{u.c.s..} \quad (4.5)$$

A comparable local compactness result cannot be obtained for $\tilde{u}_k$, since the behaviour of the right-hand side of

$$-\Delta \tilde{u}_k(x) = e^{f'/f(\eta_k(x))g'(T_k)} \tilde{v}_k(x)$$

does depend on the behaviour of $\tilde{v}_k$. Hence, in light of (4.5), (possibly passing to a subsequence), we distinguish three cases according to the behaviour of $\tilde{v}_k(0) \rightarrow \ell \in \{-\infty, c \in \mathbb{R}, +\infty\}$ as $k \rightarrow +\infty$.

4.1 Case $\tilde{v}_k(0) \rightarrow -\infty$

In this setting, by (4.5) one also has $\tilde{v}_k \rightarrow -\infty$ u.c.s., hence $-\Delta \tilde{u}_k \in [0, 1]$ u.c.s.. Recalling $\tilde{u}_k(0) = 0$ and $\tilde{u}_k \leq 0$, as before the Harnack inequality and the local elliptic regularity theory yield the existence of $\tilde{u} \in H_{loc}^2(\mathbb{R}^2)$ such that

$$\tilde{u}_k \rightarrow \tilde{u} \quad \text{u.c.s..} \quad (4.6)$$

Therefore, by Lemma 2.6 (iii),

$$\zeta_k(x) = S_k \left(1 + \underbrace{\bar{\theta}_k(x)}_{\in [0,1]} \underbrace{\frac{f'(S_k)}{f(S_k)S_k}}_{\rightarrow 0} \underbrace{\tilde{u}_k}_{\rightarrow \tilde{u}} \right) \sim S_k \quad \text{u.c.s..} \quad (4.7)$$

Moreover, we claim that

$$\frac{f'}{f}(S_k) \frac{g'}{g}(\zeta_k(x)) \sim \frac{f'(S_k)}{f(S_k)} \frac{g'(S_k)}{g(S_k)} \rightarrow b \quad \text{u.c.s.} \quad (4.8)$$

since $S_k \rightarrow +\infty$. This is clear if (1.7) is assumed, while in the case (1.8) one argues as follows. First, we have

$$\zeta_k(x) - S_k = \underbrace{\bar{\theta}_k(x)}_{\in [0,1]} \underbrace{\frac{f'(S_k)}{f(S_k)S_k}}_{\rightarrow 0} \underbrace{\tilde{u}_k}_{\rightarrow \tilde{u}} \rightarrow 0 \quad \text{u.c.s.,} \quad (4.9)$$

that is, $\zeta_k \rightarrow +\infty$ u.c.s.. Hence

$$\frac{g'}{g}(S_k) - \frac{g'}{g}(\zeta_k(x)) = \int_{\zeta_k(x)}^{S_k} \left(\frac{g'}{g} \right)'(\xi) d\xi \leq \sup_{[\zeta_k(x), S_k]} \left(\frac{g'}{g} \right)' \cdot (S_k - \zeta_k(x)) = o_k(1)$$

u.c.s., by (4.9), (H_3) and (H_4) . Therefore $g'/g(S_k) \rightarrow +\infty$ yields $g'/g(\zeta_k) \sim g'/g(S_k)$ u.c.s. and in turn (4.8). From this, (4.6), (4.8), (3.4)-(3.6), and (Λ) , after using Fatou's Lemma, one obtains

$$\begin{aligned}0 &< \int_{\mathbb{R}^2} e^{\tilde{b}\tilde{u}} dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega_k} e^{f'/f(S_k)g'/g(\zeta_k(x))\tilde{u}_k(x)} dx = \liminf_{k \rightarrow +\infty} \int_{\Omega_k} \frac{g(u_k(x_k + \lambda_k x))}{g(S_k)} dx \\ &= \liminf_{k \rightarrow +\infty} B_k \int_{\Omega_k} g(u_k(x_k + \lambda_k x)) \lambda_k^2 dx = \liminf_{k \rightarrow +\infty} \frac{g}{g'}(T_k) \int_{\Omega} g(u_k) dx \\ &\leq \Lambda \lim_{k \rightarrow +\infty} \frac{g}{g'}(T_k).\end{aligned}\quad (4.10)$$

The contradiction is immediately reached in the *non-Liouville* case (1.8) since $g'/g(T_k) \rightarrow +\infty$. On the other hand, if (1.7) is assumed, (4.10) yields

$$0 < \int_{\mathbb{R}^2} e^{b\tilde{u}} dx \leq \frac{\Lambda}{q}. \quad (4.11)$$

In this case, reasoning similarly as in (4.3), for a fixed $R > 0$ we obtain that $\|\tilde{u}_k\|_{W^{2,\bar{q}}(B_R(0))} \leq C_R$ for every $\bar{q} \in (0, 1)$, hence $\tilde{u}_k \rightarrow \tilde{u}$ in $C_{loc}^{1,\alpha}(\mathbb{R}^2)$ for all $\alpha \in (0, 1)$. Moreover, by (3.2) with our choices $x_k = 0$ and (3.6), we see that

$$\tilde{v}_k(x) - \tilde{v}_k(0) = \frac{g}{g'}(T_k) (v_k(\lambda_k x) - v_k(0)),$$

which, together with (4.4) and (1.7), implies

$$v_k(\lambda_k x) - v_k(0) \rightarrow q \bar{v}(x) \quad \text{u.c.s..}$$

Since $v_k(0) = \|v_k\|_\infty \rightarrow +\infty$, we infer that $v_k(\lambda_k x) \rightarrow +\infty$ u.c.s., and in turn that

$$\eta_k(x) = \theta_k(x)v_k(\lambda_k x) + (1 - \theta_k(x))T_k \rightarrow +\infty \quad \text{u.c.s..}$$

Hence $f'/f(\eta_k) \rightarrow p$ u.c.s. by (1.7) and so

$$-\Delta \tilde{u}_k = e^{f'/f(\eta_k(x))g'/g(T_k)} \tilde{v}_k \rightarrow 0 \quad \text{u.c.s..}$$

Since $\nabla \tilde{u}_k \rightarrow \nabla \tilde{u}$ u.c.s., this implies that $\tilde{u}$ is weakly harmonic in $\mathbb{R}^2$. Moreover, $\tilde{u}(0) = \lim_{k \rightarrow +\infty} \tilde{u}_k(0) = 0$ and $\tilde{u} \leq 0$, therefore, Liouville's theorem [38] implies $\tilde{u} \equiv 0$ in $\mathbb{R}^2$, which contradicts (4.11). We can therefore exclude that $\tilde{v}_k(0) \rightarrow -\infty$ u.c.s. and proceed with the other cases.

4.2 Case $\tilde{v}_k(0) \rightarrow c \in \mathbb{R}$

By (4.4) $\tilde{v}_k$ is bounded u.c.s. and (4.5) holds. From Lemma 2.6(iii), we then have

$$\eta_k(x) = T_k \left(1 + \underbrace{\theta_k(x)}_{\in[0,1]} \underbrace{\frac{g'(T_k)}{g(T_k)T_k}}_{\rightarrow 0} \underbrace{\tilde{v}_k}_{\leq C} \right) \sim T_k \quad \text{u.c.s.}$$

as $k \rightarrow +\infty$, and moreover,

$$\eta_k(x) - T_k = \theta_k(x) \frac{g'}{g}(T_k) \tilde{v}_k(x). \quad (4.12)$$

In order to show that $f'/f(\eta_k) \sim f'/f(T_k)$ u.c.s. as $k \rightarrow +\infty$, it is equivalent to show that $f/f'(\eta_k) \sim f/f'(T_k)$ u.c.s.. By (H_3) , (4.12), (H_b) , and (4.5), one has

$$\begin{aligned} \left| \frac{f/f'(\eta_k(x)) - f/f'(T_k)}{f/f'(T_k)} \right| &= \frac{f'}{f}(T_k) \left| \int_{T_k}^{\eta_k(x)} (f/f')'(\xi) d\xi \right| \\ &= \left(\sup_{[T_k, \eta_k(x)]} \left(\frac{f}{f'} \right)' \right) \theta_k(x) \frac{f'}{f}(T_k) \frac{g'}{g}(T_k) |\tilde{v}_k(x)| \\ &= o_k(1)(b + o_k(1))(\tilde{v}_k(0) + \bar{v}(x) + o_k(1)) = o_k(1). \end{aligned} \quad (4.13)$$

Therefore

$$-\Delta \tilde{u}_k(x) = e^{f'/f(T_k)(1+o_k(1))g'/g(T_k)} \tilde{v}_k(x) \leq C$$

which implies (4.6) and, in turn, (4.10) follows as in the previous section. Therefore, in the *non-Liouville* case (1.8) we get a contradiction. On the other hand, in the *Liouville case* (1.7), we only get (4.11). As in the previous section, $-\Delta \tilde{u}_k \rightarrow -\Delta \tilde{u}$, hence $-\Delta \tilde{u} = e^{b\tilde{v}}$ in $\mathbb{R}^2$. The same analysis can be carried out for the other equation in (3.7), so the limit $(\tilde{u}, \tilde{v})$ satisfies

$$\begin{cases} -\Delta \tilde{u} = e^{b\tilde{v}} & \text{in } \mathbb{R}^2, \\ -\Delta \tilde{v} = e^{b\tilde{u}} & \text{in } \mathbb{R}^2, \end{cases} \quad \text{with} \quad \begin{cases} \int_{\mathbb{R}^2} e^{b\tilde{u}} dx \leq \frac{\Lambda}{q}, \\ \int_{\mathbb{R}^2} e^{b\tilde{v}} dx \leq \frac{\Lambda}{p}. \end{cases} \quad (4.14)$$

Using the change of variables $U(x) := b\tilde{u}(\frac{x}{\sqrt{b}})$ and $V(x) := b\tilde{v}(\frac{x}{\sqrt{b}})$ it is easy to realise that (U, V) satisfies (4.14) with $b = 1$. Being now in the setting of [7], we can reduce the limit system to the scalar Liouville's equation. In fact, [7, Theorem 1.3] implies that $\int_{\mathbb{R}^2} e^U = 8\pi = \int_{\mathbb{R}^2} e^V$ and $U = V$ is the solution of

$$-\Delta U = e^U \quad \text{in } \mathbb{R}^2$$

given by

$$U(x) = 2 \log \left(\frac{2\sqrt{2}\lambda}{1 + \lambda^2|x|^2} \right)$$

for some $\lambda > 0$. Hence, coming back to $\tilde{u}$ and $\tilde{v}$, we infer

$$\tilde{u}(x) = \tilde{v}(x) = \frac{2}{b} \log \left(\frac{2\sqrt{2}\lambda}{1 + \lambda^2 b|x|^2} \right)$$

and, with similar computations as in (4.10),

$$\frac{8\pi}{b} = \int_{\mathbb{R}^2} e^{b\tilde{u}} dx = \lim_{R \rightarrow +\infty} \int_{B_R(0)} e^{b\tilde{u}} dx \leq \lim_{R \rightarrow +\infty} \liminf_{k \rightarrow +\infty} \frac{g}{g'}(T_k) \int_{B_{R\lambda_k}(0)} g(u_k(x)) dx,$$

hence, by (1.7), one infers

$$\lim_{R \rightarrow +\infty} \liminf_{k \rightarrow +\infty} \int_{B_{R\lambda_k}(0)} g(u_k(x)) dx \geq \frac{8\pi}{p}. \quad (4.15)$$

Since we have already observed that the only blow-up point for $(u_k)_k$ and $(v_k)_k$ is the origin, then $(u_k)_k, (v_k)_k$ are locally uniformly bounded in $\overline{B_1(0)} \setminus \{0\}$. Hence there exist u, v s.t. $u_k \rightarrow u$ and $v_k \rightarrow v$ in $L_{loc}^\tau(\overline{B_1(0)} \setminus \{0\})$, $\tau > 1$, and moreover, $(g_k) := (g(u_k))_k$ and $(f_k) := (f(v_k))_k$ are uniformly bounded in $L^1(B_1(0))$ by (Λ) . Hence there exist $\mu, \nu \in \mathcal{M}(B_1(0)) \cap L^\infty(\overline{B_1(0)} \setminus \{0\})$ such that $g_k \rightharpoonup \mu$ and $f_k \rightharpoonup \nu$ in measure, where $\mu, \nu \geq 0$ and singular at the origin, so we may write $\mu = A(x) dx + a_0 \delta_0$, $\nu = B(x) dx + b_0 \delta_0$ for some $a_0, b_0 \in \mathbb{R}^+$ and $A, B \geq 0$ in $L^1(B_1(0))$.

We first show that

$$\mu \geq \frac{8\pi}{p} \delta_0 \quad \text{and} \quad \nu \geq \frac{8\pi}{q} \delta_0. \quad (4.16)$$

Since $g_k \rightarrow \mu$ in measure in $B_1(0)$, for all $\varphi \in C_0^\infty(B_1(0))$ we have

$$\int_{B_1(0)} \varphi d\mu = \lim_{k \rightarrow +\infty} \int_{B_1(0)} \varphi g_k(x) dx = \lim_{k \rightarrow +\infty} \int_{B_1(0)} \varphi g(u_k(x)) dx. \quad (4.17)$$

Let now $t \in (0, 1)$ and choose $\varepsilon > 0$ so that $t(1 + \varepsilon) < 1$. Let $\varphi \in C_0^\infty(B_1(0))$ so that $\varphi \equiv 1$ on $B_t(0)$, $\text{supp } \varphi \subset B_{t(1+\varepsilon)}(0)$ and with values in $[0, 1]$. Then

$$\begin{aligned} \int_{B_1(0)} \varphi d\mu &= \int_{B_{t(1+\varepsilon)}(0)} \varphi d\mu = \int_{B_{t(1+\varepsilon)}(0) \setminus B_t(0)} \varphi A(x) dx + \int_{B_t(0)} d\mu \\ &\leq \int_{B_{t(1+\varepsilon)}(0) \setminus B_t(0)} A(x) dx + \int_{B_t(0)} d\mu. \end{aligned} \quad (4.18)$$

By (4.15) for a fixed $\delta > 0$, there exists $R_0 > 0$ such that for all $R \geq R_0$,

$$\frac{8\pi}{p} - \delta \leq \liminf_{k \rightarrow +\infty} \int_{B_{R\lambda_k}(0)} g(u_k) dx.$$

Fixing $R = R_0$, and since $\lambda_k \rightarrow 0$, for k large enough $R_0\lambda_k \leq t$, so $B_{R_0\lambda_k}(0) \subset B_t(0)$. Hence by (4.17)

$$\begin{aligned} \int_{B_{t(1+\varepsilon)}(0)} \varphi d\mu &= \lim_{k \rightarrow +\infty} \int_{B_{t(1+\varepsilon)}(0)} \varphi g(u_k) dx \geq \liminf_{k \rightarrow +\infty} \int_{B_t(0)} \varphi g(u_k) dx \\ &= \liminf_{k \rightarrow +\infty} \int_{B_t(0)} g(u_k) dx \geq \liminf_{k \rightarrow +\infty} \int_{B_{R_0\lambda_k}(0)} g(u_k(x)) dx \geq \frac{8\pi}{p} - \delta. \end{aligned}$$

By (4.18) we therefore obtain

$$\int_{B_{t(1+\varepsilon)}(0) \setminus B_t(0)} A(x) dx + \int_{B_t(0)} d\mu \geq \frac{8\pi}{p} - \delta.$$

Letting first $\delta \rightarrow 0$ and then $\varepsilon \rightarrow 0$, this yields

$$\int_{B_t(0)} d\mu \geq \frac{8\pi}{p},$$

which in turn, by the arbitrariness of $t \in (0, 1)$, implies $\mu \geq \frac{8\pi}{p}\delta_0$. We can argue similarly with ν and conclude that (4.16) holds. Let now Φ, Ψ be the distributional solutions of

$$\begin{cases} -\Delta\Phi = \mu & \text{in } B_1(0), \\ \Phi = 0 & \text{on } \partial B_1(0), \end{cases} \quad \text{and} \quad \begin{cases} -\Delta\Psi = \nu & \text{in } B_1(0), \\ \Psi = 0 & \text{on } \partial B_1(0). \end{cases}$$

Since

$$\begin{cases} -\Delta u_k = f(v_k) = f_k \rightarrow \nu & \text{in } B_1(0), \\ u_k = 0 & \text{on } \partial B_1(0), \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v_k = g(u_k) = g_k \rightarrow \mu & \text{in } B_1(0), \\ v_k = 0 & \text{on } \partial B_1(0), \end{cases}$$

then $u_k \rightarrow \Psi$ and $v_k \rightarrow \Phi$ in $W^{1,\sigma}(B_1(0))$ for $\sigma \in [1, 2)$ (see [30, Proposition 5.1]) and, by (4.16) and the maximum principle,

$$\Phi \geq \frac{8\pi}{p} G_B(\cdot, 0) = \frac{4}{p} \log \frac{1}{|\cdot|} \quad \text{and} \quad \Psi \geq \frac{4}{q} \log \frac{1}{|\cdot|}, \quad (4.19)$$

where G_B stands for the Green's function of $-\Delta$ with Dirichlet boundary conditions in the ball $B_1(0)$. By the symmetric role of f and g in (1.1), we may assume $p \leq q$, so we argue with Φ ; if instead $p > q$, one only needs to repeat the following argument using Ψ . By [25, Lemma 1.1] and (1.7), we know that for all $\varepsilon > 0$, there exist constants $C_\varepsilon, D_\varepsilon$ such that

$$D_\varepsilon e^{p(1-\varepsilon)t} - C_\varepsilon \leq f(t) \leq D_\varepsilon e^{p(1+\varepsilon)t} + C_\varepsilon. \quad (4.20)$$

On the one hand, by (4.19) we get

$$\int_{B_1(0)} e^{p(1-\varepsilon)\Phi} dx \geq \int_{B_1(0)} e^{4(1-\varepsilon) \log \frac{1}{|x|}} dx = 2\pi \int_0^1 \frac{d\rho}{\rho^{4(1-\varepsilon)-1}} = +\infty, \quad (4.21)$$

provided $\varepsilon > 0$ is small enough. On the other hand, by Fatou's lemma and (4.20), one has

$$\begin{aligned} \int_{B_1(0)} e^{p(1-\varepsilon)\Phi} dx &= \int_{B_1(0)} \lim_{k \rightarrow +\infty} e^{p(1-\varepsilon)v_k} dx \leq \liminf_{k \rightarrow +\infty} \int_{B_1(0)} e^{p(1-\varepsilon)v_k} dx \\ &\leq \frac{1}{D_\varepsilon} \liminf_{k \rightarrow +\infty} \int_{B_1(0)} f(v_k) dx + \frac{C_\varepsilon}{D_\varepsilon} |B_1(0)| \leq \frac{\Lambda + |B_1(0)|C_\varepsilon}{D_\varepsilon} < +\infty, \end{aligned}$$

which clearly contradicts (4.21), and eventually concludes the argument.

4.3 Case $\tilde{v}_k(0) \rightarrow +\infty$

This time the easiest setting is the Liouville. Indeed, supposing (1.7), then

$$\eta_k(x) = \underbrace{T_k}_{\rightarrow +\infty} + \underbrace{\theta_k(x)}_{\in [0,1]} \underbrace{\frac{g'}{g}(T_k)}_{\rightarrow q} \underbrace{\tilde{v}_k(x)}_{\rightarrow +\infty} \rightarrow +\infty \quad \text{u.c.s.},$$

so $f'/f(\eta_k(x)) \rightarrow p$ u.c.s.. This, together with $g'/g(T_k) \rightarrow q$, similarly to (4.10) implies

$$\frac{\Lambda}{p} \leftarrow \Lambda \frac{f}{f'}(S_k) \geq \int_{B_R(0)} e^{f'/f(\eta_k(x))g'/g(T_k)\tilde{v}_k(x)} dx \rightarrow +\infty,$$

for any $R > 0$ chosen arbitrarily, a contradiction. Suppose then from now on that (1.8) holds. Of course, also in this case we can rely on the inequality

$$\int_{B_R(0)} e^{f'/f(\eta_k(x))g'/g(T_k)\tilde{v}_k(x)} dx \leq \Lambda \frac{f}{f'}(S_k). \quad (4.22)$$

Here, our analysis branches in two subcases.

4.3.1 Subcase 1: $0 < \frac{\tilde{v}_k(0)}{T_k B_k} \leq C$ for some $C > 0$

In this case clearly $\eta_k(x) \geq T_k$ holds u.c.s. by (3.3). Aiming at finding a contradiction by means of (4.22), we need to bound from below the argument of the exponential. Using (H_5) , (4.1)-(4.4), the definition of η_k , and since $\tilde{v}_k(x) \leq \tilde{v}_k(0)$ because it is decreasing along the radius, we estimate

$$\begin{aligned} \frac{f'}{f}(\eta_k(x)) \frac{g'}{g}(T_k) \tilde{v}_k(x) &\geq \frac{f'}{f}(T_k) \frac{g'}{g}(T_k) \frac{T_k}{\eta_k(x)} \tilde{v}_k(x) \\ &\geq \frac{f'}{f}(T_k) \frac{g'}{g}(T_k) \frac{\tilde{v}_k(x)}{1 + \frac{\tilde{v}_k(x)}{T_k B_k}} \geq \frac{f'}{f}(T_k) \frac{g'}{g}(T_k) \frac{\tilde{v}_k(x)}{1 + \frac{\tilde{v}_k(0)}{T_k B_k}} \\ &= q_k \tilde{v}_k(x) = q_k \tilde{v}_k(0) + q_k \bar{v}_k(x), \end{aligned} \quad (4.23)$$

where q_k is defined as

$$q_k := \frac{f'}{f}(T_k) \frac{g'}{g}(T_k) \frac{1}{1 + \frac{\tilde{v}_k(0)}{T_k B_k}}. \quad (4.24)$$

Hence, from (4.22), since $\bar{v}_k \leq 0$ and $0 \leq q_k \leq b(1 + o_k(1))$,

$$\Lambda \frac{f}{f'}(S_k) \geq e^{q_k \tilde{v}_k(0)} \int_{B_R(0)} e^{q_k \bar{v}_k(x)} dx \geq C_R e^{q_k \tilde{v}_k(0)}, \quad (4.25)$$

where

$$C_R := |B_R(0)| e^{b(\min_{B_R(0)} \bar{v} - 1)}. \quad (4.26)$$

This yields an estimate from above for the growth of $\tilde{v}_k(0)$:

$$q_k \tilde{v}_k(0) \leq \log \left(\frac{\Lambda}{C_R} \right) + \log \left(\frac{f}{f'}(S_k) \right). \quad (4.27)$$

In order to reach a contradiction, we need to obtain an estimate from below for $\tilde{v}_k(0)$, and this is accomplished by means of the other energy estimate (4.10), that is

$$\int_{B_R(0)} e^{f'/f(S_k)g'/g(\zeta_k(x))\tilde{u}_k(x)} dx \leq \Lambda \frac{g}{g'}(T_k), \quad (4.28)$$

which is in terms of $\tilde{u}_k$, so we need to bound it from below. Exploiting the radial symmetry, and denoting by r the radial coordinate, since $\tilde{u}_k(0) = 0$ we have

$$\begin{aligned} 2\pi|\tilde{u}_k(r)| &= 2\pi(\tilde{u}_k(0) - \tilde{u}_k(r)) = -2\pi \int_0^r \tilde{u}'_k(s) \, ds = - \int_0^r \frac{|\partial B_s(0)|}{s} \partial_\nu \tilde{u}_k|_{\partial B_s(0)} \, ds \\ &= - \int_0^r \frac{1}{s} \left(\int_{\partial B_s(0)} \partial_\nu \tilde{u}_k \right) \, ds = \int_0^r \frac{1}{s} \left(\int_{B_s(0)} -\Delta \tilde{u}_k(x) \, dx \right) \, ds \\ &= \int_0^r \frac{1}{s} \left(\int_{B_s(0)} e^{f'/f(\eta_k(x))g'/g(T_k)} \tilde{v}_k(x) \, dx \right) \, ds. \end{aligned} \quad (4.29)$$

Since $\eta_k(x) \geq T_k$ u.c.s., it is easy to see by (H_4) that

$$\frac{f'}{f}(\eta_k(x)) \frac{g'}{g}(T_k) \tilde{v}_k(x) \leq \frac{f'}{f}(T_k) \frac{g'}{g}(T_k) \tilde{v}_k(x).$$

Defining

$$\gamma_k := \frac{f'}{f}(T_k) \frac{g'}{g}(T_k) \geq 0, \quad (4.30)$$

since $\bar{v}_k(x) \leq 0$, from (4.29) and (4.5), we infer

$$\begin{aligned} 2\pi|\tilde{u}_k(r)| &\leq \int_0^r \frac{1}{s} \left(\int_{B_s(0)} e^{\gamma_k \tilde{v}_k(x)} \, dx \right) \, ds = e^{\gamma_k \tilde{v}_k(0)} \int_0^r \frac{1}{s} \left(\int_{B_s(0)} e^{\gamma_k \bar{v}_k(x)} \, dx \right) \, ds \\ &\leq e^{\gamma_k \tilde{v}_k(0)} \int_0^r \frac{1}{s} |B_s(0)| \, ds = \frac{\pi r^2}{2} e^{\gamma_k \tilde{v}_k(0)}. \end{aligned}$$

By (4.25) this yields

$$-\tilde{u}_k(r) = |\tilde{u}_k(r)| \leq e^{\gamma_k \tilde{v}_k(0)} \frac{r^2}{4} \leq \left(\frac{\Lambda}{C_R} \frac{f}{f'}(S_k) \right)^{\frac{\gamma_k}{q_k}} \frac{r^2}{4}. \quad (4.31)$$

Note that, by (3.4) one has $\zeta_k(x) \leq S_k$ because $\tilde{u}_k \leq 0$. Hence, it would be nice to apply (H_4) and get $\frac{g'}{g}(\zeta_k(x)) \leq \frac{g'}{g}(S_k)$, so that, combining this with the estimate from below (4.31) (remember that $\tilde{u}_k \leq 0$), we would manage to bound from below the left-hand side of (4.28). However, (H_4) holds only for large s , and so one is first compelled to show that

$$\zeta_k(x) \rightarrow +\infty \quad \text{u.c.s.} \quad (4.32)$$

as $k \rightarrow +\infty$. We claim in fact that for any compact set $K \subset \mathbb{R}^2$ there exists $C_1 = C_1(K) > 0$ such that

$$\frac{\zeta_k(x)}{S_k} \geq C_1 \quad \text{for all } x \in K \text{ and } k \text{ large.}$$

Let us proceed by contradiction, and assume that, fixing an arbitrary $R > 0$, one finds $(\xi_k)_k \subset B_R(0)$ such that $\frac{\zeta_k(\xi_k)}{S_k} \rightarrow 0$ as $k \rightarrow +\infty$. By (3.4), this means that

$$(1 - \bar{\theta}_k(\xi_k)) + \bar{\theta}_k(\xi_k) \frac{u_k(\lambda_k \xi_k)}{S_k} \rightarrow 0.$$

Since both quantities are positive, necessarily $\bar{\theta}_k(\xi_k) \rightarrow 1$. Then, again by (3.4), one infers

$$|\tilde{u}_k(\xi_k)| = -\tilde{u}_k(\xi_k) = (1 + o_k(1)) S_k \frac{f}{f'}(S_k).$$

This, together with (4.31) for $r = |\xi_k| \leq R$, and (4.24) and (4.30), yields

$$S_k \frac{f}{f'}(S_k) \leq \frac{R^2}{4} \left(\frac{\Lambda}{C_R} \frac{f}{f'}(S_k) \right)^{1 + \frac{\tilde{v}_k(0)}{T_k B_k}} \leq \bar{C}_R \left(\frac{f}{f'}(S_k) \right)^{1 + \frac{\tilde{v}_k(0)}{T_k B_k}}$$

for some $\bar{C}_R > 0$, since we are in Subcase 1. Hence, passing to the logarithms, and noticing that $B_k = g/g'(T_k) = b^{-1}(1 + o_k(1))f'/f(T_k)$ by (H_b) for large k , one infers

$$\tilde{v}_k(0) \geq \frac{1 + o_k(1)}{b} T_k \frac{f'}{f}(T_k) \left(\log \left(\frac{f}{f'}(S_k) \right) \right)^{-1} (\log S_k - \log \bar{C}_R).$$

Combining this with (4.27) and (H_b) , yields

$$\begin{aligned} 2 \log \left(\frac{f}{f'}(S_k) \right) &\geq \log \left(\frac{\Lambda}{C_R} \right) + \log \left(\frac{f}{f'}(S_k) \right) \geq q_k \tilde{v}_k(0) \\ &\geq \frac{b(1 + o_k(1))}{1 + \frac{\tilde{v}_k(0)}{T_k B_k}} \tilde{v}_k(0) \geq \frac{b}{2(1 + C)} \tilde{v}_k(0) \\ &\geq \frac{1}{4(1 + C)} T_k \frac{f'}{f}(T_k) \left(\log \left(\frac{f}{f'}(S_k) \right) \right)^{-1} \log S_k, \end{aligned}$$

that is,

$$8(1 + C) \geq \frac{T_k f'/f(T_k)}{(\log f/f'(S_k))^2} \log S_k.$$

This is however in contradiction with Lemma 3.2, from which the ratio on the right-hand side is bounded away from zero. Thus, (4.32) is proved, and we can proceed with the argument which exploits (4.28). Indeed, by (H_4) now we obtain $\frac{g'}{g}(\zeta_k(x)) \leq \frac{g'}{g}(S_k)$, and in turn $\frac{g'}{g}(\zeta_k(x))\tilde{u}_k(x) \geq \frac{g'}{g}(S_k)\tilde{u}_k(x)$. Hence, from (4.28) using (4.31) one gets

$$\begin{aligned} \Lambda \frac{g}{g'}(T_k) &\geq \int_{B_R(0)} e^{\frac{f'}{f}(S_k) \frac{g'}{g}(S_k) \tilde{u}_k(x)} dx \geq \int_{B_R(0)} e^{-\frac{f'}{f}(S_k) \frac{g'}{g}(S_k) e^{\gamma_k \tilde{v}_k(0)} \frac{|x|^2}{4}} dx \\ &=: \int_{B_R(0)} e^{-E_k |x|^2} dx = \frac{\pi}{E_k} (1 - e^{-R^2 E_k}) = \frac{\pi}{E_k} (1 + o_k(1)) \geq \frac{\pi}{2E_k}, \end{aligned}$$

because

$$E_k := \frac{1}{4} \frac{f'}{f}(S_k) \frac{g'}{g}(S_k) e^{f'/f(T_k) g'/g(T_k) \tilde{v}_k(0)} \rightarrow +\infty$$

by (H_b) and $\tilde{v}_k(0) \rightarrow +\infty$. Therefore, by (4.25)

$$\begin{aligned} \frac{2\Lambda}{\pi} &\geq \frac{g'}{g}(T_k) E_k^{-1} = \frac{g'}{g}(T_k) \frac{4}{f'/f(S_k) g'/g(S_k)} e^{-\frac{\gamma_k}{q_k} \cdot q_k \tilde{v}_k(0)} \\ &\geq \frac{2}{b} \frac{g'}{g}(T_k) \left(\frac{\Lambda}{C_R} \frac{f}{f'}(S_k) \right)^{-\frac{\gamma_k}{q_k}} = \frac{2}{b} \left(\frac{\Lambda}{C_R} \right)^{-\frac{\gamma_k}{q_k}} \left(\frac{f'}{f}(S_k) \right)^{\frac{\gamma_k}{q_k}} \frac{g'}{g}(T_k), \end{aligned}$$

which can be rewritten as

$$\frac{b\Lambda}{\pi} \left(\frac{\Lambda}{C_R} \right)^{\frac{\gamma_k}{q_k}} \geq \left(\frac{f'}{f}(S_k) \right)^{\frac{\gamma_k}{q_k} - 1} \frac{f'}{f}(S_k) \frac{g'}{g}(T_k) \geq b(1 + o_k(1)) \left(\frac{f'}{f}(S_k) \right)^{\frac{\gamma_k}{q_k} - 1} \quad (4.33)$$

by (H_4) and Lemma 3.1. Notice that the exponent

$$\frac{\gamma_k}{q_k} - 1 = \frac{\tilde{v}_k(0)}{T_k B_k} > 0$$

by (4.24) and (4.30), and that $f'/f(S_k) \rightarrow 0$, therefore the behaviour of the right-hand side of (4.33) is different according to the following two alternatives:

$$\frac{\tilde{v}_k(0)}{T_k B_k} \log \left(\frac{f'}{f}(S_k) \right) \rightarrow \begin{cases} c \leq 0 & \text{(I),} \\ -\infty & \text{(II),} \end{cases} \quad (4.34)$$

possibly up to a subsequence.

If (4.34-I) holds, then

$$\left(\frac{f'}{f}(S_k) \right)^{\frac{\gamma_k}{q_k} - 1} = e^{(\frac{\gamma_k}{q_k} - 1) \log \left(\frac{f'}{f}(S_k) \right)} = e^{\frac{\tilde{v}_k(0)}{T_k B_k} \log \left(\frac{f'}{f}(S_k) \right)} \rightarrow C \quad (4.35)$$

as $k \rightarrow +\infty$. Hence, from (4.33) and (H_b) , (H_4) and Lemma 3.1,

$$\frac{b\Lambda}{\pi} \left(\frac{\Lambda}{C_R} \right)^{\frac{\gamma_k}{q_k}} \geq \left(\frac{f'}{f}(S_k) \right)^{\frac{\gamma_k}{q_k} - 1} \frac{f'}{f}(S_k) \frac{g'}{g}(S_k) \rightarrow Cb > 0, \quad (4.36)$$

so in order to reach a contradiction, we need that the constant C_R , defined in (4.26) and appearing on the left-hand side, diverges to $+\infty$ as $R \rightarrow +\infty$. Note, indeed, that so far the radius R was arbitrary. Since C_R depends on $\bar{v}$, which satisfies (4.2), we need to investigate first the global bounds of $\tilde{u}_k$. By (4.23) and (4.29), for $x \in \Omega_k$ and $r = |x|$,

$$\begin{aligned} -2\pi \tilde{u}_k(x) &= \int_0^r \frac{1}{s} \left(\int_{B_s(0)} e^{f'/f(\eta_k(x))g'/g(T_k)\tilde{v}_k(x)} dx \right) ds \\ &\geq e^{q_k \tilde{v}_k(0)} \int_0^r \frac{1}{s} \left(\int_{B_s(0)} e^{q_k \bar{v}_k(x)} dx \right) ds. \end{aligned} \quad (4.37)$$

Let us fix $R_0 > 0$. If $s \geq R_0$, then

$$\int_{B_s(0)} e^{q_k \bar{v}_k(x)} dx \geq \int_{B_{R_0}(0)} e^{q_k \bar{v}_k(x)} dx \geq \int_{B_{R_0}(0)} e^{b\bar{v}_k(x)} dx \rightarrow \int_{B_{R_0}(0)} e^{b\bar{v}(x)} dx =: M_0 > 0 \quad (4.38)$$

by (4.4), by recalling that $q_k \in (0, b)$ and $\bar{v}_k \leq 0$. On the other hand, if $s < R_0$ then

$$\begin{aligned} \int_{B_s(0)} e^{q_k \bar{v}_k(x)} dx &\geq \int_{B_s(0)} \left(e^{b\bar{v}_k(x)} - e^{b\bar{v}(x)} \right) dx + \int_{B_s(0)} e^{b\bar{v}(x)} dx \\ &\geq - \int_{B_s(0)} \max_{x \in B_{R_0}(0)} \left| e^{b\bar{v}_k(x)} - e^{b\bar{v}(x)} \right| dx + \int_{B_s(0)} e^{b\bar{v}(x)} dx \\ &\geq \left(-M_{k,R_0} + e^{b \min_{B_{R_0}(0)} \bar{v}} \right) \pi s^2 \geq \varepsilon_0 \pi s^2 \end{aligned} \quad (4.39)$$

for some $\varepsilon_0 > 0$, since $M_{k,R_0} := \max_{x \in B_{R_0}(0)} \left| e^{b\bar{v}_k(x)} - e^{b\bar{v}(x)} \right| \rightarrow 0$ as $k \rightarrow +\infty$ since R_0 is fixed. Therefore we may combine (4.38) and (4.39) by writing

$$\int_{B_s(0)} e^{q_k \tilde{v}_k(x)} dx \geq \varepsilon_1 \frac{\pi s^2}{1 + s^2},$$

for $\varepsilon_1 \leq \min \left\{ \varepsilon_0, \frac{M_0}{\pi} \right\}$ which holds for all $s > 0$. Thus, (4.37) becomes

$$-2\pi \tilde{u}_k(x) \geq \varepsilon_1 e^{q_k \tilde{v}_k(0)} \int_0^r \frac{\pi s}{1 + s^2} ds = e^{q_k \tilde{v}_k(0)} \frac{\varepsilon_1 \pi}{2} \log(1 + r^2),$$

which yields

$$\tilde{u}_k(x) \leq -\frac{\varepsilon_1}{4} \log(1 + |x|^2) e^{q_k \tilde{v}_k(0)}. \quad (4.40)$$

for all $x \in \Omega_k$. Since (H_b) and the boundedness of $\frac{\tilde{v}_k(0)}{T_k B_k}$ imply that $q_k \not\rightarrow 0$, then $q_k \tilde{v}_k(0) \rightarrow +\infty$, and in turn

$$\tilde{u}_k(x) \rightarrow -\infty \quad \text{as } k \rightarrow +\infty$$

for all $x \neq 0$. We transfer this information to obtain properties of $\bar{v}$ by (4.2). Arguing as in (4.29), we have

$$\begin{aligned} 2\pi |\bar{v}_k(r)| &= 2\pi (\bar{v}_k(0) - \bar{v}_k(r)) = \int_0^r \frac{1}{s} \left(\int_{B_s(0)} -\Delta \bar{v}_k(x) \, dx \right) ds \\ &= \int_0^r \frac{1}{s} \left(\int_{B_s(0)} e^{f'/f(S_k)g'/g(\zeta_k(x))} \tilde{u}_k(x) \, dx \right) ds, \end{aligned} \quad (4.41)$$

so we need to analyse the exponent on the right. Recalling $S_k \geq \zeta_k(x)$ and $\tilde{u}_k \leq 0$, by (H_5) and (4.32) one infers

$$\frac{f'}{f}(S_k) \frac{g'}{g}(\zeta_k(x)) \tilde{u}_k(x) \leq \frac{f'}{f}(S_k) \frac{g'}{g}(S_k) \frac{\zeta_k(x)}{S_k} \tilde{u}_k(x) \quad (4.42)$$

and by (3.4), (3.6), (4.31), and (4.25), one estimates

$$\frac{\zeta_k(x)}{S_k} = 1 + \bar{\theta}_k(x) \frac{\tilde{u}_k(x)}{A_k S_k} \geq 1 - \bar{\theta}_k(x) \frac{e^{\gamma_k \tilde{v}_k(0)}}{A_k S_k} \frac{r^2}{4} \geq 1 - \left(\frac{\Lambda}{C_R} \right)^{\frac{\gamma_k}{q_k}} \frac{r^2}{4} \left(\frac{f}{f'}(S_k) \right)^{\frac{\gamma_k}{q_k} - 1} \frac{1}{S_k} \rightarrow 1 \quad (4.43)$$

u.c.s. as $k \rightarrow +\infty$ by (4.35). Therefore, by (4.42), (4.43), and (4.40),

$$\begin{aligned} \frac{f'}{f}(S_k) \frac{g'}{g}(\zeta_k(x)) \tilde{u}_k(x) &\leq \frac{f'}{f}(S_k) \frac{g'}{g}(S_k) \frac{\zeta_k(x)}{S_k} \left(-\frac{\varepsilon_1}{4} \log(1 + r^2) e^{q_k \tilde{v}_k(0)} \right) \\ &\leq -(b + o_k(1)) \frac{\varepsilon_1}{4} \log(1 + |x|^2) e^{q_k \tilde{v}_k(0)} \rightarrow -\infty \end{aligned} \quad (4.44)$$

for all $x \neq 0$. By (4.44) and dominated convergence theorem ($\tilde{u}_k \leq 0$, hence $e^{f'/f(S_k)g'/g(\zeta_k(x))} \tilde{u}_k(x) \leq 1$), we get that, for any $s \in [0, r]$,

$$\lim_{k \rightarrow +\infty} \frac{1}{s} \int_{B_s(0)} e^{f'/f(S_k)g'/g(\zeta_k(x))} \tilde{u}_k(x) \, dx = 0.$$

Since

$$\frac{1}{s} \int_{B_s(0)} e^{f'/f(S_k)g'/g(\zeta_k(x))} \tilde{u}_k(x) \, dx \leq \pi s,$$

we may apply dominated convergence again and get directly from (4.41) that $\bar{v}_k(r) \rightarrow 0$ as $k \rightarrow +\infty$. In particular $\bar{v} = 0$ in $\mathbb{R}^2$ by (4.4). As a consequence the constant C_R defined in (4.26) behaves like R^2 . Therefore, we may take the limit as $R \rightarrow +\infty$ in (4.36) and finally reach the desired contradiction. This concludes our analysis of Subcase 1 under the condition (4.34-I).

Let us now consider Subcase 1 under the complementary condition (4.34-II), namely

$$\frac{\tilde{v}_k(0)}{T_k B_k} \log \left(\frac{f}{f'}(S_k) \right) \geq M$$

for any arbitrary $M > 0$, for all k large. Here the strategy resembles the one used to prove (4.32). Indeed, starting again from (4.22), by (H_5) , (3.3), and (4.4), one gets

$$\Lambda \frac{f}{f'}(S_k) \geq \int_{B_R(0)} e^{f'/f(T_k)g'/g(T_k)\frac{T_k}{\eta_k(x)}\tilde{v}_k(x)} dx \geq C_R e^{\frac{b}{1+C}\tilde{v}_k(0)} \geq C_R e^{\frac{bM}{1+C}T_k B_k (\log(f/f'(S_k)))^{-1}}.$$

Hence, applying the logarithm on both sides, one obtains

$$2 \log \left(\frac{f}{f'}(S_k) \right) \geq \log \left(\frac{\Lambda}{C_R} \right) + \log \left(\frac{f}{f'}(S_k) \right) \geq \frac{bMT_k B_k}{(1+C) \log f/f'(S_k)},$$

since $f'/f(S_k) \rightarrow 0$, that is

$$\left(\log \frac{f}{f'}(S_k) \right)^2 \geq \frac{bM}{2(1+C)} T_k B_k = \frac{bM}{2(1+C)} T_k \frac{g}{g'}(T_k),$$

from which, using (H_b) ,

$$\frac{2(1+C)}{M}(1+o_k(1)) \geq \frac{T_k f'/f(T_k)}{(\log f/f'(S_k))^2}. \quad (4.45)$$

Since M is arbitrary, this contradicts Lemma 3.2 by means of assumptions (H_6) or (H'_6) , and concludes our analysis of Subcase 1 also under assumption (4.34-II).

4.3.2 Subcase 2: $\frac{\tilde{v}_k(0)}{T_k B_k} \rightarrow +\infty$

As in Subcase 1, clearly $\eta_k(x) \geq T_k$ u.c.s., hence (4.23) holds, and so by (4.1)-(4.4) we get

$$\frac{f'}{f}(\eta_k(x)) \frac{g'}{g}(T_k) \tilde{v}_k(x) \geq \frac{f'}{f}(T_k) \frac{g'}{g}(T_k) \frac{\tilde{v}_k(0)}{1 + \frac{\tilde{v}_k(0)}{T_k B_k}} - C_R.$$

By the elementary estimate

$$\frac{1}{1+y} = \frac{1}{y} \cdot \frac{1}{1+\frac{1}{y}} \geq \frac{1}{y} \left(1 - \frac{1}{y} \right)$$

and (3.6), one ends up with

$$\begin{aligned} \frac{f'}{f}(\eta_k(x)) \frac{g'}{g}(T_k) \tilde{v}_k(x) &\geq \frac{f'}{f}(T_k) \frac{g'}{g}(T_k) T_k B_k \left(1 - \frac{T_k B_k}{\tilde{v}_k(0)} \right) - C_R \\ &= \frac{f'}{f}(T_k) T_k \left(1 - \frac{T_k B_k}{\tilde{v}_k(0)} \right) - C_R = \frac{f'}{f}(T_k) T_k (1 + o_k(1)) - C_R, \end{aligned}$$

recalling that we are supposing $\frac{\tilde{v}_k(0)}{T_k B_k} \rightarrow +\infty$. By (4.22) this yields

$$\Lambda \geq \frac{f'}{f}(S_k) \int_{B_R(0)} e^{f'/f(\eta_k(x))g'/g(T_k)\tilde{v}_k(x)} dx \geq |B_R(0)| e^{-C_R} e^{(1-\varepsilon)\frac{f'}{f}(T_k)T_k} \frac{f'}{f}(S_k). \quad (4.46)$$

Now, if we suppose that (H_6) holds, it is easy to find a contradiction. Indeed, choosing $\varepsilon = 1 - \frac{1}{a} \in (0, 1)$, where $a > 1$ is defined in (H_6) , by (H_5) and Lemma 3.1 one gets

$$\begin{aligned} \frac{\Lambda e^{C_R}}{|B_R(0)|} &\geq e^{(1-\varepsilon)\frac{f'}{f}(T_k)T_k} \frac{f'}{f}(S_k) \geq e^{(1-\varepsilon)\frac{f'}{f}(S_k)S_k} \frac{f'}{f}(S_k) \\ &\geq e^{(\log S_k)^2} \frac{f'}{f}(S_k) \geq S_k \frac{f'}{f}(S_k) \rightarrow +\infty \end{aligned}$$

by Lemma 2.6(ii).

On the other hand, under (H'_6) , again taking $\varepsilon = 1 - \frac{1}{a}$,

$$\frac{\Lambda e^{C_R}}{|B_R(0)|} \geq e^{(1-\varepsilon)\frac{f'}{f}(T_k)T_k} \frac{f'}{f}(S_k) \geq e^{(\log \log F(T_k))^2} \frac{f'}{f}(S_k) \geq \log F(T_k) \cdot \frac{f'}{f}(S_k) \rightarrow +\infty$$

by Lemma 3.1. These contradictions conclude both Subcase 2 and the whole proof of Theorem 1.1.

Remark 4. In light of the analysis in Subcases 1 and 2, we believe that (H_6) -(H'_6) can be relaxed to the following, respectively:

(H_7) there exist $t_0 > 0$ and $a > 1$, such that $t \frac{f'(t)}{f(t)} \geq a \log t$ for all $t > t_0$;

(H'_7) there exist $t_0 > 0$ and $a > 1$, such that $t \frac{f'(t)}{f(t)} \geq a \log \log F(t)$ for all $t > t_0$.

These would allow for a better bound from below on f . Indeed, with the same argument of Lemma 2.10 one easily proves that (H_7) implies $f(t) \gtrsim e^{\frac{a}{2}(\log t)^2}$ for t large, while from (H'_7) one deduces $f(t) \gtrsim e^{\frac{a}{2} \log t \log \log t}$ for t large.

Note in fact that (H_7) -(H'_7) are the right assumptions one needs in order to guarantee that the contradictory argument of Subcase 2 closes. To conclude the one in Subcase 1, and to reach the contradiction from (3.20), in assumptions (H_6) -(H'_6) a square was needed. However, we suspect that the argument in Subcase 1 is not optimal: e.g. in the step $\frac{g'}{g}(T_k) \geq \frac{g'}{g}(S_k)$ in (4.33) we loose some growth, which might be essential for an improvement of our argument.

5 Existence: Proof of Theorem 1.2

We follow the approach of [31], which is based on the *Fixed Point* (or Leray-Schauder) Index theory, and we refer to [1] for its definition and basic properties. In what follows in view of Theorem 1.2 we consider $\Omega = B_1(0)$, however, once such a priori bound is recovered in a smooth bounded domain $\Omega \subset \mathbb{R}^2$, the proof of Theorem 1.1 works without changes.

Let K be the cone of functions in $C(\overline{\Omega}) \times C(\overline{\Omega})$ which are positive in Ω and denote by S the solution operator of the system

$$\begin{cases} -\Delta u = \phi & \text{in } \Omega, \\ -\Delta v = \psi & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

namely $S(\phi, \psi) = (u, v)$. Note that S is a linear compact operator from $C(\overline{\Omega}) \times C(\overline{\Omega})$ into itself. Hence solutions of (1.1) correspond to fixed points of the map $T : (u, v) \mapsto T(u, v) := S(f(v), g(u))$, which is compact by (H_1). For $W \subset K$ relatively open such that $Tz \neq z$ for all $z \in \partial W$ we denote by $i_K(T, W)$ the fixed point index of T in W with respect to K . We are going to prove that there exist $0 < r < R$ such that $i_K(T, W_R \setminus \overline{W_r}) \neq 0$, by showing that $i_K(T, W_r) = 1$ while $i_K(T, W_R) = 0$. Here $W_s := B_s(0) \cap K$, where $B_s(0)$ for $s > 0$ denotes the ball in $C(\overline{\Omega}) \times C(\overline{\Omega})$ of radius s and centred at 0. To prove that there exists r small such that the index in W_r equals 1, we make use of the superlinearity of f and g at 0; instead, to prove the index vanishes in W_R for a large R , we exploit the a priori bound by Theorem 1.1 together with the superlinearity of the nonlinearities at ∞ .

Let us introduce the homothety $H_1(t, u, v) = tT(u, v)$ and prove that there exists $r > 0$ such that $H_1(t, u, v) \neq (u, v)$ for all $t \in [0, 1]$ if $(u, v) \in \partial W_r$. To this aim fix $\varepsilon \in (0, \lambda_1)$ and by (H_9) take $r > 0$ such that if $\|u\|_\infty + \|v\|_\infty = r$, one has $f(v) \leq \varepsilon v$ and $g(u) \leq \varepsilon u$. Introduce ϕ_1 as the first (positive)

eigenfunction of $-\Delta$ in Ω . Supposing that there exists (u, v) such that $H_1(t, u, v) = (u, v)$, namely a solution of

$$\begin{cases} -\Delta u = tf(v) & \text{in } \Omega, \\ -\Delta v = tg(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

we test the system with ϕ_1 and sum the two equations:

$$\lambda_1 \int_{\Omega} (u + v) \phi_1 = \int_{\Omega} (\nabla u + \nabla v) \nabla \phi_1 = t \int_{\Omega} (f(v) + g(u)) \phi_1 \leq \varepsilon t \int_{\Omega} (u + v) \phi_1,$$

which is a contradiction since $t \in [0, 1]$ and $\varepsilon < \lambda_1$. As a consequence,

$$i_K(T, W_r) = i_K(H_1(1, \cdot, \cdot), W_r) = i_K(H_1(0, \cdot, \cdot), W_r) = i_K(0, W_r) = 1. \quad (5.1)$$

Define now a second homothety $H_2(\mu, u, v) = S(f(v + \mu), g(u + \mu))$ where $\mu \geq 0$, the fixed point of which are solutions of

$$\begin{cases} -\Delta u = f(v + \mu) & \text{in } \Omega, \\ -\Delta v = g(u + \mu) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.2)$$

By (H_3) we know that both f and g are superlinear (see Lemma 2.7), that is, for any $M > 0$ there exists $t_0 > 0$ such that for all $t > t_0$ one has $f(t) > Mt$ and $g(t) > Mt$. Again by testing the system with ϕ_1 , we obtain

$$\begin{aligned} \int_{\Omega} f(v + \mu) \phi_1 &= \int_{\Omega} \nabla u \nabla \phi_1 = \lambda_1 \left(\int_{\{u \leq t_0\}} u \phi_1 + \int_{\{u > t_0\}} u \phi_1 \right) \\ &\leq \lambda_1 \left(t_0 \|\phi_1\|_1 + \frac{1}{M} \int_{\Omega} g(u + \mu) \phi_1 \right), \end{aligned}$$

since $\mu \geq 0$ and $\phi_1 > 0$. Similarly

$$\int_{\Omega} g(u + \mu) \phi_1 = \int_{\Omega} \nabla v \nabla \phi_1 \leq \lambda_1 \left(t_0 \|\phi_1\|_1 + \frac{1}{M} \int_{\Omega} f(v + \mu) \phi_1 \right).$$

These together yield

$$\left(1 - \frac{\lambda_1^2}{M^2}\right) \int_{\Omega} f(v + \mu) \phi_1 \leq \lambda_1 \left(1 + \frac{\lambda_1}{M}\right) t_0 \|\phi_1\|_1.$$

and

$$\left(1 - \frac{\lambda_1^2}{M^2}\right) \int_{\Omega} g(u + \mu) \phi_1 \leq \lambda_1 \left(1 + \frac{\lambda_1}{M}\right) t_0 \|\phi_1\|_1.$$

Choosing $M = 2\lambda_1$, one then finds

$$\int_{\Omega} f(v + \mu) \phi_1 \leq 2\lambda_1 t_0 \|\phi_1\|_1 \quad \text{and} \quad \int_{\Omega} g(u + \mu) \phi_1 \leq 2\lambda_1 t_0 \|\phi_1\|_1, \quad (5.3)$$

which by (H_1) and $u, v \geq 0$ imply that μ needs to be bounded from above. Hence, there exists $\bar{\mu} > 0$ such that for all $\mu > \bar{\mu}$ one has $H_2(\mu, u, v) \neq (u, v)$ in K . Moreover, for any fixed μ_0 , we note that $f(\cdot + \mu)$ and $g(\cdot + \mu)$ satisfy the assumptions (H_1) - (H'_6) uniformly for $\mu \in [0, \mu_0]$. In fact, it is evident that (H_1) - (H_4) hold; moreover, $\frac{1}{t} \frac{f}{f'}(t + \mu) = \left(1 + \frac{\mu}{t}\right) \frac{1}{t + \mu} \frac{f}{f'}(t + \mu)$, which is decreasing for large t and uniformly in $\mu \in [0, \mu_0]$. Indeed, this is a product of two eventually decreasing positive functions by

(H_5) applied to f ; the same argument applies also for g , hence (H_5) holds also for $f(\cdot + \mu)$ and $g(\cdot + \mu)$. Similarly, one also verifies (H_6) and (H'_6) : indeed, e.g.

$$\frac{t}{(\log t)^2} \frac{f'}{f}(t + \mu) = \frac{t(\log(t + \mu))^2}{(t + \mu)(\log t)^2} \cdot \frac{(t + \mu)}{(\log(t + \mu))^2} \frac{f'}{f}(t + \mu) \geq \frac{a}{2}$$

for t large, where a is the constant in (H_6) for $\mu = 0$, since the first term converges to 1 as $t \rightarrow +\infty$ uniformly in $\mu \in [0, \mu_0]$. Moreover, it is important to note that the constant Λ in Proposition 2.3 is independent of μ , provided $\mu \in [0, \mu_0]$. Indeed, inspecting its proof in [12, Theorem 1.2], it is sufficient that (5.3) holds with a bound independent of $\mu \in [0, \mu_0]$ to conclude. After all these considerations, we can thus state that Theorem 1.1 applies with a bound uniform with respect to $\mu \in [0, \mu_0]$. Therefore taking $\mu_0 = \bar{\mu} + 1 > 0$ there exists $\bar{R} = \bar{R}(\bar{\mu}) > 0$ such that all solutions of (5.2) are bounded in L^∞ norm by $\bar{R}/2$. Hence, $H_2(\mu, u, v) \neq (u, v)$ for all $(u, v) \in B_{\bar{R}}(0)^c \cap K$ and $\mu \in [0, \bar{\mu} + 1]$. Hence, combining the two information on the map H_2 , we infer

$$i_K(T, W_{\bar{R}}) = i_K(H_2(0, \cdot, \cdot), W_{\bar{R}}) = i_K(H_2(\bar{\mu} + 1, \cdot, \cdot), W_{\bar{R}}) = i_K(0, W_{\bar{R}}) = 0. \quad (5.4)$$

As a result, from (5.4) and (5.1) one deduces $i_K(T, W_R \setminus \overline{W_r}) = -1 \neq 0$, which yields the existence of a nontrivial positive solution of (1.1).

6 Open problems

1. Is it possible to show that the growth assumption (H_b) is in some sense sharp? We point out that in the scalar case the Liouville growth $t \mapsto e^t$ is sharp for *distributional solutions*, in case of a singular potential, see [6, 25].
2. Is it possible to adapt our blow-up method for (1.1) in a generic smooth bounded domain $\Omega \subset \mathbb{R}^2$? One of the main obstruction is the fact that the two components might achieve their maxima blow-up points in different points in Ω (which, however, cannot approach the boundary by Proposition 2.3 from [12]). Moreover, in order to obtain global estimates for $\tilde{u}_k$ and $\tilde{v}_k$, we relied on the radial symmetry, see e.g. (4.29) or (4.41).

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