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Richard Nutt, Roland Schnaubelt

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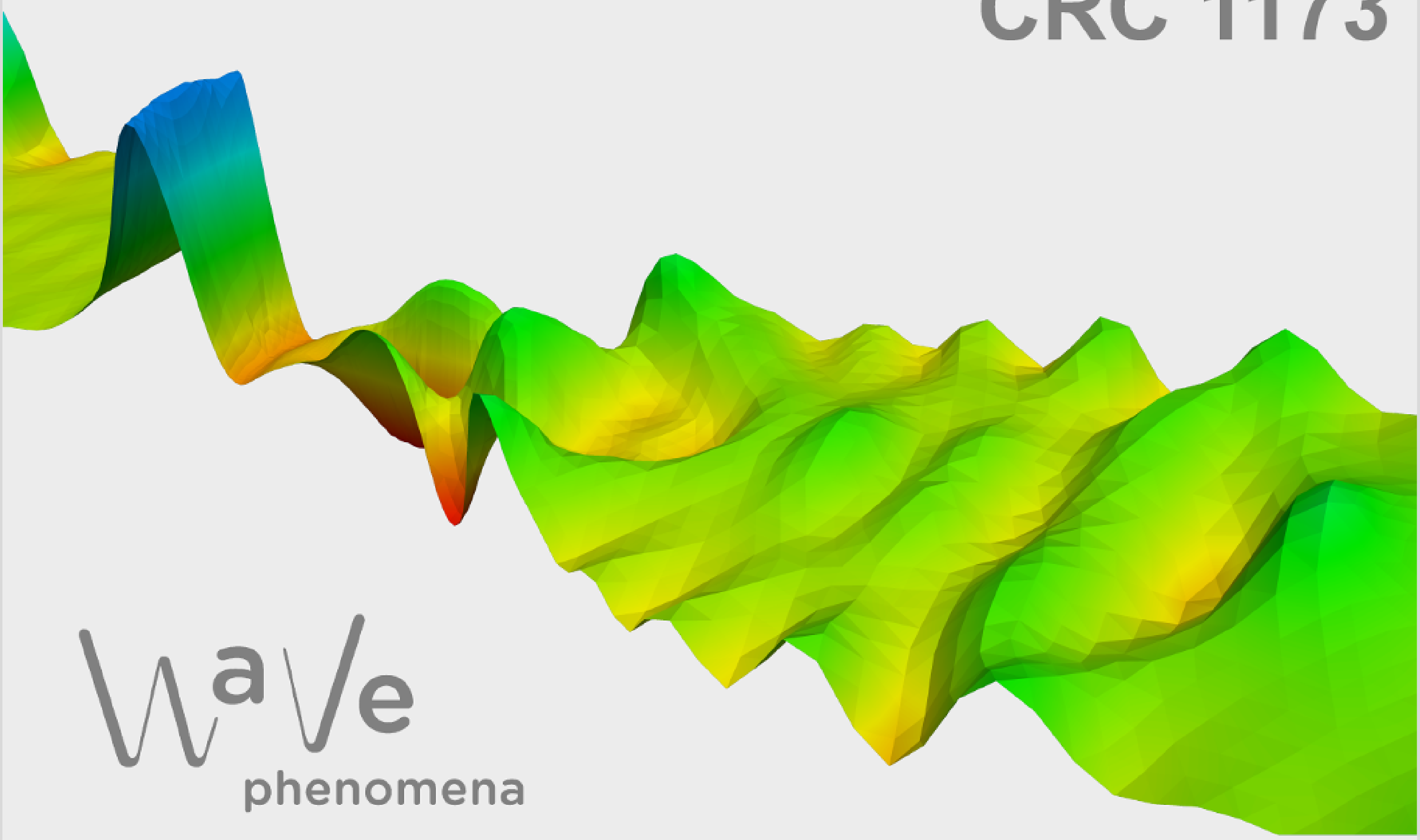

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EXPONENTIAL DECAY OF THE LINEAR MAXWELL SYSTEM DUE TO CONDUCTIVITY NEAR THE BOUNDARY

RICHARD NUTT[1] AND ROLAND SCHNAUBELT[2]

Abstract. We study the anisotropic linear Maxwell system on a bounded domain Ω with perfectly conducting boundary conditions. It is damped via a conductivity σ which is strictly positive on a collar at the boundary. We prove that solutions decay exponentially to 0, if the fields have no magnetic charges on Ω and no electric charges off the support of σ. Our approach relies on a splitting of the solution via a Helmholtz decomposition and an observability-type estimate for a related second-order system without charges, shown using Morawetz multipliers. Corresponding exact observability and controllability results are also established.

1. Introduction

The Maxwell equations are the fundamental laws of electro-magnetic theory. A non-zero conductivity σ causes dissipation of energy and thus may lead to decay of solutions. For linear anisotropic materials we show that the solutions converge exponentially to equilibria if σ is strictly positive near the boundary of the bounded domain $\Omega \subseteq \mathbb{R}^3$. It assumed that permittivity and permeability ε and μ satisfy the non-trapping condition (2.2). We allow for a multiply connected Ω and disconnected $\partial\Omega$. We obtain exponential decay to 0 if there are no magnetic charges on Ω (which holds in physics) and no electric charges on $\Omega \setminus \operatorname{supp}\sigma$. For the charge-free system with $\sigma = 0$, we also prove exact observability and controllability with respect to a collar at $\partial\Omega$. In the case of anisotropic coefficients these seem to be the first results in this context if the damping or observability region are not the full domain. Moreover, for exponential stability, so far only constant ε and μ have been treated.

The Maxwell–Ampère and Faraday equations relate the electric fields E and D with the magnetic ones B and H via

$$\partial_t D(t,x) = \operatorname{curl} H(t,x) - J(t,x), \quad \partial_t B(t,x) = -\operatorname{curl} E(t,x), \qquad t \geq 0,\ x \in \Omega,$$

where J is a current density. One has to add constitutive relations that describe the reaction of the material to the fields. We study anisotropic, instantaneous, time-independent, linear material laws

$$D(t,x) = \varepsilon(x)E(t,x), \quad B(t,x) = \mu(x)H(t,x),$$

with *permittivity* $\varepsilon \in C^1\big(\overline{\Omega}, \mathbb{R}^{3\times 3}_{\text{sym}}\big)$ and *permeability* $\mu \in C^1\big(\overline{\Omega}, \mathbb{R}^{3\times 3}_{\text{sym}}\big)$ which are uniformly positive definite. In the absence of exterior currents, Ohm's law yields

$$J(t,x) = \sigma(x)E(t,x)$$

for the non-negative *conductivity* $\sigma \in L^\infty\big(\Omega, \mathbb{R}^{3\times 3}_{\text{sym}}\big)$.

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[2]Corresponding author.

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We assume that there are no magnetic charges $\operatorname{div}(\mu H)$ and impose *perfectly conducting boundary conditions*, thus excluding boundary conductivity. In this way we arrive at the Maxwell system

$$\begin{aligned}
\partial_t\big(\varepsilon(x)E(t,x)\big) &= \operatorname{curl} H(t,x) - \sigma(x)E(t,x), \\
\partial_t\big(\mu(x)H(t,x)\big) &= -\operatorname{curl} E(t,x),
\end{aligned} \qquad t \geq 0,\ x \in \Omega, \tag{1.1}$$

$$\operatorname{div}(\mu(x)H(t,x)) = 0, \qquad t \geq 0,\ x \in \Omega, \tag{1.2}$$

$$E(t,x) \times \nu(x) = 0, \quad \nu(x) \cdot \mu(x)H(t,x) = 0, \qquad t \geq 0,\ x \in \partial\Omega, \tag{1.3}$$

$$E(0,x) = E_0(x), \quad H(0,x) = H_0(x), \qquad x \in \Omega, \tag{1.4}$$

where ν is the outer unit normal of $\partial\Omega$. See [1], [3], or [8] for a systematic treatment of the Maxwell equations from a mathematical point of view.

The anisotropy in the material laws is needed to describe non-trivial crystal structures, see e.g. [2]. Moreover, matrix-valued coefficients arise in the investigation of (even isotropic) quasilinear Maxwell systems, as in [12], [15] or [17]. We stress that the anisotropic Maxwell system is significantly stronger coupled than in the case of isotropic material laws with scalar coefficients. For instance, if the coefficients are constant, the system can be reduced to decoupled scalar wave equations for the components E (if $\operatorname{div}(\varepsilon E) = 0$) or H (if $\sigma = 0$). In the isotropic case, the equations are coupled, but only in lower order (if $\operatorname{div}(\varepsilon E) = 0$ or $\sigma = 0$). Moreover, since anisotropic material laws change the direction of the fields, the boundary conditions are much harder to handle in this setting.

The curl operator has a huge kernel consisting of gradient fields which prohibit regularity and compactness properties. Divergence conditions may counteract the kernel. Physically they are encoded in the Gaußian laws for the charges. Besides $\operatorname{div}(\mu H) = 0$, the Maxwell–Ampère law in (1.1) yields

$$\rho(t) := \operatorname{div}(\varepsilon E(t)) = \operatorname{div}(\varepsilon E_0) - \int_0^t \operatorname{div}(\sigma E(\tau))\,\mathrm{d}\tau, \qquad t \geq 0. \tag{1.5}$$

Similarly one sees that the magnetic divergence and boundary conditions in (1.2) and (1.3) are true if they are satisfied by H_0, cf. Theorem 5.2.5 in [1]. The time integral in (1.5) is a serious obstacle for the study of the long-time behavior. For isotropic coefficients one can reduce (1.5) to the scalar ODE $\partial_t \rho = -\frac{\sigma}{\varepsilon}\rho - \nabla\frac{\sigma}{\varepsilon} \cdot \varepsilon E$ for fixed $x \in \Omega$. In the anisotropic case this approach fails.

Instead of interior damping, also boundary conductivities ζ have been studied. Here one sets $\sigma = 0$ and replaces the boundary conditions (1.3) by

$$H \times \nu = \nu \times (\zeta E \times \nu),$$

see [8]. In this setting the electric charges ρ in the domain vanish if $\rho(0) = 0$. Moreover, this boundary condition can improve the trace regularity compared to (1.1)–(1.4), see [18] and the references therein. In this sense this case is simpler.

For the scalar wave equation there is a very rich and deep theory on decay properties caused by damping, whereas the corresponding theory for the Maxwell system is far less developed. We refrain from giving references for the wave case, but discuss a sample of the papers on the Maxwell equations. For scalar coefficients exponential decay of solutions was shown in [10] for boundary damping with strictly positive ζ under a non-trapping condition on a 'substarlike' domain, see [11] for the case of constant coefficients. Theorem 5.1 in [16] provides exponential decay for constant $\varepsilon, \mu \in \mathbb{R}_+$ and strictly positive scalar σ. Semilinear damping was treated in [7]. For $\sigma = 0$ and scalar coefficients, [13] establishes exact observability on a collar at $\partial\Omega$. (See [7], [10] and [16] for more results on observability and controllability.) These theorems were proven by means of Morawetz multipliers and Helmholtz decompositions. By [16], for constant $\varepsilon, \mu \in \mathbb{R}_+$ the solutions also decay exponentially if σ is strictly positive on its support and satisfies the geometric control condition from the wave case. This proof is based on microlocal analysis.

For matrix-valued coefficients, so far exponential decay has been studied only assuming strict positivity of the conductivity, see [6] and [12] for interior damping as well as [15] and [17] for the boundary case. The papers [12], [15] and [17] mainly deal with nonlinear material laws. Earlier, in [5] boundary observability on $\partial\Omega$ was shown in the anisotropic case. These papers also rely on Morawetz multipliers and Helmholtz decompositions. Weaker convergence properties have been investigated, too, here we refer to [14] for recent contributions and several references.

With the exception of Theorem 5.1 in [16], in the cited works various topological constraints are imposed, namely simple connectedness of Ω or connectedness of $\partial\Omega$, and partly even (variants of) starshapedness of Ω. The topological assumptions improve the mapping properties of the curl operator, as we recall in Lemma 2.3. Starshapedness greatly simplifies the use of Morawetz multipliers for boundary damping.

In our main result Theorem 4.1 we show the uniform exponential decay to 0 of L^2-solutions for matrix-valued C^1-coefficients assuming that $\sigma \in L^\infty$ is strictly positive on $\omega := \{\sigma > 0\}$ containing a collar at the boundary. To exclude equilibria, besides the magnetic divergence and boundary conditions we suppose that ρ vanishes on $\Omega \setminus \omega$. The domain Ω is only required to be multiply connected, in contrast to the previous literature. As in [5], [10] or [13] we assume the non-trapping condition (2.2) on ε and μ, which says that the coefficients do not decay too rapidly in radial direction. Heuristically, this property reduces backreflections, so that waves can reach the boundary in a sufficient way. In Corollary 4.8 we remove the condition on initial charges and the magnetic boundary condition. In this framework we obtain uniform exponential convergence to the set of equilibria, which are related to non-trivial charges. On the other hand, in the completely charge-free case without conductivity, Theorem 3.7 shows exact observability of the Maxwell system by the electric fields on a collar at the $\partial\Omega$ and its exact controllability by charge-free current densities supported on the collar. As noted above, so far such results with localized observations have been restricted to scalar coefficients, and for exponential stability with localized damping even to constant ε and μ.

In the next section we collect our assumptions, generation results and invertibility properties of the curl operator. The main observability-type estimate only works in the charge-free case without conductivity, see system (3.4). So we split (E, H) into the solution (V_h, W_h) of this system and other terms. To account for the charges on ω, we use a gradient field ∇p where $p \in H^1(\Omega)$ solves the elliptic PDE (2.9). One can bound $\partial_t \nabla p$ (but not ∇p) in L^2 by the dissipation term $\|\sigma^{1/2} E\|_2$ which in turn can be controlled via the energy equality. This approach goes back to [16] with $p \in H_0^1(\Omega)$ in the constant coefficient case and for connected $\partial\Omega$ and $\operatorname{supp}\sigma$. Our system (2.9) takes care of some the topological obstructions for the invertibility of curl. To deal with the others, we have to restrict the electric and magnetic fields in the Maxwell system to an invariant subsystem with finite codimension, see (2.3). The remaining part of the solution (V_i, W_i) then solves the system (3.8) that incorporates the inhomogeneity $-\sigma E - \varepsilon \nabla \partial_t p$. Such a splitting was also used in [13] without p. To correct the influence of $p(0)$ in these systems, one has to choose the initial values properly, see (3.2). Finally, for technical reasons some initial fields have to approximated by more regular ones.

In Proposition 3.2 we show the crucial observability-type estimate for V_h. Dissipation terms only occur for the electric part, so that it is useful to work with a second-order formulation involving only V_h. Moreover, the behavior of ∇p and ρ suggests to work with time derivatives. So we first estimate $\partial_t V_h$ and pass only later to V_h in Corollary 3.5 by integration. This result is then reformulated as Theorem 3.7 on observability and controllability. In a lengthy calculation also involving vector analysis, Proposition 3.2 is shown via a Morawetz multiplier and via a tailor-made multiplier using Lax–Milgram (see Lemma 3.3), as well as the energy equality for V_h. For the reasoning, it is important that $\operatorname{supp}\sigma$ contains a collar at $\partial\Omega$ and that the coefficients satisfy the non-trapping condition. For scalar coefficients similar arguments are found in [13], but it is quite sophisticated to extend them to the matrix-valued case.

In the last section we then estimate the other parts of (E, H) in several steps. First, the energy of $\partial_t(E, H)$ is bounded by that of (E, H) for which it is crucial that σ is strictly positive on its support. This was done for connected $\partial\Omega$ and ω in [16], using the auxiliar function ∇p. In our more general setting we have to proceed differently, based on a lower estimate for the curl operator in Lemma 4.2. One can control $\partial_t H$ in space-time by $\partial_t E$ and dissipation terms using the splitting of E and energy-type estimates. The inhomogeneous part (V_i, W_i) is handled just by means of Duhamel's formula, exploiting properties of the initial data of (3.8). Several of these inequalities on the time interval $[0, T]$ depend on $T > 0$. Nevertheless they can be combined into a proof of exponential stability employing a strategy that goes back to [9] in the wave case. The convergence result for data with charges then follows by a projection argument.

2. Notation, assumptions and auxiliary results

In this section we collect our main hypotheses and various basic properties. A bounded C^2-domain $\Omega \subseteq \mathbb{R}^3$ is *multiply connected* if there are disjoint C^2-surfaces $\Sigma_1, \dots, \Sigma_L$ such that $\Omega \setminus \bigcup_{l=1}^{L} \Sigma_l$ is simply connected, see [1], [3] or [4]. Simple connected Ω are considered as the special case $L = 0$.

Assumptions. Throughout, we assume the following conditions.

(H) Let $\Omega \subseteq \mathbb{R}^3$ be open, bounded and multiply connected, and $\partial\Omega \in C^2$ have connected components $\Gamma_0, \dots, \Gamma_K$. Let $\varepsilon, \mu \in C^1\big(\overline{\Omega}, \mathbb{R}^{3\times3}_{\mathrm{sym}}\big)$ be uniformly positive definite, $\sigma \in L^\infty\big(\Omega, \mathbb{R}^{3\times3}_{\mathrm{sym}}\big)$, $\omega := \{x \in \Omega \mid \sigma(x) > 0\}$ have a C^1-boundary.

The surface measure on $\partial\Omega$ or $\partial\omega$ is denoted by ς. We write $\upsilon := \Omega \setminus \overline{\omega}$ and $N_a := \{x \in \overline{\Omega} \mid \operatorname{dist}(x, \partial\Omega) < a\}$ for the collar of width $a > 0$ at $\partial\Omega$. Our core assumption for exponential stability is that there are constants $a, \sigma_0 > 0$ such that

$$N_a \subseteq \omega \quad \text{and} \quad \sigma|_\omega \geq \sigma_0 > 0. \tag{2.1}$$

In particular, this implies that $\partial\omega = \partial\upsilon \cup \partial\Omega$. $\upsilon := \Omega \setminus \overline{\omega}$ Let $m(x) = x - x_0$ for some $x_0 \in \mathbb{R}^3$. We also require the 'non-trapping' condition

$$\tilde{\varepsilon} := \varepsilon + (m \cdot \nabla)\varepsilon \geq \eta\varepsilon, \qquad \tilde{\mu} := \mu + (m \cdot \nabla)\mu \geq \eta\mu. \tag{2.2}$$

for some $\eta > 0$; i.e, the coefficients cannot decay too strongly in radial direction.

We use the L^2-based Sobolev spaces H^s on the domains and their boundaries. The spaces $H(\mathrm{curl})$ and $H(\mathrm{div})$ are the maximal domains of the distributional operators curl and div in $L^2(\Omega)$ equipped with the graph norm, respectively. (We often omit the range spaces for vector fields etc.) The kernels of these operators are written as $H(\mathrm{curl}\, 0)$ and $H(\mathrm{div}\, 0)$, and are endowed with the L^2-norm. The tangential trace $\operatorname{tr}_t v = v|_{\partial\Omega} \times \nu$ and the normal trace $\operatorname{tr}_n v = v|_{\partial\Omega} \cdot \nu$ can be extended to maps from $H(\mathrm{curl})$ and $H(\mathrm{div})$ to $H^{-1/2}(\partial\Omega)$. Their kernels are denoted by $H_{t0}(\mathrm{curl})$ and $H_{n0}(\mathrm{div})$, respectively, which are also the closure of test functions in the respective norms. We also set $H_{t0}(\mathrm{curl}\, 0) = H_{t0}(\mathrm{curl}) \cap H(\mathrm{curl}\, 0)$ and $H_{n0}(\mathrm{div}\, 0) = H_{n0}(\mathrm{div}) \cap H(\mathrm{div}\, 0)$. For $\alpha \in \{\varepsilon, \mu\}$, the operator $\operatorname{div}_\alpha$ given by

$\operatorname{div}_\alpha(v) = \operatorname{div}(\alpha v)$ with analogous notations. Finally, we need the refinements

$$H_{n0}^{\Sigma}(\operatorname{div}_\alpha 0) := \big\{u \in L^2(\Omega)^3 \;\big|\; \operatorname{div}_\alpha u = 0,\ \operatorname{tr}_n(\alpha u) = 0,\ \langle \operatorname{tr}_n(\alpha u), \mathbb{1}\rangle_{H^{-\frac{1}{2}}(\Sigma_l)},\ l \in \{1,\dots,L\}\big\},$$
$$H^{\Gamma}(\operatorname{div}_\alpha 0) := \big\{u \in L^2(\Omega)^3 \;\big|\; \operatorname{div}_\alpha u = 0,\ \langle \operatorname{tr}_n(\alpha u), \mathbb{1}\rangle_{H^{-\frac{1}{2}}(\Gamma_k)},\ k \in \{0,\dots,K\}\big\}.$$

For $\alpha = I$ we simply write $H_{n0}^{\Sigma}(\operatorname{div} 0)$ and $H^{\Gamma}(\operatorname{div} 0)$. The divergence theorem yields $H^{\Gamma}(\operatorname{div}_\alpha 0) = H(\operatorname{div}_\alpha 0)$ if $\partial\Omega$ is connected, and for simply connected Ω we have $H_{n0}^{\Sigma}(\operatorname{div}_\alpha 0) = H_{n0}(\operatorname{div}_\alpha 0)$. See [1], [3] or [4] for these and related facts.

Generation results. Let $L^2_\alpha(\Omega)$ be the space of measurable f fulfilling $\alpha^{1/2} f \in L^2(\Omega)$, endowed with the canonical norm. We define

$$X = \big\{(e,h) \in L^2_\varepsilon(\Omega) \times L^2_\mu(\Omega) \;\big|\; \varepsilon E|_\upsilon \in H^{\Gamma}(\operatorname{div} 0, \upsilon),\ \mu h \in H_{n0}^{\Sigma}(\operatorname{div} 0)\big\}, \tag{2.3}$$

where $H^{\Gamma}(\operatorname{div} 0, \upsilon)$ is the variant of $H^{\Gamma}(\operatorname{div} 0)$ on υ etc. The operator

$$D(A) = \big(H_{t0}(\operatorname{curl}) \times H(\operatorname{curl})\big) \cap X, \qquad A = \begin{pmatrix} -\varepsilon^{-1}\sigma & \varepsilon^{-1}\operatorname{curl} \\ -\mu^{-1}\operatorname{curl} & 0 \end{pmatrix}, \tag{2.4}$$

is well-defined by Theorem 6.1.4 in [1] and Proposition IX.1.3 in [4], the latter applied on υ. Note that the magnetic component of $(e,h) \in D(A)$ belongs to $H^1(\Omega)$ by Proposition 6.1 of [12], and thus to $H^1(\upsilon)$. One can then show its maximal dissipativity as in Lemma 2.1 of [14], so that A generates a contractive C_0-semigroup $T(\cdot)$ on X. Given $(E_0, H_0) \in D(A)$, we thus obtain a unique solution of (1.1)–(1.4) satisfying $\operatorname{div}(\varepsilon E(t)) = 0$ on $\Omega \setminus \overline{\omega}$ for $t \geq 0$ and

$$(E,H) \in C\big(\mathbb{R}_{\geq 0}, D(A)\big) \cap C^1\big(\mathbb{R}_{\geq 0}, X\big). \tag{2.5}$$

We assume that (E_0, H_0) belongs to $D(A)$ except for the proofs of Theorems 3.7 and 4.1, where we pass to general $(E_0, H_0) \in X$. Also in this case $(E(t), H(t)) = T(t)(E_0, H_0)$ is called a solution of the system.

Conceptually, we exclude charges in the interior of Ω away from the support of the conductivity σ. Furthermore, we assume that there are no charges on the connected components of the conductivity as in a capacitor, since these generate static electric fields that do not decay in time. By Faraday's law we also exclude the induction of currents in conductive loops along the boundaries $\partial\Sigma_i$ of the cuts of Ω, for example by an electric current passing through the 'holes' of Ω. (Think of a torus with an external current running through the middle.)

The proof of Lemma 2.1 in [14] implies that on $X_e = L^2_\varepsilon(\Omega) \times L^2_\mu(\Omega)$ the extension A_e of A with domain $H_{t0}(\operatorname{curl}) \times H(\operatorname{curl})$ generates a contraction semigroup $T_e(\cdot)$ which leaves X invariant and coincides there with $T(\cdot)$. Hence, the divergence condition on $\Omega \setminus \overline{\omega}$ for E_0, the divergence condition for H_0, and the boundary conditions on μH_0 are also invariant under the evolution for the system within X_e. Moreover, as we see in Lemma 4.7 the kernel of A_e is orthogonal to X in X_e so that our restrictions on charges and boundary conditions in (2.3) exclude the stationary fields which obstruct exponential stability.

Energies. Our arguments heavily use estimates of the energies'

$$\begin{aligned} \mathcal{E}(t) &:= \int_\Omega \big(\varepsilon E(t) \cdot E(t) + \mu H(t) \cdot H(t)\big)\,\mathrm{d}x = \|(E(t), H(t))\|_X^2, \\ \mathcal{D}(t) &:= \int_\Omega \big(\varepsilon \partial_t E(t) \cdot \partial_t E(t) + \mu \partial_t H(t) \cdot \partial_t H(t)\big)\,\mathrm{d}x = \|(\partial_t E(t), \partial_t H(t))\|_X^2, \end{aligned} \tag{2.6}$$

and in particular the following energy equality with dissipation related to σ.

Lemma 2.1. *Let* (H) *hold and* (E, H) *as in* (2.5) *solve* (1.1)*. We then obtain*

$$\mathcal{E}(s) - \mathcal{E}(t) = 2\int_s^t \int_\Omega \left|\sigma^{1/2} E\right|^2 \mathrm{d}x\, \mathrm{d}\tau,$$

$$\mathcal{D}(s) - \mathcal{D}(t) = 2\int_s^t \int_\Omega \left|\sigma^{1/2} \partial_t E\right|^2 \mathrm{d}x\, \mathrm{d}\tau, \qquad t \geq s \geq 0.$$

Proof. We only consider $\mathcal{D}$ as the estimate for $\mathcal{E}$ is shown in a similar, but simpler way. After regularizing (E_0, H_0) in $D(A^2)$, we pass to the time differentiated version of the Maxwell system (1.1). Here the energy inequality is shown in a standard way.

So, take fields $(E_0^n, H_0^n) \in D(A^2)$ converging to (E_0, H_0) in $D(A)$ with respect to the graph norm. These initial values yield solutions

$$(E^n, H^n) \in C^2\big(\mathbb{R}_{\geq 0}, X\big) \cap C^1\big(\mathbb{R}_{\geq 0}, D(A)\big) \cap C\big(\mathbb{R}_{\geq 0}, D(A^2)\big)$$

of the equations

$$\partial_t^2 E^n = \varepsilon^{-1} \operatorname{curl} \partial_t H^n - \varepsilon^{-1} \sigma \partial_t E^n, \qquad \partial_t^2 H^n = -\mu^{-1} \operatorname{curl} \partial_t E^n.$$

Then $(E^n, H^n) = T(\cdot)(E_0^n, H_0^n)$ tends to (E, H) in the space $C_b\big(\mathbb{R}_{\geq 0}, D(A)\big)$ of bounded functions, and $\partial_t(E^n, H^n) = T(\cdot)A(E_0^n, H_0^n)$ to $\partial_t(E, H)$ in $C_b\big(\mathbb{R}_{\geq 0}, X\big)$.

Let $\mathcal{D}^n = \|(\partial_t E^n, \partial_t H^n)\|_X^2$. The above system and integration by parts lead to

$$\begin{aligned}
\partial_t \mathcal{D}^n &= 2\int_\Omega \big(\partial_t E^n \cdot \varepsilon \partial_t^2 E^n + \partial_t H^n \cdot \mu \partial_t^2 H^n\big)\, \mathrm{d}x \\
&= 2\int_\Omega \big(\partial_t E^n \cdot (\operatorname{curl} \partial_t H^n - \sigma \partial_t E^n) - \partial_t H^n \cdot \operatorname{curl} \partial_t E^n\big)\, \mathrm{d}x \\
&= -2\int_\Omega \left|\sigma^{1/2} \partial_t E^n\right|^2 \mathrm{d}x,
\end{aligned}$$

since $\operatorname{tr}_t \partial_t E^n = 0$. Integrating in time, we derive

$$\mathcal{D}^n(s) - \mathcal{D}^n(t) = 2\int_s^t \int_\Omega \left|\sigma^{1/2} \partial_t E^n\right|^2 \mathrm{d}x\, \mathrm{d}s.$$

for $t \geq s \geq 0$. The result follows in the limit $n \to \infty$. □

Properties of the curl operator. We first list basic mapping properties of the gradient, divergence and curl. They can be found in Theorem 2.7 of [3] or in [4], or follow similarly.

Lemma 2.2. *In the following diagram the image of each operator is of finite codimension and contained in the kernel of the succeeding operator:*

$$H^1(\Omega) \xrightarrow{\nabla} H(\operatorname{curl}) \xrightarrow{\operatorname{curl}} H(\operatorname{div}) \xrightarrow{\operatorname{div}} L^2(\Omega),$$

$$H_c^1(\Omega) \xrightarrow{\nabla} H_{t0}(\operatorname{curl}) \xrightarrow{\operatorname{curl}} H_{n0}(\operatorname{div}) \xrightarrow{\operatorname{div}} L^2(\Omega),$$

where $H_c^1(\Omega) := \big\{f \in H^1(\Omega) \;\big|\; \operatorname{tr}_{\Gamma_k} f$ *constant for each* $k \in \{0, \dots K\; \big\} \supset H_0^1(\Omega)$.

The *cohomology spaces*

$$\mathbb{H}_1 := \big\{f \in L^2(\Omega) \;\big|\; \operatorname{curl} f = 0,\, \operatorname{div} f = 0,\, \operatorname{tr}_n = 0\big\},$$

$$\mathbb{H}_2 := \big\{f \in L^2(\Omega) \;\big|\; \operatorname{curl} f = 0,\, \operatorname{div} f = 0,\, \operatorname{tr}_t f = 0\big\}.$$

contain functions which prevent the invertibility of curl. In our setting they have finite dimensions, which are determined by topological properties of the domain Ω, see Remark 2.5 and also Sections 3.2, 6.1 and 6.2 in [1], Section 2.9 in [3], or Section IX.1.3 in [4] for more details.

The next lemma collects results on mapping properties of curl from Proposition 6.1.3, Theorem 6.1.4, Proposition 6.2.4 and Theorem 6.2.5 of [1] and from

Proposition 3, Proposition 4 and Remark 5 of Section IX.1 in [4] concerning the orthogonal complements of the cohomology spaces.

Lemma 2.3. *Let* (H) *hold. We then have the surjectivity for*

a) the 'electric' curl

$$\operatorname{curl}_E\colon H_{t0}(\operatorname{curl}) \cap H(\operatorname{div}_\varepsilon 0) \to H^{\Sigma}_{n0}(\operatorname{div} 0), \quad \text{with}$$
$$\operatorname{N}(\operatorname{curl}_E) = \mathbb{H}^{\varepsilon}_2 \coloneqq H_{t0}(\operatorname{curl} 0) \cap H(\operatorname{div}_\varepsilon 0)$$

b) and the 'magnetic' curl

$$\operatorname{curl}_H\colon H(\operatorname{curl}) \cap H_{n0}(\operatorname{div}_\mu 0) \to H^{\Gamma}(\operatorname{div} 0), \quad \text{with}$$
$$\operatorname{N}(\operatorname{curl}_H) = \mathbb{H}^{\mu}_1 \coloneqq H(\operatorname{curl} 0) \cap H_{n0}(\operatorname{div}_\mu 0).$$

Furthermore, the image sets can be characterized by

$$H^{\Sigma}_{n0}(\operatorname{div} 0) = \mathbb{H}_1^{\perp_{H_{n0}(\operatorname{div} 0)}}, \qquad H^{\Gamma}(\operatorname{div} 0) = \mathbb{H}_2^{\perp_{H(\operatorname{div} 0)}}. \tag{2.7}$$

In view of the next result we obtain the invertibility of

$$\operatorname{curl}_E\colon H_{t0}(\operatorname{curl}) \cap H^{\Gamma}(\operatorname{div}_\varepsilon 0) \to H^{\Sigma}_{n0}(\operatorname{div} 0) \qquad \text{in a) and}$$
$$\operatorname{curl}_H\colon H(\operatorname{curl}) \cap H^{\Sigma}_{n0}(\operatorname{div}_\mu 0) \to H^{\Gamma}(\operatorname{div} 0) \qquad \text{in b).}$$

We have used the following description of the weighted cohomology spaces.

Lemma 2.4. *Let* (H) *hold. For $u \in H(\operatorname{div}_\varepsilon 0)$ the field εu belongs to $\mathbb{H}_2^{\perp_{L^2}}$ if and only if u is contained in $(\mathbb{H}^{\varepsilon}_2)^{\perp_{L^2_\varepsilon}}$. Analogously, for $u \in H_{n0}(\operatorname{div}_\mu 0)$ the field μu belongs to $\mathbb{H}_1^{\perp_{L^2}}$ if and only if u is contained in $(\mathbb{H}^{\mu}_1)^{\perp_{L^2_\mu}}$.*

Proof. For the first equivalence, we assume that $\varepsilon u \in H^{\Gamma}(\operatorname{div} 0)$. Equation (IX.1.61) in [4] yields the decomposition

$$L^2(\Omega)^3 = \nabla H^1_0(\Omega) \oplus \mathbb{H}_2 \oplus \operatorname{curl} H^1(\Omega).$$

Since $H(\operatorname{div} 0)^{\perp_{L^2}} = \nabla H^1_0(\Omega)$ by (IX.1.42) in [4], our assumptions and (2.7) imply that εu is contained in $\operatorname{curl} H^1(\Omega)$; i.e., $\varepsilon u = \operatorname{curl} \Phi$ for some $\Phi \in H^1(\Omega)$. Hence, for all $h \in \mathbb{H}^{\varepsilon}_2$ we derive

$$(u, h)_{L^2_\varepsilon} = (\varepsilon u, h)_{L^2} = (\operatorname{curl} \Phi, h)_{L^2} = 0.$$

The converse implication similarly follows from the decomposition

$$L^2_\varepsilon(\Omega) = \nabla H^1_0(\Omega) \oplus_{L^2_\varepsilon} \mathbb{H}^{\varepsilon}_2 \oplus_{L^2_\varepsilon} \varepsilon^{-1} \operatorname{curl} H^1(\Omega), \tag{2.8}$$

see Propositions 6.1.1 and 6.1.12 in [1] and Proposition IX.1.3 in [4].

The second assertion is shown analogously, using Propositions 6.2.1 and 6.2.12 in [1] and Proposition IX.1.4 in [4]. □

Remark 2.5. Let (H) hold. The dimension of the first cohomology space equals the 'cutting number' L of Ω, i.e.,

$$\dim \mathbb{H}_1 = \dim \operatorname{N}(\operatorname{curl}_H) = L.$$

For simply connected Ω it is therefore trivial. Moreover, we have

$$\dim \mathbb{H}_2 = \dim \operatorname{N}(\operatorname{curl}_E) = K.$$

Thus, if $\partial\Omega$ is connected, the second cohomology space equals is trivial. See Proposition 2.8 in [3], as well as Proposition 6.1.1 and Proposition 6.2.1 in [1].

From now on we drop the subscript and simply write curl for both curl_E and curl_H. We have to control the normal trace of curls by tangential traces. To this aim we recall Lemma 4.8 of [15], see also Section 2.3 in [3].

Lemma 2.6. *For $f \in H^1(\Omega)$ we can estimate the normal trace of the curl by*

$$\|\nu \cdot \operatorname{curl} f\|_{H^{-1}(\partial\Omega)} \lesssim \|\nu \times f\|_{L^2(\partial\Omega)}.$$

Helmholtz decomposition. Our arguments rely on a splitting of the electric field $E \in C^1\big(\mathbb{R}_{\geq 0}, L^2(\Omega)\big) \cap C\big(\mathbb{R}_{\geq 0}, H_{t0}(\mathrm{curl})\big)$ into a div_ε-free and a curl-free part. The following construction is inspired by Chapter 5 in [16], where constant ε and connected $\partial\Omega$ and ω were treated using $p \in H_0^1(\Omega)$. We consider the elliptic problem

$$\begin{aligned} &\mathrm{div}(\varepsilon \nabla p) = \mathrm{div}(\varepsilon E) \quad \text{on } \Omega, \quad p = 0 \text{ on } \Gamma_0, \\ &p \text{ constant on } \Gamma_k \ \text{ and } \ \langle \mathrm{tr}_n\, \varepsilon(\nabla p - E), \mathbb{1}\rangle_{H^{-\frac{1}{2}}(\Gamma_k)} = 0, \ \ \forall\, k \in \{1, \dots, K\}. \end{aligned} \tag{2.9}$$

For the weak formulation of the problem, we define the Hilbert space

$$\mathcal{H}_c := \big\{\varphi \in H^1(\Omega) \;\big|\; \varphi = 0 \text{ on } \Gamma_0,\, \varphi \text{ constant on } \Gamma_k,\ \forall\, k \in \{1, \dots K\}\big\}$$

endowed with the norm of $H^1(\Omega)$. We introduce the form $B[u, v] := \int_\Omega \nabla u \cdot \varepsilon \nabla v \,\mathrm{d}x$ on $\mathcal{H}_c$ and set $\ell(v) = \ell_E(v) := \int_\Omega \varepsilon E \cdot \nabla v \,\mathrm{d}x$. Then the weak formulation of (2.9) reads as

$$B[p, \varphi] = \ell(\varphi) \quad \text{for all } \varphi \in \mathcal{H}_c. \tag{2.10}$$

Clearly, B and ℓ are bounded on $\mathcal{H}_c$, and B is coercive by Theorem 13.6.9 in [19]. The Lax–Milgram lemma yields a unique solution $p = p(t) \in \mathcal{H}_c$ of (2.10) with

$$\|p\|_{H^1(\Omega)} \lesssim \|\ell\|_{\mathcal{H}_c^*} = \sup_{\|\varphi\|_{\mathcal{H}_c} = 1} |\ell(\varphi)| \lesssim \|\varepsilon E\|_{L^2(\Omega)}\,. \tag{2.11}$$

We thus have bounded linear maps $L^2(\Omega) \to \mathcal{H}_c^*$; $E \mapsto \ell_E$, and $L^2(\Omega) \to \mathcal{H}_c$; $E \mapsto p$. Note that $\varepsilon E - \varepsilon \nabla p$ belongs to $H(\mathrm{div})$ with $\mathrm{div}(\varepsilon E - \varepsilon \nabla p) = 0$, since (2.10) holds in particular for $H_0^1(\Omega) \subseteq \mathcal{H}_c$. Hence, the last boundary condition in (2.9) is well-defined, and it follows by inserting $\varphi_k \in \mathcal{H}_c$ with $\varphi_k|_{\Gamma_j} = \delta_{kj}$ for $k \neq 0$ into (2.10) via

$$0 = \int_\Omega (\varepsilon E - \varepsilon \nabla p) \cdot \nabla \varphi_k \,\mathrm{d}x = \langle \mathrm{tr}_n(\varepsilon E - \varepsilon \nabla p), \mathbb{1}\rangle_{H^{-\frac{1}{2}}(\Gamma_k)}.$$

Finally, the PDE in (2.9) is understood in $\mathcal{H}_c^*$.

To pass to charge-free fields, we define

$$V := E - \nabla p \in C^1\big(\mathbb{R}_{\geq 0}, L^2(\Omega)\big) \cap C\big(\mathbb{R}_{\geq 0}, H_{t0}(\mathrm{curl})\big) \tag{2.12}$$

using Lemma 2.2 and (2.5). By the properties of p, this function satisfies

$$\mathrm{curl}\, V = \mathrm{curl}\, E, \qquad \varepsilon V \in H^\Gamma(\mathrm{div}\, 0). \tag{2.13}$$

For the last assertion, the condition $\langle \mathrm{tr}_n\, \varepsilon V, \mathbb{1}\rangle_{H^{-\frac{1}{2}}(\Gamma_k)} = 0$ is clear by (2.9) for $k > 0$. For $k = 0$ it then follows from $\mathrm{div}(\varepsilon V) = 0$ which yields $\langle \mathrm{tr}_n\, \varepsilon V, \mathbb{1}\rangle_{H^{-\frac{1}{2}}(\partial\Omega)} = 0$.

In other main arguments we use the time derivatives of p since they can be estimated solely by dissipation terms, in contrast to p in (2.11). This fact corresponds to the behavior of the charges $\mathrm{div}(\varepsilon E)$ in (1.5), where the time derivative is better suited for the study of the long-term behavior, too.

Lemma 2.7. *Let E be given by* (2.5). *Then the solution p of* (2.9) *belongs to $C^2(\mathbb{R}_{\geq 0}, \mathcal{H}_c)$ and satisfies*

$$\|\partial_t p\|_{H^1} \lesssim \|\partial_t \nabla p\|_{L^2} \lesssim \|\sigma E\|_{L^2(\omega)} \quad \text{and} \quad \big\|\partial_t^2 p\big\|_{H^1} \lesssim \big\|\partial_t^2 \nabla p\big\|_{L^2} \lesssim \|\sigma \partial_t E\|_{L^2(\omega)}\,.$$

Moreover, $\varepsilon \partial_t V$ is contained in $H^\Gamma(\mathrm{div}\, 0)$.

Proof. Because of (2.5), we can continuously differentiate $t \mapsto \ell_{E(t)}$ in $\mathcal{H}_c^*$ and thus p in $\mathcal{H}_c$. Equation (1.1) and Lemma 2.3 b) also yield

$$\partial_t \ell_{E(t)}(\varphi) = \int_\Omega \varepsilon \partial_t E \cdot \nabla \varphi \,\mathrm{d}x = -\int_\Omega \sigma E \cdot \nabla \varphi \,\mathrm{d}x \tag{2.14}$$

for $\varphi \in \mathcal{H}_c$. Estimate (2.11) then implies the first assertion, using also the Poincaré inequality in Theorem 13.6.9 in [19]. The last claim now follows from (2.13). Due to (2.14), we can differentiate in time once more and derive the remaining part. □

3. Observability estimate

Above we have split E into the charge-free part V and the curl-free part ∇p. To obtain observability results, we decompose (V, H) further into fields (V_h, W_h) and (V_i, W_i) solving a homogeneous and an inhomogeneous system, respectively. Here we combine and extend the approaches from [13] and [16]. The energy equality in Lemma 2.1 allows us to control dissipation terms, and thus the time derivatives of ∇p by Lemma 2.7. Accordingly, we first estimate $\partial_t V_h$ and thus by-pass the problem that the Gauß law (1.5) does not provide uniform bounds in t.

In the stability analysis it will be crucial that $\partial_t V_i(0)$ and $\partial_t W_i(0)$ vanish, see Lemma 4.5. To this aim, we choose suitable initial values for the inhomogeneous problem. By Lemma 2.7 the derivative $\varepsilon\partial_t V$ belongs to $H^\Gamma(\operatorname{div} 0)$, which implies

$$\sigma E + \varepsilon\partial_t \nabla p = -\varepsilon\partial_t E + \operatorname{curl} H + \varepsilon\partial_t\nabla p = -\varepsilon\partial_t V + \operatorname{curl} H \ \in H^\Gamma(\operatorname{div} 0) \tag{3.1}$$

since $\operatorname{curl} H \in H^\Gamma(\operatorname{div} 0)$ due to Lemma 2.3 b) as well as (1.2) and (1.3). Lemma 2.3 b) then provides a field $W_{i0} \in H(\operatorname{curl}) \cap H_{n0}(\operatorname{div}_\mu 0)$ which is orthogonal to $\mathrm{N}(\operatorname{curl}_H) = \mathbb{H}_1^\mu$ in L^2_μ and satisfies

$$\operatorname{curl} W_{i0} = \sigma E_0 + \varepsilon\partial_t\nabla p(0). \tag{3.2}$$

To derive the observability estimate for $\partial_t V_h$, we consider a second-order equation for our homogeneous problem using the generator

$$D(A_h) = \big(H_{t0}(\operatorname{curl}) \times H(\operatorname{curl})\big) \cap X_h, \qquad A_h = \begin{pmatrix} 0 & \varepsilon^{-1}\operatorname{curl} \\ -\mu^{-1}\operatorname{curl} & 0 \end{pmatrix} \tag{3.3}$$

without conductivity acting in the charge-free subspace

$$X_h := \big\{(v, w) \in L^2_\varepsilon(\Omega) \times L^2_\mu(\Omega) \;\big|\; \varepsilon v \in H^\Gamma(\operatorname{div} 0),\ \mu w \in H^\Sigma_{n0}(\operatorname{div} 0)\big\}.$$

of X. The operator A_h maps into X_h by Lemma 2.3. Its maximal dissipativity is a special case of that of A in (2.4) for $\sigma = 0$ and $\omega = \emptyset$. By (2.3), (2.12), (2.13) and $W_{i0} \in H_{n0}(\operatorname{div}_\mu 0)$, the fields $\big(E_0 - \nabla p(0), H_0 - W_{i0}\big)$ belong to $D(A_h)$.

We can approximate $\big(E_0 - \nabla p(0), H_0 - W_{i0}\big)$ in $D(A_h)$ by $(V_{h0}^{(n)}, W_{h0}^{(n)}) \in D(A_h^2)$ by means of Yosida approximations. For these initial fields we can derive a second-order problem. The homogeneous Maxwell system

$$\begin{aligned} \partial_t\big(\varepsilon V_h^{(n)}\big) &= \operatorname{curl} W_h^{(n)}, & \partial_t\big(\mu W_h^{(n)}\big) &= -\operatorname{curl} V_h^{(n)}, \\ \operatorname{div}\big(\varepsilon V_h^{(n)}\big) &= 0, & \operatorname{div}\big(\mu W_h^{(n)}\big) &= 0, \\ \operatorname{tr}_t V_h^{(n)} &= 0, & \operatorname{tr}_n(\mu W_h^{(n)}) &= 0, \\ V_h^{(n)}(0) &= V_{h0}^{(n)}, & W_h^{(n)}(0) &= W_{h0}^{(n)}, \end{aligned} \tag{3.4}$$

then has the solution

$$\big(V_h^{(n)}, W_h^{(n)}\big) \in C^2\big(\mathbb{R}_{\geq 0}, X_h\big) \cap C^1\big(\mathbb{R}_{\geq 0}, D(A_h)\big) \cap C\big(\mathbb{R}_{\geq 0}, D(A_h^2)\big). \tag{3.5}$$

It converges to the solution (V_h, W_h) of (3.4) with initial data $\big(E_0 - \nabla p(0), H_0 - W_{i0}\big)$ in the space $C_b^1\big(\mathbb{R}_{\geq 0}, X_h\big) \cap C_b\big(\mathbb{R}_{\geq 0}, D(A_h)\big)$. We stress that $\varepsilon V_h^{(n)}$ is divergence-free. The system (3.4) leads to the second-order equation

$$\begin{aligned} \partial_t\big(\varepsilon\partial_t V_h^{(n)}\big) &= -\operatorname{curl}\big(\mu^{-1}\operatorname{curl} V_h^{(n)}\big), \\ \operatorname{tr}_t V_h^{(n)} &= 0, \quad \operatorname{div}(\varepsilon V_h^{(n)}) = 0, \\ V_h^{(n)}(0) &= V_{h0}^{(n)}, \quad \partial_t V_h^{(n)}(0) = \varepsilon^{-1}\operatorname{curl} W_{h0}^{(n)}. \end{aligned} \tag{3.6}$$

Corresponding to (3.5), in (3.6) we assume that

$$\begin{aligned} V_h^{(n)} &\in C^2(\mathbb{R}_{\geq 0}, H^\Gamma(\operatorname{div}_\varepsilon 0)) \cap C^1\big(\mathbb{R}_{\geq 0}, H_{t0}(\operatorname{curl})\big), \\ \mu^{-1} \operatorname{curl} V_h^{(n)} &\in C\big(\mathbb{R}_{\geq 0}, H(\operatorname{curl})\big). \end{aligned} \tag{3.7}$$

To account for the missing terms in (3.4), we consider the inhomogeneous system

$$\begin{aligned} \partial_t\big(\varepsilon V_i\big) &= \operatorname{curl} W_i - \sigma E - \varepsilon \partial_t \nabla p, \\ \partial_t\big(\mu W_i\big) &= -\operatorname{curl} V_i, \\ \operatorname{div}\big(\mu W_i\big) &= 0, \\ \operatorname{tr}_t V_i &= 0, \quad \operatorname{tr}_n(\mu W_i) = 0, \\ V_i(0) &= 0, \quad W_i(0) = W_{i0}. \end{aligned} \tag{3.8}$$

Since $\big(-\varepsilon^{-1}\sigma E - \partial_t \nabla p, 0\big)$ belongs to $C^1\big(\mathbb{R}_{\geq 0}, X_h\big)$ due to (3.1) and $(0, W_{i0})$ to $D(A_h)$, this system has a unique solution

$$\big(V_i, W_i\big) \in C^1\big(\mathbb{R}_{\geq 0}, X_h\big) \cap C\big(\mathbb{R}_{\geq 0}, D(A_h)\big). \tag{3.9}$$

For later use we record that the definition of W_{i0} in (3.2) yields

$$\begin{aligned} \partial_t(\varepsilon V_i)(0) &= \operatorname{curl} W_{i0} - \sigma E_0 - \varepsilon \partial_t \nabla p(0) = 0, \\ \partial_t(\mu W_i)(0) &= -\operatorname{curl} V_i^{(n)}(0) = 0. \end{aligned} \tag{3.10}$$

Observe that the fields $\big(V_h + V_i + \nabla p, W_h + W_i\big)$ solve the original system (1.1)–(1.4) with initial values

$$\big(E_0 - \nabla p(0) + \nabla p(0), H_0 - W_{i0} + W_{i0}\big) = (E_0, H_0).$$

By uniqueness, the convergence noted after (3.5) leads to the limit

$$\big(V_h^{(n)} + V_i + \nabla p, W_h^{(n)} + W_i\big) \;\longrightarrow\; \big(V_h + V_i + \nabla p, W_h + W_i\big) = (E, H)$$

in $C_b^1\big(\mathbb{R}_{\geq 0}, X\big) \cap C_b\big(\mathbb{R}_{\geq 0}, D(A)\big)$.

To simplify notation, we drop the superscripts in this section and write $\big(V_h^{(n)}, W_h^{(n)}\big) = (V_h, W_h)$, assuming that (V_h, W_h) satisfies (3.5). The final result will then follow by the approximation argument above, see Remark 3.6. We next state the 'energy' identity for V_h.

Lemma 3.1. *Let V_h satisfy* (3.7) *and solve* (3.6)*. We then obtain*

$$\begin{aligned} &\int_\Omega \big(\varepsilon \partial_t V_h(t) \cdot \partial_t V_h(t) + \mu^{-1} \operatorname{curl} V_h(t) \cdot \operatorname{curl} V_h(t)\big)\, \mathrm{d}x \\ &\quad = \int_\Omega \big(\varepsilon \partial_t V_h(0) \cdot \partial_t V_h(0) + \mu^{-1} \operatorname{curl} V_h(0) \cdot \operatorname{curl} V_h(0)\big)\, \mathrm{d}x, \qquad t \geq 0. \end{aligned}$$

Proof. System (3.6) and integration by parts imply

$$\begin{aligned} 0 &= \int_0^t \int_\Omega \big(\partial_t^2(\varepsilon V_h) + \operatorname{curl} \mu^{-1} \operatorname{curl} V_h\big) \cdot \partial_t V_h \, \mathrm{d}x\, \mathrm{d}t \\ &= \frac{1}{2} \int_0^t \partial_t \int_\Omega \big(\partial_t(\varepsilon V_h) \cdot \partial_t V_h + \mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h\big)\, \mathrm{d}x\, \mathrm{d}t \\ &= \frac{1}{2} \Big[\int_\Omega \varepsilon \partial_t V_h(t) \cdot \partial_t V_h(t) + \mu^{-1} \operatorname{curl} V_h(t) \cdot \operatorname{curl} V_h(t)\, \mathrm{d}x \\ &\qquad - \int_\Omega \varepsilon \partial_t V_h(0) \cdot \partial_t V_h(0) + \mu^{-1} \operatorname{curl} V_h(0) \cdot \operatorname{curl} V_h(0)\, \mathrm{d}x \Big]. \end{aligned}$$

□

The following observability estimate for $\partial_t V_h$ is the core step in our arguments.

Proposition 3.2. *Assume that* (H) *and the non-trapping condition* (2.2) *hold. Let* V_h *satisfy* (3.7) *and solve* (3.6). *Then for any* $a > 0$ *there exists a time* $T_O > 0$ *such that for* $T \geq T_O$ *we have*

$$\int_\Omega \left(|\partial_t V_h(0)|^2 + |\operatorname{curl} V_h(0)|^2 \right) \mathrm{d}x \lesssim \int_0^T \int_{N_a} |\partial_t V_h|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

For the proof of Proposition 3.2 we need several auxiliary results. In the first one we use the Lax–Milgram lemma to construct a multiplier, cf. [13]. We endow the space $\mathcal{H} := H_{t0}(\operatorname{curl}) \cap H^\Gamma(\operatorname{div}_\varepsilon 0)$ with the norm $\|\cdot\|_{H(\operatorname{curl})}$. By Lemma 2.3 the operator $\operatorname{curl}_E \colon \mathcal{H} \to H^\Sigma_{n0}(\operatorname{div} 0)$ is boundedly invertible.

Lemma 3.3. *Take a map* $\tilde{\vartheta} \in L^\infty(\Omega)$ *with* $\tilde{\vartheta} = 1$ *on* $N_{a/2}$ *and* $\operatorname{supp} \tilde{\vartheta} \subseteq N_a$ *for some* $a > 0$, *and* $f \in L^2(\Omega)$. *Then there is a unique solution* $w \in \mathcal{H} \subseteq H^1(\Omega)$ *of*

$$\forall\, \psi \in \mathcal{H}: \qquad \int_\Omega \mu^{-1} \operatorname{curl} w \cdot \operatorname{curl} \psi \, \mathrm{d}x = \int_\Omega \tilde{\vartheta} f \psi \, \mathrm{d}x. \tag{3.11}$$

It satisfies

$$\|w\|_{H^1} \lesssim \|w\|_{\mathcal{H}} \lesssim \|\tilde{\vartheta} f\|_{L^2}. \tag{3.12}$$

Proof. Theorem A.6 a) of [15] (with $v = 0$ and $u \in H_{t0}(\operatorname{curl})$) yields

$$\|u\|_{H^1} \lesssim \|u\|_{L^2} + \|\operatorname{curl} u\|_{L^2} + \|\operatorname{div}(\varepsilon u)\|_{L^2}.$$

Since the divergence of εu vanishes for $u \in \mathcal{H}$ and the curl is invertible, we see that

$$\|u\|_{H^1} \lesssim \|\operatorname{curl} u\|_{L^2} \qquad \text{for } u \in \mathcal{H}.$$

Hence, the bilinear form on the left-hand side is coercive and bounded. The assertion then follows from the Lax–Milgram lemma, using that $L^2(\Omega) \hookrightarrow \mathcal{H}^\star$. ☐

For a time dependent $f \in C^1(\mathbb{R}_{\geq 0}, L^2)$ we obtain a solution $w(t)$ of (3.11) for each $t \geq 0$. The regularity of f transfers to w as shown next.

Remark 3.4. Let $f \in C^1(\mathbb{R}_{\geq 0}, L^2(\Omega))$ in Lemma 3.3. We then obtain

a) $w \in C^1(\mathbb{R}_{\geq 0}, \mathcal{H})$ and
b) $\|w(t)\|_{H^1}^2 \lesssim \int \tilde{\vartheta}\, |f(t)|^2 \, \mathrm{d}x$, $\|\partial_t w(t)\|_{H^1}^2 \lesssim \int \tilde{\vartheta}\, |\partial_t f(t)|^2 \, \mathrm{d}x$.

Proof. We first check that w is C^1. Its Lipschitz continuity follows from (3.12) and linearity via

$$\left\| \tfrac{1}{h} \big(w(t+h) - w(t) \big) \right\|_{H^1} \lesssim \left\| \tfrac{1}{h} \big(f(t+h) - f(t) \big) \right\|_{L^2}.$$

Hence, w is differentiable in $H^1(\Omega)$ for a.e. $t \geq 0$. We can now differentiate (3.11) in t a.e., obtaining

$$\int_\Omega \mu^{-1} \operatorname{curl} \partial_t w \cdot \operatorname{curl} \psi \, \mathrm{d}x = \int_\Omega \tilde{\vartheta} \partial_t f \psi \, \mathrm{d}x.$$

As above we then infer the continuity of $\partial_t w$ from (3.12), which also yields the second estimate. The first one follows from (3.12) for w. ☐

We now turn to the proof of the core result of this section.

Proof of Proposition 3.2. 1) Let $T > 0$, $\chi \in C^1(\overline{\Omega})$, and $m(x) = x - x_0$ for some $x_0 \in \mathbb{R}^3$. We multiply (3.6) with the Morawetz multiplier $\chi m \times \operatorname{curl} V_h$ obtaining

$$\begin{aligned}
0 &= \int_0^T \int_\Omega \big(\partial_t(\varepsilon \partial_t V_h) + \operatorname{curl} \mu^{-1} \operatorname{curl} V_h \big) \cdot \big(\chi m \times \operatorname{curl} V_h \big) \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_\Omega \partial_t \big((\varepsilon \partial_t V_h) \cdot (\chi m \times \operatorname{curl} V_h) \big) \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega (\varepsilon \partial_t V_h) \cdot \big(\chi m \times \operatorname{curl} \partial_t V_h \big) \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T\int_\Omega \mu^{-1}\operatorname{curl} V_h\cdot \operatorname{curl}\big(\chi m\times \operatorname{curl} V_h\big)\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\oint_{\partial\Omega} \mu^{-1}\operatorname{curl} V_h\cdot \big(\nu\times(\chi m\times \operatorname{curl} V_h)\big)\,\mathrm{d}\varsigma\,\mathrm{d}t\\
&=: I_i+I_t+I_c+I_c^{\partial}.
\end{aligned}
\tag{3.13}
$$

In the above four integrands we have to eliminate the second derivatives of V_h, which is easy for I_i. In the following we freely use standard formulas from vector analysis. Also integrating by parts and exploiting $\operatorname{tr}_t \partial_t V_h = 0$, we compute

$$
I_t=\int_0^T\int_\Omega \operatorname{curl}\partial_t V_h\cdot(\chi m\times\varepsilon\partial_t V_h)\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega \partial_t V_h\cdot\operatorname{curl}(\chi m\times\varepsilon\partial_t V_h)\,\mathrm{d}x\,\mathrm{d}t.
$$

Since $\operatorname{div}(\varepsilon V_h)=0$, the integrand can be rewritten as

$$
\begin{aligned}
&\operatorname{curl}(\chi m\times\varepsilon\partial_t V_h)\\
&\quad=\big(\varepsilon\partial_t V_h\cdot\nabla\big)(\chi m)-\chi(m\cdot\nabla)(\varepsilon\partial_t V_h)+\chi m\partial_t\operatorname{div}(\varepsilon V_h)-\varepsilon\partial_t V_h\operatorname{div}(\chi m)\\
&\quad=\big(\varepsilon\partial_t V_h\cdot\nabla\chi\big)m+\chi\varepsilon\partial_t V_h-\chi(m\cdot\nabla)(\varepsilon\partial_t V_h)-(\varepsilon\partial_t V_h)\nabla\chi\cdot m-3\chi\varepsilon\partial_t V_h.
\end{aligned}
$$

Only the middle term involves second derivatives of V_h. In this summand of the integrand we can take out the gradient via

$$
\chi(m\cdot\nabla)(\varepsilon\partial_t V_h)\cdot\partial_t V_h=\tfrac12\chi(m\cdot\nabla)(\varepsilon\partial_t V_h\cdot\partial_t V_h)+\tfrac12\chi\big((m\cdot\nabla)\varepsilon\big)\partial_t V_h\cdot\partial_t V_h
$$

by means of the product rule. The remaining problematic part of I_t is now integrated by parts resulting in

$$
\begin{aligned}
&-\int_0^T\int_\Omega\frac12\chi(m\cdot\nabla)(\varepsilon\partial_t V_h\cdot\partial_t V_h)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad=\int_0^T\int_\Omega\Big(\frac12(\nabla\chi\cdot m)(\varepsilon\partial_t V_h\cdot\partial_t V_h)+\frac32\chi(\varepsilon\partial_t V_h\cdot\partial_t V_h)\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\quad-\int_0^T\oint_{\partial\Omega}\frac12\chi\nu\cdot m(\varepsilon\partial_t V_h\cdot\partial_t V_h)\,\mathrm{d}\varsigma\,\mathrm{d}t.
\end{aligned}
$$

Recalling $\tilde\varepsilon=\varepsilon+(m\cdot\nabla)\varepsilon$ from (2.2), we conclude

$$
\begin{aligned}
I_t=&\int_0^T\int_\Omega\Big[-\frac12\chi\tilde\varepsilon\partial_t V_h\cdot\partial_t V_h-\frac12(\nabla\chi\cdot m)(\varepsilon\partial_t V_h\cdot\partial_t V_h)+\partial_t V_h\cdot(\varepsilon\partial_t V_h\cdot\nabla\chi)m\Big]\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\oint_{\partial\Omega}\frac12\chi\nu\cdot m(\varepsilon\partial_t V_h\cdot\partial_t V_h)\,\mathrm{d}\varsigma\,\mathrm{d}t.
\end{aligned}
\tag{3.14}
$$

We next treat I_c in a similar way, reformulating it as

$$
\begin{aligned}
I_c=\int_0^T\int_\Omega\mu^{-1}\operatorname{curl}V_h\cdot\big[&\operatorname{div}\operatorname{curl}(V_h)\chi m-\operatorname{div}(\chi m)\operatorname{curl}V_h\\
&+(\operatorname{curl}V_h\cdot\nabla)(\chi m)-(\chi m\cdot\nabla)\operatorname{curl}V_h\big]\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

The last term is the only problematic one. As above it is equal to

$$
\begin{aligned}
&-\int_0^T\int_\Omega(\mu^{-1}\operatorname{curl}V_h)\cdot\big((\chi m\cdot\nabla)\operatorname{curl}V_h\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\frac12\int_0^T\int_\Omega\Big[(\chi m\cdot\nabla\mu^{-1})\operatorname{curl}V_h\cdot\operatorname{curl}V_h-\chi(m\cdot\nabla)(\mu^{-1}\operatorname{curl}V_h\cdot\operatorname{curl}V_h)\Big]\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

We integrate by parts the last term and obtain

$$
I_c=\int_0^T\int_\Omega\mu^{-1}\operatorname{curl}V_h\cdot\Big((\operatorname{curl}V_h\cdot\nabla\chi)m+\chi\operatorname{curl}V_h-\frac12\operatorname{div}(\chi m)\operatorname{curl}V_h\Big)\mathrm{d}x\,\mathrm{d}t
$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^T\int_\Omega \big((\chi m\cdot\nabla)\mu^{-1}\big)\operatorname{curl}V_h\cdot\operatorname{curl}V_h\,\mathrm{d}x\,\mathrm{d}t\\
&-\frac{1}{2}\int_0^T\oint_{\partial\Omega}\chi(m\cdot\nu)(\mu^{-1}\operatorname{curl}V_h\cdot\operatorname{curl}V_h)\,\mathrm{d}\varsigma\,\mathrm{d}t\\
=&\int_0^T\int_\Omega\Big[-\frac{1}{2}\chi\tilde{\mu}_{-1}\operatorname{curl}V_h\cdot\operatorname{curl}V_h+\mu^{-1}\operatorname{curl}V_h\cdot\big((\operatorname{curl}V_h\cdot\nabla\chi)m\big)\\
&\qquad-\frac{1}{2}(m\cdot\nabla\chi)(\operatorname{curl}V_h\cdot\mu^{-1}\operatorname{curl}V_h)\Big]\mathrm{d}x\,\mathrm{d}t\\
&-\frac{1}{2}\int_0^T\oint_{\partial\Omega}\chi(m\cdot\nu)(\mu^{-1}\operatorname{curl}V_h\cdot\operatorname{curl}V_h)\,\mathrm{d}\varsigma\,\mathrm{d}t, \qquad (3.15)
\end{aligned}
$$

Note that (2.2) implies $\tilde{\mu}_{-1}\coloneqq\mu^{-1}-m\cdot\nabla\mu^{-1}\geq\eta\mu^{-1}$.

Using Lemma 2.6, the last integral in (3.13) can be rewritten as

$$
\begin{aligned}
I_c^\partial&=-\int_0^T\oint_{\partial\Omega}\mu^{-1}\operatorname{curl}V_h\cdot\big(\nu\times(\chi m\times\operatorname{curl}V_h)\big)\,\mathrm{d}\varsigma\,\mathrm{d}t\\
&=\int_0^T\oint_{\partial\Omega}\nu\cdot\big(\mu^{-1}\operatorname{curl}V_h\times(\chi m\times\operatorname{curl}V_h)\big)\,\mathrm{d}\varsigma\,\mathrm{d}t\\
&=\int_0^T\oint_{\partial\Omega}\chi(\nu\cdot m)(\mu^{-1}\operatorname{curl}V_h\cdot\operatorname{curl}V_h)\,\mathrm{d}\varsigma\,\mathrm{d}t. \qquad (3.16)
\end{aligned}
$$

Equations (3.13), (3.14), (3.15) and (3.16) then yield the core identity

$$
\begin{aligned}
&\int_0^T\int_\Omega\frac{1}{2}\chi\big(\tilde{\varepsilon}\partial_tV_h\cdot\partial_tV_h+\tilde{\mu}^{-1}\operatorname{curl}V_h\cdot\operatorname{curl}V_h\big)\,\mathrm{d}x\,\mathrm{d}t \qquad (3.17)\\
&=\Big[\int_\Omega\varepsilon\partial_tV_h\cdot(\chi m\times\operatorname{curl}V_h)\,\mathrm{d}x\Big]_0^T\\
&\quad-\int_0^T\int_\Omega\frac{1}{2}(\nabla\chi\cdot m)\big(\varepsilon\partial_tV_h\cdot\partial_tV_h+\mu^{-1}\operatorname{curl}V_h\cdot\operatorname{curl}V_h\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\int_\Omega\big((m\cdot\partial_tV_h)(\partial_t\varepsilon V_h\cdot\nabla\chi)+(m\cdot\mu^{-1}\operatorname{curl}V_h)(\operatorname{curl}V_h\cdot\nabla\chi)\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\oint_{\partial\Omega}\frac{1}{2}\chi(m\cdot\nu)\big(\mu^{-1}\operatorname{curl}V_h\cdot\operatorname{curl}V_h-\varepsilon\partial_tV_h\cdot\partial_tV_h\big)\,\mathrm{d}\varsigma\,\mathrm{d}t.
\end{aligned}
$$

2) We first take $\chi=1$. Then (3.17) and the non-trapping condition (2.2) imply

$$
\begin{aligned}
&\eta\int_0^T\int_\Omega\big(\,|\partial_tV_h|^2+|\operatorname{curl}V_h|^2\,\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\lesssim\int_\Omega|\partial_tV_h(T)|\,|\operatorname{curl}V_h(T)|\,\mathrm{d}x+\int_\Omega|\partial_tV_h(0)|\,|\operatorname{curl}V_h(0)|\,\mathrm{d}x\\
&\qquad+\int_0^T\oint_{\partial\Omega}\big(\nu\cdot m)(\mu^{-1}\operatorname{curl}V_h\cdot\operatorname{curl}V_h-\varepsilon\partial_tV_h\cdot\partial_tV_h\big)\,\mathrm{d}\varsigma\,\mathrm{d}t.
\end{aligned}
$$

Next, for $\chi\in C^1(\overline{\Omega})$ with $\chi=1$ on $\partial\Omega$ and support in $N_{a/4}$, Equation (3.17) yields

$$
\begin{aligned}
&\Big|\int_0^T\oint_{\partial\Omega}(\nu\cdot m)\big(\mu^{-1}\operatorname{curl}V_h\cdot\operatorname{curl}V_h-\varepsilon\partial_tV_h\cdot\partial_tV_h\big)\,\mathrm{d}\varsigma\,\mathrm{d}t\Big|\\
&\quad\lesssim\int_\Omega|\partial_tV_h(T)|\,|\operatorname{curl}V_h(T)|\,\mathrm{d}x+\int_\Omega|\partial_tV_h(0)|\,|\operatorname{curl}V_h(0)|\,\mathrm{d}x\\
&\qquad+\int_0^T\int_{N_{a/4}}\big(\,|\partial_tV_h|^2+|\operatorname{curl}V_h|^2\,\big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

The last two estimates lead to

$$\int_0^T\int_\Omega \big(|\partial_t V_h|^2 + |\operatorname{curl} V_h|^2\big)\,\mathrm{d}x\,\mathrm{d}t \lesssim \int_0^T\int_{N_{a/4}} \big(\,|\partial_t V_h|^2 + |\operatorname{curl} V_h|^2\,\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_\Omega |\partial_t V_h(T)|\,|\operatorname{curl} V_h(T)|\,\mathrm{d}x + \int_\Omega |\partial_t V_h(0)|\,|\operatorname{curl} V_h(0)|\,\mathrm{d}x. \quad (3.18)$$

3) In light of Lemma 3.1 we only have to estimate the curl term on $N_{a/4}$ to obtain the desired result, see (3.21). To this aim, take $\vartheta \in C^1(\overline{\Omega})$ with $\operatorname{supp}\vartheta \subseteq N_{a/2}$ and $\vartheta = 1$ on $N_{a/4}$. Equation (3.6) and integration by parts yield

$$\begin{aligned} 0 &= \int_0^T\int_\Omega (\partial_t^2(\varepsilon V_h) + \operatorname{curl}(\mu^{-1}\operatorname{curl} V_h))\cdot(\vartheta V_h)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T\int_\Omega \varepsilon\partial_t V_h\cdot\partial_t(\vartheta V_h)\,\mathrm{d}x\,\mathrm{d}t + \Big[\int_\Omega (\varepsilon\partial_t V_h)\cdot\vartheta V_h\,\mathrm{d}x\Big]_0^T \\ &\quad + \int_0^T\int_\Omega \mu^{-1}\operatorname{curl} V_h\cdot\big(\vartheta\operatorname{curl} V_h + \nabla\vartheta\times V_h\big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T\oint_{\partial\Omega} (\nu\times\mu^{-1}\operatorname{curl} V_h)\cdot(\vartheta V_h)\,\mathrm{d}\varsigma\,\mathrm{d}t. \end{aligned}$$

The boundary integral vanishes by (3.6) since

$$(\nu\times\mu^{-1}\operatorname{curl} V_h)\cdot(\vartheta V_h) = (\vartheta V_h\times\nu)\cdot(\mu^{-1}\operatorname{curl} V_h) = 0$$

on $\partial\Omega$. As $\mu^{-1} \geq \eta/\|\mu\|_\infty$, for any $\delta > 0$ we derive

$$\begin{aligned} &\int_0^T\int_{N_{a/4}} |\operatorname{curl} V_h|^2\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T\int_{N_{a/2}} \vartheta\,|\operatorname{curl} V_h|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim \delta\int_0^T\int_{N_{a/2}} |\operatorname{curl} V_h|^2\,\mathrm{d}x\,\mathrm{d}t + c_\delta\int_0^T\int_{N_{a/2}} |V_h|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{N_{a/2}} |\partial_t V_h|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_\Omega \big(\,|\partial_t V_h(T)|^2 + |V_h(T)|^2\,\big)\,\mathrm{d}x + \int_\Omega \big(\,|\partial_t V_h(0)|^2 + |V_h(0)|^2\,\big)\,\mathrm{d}x. \end{aligned}$$

We insert this inequality into (3.18) and absorb the curl term fixing a small $\delta > 0$. Using also Lemma 3.1, it follows

$$\begin{aligned} &\int_0^T\int_\Omega \big(\,|\partial_t V_h|^2 + |\operatorname{curl} V_h|^2\,\big)\,\mathrm{d}x\,\mathrm{d}t \qquad (3.19) \\ &\quad\lesssim \int_0^T\int_{N_{a/2}} \big(\,|V_h|^2 + |\partial_t V_h|^2\,\big)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \big(\,|\partial_t V_h(0)|^2 + |\operatorname{curl} V_h(0)|^2\,\big)\,\mathrm{d}x. \end{aligned}$$

4) We have to get rid of the new term involving V_h. Taking $w \in H_{t0}(\operatorname{curl})$ from Lemma 3.3 with $f = V_h$ as a multiplier, like in step 3) we derive

$$\begin{aligned} 0 &= \int_0^T\int_\Omega \big(\partial_t^2(\varepsilon V_h) + \operatorname{curl}(\mu^{-1}\operatorname{curl} V_h)\big)\cdot w\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\int_\Omega \big((\mu^{-1}\operatorname{curl} V_h)\cdot\operatorname{curl} w - \varepsilon\partial_t V_h\cdot\partial_t w\big)\,\mathrm{d}x\,\mathrm{d}t + \Big[\int_\Omega \varepsilon\partial_t V_h\cdot w\,\mathrm{d}x\Big]_0^T. \end{aligned}$$

As V_h belongs to $\mathcal{H}$ by (3.7), Equation (3.11) with $V_h = \psi$ then yields

$$0 = \int_0^T\int_\Omega \big(\tilde{\vartheta}\,|V_h|^2 - \varepsilon\partial_t V_h\cdot\partial_t w\big)\,\mathrm{d}x\,\mathrm{d}t + \Big[\int_\Omega \varepsilon\partial_t V_h\cdot w\,\mathrm{d}x\Big]_0^T.$$

The properties of $\tilde{\vartheta}$, Remark 3.4, Lemma 3.1 and Lemma 2.3 a) then imply

$$\begin{aligned}\int_0^T\int_{N_{a/2}} |V_h|^2 \,\mathrm{d}x\,\mathrm{d}t &\lesssim \delta \int_0^T\int_\Omega |\partial_t V_h|^2 \,\mathrm{d}x\,\mathrm{d}t + c_\delta \int_0^T\int_\Omega \tilde{\vartheta}\, |\partial_t V_h|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_\Omega \big(|\partial_t V_h(T)|^2 + |V_h(T)|^2\big)\mathrm{d}x + \int_\Omega \big(|\partial_t V_h(0)|^2 + |V_h(0)|^2\big)\mathrm{d}x \\ &\lesssim \delta \int_0^T\int_\Omega |\partial_t V_h|^2 \,\mathrm{d}x\,\mathrm{d}t + c_\delta \int_0^T\int_\Omega \tilde{\vartheta}\, |\partial_t V_h|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_\Omega \big(\, |\partial_t V_h(0)|^2 + |\mathrm{curl}\, V_h(0)|^2 \,\big)\,\mathrm{d}x.\end{aligned}$$

We insert this inequality in (3.19). For a fixed small $\delta > 0$, the space-time term without localization can be absorbed by the left-hand side resulting in

$$\begin{aligned}&\int_0^T\int_\Omega \big(|\partial_t V_h|^2 + |\,\mathrm{curl}\, V_h|^2\big)\,\mathrm{d}x\,\mathrm{d}t \qquad (3.20)\\ &\qquad \lesssim \int_0^T\int_{N_a} |\partial_t V_h|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \big(|\partial_t V_h(0)|^2 + |\,\mathrm{curl}\, V_h(0)|^2\big)\,\mathrm{d}x.\end{aligned}$$

5) In a final step we simplify the time integral on the left-hand side of (3.20) by means of Lemma 3.1, and obtain

$$\begin{aligned}T\int_\Omega \big(|\partial_t V_h(0)|^2 + |\,\mathrm{curl}\, V_h(0)|^2\big)\,\mathrm{d}x &\le c_0 \int_\Omega \big(|\partial_t V_h(0)|^2 + |\,\mathrm{curl}\, V_h(0)|^2\big)\,\mathrm{d}x \\ &\quad + c_1 \int_0^T\int_{N_a} |\partial_t V_h|^2 \,\mathrm{d}t\,\mathrm{d}x \qquad (3.21)\end{aligned}$$

for some constants $c_j > 0$. Taking $T > c_0$, we conclude

$$\int_\Omega \big(|\partial_t V_h(0)|^2 + |\,\mathrm{curl}\, V_h(0)|^2\big)\,\mathrm{d}x \le \frac{c_1}{T - c_0}\int_0^T\int_{N_a} |\partial_t V_h|^2 \,\mathrm{d}t\,\mathrm{d}x. \qquad \square$$

One can also show such an observability estimate for V_h instead of $\partial_t V_h$; that is, for the solutions to a homogeneous Maxwell system ($\sigma = 0$) with divergence free initial values. As in [13] we pass to an antiderivative in the proof.

Corollary 3.5. *Assume that* (H) *and the non-trapping condition* (2.2) *hold. Let* (V_h, W_h) *satisfy* (3.7) *and solve* (3.4). *For each* $a > 0$ *there exists a time* $T_O > 0$ *such that for* $T \ge T_O$ *we have*

$$\int_\Omega \big(|V_h(0)|^2 + |W_h(0)|^2\big)\,\mathrm{d}x \lesssim \int_0^T\int_{N_a} |V_h|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Since $\mu W_h(0) \in H^\Sigma_{n0}(\mathrm{div}\, 0)$, Lemma 2.3 a) yields a field u_0 in $H_{t0}(\mathrm{curl}) \cap H^\Gamma(\mathrm{div}_\varepsilon\, 0)$ with $\mathrm{curl}\, u_0 = -\mu W_h(0)$. We set

$$u(t) \coloneqq u_0 + \int_0^t V_h(s)\,\mathrm{d}s, \qquad t \ge 0,$$

obtaining $u(0) = u_0$ and $\partial_t u(0) = V_h(0)$. Note that u satisfies (3.7). We check that u solves (3.6) with different initial values. The definition of u_0 and (3.4) yield

$$\begin{aligned}\partial_t^2(\varepsilon u(t)) &= \partial_t(\varepsilon V_h(t) = \mathrm{curl}\, W_h(t) = \mathrm{curl}\,\Big(\mu^{-1}\Big(\mu W_h(0) + \int_0^t \partial_t(\mu W_h(s))\,\mathrm{d}s\Big)\Big) \\ &= -\,\mathrm{curl}\,\Big(\mu^{-1}\,\mathrm{curl}\,\Big(u_0 + \int_0^t V_h(s)\,\mathrm{d}s\Big)\Big) = -\,\mathrm{curl}\,\big(\mu^{-1}\,\mathrm{curl}\, u(t)\big),\\ \mathrm{tr}_t\, u(t) &= \mathrm{tr}_t\, u_0 + \int_0^t \mathrm{tr}_t\, V_h(s)\,\mathrm{d}s = 0,\end{aligned}$$

$$\operatorname{div}(\varepsilon u(t)) = \operatorname{div}(\varepsilon u_0) + \int_0^t \operatorname{div}(\varepsilon V_h(s))\,\mathrm{d}s = 0.$$

The corollary now follows from Proposition 3.2 as $\partial_t u = V_h$ and $\operatorname{curl} u = -\mu W_h$. $\square$

Remark 3.6. In Lemma 3.1, Proposition 3.2 and Corollary 3.5 we have assumed that $(V_h(0), W_h(0)) = \big(V_{h0}^{(n)}, W_{h0}^{(n)}\big)$ belongs to $D(A_h^2)$ and so the solutions $(V_h, W_h) = \big(V_h^{(n)}, W_h^{(n)}\big)$ satisfy (3.5). By the approximation argument discussed after (3.8), the lemma and the proposition can be extended to $(V_h(0), W_h(0))$ in $D(A_h)$. Similarly, one derives the corollary for $(V_h(0), W_h(0)) \in X_h$.

We reformulate the above corollary as (exact) observability and controllability of (1.1)–(1.4) with $\operatorname{div}(\varepsilon E) = 0$ and $\sigma = 0$. In this charge-free case we have $p = 0$ and $(V_h, W_h) = (E, H)$, see (2.9), (3.2), (3.4) and (3.8). The equivalence of observability and controllability is shown in Theorem 11.2.1 in [19], noting that A_h is skew-adjoint in $X_h = H^\Gamma(\operatorname{div}_\varepsilon 0) \times H^\Sigma_{n0}(\operatorname{div}_\mu 0)$, see (3.3).

Theorem 3.7. *Assume that* (H) *with* $\sigma = 0$ *and the non-trapping condition* (2.2) *hold. Let* $a > 0$. *Then there is a time* $T_O > 0$ *and a constant* $c > 0$ *such that for* $(E_0, H_0) \in X_h$ *we have*

$$\int_\Omega \big(|E_0|^2 + |H_0|^2\big)\,\mathrm{d}x \le c\int_0^{T_O}\int_{N_a} |E|^2\,\mathrm{d}x\,\mathrm{d}t,$$

where $(E, H) \in C(\mathbb{R}_{\ge 0}, X_h)$ *solves* (1.1)–(1.4) *with* $\sigma = 0$.

Moreover, for each $T \ge T_O$ *and* $(E_1, H_1) \in X_h$ *we find a current* $J \in L^2((0,T) \times N_a)$ *with values in* $H^\Gamma(\operatorname{div}_\varepsilon 0)$ *such that* $(\hat{E}(T), \hat{H}(T)) = (E_1, H_1)$ *for the solution* $(\hat{E}, \hat{H}) \in C(\mathbb{R}_{\ge 0}, X_h)$ *of* (1.1)–(1.4) *with* σE *replaced by* J.

4. Exponential decay

We now present the second main result of the paper.

Theorem 4.1. *Let* (H), (2.1) *and* (2.2) *hold. Then there exist constants* $M \ge 1$ *and* $\omega > 0$ *such that*

$$\|(E(t), H(t))\|_{L^2} \le M\mathrm{e}^{-\omega t}\|(E_0, H_0)\|_{L^2}, \qquad t \ge 0,$$

for the solution (E, H) *of* (1.1)–(1.4) *with initial value* $(E_0, H_0) \in X$.

Our proof relies on Lemma 4.3 which estimates the usual energy through time derivatives. This is needed to control the inhomogeneous part (V_i, W_i) via Lemma 4.5. Theorem 5.4 of [16] provides such an inequality for constant ε and μ and for connected $\partial\Omega$ and ω. The argument there is based on the splitting $E = V + \nabla p$. We could not extend this approach to our setting and proceed in a more direct way using the following estimate of E on υ.

Lemma 4.2. *For* $E \in H^\Gamma(\operatorname{div}_\varepsilon 0, \upsilon) \cap H(\operatorname{curl}, \upsilon)$, *we have*

$$\|E\|_{L^2(\upsilon)} \lesssim \|\operatorname{curl} E\|_{L^2(\upsilon)} + \|\operatorname{tr}_t E\|_{H^{-1/2}(\partial\upsilon)}.$$

Proof. By Proposition IX.1.3 of [4] the field E belongs to $\varepsilon^{-1}\operatorname{curl} H^1(\upsilon) = H^\Gamma(\operatorname{div}_\varepsilon 0, \upsilon)$, see also Lemma 2.3. In view the decomposition (2.8), we can thus compute its norm in $L^2_\varepsilon(\upsilon)$ by testing with $\varphi = \varepsilon^{-1}\operatorname{curl}\Phi$ for $\Phi \in H^1(\upsilon)$. Theorem 3.4.1 in [1] allows us to choose the vector potential Φ such that $\|\Phi\|_{H^1(\upsilon)} \lesssim \|\varepsilon\varphi\|_{L^2(\upsilon)}$. It follows

$$\|E\|_{L^2_\varepsilon(\upsilon)} = \sup_{\substack{\varphi \in H^\Gamma(\operatorname{div}_\varepsilon 0, \upsilon),\\ \|\varphi\|_{L^2_\varepsilon(\upsilon)} = 1}} \int_\upsilon E \cdot \varepsilon\varphi\,\mathrm{d}x \lesssim \sup_{\substack{\Phi \in H^1(\upsilon),\\ \|\Phi\|_{H^1(\upsilon)} = 1}} \int_\upsilon E \cdot \operatorname{curl}\Phi\,\mathrm{d}x. \tag{4.1}$$

Integration by parts then yields

$$\begin{aligned}\int_{\upsilon} E\cdot \operatorname{curl}\Phi \,\mathrm{d}x &= \int_{\upsilon} \operatorname{curl} E\cdot \Phi\,\mathrm{d}x + \langle \operatorname{tr}_t E, \Phi\rangle_{H^{-1/2}(\partial\upsilon)}\\ &\le \|\operatorname{curl} E\|_{L^2(\upsilon)}\, \|\Phi\|_{L^2(\upsilon)} + \|\operatorname{tr}_t E\|_{H^{-1/2}(\partial\upsilon)}\, \|\Phi\|_{H^1(\upsilon)}\end{aligned}$$

We infer the assertion by inserting this inequality into (4.1). □

The next proof relies on the assumption that σ either vanishes or is uniformly positive.

Lemma 4.3. *Let* (H) *and* (2.1) *hold. Take a solution* (E,H) *of* (1.1)–(1.4) *as in* (2.5)*. The energies* $\mathcal{E}$ *and* $\mathcal{D}$ *from* (2.6) *then satisfy*

$$\mathcal{E}(T) \lesssim \mathcal{D}(T), \qquad T\ge 0.$$

Proof. We adopt some ideas from the proof of Theorem 5.4 in [16]. Condition (H), the Maxwell equations (1.1), (1.3) and integration by parts lead to

$$\begin{aligned}\sigma_0 \int_\omega |E(t)|^2\,\mathrm{d}x &\le \int_\Omega \sigma E(t)\cdot E(t) = \int_\Omega \big(-\varepsilon\partial_t E(t) + \operatorname{curl} H(t)\big)\cdot E(t)\,\mathrm{d}x\\ &= -\int_\Omega \varepsilon\partial_t E(t)\cdot E(t) - \int_\Omega H(t)\cdot \mu\partial_t H(t)\,\mathrm{d}x\\ &\lesssim \sqrt{\mathcal{E}(t)}\sqrt{\mathcal{D}(t)}. \end{aligned} \tag{4.2}$$

Using also (2.5), Lemma 2.3 b) and (1.1), we then estimate the magnetic field by

$$\|H(t)\|_{L^2}^2 \lesssim \|\operatorname{curl} H(t)\|_{L^2}^2 \lesssim \|\varepsilon\partial_t E(t)\|_{L^2}^2 + \|\sigma E(t)\|_{L^2}^2 \lesssim \mathcal{D}(t) + \sqrt{\mathcal{E}(t)\mathcal{D}(t)}. \tag{4.3}$$

To control E on υ, we recall

$$\|E(t)\|_{L^2(\upsilon)} \lesssim \|\operatorname{curl} E(t)\|_{L^2(\upsilon)}^2 + \|\operatorname{tr}_t E(t)\|_{H^{-1/2}(\partial\upsilon)}\,.$$

from Lemma 4.2. Since $E\in H(\mathrm{curl})$, Proposition 2.2.32 in [1] shows that $\operatorname{tr}_{t,\partial\upsilon} E = \operatorname{tr}_{t,\partial\omega} E$ on $\partial\upsilon$. From the usual trace estimate and Equation (1.1) we thus deduce

$$\begin{aligned}\|E(t)\|_{L^2(\upsilon)} &\lesssim \|\operatorname{curl} E(t)\|_{L^2(\upsilon)} + \|\operatorname{tr}_t E(t)\|_{H^{-\frac12}(\partial\omega)} \lesssim \|\mu\partial_t H(t)\|_{L^2} + \|E(t)\|_{H(\mathrm{curl},\omega)}\\ &\lesssim \|\mu\partial_t H(t)\|_{L^2} + \|E(t)\|_{L^2(\omega)}\,.\end{aligned}$$

Combined with (4.2) and (4.3), we arrive at

$$\mathcal{E}(t) \le c_1\sqrt{\mathcal{E}(t)}\sqrt{\mathcal{D}(t)} + c_2\mathcal{D}(t) \le \tfrac12 \mathcal{E}(t) + C\mathcal{D}(t)$$

which yields the assertion. □

The time derivative of magnetic field can be estimated by the electric one using our Helmholtz decomposition. We proceed similar as in the proof of Lemma 5.2 of [16], see also Proposition 4.4 in [12]. Here and below the estimates depend on the end time, which fortunately does not cause problems in the main argument.

Lemma 4.4. *Let* (H) *hold. Take a solution* (E,H) *of* (1.1)–(1.4) *as in* (2.5)*. For* $T>0$ *we then obtain*

$$\int_0^T\int_\Omega |\partial_t H|^2\,\mathrm{d}x\,\mathrm{d}t \lesssim (1+T)\int_0^T\int_\Omega |\partial_t E|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \sigma E\cdot E\,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Fix $\Phi\in C_c^\infty((0,T))$ such that $0\le\Phi\le 1$ and $\Phi=1$ on $[\frac13 T, \frac23 T]$. In (2.12) and (2.13), we have decomposed $E = V+\nabla p$ with $V\in H_{t0}(\mathrm{curl})\cap H^\Gamma(\mathrm{div}_\varepsilon 0)$. Furthermore (after regularization) E solves the second-order problem

$$\varepsilon\partial_t^2 E = -\operatorname{curl}(\mu^{-1}\operatorname{curl} E) - \sigma\partial_t E.$$

Starting from (1.1) and integrating by parts, we then compute

$$\begin{aligned}\int_0^T\int_\Omega \Phi^2\mu\partial_t H\cdot\partial_t H\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \Phi^2\operatorname{curl}E\cdot\mu^{-1}\operatorname{curl}E\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_\Omega \Phi^2\operatorname{curl}V\cdot\mu^{-1}\operatorname{curl}E\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_\Omega \Phi^2 V\cdot\operatorname{curl}\big(\mu^{-1}\operatorname{curl}E\big)\,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_0^T\int_\Omega \Phi^2 V\cdot\big(\varepsilon\partial_t^2E+\sigma\partial_t E\big)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_\Omega \big((2\Phi\partial_t\Phi)V+\Phi^2\partial_t V\big)\cdot\varepsilon\partial_t E\,\mathrm{d}x\,\mathrm{d}t\\ &\quad -\int_0^T\int_\Omega \Phi^2 V\cdot\sigma\partial_t E\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Lemma 2.3 a) yields $\|V\|_{L^2}\lesssim\|\operatorname{curl}V\|_{L^2}$. We also insert $\partial_t V=\partial_t E-\partial_t\nabla p$. Hölder's inequality thus implies

$$\begin{aligned}\int_0^T\int_\Omega \Phi^2\mu\partial_t H\cdot\partial_t H\,\mathrm{d}x\,\mathrm{d}t \lesssim \int_0^T \Big[\,&\|\Phi'\|_\infty\big(\delta\big\|\Phi\mu^{-1/2}\operatorname{curl}V\big\|_{L^2}^2+\frac{1}{\delta}\|\partial_t E\|_{L^2}^2\big)\\ &+\big(\|\partial_t\nabla p\|_{L^2}^2+\|\partial_t E\|_{L^2}^2\big)+\big(\delta\big\|\Phi\mu^{-1/2}\operatorname{curl}V\big\|_{L^2}^2+\frac{1}{\delta}\|\sigma\partial_t E\|_{L^2}^2\big)\Big]\,\mathrm{d}t.\end{aligned}$$

As $\operatorname{curl}V=-\mu\partial_t H$ by (1.1), we can absorb the curl terms by the left-hand side. Lemma 2.7 now leads to

$$\int_0^T\int_\Omega \Phi^2\mu\partial_t H\cdot\partial_t H\,\mathrm{d}x\,\mathrm{d}t \lesssim \int_0^T\int_\Omega |\partial_t E|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega |\sigma E|^2\,\mathrm{d}x\,\mathrm{d}t.$$

By means of the energy estimates from Lemma 2.1, we conclude

$$\begin{aligned}\int_0^T\int_\Omega \mu\partial_t H\cdot\partial_t H\,\mathrm{d}x\,\mathrm{d}t &\le T\mathcal{D}(0)=T\left(\mathcal{D}(T)+2\int_0^T\sigma\partial_t E\cdot\partial_t E\right)\mathrm{d}x\,\mathrm{d}t\\ &\le 3\tfrac{T}{3}\,\mathcal{D}\left(\tfrac{2}{3}T\right)+2T\int_0^T\sigma\partial_t E\cdot\partial_t E\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim \int_{T/3}^{2T/3}\left|\mu^{\frac12}\partial_t H\right|^2\mathrm{d}x\,\mathrm{d}t+(1+T)\int_0^T|\partial_t E|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim (1+T)\int_0^T\int_\Omega|\partial_t E|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \sigma E\cdot E\,\mathrm{d}x\,\mathrm{d}t. \ \square\end{aligned}$$

We next treat the inhomogeneous part of the fields. Duhamel's formula and our choice of the initial values lead to the following estimate.

Lemma 4.5. *Let* (H) *hold and* (V_i,W_i) *as in* (3.9) *solve* (3.8). *We then obtain*

$$\int_0^T\int_\Omega\big(|\partial_t V_i|^2+|\partial_t W_i|^2\big)\,\mathrm{d}x\,\mathrm{d}t\le CT^2\int_0^T\int_\Omega|\sigma\partial_t E|^2\,\mathrm{d}x\,\mathrm{d}t\,.$$

Proof. We use the generator A_h of $T_h(\cdot)$ from (3.3). Equation (3.8) and the subsequent comments imply that

$$\begin{pmatrix}V_i(t)\\ W_i(t)\end{pmatrix}=T_h(t)\begin{pmatrix}0\\ W_{i0}\end{pmatrix}-\int_0^t T_h(s)\begin{pmatrix}\varepsilon^{-1}\sigma E(t-s)+\partial_t\nabla p(t-s)\\ 0\end{pmatrix}\mathrm{d}s.$$

Because of (2.5) and Lemma 2.7 we can differentiate this formula in L^2 with respect to t, where the resulting initial values vanish due to (3.10); i.e.,

$$\partial_t \begin{pmatrix} V_i(t) \\ W_i(t) \end{pmatrix} = -\int_0^t T_h(t-s) \begin{pmatrix} \varepsilon^{-1}\sigma\partial_t E(s) + \partial_t^2 \nabla p(s) \\ 0 \end{pmatrix} \mathrm{d}s.$$

Lemma 2.7 and Hölder's inequality now yield

$$\int_0^T \left\| \begin{pmatrix} \partial_t V_i \\ \partial_t W_i \end{pmatrix} \right\|_{L^2}^2 \mathrm{d}t \lesssim \int_0^T \left(\int_0^t \|\sigma\partial_t E\|_{L^2} \,\mathrm{d}s \right)^2 \mathrm{d}t \lesssim \int_0^T t \int_0^t \|\sigma\partial_t E\|_{L^2}^2 \,\mathrm{d}s\,\mathrm{d}t$$
$$\leq CT^2 \int_0^T \int_\Omega |\sigma\partial_t E|^2 \,\mathrm{d}x\,\mathrm{d}t. \qquad \square$$

The above estimates lead to the core inequality.

Proposition 4.6. *Assume that* (H), (2.1) *and* (2.2) *hold. Let* (E, H) *as in* (2.5) *solve* (1.1)–(1.4), *and* $T_O > 0$ *be given by Proposition 3.2. Then there exists a constant* $\gamma \in [0, 1)$ *such that*

$$\mathcal{E}(T) + \mathcal{D}(T) \leq \gamma\big(\mathcal{E}(0) + \mathcal{D}(0)\big), \qquad T \geq \max\{T_O, 1\}.$$

Proof. Lemmas 4.3 and 2.1 imply

$$T(\mathcal{E}(T) + \mathcal{D}(T)) \lesssim T\mathcal{D}(T) \lesssim \int_0^T \int_\Omega \big(|\partial_t E|^2 + |\partial_t H|^2 \big) \,\mathrm{d}x\,\mathrm{d}t.$$

We can first eliminate H be means of Lemma 4.4 via

$$T(\mathcal{E}(T) + \mathcal{D}(T)) \lesssim (1+T) \int_0^T \int_\Omega |\partial_t E|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \sigma E \cdot E \,\mathrm{d}x\,\mathrm{d}t.$$

Here we will insert the decomposition $\partial_t E = \partial_t V_h + \partial_t V_i + \partial_t \nabla p$ established before Lemma 3.1. Next, the homogeneous system (3.4) for (V_h, W_h), Lemma 3.1, and Remark 3.6 show that the integrals

$$\int_\Omega \big(\varepsilon\partial_t V_h \cdot \partial_t V_h + \mu\partial_t W_h \cdot \partial_t W_h\big) \,\mathrm{d}x = \int_\Omega \big(\varepsilon\partial_t V_h \cdot \partial_t V_h + \mu^{-1} \operatorname{curl} V_h \cdot \operatorname{curl} V_h\big) \,\mathrm{d}x$$

are constant in time. As $T \geq 1$, we thus obtain

$$T(\mathcal{E}(T) + \mathcal{D}(T)) \lesssim T \int_0^T \int_\Omega \big(|\partial_t V_h|^2 + |\partial_t V_i|^2 + |\partial_t \nabla p|^2\big) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \sigma E \cdot E \,\mathrm{d}x\,\mathrm{d}t$$
$$\lesssim T^2 \int_\Omega \big(|\partial_t V_h(0)|^2 + |\operatorname{curl} V_h(0)|^2\big) \,\mathrm{d}x$$
$$+ T \int_0^T \int_\Omega \big(|\partial_t \nabla p|^2 + |\partial_t V_i|^2\big) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \sigma E \cdot E \,\mathrm{d}x\,\mathrm{d}t.$$

The observability estimate of Proposition 3.2 and Remark 3.6 yield

$$T(\mathcal{E}(T) + \mathcal{D}(T)) \lesssim T^2 \int_0^T \int_\omega \partial_t V_h \cdot \partial_t V_h \,\mathrm{d}x\,\mathrm{d}t + T \int_0^T \int_\Omega \big(|\partial_t \nabla p|^2 + |\partial_t V_i|^2 \big) \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^T \int_\Omega \sigma E \cdot E \,\mathrm{d}x\,\mathrm{d}t$$

for $T \geq T_O$. After replacing again $V_h = E - V_i - \nabla p$, condition (2.1) leads to

$$T(\mathcal{E}(T) + \mathcal{D}(T)) \lesssim T^2 \int_0^T \int_\Omega \sigma\partial_t E(t) \cdot \partial_t E(t) \,\mathrm{d}x\,\mathrm{d}t + T^2 \int_0^T \int_\Omega \big(|\partial_t \nabla p|^2 + |\partial_t V_i|^2\big) \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^T \int_\Omega \sigma E \cdot E \,\mathrm{d}x\,\mathrm{d}t.$$

Using Lemmas 4.5, 2.7 and 2.1, we finally deduce

$$\begin{aligned}T(\mathcal{E}(T)+\mathcal{D}(T)) &\lesssim T^4\int_0^T\int_\Omega \big(\sigma E\cdot E+\sigma\partial_t E\cdot\partial_t E\big)\,\mathrm{d}x\,\mathrm{d}t\\ &\le CT^4\big(\mathcal{E}(0)-\mathcal{E}(T)+\mathcal{D}(0)-\mathcal{D}(T)\big).\end{aligned}$$

Setting $\gamma=\frac{CT^4}{CT^4+T}<1$, we conclude

$$\mathcal{E}(T)+\mathcal{D}(T)\le\gamma\big(\mathcal{E}(0)+\mathcal{D}(0)\big).\qquad\square$$

Theorem 4.1 now follows by a simple argument.

Proof of Theorem 4.1. First let $(E_0,H_0)\in D(A)$. Iterating the estimate from Proposition 4.6, we obtain constants $\tilde M\ge1$ and $\omega>0$ such that

$$\mathcal{E}(t)+\mathcal{D}(t)\le\tilde M\mathrm{e}^{-\omega t/2}\big(\mathcal{E}(0)+\mathcal{D}(0)\big)$$

for all $t\ge0$. Since $\mathcal{E}(t)=\|T(t)(E_0,H_0)\|_X^2$ and $\mathcal{D}(t)=\|AT(t)(E_0,H_0)\|_X^2$ by (2.6), we have shown that $T(\cdot)$ exponentially decays in $D(A)$ with the graph norm. This space is isomorphic to X by $(I-A)^{-1}$, so that the assertion follows. $\square$

Finally, we remove the divergence constraints in Theorem 4.1 by projecting in $X_e=L^2_\varepsilon(\Omega)\times L^2_\mu(\Omega)$ onto $\mathrm{N}(A_e)^\perp=\overline{\mathrm{R}(A_e)}$ for the extension A_e of A, see the discussion after (2.5). Here we proceed similar to [14]. Then the theorem will imply the exponential decay of the extended semigroup $T_e(\cdot)$ to the kernel $\mathrm{N}(A_e)$.

Lemma 4.7. *Let* (H) *and* (2.1) *hold. We then have*

$$\mathrm{N}(A_e)=\big\{(E,H)\in H_{t0}(\operatorname{curl}0)\times H(\operatorname{curl}0)\,\big|\,E=0\ \text{on}\ \omega\big\}=\mathrm{R}(A_e)^\perp.$$

Moreover, the orthogonal projection P onto $\mathrm{N}(A_e)$ commutes with $T_e(\cdot)$, $T_e(t)P=I$ for $t\ge1$, and $\mathrm{N}(A_e)^\perp=\overline{\mathrm{R}(A_e)}$ is contained in X.

Proof. 1) Take $w=(E,H)\in\mathrm{N}(A_e)$. This means that $\operatorname{curl}E=0$ and $\operatorname{curl}H=\sigma E$. As $\operatorname{tr}_t E=0$, integration by parts yields

$$0=(A_ew|w)_{X_e}=\int_\Omega\big(\operatorname{curl}H\cdot E-\sigma E\cdot E-\operatorname{curl}E\cdot H\big)\,\mathrm{d}x=-\int_\Omega|\sigma^{\frac12}E|^2\,\mathrm{d}x,$$

so that $\sigma E=0$ and thus $E=0$ on ω by (2.1). This shows '$\subseteq$' in the first asserted identity. The converse inclusion is clear because of $\operatorname{supp}\sigma=\overline{\omega}$.

2) Step 1) and integration by parts imply that the kernel $\mathrm{N}(A_e)$ is orthogonal to the range $\mathrm{R}(A_e)$. To show $\mathrm{N}(A_e)=\mathrm{R}(A_e)^\perp$, take $h=(f,g)\in X_e$ with

$$0=(h|A_ew)_{X_e}=\int_\Omega\big(f\cdot\operatorname{curl}H-f\cdot\sigma E-g\cdot\operatorname{curl}E\big)\,\mathrm{d}x$$

for all $w=(E,H)\in D(A_e)$. Choosing $(0,H)$ and $(E,0)$ with $E,H\in H^1_0(\Omega)$, we see that $\operatorname{curl}f=0$ and $\sigma f+\operatorname{curl}g=0$. If we insert $(0,H)$ with $H\in H^1(\Omega)$, it follows $\langle\operatorname{tr}_t f,H\rangle_{H^{-1/2}(\partial\Omega)}=0$ so that $h\in D(A_e)$. The formula in display with $h=w$ then implies $\int_\omega\sigma f\cdot f\,\mathrm{d}x=0$ which yields $f=0$ on ω and $\operatorname{curl}g=0$; i.e., $h\in\mathrm{N}(A_e)$ by step 1) as needed.

3) Hence, $\overline{\mathrm{R}(A_e)}=\mathrm{N}(A_e)^\perp$ is the kernel of P, implying $PA_e=0=A_eP$ on $D(A_e)$, and thus $PT_e(t)=T_e(t)P=P$ for $t\ge0$. Let $(f,g)=A_e(E,H)$ for some $(E,H)\in D(A_e)$. Then $g=-\mu^{-1}\operatorname{curl}E$ belongs to $H^\Sigma_{n0}(\operatorname{div}_\mu0)$ by Proposition 6.1.4 in [1], and $\varepsilon f_{|\upsilon}=\operatorname{curl}H$ to $H^\Gamma(\operatorname{div}0,\upsilon)$ because of (2.1), Proposition IX.1.3 in [4], and the density of $H^1(\upsilon)$ in $H(\operatorname{curl},\upsilon)$. As a result, $\overline{\mathrm{R}(A_e)}$ is contained in X. $\square$

Corollary 4.8. *Let* (H), (2.1)*, and* (2.2) *hold. Then there exist constants* $M' \geq 1$ *and* $\omega > 0$ *such that for* $(E_0, H_0) \in X_e = L^2_\varepsilon(\Omega) \times L^2_\mu(\Omega)$ *we have*

$$\|(T_e(t) - P)(E_0, H_0)\|_{L^2} \leq M' \mathrm{e}^{-\omega t} \|(E_0, H_0)\|_{L^2}, \qquad t \geq 0.$$

Proof. Lemma 4.7 yields $T_e(t) - P = T_e(t)(I - P) = T(t)(I - P)$ so that the result follows from Theorem 4.1. □

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[12]Karlsruhe Institute of Technology, Department of Mathematics, Englerstrasse 2, 76131 Karlsruhe, Germany

Email address: richard.nutt@kit.edu

Email address: schnaubelt@kit.edu