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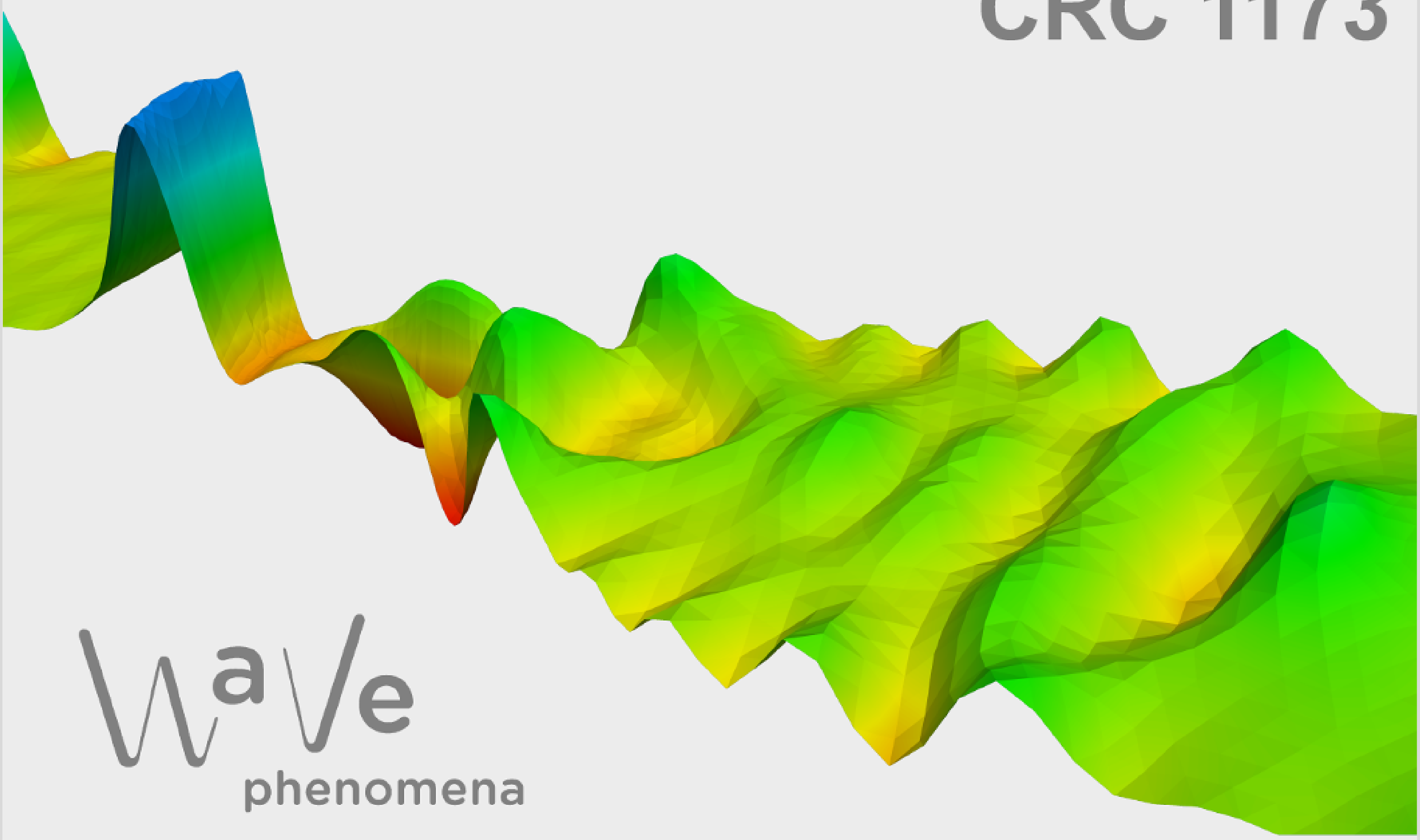

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A note on div-curl lemmas for Maxwell interface problems

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Abstract. Combining knowledge of the curl and divergence of a vector field $\mathbf{E}$ to obtain information about its spatial regularity has proven to be a very useful technique in the treatment of the Maxwell equations. We consider the interface problem, where the permittivity ε is discontinuous across a surface. An important theorem by Weber can be generalized to allow for a jump of $\varepsilon\mathbf{E}$ in normal direction across the interface. We use a Helmholtz decomposition to deduce this from the Weber result by a reduction to the task of proving higher regularity for solutions of elliptic transmission problems. For electric boundary conditions, the result was shown recently by Dohnal, Ionescu-Tira and Waurick. We extend the result to the magnetic case using similar arguments. The main goal of this note is to present the proofs in detail. In particular, we keep track of how the constants depend on the permittivity. This information is useful for approximation arguments.

1 Introduction

In the analysis of the Maxwell system and other problems one often uses so-called div-curl lemmas which control the norm of a vector field in, say, H^1 by its L^2-norm and that of its curl and divergence as well as contributions from boundary traces. See for example [2], [5], [6], [1], [4], [10], [14] and the references therein. Homogeneous interface problems were treated in [16], see Theorem 1.5 below, and inhomogeneous ones recently in [7] for the case of "electric" boundary conditions. In this note we show the magnetic version, see Theorem 1.6, using analogous methods as in [7]. Via the Helmholtz decomposition one can reduce the results to regularity properties of the elliptic transmission problems (2.5) and (2.8). These are stated without a proof in Theorem 5.2.1 of [5]. For applications to approximation arguments, cf. [13], one needs to know that the constants

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only depend on the norm and the lower bound of the coefficients. Therefore we give full proofs of these regularity results in Section 3.

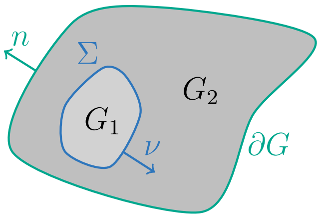

Figure 1: Schematic depiction of the domain.

Let $G = G_1 \cup \Sigma \cup G_2 \subseteq \mathbb{R}^3$ be a bounded domain consisting of two disjoint subdomains G_1 and G_2 separated by an interface $\Sigma \coloneqq \partial G_1$, satisfying $\operatorname{dist}(\Sigma, \partial G) > 0$. The results in this paper can be extended to the case of finitely many subdomains, whose boundaries all have positive distance to each other and to the boundary of G.

The outer unit normal vector on Σ, pointing outwards from G_1, is denoted by ν and the outer unit normal on ∂G by n. Let $[w] \coloneqq (w|_{G_1} - w|_{G_2})|_\Sigma$ denote the jump of a quantity w across the interface Σ. The two subdomains G_1 and G_2 have different material properties, described by the permittivity ε, which is assumed to be uniformly positive definite. For this reason we consider the following piecewise function spaces.

Definition 1.1. *Let $\ell \in \mathbb{N}_0$ and $\eta > 0$. We define*

$$\begin{aligned}
\mathcal{H}^\ell &\coloneqq \left\{ u \in L^2(G) \,\middle|\, u|_{G_i} \in H^\ell(G_i),\ i \in \{1,2\} \right\}, \\
\mathcal{C}^\ell &\coloneqq \left\{ \varepsilon : G \to \mathbb{R}^{3\times 3} \,\middle|\, \varepsilon|_{G_i} \in C^\ell(\overline{G_i})^{3\times 3},\ i \in \{1,2\} \right\}, \\
\mathcal{C}^\ell_\eta &\coloneqq \left\{ \varepsilon \in \mathcal{C}^\ell \,\middle|\, \varepsilon \text{ is symmetric, } \big(\varepsilon(x)\xi\big)\cdot \xi \geq \eta\, |\xi|^2 \text{ for all } x \in G, \xi \in \mathbb{R}^3 \right\}
\end{aligned}$$

with norms

$$\|u\|_{\mathcal{H}^\ell} \coloneqq \sum_{i=1}^{2} \|u\|_{H^\ell(G_i)}, \qquad \|\varepsilon\|_{\mathcal{C}^\ell} \coloneqq \sum_{i=1}^{2} \sum_{m,n=1}^{3} \sum_{\substack{\alpha \in \mathbb{N}_0^3, \\ |\alpha| \leq \ell}} \sup_{x \in G_i} |\partial^\alpha \varepsilon_{mn}(x)| .$$

Localization arguments used in Section 3 lead to problems on balls with the interface between the two materials described by the $x_3 = 0$ plane. We define analogous function spaces in this setting.

Definition 1.2. *Let $\ell \in \mathbb{N}_0$, $\eta > 0$ and $U \subseteq \mathbb{R}^3$ be open. We set $U_+ \coloneqq U \cap \{x_3 > 0\}$ and $U_- \coloneqq U \cap \{x_3 < 0\}$ and define*

$$\begin{aligned}
\mathcal{H}^\ell(U) &\coloneqq \left\{ u \in L^2(U) \,\middle|\, u|_{U_i} \in H^\ell(U_i),\ i \in \{+,-\} \right\}, \\
\mathcal{C}^\ell(U) &\coloneqq \left\{ \varepsilon : U \to \mathbb{R}^{3\times 3} \,\middle|\, \varepsilon|_{U_i} \in C^\ell(\overline{U_i})^{3\times 3},\ i \in \{+,-\} \right\},
\end{aligned}$$

$$\mathcal{C}^\ell_\eta(U) \coloneqq \left\{\varepsilon \in \mathcal{C}^\ell(U) \;\middle|\; \varepsilon \textit{ is symmetric, } \big(\varepsilon(x)\xi\big)\cdot\xi \ge \eta\,|\xi|^2 \textit{ for all } x \in U, \xi \in \mathbb{R}^3\right\}$$

with norms

$$\|u\|_{\mathcal{H}^\ell(U)} \coloneqq \sum_{i\in\{+,-\}} \|u\|_{H^\ell(U_i)}\ , \quad \|\varepsilon\|_{\mathcal{C}^\ell(U)} \coloneqq \sum_{i\in\{+,-\}} \sum_{m,n=1}^{3} \sum_{\substack{\alpha\in\mathbb{N}_0^3,\\ |\alpha|\le\ell}} \sup_{x\in U_i} |\partial^\alpha \varepsilon_{mn}(x)|\ .$$

For $u \in \mathcal{H}^\ell$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le \ell$, the derivative $\partial^\alpha u$ is defined piecewise by

$$\partial^\alpha u \coloneqq \begin{cases} \partial^\alpha\big(u|_{G_1}\big) & \text{on } G_1\,, \\ \partial^\alpha\big(u|_{G_2}\big) & \text{on } G_2\,. \end{cases}$$

We often omit the exponent d in function spaces like $(\mathcal{H}^\ell)^d$ and $\mathcal{H}^\ell(U)^d$ when dealing with vector fields.

The operators curl and div play a central role in the Maxwell equations.

Definition 1.3. *Let $U \subseteq \mathbb{R}^3$ be open. We define*

$$\begin{aligned} H(\operatorname{curl},U) &\coloneqq \Big\{\mathbf{E} \in L^2(U)^3 \;\Big|\; \exists\,\mathbf{F} \in L^2(U)^3\ \forall\,\Psi \in C_c^\infty(U)^3 : \\ &\qquad \int_U \mathbf{E}\cdot\operatorname{curl}\Psi\,\mathrm{d}x = \int_U \mathbf{F}\cdot\Psi\,\mathrm{d}x\Big\}\,, \\ H(\operatorname{div},U) &\coloneqq \Big\{\mathbf{E} \in L^2(U)^3 \;\Big|\; \exists\,w \in L^2(U)\ \forall\,\varphi \in C_c^\infty(U) : \\ &\qquad \int_U \mathbf{E}\cdot\nabla\varphi\,\mathrm{d}x = -\int_U w\varphi\,\mathrm{d}x\Big\}\,. \end{aligned}$$

The functions $\mathbf{F}$ *and* w *above are unique if they exist and are denoted by* $\operatorname{curl}\mathbf{E}$*, respectively* $\operatorname{div}\mathbf{E}$*. The spaces* $H(\operatorname{curl},U)$ *and* $H(\operatorname{div},U)$ *are equipped with the scalar products*

$$\begin{aligned} (\mathbf{E}|\mathbf{F})_{H(\operatorname{curl},U)} &\coloneqq (\mathbf{E}|\mathbf{F})_{L^2(U)} + (\operatorname{curl}\mathbf{E}|\operatorname{curl}\mathbf{F})_{L^2(U)}\ , \\ (\mathbf{E}|\mathbf{F})_{H(\operatorname{div},U)} &\coloneqq (\mathbf{E}|\mathbf{F})_{L^2(U)} + (\operatorname{div}\mathbf{E}|\operatorname{div}\mathbf{F})_{L^2(U)}\ . \end{aligned}$$

We also set

$$H_0(\operatorname{curl},U) \coloneqq \overline{C_c^\infty(U)^3}^{H(\operatorname{curl},U)}, \quad H_0(\operatorname{div},U) \coloneqq \overline{C_c^\infty(U)^3}^{H(\operatorname{div},U)},$$

where the closure is taken with respect to the norms in the spaces $H(\operatorname{curl},U)$*, respectively* $H(\operatorname{div},U)$*.*

The spaces $H_0(\operatorname{curl},U)$ and $H_0(\operatorname{div},U)$ incorporate conditions for the tangential respectively normal components at the boundary, see Theorems IX.1.1 and IX.1.2 in [6].

Lemma 1.4. *Let $U \subseteq \mathbb{R}^3$ be open with a compact Lipschitz boundary. Then it holds*

$$\begin{aligned} H_0(\operatorname{curl},U) &= \big\{\mathbf{E} \in H(\operatorname{curl},U) \;\big|\; n\times\mathbf{E}|_{\partial U} = 0\big\}\,, \\ H_0(\operatorname{div},U) &= \big\{\mathbf{E} \in H(\operatorname{div},U) \;\big|\; n\cdot\mathbf{E}|_{\partial U} = 0\big\}\,. \end{aligned}$$

An important result in the regularity theory for the Maxwell equations is the following theorem due to Weber, see Theorem 2.2 in [16]. In [16], the dependence of the constant on the permittivity ε is not further specified, but an inspection of the proof shows that it only depends on k, G, the constant describing the uniform positive definiteness of ε and on the norm $\|\varepsilon\|_{\mathcal{C}^{k+1}}$.

Theorem 1.5. *Let $k \in \mathbb{N}_0$, $\eta > 0$, Σ and ∂G be of class C^{k+2} and $\varepsilon \in \mathcal{C}^{k+1}_\eta$. Let $\mathbf{E} \in L^2(G)^3$ satisfy either of the cases*

a) $\mathbf{E} \in H_0(\mathrm{curl}, G)$ *and* $\varepsilon\mathbf{E} \in H(\mathrm{div}, G)$*, or*

b) $\mathbf{E} \in H(\mathrm{curl}, G)$ *and* $\varepsilon\mathbf{E} \in H_0(\mathrm{div}, G)$.

Furthermore, let $\mathrm{curl}\,\mathbf{E} \in \mathcal{H}^k$ *and* $\mathrm{div}(\varepsilon\mathbf{E}) \in \mathcal{H}^k$*. Then* $\mathbf{E}$ *is contained in* $\mathcal{H}^{k+1}$ *and it holds*

$$\|\mathbf{E}\|_{\mathcal{H}^{k+1}} \leq C\Big(\|\mathbf{E}\|_{L^2(G)} + \|\mathrm{curl}\,\mathbf{E}\|_{\mathcal{H}^k} + \|\mathrm{div}(\varepsilon\mathbf{E})\|_{\mathcal{H}^k}\Big).$$

The constant $C = C(k, \|\varepsilon\|_{\mathcal{C}^{k+1}}, \eta^{-1}, G) \geq 0$ *is nondecreasing in* $\|\varepsilon\|_{\mathcal{C}^{k+1}}$ *and* η^{-1}.

The goal of this note is to prove the following generalization. The first case is stated as Proposition 2.24 in [7] (with slightly different assumptions on the regularity of Σ, ∂G and ε). We note that $\varepsilon|_{G_i} \in C^{k,1}\left(\overline{G_i}\right)^{3\times 3}$ implies $\varepsilon|_{G_i} \in W^{1,\infty}(G_i)^{3\times 3}$, $i = 1, 2$, see Theorem 4 in Section 5.8 of [8]. Furthermore, $\mathbf{V} \in \mathcal{H}^{k+1}$ implies $\mathbf{V}|_{G_i} \in H(\mathrm{div}, G_i)$ for $i = 1, 2$, so the term $\mathrm{div}\big(\varepsilon\mathbf{E}|_{G_i}\big)$ below is well defined.

Theorem 1.6. *Let $k \in \mathbb{N}_0$, $\eta > 0$, Σ and ∂G be of class C^{k+2} and $\varepsilon \in \mathcal{C}^{k+1}_\eta$. Let $\mathbf{E} \in L^2(G)^3$ belong to either of the cases*

a) $\mathbf{E} \in H_0(\mathrm{curl}, G)$ *and* $\varepsilon\mathbf{E} + \mathbf{V} \in H(\mathrm{div}, G)$*, or*

b) $\mathbf{E} \in H(\mathrm{curl}, G)$ *and* $\varepsilon\mathbf{E} + \mathbf{V} \in H_0(\mathrm{div}, G)$

for some $\mathbf{V} \in \mathcal{H}^{k+1}$*. Furthermore, let* $\mathrm{curl}\,\mathbf{E} \in \mathcal{H}^k$ *and* $\mathrm{div}\big((\varepsilon\mathbf{E})|_{G_i}\big) \in H^k(G_i)$*,* $i = 1, 2$*. Then* $\mathbf{E}$ *is contained in* $\mathcal{H}^{k+1}$ *and it holds*

$$\|\mathbf{E}\|_{\mathcal{H}^{k+1}} \leq C\Big(\|\mathbf{E}\|_{L^2(G)} + \|\mathrm{curl}\,\mathbf{E}\|_{\mathcal{H}^k} + \sum_{i=1}^{2}\big\|\mathrm{div}\big((\varepsilon\mathbf{E})|_{G_i}\big)\big\|_{H^k(G_i)} + \|\mathbf{V}\|_{\mathcal{H}^{k+1}}\Big). \quad (1.1)$$

The constant $C = C(k, \|\varepsilon\|_{\mathcal{C}^{k+1}}, \eta^{-1}, G) \geq 0$ *is nondecreasing in* $\|\varepsilon\|_{\mathcal{C}^{k+1}}$ *and* η^{-1}.

This result is proved at the end of the paper.

2 Reduction to transmission problems

In [7], the first case of Theorem 1.6 is proved. The second one can be shown analogously. For completeness, we state detailed proofs for both cases in the following. First, a Helmholtz decomposition for the field $\mathbf{V}$ is used in the form $\varepsilon^{-1}\mathbf{V} = \nabla u + \mathbf{W}$. If the scalar potential u is sufficiently regular, the claim can be deduced from Theorem 1.5. It can be shown that u is the solution of an elliptic transmission problem. So the proof of Theorem 1.6 is reduced to regularity theory for such problems.

The following Helmholtz decompositions are taken from Lemmas 3.4 and 3.5 in [16]. We only supplement the proofs given there by stating the dependence of the relevant constants on ε and η, setting $\|\varepsilon\|_{L^\infty(G)} := \sum_{i,j=1}^{3} \|\varepsilon_{ij}\|_{L^\infty(G)}$.

Lemma 2.1. *Let G be a bounded Lipschitz domain and let $\varepsilon \in L^\infty(G)^{3\times 3}$ be symmetric and satisfy $\big(\varepsilon(x)\xi\big)\cdot\xi \geq \eta\,|\xi|^2$ for almost all $x \in G$ and all $\xi \in \mathbb{R}^3$ and some constant $\eta > 0$. There exists a constant $C = C(G) \geq 0$ depending only on G such that the following holds.*

1. *Every $\mathbf{F} \in L^2(G)^3$ can be uniquely decomposed as $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, where $\mathbf{F}_1 = \nabla f$ for some $f \in H^1_0(G)$ with*

$$\|f\|_{H^1(G)} \leq C\sqrt{\frac{\|\varepsilon\|_{L^\infty(G)}}{\eta}}\,\|\mathbf{F}\|_{L^2(G)} \tag{2.1}$$

   and $\varepsilon\mathbf{F}_2 \in H(\mathrm{div}, G)$ with $\mathrm{div}(\varepsilon\mathbf{F}_2) = 0$.

2. *Every $\mathbf{F} \in L^2(G)^3$ can be uniquely decomposed as $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, where $\mathbf{F}_1 = \nabla f$ for some $f \in H^1(G)$ with*

$$\|f\|_{H^1(G)} \leq C\sqrt{\frac{\|\varepsilon\|_{L^\infty(G)}}{\eta}}\,\|\mathbf{F}\|_{L^2(G)} \tag{2.2}$$

   and $\varepsilon\mathbf{F}_2 \in H(\mathrm{div}, G)$ with $\mathrm{div}(\varepsilon\mathbf{F}_2) = 0$ and $n\cdot(\varepsilon\mathbf{F}_2)|_{\partial G} = 0$.

Proof. Let $L^2_\varepsilon(G)^3$ denote the space $L^2(G)^3$ equipped with the scalar product given by $(\mathbf{E}|\mathbf{F})_{L^2_\varepsilon(G)} := (\varepsilon\mathbf{E}|\mathbf{F})_{L^2(G)}$. Due to the assumptions on ε, the corresponding norm is equivalent to the usual $L^2(G)^3$-norm. We define the spaces

$$X_1 := \left\{\nabla f \;\middle|\; f \in H^1_0(G)\right\}, \qquad Y_1 := \left\{\mathbf{E} \in L^2(G)^3 \;\middle|\; \mathrm{div}(\varepsilon\mathbf{E}) = 0\right\},$$
$$X_2 := \left\{\nabla f \;\middle|\; f \in H^1(G)\right\}, \qquad Y_2 := \left\{\mathbf{E} \in L^2(G)^3 \;\middle|\; \mathrm{div}(\varepsilon\mathbf{E}) = 0\,,\; n\cdot(\varepsilon\mathbf{E})|_G = 0\right\}.$$

By Lemmas 3.8 and 3.9 in [15], we have $Y_1 = X_1^\perp$ and $Y_2 = X_2^\perp$ with respect to $(\cdot|\cdot)_{L^2_\varepsilon(G)}$.

1. Let P be the orthogonal projection onto X_1 in $L^2_\varepsilon(G)^3$ and $P\mathbf{F} = \mathbf{F}_1$. It holds

$$\eta\,\|\mathbf{F}_1\|^2_{L^2(G)} \leq \|\mathbf{F}_1\|^2_{L^2_\varepsilon(G)} = \|P\mathbf{F}\|^2_{L^2_\varepsilon(G)} \leq \|\varepsilon\mathbf{F}\|_{L^2(G)}\,\|\mathbf{F}\|_{L^2(G)} \leq c\,\|\varepsilon\|_{L^\infty(G)}\,\|\mathbf{F}\|^2_{L^2(G)}\,,$$

where $c \geq 0$ is a constant independent of G, ε and η. So we have $\mathbf{F}_1 = \nabla f$ for some $f \in H_0^1(G)$ with

$$\|\nabla f\|_{L^2(G)} \leq \sqrt{\frac{c\,\|\varepsilon\|_{L^\infty(G)}}{\eta}}\,\|\mathbf{F}\|_{L^2(G)}$$

and (2.1) follows from the Poincaré inequality.

2. In the same way we find functions $f \in H^1(G)$ and $\mathbf{F}_2 \in Y_2$ satisfying $\mathbf{F} = \nabla f + \mathbf{F}_2$ and

$$\|\nabla f\|_{L^2(G)} \leq \sqrt{\frac{c\,\|\varepsilon\|_{L^\infty(G)}}{\eta}}\,\|\mathbf{F}\|_{L^2(G)}\ ,$$

where $c \geq 0$ is a constant independent of G, ε and η. We set $\overline{f} = \int_G f\,\mathrm{d}x$ and define the function $\widetilde{f} = f - \overline{f}\mathbb{1}_G$. Then it holds $\nabla\widetilde{f} = \nabla f$ and $\int_G \widetilde{f}\,\mathrm{d}x = 0$. Using the Poincaré–Wirtinger inequality, we find (2.2). □

2.1 Case a) in Theorem 1.6

We assume that the conditions of case a) in Theorem 1.6 are fulfilled. By Lemma 2.1, $\varepsilon^{-1}\mathbf{V}$ can be uniquely decomposed in $L^2(G)$ as

$$\varepsilon^{-1}\mathbf{V} = \nabla u + \mathbf{W}\,, \tag{2.3}$$

where u is contained in $H_0^1(G)$ and $\varepsilon\mathbf{W} \in H(\operatorname{div}, G)$ satisfies $\operatorname{div}(\varepsilon\mathbf{W}) = 0$. Furthermore, there exists a constant $C^{(1)} = C^{(1)}(G) \geq 0$ that $\|u\|_{H^1(G)} \leq C^{(1)}\sqrt{\frac{\|\varepsilon\|_{L^\infty(G)}}{\eta}}\,\|\mathbf{V}\|_{L^2(G)}$ holds. The identities $\operatorname{curl}(\mathbf{E} + \nabla u) = \operatorname{curl}\mathbf{E}$ and $n \times (\mathbf{E} + \nabla u)|_{\partial G} = n \times \mathbf{E}|_{\partial G} = 0$ imply that $\mathbf{E} + \nabla u$ is contained in $H_0(\operatorname{curl}, G)$. Here we have used the assumption $\mathbf{E} \in H_0(\operatorname{curl}, G)$ and the fact that the tangential trace of ∇u vanishes since $u \in H_0^1(G)$. Moreover it holds

$$\varepsilon(\mathbf{E} + \nabla u) = \varepsilon\mathbf{E} + \mathbf{V} - \varepsilon\mathbf{W} \in H(\operatorname{div}, G)\,.$$

We now assume that u has the additional regularity

$$u \in \mathcal{H}^{k+2}\,, \quad \|u\|_{\mathcal{H}^{k+2}} \leq C^{(2)}\,\|\mathbf{V}\|_{\mathcal{H}^{k+1}}\ , \tag{2.4}$$

with a constant $C^{(2)} = C^{(2)}(k, \|\varepsilon\|_{C^{k+1}}, \eta^{-1}, G) \geq 0$ nondecreasing in $\|\varepsilon\|_{C^{k+1}}$ and η^{-1}. We have

$$\mathbf{E} + \nabla u \in H_0(\operatorname{curl}, G)\,, \qquad \varepsilon(\mathbf{E} + \nabla u) \in H(\operatorname{div}, G)\,,$$
$$\operatorname{curl}(\mathbf{E} + \nabla u) \in \mathcal{H}^k\,, \qquad \operatorname{div}\big(\varepsilon(\mathbf{E} + \nabla u)\big) \in \mathcal{H}^k\,.$$

Theorem 1.5 therefore yields $\mathbf{E} + \nabla u \in \mathcal{H}^{k+1}$ as well as the existence of a constant $C^{(3)} = C^{(3)}(k, \|\varepsilon\|_{C^k}, \eta^{-1}, G) \geq 0$ nondecreasing in $\|\varepsilon\|_{C^k}$ and η^{-1} such that the estimate

$$\|\mathbf{E} + \nabla u\|_{\mathcal{H}^{k+1}} \leq C^{(3)}\Big(\|\mathbf{E} + \nabla u\|_{L^2(G)} + \|\operatorname{curl}(\mathbf{E} + \nabla u)\|_{\mathcal{H}^k} + \big\|\operatorname{div}\big(\varepsilon(\mathbf{E} + \nabla u)\big)\big\|_{\mathcal{H}^k}\Big)$$

holds. Using (2.4), we arrive at

$$\|\mathbf{E}\|_{\mathcal{H}^{k+1}} \leq \|\mathbf{E} + \nabla u\|_{\mathcal{H}^{k+1}} + \|\nabla u\|_{\mathcal{H}^{k+1}}$$

$$\leq C\left(\|\mathbf{E}\|_{L^2(G)}+\|\operatorname{curl}\mathbf{E}\|_{\mathcal{H}^k}+\sum_{i=1}^{2}\big\|\operatorname{div}\big(\varepsilon\mathbf{E}|_{G_i}\big)\big\|_{H^k(G_i)}+\|\mathbf{V}\|_{\mathcal{H}^{k+1}}\right)$$

with $C = C(k, \|\varepsilon\|_{C^{k+1}}, \eta^{-1}, G) \geq 0$ nondecreasing in $\|\varepsilon\|_{C^{k+1}}$ and η^{-1}.

So it remains to prove (2.4). As u is contained in $H_0^1(G)$, we have $u = 0$ on ∂G as well as $[u] = 0$ on Σ in the sense of traces (see Proposition 2.1.68 in [2] for the last assertion). From (2.3) and the fact that $\operatorname{div}(\varepsilon\mathbf{W}) = 0$, we conclude $-\operatorname{div}(\varepsilon\nabla u) = -\operatorname{div}\mathbf{V} =: f \in H^k(G_i)$ in G_i, $i = 1, 2$. The co-normal derivative $\nu\cdot(\varepsilon\nabla u)$ has a jump across the interface given by $[\nu\cdot\mathbf{V}] =: g \in H^{k+\frac{1}{2}}(\Sigma)$, since $[\nu\cdot(\varepsilon\mathbf{W})] = 0$ on Σ by Proposition 2.2.30 in [2]. Therefore, u is a solution to the *transmission problem* (also referred to as *diffraction problem* or *interface problem*)

$$\begin{aligned} -\operatorname{div}(\varepsilon\nabla u) &= f \quad &&\text{in } G_1\,,\\ -\operatorname{div}(\varepsilon\nabla u) &= f \quad &&\text{in } G_2\,,\\ u &= 0 \quad &&\text{on } \partial G\,,\\ [u] = 0\,,\ [\nu\cdot(\varepsilon\nabla u)] &= g \quad &&\text{on } \Sigma\,. \end{aligned} \tag{2.5}$$

We say that u is a *weak solution* of (2.5) if u is contained in $H_0^1(G)$ and satisfies

$$\int_G (\varepsilon\nabla u)\cdot\nabla\varphi\,\mathrm{d}x = \int_G f\varphi\,\mathrm{d}x + \int_\Sigma g\varphi\,\mathrm{d}\sigma$$

for all $\varphi \in H_0^1(G)$. This is also follows directly from (2.3) by multiplying with $\varepsilon\nabla\varphi$, integrating and using the divergence theorem.

2.2 Case b) in Theorem 1.6

Case b) in Theorem 1.6 similarly leads to a transmission problem, but now with with a Neumann condition on ∂G instead of a Dirichlet one. By Lemma 2.1, we can uniquely decompose $\varepsilon^{-1}\mathbf{V}$ in $L^2(G)$ as

$$\varepsilon^{-1}\mathbf{V} = \nabla u + \mathbf{W}\,, \tag{2.6}$$

where u is contained in $H^1(G)$ and $\varepsilon\mathbf{W} \in H_0(\operatorname{div}, G)$ satisfies $\operatorname{div}(\varepsilon\mathbf{W}) = 0$. In addition, there exists a constant $C^{(1)} = C^{(1)}(G) \geq 0$ such that the estimate $\|u\|_{H^1(G)} \leq C^{(1)}\sqrt{\frac{\|\varepsilon\|_{L^\infty(G)}}{\eta}}\,\|\mathbf{V}\|_{L^2(G)}$ holds. A calculation analogous to the one in Subsection 2.1 yields $\mathbf{E}+\nabla u \in H(\operatorname{curl}, G)$ and $\varepsilon(\mathbf{E}+\nabla u) \in H_0(\operatorname{div}, G)$. We again assume that u even satisfies

$$u \in \mathcal{H}^{k+2}\,, \quad \|u\|_{\mathcal{H}^{k+2}} \leq C^{(2)}\,\|\mathbf{V}\|_{\mathcal{H}^{k+1}} \tag{2.7}$$

with a constant $C^{(2)} = C^{(2)}(k, \|\varepsilon\|_{C^{k+1}}, \eta^{-1}, G) \geq 0$ nondecreasing in $\|\varepsilon\|_{C^{k+1}}$ and η^{-1}. Theorem 1.5 then again yields $\mathbf{E}+\nabla u \in \mathcal{H}^{k+1}$ and the estimate (1.1). So proving Case b) in Theorem 1.6 is reduced to showing the regularity (2.7) for solutions u of the transmission problem

$$\begin{aligned} -\operatorname{div}(\varepsilon\nabla u) &= f \quad \text{in } G_1\,, \\ -\operatorname{div}(\varepsilon\nabla u) &= f \quad \text{in } G_2\,, \\ n\cdot(\varepsilon\nabla u) &= h \quad \text{on } \partial G\,, \\ [u]=0\,,\ [\nu\cdot(\varepsilon\nabla u)] &= g \quad \text{on } \Sigma\,, \end{aligned} \tag{2.8}$$

where f and g are defined as in (2.5), and $h := n\cdot\mathbf{V}|_{\partial G} \in H^{k+\frac{1}{2}}(\partial G)$. In contrast to (2.5), the boundary condition on ∂G is of Neumann type because we only have $u \in H^1(G)$ instead of $u \in H^1_0(G)$. A weak solution to (2.8) is a function $u \in H^1(G)$ satisfying

$$\int_G (\varepsilon\nabla u)\cdot\nabla\varphi \,\mathrm{d}x = \int_G f\varphi\,\mathrm{d}x + \int_\Sigma g\varphi\,\mathrm{d}\sigma + \int_{\partial G} h\varphi\,\mathrm{d}\sigma$$

for all $\varphi \in H^1(G)$.

The regularity theory for transmission problems is known and has been studied by several authors, see for example [5], [12] and [3]. However, since we were not able to find detailed proofs in our setting, we present one in the following.

3 Regularity theory for transmission problems

In this section, we study the regularity properties of weak solutions of (2.5) and (2.8). We start with the case $k = 0$ and $g = 0$, $h = 0$. The proof follows the proof of Theorem 4 of Section 6.3 in [8] and uses the method of difference quotients. For $h \in \mathbb{R}\setminus\{0\}$ and $\ell \in \{1,2,3\}$ we define the operator D^h_ℓ by

$$D^h_\ell u(x) = \frac{1}{h}\big(u(x+h\mathrm{e}_\ell) - u(x)\big)\,.$$

As a first step, we consider a small neighbourhood of a point on the interface. After a coordinate transform, this can be assumed to have the form of a ball, with the interface described by the $x_3 = 0$ plane. In the proof of Proposition 3.5 below, a localization argument is used and the cut-off functions lead to extra terms in the equation. In order to deal with these, we introduce the additional term $\mathbf{F}\cdot\nabla\varphi$ below.

Lemma 3.1. *Let $\eta > 0$, $0 < s < r$, $U := B(0,r) \subseteq \mathbb{R}^3$, $V := B(0,s) \subseteq \mathbb{R}^3$ and $\varepsilon \in \mathcal{C}^1_\eta(U)$. Let $f \in L^2(U)$, $\mathbf{F} \in \mathcal{H}^1(U)^3$, and let $u \in H^1(U)$ satisfy*

$$\int_U (\varepsilon\nabla u)\cdot\nabla\varphi\,\mathrm{d}x = \int_U (f\varphi + \mathbf{F}\cdot\nabla\varphi)\,\mathrm{d}x \tag{3.1}$$

for all $\varphi \in H^1_0(U)$. Then $\partial_\ell u \in H^1(V)$ for $\ell = 1,2$ and $u \in \mathcal{H}^2(V)$. In addition, there exists a constant $C = C(\|\varepsilon\|_{\mathcal{C}^1(U)}, \eta^{-1}, r, s) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^1(U)}$ and η^{-1} such that

$$\|\partial_\ell u\|_{H^1(V)} \le C\left(\|f\|_{L^2(U)} + \|\mathbf{F}\|_{\mathcal{H}^1(U)} + \|u\|_{H^1(U)}\right)\,,\ \ell = 1,2\,, \tag{3.2}$$

$$\|u\|_{\mathcal{H}^2(V)} \le C\left(\|f\|_{L^2(U)} + \|\mathbf{F}\|_{\mathcal{H}^1(U)} + \|u\|_{H^1(U)}\right) \tag{3.3}$$

hold.

Proof. Let $W := B(0, \frac{r+s}{2})$, $\zeta \in C_c^\infty(\mathbb{R}^3)$ with $\zeta = 1$ on V, $\operatorname{supp} \zeta \subseteq W$ and $0 \leq \zeta \leq 1$. Furthermore, let $\ell \in \{1, 2\}$ and $h \in \mathbb{R} \setminus \{0\}$ with $|h| < \frac{r-s}{4}$. The function

$$\varphi := -D_\ell^{-h} \left(\zeta^2 D_\ell^h u \right) \tag{3.4}$$

is then contained in $H_0^1(U)$. Inserting this into (3.1) leads to $A = B$ with the terms

$$A := \sum_{i,j=1}^{3} \int_U \varepsilon_{ij} \partial_i u \partial_j \varphi \, \mathrm{d}x \,, \quad B := \int_U (f\varphi + \mathbf{F} \cdot \nabla \varphi) \, \mathrm{d}x \,. \tag{3.5}$$

To estimate A, we first substitute $y = x - he_\ell$ to obtain

$$A = \sum_{i,j=1}^{3} \int_U D_\ell^h \left(\varepsilon_{ij} \partial_i u \right) \partial_j \left(\zeta^2 D_\ell^h u \right) \, \mathrm{d}y \,. \tag{3.6}$$

Using the notion $v^h(x) := v(x + he_\ell)$ and the product rule $D_\ell^h(vw) = v^h D_\ell^h w + w D_\ell^h v$, we derive $A = A_1 + A_2$ with

$$A_1 := \sum_{i,j=1}^{3} \int_U \varepsilon_{ij}^h \left(D_\ell^h \partial_i u \right) \left(D_\ell^h \partial_j u \right) \zeta^2 \, \mathrm{d}x \,,$$

$$A_2 := \sum_{i,j=1}^{3} \int_U \Big(2\varepsilon_{ij}^h \zeta \partial_j \zeta \left(D_\ell^h \partial_i u \right) D_\ell^h u + \left(D_\ell^h \varepsilon_{ij} \right) \partial_i u D_\ell^h (\partial_j u) \zeta^2$$

$$+ 2 \left(D_\ell^h \varepsilon_{ij} \right) \partial_i u \zeta \partial_j \zeta D_\ell^h u \Big) \, \mathrm{d}x \,.$$

Since ε is uniformly positive definite, we have

$$A_1 \geq \eta \int_U \zeta^2 \left| D_\ell^h \nabla u \right|^2 \, \mathrm{d}x \,. \tag{3.7}$$

To estimate A_2, we use the boundedness of ε_{ij}, ζ and $\partial_j \zeta$ on U as well as the fact that $\left| D_\ell^h \varepsilon_{ij}(x) \right|$ is bounded by a constant independent of h and x for all $i, j \in \{1, 2, 3\}$, since $\varepsilon|_{U_\pm}$ is Lipschitz continuous on $U_\pm$. Here it is crucial that $\ell \neq 3$. So there exists a constant $c = c(r, s) > 0$ independent of ε and η such that with $C_\varepsilon^{(1)} := c \left\| \varepsilon \right\|_{\mathcal{C}^1(U)}$ it holds

$$|A_2| \leq C_\varepsilon^{(1)} \int_W \zeta \left(\left| D_\ell^h \nabla u \right| \left| D_\ell^h u \right| + \left| D_\ell^h \nabla u \right| |\nabla u| + \left| D_\ell^h u \right| |\nabla u| \right) \, \mathrm{d}x \,.$$

(Below we often do not state the dependence of the constants on r, s.) This implies for every $\delta > 0$ the estimate

$$|A_2| \leq \delta \int_W \zeta^2 \left| D_\ell^h \nabla u \right|^2 \, \mathrm{d}x + \left(\frac{{C_\varepsilon^{(1)}}^2}{2\delta} + \frac{C_\varepsilon^{(1)}}{2} \right) \int_W \left(|\nabla u|^2 + \left| D_\ell^h u \right|^2 \right) \, \mathrm{d}x \,. \tag{3.8}$$

By Theorem 3 of Section 5.8 in [8], there exists a constant $C^{(2)} > 0$ such that

$$\int_W \left|D_\ell^h u\right|^2 \,\mathrm{d}x \le C^{(2)} \int_U |\nabla u|^2 \,\mathrm{d}x \tag{3.9}$$

holds for all $h \in \mathbb{R}\setminus\{0\}$ with $|h| < \frac{r-s}{4}$. Choosing $\delta = \frac{\eta}{2}$, we obtain from (3.8) the bound

$$|A_2| \le \frac{\eta}{2}\int_U \zeta^2 \left|D_\ell^h \nabla u\right|^2 \,\mathrm{d}x + C_\varepsilon^{(3)} \int_U |\nabla u|^2 \,\mathrm{d}x$$

with

$$C_{\varepsilon,\eta}^{(3)} := \left(1 + C^{(2)}\right)\left(\frac{{C_\varepsilon^{(1)}}^2}{\eta} + \frac{C_\varepsilon^{(1)}}{2}\right).$$

Together with (3.7), this yields

$$A \ge \frac{\eta}{2}\int_U \zeta^2 \left|D_\ell^h \nabla u\right|^2 \,\mathrm{d}x - C_{\varepsilon,\eta}^{(3)} \int_U |\nabla u|^2 \,\mathrm{d}x\,. \tag{3.10}$$

We now turn to $B = B_1 + B_2$ with $B_1 := \int_U f\varphi \,\mathrm{d}x$ and $B_2 := \int_U \mathbf{F}\cdot\nabla\varphi \,\mathrm{d}x$. Again using (3.9), we derive the estimate

$$\begin{aligned}\int_U |\varphi|^2 \,\mathrm{d}x &\le C^{(2)} \int_W \left|\nabla(\zeta^2 D_\ell^h u)\right|^2 \,\mathrm{d}x \le \widetilde{C}^{(4)} \int_W \left(\left|D_\ell^h u\right|^2 + \zeta^2 \left|D_\ell^h \nabla u\right|^2\right) \mathrm{d}x \\ &\le C^{(4)} \int_U \left(|\nabla u|^2 + \zeta^2 \left|D_\ell^h \nabla u\right|^2\right) \mathrm{d}x\end{aligned}$$

for some constants $\widetilde{C}^{(4)}, C^{(4)} > 0$. Therefore it holds

$$\begin{aligned}|B_1| &\le \alpha \int_U \left(|\nabla u|^2 + \zeta^2 \left|D_\ell^h \nabla u\right|^2\right) \mathrm{d}x + \frac{C^{(4)}}{4\alpha}\int_U f^2 \,\mathrm{d}x \\ &\le \alpha \int_U \zeta^2 \left|D_\ell^h \nabla u\right|^2 \,\mathrm{d}x + \frac{C^{(4)}}{\alpha}\int_U \left(f^2 + |\nabla u|^2\right) \mathrm{d}x\end{aligned} \tag{3.11}$$

for all $\alpha > 0$ with $\alpha^2 \le C^{(4)}$. The same substitution used to obtain (3.6) yields

$$B_2 = \int_W D_\ell^h \mathbf{F} \cdot \nabla(\zeta^2 D_\ell^h u) \,\mathrm{d}x\,.$$

Let $\beta > 0$. As for A_2, we apply (3.9) and estimate

$$\begin{aligned}|B_2| &\le \frac{\beta}{2}\int_W \left|\nabla(\zeta^2 D_\ell^h u)\right|^2 \,\mathrm{d}x + \frac{1}{2\beta}\int_W \left|D_\ell^h \mathbf{F}\right|^2 \,\mathrm{d}x \\ &\le \beta\left(\int_W \left|2\zeta D_\ell^h u \nabla\zeta\right|^2 \,\mathrm{d}x + \int_W \left|\zeta^2 D_\ell^h \nabla u\right|^2 \,\mathrm{d}x\right) + \frac{1}{2\beta}\int_W \left|D_\ell^h \mathbf{F}\right|^2 \,\mathrm{d}x \\ &\le \beta C^{(5)}\left(\int_U |\nabla u|^2 \,\mathrm{d}x + \int_U \zeta^2 \left|D_\ell^h \nabla u\right|^2 \,\mathrm{d}x\right) + \frac{1}{2\beta}\int_W \left|D_\ell^h \mathbf{F}\right|^2 \,\mathrm{d}x\end{aligned} \tag{3.12}$$

for some constant $C^{(5)} > 0$. To deal with the last term, we use a variant of (3.9) on W_+ and W_-: There exists a constant $C^{(6)} > 0$ such that

$$\int_W \left|D_\ell^h \mathbf{F}\right|^2 \,\mathrm{d}x \le C^{(6)} \left\|\mathbf{F}\right\|^2_{\mathcal{H}^1(U)} \tag{3.13}$$

holds for all $h \in \mathbb{R} \setminus \{0\}$ with $|h| < \frac{r-s}{4}$, see the comments at the end of Section 5.8.2 of [8]. Inequalities (3.10), (3.11), (3.12) and (3.13) lead to

$$\begin{aligned}
&\left(\frac{\eta}{2} - \alpha - \beta C^{(5)}\right) \int_U \zeta^2 \left|D_\ell^h \nabla u\right|^2 \,\mathrm{d}x \\
&\quad \le C^{(3)}_{\varepsilon,\eta} \int_U |\nabla u|^2 \,\mathrm{d}x + \frac{C^{(4)}}{\alpha} \int_U \left(f^2 + |\nabla u|^2\right) \mathrm{d}x + \beta C^{(5)} \int_U |\nabla u|^2 \,\mathrm{d}x \\
&\qquad + \frac{C^{(6)}}{2\beta} \left\|\mathbf{F}\right\|^2_{\mathcal{H}^1(U)} .
\end{aligned}$$

We choose $\alpha = \min\left\{\sqrt{C^{(4)}}, \frac{\eta}{8}\right\} > 0$ and $\beta = \frac{\eta}{8C^{(5)}} > 0$ and conclude that there exists a constant $C^{(7)}_{\varepsilon,\eta} = C^{(7)}_{\varepsilon,\eta}(\|\varepsilon\|_{\mathcal{C}^1(U)}, \eta^{-1}) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^1(U)}$ and η^{-1} such that

$$\left\|D_\ell^h \nabla u\right\|_{L^2(V)} \le \left\|\zeta D_\ell^h \nabla u\right\|_{L^2(U)} \le C^{(7)}_{\varepsilon,\eta} \left(\|f\|_{L^2(U)} + \|\mathbf{F}\|_{\mathcal{H}^1(U)} + \|u\|_{H^1(U)}\right)$$

holds for all $h \in \mathbb{R} \setminus \{0\}$ with $|h| < \frac{r-s}{4}$. The proof of Theorem 3 in Section 5.8.2 of [8] now implies that $\partial_\ell u$ is contained in $H^1(V)$ for $\ell = 1, 2$ and u satisfies

$$\sum_{\substack{m,\ell=1,\\ m+\ell<6}}^{3} \|\partial_{m\ell} u\|_{L^2(V)} \le C^{(7)}_{\varepsilon,\eta} \left(\|f\|_{L^2(U)} + \|\mathbf{F}\|_{\mathcal{H}^1(U)} + \|u\|_{H^1(U)}\right) . \tag{3.14}$$

So we have proved (3.2) and it remains to consider $\partial_{33} u$. For any $\psi \in H^1_0(U_+)$, we obtain

$$\int_{U_+} (\varepsilon \nabla u) \cdot \nabla \psi \,\mathrm{d}x = \int_{U_+} (f - \operatorname{div} \mathbf{F}) \psi \,\mathrm{d}x \tag{3.15}$$

from (3.1). Since $\varepsilon|_{U_+} \in C^1(\overline{U_+})^{3\times 3}$ and $(f - \operatorname{div}\mathbf{F})|_{U_+} \in L^2(U_+)$, we can apply Theorem 4.9 in [9] to deduce that $u|_{U_+} \in H^2_{\mathrm{loc}}(U_+)$ satisfies $-\operatorname{div}(\varepsilon \nabla u) = f - \operatorname{div}\mathbf{F}$ almost everywhere in U_+. Therefore

$$\varepsilon_{33} \partial_{33} u = -\sum_{\substack{m,\ell=1,\\ m+\ell<6}}^{3} \varepsilon_{m\ell} \partial_{m\ell} u - \sum_{m,\ell=1}^{3} (\partial_m \varepsilon_{m,\ell}) \partial_\ell u - f + \operatorname{div}\mathbf{F}$$

holds almost everywhere in U_+. The matrix ε is uniformly positive definite, so $\varepsilon_{33}(x) \ge \eta > 0$ for all $x \in U$ and we obtain

$$|\partial_{33} u| \le C^{(8)}_{\varepsilon,\eta} \left(\sum_{\substack{m,\ell=1,\\ m+\ell<6}}^{3} |\partial_{m\ell} u| + |\nabla u| + |f| + |\operatorname{div}\mathbf{F}| \right)$$

almost everywhere in U_+ for some constant $C^{(8)}_{\varepsilon,\eta} = C^{(8)}_{\varepsilon,\eta}(\|\varepsilon\|_{\mathcal{C}^1(U)}, \eta^{-1}) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^1(U)}$ and η^{-1}, with an analogous result on U_-. This estimate and (3.14) yield $u \in \mathcal{H}^2(V)$ and (3.3). □

We can now perform an induction to prove higher piecewise regularity of the solution in the setting of Lemma 3.1, provided ε, f and $\mathbf{F}$ are sufficiently regular.

Lemma 3.2. *Let $k \in \mathbb{N}_0$, $\eta > 0$ and $s, r \in \mathbb{R}$ with $0 < s < r$. Let $U := B(0,r) \subseteq \mathbb{R}^3$, $V := B(0,s) \subseteq \mathbb{R}^3$ and $\varepsilon \in \mathcal{C}^{k+1}_\eta(U)$. Let $f \in \mathcal{H}^k(U)$, $\mathbf{F} \in \mathcal{H}^{k+1}(U)^3$ and let $u \in H^1(U)$ satisfy (3.1) for all $\varphi \in H^1_0(U)$. Then $u \in \mathcal{H}^{k+2}(V)$ and there exists a constant $C = C(k, \|\varepsilon\|_{\mathcal{C}^{k+1}(U)}, \eta^{-1}, r, s) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{k+1}(U)}$ and η^{-1} such that*

$$\|u\|_{\mathcal{H}^{k+2}(V)} \le C \left(\|f\|_{\mathcal{H}^k(U)} + \|\mathbf{F}\|_{\mathcal{H}^{k+1}(U)} + \|u\|_{H^1(U)} \right)$$

holds. Furthermore, $\partial^\alpha u \in H^1(V)$ with

$$\|\partial^\alpha u\|_{H^1(V)} \le C \left(\|f\|_{\mathcal{H}^k(U)} + \|\mathbf{F}\|_{\mathcal{H}^{k+1}(U)} + \|u\|_{H^1(U)} \right)$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le k+1$ and $\alpha_3 = 0$.

Proof. The proof is based on the proof of Theorem 5 in Section 6.3 of [8]. We use an induction and prove $u \in \mathcal{H}^{m+2}(V)$ as well a corresponding estimate for $m \in \{0, \dots, k\}$. The case $m = 0$ follows from Lemma 3.1. Let $m < k$ and assume that the claim holds for m instead of k. We fix the radius $t = \frac{r+s}{2}$ and set $W := B(0,t)$. By assumption, we have $u \in \mathcal{H}^{m+2}(W)$ and $\partial^\alpha u \in H^1(W)$ with

$$\begin{aligned} \|u\|_{\mathcal{H}^{m+2}(W)} &\le C_{m,\varepsilon,\eta} \left(\|f\|_{\mathcal{H}^m(U)} + \|\mathbf{F}\|_{\mathcal{H}^{m+1}(U)} + \|u\|_{H^1(U)} \right), \\ \|\partial^\alpha u\|_{H^1(W)} &\le C_{m,\varepsilon,\eta} \left(\|f\|_{\mathcal{H}^m(U)} + \|\mathbf{F}\|_{\mathcal{H}^{m+1}(U)} + \|u\|_{H^1(U)} \right) \end{aligned} \tag{3.16}$$

for all $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le m+1$, $\alpha_3 = 0$, where $C_{m,\varepsilon,\eta} = C_{m,\varepsilon,\eta}(m, \|\varepsilon\|_{\mathcal{C}^{m+1}(U)}, \eta^{-1}) > 0$ is a constant nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+1}(U)}$ and η^{-1}. Let $\widetilde{\varphi} \in C_c^\infty(W)$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = m+1$ and $\alpha_3 = 0$. We set $\varphi := (-1)^{|\alpha|} \partial^\alpha \widetilde{\varphi} \in C_c^\infty(W)$, $\widetilde{u} := \partial^\alpha u \in H^1(W)$ and

$$\widetilde{\mathbf{F}} := \partial^\alpha \mathbf{F} - \sum_{\substack{\beta \le \alpha, \\ \beta \ne \alpha}} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \varepsilon \partial^\beta \nabla u \in \mathcal{H}^1(W)^3 .$$

Clearly, $\widetilde{f} := \partial^\alpha f$ (defined piecewise) exists and is contained in $L^2(W)$. The functions $\widetilde{f}$ and $\widetilde{\mathbf{F}}$ satisfy

$$\|\widetilde{f}\|_{L^2(W)} \le \|f\|_{\mathcal{H}^{m+1}(W)}, \quad \|\widetilde{\mathbf{F}}\|_{\mathcal{H}^1(W)} \le C^{(1)}_\varepsilon \left(\|\mathbf{F}\|_{\mathcal{H}^{m+2}(W)} + \|u\|_{\mathcal{H}^{m+2}(W)} \right) \tag{3.17}$$

for $C^{(1)}_\varepsilon := c \|\varepsilon\|_{\mathcal{C}^{m+2}(U)}$ with a constant $c > 0$. We insert φ into (3.1) and integrate by parts to obtain

$$\int_W (\varepsilon \nabla \widetilde{u}) \cdot \nabla \widetilde{\varphi} \, \mathrm{d}x = \int_W \left(\widetilde{f} \widetilde{\varphi} + \widetilde{\mathbf{F}} \cdot \nabla \widetilde{\varphi} \right) \mathrm{d}x ,$$

noting that no boundary integrals involving the interface $W \cap \{x_3 = 0\}$ appear since $\alpha_3 = 0$. By density the above equation holds for all $\widetilde{\varphi} \in H_0^1(W)$. Lemma 3.1 now implies $\widetilde{u} \in \mathcal{H}^2(V)$ and $\partial_\ell \widetilde{u} \in H^1(V)$ with

$$\|\widetilde{u}\|_{\mathcal{H}^2(V)} \leq C^{(2)}_{\varepsilon,\eta} \left(\|\widetilde{f}\|_{L^2(W)} + \|\widetilde{\mathbf{F}}\|_{\mathcal{H}^1(W)} + \|\widetilde{u}\|_{H^1(W)} \right) ,$$
$$\|\partial_\ell \widetilde{u}\|_{H^1(V)} \leq C^{(2)}_{\varepsilon,\eta} \left(\|\widetilde{f}\|_{L^2(W)} + \|\widetilde{\mathbf{F}}\|_{\mathcal{H}^1(W)} + \|\widetilde{u}\|_{H^1(W)} \right)$$

for $\ell = 1, 2$ and some constant $C^{(2)}_{\varepsilon,\eta} = C^{(2)}_{\varepsilon,\eta}(\|\varepsilon\|_{\mathcal{C}^1(U)}, \eta^{-1}, r, s) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^1(U)}$ and η^{-1}. Using (3.16) and (3.17), we deduce that $u|_{V_\pm}$ has weak derivatives $\partial^\beta u|_{V_\pm} \in L^2(V_\pm)$ for all $\beta \in \mathbb{N}_0^3$ with $|\beta| = m + 3$ and $\beta_3 \in \{0, 1, 2\}$, satisfying

$$\left\|\partial^\beta u|_{V_+}\right\|_{L^2(V_+)} + \left\|\partial^\beta u|_{V_-}\right\|_{L^2(V_-)} \leq C^{(3)}_{\varepsilon,\eta} \left(\|f\|_{\mathcal{H}^{m+1}(U)} + \|\mathbf{F}\|_{\mathcal{H}^{m+2}(U)} + \|u\|_{H^1(U)} \right) \quad (3.18)$$

and additionally, $\partial^\mu u \in H^1(V)$ with

$$\|\partial^\mu u\|_{H^1(V)} \leq C^{(3)}_{\varepsilon,\eta} \left(\|f\|_{\mathcal{H}^{m+1}(U)} + \|\mathbf{F}\|_{\mathcal{H}^{m+2}(U)} + \|u\|_{H^1(U)} \right)$$

for all $\mu \in \mathbb{N}_0^3$ with $|\mu| \leq m + 2$ and $\mu_3 = 0$, where $C^{(3)}_{\varepsilon,\eta} = C^{(3)}_{\varepsilon,\eta}(\|\varepsilon\|_{\mathcal{C}^{m+2}(U)}, \eta^{-1}) > 0$ is nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+2}(U)}$ and η^{-1}. It remains to show (3.18) also for $\beta_3 \in \{3, \dots, m + 3\}$. For this we use a second induction on β_3 (for fixed $m < k$) and assume that (3.18) is true for all $\beta \in \mathbb{N}_0^3$ with $|\beta| = m + 3$ and $\beta_3 \in \{0, \dots, j\}$ for some $j \in \{2, \dots, m + 2\}$ (possibly with a different constant than $C^{(3)}_{\varepsilon,\eta}$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+2}(U)}$ and η^{-1}). So let $\beta \in \mathbb{N}_0^3$ with $|\beta| = m + 3$ and $\beta_3 = j + 1$. We write $\beta = \gamma + (0, 0, 2)$ and argue as in the last step of the proof of Lemma 3.1: By Theorem 4.11 in [9], $u|_{U_\pm}$ is contained in $H^{m+3}_{\text{loc}}(U_\pm)$ and satisfies $-\operatorname{div}(\varepsilon \nabla u) = f - \operatorname{div} \mathbf{F}$ almost everywhere in $U_\pm$. Applying ∂^γ leads to

$$\varepsilon_{33} \partial^\beta u = -\partial^\gamma f + \partial^\gamma \operatorname{div} \mathbf{F} + R ,$$

where

$$R := \partial^\gamma \left(\sum_{\substack{i,j=1,\\ i+j<6}}^{3} \varepsilon_{ij} \partial_{ij} u + \sum_{i,j=1}^{3} \partial_i \varepsilon_{ij} \partial_j u \right)$$

contains only terms with derivatives $\partial^\nu u$ with $|\nu| \leq m + 3$ and $\nu_3 \leq j$. We can now use that $\varepsilon_{33}(x) \geq \eta > 0$ for all $x \in U$ and the induction assumption to obtain $\partial^\beta u|_{V_\pm} \in L^2(V_\pm)$ and

$$\left\|\partial^\beta u|_{V_+}\right\|_{L^2(V_+)} + \left\|\partial^\beta u|_{V_-}\right\|_{L^2(V_-)} \leq C^{(4)}_{\varepsilon,r,s} \left(\|f\|_{\mathcal{H}^{m+1}(U)} + \|\mathbf{F}\|_{\mathcal{H}^{m+2}(U)} + \|u\|_{H^1(U)} \right)$$

for some constant $C^{(4)}_{\varepsilon} = C^{(4)}_{\varepsilon}(\|\varepsilon\|_{\mathcal{C}^{m+2}(U)}, \eta^{-1}) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+2}(U)}$ and η^{-1}. This completes the induction on j, and we conclude $u \in \mathcal{H}^{m+3}(V)$ and

$$\|u\|_{\mathcal{H}^{m+3}(V)} \leq C^{(5)}_{m,\varepsilon} \left(\|f\|_{\mathcal{H}^{m+1}(U)} + \|\mathbf{F}\|_{\mathcal{H}^{m+2}(U)} + \|u\|_{H^1(U)} \right)$$

for some constant $C^{(5)}_{m,\varepsilon} = C^{(5)}_{\varepsilon}(\|\varepsilon\|_{\mathcal{C}^{m+2}(U)}, \eta^{-1}) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+2}(U)}$ and η^{-1}, finishing the proof. ☐

We need similar statements for a neighbourhood of a point on the boundary ∂G, both for Dirichlet and Neumann boundary conditions. After a coordinate transform, we can assume the intersection of the neighbourhood with G to be a half-ball. In the case of Neumann boundary conditions, we can assume u to vanish near the curved part of the half-ball, since this is true for the localized functions used in the proof of Proposition 3.5.

Lemma 3.3. *Let $0 < s < r$, $U_+ := B(0,r) \cap \{x_3 > 0\} \subseteq \mathbb{R}^3$, $V_+ := B(0,s) \cap \{x_3 > 0\} \subseteq \mathbb{R}^3$, $\varepsilon \in C^1(\overline{U_+})^{3\times 3}$ be symmetric and satisfy $\big(\varepsilon(x)\xi\big) \cdot \xi \geq \eta\,|\xi|^2$ for all $x \in U_+$, $\xi \in \mathbb{R}^3$ and some constant $\eta > 0$. Let $f \in L^2(U_+)$, $\mathbf{F} \in H^1(U_+)^3$, and assume that one of the following two conditions is true.*

a) The function $u \in H^1_0(U_+)$ satisfies

$$\int_{U_+} (\varepsilon \nabla u) \cdot \nabla \varphi \,\mathrm{d}x = \int_{U_+} (f\varphi + \mathbf{F} \cdot \nabla \varphi)\,\mathrm{d}x \tag{3.19}$$

for all $\varphi \in H^1_0(U_+)$.

b) The function $u \in H^1(U_+)$ satisfies $u = 0$ on $U_+ \setminus V_+$ and (3.19) for all $\varphi \in H^1(U_+)$.

Then $u \in H^2(V_+)$ and there exists a constant $C = C(\|\varepsilon\|_{C^1(U_+)}, \eta^{-1}, r, s) > 0$ nondecreasing in $\|\varepsilon\|_{C^1(U_+)}$ and η^{-1} such that

$$\|u\|_{H^2(V_+)} \leq C \left(\|f\|_{L^2(U_+)} + \|\mathbf{F}\|_{H^1(U_+)} + \|u\|_{H^1(U_+)} \right)$$

holds.

Proof. The proof is basically the same as for Lemma 3.1. We set $W_+ := B\left(0, \frac{r+s}{2}\right) \cap \{x_3 > 0\}$ and choose a function $\zeta \in C^\infty_c(\mathbb{R}^3)$ with $\zeta = 1$ on V_+ and $\operatorname{supp}\zeta \subseteq W_+$. Let $\ell \in \{1,2\}$ and φ be defined by (3.4). In the first case of the statement, both u and φ are contained in $H^1_0(U_+)$, while in the second case, they are only elements of $H^1(U_+)$. We insert φ into (3.19). In both cases, we obtain $A = B$ with A and B given by (3.5). The estimates for A and B now work the in same way as in the proof of Lemma 3.1, the only difference being that (3.13) is replaced by

$$\int_{W_+} \left|D^h_\ell \mathbf{F}\right|^2 \mathrm{d}x \leq C^{(6)} \|\mathbf{F}\|^2_{H^1(U_+)} . \qquad ☐$$

As for Lemma 3.2, an induction argument can be used to obtain higher regularity of the solution under appropriate assumptions.

Lemma 3.4. *Let $k \in \mathbb{N}_0$, $0 < s < r$, $U_+ := B(0,r) \cap \{x_3 > 0\} \subseteq \mathbb{R}^3$, $V_+ := B(0,s) \cap \{x_3 > 0\} \subseteq \mathbb{R}^3$, $\varepsilon \in C^{k+1}(\overline{U_+})^{3\times 3}$ be symmetric and satisfy $\big(\varepsilon(x)\xi\big) \cdot \xi \geq \eta\,|\xi|^2$ for all $x \in U_+$, $\xi \in \mathbb{R}^3$ and some constant $\eta > 0$. Let $f \in H^k(U_+)$, $\mathbf{F} \in H^{k+1}(U_+)^3$, and assume that one the following two conditions is true.*

a) The function $u \in H_0^1(U_+)$ satisfies (3.19) for all $\varphi \in H_0^1(U_+)$.

b) The function $u \in H^1(U_+)$ satisfies $u = 0$ on $U_+ \backslash V_+$ and (3.19) for all $\varphi \in H^1(U_+)$.

Then $u \in H^{k+2}(V_+)$ and there exists a constant $C = C(k, \|\varepsilon\|_{C^{k+1}(U_+)}, \eta^{-1}, r, s) > 0$ nondecreasing in $\|\varepsilon\|_{C^{k+1}(U_+)}$ and η^{-1} such that

$$\|u\|_{H^{k+2}(V_+)} \leq C \left(\|f\|_{H^k(U_+)} + \|\mathbf{F}\|_{H^{k+1}(U_+)} + \|u\|_{H^1(U_+)} \right)$$

holds.

Proof. The proof is analogous to the one of Lemma 3.2: We prove inductively $u \in H^{m+2}(V_+)$ and the corresponding estimate for $m \in \{0, \dots, k\}$. The case $m = 0$ is Lemma 3.3. So let $m < k$ and assume that the claim holds for m instead of k. We set $W_+ := B(0,t) \cap \{x_3 > 0\}$ for some $t = \frac{r+s}{2}$ and have $u \in H^{m+2}(W_+)$ as well as

$$\|u\|_{H^{m+2}(W_+)} \leq C_{m,\varepsilon,\eta} \left(\|f\|_{H^m(U_+)} + \|\mathbf{F}\|_{H^{m+1}(U_+)} + \|u\|_{H^1(U_+)} \right)$$

for some constant $C_{m,\varepsilon,\eta} = C_{m,\varepsilon,\eta}(m, \|\varepsilon\|_{C^{m+1}(U_+)}, \eta^{-1}) > 0$ nondecreasing in $\|\varepsilon\|_{C^{m+1}(U_+)}$ and η^{-1}.

a) We start with the first case. Let $\widetilde{\varphi} \in C_c^\infty(W_+)$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = m+1$ and $\alpha_3 = 0$. We define φ, $\widetilde{u}$, $\widetilde{f}$ and $\widetilde{\mathbf{F}}$ as in the proof of Lemma 3.2 and obtain

$$\int_{W_+} (\varepsilon \nabla \widetilde{u}) \cdot \nabla \widetilde{\varphi} \, \mathrm{d}x = \int_{W_+} \left(\widetilde{f} \widetilde{\varphi} + \widetilde{\mathbf{F}} \cdot \nabla \widetilde{\varphi} \right) \mathrm{d}x \,. \tag{3.20}$$

Lemma 3.3 yields $\widetilde{u} \in H^2(V_+)$ and

$$\|\widetilde{u}\|_{H^2(V_+)} \leq C_{\varepsilon,\eta}^{(1)} \left(\|\widetilde{f}\|_{L^2(W_+)} + \|\widetilde{\mathbf{F}}\|_{H^1(W_+)} + \|\widetilde{u}\|_{H^1(W_+)} \right)$$

for some constant $C_{\varepsilon,\eta}^{(1)} = C_{\varepsilon,\eta}^{(1)}(\|\varepsilon\|_{C^1(U_+)}, \eta^{-1}) > 0$ nondecreasing in $\|\varepsilon\|_{C^1(U_+)}$ and η^{-1}. It follows $\partial^\beta u \in L^2(V_+)$ with

$$\|\partial^\beta u\|_{L^2(V_+)} \leq C_{m,\varepsilon,\eta}^{(2)} \left(\|f\|_{H^{m+1}(U_+)} + \|\mathbf{F}\|_{H^{m+2}(U_+)} + \|u\|_{H^1(U_+)} \right)$$

for all $\beta \in \mathbb{N}_0^3$ with $|\beta| = m+3$ and $\beta_3 \in \{0,1,2\}$ and some constant $C_{m,\varepsilon,\eta}^{(2)} = C_{m,\varepsilon,\eta}^{(2)}(m, \|\varepsilon\|_{C^{m+2}(U_+)}, \eta^{-1}) > 0$ nondecreasing in $\|\varepsilon\|_{C^{m+2}(U_+)}$ and η^{-1}. As in the proof of Lemma 3.2, an induction over $\beta_3 \in \{0, \dots, m+3\}$, using that ε is uniformly positive definite, leads to $u \in H^{m+3}(V_+)$ with

$$\|u\|_{\mathcal{H}^{m+3}(V_+)} \leq C_{m,\varepsilon,\eta}^{(3)} \left(\|f\|_{\mathcal{H}^{m+1}(U_+)} + \|\mathbf{F}\|_{\mathcal{H}^{m+2}(U_+)} + \|u\|_{H^1(U_+)} \right)$$

for some constant $C_{m,\varepsilon,\eta}^{(3)} = C_{m,\varepsilon,\eta}^{(3)}(\|\varepsilon\|_{C^{m+1,1}(U_+)}, \eta^{-1}) > 0$ nondecreasing in $\|\varepsilon\|_{C^{m+1,1}(U_+)}$ and η^{-1}.

b) The second case works in the same way, but instead of $\widetilde{\varphi} \in C_c^\infty(W_+)$, we take $\widetilde{\varphi} \in C_c^\infty(\overline{W_+}) := \left\{ \psi|_{W_+} \mid \psi \in C_c^\infty(\mathbb{R}^3) \right\}$. Due to $\alpha_3 = 0$ and the assumption that $u = 0$ on $U_+ \setminus V_+$, no boundary integrals appear in the derivation of (3.20). Since $C_c^\infty(\overline{W_+})$ is dense in $H^1(W_+)$ by Theorem 3.6 in [17], equation (3.20) holds for all $\widetilde{\varphi} \in H^1(W_+)$ and we can use the second case of Lemma 3.3 to obtain $\widetilde{u} \in H^2(V_+)$ with the corresponding estimate. The rest of the proof is the same as in case a). □

We now consider the the transmission problems (2.5) and (2.8) for the case $g = 0$ and $h = 0$. Under appropriate assumptions on the domain, ε and f, weak solutions have additional regularity.

Proposition 3.5. *Let $k \in \mathbb{N}_0$, $\eta > 0$, Σ and ∂G be of class C^{k+2} and $\varepsilon \in \mathcal{C}_\eta^{k+1}$. Let $f \in \mathcal{H}^k$ and one the following two conditions be true.*

a) The function $u \in H_0^1(G)$ satisfies

$$\int_G (\varepsilon \nabla u) \cdot \nabla \varphi \, \mathrm{d}x = \int_G f \varphi \, \mathrm{d}x \tag{3.21}$$

for all $\varphi \in H_0^1(G)$.

b) The function $u \in H^1(G)$ satisfies (3.21) for all $\varphi \in H^1(G)$.

Then $u \in \mathcal{H}^{k+2}$ and there exists a constant $C = C(k, \|\varepsilon\|_{\mathcal{C}^{k+1}}, \eta^{-1}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{k+1}}$ and η^{-1} such that

$$\|u\|_{\mathcal{H}^{k+2}} \leq C \left(\|f\|_{\mathcal{H}^k} + \|u\|_{H^1(G)} \right)$$

holds.

Proof. We first show $u \in \mathcal{H}^2$ and a corresponding estimate and perform an iteration argument at the end.

1) *(Regularity near the boundary)* Let $x^0 \in \partial G$. Then there exists a radius $R > 0$ and a function $\gamma \in C^{k+2}(\mathbb{R}^2, \mathbb{R})$ such that $B(x^0, R) \cap \Sigma = \emptyset$ and - possibly after an orthogonal transformation of the coordinate system - it holds

$$G \cap B(x^0, R) = \left\{ x \in B(x^0, R) \mid x_3 > \gamma(x_1, x_2) \right\} .$$

We now perform a coordinate transform from x to $y = \Phi(x)$ in order to "flatten" the boundary ∂G: Let $\Phi, \Psi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\Phi(x) = \begin{pmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ x_3 - \gamma(x_1, x_2) \end{pmatrix} , \quad \Psi(y) = \begin{pmatrix} y_1 + x_1^0 \\ y_2 + x_2^0 \\ y_3 + \gamma(y_1 + x_1^0, y_2 + x_2^0) \end{pmatrix} .$$

Then we have $\Phi(x^0) = 0$ and $\Phi, \Psi \in C^{k+2}(\mathbb{R}^3, \mathbb{R}^3)$ satisfy $\Phi = \Psi^{-1}$ as well as $\det \Phi' = \det \Psi' = 1$. It is possible to find a radius $r > 0$ such that $\widetilde{U}_+ := B(0, r) \cap \{y_3 > 0\}$

is contained in $\Phi\big(G \cap B(x^0, R)\big)$. We choose some $s \in (0, r)$ and set $\widetilde{V} := B(0, s)$, $\widetilde{V}_+ := \widetilde{V} \cap \{y_3 > 0\}$, $U_+ := \Psi(\widetilde{U}_+)$, $V := \Psi(\widetilde{V})$ and $V_+ := \Psi(\widetilde{V}_+)$. We note that V is an open neighbourhood of x^0 and set

$$\widetilde{\varepsilon}_{ij}(y) := \sum_{r,s=1}^{3} \varepsilon_{rs}\big(\Psi(y)\big)\partial_r\Phi_i\big(\Psi(y)\big)\partial_s\Phi_j\big(\Psi(y)\big)$$

for all $i, j \in \{1, 2, 3\}$ and all $y \in \widetilde{U}_+$. The symmetric matrix $\widetilde{\varepsilon}$ is contained in the space $C^{k+1}\big(\overline{\widetilde{U}_+}\big)^{3\times 3}$ with $\|\widetilde{\varepsilon}\|_{C^{k+1}(\widetilde{U}_+)} \le c\,\|\varepsilon\|_{\mathcal{C}^{k+1}}$ for some constant $c = c(G) > 0$ and for any $y \in \widetilde{U}_+$ and $\xi \in \mathbb{R}^3$, we have

$$\big(\widetilde{\varepsilon}(y)\xi\big)\cdot\xi = \big(\varepsilon\big(\Psi(y)\big)\chi\big)\cdot\chi \ge \eta\,|\chi|^2 \ge \widetilde{c}\eta\,|\xi|^2\,,$$

where $\chi := \Phi'\big(\Psi(y)\big)^{\mathsf{T}}\xi$ and $\widetilde{c} = \widetilde{c}(G) > 0$. Let $\theta \in C_c^\infty(\mathbb{R}^3)$ with $\operatorname{supp}\theta \subseteq V$ and define $v := \theta u$. We treat the two cases from the statement of the theorem separately.

a) In the first case, the function v is an element of $H_0^1(U_+)$ and satisfies

$$\int_{U_+} (\varepsilon\nabla v)\cdot\nabla\varphi\,\mathrm{d}x = \int_{U_+} (g\varphi + \mathbf{F}\cdot\nabla\varphi)\,\mathrm{d}x \tag{3.22}$$

for all $\varphi \in H_0^1(U_+)$, where

$$g := \theta f - (\varepsilon\nabla u)\cdot\nabla\theta \in L^2(U_+)\,, \quad \mathbf{F} := u\varepsilon\nabla\theta \in H^1(U_+)^3\,. \tag{3.23}$$

This follows from (3.21) and the product rule. Let $\widetilde{\varphi} \in H_0^1(\widetilde{U}_+)$. We define $\varphi(x) := \widetilde{\varphi}\big(\Phi(x)\big)$ for all $x \in U_+$ and $\widetilde{v}(y) := v\big(\Psi(y)\big)$ for all $y \in \widetilde{U}_+$. By Theorem 4.1 in [17], we have $\varphi \in H_0^1(U_+)$ and $\widetilde{v} \in H_0^1(\widetilde{U}_+)$. We further define $\widetilde{g}$ and $\widetilde{\mathbf{F}}$ on $\widetilde{U}_+$ by $\widetilde{g}(y) := g\big(\Psi(y)\big)$ and $\widetilde{F}_i(y) := \mathbf{F}\big(\Psi(y)\big)\cdot\nabla\Phi_i\big(\Psi(y)\big)$, $i = 1, 2, 3$. It holds $\widetilde{g} \in L^2(\widetilde{U}_+)$ and $\widetilde{\mathbf{F}} \in H^1(\widetilde{U}_+)^3$. Using the chain rule, $\Phi'\big(\Psi(y)\big) = \big(\Psi'(y)\big)^{-1}$ and the transformation rule, we obtain

$$\begin{aligned}\int_{\widetilde{U}_+} (\widetilde{\varepsilon}\nabla\widetilde{v})\cdot\nabla\widetilde{\varphi}\,\mathrm{d}y &= \int_{U_+} (\varepsilon\nabla v)\cdot\nabla\varphi\,\mathrm{d}x = \int_{U_+} (g\varphi + \mathbf{F}\cdot\nabla\varphi)\,\mathrm{d}x \\ &= \int_{\widetilde{U}_+} \Big(\widetilde{g}\widetilde{\varphi} + \widetilde{\mathbf{F}}\cdot\nabla\widetilde{\varphi}\Big)\,\mathrm{d}y\,.\end{aligned}$$

Lemma 3.3 now implies $\widetilde{v} \in H^2\big(\widetilde{V}_+\big)$ and the estimate

$$\|\widetilde{v}\|_{H^2(\widetilde{V}_+)} \le \widetilde{C}_{\varepsilon,\eta}\left(\|\widetilde{g}\|_{L^2(\widetilde{U}_+)} + \big\|\widetilde{\mathbf{F}}\big\|_{H^1(\widetilde{U}_+)} + \|\widetilde{v}\|_{H^1(\widetilde{U}_+)}\right)$$

for some constant $\widetilde{C}_{\varepsilon,\eta} = \widetilde{C}_{\varepsilon,\eta}(\|\varepsilon\|_{\mathcal{C}^1}, \eta^{-1}, r, s) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^1}$ and η^{-1}. We deduce $v \in H^2(V_+)$ and

$$\|v\|_{H^2(V_+)} \le C_{\varepsilon,\eta,G}\left(\|f\|_{L^2(G)} + \|u\|_{H^1(G)}\right)$$

for some constant $C_{\varepsilon,\eta,G} = C_{\varepsilon,\eta,G}(\|\varepsilon\|_{\mathcal{C}^1}, \eta^{-1}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^1}$ and η^{-1}, see Theorem 4.1 in [17].

b) In the second case, $v \in H^1(U_+)$ satisfies (3.22) for all $\varphi \in H^1(U_+)$, where g and $\mathbf{F}$ are again given by (3.23). Let $\widetilde{\varphi} \in H^1(\widetilde{U}_+)$. We define φ, $\widetilde{v}$ and $\widetilde{g}$ and $\widetilde{\mathbf{F}}$ as in Case a). Then we have $\varphi \in H^1(U_+)$, $\widetilde{v} \in H^1(\widetilde{U}_+)$, $\widetilde{g} \in L^2(\widetilde{U}_+)$, $\widetilde{\mathbf{F}} \in H^1(\widetilde{U}_+)^3$. As above, we obtain

$$\int_{\widetilde{U}_+} (\widetilde{\varepsilon}\nabla\widetilde{v}) \cdot \nabla\widetilde{\varphi} \, \mathrm{d}y = \int_{\widetilde{U}_+} \Big(\widetilde{g}\widetilde{\varphi} + \widetilde{\mathbf{F}} \cdot \nabla\widetilde{\varphi}\Big) \, \mathrm{d}y \,.$$

Again, Lemma 3.3 implies $\widetilde{v} \in H^2(\widetilde{V}_+)$ and the estimate

$$\|\widetilde{v}\|_{H^2(\widetilde{V}_+)} \leq \widetilde{C}_{\varepsilon,\eta} \Big(\|\widetilde{g}\|_{L^2(\widetilde{U}_+)} + \|\widetilde{\mathbf{F}}\|_{H^1(\widetilde{U}_+)} + \|\widetilde{v}\|_{H^1(\widetilde{U}_+)} \Big)$$

for some constant $\widetilde{C}_{\varepsilon,\eta} = \widetilde{C}_{\varepsilon,\eta}(\|\varepsilon\|_{\mathcal{C}^1}, \eta^{-1}) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^1}$ and η^{-1}. Therefore we have $v \in H^2(V_+)$ and

$$\|v\|_{H^2(V_+)} \leq C_{\varepsilon,\eta,G} \Big(\|f\|_{L^2(G)} + \|u\|_{H^1(G)} \Big)$$

for some constant $C_{\varepsilon,\eta,G} = C_{\varepsilon,\eta,G}(\|\varepsilon\|_{\mathcal{C}^1}, \eta^{-1}) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^1}$ and η^{-1}.

2) *(Regularity near the interface)* Now let $x^0 \in \Sigma$. Since $\operatorname{dist}(\Sigma, \partial G) > 0$, we can find a radius $R > 0$ such that $B(x^0, R) \cap \partial G = \emptyset$ and - possibly after an orthogonal transformation of the coordinate system - we have

$$G_1 \cap B(x^0, R) = \big\{ x \in B(x^0, R) \,\big|\, x_3 > \gamma(x_1, x_2) \big\} \,,$$
$$G_2 \cap B(x^0, R) = \big\{ x \in B(x^0, R) \,\big|\, x_3 < \gamma(x_1, x_2) \big\} \,,$$

where $\gamma \in C^{k+2}(\mathbb{R}^2, \mathbb{R}^2)$. We define Φ and Ψ as in Step 1) and find a radius $r > 0$ such that $\widetilde{U} := B(0, r)$ is contained in $\Phi\big(G \cap B(x^0, R)\big)$. We set $U := \Psi(\widetilde{U})$,

$$\widetilde{U}_+ := \widetilde{U} \cap \{y_3 > 0\} \,, \quad \widetilde{U}_- := \widetilde{U} \cap \{y_3 < 0\} \,,$$
$$U_1 := \Psi\big(\widetilde{U}_+\big) \,, \qquad U_2 := \Psi\big(\widetilde{U}_-\big)$$

and note that $U_i = U \cap G_i$ for $i = 1, 2$. Additionally, let $s \in (0, r)$ and set $\widetilde{V} := B(0, s)$, $V := \Psi(\widetilde{V})$ and

$$\widetilde{V}_+ := \widetilde{V} \cap \{y_3 > 0\} \,, \quad \widetilde{V}_- := \widetilde{V} \cap \{y_3 < 0\} \,,$$
$$V_1 := \Psi\big(\widetilde{V}_+\big) \,, \qquad V_2 := \Psi\big(\widetilde{V}_-\big) \,.$$

Let $\theta \in C_c^\infty(\mathbb{R}^3)$ with $\operatorname{supp}\theta \subseteq V$ and set $v := \theta u \in H_0^1(U)$. Using (3.21), we obtain

$$\int_U (\varepsilon\nabla v) \cdot \nabla\varphi \, \mathrm{d}x = \int_U (g\varphi + \mathbf{F} \cdot \nabla\varphi) \, \mathrm{d}x$$

for all $\varphi \in H_0^1(U)$, where

$$g := \theta f - (\varepsilon\nabla u) \cdot \nabla\theta \in L^2(U) \,, \quad \mathbf{F} := u\varepsilon\nabla\theta \in L^2(U)^3 \,.$$

As $\varepsilon|_{G_i} \in C^{k+1}\left(\overline{G_i}\right)^{3\times 3}$ for $i = 1, 2$, we have $\mathbf{F}|_{U_i} \in H^1(U_i)^3$ for $i = 1, 2$ and the estimate

$$\|\mathbf{F}\|_{H^1(U_1)} + \|\mathbf{F}\|_{H^1(U_2)} \leq c_\varepsilon \|u\|_{H^1(G)}$$

for $c_\varepsilon := c\,\|\varepsilon\|_{\mathcal{C}^1}$ with a constant $c > 0$. Let $\widetilde{\varphi} \in H^1_0(\widetilde{U})$ and let φ, $\widetilde{\varepsilon}$, $\widetilde{v}$, $\widetilde{g}$ and $\widetilde{\mathbf{F}}$ be defined as in Step 1). Then we have $\varphi \in H^1_0(U)$, $\widetilde{\varepsilon}|_{\widetilde{U}_\pm} \in C^{k+1}(\overline{\widetilde{U}_\pm})$, $\widetilde{v} \in H^1_0(\widetilde{U})$, $\widetilde{g} \in L^2(\widetilde{U})$, $\widetilde{\mathbf{F}} \in L^2(\widetilde{U})^3$ and $\widetilde{\mathbf{F}}|_{\widetilde{U}_\pm} \in H^1(\widetilde{U}_\pm)^3$. As in Step 1), we deduce

$$\int_{\widetilde{U}} (\widetilde{\varepsilon}\nabla\widetilde{v}) \cdot \nabla\widetilde{\varphi}\,\mathrm{d}y = \int_U (\varepsilon\nabla v) \cdot \nabla\varphi\,\mathrm{d}x = \int_U \left(g\varphi + \mathbf{F}\cdot\nabla\varphi\right)\mathrm{d}x = \int_{\widetilde{U}} \left(\widetilde{g}\widetilde{\varphi} + \widetilde{\mathbf{F}}\cdot\nabla\widetilde{\varphi}\right)\mathrm{d}y\,.$$

Again, $\widetilde{\varepsilon}$ is symmetric and uniformly positive definite on $\widetilde{U}$. Lemma 3.1 therefore yields $\widetilde{v}|_{\widetilde{V}_\pm} \in H^2(\widetilde{V}_\pm)$ and a positive constant $\widetilde{C}_{\varepsilon,\eta,G} = \widetilde{C}_{\varepsilon,\eta,G}(\|\varepsilon\|_{\mathcal{C}^1}, \eta^{-1}, G)$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^1}$ and η^{-1} such that

$$\|\widetilde{v}\|_{H^2(\widetilde{V}_+)} + \|\widetilde{v}\|_{H^2(\widetilde{V}_-)} \leq \widetilde{C}_{\varepsilon,\eta,G}\left(\|\widetilde{g}\|_{L^2(\widetilde{U})} + \left\|\widetilde{\mathbf{F}}\right\|_{H^1(\widetilde{U}_+)} + \left\|\widetilde{\mathbf{F}}\right\|_{H^1(\widetilde{U}_-)} + \|\widetilde{v}\|_{H^1(\widetilde{U})}\right)$$

holds. This implies $v|_{V_i} \in H^2(V_i)$ for $i = 1, 2$ and

$$\|v\|_{H^2(V_1)} + \|v\|_{H^2(V_2)} \leq C_{\varepsilon,\eta,G}\left(\|f\|_{L^2(G)} + \|u\|_{H^1(G)}\right)$$

and some constant $C_{\varepsilon,\eta,G} = C_{\varepsilon,\eta,G}(\|\varepsilon\|_{\mathcal{C}^1}, \eta^{-1}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^1}$ and η^{-1}.

3) *(Regularity in the interior)* Let $i \in \{1, 2\}$, $x^0 \in G_i$ and $r > 0$ such that $U := B(x^0, r) \subseteq G_i$. We choose some $s \in (0, r)$, set $V := B(x^0, s)$, choose a test function $\theta \in C_c^\infty(\mathbb{R}^3)$ with $\operatorname{supp}\theta \subseteq V$ and define $v := \theta u \in H^1_0(U)$. Using (3.21) and the divergence theorem, we obtain

$$\int_U (\varepsilon\nabla v)\cdot\nabla\varphi\,\mathrm{d}x = \int_U \left(g\varphi + \mathbf{F}\cdot\nabla\varphi\right)\mathrm{d}x$$

for all $\varphi \in H^1_0(U)$, where $g \in L^2(U)$ and $\mathbf{F} \in H^1(U)$ are defined as in (3.23). Theorem 4.9 in [9] now implies $v \in H^2(V)$ and

$$\|v\|_{H^2(V)} \leq \widetilde{C}_{\varepsilon,G}\left(\|g\|_{L^2(U)} + \|\mathbf{F}\|_{H^1(U)} + \|v\|_{L^2(U)}\right) \leq C_{\varepsilon,G}\left(\|f\|_{L^2(G)} + \|u\|_{H^1(G)}\right)$$

for positive constants $\widetilde{C}_{\varepsilon,G} = \widetilde{C}_{\varepsilon,G}(\|\varepsilon\|_{\mathcal{C}^1}, G)$ and $C_{\varepsilon,G} = C_{\varepsilon,G}(\|\varepsilon\|_{\mathcal{C}^1}, G)$.

4) *(Localization)* We cover the compact set $\overline{G}$ by a finite number of open sets V from Steps 1)-3): $\overline{G} \subseteq V_1 \cup \cdots \cup V_{K+L+M+N}$, where $V_1, \ldots, V_K$ and $V_K + 1, \ldots, V_{K+L}$ are as in Step 3) and are contained in G_1 respectively G_2, while $V_{K+L+1}, \ldots, V_{K+L+M}$ are as in Step 2) and cover Σ, and $V_{K+L+M+1}, \ldots, V_{K+L+M+N}$ are as in Step 1) and cover ∂G. Let $\theta_j \in C_c^\infty(\mathbb{R}^3)$, $j \in \{1, \ldots, J\}$, $J := K+L+M+N$, be a corresponding partition of unity, i.e., $\operatorname{supp}\theta_j \subseteq V_j$ and $\sum_{j=1}^{K+L+M+N} \theta_j = 1$ on $\overline{G}$. We set $v_j := \theta_j u$ for $j \in \{1, \ldots, J\}$

and $V_{K+L+m,i} := V_{K+L+m} \cap G_i$ for $m \in \{1, \dots, M\}$ and $i \in \{1,2\}$. Since $u = \sum_{j=1}^J v_j$, $\operatorname{supp} v_j \subseteq V_j$ and

$$G_1 \subseteq \left(\bigcup_{k=1}^{K} V_k \right) \cup \left(\bigcup_{m=1}^{M} V_{K+L+m,1} \right) ,$$
$$G_2 \subseteq \left(\bigcup_{\ell=1}^{L} V_{K+\ell} \right) \cup \left(\bigcup_{m=1}^{M} V_{K+L+m,2} \right) \cup \left(\bigcup_{n=1}^{N} V_{K+M+L+n} \right) ,$$

the claim for the case $k = 0$ follows from Steps 1)-3). (Note that in this step we can fix r and s above for each V_i only depending on G.)

5) *(Iteration)* If $k \geq 1$, we can iterate the above steps: Let $m \in \{0, \dots, k-1\}$ and u be contained in $\mathcal{H}^{m+2}$ with

$$\|u\|_{\mathcal{H}^{m+2}} \leq C_{m,\varepsilon,\eta,G} \left(\|f\|_{\mathcal{H}^m} + \|u\|_{H^1(G)} \right)$$

for some constant $C_{m,\varepsilon,\eta,G} = C_{m,\varepsilon,\eta,G}(m, \|\varepsilon\|_{\mathcal{C}^{m+1}}, \eta^{-1}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+1}}$ and η^{-1}. Then we see that in both cases of Step 1), the functions g and $\mathbf{F}$ satisfy $g \in H^{m+1}(U_+)$, $\mathbf{F} \in H^{m+2}(U_+)$ and

$$\|g\|_{H^{m+1}(U_+)} \leq C^{(1)}_{m,\varepsilon,G} \left(\|f\|_{H^{m+1}(U_+)} + \|u\|_{H^{m+2}(U_+)} \right) ,$$
$$\|\mathbf{F}\|_{H^{m+2}(U_+)} \leq C^{(1)}_{m,\varepsilon,G} \|u\|_{H^{m+2}(U_+)}$$

for some constant $C^{(1)}_{\varepsilon,G} = C^{(1)}_{m,\varepsilon,G}(m, \|\varepsilon\|_{\mathcal{C}^{m+2}}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+2}}$. Then we can use Lemma 3.4 instead of Lemma 3.3 to obtain

$$\|v\|_{H^{m+3}(V_+)} \leq \widetilde{C}^{(1)}_{m,\varepsilon,\eta,G} \left(\|f\|_{\mathcal{H}^{m+1}} + \|u\|_{H^1(G)} \right)$$

for the function v from Step 1) and $\widetilde{C}^{(1)}_{m,\varepsilon,\eta,G} = \widetilde{C}^{(1)}_{m,\varepsilon,\eta,G}(m, \|\varepsilon\|_{\mathcal{C}^{m+2}}, \eta^{-1}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+2}}$ and η^{-1}.

In Step 2), we analogously have $g|_{U_i} \in H^{m+1}(U_i)$, $\mathbf{F}|_{U_i} \in H^{m+2}(U_i)^3$ and

$$\|g\|_{H^{m+1}(U_i)} \leq C^{(2)}_{m,\varepsilon,G} \left(\|f\|_{H^{m+1}(U_i)} + \|u\|_{H^{m+2}(U_i)} \right) , \quad \|\mathbf{F}\|_{H^{m+2}(U_i)} \leq C^{(2)}_{m,\varepsilon,G} \|u\|_{H^{m+2}(U_i)}$$

for $i = 1, 2$ and some constant $C^{(2)}_{m,\varepsilon,G} = C^{(1)}_{m,\varepsilon,G}(m, \|\varepsilon\|_{\mathcal{C}^{m+2}}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+2}}$. Now we use Lemma 3.2 instead of Lemma 3.1 and get for v from Step 2) the estimate

$$\|v\|_{H^{m+3}(V_1)} + \|v\|_{H^{m+3}(V_2)} \leq \widetilde{C}^{(2)}_{m,\varepsilon,\eta,G} \left(\|f\|_{\mathcal{H}^{m+1}} + \|u\|_{H^1(G)} \right)$$

for some constant $\widetilde{C}^{(2)}_{m,\varepsilon,\eta,G} = \widetilde{C}^{(2)}_{m,\varepsilon,\eta,G}(m, \|\varepsilon\|_{\mathcal{C}^{m+2}}, \eta^{-1}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+2}}$ and η^{-1}.

For the interior regularity in Step 3), Theorem 4.11 in [9] yields

$$\|v\|_{H^{m+3}(V)} \leq C^{(3)}_{m,\varepsilon,\eta,G} \left(\|f\|_{\mathcal{H}^{m+1}} + \|u\|_{H^1(G)} \right)$$

for v and V from Step 3) and some constant $C^{(3)}_{m,\varepsilon,\eta,G} = C^{(3)}_{m,\varepsilon,\eta,G}(m, \|\varepsilon\|_{\mathcal{C}^{m+2}}, \eta^{-1}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+2}}$ and η^{-1}.

The localization from Step 4) then works in the same way and finally yields $u \in \mathcal{H}^{m+3}$ with

$$\|u\|_{\mathcal{H}^{m+3}} \le \widetilde{C}_{m,\varepsilon,,\eta,G} \left(\|f\|_{\mathcal{H}^{m+1}} + \|u\|_{H^1(G)} \right)$$

for some constant $\widetilde{C}_{m,\varepsilon,\eta,G} = \widetilde{C}_{m,\varepsilon,\eta,G}(m, \|\varepsilon\|_{\mathcal{C}^{m+2}}, \eta^{-1}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{m+2}}$ and η^{-1}. This completes the proof. □

We can now prove the desired regularity properties for solutions of the transmission problems (2.5) and (2.8) by reducing the problems to the corresponding ones with homogeneous data on the interface and boundary and applying Proposition 3.5.

Theorem 3.6. *Let $k \in \mathbb{N}_0$, $\eta > 0$, Σ and ∂G be of class C^{k+2} and $\varepsilon \in \mathcal{C}^{k+1}_\eta$. Let $f \in \mathcal{H}^k$, $g \in H^{k+\frac{1}{2}}(\Sigma)$ and $h \in H^{k+\frac{1}{2}}(\partial G)$. Then there exists a constant $C = C(k, \|\varepsilon\|_{\mathcal{C}^{k+1}}, \eta^{-1}, G)$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{k+1}}$ and η^{-1} such that the following two statements hold.*

a) Let u be a weak solution of

$$\begin{aligned} -\operatorname{div}(\varepsilon \nabla u) &= f \quad \text{in } G_1\,, \\ -\operatorname{div}(\varepsilon \nabla u) &= f \quad \text{in } G_2\,, \\ u &= 0 \quad \text{on } \partial G\,, \\ [u] = 0\,,\ [\nu \cdot (\varepsilon \nabla u)] &= g \quad \text{on } \Sigma\,, \end{aligned} \tag{3.24}$$

meaning that $u \in H^1_0(G)$ satisfies

$$\int_G (\varepsilon \nabla u) \cdot \nabla \varphi \,\mathrm{d}x = \int_G f\varphi \,\mathrm{d}x + \int_\Sigma g\varphi \,\mathrm{d}\sigma \tag{3.25}$$

for all $\varphi \in H^1_0(G)$. Then $u \in \mathcal{H}^{k+2}$ with

$$\|u\|_{\mathcal{H}^{k+2}} \le C \left(\|f\|_{\mathcal{H}^k} + \|g\|_{H^{k+\frac{1}{2}}(\Sigma)} + \|u\|_{H^1(G)} \right) .$$

b) Let u be a weak solution of

$$\begin{aligned} -\operatorname{div}(\varepsilon \nabla u) &= f \quad \text{in } G_1\,, \\ -\operatorname{div}(\varepsilon \nabla u) &= f \quad \text{in } G_2\,, \\ n \cdot (\varepsilon \nabla u) &= h \quad \text{on } \partial G\,, \\ [u] = 0\,,\ [\nu \cdot (\varepsilon \nabla u)] &= g \quad \text{on } \Sigma\,, \end{aligned}$$

meaning that $u \in H^1(G)$ satisfies

$$\int_G (\varepsilon \nabla u) \cdot \nabla \varphi \,\mathrm{d}x = \int_G f\varphi \,\mathrm{d}x + \int_\Sigma g\varphi \,\mathrm{d}\sigma + \int_{\partial G} h\varphi \,\mathrm{d}\sigma \tag{3.26}$$

for all $\varphi \in H^1(G)$. Then $u \in \mathcal{H}^{k+2}$ with

$$\|u\|_{\mathcal{H}^{k+2}} \le C \left(\|f\|_{\mathcal{H}^k} + \|g\|_{H^{k+\frac{1}{2}}(\Sigma)} + \|h\|_{H^{k+\frac{1}{2}}(\partial G)} + \|u\|_{H^1(G)} \right) .$$

Proof. a) We begin with the first case and claim that $a := \big((\varepsilon\nu)\cdot\nu\big)^{-1} g$ is well defined and contained in $H^{k+\frac{1}{2}}(\Sigma)$. The normal vector field ν can be extended to an open neighbourhood U of Σ. We set $V := U \cap G_1$. By choosing U small enough, we can ensure that there exists a constant $c > 0$ such that the extension $\widetilde{\nu} \in C^{k+1}(V)$ satisfies $|\widetilde{\nu}| \geq c$ on V. Since ε is uniformly positive definite, the function $b := (\varepsilon\widetilde{\nu})\cdot\widetilde{\nu} \in C^{k+1}(V)$ satisfies $b \geq \eta c^2 > 0$ on V. Therefore b^{-1} is well defined and contained in $C^{k+1}(V)$ with $\|b^{-1}\|_{C^{k+1}(V)} \leq c(\|\varepsilon\|_{\mathcal{C}^{k+1}}, \eta^{-1})$, where c is nondecreasing. We now extend g to $\widetilde{g} \in H^{k+1}(V)$ using Theorem 1.5.1.2 in [11]. Then $b\widetilde{g} \in H^{k+1}(V)$ by Theorem 1.4.1.1 in [11]. Let $\operatorname{tr} : H^{k+1}(V) \to H^{k+\frac{1}{2}}(\Sigma)$ denote the trace on Σ. There exists a sequence $(\widetilde{g}_n)$ in $C_c^\infty\left(\overline{V}\right)$ converging to $\widetilde{g}$ in $H^{k+1}(V)$, which implies $b^{-1}\widetilde{g}_n \to b^{-1}\widetilde{g}$ in $H^{k+1}(V)$ as $n \to \infty$. So we conclude

$$a = \operatorname{tr}\left(b^{-1}\widetilde{g}\right) = \lim_{n\to\infty} \operatorname{tr}\left(b^{-1}\widetilde{g}_n\right) \in H^{k+\frac{1}{2}}(\Sigma)\,.$$

By Theorem 1.5.1.2 in [11], there exists a function $\widetilde{w} \in H^{k+2}(G_1)$ such that $\widetilde{w}|_\Sigma = 0$, $\partial_\nu \widetilde{w}|_\Sigma = a$ and

$$\|\widetilde{w}\|_{H^{k+2}(G_1)} \leq C^{(1)}_{k,G_1} \|a\|_{H^{k+\frac{1}{2}}(\Sigma)} \leq C^{(2)}_{k,\varepsilon,\eta,G_1} \|g\|_{H^{k+\frac{1}{2}}(\Sigma)} \tag{3.27}$$

for some constants $C^{(1)}_{k,G_1}, C^{(2)}_{k,\varepsilon,\eta,G_1} > 0$, where $C^{(2)}_{k,\varepsilon,\eta,G_1} = C^{(2)}_{k,\varepsilon,\eta,G_1}(k, \|\varepsilon\|_{\mathcal{C}^{k+1}}, \eta^{-1}, G)$ is nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{k+1}}$ and η^{-1}, and $\partial_\nu\widetilde{w}|_\Sigma$ denotes the normal derivative of $\widetilde{w}$ on Σ. Since $\widetilde{w}$ vanishes on Σ (in the trace sense), we can extend it by zero to a function $w \in H^1_0(G) \cap \mathcal{H}^{k+2}$. We note that w is constant on Σ, so it holds $\nabla w = \nu\partial_\nu w$ on Σ which implies $[\nu\cdot(\varepsilon\nabla w)] = g$ on Σ. We now set $v := u - w \in H^1_0(G)$. Let $\varphi \in H^1_0(G)$. Using the divergence theorem, we obtain from (3.25) the identity

$$\int_G (\varepsilon\nabla v)\cdot\nabla\varphi \,\mathrm{d}x = \int_G \big(f + \operatorname{div}(\varepsilon\nabla w)\big)\varphi \,\mathrm{d}x\,.$$

As $f + \operatorname{div}(\varepsilon\nabla w)$ is contained in $\mathcal{H}^k(G)$, where $\operatorname{div}(\varepsilon\nabla w)$ is uderstood piecewiese, Proposition 3.5 yields $v \in \mathcal{H}^{k+2}$ with

$$\|v\|_{\mathcal{H}^{k+2}} \leq C^{(3)}_{k,\varepsilon,\eta,G}\left(\|f + \operatorname{div}(\varepsilon\nabla w)\|_{\mathcal{H}^k} + \|v\|_{H^1(G)}\right)$$

for some constant $C^{(3)}_{k,\varepsilon,\eta,G} = C^{(3)}_{k,\varepsilon,\eta,G}(k, \|\varepsilon\|_{\mathcal{C}^{k+1}}, \eta^{-1}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{k+1}}$ and η^{-1}. Using

$$\|\operatorname{div}(\varepsilon\nabla w)\|_{\mathcal{H}^k} \leq C^{(4)}_\varepsilon \|w\|_{\mathcal{H}^{k+2}}\,, \quad \|u\|_{\mathcal{H}^\ell} \leq \|v\|_{\mathcal{H}^\ell} + \|w\|_{\mathcal{H}^\ell}$$

for $\ell \leq k+2$ and (3.27), we conclude $u \in \mathcal{H}^{k+2}$ with

$$\|u\|_{\mathcal{H}^{k+2}} \leq C^{(5)}_{k,\varepsilon,\eta,G}\left(\|f\|_{\mathcal{H}^k} + \|g\|_{H^{k+\frac{1}{2}}(\Sigma)} + \|u\|_{H^1(G)}\right)\,,$$

where $C^{(4)}_\varepsilon = C^{(4)}_\varepsilon(\|\varepsilon\|_{\mathcal{C}^{k+1}})$ and $C^{(5)}_{k,\varepsilon,\eta,G} = C^{(5)}_{k,\varepsilon,\eta,G}(k, \|\varepsilon\|_{\mathcal{C}^{k+1}}, \eta^{-1}, G)$ are positive constants nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{k+1}}$ and η^{-1}.

b) For the second case, we use the same function w to handle g and proceed similarly for the inhomogeneity h. We find in the same way a function $q \in H^{k+2}(G)$ with $q|_{\partial G} = 0$, $\partial_n q|_{\partial G} = \big(n \cdot (\varepsilon n)\big)^{-1} h$ and

$$\|q\|_{H^{k+2}(G)} \leq C^{(6)}_{k,\varepsilon,\eta,G} \|h\|_{H^{k+\frac{1}{2}}(\Sigma)} \tag{3.28}$$

for some constant $C^{(6)}_{k,\varepsilon,\eta,G} = C^{(6)}_{k,\varepsilon,\eta,G}(k, \|\varepsilon\|_{\mathcal{C}^{k+1}}, \eta^{-1}, G) > 0$ nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{k+1}}$ and η^{-1}. We set

$$U_1 := \left\{x \in \mathbb{R}^3 \;\middle|\; \operatorname{dist}(x, \partial G) < \frac{r}{4}\right\}, \quad U_2 := \left\{x \in \mathbb{R}^3 \;\middle|\; \operatorname{dist}(x, G_1) < \frac{r}{4}\right\},$$

where $r := \operatorname{dist}(\Sigma, \partial G) > 0$, and choose a cut-off function $\theta \in C_c^\infty(\mathbb{R}^3)$ with $\theta = 1$ on U_1 and $\theta = 0$ on U_2. The function $v := u - w - \theta q \in H^1(G)$ satisfies for any $\varphi \in H^1(G)$ the equation

$$\int_G (\varepsilon \nabla v) \cdot \nabla \varphi \, \mathrm{d}x = \int_G \big(f + \operatorname{div}(\varepsilon \nabla w) + \operatorname{div}\big(\varepsilon \nabla(\theta q)\big)\big) \varphi \, \mathrm{d}x,$$

which follows from (3.26) and the properties of w and q. We can now apply the second part of Proposition 3.5 and conclude in the same way as in Step a) that u is contained in $\mathcal{H}^{k+2}$ with

$$\|u\|_{\mathcal{H}^{k+2}} \leq C_{k,\varepsilon,\eta,G} \left(\|f\|_{\mathcal{H}^k} + \|g\|_{H^{k+\frac{1}{2}}(\Sigma)} + \|h\|_{H^{k+\frac{1}{2}}(\partial G)} + \|u\|_{H^1(G)}\right),$$

where $C_{k,\varepsilon,\eta,G} = C_{k,\varepsilon,\eta,G}(k, \|\varepsilon\|_{\mathcal{C}^{k+1}}, \eta^{-1}, G) > 0$ is nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{k+1}}$ and η^{-1}. □

We can now combine Theorem 3.6 with the calculations of Section 2 to finally prove Theorem 1.6.

Proof of Theorem 1.6. a) We start with the first case of Theorem 1.6. By Section 2.1, it suffices to show that a weak solution u of the transmission problem (3.24) with $f = -\operatorname{div} \mathbf{V}$ (understood piecewise) and $g = [\nu \cdot \mathbf{V}]$ and the property $\|u\|_{H^1(G)} \leq C^{(1)}_{k,\varepsilon,\eta,G} \|\mathbf{V}\|_{L^2(G)}$ has the additional regularity $u \in \mathcal{H}^{k+2}$ and satisfies the estimate $\|u\|_{\mathcal{H}^{k+2}} \leq C^{(2)}_{k,\varepsilon,\eta,G} \|\mathbf{V}\|_{\mathcal{H}^{k+1}}$, where $C^{(i)}_{k,\varepsilon,\eta,G} = C^{(i)}_{k,\varepsilon,\eta,G}(k, \|\varepsilon\|_{\mathcal{C}^{k+1}}, \eta^{-1}, G) > 0$ are nondecreasing in $\|\varepsilon\|_{\mathcal{C}^{k+1}}$ and η^{-1} for $i = 1, 2$. This is a direct consequence of the first case of Theorem 3.6.

b) The second case of Theorem 1.6 follows analogously, using Section 2.2 and the second case of Theorem 3.6. □

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