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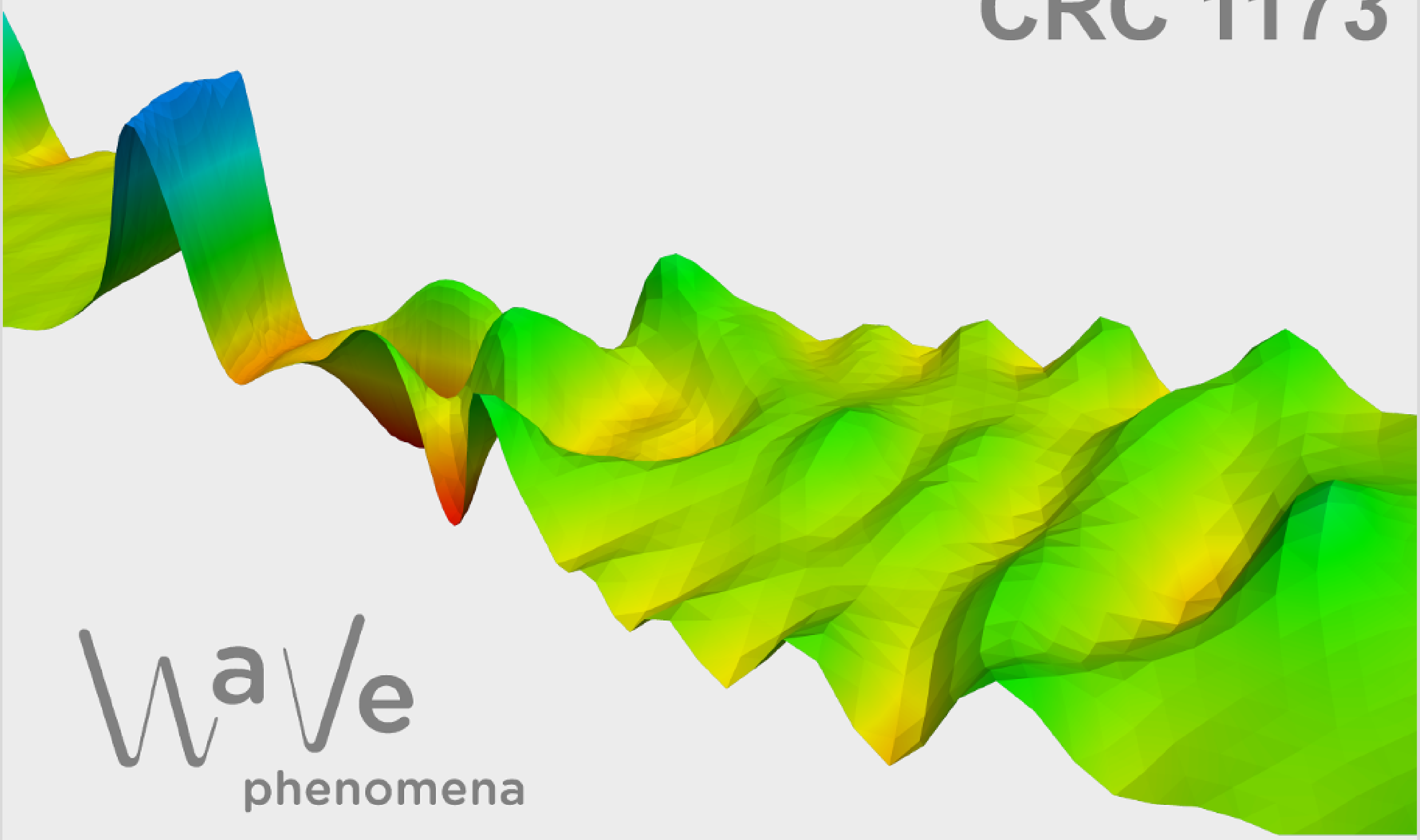

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# LOCAL WELLPOSEDNESS OF MAXWELL SYSTEMS WITH RETARDED MATERIAL LAWS IN LOW REGULARITY

CHRISTOPHER BRESCH AND ROLAND SCHNAUBELT

*Dedicated to Guiseppe Da Prato, in admiration of his manifold seminal contributions to the theory of evolution equations*

Abstract. We develop a complete local wellposedness theory for a Maxwell system on $\mathbb{R}^3$ and a large class of nonlinear material laws which are nonlocal in time. Such constitutive relations are typical for nonlinear optics. The problem was treated before in the Sobolev space $H^s$ for $s > 3/2$ by means of energy methods. Using a recently shown Strichartz estimate, we can lower this level of regularity to $s > 1$. In this context 'charge-type' terms would spoil the analysis. We avoid them by the Helmholtz projection for the divergence operator with coeffients, which requires mapping properties of the projection also in $H^{\alpha,q}$ with $q \neq 2$.

## 1. Introduction

Electromagnetic theory is based on the Maxwell equations. They contain constitutive relations that describe the interaction of the material and the fields, where retarded material laws play an important role in optics, for instance. We treat a large class of nonlinear relations, as discussed in [4], [6] or [9]. If the coefficients of the nonlinearity are differentiable in time (which is true in the standard models), the system is of semilinear nature though nonlocal in time. One can treat it using energy methods in the Sobolev space $H^s$ with $s > \frac{3}{2}$, since this is a Banach algebra. On the spatial domain $\mathbb{R}^3$, existence and uniqueness of solutions was shown in [2] in such a framework, see also [7] for recent results on domains for $s = 2$ and [9] for the linear case. In this paper we study the full space case and establish a complete local wellposedness theory within $H^s$ for $s \in (1, \frac{3}{2}]$. This reduction of the regularity level is caused by dispersive methods, using a Strichartz estimate from the current paper [16]. For instantaneous material laws, which lead to quasilinear problems, related progress was achieved in [17] in the 2D setting and in [16], [18] for certain 3D cases.

The system of macroscopic Maxwell equations is given by

$$\partial_t \mathbf{D} = \operatorname{curl} \mathbf{H} - \mathbf{J}, \quad \partial_t \mathbf{B} = -\operatorname{curl} \mathbf{E}, \qquad t \geq 0, \quad x \in \mathbb{R}^3, \tag{1.1}$$

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where we use a unit system for which the constants $\varepsilon_0$ and $\mu_0$ are 1. The two Gauß laws

$$\operatorname{div} \mathbf{D} = \rho, \quad \operatorname{div} \mathbf{B} = 0$$

for the charge $\rho$ can be absorbed into the initial conditions $\operatorname{div} \mathbf{D}(0) = \rho(0)$ and $\operatorname{div} \mathbf{B}(0) = 0$ since solutions of (1.1) satisfy

$$\operatorname{div} \mathbf{D}(t) = \operatorname{div} \mathbf{D}(0) - \int_0^t \operatorname{div} \mathbf{J}(\tau)\, \mathrm{d}\tau, \qquad \operatorname{div} \mathbf{B}(t) = \operatorname{div} \mathbf{B}(0)\ (= 0) \quad (1.2)$$

because of $\operatorname{div} \operatorname{curl} = 0$. The electric displacement $\mathbf{D} = \mathbf{E} + \mathbf{P}$ is related to the electric field $\mathbf{E}$ by the polarisation $\mathbf{P}$ describing the density of electric dipoles in the medium. Analogously, the magnetic field $\mathbf{H} = \mathbf{B} - \mathbf{M}$ is given by the magnetic induction $\mathbf{B}$ and the magnetisation $\mathbf{M}$ caused by the density of magnetic dipoles. Moreover, we assume that the free current density is of the form $\mathbf{J} = \sigma \mathbf{E} + \mathbf{J}_0$, where $\sigma$ denotes the conductivity of the medium and $\mathbf{J}_0$ is a given, externally applied current density.

So far, (1.1) is underdetermined and has to be complemented by *material laws* specifying the dependence of $\mathbf{P}$ and $\mathbf{M}$ on the fields. In this work we study nonlinear material laws which include retardation effects, such as

$$\mathbf{P}(\mathbf{E})(t,x) = \chi_{\mathrm{e}}(x)\mathbf{E}(t,x) + \int_{-\infty}^{t}\int_{-\infty}^{t} R(t-r, t-r', x)[\mathbf{E}(r,x), \mathbf{E}(r',x)]\, \mathrm{d}r\, \mathrm{d}r'$$

for $t \geq 0$, $x \in \mathbb{R}^3$, and a bilinear *response function* $R(t,t',x) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. Actually we treat finite sums of analogous $n$-linear terms and also allow for a nonlinear and retarded magnetization, where all nonlocal terms may depend on $(\mathbf{E}, \mathbf{H}) = u$. We assume that the kernels are $C^2$ or $H^2$ in $x$ and $W^{1,1} \cap W^{1,\infty}$ in time, roughly speaking. See Section 2 for the details, and [4] or [6] for the background in nonlinear optics, for instance. We stress that the retarded terms act on history functions $u_t : r \mapsto u(t+\tau)$ for $\tau \leq 0$. So one has to prescribe the prehistory $u_{\mathrm{h}}(\tau)$ for $\tau \leq 0$ in (1.1).

Under our assumptions, in (1.1) one can differentiate the nonlocal terms in time without applying $\partial_t$ to the fields. One can then control the resulting terms in $H^s(\mathbb{R}^3)$ for $s > \frac{3}{2}$ since this space embeds into $L^\infty(\mathbb{R}^3)$, see [2]. To lower the regularity level to $s \in (1, \frac{3}{2}]$, we involve the Sobolev spa $H^{\alpha,q}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ with $\alpha > \frac{3}{q}$ and large $q$. This is only possible if one exploits dispersive behavior, namely Strichartz estimates, as the linear part does not leave invariant such spaces unless $q = 2$. Recently in [16] such estimates were shown for scalar permittivity $\varepsilon := 1 + \chi_{\mathrm{e}}$ and permeability $\mu = 1 + \chi_{\mathrm{m}}$, see [17] for the 2D case. Here we need $C_b^2$ coefficients to prevent a regularity loss, see [16]. At least for the 2D case these results are sharp, [17]. We are only aware of one earlier paper [8] on Strichartz estimates for the Maxwell system, where the coefficients were assumed to be smooth and constant outside a compact set. For matrix-valued coefficients the available theory in 3D is still restricted to special cases, see [16] and [18].

In the next section we discuss the basic framework and the material laws. Here the analysis is quite involved because of the nonlocality and complex structure of the nonlinear terms, and also since we have to work with the norms of both $H^s$ and $H^{\alpha,q}$ in the estimates. In Section 3 we reformulate the system (1.1) as the retarded evolution equation (3.1). Moreover, we recall that the linear and instantaneous part is governed by a $C_0$-group in $H^s$. Then we discuss the Strichartz estimate (3.6) for the linear and instantaneous Maxwell system from [16]. It bounds the norm of the fields in $L^p((0,T),H^{-\gamma,q})$ by the $L^2$-norm of the initial value and the norm of the inhomogeneity in $L^1((0,T),L^2)$, where $p,q\geq 2$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and $\gamma = 1-\frac{2}{q}$. One thus increases the spatial integrability at the prize of a loss in regularity, compared to the energy estimate. Here the endpoint $q=\infty$ has to be excluded, see [20]. This fact prevents us to reach the regularity level $H^1$ in the local wellposedness theory. In contrast to the analogous results for the wave equation, see [21] and [22], the Strichartz estimate (3.6) contains also charge terms on the right. Extra regularity of $\rho$ is needed to counteract the large kernel of curl. In Theorem 3.4 we shift the regularity level from $L^2$ to $H^s$ with $s\in(1,\frac{3}{2}]$ and to $H^{s-\gamma,q}$, by means of lengthy commutator arguments.

Our main arguments are based on linearization and perturbation arguments, treating the nonlocal part as an inhomogeneity. It thus causes additional charges compared with the unperturbed system, which would spoil the final result. As a remedy we apply the Helmholtz projection $Q$ mapping onto the kernel $N(\mathrm{curl})\times N(\mathrm{curl})$ along $N(\mathrm{div}(\varepsilon\,\bullet))\times N(\mathrm{div}(\mu\,\bullet))$. This splits the problem into a 'charge free' part and a part without curls, which are treated by the Strichartz inequality respectively direct estimates. For this we need mapping peoperties of $Q$ also in $H^{\alpha,q}(\mathbb{R}^3)$ that we establish in Section 4, based on elliptic regularity and a bootstrapping argument.

Based on these preparations, in the last section we then establish local wellposedness of the Maxwell problem (3.1) in $H^s$ for $s\in(1,\frac{3}{2}]$, including a blow-up criterion and continuous dependence on data. We also show that our solutions coincide with those obtained for more regular data with the maximal existence times, if the relevant conditions are met, see Proposition 5.7. This fact is crucial if one has to approximate data by regular one to justify, e.g., energy estimates. In all the arguments we have to take care of the influence of the prehistory and work in paces $C(J,H^s)\cap L^p(J,H^{\alpha,q})$.

## 2. Function spaces and constitutive relations

We use the standard (complex) function spaces $C_b^k(\mathbb{R}^3)$ and $C_b^r(\mathbb{R}^3)$ of bounded continuously differentiable and Hölder continuous functions for $k\in\mathbb{N}_0$ and $r\in\mathbb{R}_{>0}\setminus\mathbb{N}$. Moreover, we work with usual Lebesgue and Sobolev spaces $L^p$ and $W^{k,p}$, and with the maximal domains $H(\mathrm{curl})$ and $H(\mathrm{div})$ of curl and div in $L^2(\mathbb{R}^3)^3$. We write $a\lesssim b$ if $a\leq cb$ for some constant $c>0$ etc. The Fourier transform is given by $\mathcal{F}u(\xi)=\widehat{u}(\xi)=(2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3}u(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x$ for $f\in L^1(\mathbb{R}^3)$, and $\mathcal{F}$ also denotes its extension to the space of tempered distributions $\mathcal{S}^*(\mathbb{R}^3)$.

In the context of homogeneous fractional derivatives we also use the subspace $\mathcal{S}_h^*(\mathbb{R}^3)$ of $u\in\mathcal{S}^*(\mathbb{R}^3)$ such that $\|\theta(\lambda D)u\|_{L^\infty}\to 0$ as $\lambda\to\infty$ for any $\theta\in C_c^\infty(\mathbb{R}^3)$, where $\theta(\lambda D)u:=\mathcal{F}^{-1}\big(\theta(\lambda\cdot)\widehat{u}\big)$. We note that $u\in\mathcal{S}^*(\mathbb{R}^3)$ with $\widehat{u}\in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ belong to $\mathcal{S}_h^*(\mathbb{R}^3)$. See p.22 in [3]. We need the fractional derivatives $\langle\nabla\rangle^s u:=\mathcal{F}^{-1}\big(\langle\xi\rangle^s\,\widehat{u}\big)$ and $|\nabla|^s u:=\mathcal{F}^{-1}\big(|\xi|^s\,\widehat{u}\big)$ for $s\in\mathbb{R}$, $u\in\mathcal{S}^*(\mathbb{R}^3)$, resp. $u\in\mathcal{S}_h^*(\mathbb{R}^3)$. Here $\langle\xi\rangle^s$ stands for the map $\xi\mapsto\langle\xi\rangle^s=(1+|\xi|^2)^{\frac{s}{2}}$ etc.

The fractional Sobolev spaces are then given by

$$H^{s,p}(\mathbb{R}^3) = \big\{u \in \mathcal{S}^*(\mathbb{R}^3) \;\big|\; \langle\nabla\rangle^s u \in L^p(\mathbb{R}^3)\big\}$$

for $s \in \mathbb{R}$ and $p \in (1,\infty)$, endowed with the canonical norm, where we put $H^s(\mathbb{R}^3) := H^{s,2}(\mathbb{R}^3)$. For $s \in \mathbb{N}_0$, it is known that $H^{s,p}(\mathbb{R}^3)$ coincides with the usual Sobolev space.

The following conditions on parameters will be used thoughout the paper. Triples $(p,q,\gamma)$ satisfying the first statement are called *strict wave admissible.*

**Assumption 2.1.** *Let $p, q \in (2,\infty)$ fulfill $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and set $\gamma = 1-\frac{2}{q} \in (0,1)$. Let $s \in \big(1,\frac{3}{2}\big]$ with $s > 1+\frac{1}{q}$ and define $\alpha = s-\gamma \in \big(\frac{3}{q},\frac{3}{2}\big)$.*

Note that $H^{\alpha,q}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ by Sobolev's embedding which is crucial in the following. The next two lemmas are frequently used to handle products appearing in our model for the constitutive relations below. They follow from Sobolev's embedding, interpolation, and Theorem 4.6.4.2 in [15], for instance.

**Lemma 2.2.** *Let Assumption 2.1 be true, $m \in H^2(\mathbb{R}^3) + C_b^2(\mathbb{R}^3)$, $u \in H^s(\mathbb{R}^3)$, and $v \in H^{\alpha,q}(\mathbb{R}^3)$. Then the products $mu$ and $mv$ belong to $H^s(\mathbb{R}^3)$ and $H^{\alpha,q}(\mathbb{R}^3)$, respectively, and they satisfy*

$$\|mu\|_{H^s} \lesssim \|m\|_{H^2+C_b^2}\|u\|_{H^s}, \qquad \|mv\|_{H^{\alpha,q}} \lesssim \|m\|_{H^2+C_b^2}\|v\|_{H^{\alpha,q}}.$$

**Lemma 2.3.** *Assumption 2.1 be true, $u, v \in H^s(\mathbb{R}^3)\cap H^{\alpha,q}(\mathbb{R}^3)$, and $\tilde u, \tilde v \in H^{\alpha,q}(\mathbb{R}^3)$. Then $uv \in H^s(\mathbb{R}^3)$, $\tilde u\tilde v \in H^{\alpha,q}(\mathbb{R}^3)$, and we have*

$$\|uv\|_{H^s} \lesssim \|u\|_{H^s}\|v\|_{H^{\alpha,q}} + \|u\|_{H^{\alpha,q}}\|v\|_{H^s}, \quad \|\tilde u\tilde v\|_{H^{\alpha,q}} \lesssim \|\tilde u\|_{H^{\alpha,q}}\|\tilde v\|_{H^{\alpha,q}}.$$

We now introduce the function space in which we obtain solutions to the Maxwell system in Section 5. For parameters $s$, $p$, $q$, and $\alpha$ satisfying the conditions of Assumption 2.1 and $b \in \mathbb{R}$, we define

$$Z^{s,p}_{\alpha,q}(b) := C_b\big((-\infty,b], H^s(\mathbb{R}^3)^6\big) \cap L^p\big((-\infty,b), H^{\alpha,q}(\mathbb{R}^3)^6\big),$$

with the canonical norm. We often write just $Z(b)$.

*Constitutive relations* (or *material laws*) describe the interaction of physical systems with electromagnetic fields. We choose $\mathbf{E}$ and $\mathbf{H}$ as variables, and for polarization and magnetization we impose a relation of the form

$$\begin{aligned} \mathbf{P}(\mathbf{E},\mathbf{H})(t,x) &= \chi_{\mathrm{e}}(x)\mathbf{E}(t,x) + \widetilde{\mathbf{P}}(\mathbf{E},\mathbf{H})(t,x),\\ \mathbf{M}(\mathbf{E},\mathbf{H})(t,x) &= \chi_{\mathrm{m}}(x)\mathbf{H}(t,x) + \widetilde{\mathbf{M}}(\mathbf{E},\mathbf{H})(t,x), \end{aligned} \tag{2.1}$$

where $\chi_{\mathrm{e}}$ and $\chi_{\mathrm{m}}$ are the *electric* respectively *magnetic susceptibilities*, which we assume to be scalar-valued. The terms $\chi_{\mathrm{e}}\mathbf{E}$ and $\chi_{\mathrm{m}}\mathbf{H}$ describe a linear, isotropic, local-in-space and instantaneous response of the medium to fields $\mathbf{E}$ and $\mathbf{H}$. We focus on the nonlinear, anisotropic and retarded contributions $\widetilde{\mathbf{P}}(\mathbf{E},\mathbf{H})$ and $\widetilde{\mathbf{M}}(\mathbf{E},\mathbf{H})$. In many cases, $\mathbf{P}$ can be regarded as independent of $\mathbf{H}$ and $\mathbf{M}$ as independent of $\mathbf{E}$, leading to a standard model in nonlinear optics (if $\widetilde{\mathbf{M}}=0$), see [4], [6], and [9]. This is not true in so-called bianisotropic materials which exhibit a coupling of electric and magnetic effects. In our model we allow for such couplings in the retarded terms of (2.1). Introducing the *permittivity* $\varepsilon := 1+\chi_{\mathrm{e}}$ and *permeability* $\mu := 1+\chi_{\mathrm{m}}$, we obtain

$$\mathbf{D} = \varepsilon\mathbf{E} + \widetilde{\mathbf{P}}(\mathbf{E},\mathbf{H}), \quad \mathbf{B} = \mu\mathbf{H} + \widetilde{\mathbf{M}}(\mathbf{E},\mathbf{H}).$$

We combine $\mathbf{E}$ and $\mathbf{H}$ into a variable $u = (\mathbf{E}, \mathbf{H})$. The retarded part is given by $N$ summands

$$(\widetilde{\mathbf{P}}(\mathbf{E}, \mathbf{H}), \widetilde{\mathbf{M}}(\mathbf{E}, \mathbf{H})) =: \Gamma(u) = \sum_{n=1}^{N} \Gamma^{(n)}(u),$$

which can be written in components as

$$\begin{aligned}&\Gamma^{(n)}_{j_0}(u)(t,x) \qquad (2.2)\\ &= \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} R^{(n)}_{j_0 j_1 \cdots j_n}(t - r_1, \ldots, t - r_n, x) u_{j_1}(r_1, x) \ldots u_{j_n}(r_n, x) \, \mathrm{d}r_1 \ldots \mathrm{d}r_n.\end{aligned}$$

for $j_0 \in \{1, \ldots, 6\}$. (We use the Einstein convention of summing over repeated indices.) Here $R^{(n)} = \big(R^{(n)}_{j_o j_1 \cdots j_n}\big) : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}^{6^{(n+1)}}$ is called the $n$th order *response function* and $R^{(n)}(\tau_1, \ldots, \tau_n, x)$ is a tensor of rank $n+1$ for each $(\tau_1, \ldots, \tau_n, x) \in \mathbb{R}^n_{\geq 0} \times \mathbb{R}^3$. It weights the contributions of the electromagnetic fields at times $\tau_1, \ldots, \tau_n$ to the polarisation and magnetisation at time $t$. The material response described by (2.2) is local in space, but nonlocal in time. Due to causality, the integrals run only up to time $t$. The model incorporates time invariance, since the response functions only depend on time differences $t - r_i$. By a substitution, we can also write

$$\Gamma^{(n)}_{j_0}(u)(t,x) = \int_{\mathbb{R}^n_{\geq 0}} R^{(n)}_{j_0 j_1 \cdots j_n}(\tau, x) \prod_{m=1}^{n} u_{j_m}(t - \tau_m, x) \, \mathrm{d}\tau. \qquad (2.3)$$

From now on, we omit the spatial variable $x$. On the material coefficients and the external current density $\mathbf{J}_0$ we impose the following assumptions. While the conductivity $\sigma$ is matrix-valued with entries in $H^2(\mathbb{R}^3) + C^2_b(\mathbb{R}^3)$, permittivity $\varepsilon$ and permeability $\mu$ have to be strictly positive scalars and belong to $C^2_b(\mathbb{R}^3)$. The response functions are assumed to be twice differentiable in space and once in time, with a decaying memory described by a map $\phi$. An additional function $\phi_{\sup}$ defined by shifts of $\phi$ is needed in the proof of the continuity of $\Gamma$, see Lemma 2.8.

**Assumption 2.4.** *Let Assumption 2.1 be true, let $\eta > 0$, and $\varepsilon, \mu \in C^2_b(\mathbb{R}^3)$ be real-valued functions satisfying $\varepsilon(x), \mu(x) \geq \eta$ for all $x \in \mathbb{R}^3$. Let $\sigma$ be contained in $H^2(\mathbb{R}^3)^{3\times 3} + C^2_b(\mathbb{R}^3)^{3\times 3}$ and $\mathbf{J}_0$ in $L^1_{\mathrm{loc}}\big([0,\infty), H^s(\mathbb{R}^3)^3 \cap H^{\alpha,q}(\mathbb{R}^3)^3\big)$.*

*Also, $R^{(n)}_{j_0 \cdots j_n} \in C^1\big(\mathbb{R}^n_{\geq 0}, H^2(\mathbb{R}^3) + C^2_b(\mathbb{R}^3)\big)$ is real-valued and bounded by*

$$\max\Big\{\big\|R^{(n)}_{j_0 \cdots j_n}(\tau)\big\|_{H^2 + C^2_b}, \big\|\partial_{\tau_\ell} R^{(n)}_{j_0 \cdots j_n}(\tau)\big\|_{H^2 + C^2_b}\Big\} \leq \prod_{m=1}^{n} \phi(\tau_m)$$

*for a map $\phi \in L^1\big((0,\infty)\big) \cap C_b\big([0,\infty)\big)$ and all $n \in \{1, \ldots, N\}$, $\ell \in \{1, \ldots, n\}$, $j_0, \ldots, j_n \in \{1, \cdots, 6\}$, $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n_{\geq 0}$. Finally, the function $\phi_{\sup} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined by*

$$\phi_{\sup}(t) = \sup\big\{\phi(t+h) \,\big|\, -1 \leq h \leq 1, t + h \geq 0\big\}$$

*is contained in $L^1\big((0,\infty)\big) \cap L^\infty\big((0,\infty)\big)$.*

We note that Assumption 2.4 implies that $\varepsilon^{-1}$ and $\mu^{-1}$ are also contained in $C_b^2(\mathbb{R}^3)$. The parameters $\varepsilon$ and $\mu$ are combined into the matrix

$$\kappa := \begin{pmatrix} \varepsilon I_{3\times 3} & 0 \\ 0 & \mu I_{3\times 3} \end{pmatrix}.$$

We define the space $L^2_{\varepsilon,\mu}(\mathbb{R}^3)^6$ as $L^2(\mathbb{R}^3)^6$ endowed with the scalar product

$$\Big((\mathbf{E},\mathbf{H})\Big|(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}})\Big)_{L^2_{\varepsilon,\mu}} := \Big(\varepsilon\mathbf{E}\Big|\widetilde{\mathbf{E}}\Big)_{L^2(\mathbb{R}^3)^3} + \Big(\mu\mathbf{H}\Big|\widetilde{\mathbf{H}}\Big)_{L^2(\mathbb{R}^3)^3},$$

where $(\mathbf{F}|\mathbf{G})_{L^2(\mathbb{R}^3)^3} := \int_{\mathbb{R}^3} \mathbf{F}\cdot\overline{\mathbf{G}}\,\mathrm{d}x$ is the usual scalar product on $L^2(\mathbb{R}^3)^3$.

An often used model to describe the optical properties of materials is the Lorentz oscillator model, in which electrons bound to atoms are treated as damped harmonic oscillators, see Section 7.5 of [11]. This model can be generalized by including anharmonic and anisotropic terms in the potential. One obtains expressions for the response functions that include trigonometric polynomials times decaying exponentials and that satisfy Assumption 2.4 for a function $\phi$ of the form $K\mathrm{e}^{-\gamma t}$ with $K,\gamma > 0$, see e.g. Appendix A.2 of [5].

Since (1.1) contains $\partial_t\mathbf{D}$ and $\partial_t\mathbf{B}$, we need an expression for $\partial_t\Gamma(u)$. A formal differentiation of (2.2) (justified in Lemma 2.11) yields

$$\begin{aligned}
&\sum_{\ell=1}^{n}\Bigg[\int_0^\infty\!\!\cdots\int_0^\infty \partial_{\tau_\ell}R^{(n)}_{j_0\dots j_n}(\tau)u_{j_1}(t-\tau_1)\dots u_{j_n}(t-\tau_n)\,\mathrm{d}\tau_1\dots\mathrm{d}\tau_n \\
&+\int_0^\infty\!\!\cdots\int_0^\infty R^{(n)}_{j_0\dots j_n}(\tau_1,\dots,\tau_{\ell-1},0,\tau_{\ell+1},\dots,\tau_n)u_{j_1}(t-\tau_1)\dots u_{j_{\ell-1}}(t-\tau_{\ell-1}) \\
&\qquad\cdot u_{j_\ell}(t)u_{j_{\ell+1}}(t-\tau_{\ell+1})\dots u_{j_n}(t-\tau_n)\,\mathrm{d}\tau_1\dots\mathrm{d}\tau_{\ell-1}\,\mathrm{d}\tau_{\ell+1}\dots\mathrm{d}\tau_n\Bigg] \\
&= \sum_{\ell=1}^{n}\int_{\mathbb{R}^n_{>0}}\partial_{\tau_\ell}R^{(n)}_{j_0\dots j_n}(\tau)\prod_{m=1}^{n}u_{j_m}(t-\tau_m)\,\mathrm{d}\tau + \int_{\partial\mathbb{R}^n_{>0}}R^{(n)}_{j_0\dots j_n}(\tau)\prod_{m=1}^{n}u_{j_m}(t-\tau_m)\,\mathrm{d}\tau \\
&=: Y^{(n)}_{j_0}(u)(t).
\end{aligned} \tag{2.4}$$

We set $Y(u) := \sum_{n=1}^N Y^{(n)}(u)$ with $Y^{(n)}(u)$ given by (2.4) for $j_0 \in \{1,\dots,6\}$.

In the remainder of this section we study properties of the retarded part of the material law and show that $\partial_t\Gamma(u) = Y(u)$ holds in a weak sense. First a useful formula is recalled, using the standard convention that empty sums and products evaluate to zero respectively one.

**Lemma 2.5.** *Let $v_i$ and $w_i$ be elements of a commutative ring for $i \in \{1,\dots,n\}$ and $n\in\mathbb{N}$. Then it holds*

$$\prod_{m=1}^{n} v_m - \prod_{m=1}^{n} w_m = \sum_{\ell=1}^{n}\Bigg(\prod_{m=1}^{\ell-1} v_m\,(v_\ell - w_\ell)\prod_{m=\ell+1}^{n} w_m\Bigg).$$

One can check that the expressions (2.3) and (2.4) are well-defined elements of $H^{\alpha,q}(\mathbb{R}^3)$ for a.e. $t<b$ and suitable functions $u$.

**Lemma 2.6.** *Let Assumption 2.4 be true, $b\in\mathbb{R}$ and $u$ be contained in $L^p\big((-\infty,b),H^{\alpha,q}(\mathbb{R}^3)^6\big)$. Take $n\in\{1,\dots,N\}$ and $j_0\in\{1,\dots,6\}$. Then* *the maps* $F_{j_0}^{(n,t)}, G_{j_0}^{(n,t)} : \mathbb{R}_{>0}^n \to H^{\alpha,q}(\mathbb{R}^3)$ *defined by*

$$F_{j_0}^{(n,t)}(\tau) = R_{j_0\dots j_n}^{(n)}(\tau) \prod_{m=1}^n u_{j_m}(t-\tau_m),$$

$$G_{j_0}^{(n,t)}(\tau) = \sum_{\ell=1}^n \partial_{\tau_\ell} R_{j_0\dots j_n}^{(n)}(\tau) \prod_{m=1}^n u_{j_m}(t-\tau_m)$$

*are Bochner integrable for every* $t \in (-\infty, b)$. *The map* $H_{j_0}^{(n,t)} : \partial\mathbb{R}_{>0}^n \to H^{\alpha,q}(\mathbb{R}^3)$ *defined by*

$$H_{j_0}^{(n,t)}(\tau) = R_{j_0\dots j_n}^{(n)}(\tau) \prod_{m=1}^n u_{j_m}(t-\tau_m)$$

*is Bochner integrable for a.e.* $t \in (-\infty, b)$.

Here the measurability is shown approximating by simple functions and using Lemmas 2.2, 2.3 and 2.5. The latter also yield the integrability via

$$\begin{aligned}\int_{\mathbb{R}_{>0}^n} \left\| F_{j_0}^{(n,t)}(\tau) \right\|_{H^{\alpha,q}} \mathrm{d}\tau &\lesssim \sum_{j_1,\dots,j_n=1}^6 \prod_{m=1}^n \int_0^\infty \phi(\tau_m) \left\| u_{j_m}(t-\tau_m) \right\|_{H^{\alpha,q}} \mathrm{d}\tau_m \\ &\le \|\phi\|_{L^{p'}((0,\infty))}^n \|u\|_{L^p((-\infty,b),H^{\alpha,q})}^n, \end{aligned} \tag{2.5}$$

and analogously for the other maps.

Since our spaces are separable, strong measurability of a map $f$ is equivalent to that of $\langle f, \phi \rangle$ for all test functions $\phi$, say. So we can transfer measurability from $H^{\alpha,q}(\mathbb{R}^3)$ to $H^s(\mathbb{R}^3)$ based on Lemma 2.6. The integrability in $H^s$ is then shown by Lemmas 2.2 and 2.3 and Hölder's inequality via, e.g.,

$$\begin{aligned}&\int_{\mathbb{R}_{>0}^n} \left\| F_{j_0}^{(n,t)}(\tau) \right\|_{H^s} \mathrm{d}\tau \qquad (2.6)\\ &\lesssim \sum_{j_1,\dots,j_n=1}^6 \sum_{\ell=1}^n \int_{\mathbb{R}_{>0}^n} \phi(\tau_\ell) \left\| u_{j_\ell}(t-\tau_\ell) \right\|_{H^s} \prod_{m=1, m\neq \ell}^n \phi(\tau_m) \| u_{j_m}(t-\tau_m) \|_{H^{\alpha,q}} \mathrm{d}\tau \\ &\le \|\phi\|_{L^1((0,\infty))} \|\phi\|_{L^{p'}((0,\infty))}^{n-1} \sup_{r \le b} \|u(r)\|_{H^s} \|u\|_{L^p((-\infty,b),H^{\alpha,q})}^{n-1}. \end{aligned}$$

**Corollary 2.7.** *Let Assumption 2.4 be true,* $b \in \mathbb{R}$, $u \in Z(b)$, $n \in \{1, \dots, N\}$, *and* $j_0 \in \{1, \dots, 6\}$. *Then the maps* $F_{j_0}^{(n,t)}, G_{j_0}^{(n,t)} : \mathbb{R}_{>0}^n \to H^s(\mathbb{R}^3)$ *defined as in Lemma 2.6 are Bochner integrable for every* $t \in (-\infty, b)$. *The map* $H_{j_0}^{(n,t)} : \partial\mathbb{R}_{>0}^n \to H^s(\mathbb{R}^3)$ *defined as in Lemma 2.6 is Bochner integrable for a.e.* $t \in (-\infty, b)$.

We next show boundedness and continuity in time of the nonlinearity.

**Lemma 2.8.** *Let Assumption 2.4 be true,* $b \in \mathbb{R}$, $u \in Z(b)$ *and* $n \in \{1, \dots, N\}$. *Then* $\Gamma(u)$ *is contained in the space* $C_b\big((-\infty, b], H^{\alpha,q}(\mathbb{R}^3)^6 \cap H^s(\mathbb{R}^3)^6\big)$ *and satisfies*

$$\sup_{t \le b} \left\| \Gamma^{(n)}(u)(t) \right\|_{H^{\alpha,q}} \lesssim \|u\|_{L^p((-\infty,b),H^{\alpha,q})}^n,$$

$$\sup_{t\leq b}\left\|\Gamma^{(n)}(u)(t)\right\|_{H^s}\lesssim \sup_{t\leq b}\|u(t)\|_{H^s}\,\|u\|^{n-1}_{L^p((-\infty,b),H^{\alpha,q})}.$$

*Proof.* Let $n\in\{1,\dots,N\}$, $j_0\in\{1,\dots,6\}$, and $t\leq b$. The boundedness of $\Gamma(u)$ follows as in (2.5) and (2.6) by

$$\left\|\Gamma^{(n)}_{j_0}(u)(t)\right\|_{H^{\alpha,q}}\lesssim \|\phi\|^n_{L^{p'}((0,\infty))}\,\|u\|^n_{L^p((-\infty,b),H^{\alpha,q})},$$
$$\left\|\Gamma^{(n)}_{j_0}(u)(t)\right\|_{H^s}\lesssim \|\phi\|_{L^1((0,\infty))}\|\phi\|^{n-1}_{L^{p'}((0,\infty))}\sup_{\tau\leq b}\|u(\tau)\|_{H^s}\|u\|^{n-1}_{L^p((-\infty,b),H^{\alpha,q})}.$$

We now prove continuity from the right. (The other case is done similarly.) Let $h\in[0,1]$ with $t+h\leq b$. We set $W_t=(-\infty,t)^n$ and write $k^{(n)}_{j_0\cdots j_n}(t,h,r)$ for the norm

$$\big\|R^{(n)}_{j_0\cdots j_n}(t+h-r_1,\dots,t+h-r_n)-R^{(n)}_{j_0\cdots j_n}(t-r_1,\dots,t-r_n)\big\|_{H^2+C_b^2}.$$

As above, Lemmas 2.2 and 2.3 yield

$$\begin{aligned}
&\left\|\Gamma^{(n)}_{j_0}(u)(t+h)-\Gamma^{(n)}_{j_0}(u)(t)\right\|_{H^{\alpha,q}}\\
&\lesssim \sum_{j_1,\dots,j_n=1}^{6}\Bigg[\int_{W_t}k^{(n)}_{j_0\cdots j_n}(t,h,r)\prod_{m=1}^{n}\|u_{j_m}(r_m)\|_{H^{\alpha,q}}\,\mathrm{d}r\\
&+\int_{W_{t+h}\setminus W_t}\big\|R^{(n)}_{j_0\cdots j_n}(t+h-r_1,\dots,t+h-r_n)\big\|_{H^2+C_b^2}\prod_{m=1}^{n}\|u_{j_m}(r_m)\|_{H^{\alpha,q}}\,\mathrm{d}r\Bigg]\\
&\lesssim f_1(h)\,\|u\|^n_{L^p((-\infty,b),H^{\alpha,q})}+\sum_{j_1,\dots,j_n=1}^{6}\sum_{k=1}^{n}\int_t^{t+h}\phi(t+h-r_k)\|u_{j_k}(r_k)\|_{H^{\alpha,q}}\,\mathrm{d}r_k\\
&\qquad\cdot\prod_{m=1,m\neq k}^{n}\int_{-\infty}^{t+h}\phi(t+h-r_m)\,\|u_{j_m}(r_m)\|_{H^{\alpha,q}}\,\mathrm{d}r_m\\
&\lesssim\Big(f_1(h)+h^{\frac{1}{p'}}\,\|\phi\|_{L^\infty((0,\infty))}\,\|\phi\|^{n-1}_{L^{p'}((0,\infty))}\Big)\,\|u\|^n_{L^p((-\infty,b),H^{\alpha,q})}
\end{aligned}$$

with

$$f_1(h):=\sum_{j_1,\dots,j_n=1}^{6}\Big(\int_{W_t}k^{(n)}_{j_0\cdots j_n}(t,h,r)^{p'}\,\mathrm{d}r\Big)^{\frac{1}{p'}}.$$

We have $f_1(h)\to 0$ as $h\to 0^+$ by Lebesgue's theorem, using the continuity in time of the response functions as well as the estimate

$$k^{(n)}_{j_0\cdots j_n}(t,h,r)\leq 2\prod_{m=1}^{n}\phi_{\sup}(t-r_m) \tag{2.7}$$

and Assumption 2.4. The calculation in $H^s(\mathbb{R}^3)$ proceeds similarly with some modfications as in (2.6). We omit the details. □

We now turn to the question in which sense the formal differentiation of $\Gamma(u)$ in (2.4) can be justified. As a first step, using the algebra property of $H^2(\mathbb{R}^3)$, we can differentiate sufficiently regular $u$ in a classical sense. The proof is rather standard (though tedious) and a minor modification of Lemma 3.17 in [5], so that it is not presented here.

**Lemma 2.9.** *Let Assumption 2.4 hold, $b\in\mathbb{R}$, and $u\in C_b\big((-\infty,b],H^2(\mathbb{R}^3)^6\big)$. Then $\Gamma(u)$ belongs to $C^1\big((-\infty,b),H^2(\mathbb{R}^3)^6\big)$ with $\partial_t\Gamma(u)=Y(u)$, see* (2.4).

In our setting, $u$ is less regular than $H^2$ and we only obtain $\partial_t\Gamma(u)=Y(u)$ in a weak sense. As a preparation for proving this in Lemma 2.11, we state estimates that easily follow from Lemma 2.8 and the variants of inequality (2.6) for $G_{j_0}^{(n,t)}$ and $H_{j_0}^{(n,t)}$.

**Lemma 2.10.** *Let Assumption 2.4 be true, $a,b\in\mathbb{R}$ with $a<b$, $J=(a,b)$, $u\in Z(b)$, and $n\in\{1,\dots,N\}$. Then $\Gamma(u)$ and $Y(u)$ are contained in the space $L^1\big((a,b),H^s(\mathbb{R}^3)^6\big)$ and satisfy*

$$\left\|\Gamma^{(n)}(u)\right\|_{L^1(J,H^s)} \lesssim (b-a)\sup_{\tau\le b}\|u(\tau)\|_{H^s}\,\|u\|^{n-1}_{L^p((-\infty,b),H^{\alpha,q})}, \tag{2.8}$$
$$\left\|Y^{(n)}(u)\right\|_{L^1(J,H^s)} \lesssim \big[(b-a)+(b-a)^{\frac{1}{p'}}\big]\sup_{\tau\le b}\|u(\tau)\|_{H^s}\|u\|^{n-1}_{L^p((-\infty,b),H^{\alpha,q})}.$$

We can now differentiate the polarisation and magnetisation in time.

**Lemma 2.11.** *Let Assumption 2.4 hold, $a,b\in\mathbb{R}$ with $a<b$ and $u\in Z(b)$. Then $\Gamma(u)$ belongs to $W^{1,1}\big((a,b),H^s(\mathbb{R}^3)^6\big)$ with $\partial_t\Gamma(u)=Y(u)$, see* (2.4).

*Proof.* We use mollifiers $\chi$ in $\mathbb{R}^3$ and define $u^{(k)}(t)=\chi_{1/k}*u(t)$ for $t\le b$ and $k\in\mathbb{N}$. Since $u^{(k)}\in C_b\big((-\infty,b],H^2(\mathbb{R}^3)^6\big)$, Lemma 2.9 implies that $\Gamma(u^{(k)})$ is contained in $C^1\big((-\infty,b),H^2(\mathbb{R}^3)^6\big)$ with $\partial_t\Gamma\left(u^{(k)}\right)=Y\left(u^{(k)}\right)$. It thus remains to show the limits $\Gamma\left(u^{(k)}\right)\to\Gamma(u)$ and $Y\left(u^{(k)}\right)\to Y(u)$ in $L^1\big((a,b),H^s(\mathbb{R}^3)^6\big)$ as $k\to\infty$. Let $n\in\{1,\dots,N\}$ and $j_0\in\{1,\dots,6\}$.

1) We start with the convergence of $\left(\Gamma\left(u^{(k)}\right)\right)_k$. Using Lemmas 2.2 and 2.5, we obtain

$$\left\|\Gamma_{j_0}^{(n)}\left(u^{(k)}\right)(t)-\Gamma_{j_0}^{(n)}(u)(t)\right\|_{H^s} \lesssim \sum_{\ell=1}^{n}\int_{\mathbb{R}^n_{>0}}\left\|R^{(n)}_{j_0\dots j_n}(\tau)\right\|_{H^2+C_b^2}$$
$$\cdot\left\|\Big(\prod_{m=1}^{\ell-1}u^{(k)}_{j_m}(t-\tau_m)\Big)\big(u^{(k)}_{j_\ell}(t-\tau_\ell)-u_{j_\ell}(t-\tau_\ell)\big)\prod_{m=\ell+1}^{n}u_{j_m}(t-\tau_m)\right\|_{H^s}\mathrm{d}\tau$$

for a.e. $t<b$. The products of the fields can be estimated using Lemma 2.3. This leads to

$$\big\|\Gamma^{(n)}\big(u^{(k)}\big)(t)-\Gamma^{(n)}(u)(t)\big\|_{H^s}\lesssim\sum_{i=1}^{3}\int_{\mathbb{R}^n_{>0}}I_i^{(n,k)}(t,\tau)\,\mathrm{d}\tau,$$
$$I_1^{(n,k)}(t,\tau):=\prod_{j=1}^{n}\phi(\tau_j)\sum_{\ell=1}^{n}\Bigg[\sum_{i=1}^{\ell-1}\left\|u^{(k)}(t-\tau_i)\right\|_{H^s}\prod_{m=1,m\neq i}^{\ell-1}\left\|u^{(k)}(t-\tau_m)\right\|_{H^{\alpha,q}}$$
$$\cdot\left\|u^{(k)}(t-\tau_\ell)-u(t-\tau_\ell)\right\|_{H^{\alpha,q}}\prod_{m=\ell+1}^{n}\|u(t-\tau_m)\|_{H^{\alpha,q}}\Bigg],$$
$$I_2^{(n,k)}(t,\tau):=\prod_{j=1}^{n}\phi(\tau_j)\sum_{\ell=1}^{n}\Bigg[\prod_{m=1}^{\ell-1}\left\|u^{(k)}(t-\tau_m)\right\|_{H^{\alpha,q}}$$

$$\cdot \left\| u^{(k)}(t-\tau_\ell) - u(t-\tau_\ell) \right\|_{H^s} \prod_{m=\ell+1}^{n} \|u(t-\tau_m)\|_{H^{\alpha,q}} \Big],$$

$$I_3^{(n,k)}(t,\tau) := \prod_{j=1}^{n} \phi(\tau_j) \sum_{\ell=1}^{n} \Big[ \sum_{i=\ell+1}^{n} \|u(t-\tau_i)\|_{H^s} \prod_{m=1}^{\ell-1} \left\| u^{(k)}(t-\tau_m) \right\|_{H^{\alpha,q}}$$
$$\cdot \left\| u^{(k)}(t-\tau_\ell) - u(t-\tau_\ell) \right\|_{H^{\alpha,q}} \prod_{m=\ell+1, m\neq i}^{n} \|u(t-\tau_m)\|_{H^{\alpha,q}} \Big].$$

In the case $n = 1$, both $I_1^{(1,k)}$ and $I_3^{(1,k)}$ vanish. The properties of mollifiers show that $I_i^{(n,k)}(t,\tau)$ converges to 0 as $k \to \infty$ for a.e. $(t,\tau) \in (-\infty, b) \times \mathbb{R}^n_{>0}$, all $j \in \{1,2,3\}$. We further estimate

$$I_1^{(n,k)}(t,\tau) \le 2 \sum_{\ell=1}^{n} \sum_{i=1}^{\ell-1} \phi(\tau_i) \sup_{r\le b} \|u(r)\|_{H^s} \prod_{m=1, m\neq i}^{n} \phi(\tau_m) \|u(t-\tau_m)\|_{H^{\alpha,q}},$$

$$I_2^{(n,k)}(t,\tau) \le 2 \sum_{\ell=1}^{n} \phi(\tau_\ell) \sup_{r\le b} \|u(r)\|_{H^s} \prod_{m=1, m\neq \ell}^{n} \phi(\tau_m) \|u(t-\tau_m)\|_{H^{\alpha,q}},$$

$$I_3^{(n,k)}(t,\tau) \le 2 \sum_{\ell=1}^{n} \sum_{i=\ell+1}^{n} \phi(\tau_i) \sup_{r\le b} \|u(r)\|_{H^s} \prod_{m=1, m\neq i}^{n} \phi(\tau_m) \|u(t-\tau_m)\|_{H^{\alpha,q}}.$$

The integrals over $\mathbb{R}^n_{>0}$ of the terms $\widetilde{I}_j^{(n)}(t,\tau)$ on the right are bounded by

$$\|\phi\|_{L^1((0,\infty))} \|\phi\|^{n-1}_{L^{p'}((0,\infty))} \sup_{r\le b} \|u(r)\|_{H^s} \|u\|^{n-1}_{L^p((-\infty,b),H^{\alpha,q})} < \infty$$

for a.e. $t < b$ and all $j \in \{1,2,3\}$ due to Hölder, so that $\Gamma^{(n)}(u^{(k)})$ tends to $\Gamma^{(n)}(u)$ pointwise in $H^s$ as $k \to \infty$ by dominated convergence. Since also

$$\left\| \Gamma^{(n)}(u^{(k)})(t) - \Gamma^{(n)}(u)(t) \right\|_{H^s} \lesssim \sup_{r\le b} \|u(r)\|_{H^s} \|u\|^{n-1}_{L^p((-\infty,b),H^{\alpha,q})},$$

Lebesgue's theorem implies the convergence in $L^1\big((a,b), H^s(\mathbb{R}^3)^6\big)$.

2) We now prove $Y^{(n)}(u^{(k)}) \to Y^{(n)}(u)$ in $L^1\big((a,b), H^s(\mathbb{R}^3)^6\big)$ as $k \to \infty$. The convergence of the summands involving $\partial_{\tau_\ell} R^{(n)}$ is shown as in step 1). The remaining ones have to be treated a bit differently. As above we estimate

$$\left\| \mathbb{R}^{(n)}_{j_0\cdots j_n}(\tau) \Big( \prod_{m=1}^{n} u^{(k)}_{j_m}(t-\tau_m) - \prod_{m=1}^{n} u_{j_m}(t-\tau_m) \Big) \right\|_{H^s} \lesssim \sum_{i=1}^{3} I_i^{(n,k)}(t,\tau)$$

for a.e. $t < b$. In the case $n = 1$, we have $I_1^{(1,k)} = I_3^{(1,k)} = 0$ for all $k \in \mathbb{N}$ and

$$\int_{\partial\mathbb{R}_{>0}} \widetilde{I}_2^{(1)}(t,\tau)\,\mathrm{d}\tau = \widetilde{I}_2^{(1)}(t,0) \lesssim \sup_{r\le b} \|u(r)\|_{H^s} < \infty. \tag{2.9}$$

For $n > 1$, $j \in \{1,2,3\}$ and a.e. $t < b$ it holds

$$\int_{\partial\mathbb{R}^n_{>0}} \widetilde{I}_j^{(n)}(t,\tau)\,\mathrm{d}\tau \tag{2.10}$$
$$\lesssim \sup_{r\le b} \|u(r)\|_{H^s} \Big( \|u\|_{L^p((-\infty,b),H^{\alpha,q})} + \|u(t)\|_{H^{\alpha,q}} \Big) \|u\|^{n-2}_{L^p((-\infty,b),H^{\alpha,q})} < \infty.$$

As in step 1), Lebesgue's theorem now implies

$$\int_{\partial\mathbb{R}^n_{>0}} R^{(n)}_{j_0\dots j_n}(\tau)\prod_{m=1}^{n} u^{(k)}_{j_m}(t-\tau_m)\,\mathrm{d}\tau \longrightarrow \int_{\partial\mathbb{R}^n_{>0}} R^{(n)}_{j_0\dots j_n}(\tau)\prod_{m=1}^{n} u_{j_m}(t-\tau_m)\,\mathrm{d}\tau$$

in $H^s(\mathbb{R}^3)$. Using (2.9) and (2.10) for the majorant, we arrive at the limit

$$\int_{\partial\mathbb{R}^n_{>0}} R^{(n)}_{j_0\dots j_n}(\tau)\prod_{m=1}^{n} u^{(k)}_{j_m}(\bullet-\tau_m)\,\mathrm{d}\tau \longrightarrow \int_{\partial\mathbb{R}^n_{>0}} R^{(n)}_{j_0\dots j_n}(\tau)\prod_{m=1}^{n} u_{j_m}(\bullet-\tau_m)\,\mathrm{d}\tau$$

in $L^1\big((a,b),H^s(\mathbb{R}^3)\big)$ as $k\to\infty$, again by Lebesgue's theorem. □

Let Assumption 2.4 hold. Lemma 2.11 motivates to define the map $F : Z(0)\to H^s(\mathbb{R}^3)^6$ given by $F=\sum_{n=1}^N F^{(n)}$ for $F^{(n)} : Z(0)\to H^s(\mathbb{R}^3)^6$ and

$$F^{(n)}_{j_0}(u) = -\nu_{j_0}\Big[\sum_{\ell=1}^{n}\int_{\mathbb{R}^n_{>0}} \partial_{\tau_\ell} R^{(n)}_{j_0\dots j_n}(\tau)\prod_{k=1}^{n} u_{j_k}(-\tau_k)\,\mathrm{d}\tau + \int_{\partial\mathbb{R}^n_{>0}} R^{(n)}_{j_0\dots j_n}(\tau)\prod_{k=1}^{n} u_{j_k}(-\tau_k)\,\mathrm{d}\tau\Big]$$

for $j_0\in\{1,\dots,6\}$, where $\nu_{j_0}=\varepsilon^{-1}$ if $j_0\le 3$ and $\nu_{j_0}=\mu^{-1}$ if $j_0\ge 4$. If $u=(\mathbf{E},\mathbf{H})\in Z(b)$ for some $b\in\mathbb{R}$, in (1.1) we can thus write

$$-\partial_t\begin{pmatrix}\varepsilon^{-1}\widetilde{\mathbf{P}}(\mathbf{E},\mathbf{H})(t)\\ \mu^{-1}\widetilde{\mathbf{M}}(\mathbf{E},\mathbf{H})(t)\end{pmatrix} = -\kappa^{-1}\partial_t\Gamma(u)(t) = F(u_t) \tag{2.11}$$

for a.e. $t<b$. Here we use the *history function* $u_t\in Z(0)$ given by $u_t(\tau):=u(t+\tau)$ for all $\tau\le 0$.

Since $\varepsilon^{-1}\in C^2_b(\mathbb{R}^3)$, Lemmas 2.2 and 2.11 show that $F(u_\bullet): t\mapsto F(u_t)$ is contained in $L^1_{\mathrm{loc}}\big((-\infty,b),H^s(\mathbb{R}^3)^6\big)$. The next result collects estimates for $F$ needed in our main wellposedness results.

**Lemma 2.12.** *Let Assumption 2.4 be true. Let $r,b>0$ and $u,\widetilde{u}\in Z(b)$ satisfy $\|u\|_{Z(b)}\le r$ and $\|\widetilde{u}\|_{Z(b)}\le r$. Then we have the estimates*

$$\|F(u_\bullet)\|_{L^1((0,b),H^s)} \lesssim \Big(b+b^{\frac{1}{p'}}\Big)\left(1+r^N\right),$$

$$\|F(u_\bullet)-F(\widetilde{u}_\bullet)\|_{L^1((0,b),H^{\alpha,q})} \lesssim \Big(b+b^{\frac{1}{p'}}\Big)\left(1+r^{N-1}\right)\|u-\widetilde{u}\|_{Z(b)},$$

$$\|F(u_\bullet)-F(\widetilde{u}_\bullet)\|_{L^1((0,b),H^s)} \lesssim \Big(b+b^{\frac{1}{p'}}\Big)\left(1+r^{N-1}\right)\|u-\widetilde{u}\|_{Z(b)}.$$

*Proof.* The first estimate follows directly from (2.8). We proceed similarly to the proof of Lemma 2.11 for the other two statements, focusing on the more complicated case $H^s$. (The Banach algebra $H^{\alpha,q}$ can be treated more easily.) Let $0\le t\le b$ and $n\in\{1,\dots,N\}$. We compute

$$\Big\|F^{(n)}(u_t)-F^{(n)}(\widetilde{u}_t)\Big\|_{H^s} \lesssim \sum_{i=1}^{3}\int_{\mathbb{R}^n_{>0}} I_i(t,\tau)\,\mathrm{d}\tau + \sum_{i=1}^{3}\int_{\partial\mathbb{R}^n_{>0}} I_i(t,\tau)\,\mathrm{d}\tau,$$

$$I_1(t,\tau) := \sum_{\ell=1}^{n}\sum_{i=1}^{\ell-1}\Big[\phi(\tau_i)\,\|u(t-\tau_i)\|_{H^s}\prod_{m=1,m\neq i}^{\ell-1}\phi(\tau_m)\,\|u(t-\tau_m)\|_{H^{\alpha,q}}$$

$$\cdot \phi(\tau_\ell) \left\| u(t-\tau_\ell) - \widetilde{u}(t-\tau_\ell) \right\|_{H^{\alpha,q}} \prod_{m=\ell+1}^{n} \phi(\tau_m) \left\| \widetilde{u}(t-\tau_m) \right\|_{H^{\alpha,q}} \Big],$$

$$I_2(t,\tau) := \sum_{\ell=1}^{n} \Big[ \prod_{m=1}^{\ell-1} \phi(\tau_m) \left\| u(t-\tau_m) \right\|_{H^{\alpha,q}} \phi(\tau_\ell) \left\| u(t-\tau_\ell) - \widetilde{u}(t-\tau_\ell) \right\|_{H^s}$$
$$\cdot \prod_{m=\ell+1}^{n} \phi(\tau_m) \left\| \widetilde{u}(t-\tau_m) \right\|_{H^{\alpha,q}} \Big],$$

$$I_3(t,\tau) := \sum_{\ell=1}^{n} \sum_{i=\ell+1}^{n} \Big[ \phi(\tau_i) \left\| \widetilde{u}(t-\tau_i) \right\|_{H^s} \prod_{m=1}^{\ell-1} \phi(\tau_m) \left\| u(t-\tau_m) \right\|_{H^{\alpha,q}}$$
$$\cdot \phi(\tau_\ell) \left\| u(t-\tau_\ell) - \widetilde{u}(t-\tau_\ell) \right\|_{H^{\alpha,q}} \prod_{m=\ell+1, m\neq i}^{n} \phi(\tau_m) \left\| \widetilde{u}(t-\tau_m) \right\|_{H^{\alpha,q}} \Big].$$

If $n = 1$, we have $I_1 = 0 = I_3$ and $I_2(t,\tau_1) = \phi(\tau_1) \left\| u(t-\tau_1) - \widetilde{u}(t-\tau_1) \right\|_{H^s}$. So the claim is clear. For $n > 1$ we deduce

$$\sum_{j=1}^{3} \int_{\mathbb{R}^n_{>0}} I_j(t,\tau) \, \mathrm{d}\tau \lesssim r^{n-1} \left\| u - \widetilde{u} \right\|_{Z(b)},$$

again by means of Hölder's inequality. The integral over $\partial\mathbb{R}^n_{>0}$ leads to the six cases where $u, \widetilde{u}$ and $u - \widetilde{u}$ are evaluated on the boundary either in the $H^s$- or the $H^{\alpha,q}$-norm. This results in

$$\sum_{j=1}^{3} \int_{\partial\mathbb{R}^n_{>0}} I_j(t,\tau) \, \mathrm{d}\tau \lesssim r^{n-1} \left\| u - \widetilde{u} \right\|_{Z(b)} + r^{n-2} \left\| u(t) \right\|_{H^{\alpha,q}} \left\| u - \widetilde{u} \right\|_{Z(b)}$$
$$+ r^{n-2} \left\| \widetilde{u}(t) \right\|_{H^{\alpha,q}} \left\| u - \widetilde{u} \right\|_{Z(b)} + r^{n-1} \left\| u(t) - \widetilde{u}(t) \right\|_{H^{\alpha,q}}.$$

Hölder's inequality in time then implies the assertion. □

## 3. The linear part and Strichartz estimate

We write the Maxwell system (1.1) as the retarded evolution equation

$$\begin{aligned} u'(t) &= (A+B)u(t) + F(u_t) + g(t), \quad t \geq 0, \\ u(t) &= u_{\mathrm{h}}(t), \quad t \leq 0, \end{aligned} \tag{3.1}$$

with the *Maxwell operator*

$$A = \begin{pmatrix} 0 & \frac{1}{\varepsilon} \operatorname{curl} \\ -\frac{1}{\mu} \operatorname{curl} & 0 \end{pmatrix} \qquad \text{and the perturbation} \qquad B = \begin{pmatrix} -\frac{1}{\varepsilon}\sigma & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

The external current density $\mathbf{J}_0$ from Assumption 2.4 leads to the inhomogeneity $g : [0,\infty) \to \mathbb{R}^6$ defined by

$$g(t) := \begin{pmatrix} -\varepsilon^{-1}\mathbf{J}_0(t) \\ 0 \end{pmatrix}, \tag{3.3}$$

which belongs to $L^1_{\mathrm{loc}}\big([0,\infty), H^s(\mathbb{R}^3)^6\big) \cap L^p_{\mathrm{loc}}\big([0,\infty), H^{\alpha,q}(\mathbb{R}^3)^6\big)$. Since the nonlinear term $F(u_t)$ at time $t$ depends on all values $u(r)$ for $r \leq t$, the initial condition $u(0)$ is not sufficient, but instead a whole history function $u_{\mathrm{h}}$ has to be prescribed.

To treat (3.1), we use the strongly continuous group generated by $A_\sigma = A + B$. On $L^2_{\varepsilon,\mu}(\mathbb{R}^3)^6$, the operator $A$ is equipped with the domain $\mathrm{D}(A) = H(\mathrm{curl}) \times H(\mathrm{curl})$. It is well known to be skew-adjoint and therefore generates a unitary $C_0$-group $S(\cdot)$ by Stone's theorem, cf. [1]. Since $B$ is bounded on $L^2_{\varepsilon,\mu}(\mathbb{R}^3)^6$, the sum $A_\sigma$ on $\mathrm{D}(A)$ generates a $C_0$-group $S_\sigma(\cdot)$, also on $L^2(\mathbb{R}^3)^6$. These groups can be restricted to $H^s(\mathbb{R}^3)^6$.

**Lemma 3.1.** *Let Assumption 2.4 be true. Then the restrictions $S(\cdot)|_{H^s}$ and $S_\sigma(\cdot)|_{H^s}$ to $H^s(\mathbb{R}^3)^6$ are again $C_0$-groups which are generated by the part $A|_{H^s}$ of $A$ in $H^s(\mathbb{R}^3)^6$, respectively $A_\sigma|_{H^s}$.*

*Proof.* We use the isomorphism $L = I - \Delta : H^2(\mathbb{R}^3)^6 \to L^2(\mathbb{R}^3)^6$. Since the coefficients belong to $C_b^2$, the operator $LA_\sigma L^{-1}$ on $\mathrm{D}(A)$ is a bounded perturbation of $A_\sigma$. Standard semigroup theory now yields that $S_\sigma(\cdot)|_{H^2}$ is a $C_0$-group generated by $A_\sigma|_{H^2}$, cf. Theorems 4.5.5 and 4.5.8 in [14]. The assertion then follows by interpolating between $L^2(\mathbb{R}^3)^6$ and $H^2(\mathbb{R}^3)^6$. □

In the following we mostly omit the restriction symbols. In the crucial fixed-point argument of Lemma 5.1 we regard the nonlinearity in (3.1) as an inhomogenity (by freezing $u$), in which we also absorb the term $Bu$. So we consider the evolution equation

$$u'(t) = Au(t) + f(t), \quad t \geq 0, \qquad u(0) = u_0. \tag{3.4}$$

Let $f \in L^1\big((0,T), H^s(\mathbb{R}^3)^6\big)$ for some $T > 0$ and $u_0 \in H^s(\mathbb{R}^3)^6$. Then the problem (3.4) has a unique mild solution $u \in C\big([0,T], H^s(\mathbb{R}^3)^6\big)$ given by

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau)\,\mathrm{d}\tau, \quad t \in [0,T]. \tag{3.5}$$

In contrast to (3.3), here we allow for nonzero 'magnetic' components in $f$ since they appear in the analysis.

*Remark* 3.2. In the above situation, note that the mild solution $u$ is continuous in $H^s(\mathbb{R}^3)^6$. Since this space is contained in $\mathrm{D}\left(A|_{H^{s-1}}\right)$, Theorem 8.1.3 in [25] implies that $u$ belongs to $W^{1,1}\big((0,T), H^{s-1}\big)$ and solves (3.4) in $H^{s-1}(\mathbb{R}^3)^6$ for a.e. $t \geq 0$. It is called a *strong* $H^s$-solution. Conversely, a strong solution satisfies (3.5) by Theorem 8.1.1 of [25].

To control the nonlinearity $F$ in (3.1), we need the space $H^{\alpha,q}$. However, the above groups do not leave invariant $L^q$-spaces for $q \neq 2$. To overcome this fundamental difficulty in wave-type problems, Strichartz estimates are a powerful tool. We use the recent result Theorem 1.3 of [16] for the isotropic linear Maxwell system due to Schippa, which we state in a simplified version.

**Theorem 3.3.** *Let $\varepsilon, \mu \in C^1\big(\mathbb{R} \times \mathbb{R}^3, \mathbb{R}\big)$ satisfy $\partial^\alpha(\varepsilon,\mu) \in L^1\big(\mathbb{R}, L^\infty(\mathbb{R}^3)^2\big)$ for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha| = 2$ and $\eta \leq \varepsilon(t,x), \mu(t,x) \leq \eta^{-1}$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^3$ and some $\eta > 0$. Set*

$$\kappa = \begin{pmatrix} \varepsilon I_{3\times3} & 0 \\ 0 & \mu I_{3\times3} \end{pmatrix}, \quad A_{\mathrm{co}} := \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix}, \quad L := \partial_t - A_{\mathrm{co}}\kappa^{-1},$$
$$v = (v_1, v_2) \quad \textit{with } v_1, v_2 : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \rho = \big(\mathrm{div}(v_1), \mathrm{div}(v_2)\big).$$

*Let* $p, q \in [2,\infty]$ *satisfy* $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ *and* $(p,q)\neq(2,\infty)$, $\gamma := \frac{3}{2}-\frac{3}{q}-\frac{1}{p}$, *and* $T\in(0,1]$. *Then we have the estimate*

$$\big\||\nabla|^{-\gamma} v\big\|_{L^p((0,T),L^q)} \lesssim \|v\|_{L^\infty((0,T),L^2)} + \|Lv\|_{L^1((0,T),L^2)} + \big\||\nabla|^{-\frac{1}{2}}\rho(0)\big\|_{L^2} + \big\||\nabla|^{-\frac{1}{2}}\partial_t\rho\big\|_{L^1((0,T),L^2)}, \tag{3.6}$$

*provided the right-hand side is finite.*

Observe that for the triple $(p,q,\gamma)=(\infty,2,0)$ the result holds trivially. In the above estimate one controls increased space integrability $q>2$, but has to pay a prize in regularity and time regularity. The conditions on the exponents cannot be improved (if $v(0)\neq 0$), in particular the endpoint case $(2,\infty,1)$ is forbidden, cf. [20]. We note that the 'charge' $\rho$ is given by the data $v(0)$ and $Lv$, cf. (1.2). Moreover, it is needed to counteract the huge kernel of $A_{\mathrm{co}}$ containing functions $\kappa(\nabla\varphi,\nabla\psi)^\top$ which may belong to $L^2\setminus H^{-\gamma,q}$.

We next bring the estimate (3.6) into a form more suited to our problem, given by the next theorem. One could replace $T\le 1$ by any $T>0$ obtaining a constant $C_{\mathrm{Str}}(T)$ nondecreasing in $T$.

**Theorem 3.4.** *Let Assumption 2.4 be true,* $T\in(0,1]$, $u_0\in H^s(\mathbb{R}^3)^6$, *and* $f\in L^1\big((0,T),H^s(\mathbb{R}^3)^6\big)$. *Then the mild solution* $u=(\mathbf{E},\mathbf{H})$ *of* (3.4) *is contained in the space* $L^p\big((0,T),H^{\alpha,q}(\mathbb{R}^3)^6\big)$ *and satisfies*

$$\|u\|_{L^p((0,T),H^{\alpha,q})} \le C_{\mathrm{Str}}\big(\|u_0\|_{H^s} + \|f\|_{L^1((0,T),H^s)} + \|\rho(0)\|_{H^{s-\frac{1}{2}}} + \|\partial_t\rho\|_{L^1((0,T),H^{s-\frac{1}{2}})}\big), \tag{3.7}$$

*if the norms of* $\rho=\big(\operatorname{div}(\varepsilon\mathbf{E}),\operatorname{div}(\mu\mathbf{H})\big)$ *are finite.*

*Proof.* We consider $\varepsilon$ and $\mu$ as constant functions on $[0,T]$ and extend them to maps satisfying the conditions of Theorem 3.3.

1) We first replace the homogeneous fractional derivatives by Sobolev spaces, cf. §3.2.1 in [16]. To this aim, we use the Fourier cut-off $S_0=\mathcal{F}^{-1}\chi\mathcal{F}$ for a smooth function $\chi$ being 1 on $B(0,1)$ and with support in $B(0,2)$. Since $-\gamma-\frac{3}{q}=\frac{1}{p}-\frac{3}{2}$, Sobolev's embedding and Plancherel imply

$$\|S_0 v\|_{L^p((0,T),H^{-\gamma,q})} \lesssim \|\langle\xi\rangle^{\frac{1}{p}}\chi\widehat{v}\|_{L^p((0,T),L^2)} \lesssim T^{\frac{1}{p}}\|v\|_{L^\infty((0,T),L^2)},$$

Note that $S_0$ commutes with div, $\langle\nabla\rangle^\theta$ and $\partial_t$, that $S_0$ is bounded on $L^2(\mathbb{R}^3)^6$ and that $|\xi|\eqsim\langle\xi\rangle$ on $\mathbb{R}^3\setminus B(0,1)$. By (3.6) and Plancherel the high-frequency part $w=v-S_0v$ can thus be estimated via

$$\||\nabla|^{-\gamma} w\|_{L^p((0,T),L^q)} \lesssim \|w\|_{L^\infty(0,T),L^2)} + \|Lw\|_{L^1((0,T),L^2)} + T^{\frac{1}{2}}\big(\|\rho(0)\|_{H^{-\frac{1}{2}}} + \|\partial_t\rho\|_{L^1((0,T),H^{-\frac{1}{2}})}\big).$$

Note that $[S_0,L]=(A_{\mathrm{co}}\kappa^{-1})S_0+[\kappa^{-1},A_{\mathrm{co}}S_0]$. Since $\kappa^{-1}$ is Lipschitz, the second commutator is bounded on $L^2(\mathbb{R}^3)^6$ by Proposition 4.1.A in [23]. Hence (3.6) is also true with inhomogeneous fractional derivatives.

2) Next we pass to $u=(\mathbf{E},\mathbf{H})$ and to $H^s$. We set $\widetilde{v}=\kappa^{-1}v$. Since $\varepsilon,\mu\in C_b^2(\mathbb{R}^3)$, $\kappa^{-1}$ is contained in $\mathcal{B}\big(H^{\gamma,q'}(\mathbb{R}^3)^6\big)$ by interpolation, and duality then implies $\kappa^{-1}\in\mathcal{B}\big(H^{-\gamma,q}(\mathbb{R}^3)^6\big)$. So step 1) yields the estimate

$$\|\widetilde{v}\|_{L^p((0,T),H^{-\gamma,q})} \lesssim \|\widetilde{v}\|_{L^\infty((0,T),L^2)} + \|L(\kappa\widetilde{v})\|_{L^1((0,T),L^2)}$$

$$+ \|\widetilde{\rho}(0)\|_{H^{-\frac{1}{2}}} + \|\partial_t \widetilde{\rho}\|_{L^1((0,T),H^{-\frac{1}{2}})} \tag{3.8}$$

for $\widetilde{\rho} := \big(\operatorname{div}(\varepsilon \widetilde{v}_1), \operatorname{div}(\mu \widetilde{v}_2)\big)$, $\widetilde{v} = (\widetilde{v}_1, \widetilde{v}_2)$, omitting the dependence on $T \le 1$.

Let $u_0 \in H^s(\mathbb{R}^3)^6$, $f = (\mathbf{K}, \mathbf{L}) \in L^1\big((0,T), H^s(\mathbb{R}^3)^{3+3}\big)$, and $u = (\mathbf{E}, \mathbf{H})$ be the mild solution of (3.4). By Remark 3.2 it is a strong $H^s$-solution. By (3.5), the map $w := \langle \nabla \rangle^s u$ satisfies the energy-type estimate

$$\|u\|_{L^\infty((0,T),H^s)} = \|w\|_{L^\infty((0,T),L^2)} \lesssim \|u_0\|_{H^s} + \|f\|_{L^1((0,T),H^s)} . \tag{3.9}$$

This estimate, (3.8) and $\alpha = s - \gamma$ from Assumption 2.1 imply

$$\begin{aligned} \|u\|_{L^p((0,T),H^{\alpha,q})} = \|w\|_{L^p((0,T),H^{-\gamma,q})} &\lesssim \|u_0\|_{H^s} + \|f\|_{L^1((0,T),H^s)} \\ &+ \|L(\kappa w)\|_{L^1((0,T),L^2)} + \|\rho_s(0)\|_{H^{-\frac{1}{2}}} + \|\partial_t \rho_s\|_{L^1((0,T),H^{-\frac{1}{2}})} \end{aligned} \tag{3.10}$$

with $L(\kappa w) = \kappa(\partial_t w - \kappa^{-1} A_{\mathrm{co}} w)$ and $\rho_s := \big(\operatorname{div}(\varepsilon \langle \nabla \rangle^s \mathbf{E}), \operatorname{div}(\mu \langle \nabla \rangle^s \mathbf{H})\big)$.

As $\langle \nabla \rangle^s \in \mathcal{B}\big(H^{s-1}, H^{-1}\big)$, $\partial_t w$ belongs to $L^1\big((0,T), H^{-1}(\mathbb{R}^3)^6\big)$. Equation (3.4) leads to

$$\partial_t w = \langle \nabla \rangle^s \big(\kappa^{-1} A_{\mathrm{co}} u + f\big) = \kappa^{-1} A_{\mathrm{co}} \langle \nabla \rangle^s u + \big[\langle \nabla \rangle^s, \kappa^{-1}\big] A_{\mathrm{co}} u + \langle \nabla \rangle^s f. \tag{3.11}$$

Theorem 1.4 in [13] yields the commutator estimate

$$\|\langle \nabla \rangle^\tau (\phi\psi) - \phi \langle \nabla \rangle^\tau \psi\|_{L^2} \lesssim \|\langle \nabla \rangle^\tau \phi\|_{L^\infty} \|\psi\|_{L^2} + \|\nabla \phi\|_{L^\infty} \big\|\langle \nabla \rangle^{\tau-1} \psi\big\|_{L^2}. \tag{3.12}$$

for $\tau > 0$ and $\tau \neq 1$. We further have

$$\|\langle \nabla \rangle^\tau (\phi\psi) - \phi \langle \nabla \rangle^\tau \psi\|_{L^2} \lesssim \|\phi\|_{W^{1,\infty}} \|\psi\|_{L^2} \tag{3.13}$$

for $\tau \in (0,1]$ by Proposition 4.1.A in [23]. Remark 2.2.2.3 and Theorems 2.3.8 and 2.5.7 in [24] imply that $\langle \nabla \rangle^\tau$ is a bounded operator from $C_b^r(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$ for $r > \tau$. With $\tau = s$, inequality (3.12) shows the boundedness of the commutators $[\langle \nabla \rangle^s, \nu]$ and $\big[\langle \nabla \rangle^s, \nu^{-1}\big]$ from $H^{s-1}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ for $\nu \in \{\varepsilon, \mu\}$ as $s \le \frac{3}{2}$ and $\nu \in C_b^2$. So (3.11) and (3.9) lead to

$$\begin{aligned} \|L(\kappa w)\|_{L^1((0,T),L^2)} &\lesssim \big\|\partial_t w - \kappa^{-1} A_{\mathrm{co}} w\big\|_{L^1((0,T),L^2)} \\ &\lesssim \|A_{\mathrm{co}} u\|_{L^1((0,T),H^{s-1})} + \|f\|_{L^1((0,T),H^s)} \lesssim \|u_0\|_{H^s} + \|f\|_{L^1((0,T),H^s)} . \end{aligned} \tag{3.14}$$

We now turn to the terms involving $\rho_s$, treating only the first component. (The second one does not differ.) Here the commutators $[\langle \nabla \rangle^s, \partial_i \varepsilon]$ and $\big[\langle \nabla \rangle^s, \partial_i \frac{1}{\varepsilon}\big]$ occur for $i \in \{1,2,3\}$. To avoid additional regularity of $\varepsilon$ and $\mu$, we have to exploit that $\rho_s$ is estimated only in $H^{-\frac{1}{2}}$. Observe that

$$\begin{aligned} \langle \nabla \rangle^{-\frac{1}{2}} \big[\langle \nabla \rangle^s, m\big] = \big[\langle \nabla \rangle^{s-\frac{1}{2}}, m\big] + \big[m, \langle \nabla \rangle^{\frac{1}{2}}\big] \langle \nabla \rangle^{s-1} \\ + \langle \nabla \rangle^{-\frac{1}{2}} \big[\langle \nabla \rangle, m\big] \langle \nabla \rangle^{s-1} \end{aligned}$$

for $m = \partial_i \varepsilon$ or $\partial_i \frac{1}{\varepsilon}$ in $C_b^1$. Since $s \le \frac{3}{2}$, the commutators on the right are $L^2$-bounded thanks to (3.13). Thus $\big[\langle \nabla \rangle^s, m\big] : H^{s-1} \to H^{-\frac{1}{2}}$ is continuous.

The charge can be written in the form

$$\operatorname{div}(\varepsilon \langle \nabla \rangle^s \mathbf{E}) = \langle \nabla \rangle^s \operatorname{div}(\varepsilon \mathbf{E}) + [\varepsilon, \langle \nabla \rangle^s] \operatorname{div} \mathbf{E} + [\nabla \varepsilon, \langle \nabla \rangle^s] \cdot \mathbf{E}$$

In $H^{-\frac{1}{2}}$ these terms are estimated by $\|\operatorname{div}(\varepsilon \mathbf{E})\|_{H^{s-\frac{1}{2}}} + \|\mathbf{E}\|_{H^s}$; i.e.,

$$\|\rho_s(0)\|_{H^{-\frac{1}{2}}} \lesssim \|\rho(0)\|_{H^{s-\frac{1}{2}}} + \|u_0\|_{H^s} . \tag{3.15}$$

Next, working in $H^{-2}(\mathbb{R}^3)$ and using (3.11), we obtain

$$\begin{aligned}\partial_t\big(\rho_s\big)_1 &= \operatorname{div}\left(\varepsilon\,\langle\nabla\rangle^s\,\partial_t\mathbf{E}\right)\\ &= \operatorname{div}\Big(\operatorname{curl}\langle\nabla\rangle^s\,\mathbf{H} + \varepsilon\left[\langle\nabla\rangle^s, \varepsilon^{-1}\right]\operatorname{curl}\mathbf{H} + \langle\nabla\rangle^s\,\varepsilon\mathbf{K} + \left[\varepsilon, \langle\nabla\rangle^s\right]\mathbf{K}\Big)\\ &= \nabla\varepsilon\cdot\big(\left[\langle\nabla\rangle^s, \varepsilon^{-1}\right]\operatorname{curl}\mathbf{H}\big) + \varepsilon\left[\langle\nabla\rangle^s, \nabla\tfrac{1}{\varepsilon}\right]\cdot\operatorname{curl}\mathbf{H}\\ &\qquad + \langle\nabla\rangle^s\operatorname{div}(\varepsilon\mathbf{K}) + \left[\nabla\varepsilon, \langle\nabla\rangle^s\right]\cdot\mathbf{K} + \left[\varepsilon, \langle\nabla\rangle^s\right]\operatorname{div}\mathbf{K}.\end{aligned}$$

Let $E_r = L^1((0,T), H^r)$. The properties of the commutators, (3.5) and the equation $\operatorname{div}(\varepsilon\mathbf{K}) = \partial_t\operatorname{div}(\varepsilon\mathbf{E}) - \operatorname{div}\operatorname{curl}\mathbf{H}$ imply

$$\begin{aligned}\left\|\nabla\varepsilon\cdot\big(\left[\langle\nabla\rangle^s, \varepsilon^{-1}\right]\operatorname{curl}\mathbf{H}\big)\right\|_{E_{-\frac{1}{2}}} &\lesssim \left\|\big(\left[\langle\nabla\rangle^s, \varepsilon^{-1}\right]\operatorname{curl}\mathbf{H}\big)\right\|_{E_{-\frac{1}{2}}}\\ &\lesssim \|\operatorname{curl}\mathbf{H}\|_{E_{s-1}} \lesssim \|u\|_{E_s} \lesssim \|u_0\|_{H^s} + \|f\|_{E_s},\\ \left\|\varepsilon\left[\langle\nabla\rangle^s, \nabla\tfrac{1}{\varepsilon}\right]\cdot\operatorname{curl}\mathbf{H}\right\|_{E_{-\frac{1}{2}}} &\lesssim \|\operatorname{curl}\mathbf{H}\|_{E_{s-1}} \lesssim \|u_0\|_{H^s} + \|f\|_{E_s},\\ \|\langle\nabla\rangle^s\operatorname{div}(\varepsilon\mathbf{K})\|_{E_{-\frac{1}{2}}} &= \|\operatorname{div}(\varepsilon\mathbf{K})\|_{E_{s-\frac{1}{2}}} = \|\partial_t\rho\|_{E_{s-\frac{1}{2}}},\\ \|[\nabla\varepsilon, \langle\nabla\rangle^s]\cdot\mathbf{K}\|_{E_{-\frac{1}{2}}} &\lesssim \|\mathbf{K}\|_{E_{s-1}} \le \|f\|_{E_s}\\ \|[\varepsilon, \langle\nabla\rangle^s]\operatorname{div}\mathbf{K}\|_{E_{-\frac{1}{2}}} &\lesssim \|\operatorname{div}\mathbf{K}\|_{E_{s-1}} \le \|f\|_{E_s}.\end{aligned}$$

We arrive at

$$\|\partial_t\rho_s\|_{L^1((0,T),H^{-\frac{1}{2}})} \lesssim \|u_0\|_{H^s} + \|f\|_{L^1((0,T),H^s)} + \|\partial_t\rho\|_{L^1((0,T),H^{s-\frac{1}{2}})}. \tag{3.16}$$

Formulas (3.10), (3.14), (3.15) and (3.16) lead to (3.7). □

Let $\widetilde{\mathbf{M}} = 0$ for simplicity. Then $\operatorname{div}(\mu\mathbf{H}) = \operatorname{div}\mathbf{B}$ describes the magnetic charges and vanishes. But $\operatorname{div}(\varepsilon\mathbf{E})$ does not describe the free electric charges, since the contribution $\widetilde{\mathbf{P}}$ of the polarisation to the $\mathbf{D}$-field is missing. Handling these 'charge-like' terms is difficult, so in Section 5 we use a projection operator to split our problem into a part involving $\rho$ and a 'charge-free' part. The first one is handled directly and for the second one we can use the Strichartz estimate. This projection operator is the topic of the next section.

## 4. Helmholtz projection

To deal with the divergence and curl operators, we first define the spaces

$$C^\infty_{c,\sigma}(\mathbb{R}^3) = \left\{f \in C^\infty_c(\mathbb{R}^3)^3 \;\middle|\; \operatorname{div} f = 0\right\}, \quad \nabla C^\infty_c(\mathbb{R}^3) = \left\{\nabla\phi \;\middle|\; \phi \in C^\infty_c(\mathbb{R}^3)\right\},$$

$$L^2_\sigma(\mathbb{R}^3) = \overline{C^\infty_{c,\sigma}(\mathbb{R}^3)}^{L^2}, \quad G(\mathbb{R}^3) = \left\{f \in L^2(\mathbb{R}^3)^3 \;\middle|\; \exists\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^3) : f = \nabla\phi\right\},$$

and collect known results on the Helmholtz decomposition of $L^2(\mathbb{R}^3)^3$ into divergence-free functions and gradients of scalar maps, cf. Lemmas II.2.5.1 and II.2.5.4 in [19].

**Lemma 4.1.** *We have* $L^2(\mathbb{R}^3)^3 = L^2_\sigma(\mathbb{R}^3) \oplus_\perp G(\mathbb{R}^3)$ *and*

$$L^2_\sigma(\mathbb{R}^3) = \left\{f \in L^2(\mathbb{R}^3)^3 \;\middle|\; \operatorname{div} f = 0\right\} = N(\operatorname{div}), \quad G(\mathbb{R}^3) = \overline{\nabla C^\infty_c(\mathbb{R}^3)}^{L^2}.$$

The next lemma states that curl-free functions are given by gradients.

**Lemma 4.2.** *It holds* $G(\mathbb{R}^3) = \left\{v \in L^2(\mathbb{R}^3)^3 \,\middle|\, \operatorname{curl} v = 0\right\} = N(\operatorname{curl})$. *Moreover, $\phi$ in $G(\mathbb{R}^3)$ also belongs to $\mathcal{S}^*_h(\mathbb{R}^3)$ and $\widehat{\phi}$ to $L^1_{\mathrm{loc}}(\mathbb{R}^3)$.*

*Proof.* Since $\operatorname{curl}\nabla = 0$, we have $G(\mathbb{R}^3) \subseteq N(\operatorname{curl})$ which implies $N(\operatorname{curl})^\perp \subseteq G(\mathbb{R}^3)^\perp$. Let $f \in N(\operatorname{div})$ and $g \in N(\operatorname{curl})$. We use mollifiers and define $g_n := \chi_{\frac{1}{n}} * g \in C^\infty(\mathbb{R}^3)^3 \cap L^\infty(\mathbb{R}^3)^3$ for $n \in \mathbb{N}$. Then we have $g_n \to g$ in $L^2(\mathbb{R}^3)^3$ as $n \to \infty$. Since $\operatorname{curl} g_n = \chi_{\frac{1}{n}} * \operatorname{curl} g = 0$, it is well known that there is a sequence $(\phi_n)$ in $C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)^3$ with $g_n = \nabla\phi_n$ for all $n \in \mathbb{N}$. Lemma 4.1 provides a sequence $(f_n)$ in $C_c^\infty(\mathbb{R}^3)^3 \cap N(\operatorname{div})$ with limit $f$ in $L^2(\mathbb{R}^3)^3$. We infer

$$\int_{\mathbb{R}^3} f \cdot \overline{g}\,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^3} f_n \cdot \nabla\overline{\phi_n}\,\mathrm{d}x = -\lim_{n\to\infty} \int_{\mathbb{R}^3} \operatorname{div}(f_n)\overline{\phi_n}\,\mathrm{d}x = 0,$$

and hence $N(\operatorname{div}) \subseteq N(\operatorname{curl})^\perp$. The Helmholtz decomposition then yields

$$N(\operatorname{curl})^\perp \subseteq G(\mathbb{R}^3)^\perp = N(\operatorname{div}) \subseteq N(\operatorname{curl})^\perp,$$

from which we conclude $N(\operatorname{curl}) = G(\mathbb{R}^3)$.

Note that $\mathrm{i}\xi_k\mathcal{F}\phi_n$ converges to $\widehat{g}_k$ in $L^2(\mathbb{R}^3)$ as $n \to \infty$, and thus also pointwise a.e. and with a pointwise majorant $h \in L^2(\mathbb{R}^3)$ for $k \in \{1,2,3\}$, after passing to a subsequence. Hence, $\mathcal{F}\phi_n$ is bounded by $3h/|\xi|_1 \in L^1(B)$ on $B = B(0,1)$ and by $3h$ on $\mathbb{R}^3 \setminus B$. Using dominated convergence, we see that $\mathcal{F}\phi_n$ tends to $-\mathrm{i}\widehat{g}_k/\xi_k =: \psi$ in $\mathcal{S}^\star(\mathbb{R}^3) \cap L^1_{\mathrm{loc}}(\mathbb{R}^3)$. For $\phi = \mathcal{F}^{-1}\psi$ we infer $\nabla\phi = g$ and $\phi \in \mathcal{S}_h^*(\mathbb{R}^3)$, see p.22 in [3]. □

For the remainder of this section, let $\theta \in W^{1,\infty}(\mathbb{R}^3)$ be bounded from below by some positive constant $\eta$. We define $L^2_\theta(\mathbb{R}^3)^3$ as the space $L^2(\mathbb{R}^3)^3$ equipped with the weighted scalar product

$$(f|g)_{L^2_\theta} := \int_{\mathbb{R}^3} \theta f \cdot \overline{g}\,\mathrm{d}x.$$

For $V \subseteq L^2(\mathbb{R}^3)^3$ we denote by $V^{\perp_\theta}$ the orthogonal complement of $V$ with respect to $(\cdot|\cdot)_{L^2_\theta}$. The next lemma yields a Helmholtz decomposition with respect to this weighted scalar product, replacing $\operatorname{div}(f)$ by the expression $\operatorname{div}(\theta f)$. We define the operators $\operatorname{div}_\theta : H(\operatorname{div}) \to L^2(\mathbb{R}^3)$ and $\mathcal{A}_\theta : H(\operatorname{curl}) \to L^2(\mathbb{R}^3)^3$ by $\operatorname{div}_\theta(f) = \operatorname{div}(\theta f)$ and $\mathcal{A}_\theta = \theta^{-1}\operatorname{curl}$. (Note that $\operatorname{div}_\theta(f) = \theta \operatorname{div} f + \nabla\theta \cdot f$).

**Lemma 4.3.** *It holds* $N(\operatorname{curl})^{\perp_\theta} = N(\operatorname{div}_\theta)$.

*Proof.* Let $u \in N(\operatorname{curl})^{\perp_\theta}$ and $\varphi \in C_c^\infty(\mathbb{R}^3)$. Since $\operatorname{curl}\nabla\varphi = 0$, it follows

$$\int_{\mathbb{R}^3} \theta u \cdot \nabla\overline{\varphi}\,\mathrm{d}x = (u|\nabla\varphi)_{L^2_\theta} = 0,$$

which implies $\theta u \in H(\operatorname{div})$ and $\operatorname{div}_\theta(u) = 0$.

Conversely, let $u \in N(\operatorname{div}_\theta)$ and $v \in N(\operatorname{curl})$. Lemmas 4.1 and 4.2 provide a sequence $(\phi_n)$ in $C_c^\infty(\mathbb{R}^3)$ with $\nabla\phi_n \to v$ in $L^2(\mathbb{R}^3)^3$. So we obtain

$$(u|v)_{L^2_\theta} = \lim_{n\to\infty} \int_{\mathbb{R}^3} \theta u \cdot \nabla\overline{\phi_n}\,\mathrm{d}x = -\lim_{n\to\infty} \int_{\mathbb{R}^3} \operatorname{div}(\theta u)\overline{\phi_n}\,\mathrm{d}x = 0. \qquad \square$$

Let $\mathcal{Q}_\theta : L^2(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)^3$ be the orthogonal projection with respect to $(\cdot|\cdot)_{L^2_\theta}$ onto $N(\operatorname{curl}) = N(\mathcal{A}_\theta)$. The orthogonal projection onto $N(\mathcal{A}_\theta)^{\perp_\theta}$ is thus given by $\widetilde{\mathcal{Q}}_\theta := I - \mathcal{Q}_\theta$. We collect basic properties.

**Lemma 4.4.** *We have* $N(\mathcal{Q}_\theta) = R(\widetilde{\mathcal{Q}}_\theta) = N(\operatorname{div}_\theta)$ *and* $R(\mathcal{Q}_\theta) = N(\widetilde{\mathcal{Q}}_\theta) = N(\operatorname{curl})$.

In the sequel, we need boundedness properties of $\mathcal{Q}_\theta$ on our Sobolev spaces. The next lemma states that $\mathcal{Q}_\theta$ can be restricted to a bounded linear operator on $H^s(\mathbb{R}^3)^3$ as well as to a bounded linear operator from $L^2(\mathbb{R}^3)^3 \cap H^{\alpha,q}(\mathbb{R}^3)^3$ to $H^{\alpha,q}(\mathbb{R}^3)^3$. We denote these restrictions also by $\mathcal{Q}_\theta$. We do not know whether one can discard $L^2$ here, but this matter plays no role below (where we only need $\alpha, s \in [0,2]$). For smooth $\theta > 0$ being 1 outside a compact set, the arguments in §4.3 of [8] imply boundedness of $Q_\theta$ on $H^{\alpha,q}$ and $H^s$. They involve the theory of Fourier integral and pseudodifferential operators, which we want to avoid.

**Lemma 4.5.** *Let* $\eta > 0$, $k \in \mathbb{N}$, $q \in [2,\infty)$ *and* $s \in [0,k]$. *Let* $\theta \in C_b^k(\mathbb{R}^3)$ *satisfy* $\theta(x) \ge \eta$ *for all* $x \in \mathbb{R}^3$. *Then we have*

$$\|\mathcal{Q}_\theta v\|_{H^{s,q}} \lesssim \|v\|_{H^{s,q}} + \|v\|_{L^2}, \tag{4.1}$$

$$\|\mathcal{Q}_\theta v\|_{H^s} \lesssim \|v\|_{H^s} \tag{4.2}$$

*for all* $v \in \mathcal{S}(\mathbb{R}^3)^3$.

*Proof.* Let $v \in \mathcal{S}(\mathbb{R}^3)^3$. Lemmas 4.1 and 4.2 provide a function $\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^3) \cap \mathcal{S}_h^*(\mathbb{R}^3)$ with $\mathcal{Q}_\theta v = \nabla\phi$ and $\widehat{\phi} \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. By the Sobolev embedding, $\phi$ belongs to $L^6(\mathbb{R}^3)$ and $\|\phi\|_{L^6} \lesssim \|\mathcal{Q}_\theta v\|_{L^2} \lesssim \|v\|_{L^2}$. (See Theorem 1.38 in [3].)

We define the differential operator $\mathcal{L}$ by $\mathcal{L}u = \operatorname{div}(\theta\nabla u)$ and observe that

$$\mathcal{L}\phi = \operatorname{div}(\theta\mathcal{Q}_\theta v) = \operatorname{div}(\theta v) - \operatorname{div}\big(\theta(I - \mathcal{Q}_\theta)v\big) = \operatorname{div}(\theta v) =: g \in H^{-1,6}(\mathbb{R}^3).$$

The main theorem in [12] shows that $\mathcal{L} - \lambda$ is an isomorphism from $H^{1,6}(\mathbb{R}^3)$ to $H^{-1,6}(\mathbb{R}^3)$ for some $\lambda \ge 0$. So $\phi = (\mathcal{L} - \lambda)^{-1}(g - \lambda\phi)$ lies in $H^{1,6}(\mathbb{R}^3)$ and

$$\|\phi\|_{H^{1,6}} \lesssim \|g\|_{H^{-1,6}} + \|\phi\|_{H^{-1,6}} \lesssim \|v\|_{L^6} + \|\phi\|_{L^6},$$

which implies $\|\mathcal{Q}_\theta v\|_{L^6} \lesssim \|\phi\|_{H^{1,6}} \lesssim \|v\|_{L^6} + \|v\|_{L^2}$. Hence, $\mathcal{Q}_\theta \in \mathcal{B}(L^2(\mathbb{R}^3)^3)$ restricts to a bounded linear operator from $L^2(\mathbb{R}^3)^3 \cap L^6(\mathbb{R}^3)^3$ to $L^6(\mathbb{R}^3)^3$.

Let $r \in [2,6]$ and take $\beta \in [0,1]$ with $\frac{1}{r} = \frac{\beta}{2} + \frac{1-\beta}{6}$. It follows

$$\|\mathcal{Q}_\theta v\|_{L^r} \le \|\mathcal{Q}_\theta v\|_{L^2}^{\beta} \|\mathcal{Q}_\theta v\|_{L^6}^{1-\beta} \lesssim \|v\|_{L^2} + \|v\|_{L^6}. \tag{4.3}$$

Now we choose some $\delta \in (0,1]$ and set $r := 3 - \delta$. Since $\nabla\phi \in L^r(\mathbb{R}^3)$ and $\phi \in \mathcal{S}_h^*(\mathbb{R}^3)$, by Proposition 2.2 in [10] the map $\phi$ is contained in $L^q(\mathbb{R}^3)$ for $q = \frac{9}{\delta} - 3$, and hence

$$\|\phi\|_{L^q} \lesssim \|\mathcal{Q}_\theta v\|_{L^r} \lesssim \|v\|_{L^2} + \|v\|_{L^6}.$$

As a result, $\phi$ belongs to $L^q(\mathbb{R}^3)$ for all $q \in [6,\infty)$. By [12], the operator $\mathcal{L} - \widetilde{\lambda}$ is an isomorphism from $H^{1,q}(\mathbb{R}^3)$ to $H^{-1,q}(\mathbb{R}^3)$ for some $\widetilde{\lambda} = \widetilde{\lambda}(q) \ge 0$. Therefore $\phi = (\mathcal{L} - \widetilde{\lambda})^{-1}\big(g - \widetilde{\lambda}\phi\big) \in H^{1,q}(\mathbb{R}^3)$ is bounded by

$$\|\phi\|_{H^{1,q}} \lesssim \|g\|_{H^{-1,q}} + \|\phi\|_{H^{-1,q}} \lesssim \|v\|_{L^q} + \|\phi\|_{L^q} \lesssim \|v\|_{L^q} + \|v\|_{L^2} \tag{4.4}$$

for all $q \in [6,\infty)$. This inequality and (4.3) imply $\|\mathcal{Q}_\theta v\|_{L^q} \lesssim \|v\|_{L^q} + \|v\|_{L^2}$ for all $q \in [2,\infty)$ and therefore (4.1) for $s = 0$.

Now let $s \in (0,1)$. We set $\psi_s := \langle\nabla\rangle^s \phi$ and have $\|\psi_s\|_{L^q} \lesssim \|\phi\|_{H^{1,q}}$. By the first part of Theorem 1.4 in [13], it holds $\langle\nabla\rangle^s (\theta\nabla\phi) = \theta \langle\nabla\rangle^s \nabla\phi + R$ with a remainder term $R$ satisfying

$$\|R\|_{L^q} \lesssim \|\langle\nabla\rangle^s \theta\|_{L^\infty} \|\nabla\phi\|_{L^q} + \|\nabla\theta\|_{L^\infty} \left\|\langle\nabla\rangle^{s-1} \nabla\phi\right\|_{L^q} \lesssim \|\phi\|_{H^{1,q}}.$$

Because of $\operatorname{div}(\theta\mathcal{Q}_\theta v) = \operatorname{div}(\theta v)$, we obtain the equation

$$\operatorname{div}(\theta\nabla\psi_s) = \langle\nabla\rangle^s \operatorname{div}(\theta\nabla\phi) - \operatorname{div}(R) = \langle\nabla\rangle^s \operatorname{div}(\theta v) - \operatorname{div}(R) =: g_s.$$

We can bound $g_s$ in $H^{-1,q}(\mathbb{R}^3)$ by $c\big(\|v\|_{H^{s,q}} + \|\phi\|_{H^{1,q}}\big)$. As above and using (4.4), we deduce

$$\|\psi_s\|_{H^{1,q}} \lesssim \|g_s\|_{H^{-1,q}} + \|\psi_s\|_{H^{-1,q}} \lesssim \|v\|_{H^{s,q}} + \|\phi\|_{H^{1,q}} \lesssim \|v\|_{H^{s,q}} + \|v\|_{L^2},$$
$$\|\mathcal{Q}_\theta v\|_{H^{s,q}} = \|\langle\nabla\rangle^s \nabla\phi\|_{L^q} \lesssim \|\psi_s\|_{H^{1,q}} \lesssim \|v\|_{H^{s,q}} + \|v\|_{L^2}.$$

The general result is then proved via induction, invoking the second part of Theorem 1.4 in [13] at the end. □

Now we define the projection $Q$ for the Maxwell problem on $L^2_{\varepsilon,\mu}(\mathbb{R}^3)^6$ by

$$Q\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}\mathcal{Q}_\varepsilon\mathbf{E}\\ \mathcal{Q}_\mu\mathbf{H}\end{pmatrix}$$

for $(\mathbf{E},\mathbf{H}) \in L^2(\mathbb{R}^3)^6$, and we set $\widetilde{Q} := I - Q$. The needed properties of $Q$ and $\widetilde{Q}$ follow from the above results on $\mathcal{Q}_\theta$. The statement on $S(\cdot)$ is a consequence of the resolvent approximation.

**Lemma 4.6.** *Let $\eta > 0$, $q \in [2,\infty)$ and $s \in [0,2]$. Assume that $\varepsilon, \mu \in C^2_b(\mathbb{R}^3)$ satisfy $\varepsilon(x), \mu(x) \geq \eta$ for all $x \in \mathbb{R}^3$. Let $Q$ be defined as above, $A$ by (3.2), and $S(\cdot)$ denote the group generated by $A$.*

*Then the operator $Q$ is the orthogonal projection onto $N(A)$ in $L^2_{\varepsilon,\mu}(\mathbb{R}^3)^6$ with respect to $(\cdot|\cdot)_{L^2_{\varepsilon,\mu}}$. It satisfies*

$$N(Q) = R(\widetilde{Q}) = \big\{(\mathbf{E},\mathbf{H}) \in L^2(\mathbb{R}^3)^6 \;\big|\; \operatorname{div}(\varepsilon\mathbf{E}) = \operatorname{div}(\mu\mathbf{H}) = 0\big\},$$
$$R(Q) = N(\widetilde{Q}) = \big\{(\mathbf{E},\mathbf{H}) \in L^2(\mathbb{R}^3)^6 \;\big|\; \operatorname{curl}\mathbf{E} = \operatorname{curl}\mathbf{H} = 0\big\}$$

*as well as $QAu = AQu = 0$ for all $u \in \mathrm{D}(A)$ and $QS(t) = S(t)Q$ for $t \in \mathbb{R}$.*

*Furthermore, $Q|_{\mathcal{S}}$ and $\widetilde{Q}|_{\mathcal{S}}$ can be uniquely extended to bounded linear operators on $H^s(\mathbb{R}^3)^6$ and from $L^2(\mathbb{R}^3)^6 \cap H^{s,q}(\mathbb{R}^3)^6$ to $H^{s,q}(\mathbb{R}^3)^6$.*

By the above lemma we can simplify the Strichartz estimate (3.7) if the data are 'charge-free'.

**Corollary 4.7.** *In the setting of Theorem 3.4, let $Qu_0 = 0$ and $Qf = 0$. Then the mild solution $u$ of (3.4) is contained in $L^p\big((0,T), H^{\alpha,q}(\mathbb{R}^3)^6\big)$ and satisfies $Qu = 0$ by (3.5) as well as the estimate*

$$\|u\|_{L^p((0,T),H^{\alpha,q})} \leq C_{\mathrm{Str}}\big(\|u_0\|_{H^s} + \|f\|_{L^1((0,T),H^s)}\big). \tag{4.5}$$

## 5. Local wellposedness

Our goal is to show local wellposedness for (3.1). (Recall the definitions made before and after this equation.) Let Assumption 2.4 be true, $J$ be an interval with $\sup J > 0$ which contains $(-\infty, 0]$, and $u_{\mathrm{h}}$ belong to $Z(0)$. Set $J_+ = J \cap \mathbb{R}_{>0}$. We say that $u = (\mathbf{E}, \mathbf{H})$ is a *mild solution* of (3.1) on $J$ if $u$ is an element of $C\big(J, H^s(\mathbb{R}^3)^6\big) \cap L^p\big((-\infty, b), H^{\alpha,q}(\mathbb{R}^3)^6\big)$ for all $b < \sup J$ and satisfies Duhamel's formula

$$u(t) = \begin{cases} S_\sigma(t)\big(u_{\mathrm{h}}(0)\big) + \int_0^t S_\sigma(t-\tau)\big(F(u_\tau) + g(\tau)\big)\,\mathrm{d}\tau, & t \in J_+, \\ u_{\mathrm{h}}(t), & t \le 0. \end{cases} \tag{5.1}$$

Using standard perturbation theory for semigroups, one can equivalently require

$$u(t) = \begin{cases} S(t)\big(u_{\mathrm{h}}(0)\big) + \int_0^t S(t-\tau)\big(Bu(\tau) + F(u_\tau) + g(\tau)\big)\,\mathrm{d}\tau, & t \in J_+, \\ u_{\mathrm{h}}(t), & t \le 0. \end{cases}$$

We stress that a mild solution belongs to $W^{1,1}\big(J_+, H^{s-1}(\mathbb{R}^3)^6\big)$ and satisfies (3.1) for a.e. $t \ge 0$, see Remark 3.2. In particular, concatenating and shifting mild solutions lead again to mild solutions.

If we applied the Strichartz estimate (3.7) to the mild solution $u$ of (3.1), the charge contributions on its right-hand side would spoil the resulting local wellposedness theory. As a remedy we split the problem into two parts, using the projection onto 'charge-free' fields from Lemma 4.6. Set $u = v + w$ with $v := Qu$ and $w := \widetilde{Q}u$. Since $Q$ commutes with $A$ and maps into $N(A)$ the functions $v$ and $w$ are mild solutions of the two sub-problems

$$\begin{aligned} v'(t) &= Q\big(Bu(t) + F(u_t) + g(t)\big), \quad t \ge 0, \\ v(t) &= Qu_{\mathrm{h}}(t), \quad t \le 0, \end{aligned} \tag{5.2}$$

respectively

$$\begin{aligned} w'(t) &= Aw(t) + \widetilde{Q}\big(Bu(t) + F(u_t) + g(t)\big), \quad t \ge 0, \\ w(t) &= \widetilde{Q}u_{\mathrm{h}}(t), \quad t \le 0, \end{aligned} \tag{5.3}$$

As $\widetilde{Q}$ projects onto 'charge-free' fields, the Strichartz estimate from Corollary 4.7 can be used for $w$. In the sub-problem for $v$, we can integrate the nonlinearity $F(u_t)$ directly by means of (2.11) and Lemmas 2.2 and 2.11, leading to

$$v(t) = Qu_{\mathrm{h}}(0) + \int_0^t Q\big(Bu(\tau) + g(\tau)\big)\,\mathrm{d}\tau + Q\kappa^{-1}\big(\Gamma(u)(0) - \Gamma(u)(t)\big). \tag{5.4}$$

Additionally we have to require that the curl-free part $Qu_{\mathrm{h}}(0)$ at the initial time lies in $H^{\alpha,q}(\mathbb{R}^3)^6$. As we see in Lemma 5.1, $Qu_{\mathrm{h}}(0) \in H^{\alpha,q}(\mathbb{R}^3)^6$ implies the additional property $Qu \in C\big(J \cap [0,\infty), H^{\alpha,q}(\mathbb{R}^3)^6\big)$. This is needed for the construction of a mild solution on a maximal time interval, where we restart problem (3.1) using the shifted solution as a new initial history $u_{\mathrm{h}}$.

The next lemma yields a local solution by means of the core fixed-point argument. With the constant $\widetilde{C}_{\mathrm{Str}}$ from (5.10). we set

$$M := \sup_{0 \le t \le 1} \|S(t)\|_{\mathcal{B}(H^s)} \ge 1, \qquad K := 1 + M + \widetilde{C}_{\mathrm{Str}}. \tag{5.5}$$

**Lemma 5.1.** *Let Assumption 2.4 be true and* $r_0 > 0$. *Then there exists a time* $b_0 = b_0(r_0) \in (0,1]$ *such that for each* $u_{\mathrm{h}} \in Z(0)$ *with* $Qu_{\mathrm{h}}(0) \in H^{\alpha,q}(\mathbb{R}^3)^6$ *and* $\|Qu_{\mathrm{h}}(0)\|_{H^{\alpha,q}} + \|u_{\mathrm{h}}\|_{Z(0)} + \|g\|_{L^1((0,1),H^s\cap H^{\alpha,q})} \le r_0$ *there is a mild solution* $u \in Z(b_0)$ *of* (3.1) *on* $(-\infty, b_0]$. *It is the only mild solution of* (3.1) *on* $(-\infty, b_0]$ *satisfying* $\|u\|_{Z(b_0)} \le 1 + Kr_0$ *for* $K$ *from* (5.5), *and* $Qu$ *is contained in* $C\big([0,b_0], H^{\alpha,q}(\mathbb{R}^3)^6\big)$. *For each* $b \in (0, b_0]$, *the restriction of* $u$ *to* $(-\infty, b]$ *is the unique mild solution on* $(-\infty, b]$ *with* $\|u\|_{Z(b)} \le 1 + Kr_0$.

*Proof.* Let $r_0 > 0$ and $u_{\mathrm{h}} \in Z(0)$ with $Qu_{\mathrm{h}}(0) \in H^{\alpha,q}$ and $\|Qu_{\mathrm{h}}(0)\|_{H^{\alpha,q}} + \|u_{\mathrm{h}}\|_{Z(0)} + \|g\|_{L^1((0,1),H^s\cap H^{\alpha,q})} \le r_0$. We set $r := 1 + Kr_0$. For a time $b \in (0,1]$ (specified later) we define the space

$$Z(b,r) := \big\{u \in Z(b) \;\big|\; u|_{(-\infty,0]} = u_{\mathrm{h}},\ \|u\|_{Z(b)} \le r\big\}$$

equipped with the complete metric induced by the norm of $Z(b)$. On $Z(b,r)$ we introduce the fixed-point map $\Phi = \Phi_{u_{\mathrm{h}},g}$ by

$$\Phi(u)(t) = \begin{cases} S(t)u_{\mathrm{h}}(0) + \int_0^t S(t-\tau)\big(Bu(\tau) + F(u_\tau) + g(\tau)\big)\,\mathrm{d}\tau, & 0 < t \le b, \\ u_{\mathrm{h}}(t), & t \le 0. \end{cases} \tag{5.6}$$

As above, we split $\Phi$ into $\Phi_1 + \Phi_2$ given by

$$\Phi_1(u)(t) = \begin{cases} Qu_{\mathrm{h}}(0) + \int_0^t Q\big(Bu(\tau) + F(u_\tau) + g(\tau)\big)\,\mathrm{d}\tau, & 0 < t \le b, \\ Qu_{\mathrm{h}}(t), & t \le 0, \end{cases}$$

$$\Phi_2(u)(t) = \begin{cases} S(t)\widetilde{Q}u_{\mathrm{h}}(0) + \int_0^t S(t-\tau)\widetilde{Q}\big(Bu(\tau)+F(u_\tau)+g(\tau)\big)\mathrm{d}\tau, & 0<t\le b, \\ \widetilde{Q}u_{\mathrm{h}}(t), & t \le 0. \end{cases}$$

Let $u = (\mathbf{E},\mathbf{H}), \tilde{u} = (\widetilde{\mathbf{E}},\widetilde{\mathbf{H}}) \in Z(b,r)$ and $f := Bu + F(u_\bullet) + g \in L^1((0,b), H^s)$.

1) We first show that $\Phi$ maps into $Z(b,r)$. Clearly, $\Phi(u)$ is contained in $C\big((-\infty,b], H^s\big)$. Set $E_s(b) = L^1((0,b),H^s)$. By Lemma 2.12, there exists a constant $C' > 0$ such that

$$\begin{aligned} \|f\|_{L^1((0,b),H^s)} &\le b\,\|B\|_{\mathcal{B}(H^s)} \sup_{0\le t\le b} \|u(t)\|_{H^s} + \|F(u_\bullet)\|_{E_s(b)} + \|g\|_{E_s(b)} \\ &\le C'(1+r^N)\big(b + b^{\frac{1}{p'}}\big) + \|g\|_{E_s(b)} =: \omega(b) + \|g\|_{E_s(b)}\,. \end{aligned}$$

Hence, $\Phi(u)$ belongs to $C_b\big((-\infty,b], H^s(\mathbb{R}^3)^6\big)$ with

$$\sup_{t\le b} \|\Phi(u)(t)\|_{H^s} \le M\big(r_0 + \omega(b)\big). \tag{5.7}$$

For $L^p\big((-\infty,b), H^{\alpha,q}\big)$, we start with $\Phi_1(u)$. As in (5.4), we can write

$$\Phi_1(u)(t) = Qu_{\mathrm{h}}(0) + \int_0^t QB\big(u(\tau)+g(\tau)\big)\mathrm{d}\tau + Q\kappa^{-1}\big(\Gamma(u)(0) - \Gamma(u)(t)\big) \tag{5.8}$$

for $t \in (0,b)$. Since $b \le 1$, Lemmas 2.2, 2.8, 4.6 and Hölder's inequality yield

$$\begin{aligned} &\Big\| \int_0^t QB\big(u(\tau) + g(\tau)\big)\,\mathrm{d}\tau \Big\|_{H^{\alpha,q}} \\ &\lesssim \|B\|_{C_b^2} \|Q\|_{\mathcal{B}(L^2\cap H^{\alpha,q}, H^{\alpha,q})} \big(\|u\|_{L^1((0,t),H^s\cap H^{\alpha,q})} + \|g\|_{L^1((0,t),H^s\cap H^{\alpha,q})}\big) \\ &\lesssim r + \|g\|_{L^1((0,t),H^s\cap H^{\alpha,q})}\,, \end{aligned}$$

$$\begin{aligned}&\big\|Q\kappa^{-1}\big(\Gamma(u)(0)-\Gamma(u)(t)\big)\big\|_{H^{\alpha,q}}\\&\quad\lesssim \|\Gamma(u)(0)-\Gamma(u)(t)\|_{L^2}+\|\Gamma(u)(0)-\Gamma(u)(t)\|_{H^{\alpha,q}}\lesssim 1+r^N,\end{aligned}$$

using also the assumption on $u_{\mathrm{h}}$. We thus obtain

$$\|\Phi_1(u)\|_{L^p((0,b),H^{\alpha,q})}\le \widehat{C}b^{\frac{1}{p}}\left(1+r_0+r^N\right) \tag{5.9}$$

for a constant $\widehat{C}>0$. For $\Phi_2(u)$ we use the Strichartz estimate. By Lemmas 2.2, 2.12 and 4.6, $\widetilde{Q}f$ belongs to $L^1\big((0,b),H^s\big)$ and satisfies

$$\big\|\widetilde{Q}f\big\|_{E_s(b)}\le \tfrac{L}{2}\big((b+b^{\frac{1}{p'}})(1+r^N)+\|g\|_{E_s(b)}\big)\le L\big(b^{\frac{1}{p'}}(1+r^N)+\|g\|_{E_s(b)}\big)$$

for a constant $L>0$. Corollary 4.7 then shows $\Phi_2(u)\in L^p\big((0,b),H^{\alpha,q}\big)$ and

$$\begin{aligned}\|\Phi_2(u)\|_{L^p((0,b),H^{\alpha,q})}&\le C_{\mathrm{Str}}\big(\big\|\widetilde{Q}u_{\mathrm{h}(0)}\big\|_{H^s}+L\big(b^{\frac{1}{p'}}(1+r^N)+\|g\|_{E_s(b)}\big)\big)\\&\le \widetilde{C}_{\mathrm{Str}}\big(r_0+b^{\frac{1}{p'}}(1+r^N)\big).\end{aligned} \tag{5.10}$$

Involving also $u_{\mathrm{h}}$, estimates (5.7), (5.9) and (5.10) lead to

$$\|\Phi(u)\|_{Z(b)}\le \big(1+M+\widehat{C}b^{\frac{1}{p}}+\widetilde{C}_{\mathrm{Str}}\big)r_0+\big(\widehat{C}b^{\frac{1}{p}}+\widetilde{C}_{\mathrm{Str}}b^{\frac{1}{p'}}\big)(1+r^N)+M\omega(b).$$

Since the right-hand side converges to $(1+M+\widetilde{C}_{\mathrm{Str}})r_0=Kr_0$ as $b\to 0$, there exists a time $b_0^{(1)}(r_0)\in(0,1]$ with $\|\Phi(u)\|_{Z(b)}\le r$ for all $b\in(0,b_0^{(1)}(r_0)]$

2) We now prove that the map $\Phi$ is a strict contraction on $Z(b,r)$ for sufficiently small $b$. It holds

$$\Phi(u)(t)-\Phi(\widetilde{u})(t)=\begin{cases}\int_0^t S(t-\tau)\big(B(u(\tau)-\widetilde{u}(\tau))+F(u_\tau)-F(\widetilde{u}_\tau)\big)\mathrm{d}\tau, & 0<t\le b,\\ 0, & t\le 0.\end{cases}$$

Observe that

$$\left\|\int_0^t S(t-\tau)B\big(u(\tau)-\widetilde{u}(\tau)\big)\,\mathrm{d}\tau\right\|_{H^s}\lesssim b\,\|u-\widetilde{u}\|_{Z(b)}. \tag{5.11}$$

Together with the properties of $F$ in Lemma 2.12, we infer

$$\|\Phi(u)(t)-\Phi(\widetilde{u})(t)\|_{H^s}\lesssim M\big(b+b^{\frac{1}{p'}}\big)(1+r^{N-1})\,\|u-\widetilde{u}\|_{Z(b)} \tag{5.12}$$

for all $t\in[0,b]$. For $L^p((0,b),H^{\alpha,q})$, the part $\Phi_1=Q\Phi$ satisfies

$$\Phi_1(u)(t)-\Phi_1(\widetilde{u})(t)=\begin{cases}\int_0^t Q\big(B(u(\tau)-\widetilde{u}(\tau))+F(u_\tau)-F(\widetilde{u}_\tau)\big)\mathrm{d}\tau, & 0<t\le b,\\ 0, & t\le 0.\end{cases}$$

As above, Lemma 2.12 yields

$$\|\Phi_1(u)-\Phi_1(\widetilde{u})\|_{L^p((0,b),H^{\alpha,q})}\lesssim \big(b^{1+\frac{1}{p}}+b\big)(1+r^{N-1})\,\|u-\widetilde{u}\|_{Z(b)}. \tag{5.13}$$

Concerning $\Phi_2=\widetilde{Q}\Phi$, Corollary 4.7, estimate (5.11), and Lemma 2.12 imply

$$\|\Phi_2(u)-\Phi_2(\widetilde{u})\|_{L^p((0,b),H^{\alpha,q})}\lesssim C_{\mathrm{Str}}\big(b+b^{\frac{1}{p'}}\big)(1+r^{N-1})\,\|u-\widetilde{u}\|_{Z(b)}. \tag{5.14}$$

Estimates (5.12), (5.13) and (5.14) lead to a time $b_0^{(2)}\in(0,1]$ with

$$\|\Phi(u)-\Phi(\widetilde{u})\|_{Z(b)}\le \tfrac{1}{2}\,\|u-\widetilde{u}\|_{Z(b)}$$

for all $b \in \big(0, b_0^{(2)}\big]$. So with $b_0 := \min\big\{b_0^{(1)}, b_0^{(2)}\big\}$, the map $\Phi$ is a strict contraction on $Z(b_0, r)$ and Banach's fixed-point theorem yields a unique $u \in Z(b_0, r)$ satisfying $\Phi(u) = u$. The variants with $b \le b_0$ are clear.

Finally $Qu$ belongs to $C\big([0, b_0], H^{\alpha,q}(\mathbb{R}^3)^6\big)$ because of (5.8), the assumption $Qu_{\mathrm{h}}(0) \in H^{\alpha,q}(\mathbb{R}^3)^6$ and the mapping properties of the involved maps, see Lemmas 2.2, 2.8 and 4.6. □

The mild solution in Lemma 5.1 is only unique under a condition on its size. We now show that mild solutions in $Z(b)$ are unique unconditionally.

**Lemma 5.2.** *Let Assumption 2.4 be true,* $u_{\mathrm{h}} \in Z(0)$, $Qu_{\mathrm{h}}(0) \in H^{\alpha,q}(\mathbb{R}^3)^6$, *and* $u \in Z(T_1)$, $v \in Z(T_2)$ *be mild solutions of* (3.1) *on* $(-\infty, T_1]$ *respectively* $(-\infty, T_2]$. *We then obtain* $u = v$ *on* $(-\infty, T_3]$ *with* $T_3 = \min\{T_1, T_2\}$.

*Proof.* Without loss of generality, let $T_1 \le T_2$. We define

$$\hat{t} := \sup\big\{t \le T_1 \;\big|\; u(\tau) = v(\tau) \text{ for all } \tau \le t\big\}.$$

Then we have $\hat{t} \ge 0$, and $u(\hat{t}) = v(\hat{t})$ by continuity. Suppose that $\hat{t} < T_1$. The functions $\hat{u} := u(\bullet + \hat{t})$ and $\hat{v} := v(\bullet + \hat{t})$ are mild solutions on $(-\infty, T_1 - \hat{t}]$ if $u_{\mathrm{h}}$ is replaced by $u(\bullet + \hat{t})$ and $g$ by $g(\bullet + \hat{t})$. We set

$$\hat{r}_0 := \max\Big\{\|u_{\mathrm{h}}\|_{Z(\hat{t})} + \|g\|_{L^1((\hat{t},1+\hat{t}),H^s\cap H^{\alpha,q})}, \big\|Qu_{\mathrm{h}}(\hat{t})\big\|_{H^{\alpha,q}}\Big\}.$$

Lemma 5.1 yields a time $\hat{b}_0 = b_0(\hat{r}_0) > 0$ and a mild solution $w$ of

$$\begin{aligned} w'(t) &= (A+B)w(t) + F(w_t) + g(t+\hat{t}), \quad t \ge 0, \\ w(t) &= u(t+\hat{t}), \quad t \le 0, \end{aligned}$$

on $(-\infty, \hat{b}_0]$ and it is the only one with $\|w\|_{Z(\hat{b}_0)} \le 1 + K\hat{r}_0$. There is also a time $b_1 \in (0, \hat{b}_0]$ with $\hat{t} + b_1 \le T_1$ and $\|\hat{u}\|_{Z(b_1)}, \|\hat{v}\|_{Z(b_1)} \le 1 + K\hat{r}_0$. It then follows $\hat{u} = w = \hat{v}$ on $(-\infty, b_1]$, which contradicts the definition of $\hat{t}$. □

We next show that solutions do not depend on the parameters $(s, p, q, \alpha)$ as far as the assumptions are met.

**Lemma 5.3.** *Let the conditions in Assumption 2.4 be true for* $(s, p, q, \alpha)$ *and* $(\overline{s}, \overline{p}, \overline{q}, \overline{\alpha})$. *Let* $u_{\mathrm{h}}$ *belong to* $Z^{s,p}_{\alpha,q}(0) \cap Z^{\overline{s},\overline{p}}_{\overline{\alpha},\overline{q}}(0)$ *with* $Qu_{\mathrm{h}}(0) \in H^{\alpha,q}(\mathbb{R}^3)^6 \cap H^{\overline{\alpha},\overline{q}}(\mathbb{R}^3)^6$. *Let* $u \in Z^{s,p}_{\alpha,q}(T_1)$ *and* $\overline{u} \in Z^{\overline{s},\overline{p}}_{\overline{\alpha},\overline{q}}(T_2)$ *be mild solutions of* (3.1) *on* $(-\infty, T_1]$, *resp.* $(-\infty, T_2]$. *Then* $u = \overline{u}$ *on* $(-\infty, T_3]$ *for* $T_3 = \min\{T_1, T_2\}$.

*Proof.* We proceed as in the proof of Lemma 5.2, assume $T_1 \le T_2$ and define $\hat{t}$ and $\hat{r}_0$ analogously for $u$, $\overline{u}$, $u_{\mathrm{h}}$, and $g$ in these spaces. The fixed-point argument in the proof of Lemma 5.1 also works if $Z(b)$ is replaced by $Z^{s,p}_{\alpha,q}(b) \cap Z^{\overline{s},\overline{p}}_{\overline{\alpha},\overline{q}}(b)$ and $H^{\alpha,q}(\mathbb{R}^3)^6$ by $H^{\alpha,q}(\mathbb{R}^3)^6 \cap H^{\overline{\alpha},\overline{q}}(\mathbb{R}^3)^6$. This yields a solution in this space for $t > \hat{t}$ which extends $u$ and $\overline{u}$, violating the definition of $\hat{t}$. □

Lemma 5.2 allows us to define the *maximal existence time*

$$t^+(u_{\mathrm{h}}, g) := \sup\big\{b > 0 \;\big|\; \exists \text{ a mild solution } u^b \in Z(b) \text{ of (3.1) on } (-\infty, b]\big\}.$$

The interval $J^+(u_{\mathrm{h}}, g) := \big(-\infty, t^+(u_{\mathrm{h}}, g)\big)$ is called the *maximal existence interval.* By means of uniqueness, setting $u(t) := u^b(t)$ for $t \le b < t^+$ we obtain a mild solution of (3.1) on $J^+(u_{\mathrm{h}}, g)$, called *maximal mild solution.*

We can now state the existence and uniqueness of a maximal mild solution and give a blow-up condition.

**Theorem 5.4.** *Let Assumption 2.4 be true. Let* $u_\mathrm{h} \in Z(0)$ *satisfy* $Qu_\mathrm{h}(0) \in H^{\alpha,q}(\mathbb{R}^3)^6$. *Then the following assertions hold.*

*(1) There exists a unique mild solution* $u$ *of* (3.1) *on* $J^+(u_\mathrm{h}, g)$ *such that* $Qu \in C\big(J^+(u_\mathrm{h}, g), H^{\alpha,q}(\mathbb{R}^3)^6\big)$.

*(2) If* $t^+(u_\mathrm{h}, g) < \infty$, *then there exists a sequence* $(t_k)$ *in* $\big(0, t^+(u_\mathrm{h}, g)\big)$ *with* $t_k \to t^+(u_\mathrm{h}, g)$ *and*

$$\|u(t_k)\|_{H^s} + \|u\|_{L^p((-\infty,t_k),H^{\alpha,q})} \longrightarrow \infty, \quad k \to \infty. \tag{5.15}$$

*Proof.* The first statement is a consequence of Lemmas 5.1 and 5.2 and the definition of the maximal existence interval.

We prove the second assertion by contradiction. Let $t^+ := t^+(u_\mathrm{h}, g) < \infty$, $u = (\mathbf{E}, \mathbf{H})$, and suppose (5.15) is false. By monotone convergence, we obtain

$$C := \sup_{t<t^+} \|u(t)\|_{H^s} + \|u\|_{L^p((-\infty,t^+),H^{\alpha,q})} < \infty.$$

Let $t < t^+$. Formula (5.4) and Lemma 2.8 yield

$$\|Qu(t)\|_{H^{\alpha,q}} \lesssim \|Qu_\mathrm{h}(0)\|_{H^{\alpha,q}} + \big(t^+\big)^{\frac{1}{p'}} \|u\|_{L^p((0,t^+),H^{\alpha,q})} + \|g\|_{L^1(J^+,H^{\alpha,q})}$$
$$+ \sum_{n=1}^{N} \Big[ \|u\|^n_{L^p(J^+,H^{\alpha,q})} + \|u\|^{n-1}_{L^p(J^+,H^{\alpha,q})} \sup_{\tau\le t^+} \|u(\tau)\|_{H^s} \Big] < \infty.$$

Let $\widetilde{C}$ be the constant on the right and $(t_k)$ be a sequence in $(0, t^+)$ with $t_k \to t^+$ as $k \to \infty$. We set $\widetilde{r}_0 := \max\{C, \widetilde{C}\}$ and define sequences $(f_k)$ and $(g_k)$ of functions by $f_k(t) := u(t + t_k)$ and $g_k(t) := g(t + t_k)$ for $t < t^+ - t_k$ and $k \in \mathbb{N}$. Then we have $\|f_k\|_{Z(0)} \le \widetilde{r}_0$ and $\|Qf_k(0)\|_{H^{\alpha,q}} \le r_0$ for all $k \in \mathbb{N}$. Lemma 5.1 provides a time $\tilde{b}_0 = b_0(\widetilde{r}_0) > 0$, independent of $k$, such that

$$v'(t) = (A + B)v(t) + F(v_t) + g_k(t), \quad t \ge 0, \qquad v(t) = f_k(t), \quad t \le 0,$$

has a mild solution $v_k$ on $(-\infty, \tilde{b}_0]$ for all $k \in \mathbb{N}$. We now pick $k \in \mathbb{N}$ with $t_k + \tilde{b}_0 > t^+$ and obtain a mild solution of (3.1) on the interval $(-\infty, t_k + \tilde{b}_0]$, contradicting the definition of $t^+$. □

*Remark* 5.5. Observe that the coefficients are real-valued and that $Q$ and the fractional derivatives appearing in the Strichartz estimate leave invariant real-valued functions. Let the data $u_\mathrm{h}$ and $g$ be real. Then Lemma 5.1 can also be shown in spaces of real-valued functions, and the solution is real.

The next result provides continuous dependence on the initial data $u_\mathrm{h}$ and the inhomogenity $g$.

**Theorem 5.6.** *Let Assumption 2.4 be true,* $u_\mathrm{h} \in Z(0)$, *and satisfy* $Qu_\mathrm{h}(0) \in H^{\alpha,q}(\mathbb{R}^3)^6$. *Let* $u$ *be the maximal mild solution of* (3.1) *on* $\big(-\infty, t^+(u_\mathrm{h}, g)\big)$ *and let* $b \in \big(0, t^+(u_\mathrm{h}, g)\big)$. *Then there exist constants* $\delta = \delta(b, u_\mathrm{h}, g) > 0$ *and* $C = C(b, u_\mathrm{h}, g) > 0$ *such that for* $v_\mathrm{h}, w_\mathrm{h} \in Z(0)$ *with* $Qv_\mathrm{h}(0), Qw_\mathrm{h}(0) \in H^{\alpha,q}(\mathbb{R}^3)^6$ *and* $d, e \in L^1_{\mathrm{loc}}\big([0,\infty), H^s(\mathbb{R}^3)^6 \cap H^{\alpha,q}(\mathbb{R}^3)^6\big)$ *satisfying*

$$\|u_\mathrm{h} - v_\mathrm{h}\|_{Z(0)} + \|Q(u_\mathrm{h}(0) - v_\mathrm{h}(0))\|_{H^{\alpha,q}} + \|g - d\|_{L^1((0,b),H^s\cap H^{\alpha,q})} \le \delta,$$
$$\|u_\mathrm{h} - w_\mathrm{h}\|_{Z(0)} + \|Q(u_\mathrm{h}(0) - w_\mathrm{h}(0))\|_{H^{\alpha,q}} + \|g - e\|_{L^1((0,b),H^s\cap H^{\alpha,q})} \le \delta,$$

*we have* $\min\{t^+(v_{\rm h},d),t^+(w_{\rm h},e)\}>b$ *and*

$$\|v-w\|_{Z(b)}+\sup_{0\le t\le b}\|Q\,(v(t)-w(t))\|_{H^{\alpha,q}}$$
$$\le C\big(\|v_{\rm h}-w_{\rm h}\|_{Z(0)}+\|Q\,(v_{\rm h}(0)-w_{\rm h}(0))\|_{H^{\alpha,q}}+\|d-e\|_{L^1((0,b),H^s\cap H^{\alpha,q})}\big),$$

*where* $v$ *and* $w$ *are the mild solutions of* (3.1) *for* $v_{\rm h}$ *and* $d$*, resp.* $w_{\rm h}$ *and* $e$*.*

*Proof.* We set $b'=\max\{b,1\}$, $M_b:=\sup_{0\le t\le b'}\|S(t)\|_{\mathcal{B}(H^s)}$ and

$$\overline{r}_0:=1+\|u\|_{Z(b)}+\|g\|_{L^1((0,b'),H^s\cap H^{\alpha,q})}+\sup_{0\le t\le b}\|Qu(t)\|_{H^{\alpha,q}}.$$

1) Let $\delta_1\in(0,1)$ and $v_{\rm h}$, $w_{\rm h}$, $d$, and $e$ be as in the claim, with $\delta$ replaced by $\delta_1$. In particular, we have the estimates

$$\overline{r}_0\ge\max\big\{\|v_{\rm h}\|_{Z(0)}+\|d\|_{L^1((0,b),H^s\cap H^{\alpha,q})},\|w_{\rm h}\|_{Z(0)}+\|e\|_{L^1((0,b),H^s\cap H^{\alpha,q})}\big\},$$
$$\overline{r}_0\ge\max\{\|Qv_{\rm h}(0)\|_{H^{\alpha,q}},\|Qw_{\rm h}(0)\|_{H^{\alpha,q}}\}.$$

Lemma 5.1 thus yields a time $\bar{b}_0:=b_0(\overline{r}_0)\in(0,1]$ smaller than $b$, $t^+(v_{\rm h},d)$ and $t^+(w_{\rm h},e)$ such that $\|v\|_{Z(\bar{b}_0)},\|w\|_{Z(\bar{b}_0)}\le 1+K\overline{r}_0=:\overline{r}$ and $v=\Phi_{v_{\rm h},d}(v)$ and $w=\Phi_{w_{\rm h},e}(w)$ on $[0,\bar{b}_0]$ with the fixed-point map from (5.6). Let $0\le\theta\le t\le\bar{b}_0$. Observe that

$$v(\theta)-w(\theta)=\Phi_{v_{\rm h},d}(v)(\theta)-\Phi_{v_{\rm h},d}(w)(\theta)+\Phi_{v_{\rm h},d}(w)(\theta)-\Phi_{w_{\rm h},e}(w)(\theta).\quad(5.16)$$

The first difference on the right is equal to

$$\int_0^\theta S(\theta-\tau)\big(B(v(\tau)-w(\tau))+F(v_\tau)-F(w_\tau)\big)\,\mathrm{d}\tau.\quad(5.17)$$

We write $E_s(\tau)=L^1((0,\tau),H^s)$ and $E_{s,\alpha}(\tau)=L^1((0,\tau),H^s\cap H^{\alpha,q})$. Since $t\le\bar{b}_0\le 1$, Lemmas 2.2 and 2.12 then imply

$$\begin{aligned}&\big\|\Phi_{v_{\rm h},d}(v)(\theta)-\Phi_{v_{\rm h},d}(w)(\theta)\big\|_{H^s} &(5.18)\\ &\qquad\le M_b\big(C_1\|B\|_{C_b^2}\,t\sup_{0\le\theta\le t}\|v(\theta)-w(\theta)\|_{H^s}+\big\|F(v_\bullet)-F(w_\bullet)\big\|_{E_s(t)}\big)\\ &\qquad\le C_{1,b}t^{\frac{1}{p'}}(1+\overline{r}^{N-1})\,\|v-w\|_{Z(t)}\end{aligned}$$

with a constant $C_{1,b}>0$. The term $\Phi_{v_{\rm h},d}(w)(\theta)-\Phi_{w_{\rm h},e}(w)(\theta)$ is written as

$$D_2(\theta)=S(\theta)\big(v_{\rm h}(0)-w_{\rm h}(0)\big)+\int_0^\theta S(\theta-\tau)\big(d(\tau)-e(\tau)\big)\mathrm{d}\tau,\quad(5.19)$$

and can be estimated by

$$\|D_2(\theta)\|_{H^s}\le M_b\big(\|v_{\rm h}(0)-w_{\rm h}(0)\|_{H^s}+\|d-e\|_{E_s(\theta))}\big).\quad(5.20)$$

As $v(t)-w(t)=v_{\rm h}(t)-w_{\rm h}(t)$ for $t\le 0$, inequalities (5.18) and (5.20) lead to

$$\begin{aligned}&\sup_{\theta\le t}\|v(\theta)-w(\theta)\|_{H^s} &(5.21)\\ &\qquad\le C_{1,b}t^{\frac{1}{p'}}(1+\overline{r}^{N-1})\,\|v-w\|_{Z(t)}+M_b\big(\|v_{\rm h}-w_{\rm h}\|_{Z(0)}+\|d-e\|_{E_s(t)}\big).\end{aligned}$$

We turn to the estimate for $v-w$ in $L^p\big((0,t),H^{\alpha,q}(\mathbb{R}^3)^6\big)$. Like in (5.13) and (5.14), one controls the term (5.17) via

$$\|\Phi_{v_{\rm h},d}(v)-\Phi_{v_{\rm h},d}(w)\|_{L^p((0,t),H^{\alpha,q})}\le C_2t^{\frac{1}{p'}}\left(1+\overline{r}^{N-1}\right)\|v-w\|_{Z(t)}\quad(5.22)$$

for a constant $C_2 > 0$. To treat (5.19), we again use the projection $Q$ and set

$$\chi(t) := QD_2(\theta) = Q\big(v_\mathrm{h}(0) - w_\mathrm{h}(0)\big) + \int_0^t Q\big(d(\tau) - e(\tau)\big)\,\mathrm{d}\tau,$$
$$\psi(t) := \widetilde{Q}D_2(\theta) = \widetilde{Q}S(t)\big(v_\mathrm{h}(0) - w_\mathrm{h}(0)\big) + \int_0^t \widetilde{Q}S(t-\tau)\big(d(\tau) - e(\tau)\big)\,\mathrm{d}\tau.$$

The first term is bounded by

$$\|\chi\|_{L^p((0,t),H^{\alpha,q})} \le \big\|Q\big(v_\mathrm{h}(0) - w_\mathrm{h}(0)\big)\big\|_{H^{\alpha,q}} + \|Q\|\ \|d-e\|_{L^1((0,t),H^s\cap H^{\alpha,q})}\,.$$

Since $\psi$ solves

$$\psi'(t) = A\psi(t) + d(t) - e(t),\ t \ge 0, \quad \psi(0) = \widetilde{Q}\big(v_\mathrm{h}(0) - w_\mathrm{h}(0)\big),$$

we can apply the Strichartz inquality from Corollary 4.7 and infer

$$\|\psi\|_{L^p((0,t),H^{\alpha,q})} \lesssim C_\mathrm{Str}\Big(\|v_\mathrm{h} - w_\mathrm{h}\|_{Z(0)} + \|d-e\|_{L^1((0,t),H^s)}\Big)\,.$$

It follows

$$\begin{aligned}\|\Phi_{v_\mathrm{h},d}(w) - \Phi_{w_\mathrm{h},e}(w)\|_{L^p((0,t),H^{\alpha,q})} &\le \|\chi\|_{L^p((0,t),H^{\alpha,q})} + \|\psi\|_{L^p((0,t),H^{\alpha,q})}\\ &\le C_3\big(\,\|Q(v_\mathrm{h}(0) - w_\mathrm{h}(0))\|_{H^{\alpha,q}} + \|v_\mathrm{h} - w_\mathrm{h}\|_{Z(0)} + \|d-e\|_{E_{s,\alpha}(t)}\,\big) \quad (5.23)\end{aligned}$$

for a constant $C_3 > 0$. Formulas (5.16), (5.22) and (5.23) yield

$$\begin{aligned}\|v-w\|_{L^p((-\infty,t),H^{\alpha,q})} &\le C_2 t^{\frac{1}{p'}}\left(1+\overline{r}^{N-1}\right)\|v-w\|_{Z(t)} \qquad (5.24)\\ &+ (\|Q\| + C_3)\big[\,\big\|Q\big(v_\mathrm{h}(0) - w_\mathrm{h}(0)\big)\big\|_{H^{\alpha,q}} + \|v_\mathrm{h} - w_\mathrm{h}\|_{Z(0)} + \|d-e\|_{E_{s,\alpha}(t)}\,\big].\end{aligned}$$

Since $Q(v(t) - w(t))$ is given by

$$Q(v_\mathrm{h}(0) - w_\mathrm{h}(0)) + \int_0^t Q\big(B(v(\tau) - w(\tau)) + F(v_\tau) - F(w_\tau) + d(\tau) - e(\tau)\big)\mathrm{d}\tau,$$

Lemma 2.12 implies

$$\begin{aligned}\big\|Q\big(v(t) - w(t)\big)\big\|_{H^{\alpha,q}} &\le \big\|Q\big(v_\mathrm{h}(0) - w_\mathrm{h}(0)\big)\big\|_{H^{\alpha,q}}\\ &\quad + C_4\big(t^{\frac{1}{p'}}(1+\overline{r}^{N-1})\,\|v-w\|_{Z(t)} + \|d-e\|_{E_{s,\alpha}(t)}\,\big)\end{aligned}$$

for a constant $C_4 > 0$. Together with (5.21) and (5.24), we arrive at

$$\begin{aligned}\|v-w\|_{Z(t)} + \sup_{0\le\tau\le t}\big\|Q\big(v(\tau) - w(\tau)\big)\big\|_{H^{\alpha,q}} &\le C_{5,b,\overline{r}_0}\,t^{\frac{1}{p'}}\,\|v-w\|_{Z(t)}\\ &+ C_{6,b}\big(\,\|v_\mathrm{h} - w_\mathrm{h}\|_{Z(0)} + \big\|Q\big(v_\mathrm{h}(0) - w_\mathrm{h}(0)\big)\big\|_{H^{\alpha,q}} + \|d-e\|_{E_{s,\alpha}(t)}\,\big)\end{aligned}$$

for constants $C_{5,b,\overline{r}_0} > 0$ and $C_{6,b} > 1$. The first term on the right can be absorbed for small $t$. For $t \in [0, b_1]$ and some $b_1 := b_1(\overline{r}_0) \in (0, \bar{b}_0]$, we obtain

$$\begin{aligned}\|v-w\|_{Z(t)} &+ \sup_{0\le\tau\le t}\big\|Q\big(v(\tau) - w(\tau)\big)\big\|_{H^{\alpha,q}} \qquad (5.25)\\ &\le 2C_{6,b}\big(\,\|v_\mathrm{h} - w_\mathrm{h}\|_{Z(0)} + \big\|Q\big(v_\mathrm{h}(0) - w_\mathrm{h}(0)\big)\big\|_{H^{\alpha,q}} + \|d-e\|_{E_{s,\alpha}(t)}\,\big).\end{aligned}$$

2) If $b_1 > b$, we have $\min\{t^+(v_\mathrm{h}), t^+(w_\mathrm{h})\} > \bar{b}_0 \ge b_1 > b$ and the proof is complete with $\delta := \delta_1 \in (0,1)$ and $C := 2C_{6,b}$. If $b_1 \le b$ we restrict $\delta_1$ to be smaller than $(2C_{6,b})^{-1}$. Using the special case $w_\mathrm{h} = u_\mathrm{h}$ and $e = g$ in (5.25), we obtain $\|v\|_{Z(b_1)} \le \overline{r}_0$ as well as $\|Qv(b_1)\|_{H^{\alpha,q}} \le \overline{r}_0$. Analogously, it holds $\|w\|_{Z(b_1)} \le \overline{r}_0$ and $\|Qw(b_1)\|_{H^{\alpha,q}} \le \overline{r}_0$. Thus we can iterate the above

procedure. Here it is crucial that the step size $b_1$ depends only on $\overline{r}_0$ and $b$. Applying Lemma 5.1 to the shifted data $\widetilde{v}_{\rm h} := v(\cdot + b_1)$, $\widetilde{w}_{\rm h} := w(\cdot + b_1)$, $\widetilde{d} := d(\cdot + b_1)$ and $\widetilde{e} := e(\cdot + b_1)$, we find mild solutions $\widetilde{v} = \Phi_{\widetilde{v}_{\rm h},\widetilde{d}}(\widetilde{v})$ and $\widetilde{w} = \Phi_{\widetilde{w}_{\rm h},\widetilde{e}}(\widetilde{w})$ on $[0, \bar{b}_0]$ satisfying $\max\{\|\widetilde{v}\|_{Z(\bar{b}_0)}, \|\widetilde{w}\|_{Z(\bar{b}_0)}\} \le \overline{r}$. The same estimates as above then yield

$$\begin{aligned}\|\widetilde{v} - \widetilde{w}\|_{Z(t)} &+ \sup_{0\le\tau\le t} \big\|Q\big(\widetilde{v}(\tau) - \widetilde{w}(\tau)\big)\big\|_{H^{\alpha,q}} \\ &\le 2C_{6,b}\big( \|\widetilde{v} - \widetilde{w}\|_{Z(0)} + \big\|Q\big(\widetilde{v}_{\rm h}(0) - \widetilde{w}_{\rm h}(0)\big)\big\|_{H^{\alpha,q}} + \big\|\widetilde{d} - \widetilde{e}\big\|_{E_{s,\alpha}(t)}\big)\end{aligned}$$

for all $t \in [0, b_1]$. Shifting back, together with (5.25) we conclude

$$\begin{aligned}\|v - w\|_{Z(t)} &+ \sup_{0\le\tau\le t} \big\|Q\big(v_{\rm h}(\tau) - w_{\rm h}(\tau)\big)\big\|_{H^{\alpha,q}} \\ &\le 6C_{6,b}^2\big( \|v - w\|_{Z(0)} + \big\|Q\big(v(0) - w(0)\big)\big\|_{H^{\alpha,q}} + \|d - e\|_{E_{s,\alpha}(t)} \big)\end{aligned}$$

for all $t \in [0, 2b_1]$. If $2b_1 > b$ the proof is complete with $\delta := \delta_1 \in \big(0, (2C_{6,b})^{-1}\big)$ and $C := 6C_{6,b}^2$. Otherwise we iterate again. This procedure terminates after a finite number of steps, depending on $\overline{r}_0$. Hence, the constants $\delta$ and $C$ depend on $u_{\rm h}$, $g$, and $b$. □

Let Assumption 2.4 be true. We look at data $u_{\rm h}$ and $g$ with additional regularity, namely $u_{\rm h} \in C_b\big((-\infty, 0], H^{\tilde{s}}(\mathbb{R}^3)^6\big) \cap L^p\big((-\infty, 0), H^{\alpha,q}(\mathbb{R}^3)^6\big)$ with $Qu_{\rm h}(0) \in H^{\alpha,q}(\mathbb{R}^3)^6$ and $g \in L^1_{\rm loc}\big([0,\infty), H^{\tilde{s}}(\mathbb{R}^3)^6\big) \cap L^p_{\rm loc}\big([0,\infty), H^{\alpha,q}(\mathbb{R}^3)^6\big)$ for some $\frac{3}{2} < \tilde{s} \le 2$. Theorem 5.4 provides a maximal mild solution in the space $C\big((-\infty, t_s^+), H^s(\mathbb{R}^3)^6\big) \cap L^p(-\infty, t_s^+), H^{\alpha,q}(\mathbb{R}^3)^6\big)$. On the other hand, the Banach algebra structure of $H^{\tilde{s}}(\mathbb{R}^3)$ allows to prove local wellposedness in $C_b\big((-\infty, 0], H^{\tilde{s}}(\mathbb{R}^3)^6\big)$, without using the Strichartz estimate, cf. [2]. So we also have a maximal mild solution $\widetilde{u}$ of (3.1) in $C\big((-\infty, t_{\tilde{s}}^+), H^{\tilde{s}}(\mathbb{R}^3)^6\big)$. Here we write $t_s^+$ and $t_{\tilde{s}}^+$ to distinguish between the two maximal existence times. The next proposition shows that the two solutions coincide on the intersection of their maximal existence intervals and that these intervals are the same if $s$ is close to 1 or $\tilde{s} = 2$. For simplicity, we restrict to $\tilde{s} \in (\frac{3}{2}, 2]$. Larger values of $\tilde{s}$ can be treated as well, but require higher regularity assumptions on coefficients and on $\mathbf{J}_0$.

**Proposition 5.7.** *Let Assumption 2.4 be true, $\frac{3}{2} < \tilde{s} \le 2$ and $u_{\rm h}$ be contained in $C_b\big((-\infty, 0], H^{\tilde{s}}(\mathbb{R}^3)^6\big) \cap L^p\big((-\infty, 0), H^{\alpha,q}(\mathbb{R}^3)^6\big)$ with $Qu_{\rm h}(0) \in H^{\alpha,q}(\mathbb{R}^3)^6$. Let $\mathbf{J}_0$ also belong to $L^1_{\rm loc}\big([0,\infty), H^{\tilde{s}}(\mathbb{R}^3)^3\big)$. Then we have $t_{\tilde{s}}^+ \ge t_s^+$ and the above mentioned solutions $u$ and $\widetilde{u}$ coincide for $t < t_s^+$. If additionally $\tilde{s} \ge s + \frac{1}{2} - \frac{1}{q}$, then it holds $t_{\tilde{s}}^+ = t_s^+$ and $H^{\tilde{s}} \hookrightarrow H^{\alpha,q}$.*

*Proof.* 1) Set $\overline{\alpha} = \min\{\alpha, \tilde{s} - \frac{3}{2} + \frac{3}{q}\} > \frac{3}{q}$ and $\overline{s} := \overline{\alpha} + 1 - \frac{2}{q} \le s$. We thus have the embeddings $H^{\alpha,q}(\mathbb{R}^3) \hookrightarrow H^{\overline{\alpha},q}(\mathbb{R}^3)$, $H^s(\mathbb{R}^3) \hookrightarrow H^{\overline{s}}(\mathbb{R}^3)$ and $H^{\tilde{s}}(\mathbb{R}^3) \hookrightarrow H^{\overline{\alpha},q}(\mathbb{R}^3)$. Hence, $u_{\rm h}$ is contained in $Z_{\overline{\alpha},q}^{\overline{s},p}(0)$ with $Qu_{\rm h}(0) \in H^{\overline{\alpha},q}(\mathbb{R}^3)^6$ and for any $T < \min\{t_s^+, t_{\tilde{s}}^+\}$, both $u$ and $\widetilde{u}$ belong to $Z_{\overline{\alpha},q}^{\overline{s},p}(T)$. Therefore it holds $u(t) = \widetilde{u}(t)$ for all $t < \min\{t_s^+, t_{\tilde{s}}^+\}$ by Lemma 5.2.

2) Let $b \in (0, t_s^+)$ and assume $t_{\tilde{s}}^+ < b$. The blow-up condition in $H^{\tilde{s}}(\mathbb{R}^3)^6$ then provides a sequence $(t_k)$ in $(0, t_{\tilde{s}}^+)$ satisfying $t_k \to t_{\tilde{s}}^+$ and $\|u(t_k)\|_{H^{\tilde{s}}} \to$

$\infty$ as $k \to \infty$. We set $r := \|u\|_{Z^{s,p}_{\alpha,q}(b)} < \infty$. Lemma 2.3 also holds if $H^s(\mathbb{R}^3)$ is replaced by $H^{\tilde{s}}(\mathbb{R}^3)$. So as in Lemma 2.10, we can compute

$$\|F(u_\tau)\|_{H^{\tilde{s}}} \lesssim_r \sup_{\theta\le\tau} \|u(\theta)\|_{H^{\tilde{s}}} + \|u(\tau)\|_{H^{\alpha,q}} \int_0^\infty \phi(\theta)\, \|u(\tau-\theta)\|_{H^{\tilde{s}}} \,\mathrm{d}\theta$$

for all $\tau \in [0, t^+_{\tilde{s}})$. We set $e(t) := \sup_{\tau\le t} \|u(\tau)\|_{H^{\tilde{s}}}$ for $t \in [0, t^+_{\tilde{s}})$ and estimate Duhamel's formula by

$$\begin{aligned}\|u(t)\|_{H^{\tilde{s}}} &\lesssim \|u_{\mathrm{h}}(0)\|_{H^{\tilde{s}}} + \int_0^t \|F(u_\tau)\|_{H^{\tilde{s}}} \,\mathrm{d}\tau + \|g\|_{L^1((0,b),H^{\tilde{s}})} \\ &\lesssim_r \|u_{\mathrm{h}}(0)\|_{H^{\tilde{s}}} + \int_0^t e(\tau)\,\mathrm{d}\tau + \int_0^t \|u(\tau)\|_{H^{\alpha,q}} \int_0^\infty \phi(\theta)\, \|u(\tau-\theta)\|_{H^{\tilde{s}}} \,\mathrm{d}\theta\,\mathrm{d}\tau \\ &\quad + \|g\|_{L^1((0,b),H^{\tilde{s}})}.\end{aligned}$$

Observe that

$$\begin{aligned}\int_0^\infty \phi(\theta)\, \|u(\tau-\theta)\|_{H^{\tilde{s}}} \,\mathrm{d}\theta &\le \|\phi\|_{L^1((0,\infty))}\, e(0) + \|\phi\|_{L^\infty((0,\infty))} \int_0^\tau e(\theta)\,\mathrm{d}\theta \\ &\lesssim 1 + \int_0^t e(\theta)\,\mathrm{d}\theta.\end{aligned}$$

This inequality and $\int_0^t \|u(\tau)\|_{H^{\alpha,q}} \,\mathrm{d}\tau \le b^{\frac{1}{p'}} r$ lead to

$$e(t) \le c + d \int_0^t e(\tau)\,\mathrm{d}\tau$$

for all $t \in \big[0, t^+_{\tilde{s}}\big)$ with positive constants $c = c(b,r)$ and $d = d(b,r)$. Gronwall's inequality now implies $e(t_k) \le c e^{d t_k} \le c e^{d t^+_{\tilde{s}}} < \infty$ as $k \to \infty$ which contradicts the blow-up condition. So we have $t^+_{\tilde{s}} \ge b$ and since $b \in (0, t^+_s)$ is arbitrary, we conclude that $t^+_{\tilde{s}} \ge t^+_s$.

3) Let $\tilde{s} \ge s + \frac{1}{2} - \frac{1}{q}$. Assumption 2.1 then yields $\tilde{s} - \frac{3}{2} \ge \alpha - \frac{3}{q}$, so that $H^{\tilde{s}}(\mathbb{R}^3) \hookrightarrow H^{\alpha,q}(\mathbb{R}^3)$ by Sobolev's embedding. In particular, $Qu_{\mathrm{h}}(0)$ belongs to $H^{\alpha,q}(\mathbb{R}^3)^6$. Let $b \in \big(0, t^+_{\tilde{s}}\big)$ and suppose $t^+_s < b$. Theorem 5.4 provides a sequence $(t_k)$ in $(0, t^+_s)$ satisfying $t_k \to t^+_s$ and $w_k := \|u(t_k)\|_{H^s} + \|u\|_{L^p((-\infty,t_k),H^{\alpha,q})} \to \infty$ as $k \to \infty$. But we can estimate $w_k$ by

$$C \sup_{\tau\le b} \|u(\tau)\|_{H^{\tilde{s}}} + C b^{\frac{1}{p}} \sup_{0\le\tau\le b} \|u(\tau)\|_{H^{\tilde{s}}} < \infty$$

for all $k \in \mathbb{N}$, where $C > 0$ is a constant independent of $k$. Therefore we have $t^+_s \ge b$ and since $b \in (0, t^+_{\tilde{s}})$ is arbitrary, the claim follows. □

## References


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