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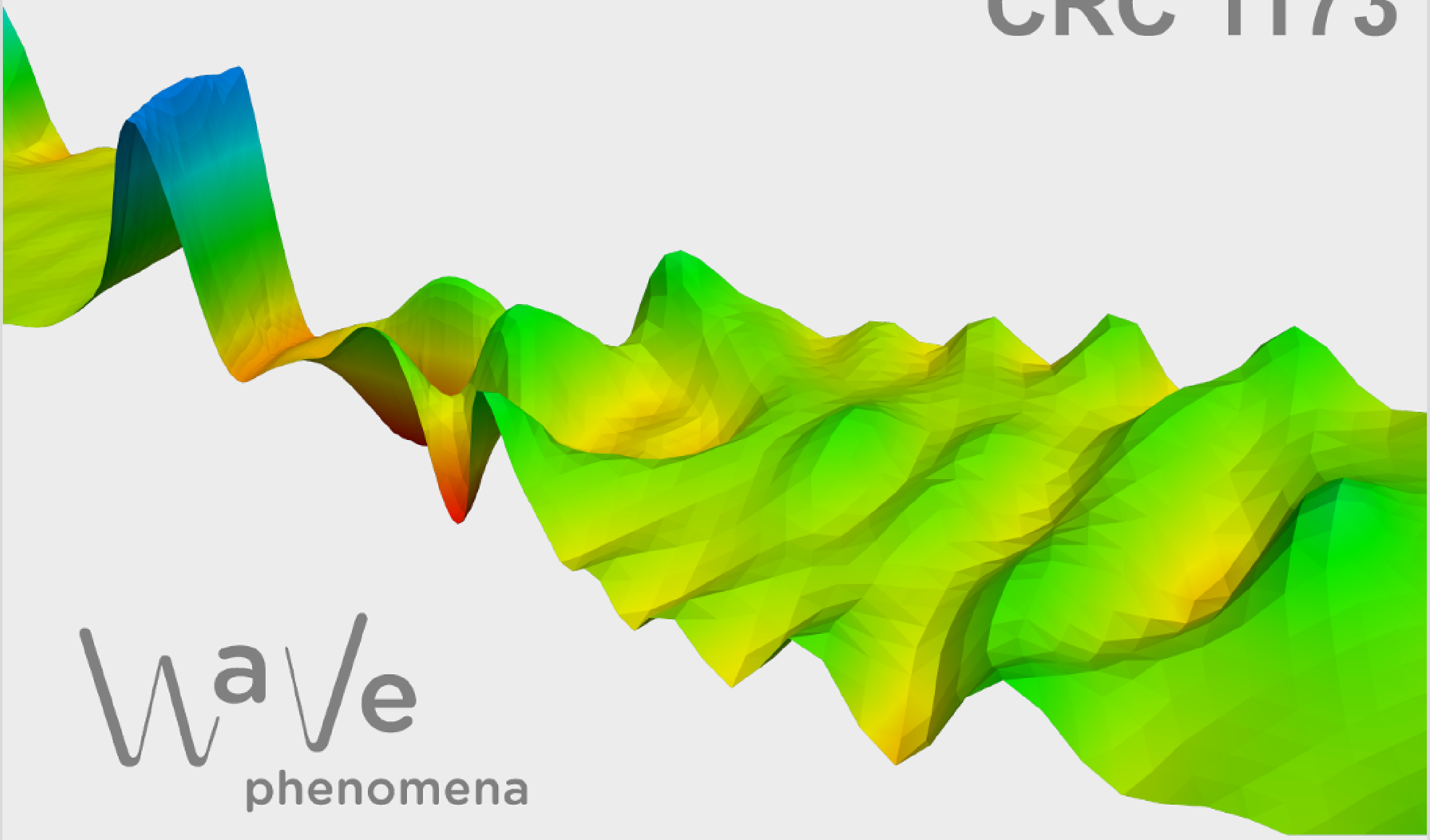

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RADIAL SYMMETRY OF POSITIVE SOLUTIONS TO QUASILINEAR HARDY-SOBOLEV DOUBLY CRITICAL SYSTEMS

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Abstract. The aim of this paper is to prove radial symmetry results for positive weak solutions with finite energy to the following quasilinear doubly critical system

$$\begin{cases} -\Delta_p u &= \gamma \dfrac{u^{p-1}}{|x|^p} + u^{p^*-1} + \nu\alpha u^{\alpha-1}v^\beta \quad \text{in} \quad \mathbb{R}^n \\ -\Delta_p v &= \gamma \dfrac{v^{p-1}}{|x|^p} + v^{p^*-1} + \nu\beta u^{\alpha}v^{\beta-1} \quad \text{in} \quad \mathbb{R}^n, \end{cases}$$

where $1 < p < n$, $\gamma \in [0, \Lambda_{n,p})$ with $\Lambda_{n,p} = [(n-p)/p]^p$, $\alpha, \beta > 1$ such that $\alpha+\beta = p^* = np/(n-p)$ and $\nu > 0$.

1. Introduction

In this paper, we investigate qualitative properties of solutions to the following doubly critical system involving the p-Laplace operator:

$$(\mathcal{S}^*) \qquad \begin{cases} -\Delta_p u &= \gamma \dfrac{u^{p-1}}{|x|^p} + u^{p^*-1} + \nu\alpha u^{\alpha-1}v^\beta \quad \text{in} \quad \mathbb{R}^n \\ -\Delta_p v &= \gamma \dfrac{v^{p-1}}{|x|^p} + v^{p^*-1} + \nu\beta u^{\alpha}v^{\beta-1} \quad \text{in} \quad \mathbb{R}^n \\ u, v > 0 & \qquad\qquad\qquad\qquad \text{in} \quad \mathbb{R}^n \setminus \{0\} \\ u, v \in \mathcal{D}^{1,p}(\mathbb{R}^n), \end{cases}$$

where $\gamma \in [0, \Lambda_{n,p})$ with $\Lambda_{n,p} = [(n-p)/p]^p$ representing the optimal constant in Hardy's inequality for $n > p$. We note that this system is well posed once one assumes that both $\alpha, \beta > 1$ are real parameters that satisfy

$$\alpha + \beta = p^*.$$

Here, $p^* = np/(n-p)$ denotes the critical Sobolev exponent, $\nu > 0$ is the coupling parameter. By the previous homogeneity condition we deduce that $p^* > 2$. Thus, here and in all the paper we will assume that

$$\frac{2n}{n+2} < p < n.$$

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We point out that $\mathcal{D}^{1,p}(\mathbb{R}^n)$ is the completion of $\mathcal{C}_c^\infty(\mathbb{R}^n)$, the space of smooth functions with compact support, with respect to the norm

$$\|u\| := \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx\right)^{\frac{1}{p}}.$$

It is well known, by standard regularity theory (see e.g. [19, 41]), it follows that solutions to ($\mathcal{S}^*$) are of class $\mathcal{C}^{1,\alpha}$ far from the origin.

Our main objective is to prove radial symmetry of solutions to system ($\mathcal{S}^*$). When $\gamma = \nu = 0$, the system reduces to the standard p-Sobolev critical equation

$$\begin{cases} -\Delta_p u = u^{p^*-1} & \text{in} \quad \mathbb{R}^n \\ u > 0 & \text{in} \quad \mathbb{R}^n. \end{cases} \tag{1.1}$$

It is well-known that the Aubin-Talenti bubbles, defined as

$$\mathcal{V}_{\lambda,x_0}(x) := \left[\frac{\lambda^{\frac{1}{p-1}} \sqrt[p]{n\left(\frac{n-p}{p-1}\right)^{p-1}}}{\lambda^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}}}\right]^{\frac{n-p}{p}} \tag{1.2}$$

is a family of positive radial solutions to (1.1), with $\lambda > 0$, and $x_0 \in \mathbb{R}^n$. These functions achieve equality in the Sobolev inequality in $\mathbb{R}^n$. The situation in case $p = 2$ is fully understood. In fact, in [27], it has been shown that any positive solution u to (1.1) that satisfies the decay assumption $u(x) = \mathcal{O}(1/|x|^m)$ at infinity for $m > 0$ is radially symmetric and decreasing around some point $x_0 \in \mathbb{R}^n$, i.e. $u(x) = \mathcal{V}_{\lambda,x_0}(x)$ (with $p = 2$). The authors used a refined version of the moving planes technique taking into account the lack of compactness in the whole space $\mathbb{R}^n$, developed by themselves in bounded domains in [26]. The problem was completely solved in the seminal paper by Caffarelli, Gidas and Spruck [9], where the authors classified all the solutions to (1.1) with $p = 2$ and without an apriori finite energy assumption. Exploiting the Kelvin transformation, they showed that the moving planes procedure can start. Finally, they proved that any solution $u \in H^1_{loc}(\mathbb{R}^n)$ to (1.1) is of the form (1.2), hence solutions are unique up to rescaling and translations.

The situation in the case $p \neq 2$ is much more involved, since the approach with Kelvin transformation is not applicable for this equation. Under the finite energy assumption, i.e. $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, the classification of all the positive solutions to (1.1) has been in fact an open and challenging problem recently solved in a series of papers by Damascelli, Merchán, Montoro and Sciunzi [16] ($2n/(n+2) < p < 2$), Vétois [43] ($1 < p < 2$) and Sciunzi [36] ($2 < p < n$). In particular, in these articles it was proved that any positive solution to (1.1) that belongs to $\mathcal{D}^{1,p}(\mathbb{R}^n)$ is of the form (1.2). More recently, Ciraolo, Figalli and Roncoroni [12] used a strategy based on integral estimates to extend the classification of positive $\mathcal{D}^{1,p}$-solutions to a class of anisotropic p-Laplace-type equations in convex cones. A similar approach was used in [10] to obtain new classification results for positive, weak solutions to (1.1) which are not a priori in $\mathcal{D}^{1,p}(\mathbb{R}^n)$. In particular, in [10], the authors managed to obtain the complete classification of positive, weak solutions to (1.1) in the case where $n = 2$ and $1 < p < 2$ or $n = 3$ and $3/2 < p < 2$. The method was recently improved by [34] who managed to extend this result to the case where $n \geq 3$ and $p \geq (n+1)/3$ and by Vetóis [44] for $n \geq 4$ and $p > p_n$ for some number $p_n \in (n/3, (n+1)/3)$ such that $p_n \sim n/3 + 1/n$.

Now, we focus our attention on the case $\gamma \neq 0$ and $\nu = 0$; in this setting, system ($\mathcal{S}^*$) reduces to the Hardy-Sobolev doubly critical equation

$$\begin{cases} -\Delta_p u = \gamma \dfrac{u^{p-1}}{|x|^p} + u^{p^*-1} & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n \setminus \{0\}. \end{cases} \tag{1.3}$$

Let us now turn to the case $p = 2$ so that $\Lambda_{n,2}$ is the best constant in the Hardy-Sobolev inequality for $p = 2$. The main contribution in the semilinear case is due to Terracini in [40], where the author, by means of variational arguments and the concentration compactness principle, showed the existence of solutions to a more general version of the equation (1.3). Making use of the Kelvin transformation and of the moving plane technique, she was able to prove that any positive solution with finite energy to (1.3) is radially symmetric about the origin. Finally, thanks to a detailed ODE's analysis, she gave a complete classification of the solutions to (1.3), given by

$$\mathcal{U}_\lambda(x) = \lambda^{\frac{2-n}{2}} \mathcal{U}\left(\frac{x}{\lambda}\right) \qquad \text{with} \qquad \mathcal{U}(x) = \frac{A(n,\gamma)}{|x|^\mu \left(1 + |x|^{2-\frac{4\mu}{n-2}}\right)^{\frac{n-2}{2}}}, \tag{1.4}$$

where

$$\mu := \frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 - \gamma}, \qquad A(n,\gamma) := \left(\frac{n(n-2-2\mu)^2}{n-2}\right)^{\frac{n-2}{4}},$$

and $\lambda > 0$ is a scaling factor. Obviously, $\mathcal{U}_\lambda = \mathcal{V}_{\lambda,0}$ if $\gamma = 0$.

The case of p-Lapalce operator was firstly treated in [1], where the authors studied the existence of solutions to (1.3) and a very fine ODE's analysis was carried out. The radial symmetry of the solutions there was an assumption. In [33] the authors proved that each solution to (1.3) with finite energy is radially symmetric (and radially decreasing) about the origin. Having in force the radial symmetry of the solution it is easy to derive the ordinary differential equation associated to the PDE and fulfilled by the solution $u = u(r)$ and, hence, to apply the results in [1]. In particular, they proved also uniqueness up to scaling of the *radial* solutions as well as the asymptotic behavior are proved showing in particular that, a *radial* solution (1.3), satisfy the following properties:

$$\begin{aligned} &\mathcal{U}_{p,\lambda}(x) = \lambda^{\frac{p-n}{p}} \mathcal{U}_p\left(\frac{x}{\lambda}\right) \qquad \text{with} \quad \mathcal{U}_p(x) = u(r) \\ &\lim_{r\to 0^+} r^{\mu_1}\mathcal{U}_p(r) = C_1, \qquad \lim_{r\to+\infty} r^{\mu_2}\mathcal{U}_p(r) = C_2, \\ &\lim_{r\to 0^+} r^{\mu_1+1}|\mathcal{U}_p'(r)| = C_1\mu_1, \qquad \lim_{r\to+\infty} r^{\mu_2+1}|\mathcal{U}_p'(r)| = C_2\mu_2, \end{aligned}$$

for some positive constants C_1, C_2. Obviously, $\mathcal{U}_{p,\lambda} = \mathcal{U}_\lambda$ and $\mathcal{U}_p = \mathcal{U}$ if $p = 2$. Here and in the sequel $\mu_1, \mu_2 \in [0, +\infty)$, $\mu_1 < \mu_2$ are two roots of the equation

$$\mu^{p-2}\left[(p-1)\mu^2 - (n-p)\mu\right] + \gamma = 0.$$

We remark (for later use) that

$$0 \leqslant \mu_1 < \frac{n-p}{p} < \mu_2 \leqslant \frac{n-p}{p-1}.$$

Note that when $p = 2$ then $\mu_1 = \tau$ and $\mu_2 = n - 2 - \tau$. Instead, when $p \neq 2$ but $\gamma = 0$ then $\mu_1 = 0$ and $\mu_2 = \dfrac{n-p}{p-1}$.

We now turn our attention to the study of ($\mathcal{S}^*$). Nonlinear Schrödinger problems, particularly those of Gross–Pitaevskii type, seems to have strong connections to various physical phenomena. These kind of systems appear naturally in the Hartree–Fock theory for double condensates specifically in binary mixtures of Bose Einstein condensates occupying different hyperfine states while spatially overlapping (see [24] and [25] for comprehensive details). The solitary-wave solutions to coupled Gross–Pitaevski equations satisfy the following system:

$$\begin{cases} -\Delta u + V(x)u = \mu u^{2q-1} + \nu u^{q-1}v^q & \text{in } \mathbb{R}^n \\ -\Delta v + V(x)v = \mu v^{2q-1} + \nu u^q v^{q-1} & \text{in } \mathbb{R}^n, \end{cases} \tag{1.5}$$

where V represents the potential of the system and $1 < q \leq \dfrac{2^*}{2}$. This formulation is commonly referred to as the Bose–Einstein condensate system. For subcritical regimes, we refer to [4], [5], [30], [38], and [39] for existence and multiplicity results under various assumptions on V and ν.

In the critical case where $q = \dfrac{2^*}{2}$, if V is a non-zero constant, system (1.5) admits only the trivial solution $(0,0)$, that is a consequence of the Pohozaev identity. When $V = 0$, [35] demonstrated the uniqueness of ground states under specific parameter conditions for generalized systems. Conversely, [13] examined the competitive setting ($\nu < 0$), establishing the existence of infinitely many fully nontrivial solutions that are not conformally equivalent.

In [21], the authors treated the case of Hardy-type potential $V = -\dfrac{\gamma}{|x|^2}$. With this choice of V, the authors were able to study problem ($\mathcal{S}^*$) when $p = 2$. What they have proved in [21] is that any $(u,v) \in \mathcal{D}^{1,2}(\mathbb{R}^n) \times \mathcal{D}^{1,2}(\mathbb{R}^n)$ solution to ($\mathcal{S}^*$) is of synchronized type:

$$(u,v) = (c_1 \mathcal{U}_{\lambda_0}, c_2 \mathcal{U}_{\lambda_0}),$$

where $\mathcal{U}_\lambda$ is given in (1.4), $\lambda_0 > 0$ and c_1, c_2 are any positive constants satisfying the system

$$\begin{cases} c_1^{2^*-2} + \nu\alpha c_1^{\alpha-2} c_2^{\beta} = 1 \\ c_2^{2^*-2} + \nu\beta c_1^{\alpha} c_2^{\beta-2} = 1. \end{cases}$$

This result was new also in the case $\gamma = 0$. Under that assumption the explicit solutions in (1.4) reduce to those of (1.2).

Doubly critical problems have received significant attention in recent years. The pioneering work of [2] investigated general Hardy-Sobolev type systems making use of variational techniques. In cooperative regimes ($\nu > 0$), they established the existence of ground and bound states contingent on parameters α, β, dimension n, and a potential function h in the coupling term. We also refer to [11, 14] for further existence results in the critical regime.

The primary objective of this paper is to show that any positive solutions, with finite energy, to problem ($\mathcal{S}^*$) is radially symmetric and radially decreasing around the origin, in the same spirit of [21].

Thus our main result is given by the following:

Theorem 1.1. *Let $(u,v) \in \mathcal{D}^{1,p}(\mathbb{R}^n) \times \mathcal{D}^{1,p}(\mathbb{R}^n)$ be a solution to ($\mathcal{S}^*$). Then (u,v) is radially symmetric and radially decreasing around the origin.*

Remark 1.2. *This result holds (and is new) also in the case $\gamma = 0$.*

The proof of the radial symmetry of the solutions is based on a fine adaptation of the moving plane procedure for quasilinear elliptic problems. This procedure goes back to the seminal papers of Alexandrov and Serrin [3, 6, 26, 37]. The situation in the quasilinear case is much more involved, and in the same spirit of [33, 36] we first recover asymptotic estimates for the solutions and its gradient. Part of this estimate are due to [28]. The missing part is proved in this paper, and it was inspired by [23]. Regarding elliptic systems, the moving plane technique was adapted by Troy in [42], where the cooperative case is analyzed. The procedure was also applied for semilinear systems in the half-space in [18] and in the whole space by Busca and Sirakov in [8]. There are several results on the qualitative properties of solutions for cooperative elliptic systems; in particular, we refer to [20, 21] for a complete analysis in the semilinear case and to [7, 22] for the quasilinear case.

2. Preliminaries and asymptotic behavior of solutions

Notation. Generic fixed and numerical constants will be denoted by C (with subscript in some case), which may vary line by line, even within a single formula. By $|A|$ we will denote the Lebesgue measure of a measurable set A.

2.1. Preliminary results.

The aim of this section is to give some preliminary technical lemmas and theorems that will be crucial in the proof of the main result.

The first result, for which we refer to [32], summarizes some Caffarelli-Kohn-Nirenberg type inequalities.

Theorem 2.1 (Proposition 1.1 of [32])**.** *Let $r_1 \geqslant 1$, $r_2 > 0$, $\bar{\kappa}, \tilde{\kappa} \in \mathbb{R}$ such that*

$$\frac{1}{r_2} + \frac{\tilde{\kappa}}{n} = \frac{1}{r_1} + \frac{\bar{\kappa} - 1}{n},$$

and with

$$0 \leqslant \bar{\kappa} - \tilde{\kappa} \leqslant 1.$$

If $\dfrac{1}{r_2} + \dfrac{\tilde{\kappa}}{n} > 0$ then for any $u \in \mathcal{C}_c^1(\mathbb{R}^n)$ it holds

$$\left(\int_{\mathbb{R}^n} |x|^{\tilde{\kappa} r_2} |u|^{r_2} dx \right)^{\frac{1}{r_2}} \leqslant C \left(\int_{\mathbb{R}^n} |x|^{r_1 \bar{\kappa}} |\nabla u|^{r_1} dx \right)^{\frac{1}{r_1}}, \tag{2.1}$$

where C is a positive constant independent of u. If $\dfrac{1}{r_2} + \dfrac{\tilde{\kappa}}{n} < 0$, then (2.1) holds for any $u \in \mathcal{C}_c^1(\mathbb{R}^n \setminus \{0\})$.

Remark 2.2. *As pointed out in [33], the same result works in the case of u defined in exterior domains with the right decay properties.*

We point out that the natural weight associated to our problem is $|\nabla u|^{p-2}$. However, the presence of Hardy potential forces us to give an estimate from below for an expression which involves this weight. In particular, we state here a result whose proof can be found in [33].

Lemma 2.3. [33, Lemma 2.1] *Let u, v be positive and $\mathcal{C}^1$-functions in a neighbourhood of some point $x_0 \in \mathbb{R}^n$. Then it holds*

$$\begin{aligned} &|\nabla u|^{p-2} \nabla u \cdot \nabla \left(u - \frac{v^p}{u^p} u \right) + |\nabla v|^{p-2} \nabla v \cdot \nabla \left(v - \frac{u^p}{v^p} v \right) \\ &\geqslant C_p \min\{v^p, u^p\} \left(|\nabla \log u| + |\nabla \log v| \right)^{p-2} |\nabla \log u - \nabla \log v|^2, \end{aligned} \tag{2.2}$$

near x_0 for some constant C_p depending only on p.

2.2. Decay and asymptotic behavior.

This section is dedicated to decay estimates for positive weak solutions of $(\mathcal{S}^*)$ and their gradient, which we will be used in the application of the moving plane method. The moving plane procedure is strongly related to suitable comparison principles. When the domain is the whole space, considering problems with a source term involving the Hardy potential, weak comparison principles are naturally related to the use of Hardy-type inequalities that involve the classical radial weights.

Now, we state the upper and lower estimates of the solutions and the upper one of their gradient, contained in [28], in our case.

Theorem 2.4. (*Theorem* 1.1, [28]) *Let* $(u, v) \in \mathcal{D}^{1,p}(\mathbb{R}^n) \times \mathcal{D}^{1,p}(\mathbb{R}^n)$ *be a weak solution of* ($\mathcal{S}^*$) *with* $1 < p < n$, $0 \leq \gamma < \Lambda_{n,p}$, $\alpha + \beta = p^*$. *Then there exist positive constants* $0 < R_0 < 1 < R_1$ *depending on* $n, p, \gamma, \nu, \alpha, \beta$, u *and* v, *such that*

$$\frac{c_1^u}{|x|^{\mu_1}} \leq u(x) \leq \frac{C_1^u}{|x|^{\mu_1}}, \quad \frac{c_1^v}{|x|^{\mu_1}} \leq v(x) \leq \frac{C_1^v}{|x|^{\mu_1}} \qquad x \in B_{R_0} \setminus \{0\}, \tag{2.3}$$

and

$$\frac{c_2^u}{|x|^{\mu_2}} \leq u(x) \leq \frac{C_2^u}{|x|^{\mu_2}}, \quad \frac{c_2^v}{|x|^{\mu_2}} \leq v(x) \leq \frac{C_2^v}{|x|^{\mu_2}} \qquad x \in (B_{R_1})^c, \tag{2.4}$$

where μ_1, μ_2 *are the unique solutions, see Remark* 2.5, *of*

$$\mu^{p-2}[(p-1)\mu^2 - (n-p)\mu] + \gamma = 0, \tag{2.5}$$

C_1^u, C_2^u, C_1^v, C_2^v *are positive constants depending on* $n, p, \gamma, \nu, \alpha, \beta$ *and* u, c_1^u, c_1^v *are positive constants depending on* $n, p, \gamma, R_0, \mu_1, \nu, \alpha, \beta$ *and* u *or* v, *respectively,* c_2^u, c_2^v *are positive constants depending on* $n, p, \gamma, R_1, \mu_2, \nu, \alpha, \beta$ *and* u *or* v, *respectively.*

Note that, since the constants in front of Hardy's terms coincide, then the further assumptions ($\mathcal{H}_2$) and ($\mathcal{H}_4$) in [28] trivially hold.

Remark 2.5. *Note that, by* $0 \leq \gamma < \Lambda_{n,p}$, *then* (2.5) *admits two solutions* μ_1, μ_2 *such that*

$$0 \leq \mu_1 < \frac{n-p}{p} < \mu_2 \leq \frac{n-p}{p-1},$$

since $(n-p)/p$ *is the point of the (negative) minimum of the function related to* (2.5).

Theorem 2.6. (*Theorem* 1.3, [28]) *Let* $(u, v) \in \mathcal{D}^{1,p}(\mathbb{R}^n) \times \mathcal{D}^{1,p}(\mathbb{R}^n)$ *be a weak solution of* ($\mathcal{S}^*$) *with* $1 < p < n$, $0 \leq \gamma < \Lambda_{n,p}$ *and* $\alpha + \beta = p^*$. *Then there exist positive constants* $\tilde{C}^u$, $\tilde{C}^v$ *depending on* $n, p, \gamma, \nu, \alpha, \beta$ *and* u *or* v, *respectively, such that*

$$|\nabla u(x)| \leq \frac{\tilde{C}^u}{|x|^{\mu_1+1}}, \quad |\nabla v(x)| \leq \frac{\tilde{C}^v}{|x|^{\mu_1+1}} \qquad x \in B_{R_0}, \tag{2.6}$$

and

$$|\nabla u(x)| \leq \frac{\tilde{C}^u}{|x|^{\mu_2+1}}, \quad |\nabla v(x)| \leq \frac{\tilde{C}^v}{|x|^{\mu_2+1}} \qquad x \in (B_{R_1})^c, \tag{2.7}$$

where μ_1, μ_2 *are roots of* (2.5) *as in Theorem* 2.4, *and* $0 < R_0 < 1 < R_1$ *are constants depending on* $n, p, \gamma, \nu, \alpha, \beta$, u *and* v.

We end this section by stating and proving the lower estimate of the gradients of solutions of ($\mathcal{S}^*$), taking inspiration from [23].

Theorem 2.7. *Let* $(u, v) \in \mathcal{D}^{1,p}(\mathbb{R}^n) \times \mathcal{D}^{1,p}(\mathbb{R}^n)$ *be a weak solution of* ($\mathcal{S}^*$) *with* $1 < p < n$, $0 \leq \gamma < \Lambda_{n,p}$ *and* $\alpha + \beta = p^*$. *Then there exist positive constants* $\tilde{c}^u$, $\tilde{c}^v$ *depending on* $n, p, \gamma, \nu, \alpha, \beta$ *and* u *or* v, *respectively, such that*

$$|\nabla u(x)| \geq \frac{\tilde{c}^u}{|x|^{\mu_1+1}}, \quad |\nabla v(x)| \geq \frac{\tilde{c}^v}{|x|^{\mu_1+1}} \qquad x \in B_{R_2}, \tag{2.8}$$

and

$$|\nabla u(x)| \geq \frac{\tilde{C}^u}{|x|^{\mu_2+1}}, \quad |\nabla v(x)| \geq \frac{\tilde{C}^v}{|x|^{\mu_2+1}} \qquad x \in (B_{R_3})^c, \tag{2.9}$$

where μ_1, μ_2 *are roots of* (2.5) *as in Theorem* 2.4, *and* $0 < R_2 < 1 < R_3$ *are constants depending on* $n, p, \gamma, \nu, \alpha, \beta$, u *and* v.

Before giving the proof, we have to state a preliminary lemma that will be crucial in the sequel but without proving it since it is the Euclidean case of Theorem 4.1 in [23] where the Finsler setting is treated (more precisely, instead of the Euclidean norm, a more general function H° appears).

Lemma 2.8. *Let $w \in \mathcal{C}^{1,\alpha}_{loc}(\mathbb{R}^n \setminus \{0\})$ be a positive weak solution of the equation*

$$-\Delta_p w - \frac{\gamma}{|x|^p} w^{p-1} = 0 \qquad \text{in } \mathbb{R}^n \setminus \{0\},$$

where $0 \le \gamma < \Lambda_{n,p}$. Assume that there exist positive constants C_1, c_1 such that

$$\frac{c_1}{|x|^{\mu_1}} \leqslant w(x) \leqslant \frac{C_1}{|x|^{\mu_1}}, \qquad \forall x \in \mathbb{R}^n \setminus \{0\}, \tag{2.10}$$

and suppose that there exists a positive constant $\hat{C}_1$ such that

$$|\nabla w(x)| \leqslant \frac{\hat{C}_1}{|x|^{\mu_1+1}}, \qquad \forall x \in \mathbb{R}^n \setminus \{0\}, \tag{2.11}$$

then

$$w(x) = \frac{\bar{c}_1}{|x|^{\mu_1}}, \text{ with } \bar{c}_1 := \limsup_{|x|\to 0} |x|^{\mu_1} w(x),$$

On the other hand, suppose that there exist positive constants $\tilde{C}_1$ and $\tilde{c}_1$ such that

$$\frac{\tilde{c}_1}{|x|^{\mu_2}} \leqslant w(x) \leqslant \frac{\tilde{C}_1}{|x|^{\mu_2}}, \qquad \forall x \in \mathbb{R}^n \setminus \{0\},$$

and suppose that there exists a positive constant $\overline{C}_1$ such that

$$|\nabla w(x)| \leqslant \frac{\overline{C}_1}{|x|^{\mu_2+1}}, \qquad \forall x \in \mathbb{R}^n \setminus \{0\},$$

then

$$w(x) = \frac{\check{c}_1}{|x|^{\mu_2}}, \text{ with } \check{c}_1 := \limsup_{|x|\to +\infty} |x|^{\mu_2} w(x).$$

Now we are ready to prove the lower bound estimates of the gradients of solutions of ($\mathcal{S}^*$).

Proof of Theorem 2.7. We will prove (2.9) since (2.8) can be proved in a similar way. Suppose by contradiction that there exist sequences of points x_m and x'_m such that $|x_m|, |x'_m| \to +\infty$ and

$$|x_m|^{\mu_2+1} |\nabla u(x_m)| \to 0 \qquad \text{for } m \to +\infty, \tag{2.12}$$

$$|x'_m|^{\mu_2+1} |\nabla v(x'_m)| \to 0 \qquad \text{for } m \to +\infty. \tag{2.13}$$

Let $R_m := |x_m|$, $R'_m := |x'_m|$ and consider

$$w_m(x) := R_m^{\mu_2} u(R_m x), \qquad z_m(x) := (R'_m)^{\mu_2} v(R_m x).$$

For $0 < a < A$ fixed and m sufficiently large, from Theorem 2.4, relabeling the constants, we have

$$\frac{c}{A^{\mu_2}} \leqslant w_n(x) \leqslant \frac{C}{a^{\mu_2}}, \quad \frac{c}{A^{\mu_2}} \leqslant z_n(x) \leqslant \frac{C}{a^{\mu_2}} \qquad \text{in } \overline{B_A \setminus B_a}.$$

Furthermore, recalling the estimate from above of the gradients of the weak solution (u, v) in Theorem 2.6, we get

$$|\nabla w_m(x)| \leqslant \frac{\tilde{C}^{w_m}}{a^{\mu_2+1}}, \quad |\nabla z_m(x)| \leqslant \frac{\tilde{C}^{z_m}}{a^{\mu_2+1}} \quad \text{in } \overline{B_A \setminus B_a}.$$

By [29], w_m and z_m are also uniformly bounded in $\mathcal{C}^{1,\eta}(K)$, for $0 < \eta < 1$ and for any compact set $K \subset B_A \setminus B_a$. Moreover

$$w_m(x) \to w_{a,A}, \quad z_m(x) \to z_{a,A} \qquad \text{in } B_A \setminus B_a,$$

in the norm $\mathcal{C}^{1,\eta'}$, for $0<\eta'<\eta$. Moreover, since (w_m,z_m) weakly solves

$$\begin{cases} -\Delta_p w_m - \gamma\dfrac{w_m^{p-1}}{|x|^p} = R_m^{\mu_2(p-1)+p-\mu_2(p^*-1)}(w_m^{p^*-1}+\nu\alpha w_m^{\alpha-1}z_m^{\beta}) & \text{in} \quad B_A\setminus B_a \\ -\Delta_p z_m - \gamma\dfrac{z_m^{p-1}}{|x|^p} = R_m^{\mu_2(p-1)+p-\mu_2(p^*-1)}(z_m^{p^*-1}+\nu\beta w_m^{\alpha}z_m^{\beta-1}) & \text{in} \quad B_A\setminus B_a \end{cases}$$

and $w_m \rightharpoonup w_{a,A}$, $z_m \rightharpoonup z_{a,A}$ in $\mathcal{D}^{1,p}(B_A\setminus B_a)$, we deduce that

$$\begin{cases} -\Delta_p w_{a,A} - \gamma\dfrac{w_{a,A}^{p-1}}{|x|^p} = 0 & \text{in} \quad B_A\setminus B_a \\ -\Delta_p z_{a,A} - \gamma\dfrac{z_{a,A}^{p-1}}{|x|^p} = 0 & \text{in} \quad B_A\setminus B_a \end{cases}$$

Now we take $a_j=1/j$ and $A_j=j$, for $j\in\mathbb{N}$ and we construct w_{a_j,A_j}, z_{a_j,A_j} as above. For j goes to infinity, using a standard diagonal process, we construct a limiting profile (w_∞,z_∞) so that

$$\begin{cases} -\Delta_p w_\infty - \gamma\dfrac{w_\infty^{p-1}}{|x|^p} = 0 & \text{in} \quad \mathbb{R}^n\setminus\{0\} \\ -\Delta_p z_\infty - \gamma\dfrac{z_\infty^{p-1}}{|x|^p} = 0 & \text{in} \quad \mathbb{R}^n\setminus\{0\} \end{cases}$$

with $w_\infty\equiv w_{a_j,A_j}$ and $z_\infty\equiv z_{a_j,A_j}$ in $B_{A_j}\setminus B_{a_j}$.

Since both w_∞ and z_∞ satisfy the assumptions (2.10) and (2.11) of the Lemma 2.8, we get

$$w_\infty(x)=\frac{\bar c_1}{|x|^{\mu_2}}, \qquad z_\infty(x)=\frac{\bar c_2}{|x|^{\mu_2}},$$

where $\bar c_1 := \limsup\limits_{|x|\to 0}|x|^{\mu_2}w_\infty(x)$, $\bar c_2 := \limsup\limits_{|x|\to 0}|x|^{\mu_2}z_\infty(x)$.

Now we set $y_m=x_m/R_m$ and $y'_m=x'_m/R_m$, and by (2.12) and (2.13), we deduce that $|\nabla w_m(y_m)|$ and $|\nabla z_m(y'_m)|$ tend to zero as R_m tends to infinity. This fact and the uniform convergence of the gradients imply that there exist $\overline{y},\overline{y}'\in\partial B_1$ such that $y_m\to\overline{y}$, $y'_m\to\overline{y}'$

$$|\nabla w_\infty(\overline{y})|=0=|\nabla z_\infty(\overline{y}')|.$$

This is a contradiction since the functions w_∞ and z_∞ have no critical points. $\square$

3. Radial symmetry of the solutions

To prove the symmetry and monotonicity result, we need a fine version of the moving plane technique that is inspired by [33]. To exploit it we need the following notations. We will study the symmetry of the solutions in the η-direction for any $\eta\in\mathbb{S}^{n-1}$ (i.e. $|\eta|=1$). Since the problem is invariant up to rotations we fix $\eta=e_1$ and we let

$$\begin{gathered} T_\lambda=\{x\in\mathbb{R} \,:\, x_1=\lambda\}, \\ \Sigma_\lambda=\{x\in\mathbb{R} \,:\, x_1<\lambda\}, \\ x_\lambda=R_\lambda(x)=(2\lambda-x_1,x')\in\mathbb{R}\times\mathbb{R}^{n-1}, \\ u_\lambda(x)=u(x_\lambda), \\ v_\lambda(x)=v(x_\lambda). \end{gathered}$$

We are now able to prove our symmetry result.

First of all we underline that it is easy to see that (u_λ, v_λ) solves

$$(\mathcal{S}_\lambda^*) \qquad \begin{cases} -\Delta_p u_\lambda = \dfrac{\gamma}{|x_\lambda|^p} u_\lambda^{p-1} + u_\lambda^{p^*-1} + \nu\alpha u_\lambda^{\alpha-1} v_\lambda^{\beta} & \text{in} \quad \mathbb{R}^n \\ -\Delta_p v_\lambda = \dfrac{\gamma}{|x_\lambda|^p} v_\lambda^{p-1} + v_\lambda^{p^*-1} + \nu\beta u_\lambda^{\alpha} v_\lambda^{\beta-1} & \text{in} \quad \mathbb{R}^n \\ u_\lambda, v_\lambda > 0 & \text{in} \quad \mathbb{R}^n. \end{cases}$$

In what follows we set

$$\Lambda_0 := \{\lambda < 0 \, : \, u \leqslant u_t, v \leqslant v_t \text{ in } \Sigma_t, \; \forall \, t \leqslant \lambda\}.$$

If $\Lambda_0 \neq \emptyset$ we denote by $\lambda_0 := \sup \Lambda_0$.

Proof of Theorem 1.1. We prove the result studying, sometimes in different approaches, the singular case $1 < p < 2$ and the degenerate case $p > 2$ as in [33]. For $p = 2$ we refer to [21]. We divide the proof in two steps.

Step 1: $\Lambda_0 \neq \emptyset$.

To prove that $\Lambda_0 \neq \emptyset$, it is necessary to prove the existence of $\lambda < 0$ with $|\lambda|$ sufficiently large such that $u \leqslant u_t$ and $v \leqslant v_t$ in Σ_t for every $t \leqslant \lambda$.
We denote by R_0 and R_1 the radii given by (2.3) and (2.6), (2.4) and (2.7), while by R_2 and R_3 the radii given by (2.8), (2.9) for the entire proof, and we firstly observe that for $|\overline{\lambda}| > \max(R_1, R_3)$ one has, by (2.3) and (2.4), that there exists $\tilde{R}_0 := \tilde{R}_0(\overline{\lambda})$ such that $\tilde{R}_0 < R_0$, $B_{\tilde{R}_0}(0_{\overline{\lambda}}) \subset \Sigma_{\overline{\lambda}}$ and

$$\sup_{x \in B_{\tilde{R}_0}(0_{\overline{\lambda}})} u(x) < \inf_{x \in B_{\tilde{R}_0}(0_{\overline{\lambda}})} u_{\overline{\lambda}}(x) \quad \text{and} \quad \sup_{x \in B_{\tilde{R}_0}(0_{\overline{\lambda}})} v(x) < \inf_{x \in B_{\tilde{R}_0}(0_{\overline{\lambda}})} v_{\overline{\lambda}}(x).$$

Therefore, exploiting again (2.3) and (2.4), we deduce that for every $\lambda < \overline{\lambda}$ one has

$$\sup_{x \in B_{\tilde{R}_0}(0_\lambda)} u(x) \leq \inf_{x \in B_{\tilde{R}_0}(0_\lambda)} u_\lambda(x) \quad \text{and} \quad \sup_{x \in B_{\tilde{R}_0}(0_\lambda)} v(x) \leq \inf_{x \in B_{\tilde{R}_0}(0_\lambda)} v_\lambda(x),$$

which gives that $u < u_\lambda$ and $v < v_\lambda$ in $B_{\tilde{R}_0}(0_\lambda) \subset \Sigma_\lambda$ for every $\lambda \leqslant \overline{\lambda}$ and with $\tilde{R}_0$ independent of λ.

Now, let us consider $R > 0$ and a cut-off function $\varphi_R \in \mathcal{C}_0^\infty(B_{2R}(0))$ such that $0 \leqslant \varphi_R \leqslant 1$, $\varphi_R \equiv 1$ on $B_R(0)$ and $|\nabla \varphi_R| \leqslant 2/R$. We denote by $\Sigma'_\lambda = \Sigma_\lambda \setminus B_{\tilde{R}_0}(0_\lambda)$ and by $\hat{B}_\rho := B_\rho(0) \cap \Sigma'_\lambda$ for $\rho > 0$. Finally, we set

$$\begin{aligned} \xi_\lambda(x) &:= u(x) - u_\lambda(x) \quad \text{and} \quad \zeta_\lambda := v(x) - v_\lambda(x), \\ \xi_{p,\lambda}(x) &:= u^p(x) - u_\lambda^p(x) \quad \text{and} \quad \zeta_{p,\lambda}(x) := v^p(x) - v_\lambda^p(x), \end{aligned}$$

and we recall that for every $\lambda < -\tilde{R}_0$ we have

$$u, v \in L^{p^*}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda) \cap \mathcal{C}^0(\overline{\Sigma_\lambda}).$$

since $u, v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ and by Theorems 2.4 and 2.6. If $\theta > \max\{2, p\}$ and $\lambda \leqslant \overline{\lambda}$, we consider

$$\begin{aligned} \Phi_{1,\lambda} &= \varphi_R^\theta u^{1-p} \xi_{p,\lambda}^+ \chi_{\Sigma_\lambda}, \qquad & \Phi_{2,\lambda} &= \varphi_R^\theta u_\lambda^{1-p} \xi_{p,\lambda}^+ \chi_{\Sigma_\lambda}, \\ \Psi_{1,\lambda} &= \varphi_R^\theta v^{1-p} \zeta_{p,\lambda}^+ \chi_{\Sigma_\lambda}, \qquad & \Psi_{2,\lambda} &= \varphi_R^\theta v_\lambda^{1-p} \zeta_{p,\lambda}^+ \chi_{\Sigma_\lambda}, \end{aligned}$$

We remark that $\operatorname{supp}(\Phi_{j,\lambda})$, $\operatorname{supp}(\Psi_{j,\lambda}) \subset \hat{B}_{2R}$ for $j = 1, 2$. Hence, we take $\Phi_{1,\lambda}$ and $\Psi_{1,\lambda}$ as a test function in both equation of $(\mathcal{S}^*)$ respectively, while $\Phi_{2,\lambda}$ and $\Psi_{2,\lambda}$ in $(\mathcal{S}_\lambda^*)$ respectively. Then, we subtract the first equation of ($\mathcal{S}^*$) with the first of ($\mathcal{S}^*_\lambda$), and we do the same for both the second equations. Hence, one gets

$$
\begin{aligned}
&\int_{\hat{B}_{2R}} \left(|\nabla u|^{p-2}\nabla u\cdot\nabla\Phi_{1,\lambda}-|\nabla u_\lambda|^{p-2}\nabla u_\lambda\cdot\nabla\Phi_{2,\lambda}\right)\,dx\\
&=\gamma\int_{\hat{B}_{2R}}\left(\frac{1}{|x|^p}-\frac{1}{|x_\lambda|^p}\right)\varphi_R^\theta\xi_{p,\lambda}^+\,dx+\int_{\hat{B}_{2R}}(u^{p^*-p}-u_\lambda^{p^*-p})\varphi_R^\theta\xi_{p,\lambda}^+\,dx\\
&\quad+\nu\alpha\int_{\hat{B}_{2R}}(u^{\alpha-p}v^\beta-u_\lambda^{\alpha-p}v_\lambda^\beta)\varphi_R^\theta\zeta_{p,\lambda}^+\,dx,
\end{aligned}\tag{3.1}
$$

$$
\begin{aligned}
&\int_{\hat{B}_{2R}} \left(|\nabla v|^{p-2}\nabla v\cdot\nabla\Psi_{1,\lambda}-|\nabla v_\lambda|^{p-2}\nabla v_\lambda\cdot\nabla\Psi_{2,\lambda}\right)\,dx\\
&=\gamma\int_{\hat{B}_{2R}}\left(\frac{1}{|x|^p}-\frac{1}{|x_\lambda|^p}\right)\varphi_R^\theta\zeta_{p,\lambda}^+\,dx+\int_{\hat{B}_{2R}}(v^{p^*-p}-v_\lambda^{p^*-p})\varphi_R^\theta\zeta_{p,\lambda}^+\,dx\\
&\quad+\nu\beta\int_{\hat{B}_{2R}}(u^{\alpha}v^{\beta-p}-u_\lambda^{\alpha}v_\lambda^{\beta-p})\varphi_R^\theta\zeta_{p,\lambda}^+\,dx,
\end{aligned}\tag{3.2}
$$

and, since $|x|\geqslant|x_\lambda|$ in Σ_λ, one has that the first terms on the right hand side of (3.1) and (3.2) are negative. Using the very definition of the test functions, we get

$$
\begin{aligned}
\mathcal{I}_1:=&\int_{\hat{B}_{2R}}\varphi_R^\theta\left(|\nabla u|^{p-2}\nabla u\cdot\nabla(u^{1-p}\xi_{p,\lambda}^+)-|\nabla u_\lambda|^{p-2}\nabla u_\lambda\cdot\nabla(u_\lambda^{1-p}\xi_{p,\lambda}^+)\right)\,dx\\
\leqslant&-\theta\int_{\hat{B}_{2R}}\varphi_R^{\theta-1}u^{1-p}\xi_{p,\lambda}^+|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi_R\,dx\\
&+\theta\int_{\hat{B}_{2R}}\varphi_R^{\theta-1}u_\lambda^{1-p}\xi_{p,\lambda}^+|\nabla u_\lambda|^{p-2}\nabla u_\lambda\cdot\nabla\varphi_R\,dx\\
&+\int_{\hat{B}_{2R}}\left(u^{p^*-p}-u_\lambda^{p^*-p}\,dx\right)\varphi_R^\theta\xi_{p,\lambda}^+\,dx+\nu\alpha\int_{\hat{B}_{2R}}\left(u^{\alpha-p}v^\beta-u_\lambda^{\alpha-p}v_\lambda^\beta\right)\varphi_R^\theta\xi_{p,\lambda}^+\,dx\\
=:&\ \mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5.
\end{aligned}\tag{3.3}
$$

Analogously, we obtain a similar estimate on (3.2)

$$
\begin{aligned}
\mathcal{E}_1:=&\int_{\hat{B}_{2R}}\varphi_R^\theta\left(|\nabla v|^{p-2}\nabla v\cdot\nabla(v^{1-p}\zeta_{p,\lambda}^+)-|\nabla v_\lambda|^{p-2}\nabla v_\lambda\cdot\nabla(v_\lambda^{1-p}\zeta_{p,\lambda}^+)\right)\,dx\\
\leqslant&-\theta\int_{\hat{B}_{2R}}\varphi_R^{\theta-1}v^{1-p}\zeta_{p,\lambda}^+|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi_R\,dx\\
&+\theta\int_{\hat{B}_{2R}}\varphi_R^{\theta-1}v_\lambda^{1-p}\zeta_{p,\lambda}^+|\nabla v_\lambda|^{p-2}\nabla v_\lambda\cdot\nabla\varphi_R\,dx\\
&+\int_{\hat{B}_{2R}}\left(v^{p^*-p}-v_\lambda^{p^*-p}\right)\varphi_R^\theta\zeta_{p,\lambda}^+\,dx+\nu\beta\int_{\hat{B}_{2R}}\left(u^{\alpha}v^{\beta-p}-u_\lambda^{\alpha}v_\lambda^{\beta-p}\right)\varphi_R^\theta\zeta_{p,\lambda}^+\,dx\\
=:&\ \mathcal{E}_2+\mathcal{E}_3+\mathcal{E}_4+\mathcal{E}_5.
\end{aligned}\tag{3.4}
$$

In the sequel, we will focus our attention only on (3.3), since the estimates on (3.4) will be very similar; we start by estimating $\mathcal{I}_1$. By using (2.2) it yields that for $p>2$ one has

$$\begin{aligned}
\mathcal{I}_1 &= \int_{\hat{B}_{2R}} \varphi_R^\theta \left(|\nabla u|^{p-2}\nabla u\cdot\nabla(u^{1-p}\xi^+_{p,\lambda}) - |\nabla u_\lambda|^{p-2}\nabla u_\lambda\cdot\nabla(u_\lambda^{1-p}\xi^+_{p,\lambda})\right) dx\\
&= \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta \left(|\nabla u|^{p-2}\nabla u\cdot\nabla\left(u-\frac{u_\lambda^p}{u^p}u\right) + |\nabla u_\lambda|^{p-2}\nabla u_\lambda\cdot\nabla\left(u_\lambda-\frac{u^p}{u_\lambda^p}u_\lambda\right)\right) dx\\
&\geqslant C_p \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta u_\lambda^p\left(|\nabla\log u|+|\nabla\log u_\lambda|\right)^{p-2}|\nabla\log u-\nabla\log u_\lambda|^2\,dx\\
&\geqslant C_p \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta \left(\frac{u_\lambda}{u}\right)^p u^2\left(|\nabla u|+|\nabla u_\lambda|\right)^{p-2}|\nabla\log u-\nabla\log u_\lambda|^2\,dx\\
&\geqslant c_1 \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta u^2\left(|\nabla u|+|\nabla u_\lambda|\right)^{p-2}|\nabla\log u-\nabla\log u_\lambda|^2\,dx,
\end{aligned} \tag{3.5}$$

while for $1<p<2$ we obtain

$$\begin{aligned}
\mathcal{I}_1 &\geqslant C_p \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta u_\lambda^p \frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla\log u|+|\nabla\log u_\lambda|)^{2-p}}\,dx\\
&\geqslant C_p \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta u_\lambda^2 \frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}\,dx.
\end{aligned} \tag{3.6}$$

We point out that in the last line of (3.5) we used that

$$\frac{u_\lambda}{u}\geqslant\tilde{c}\quad\text{in }\Sigma_\lambda, \tag{3.7}$$

and $c_1:=C_p\tilde{c}^p$. Indeed if $x\in\Sigma_\lambda\setminus B_{R_1}(0_\lambda)$ then from (2.3) and (2.4) one has (recall that $|x|\geqslant|x_\lambda|$)

$$\frac{u_\lambda}{u}\geqslant\tilde{c}_1\frac{|x|^{\mu_2}}{|x_\lambda|^{\mu_2}}\geqslant\tilde{c}_1.$$

where $\tilde{c}_1=c_2^{u_\lambda}/C_2^{u_\lambda}$. Otherwise, if $x\in\Sigma_\lambda\cap B_{R_1}(0_\lambda)$ then

$$\frac{u_\lambda}{u}\geqslant\tilde{c}_1|\bar{\lambda}|^{\mu_2}\inf_{x\in B_{R_1}(0)}u(x)\geqslant\tilde{c}_2,$$

where we take $\tilde{c}=\min(\tilde{c}_1,\tilde{c}_2)$ and $|\bar{\lambda}|>R_1$. Now, using Schwarz inequality, (2.4) and (2.7), we deduce that

$$\begin{aligned}
\mathcal{I}_2 &= -\theta\int_{\hat{B}_{2R}} \varphi_R^{\theta-1}u^{1-p}\xi^+_{p,\lambda}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi_R\,dx\\
&\leqslant \theta\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^{\theta-1}u\left(1-\left(\frac{u_\lambda}{u}\right)^p\right)|\nabla u|^{p-1}|\nabla\varphi_R|\,dx\\
&\leqslant \frac{2\theta}{R}\int_{\hat{B}_{2R}\setminus\hat{B}_R} u|\nabla u|^{p-1}\,dx \leqslant \frac{C}{R}\int_{\hat{B}_{2R}\setminus\hat{B}_R}\frac{1}{|x|^{(\mu_2+1)(p-1)+\mu_2}}\,dx\leqslant\frac{C}{R^\kappa}
\end{aligned} \tag{3.8}$$

where

$$\kappa:=p\mu_2+p-n \tag{3.9}$$

which is strictly positive since $\mu_2 > \dfrac{n-p}{p}$. Now we want to estimate $\mathcal{I}_3$; using (2.4) and (3.7), we are able to deduce that

$$
\begin{aligned}
\mathcal{I}_3 &= \theta \int_{\hat{B}_{2R}} \varphi_R^{\theta-1} u_\lambda^{1-p} \xi_{p,\lambda}^+ |\nabla u_\lambda|^{p-2} \nabla u_\lambda \cdot \nabla \varphi_R \, dx \\
&\leqslant C \int_{\hat{B}_{2R}\setminus \hat{B}_R} \theta \varphi_R^{\theta-1} u_\lambda^{1-p} \left(u^p - u_\lambda^p\right)^+ |\nabla u_\lambda|^{p-1} |\nabla \varphi_R| \, dx \\
&\leqslant \frac{2}{R} \int_{(\hat{B}_{2R}\setminus \hat{B}_R)\cap\{u\geqslant u_\lambda\}} u_\lambda \left(\left(\frac{u}{u_\lambda}\right)^p - 1\right) |\nabla u_\lambda|^{p-1} \, dx \\
&\leqslant \frac{2}{R} \int_{(\hat{B}_{2R}\setminus \hat{B}_R)\cap\{u\geqslant u_\lambda\}} u_\lambda \left(\frac{u}{u_\lambda}\right)^p |\nabla u_\lambda|^{p-1} \, dx \leqslant \frac{C}{R} \int_{(\hat{B}_{2R}\setminus \hat{B}_R)\cap\{u\geqslant u_\lambda\}} u |\nabla u_\lambda|^{p-1} \, dx \\
&\leqslant \frac{C}{R} \left(\int_{\hat{B}_{2R}\setminus \hat{B}_R} |\nabla u_\lambda|^p \, dx \right)^{\frac{p-1}{p}} \left(\int_{\hat{B}_{2R}\setminus \hat{B}_R} u^p \, dx \right)^{\frac{1}{p}} dx \\
&\leqslant \frac{C}{R} \left(\int_{\hat{B}_{2R}\setminus \hat{B}_R} \frac{1}{|x|^{\mu_2 p}} \, dx \right)^{\frac{1}{p}} \leqslant \frac{C}{R^{\frac{\kappa}{p}}}.
\end{aligned} \tag{3.10}
$$

For the term $\mathcal{I}_4$ we first note that

$$
\begin{aligned}
\mathcal{I}_4 &= \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \left(u^{p^*-p} - u_\lambda^{p^*-p}\right) \varphi_R^\theta \xi_{p,\lambda}^+ \, dx = \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \left(\frac{u^{p^*-1}}{u^{p-1}} - \frac{u_\lambda^{p^*-1}}{u_\lambda^{p-1}}\right) \varphi_R^\theta \xi_{p,\lambda}^+ \, dx \\
&\leqslant \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \frac{1}{u^{p-1}} \left(u^{p^*-1} - u_\lambda^{p^*-1}\right) \varphi_R^\theta \xi_{p,\lambda}^+ \, dx,
\end{aligned}
$$

recalling that $p^* \geq 2$, then applying twice Mean Value Theorem and using (2.4) we deduce that

$$
\mathcal{I}_4 \leqslant c_p \int_{\hat{B}_{2R}} u^{p^*-2} \varphi_R^\theta (\xi_\lambda^+)^2 \, dx \leqslant c_p \int_{\hat{B}_{2R}} \frac{1}{|x|^{\mu_2(p^*-2)}} \varphi_R^\theta (\xi_\lambda^+)^2 \, dx. \tag{3.11}
$$

Let us consider the real function $g(t) = \log(a + t(b-a))$ where $0 < a \leq b$, then making use of Taylor's formula, we get

$$
\log b = \log a + (b-a) \int_0^1 \frac{1}{a + t(b-a)},
$$

and since $t \in [0,1]$, we easily obtain

$$
b - a = \frac{\log b - \log a}{\displaystyle\int_0^1 \frac{1}{a+t(b-a)}} \leqslant b(\log b - \log a). \tag{3.12}
$$

Choosing $b = u$ and $a = u_\lambda$ in (3.12), and thanks to (3.11) we deduce

$$
\begin{aligned}
\mathcal{I}_4 &\leqslant c_p \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \frac{1}{|x|^{\mu_2(p^*-2)}} \varphi_R^\theta (u - u_\lambda)^2 \, dx \\
&\leqslant C \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \frac{1}{|x|^{\mu_2(p^*-2)}} \varphi_R^\theta u^2 \left(\log u - \log u_\lambda\right)^2 \, dx \\
&\leqslant C \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \frac{1}{|x|^{\mu_2 p^*}} \varphi_R^\theta \left((\log u - \log u_\lambda)^+\right)^2 \, dx.
\end{aligned}
$$

Furthermore, we can rewrite the previous estimate as follows

$$
\begin{aligned}
\mathcal{I}_4 &\leqslant C \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \frac{1}{|x|^{\kappa^*-2\vartheta+2}} \left(\varphi_R^{\frac{\theta}{2}}(\log u - \log u_\lambda)^+\right)^2 dx \\
&\leqslant \frac{C}{|\lambda|^{\kappa^*}} \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} |x|^{2\vartheta-2} \left(\varphi_R^{\frac{\theta}{2}}(\log u - \log u_\lambda)^+\right)^2 dx,
\end{aligned}
\tag{3.13}
$$

where

$$
\kappa^* := \mu_2(p^*-p)-p, \qquad 2\vartheta := -[(\mu_2+1)(p-2)+2\mu_2].
$$

We note that $\kappa^*-2\vartheta+2=\mu_2 p^*$, and $\kappa^*>0$ since $\mu_2>\dfrac{n-p}{p}$. Applying Theorem 2.1 to the right hand side of (3.13), where $r_1=r_2=2$ which implies that

$$
\tilde{\kappa} := \bar{\kappa}-1 = -\frac{(\mu_2+1)p}{2}
$$

and that

$$
\frac{1}{2}+\frac{\tilde{\kappa}}{n} = \frac{n-\mu_2 p - p}{2n} < 0
$$

since $\mu_2 > \dfrac{n-p}{p}$. Hence, we obtain

$$
\mathcal{I}_4 \leqslant \frac{C}{|\lambda|^{\kappa^*}} \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} |x|^{2\vartheta} |\nabla(\varphi_R^{\frac{\theta}{2}}(\log u - \log u_\lambda)^+)|^2 \, dx,
\tag{3.14}
$$

and, in order to give an estimate of the right hand side of (3.14), we have to distinguish two cases: $p>2$ and $1<p<2$. Let us consider the first case, $p>2$ in (3.14). By using (2.9) and (3.7), we have

$$
\begin{aligned}
\mathcal{I}_4 \leqslant& \frac{C}{|\lambda|^{\kappa^*}} \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \frac{1}{|x|^{(\mu_2+1)(p-2)}} \varphi_R^{\theta} u^2 |\nabla \log u - \nabla \log u_\lambda|^2 \, dx \\
&+ \frac{C}{|\lambda|^{\kappa^*}} \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} |x|^{2\vartheta} \left(\log u - \log u_\lambda\right)^2 |\nabla \varphi_R|^2 \, dx \\
\leqslant& \frac{C}{|\lambda|^{\kappa^*}} \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^{\theta} u^2 |\nabla u|^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 \, dx + \frac{C}{|\lambda|^{\kappa^*} R^2} \int_{\hat{B}_{2R}\setminus \hat{B}_R} |x|^{2\theta} \, dx \\
\leqslant& \frac{C}{|\lambda|^{\kappa^*}} \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^{\theta} u^2 \left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2} |\nabla \log u - \nabla \log u_\lambda|^2 \, dx + \frac{C}{|\lambda|^{\kappa^*} R^{\kappa}}
\end{aligned}
\tag{3.15}
$$

where κ is defined in (3.9). The evaluation of $\mathcal{I}_5$ is a delicate issue within this argument. Note that, in particular, we could have either $\alpha-p<1$, or $\beta-p<1$, or both. In all these cases we have to take into account the estimate (3.7) which gives us

$$
\tilde{c} \leqslant \frac{u_\lambda}{u} \leqslant 1 \qquad \text{in } \Sigma_\lambda \cap \operatorname{supp}(\xi_\lambda^+).
\tag{3.16}
$$

Now we are ready to estimate the coupling term $\mathcal{I}_5$.

$$
\begin{aligned}
\mathcal{I}_5 =& \nu\alpha \int_{\hat{B}_{2R}} \left(u^{\alpha-p} v^\beta - u_\lambda^{\alpha-p} v_\lambda^\beta\right) \varphi_R^\theta \xi_{p,\lambda}^+ \, dx \\
=& \nu\alpha \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \left(u^{\alpha-p} v^\beta - u_\lambda^{\alpha-p} v_\lambda^\beta\right) \varphi_R^\theta (u^p - u_\lambda^p) \, dx \\
\leqslant& \nu\alpha \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \left(u^{\alpha-p} - u_\lambda^{\alpha-p}\right) v^\beta \varphi_R^\theta (u^p - u_\lambda^p) \, dx \\
&+ \nu\alpha \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda,\ v>v_\lambda\}} u_\lambda^{\alpha-p} \left(v^\beta - v_\lambda^\beta\right) \varphi_R^\theta (u^p - u_\lambda^p) \, dx =: \mathcal{I}_{5,1} + \mathcal{I}_{5,2},
\end{aligned}
\tag{3.17}
$$

where in the last inequality we exploited the fact that our system is cooperative and that all the integrals are computed in support of $(u-u_\lambda)^+$. We start by estimating $\mathcal{I}_{5,1}$, i.e. thanks to the Mean Value Theorem we get

$$\mathcal{I}_{5,1} \leqslant \begin{cases} 0 & \text{if} \quad \alpha-p\leqslant 0 \\ \nu\alpha p(\alpha-p)\displaystyle\int_{\hat{B}_{2R}} u_\lambda^{\alpha-p-1}u^{p-1}v^\beta(\xi_\lambda^+)^2\varphi_R^\theta\,dx & \text{if} \quad 0<\alpha-p<1 \\ \nu\alpha p\displaystyle\int_{\hat{B}_{2R}} u^{p-1}v^\beta(\xi_\lambda^+)^2\varphi_R^\theta\,dx & \text{if} \quad \alpha-p=1 \\ \nu\alpha p(\alpha-p)\displaystyle\int_{\hat{B}_{2R}} u^{\alpha-2}v^\beta(\xi_\lambda^+)^2\varphi_R^\theta\,dx & \text{if} \quad \alpha-p>1 \end{cases}$$

The case $0<\alpha-p<1$ needs of another manipulation. Using (3.16) and again $c_2^u \le |x|^{\mu_2}u \le C_2^u$ and $c_2^v \le |x|^{\mu_2}v \le C_2^v$, we get

$$\begin{aligned}\mathcal{I}_{5,1} &= \nu\alpha p(\alpha-p)\int_{\hat{B}_{2R}}\left(\frac{u}{u_\lambda}\right)^{\alpha-p-1} u^{\alpha-p-1}u^{p-1}v^\beta(\xi_\lambda^+)^2\varphi_R^\theta\,dx \\ &\leqslant \nu\alpha p(\alpha-p)(C_2^v)^\beta\int_{\hat{B}_{2R}} u^{\alpha-2}\frac{1}{|x|^{\mu_2\beta}}(\xi_\lambda^+)^2\varphi_R^\theta\,dx \\ &\leqslant \nu\alpha\check{C}\int_{\hat{B}_{2R}} u^{p^*-2}(\xi_\lambda^+)^2\varphi_R^\theta\,dx.\end{aligned}$$

Thus, we have

$$\mathcal{I}_{5,1} \leqslant \begin{cases} 0 & \text{if} \quad \alpha-p\leqslant 0 \\ \nu\alpha\check{\check{C}}\displaystyle\int_{\hat{B}_{2R}} u^{p^\star-2}(\xi_\lambda^+)^2\varphi_R^\theta\,dx & \text{if} \quad \alpha-p>0. \end{cases} \tag{3.18}$$

From now on, for $p>2$, we repeat verbatim the computation from (3.11) to (3.15) and we get

$$\begin{aligned}\mathcal{I}_{5,1} \leqslant& \nu\alpha\check{\check{C}}\int_{\hat{B}_{2R}} u^{p^\star-2}(\xi_\lambda^+)^2\varphi_R^\theta\,dx \\ \leqslant& \frac{\nu\alpha\bar{\bar{C}}}{|\lambda|^{\kappa^*}}\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta u^2\left(|\nabla u|+|\nabla u_\lambda|\right)^{p-2}|\nabla\log u-\nabla\log u_\lambda|^2\,dx + \frac{\nu\alpha\bar{\bar{C}}}{|\lambda|^{\kappa^*}R^\kappa}.\end{aligned} \tag{3.19}$$

The term $\mathcal{I}_{5,2}$ is much more standard. Thanks to the cooperativity of the system and the Mean Value Theorem we get

$$\begin{aligned}\mathcal{I}_{5,2} \leqslant& \nu\alpha\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda,\ v>v_\lambda\}} u_\lambda^{\alpha-p}\left(v^\beta-v_\lambda^\beta\right)\varphi_R^\theta(u^p-u_\lambda^p)\,dx \\ \leqslant& \nu\alpha\beta p\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda,\ v>v_\lambda\}}\left(\frac{u_\lambda}{u}\right)^{\alpha-p} u^{\alpha-1}v^{\beta-1}\zeta_\lambda^+\xi_\lambda^+\varphi_R^\theta\,dx \\ \leqslant& \nu\alpha\beta pC\int_{\hat{B}_{2R}} u^{\alpha-1}v^{\beta-1}\zeta_\lambda^+\xi_\lambda^+\varphi_R^\theta\,dx,\end{aligned} \tag{3.20}$$

where in the last inequality we also used (3.16) in the case $\alpha-p<0$. Applying first (2.4) to the right hand side of (3.20) and then Young's inequality we get

$$\begin{aligned}\mathcal{I}_{5,2} \leqslant& \nu\alpha\beta pC\int_{\hat{B}_{2R}}\frac{1}{|x|^{\mu_2(p^*-2)}}\zeta_\lambda^+\xi_\lambda^+\varphi_R^\theta\,dx, \\ \leqslant& \frac{\nu\alpha\beta pC}{2}\int_{\hat{B}_{2R}}\frac{1}{|x|^{\mu_2(p^*-2)}}(\xi_\lambda^+)^2\varphi_R^\theta\,dx + \frac{\nu\alpha\beta pC}{2}\int_{\hat{B}_{2R}}\frac{1}{|x|^{\mu_2(p^*-2)}}(\zeta_\lambda^+)^2\varphi_R^\theta\,dx.\end{aligned} \tag{3.21}$$

Now in order to estimate the right hand side of (3.21) we have to repeat verbatim the computation from (3.11) to (3.15), for $p > 2$, and thus we obtain

$$
\begin{aligned}
\mathcal{I}_{5,2} \leqslant & \frac{\nu\alpha\beta pC}{2}\int_{\hat{B}_{2R}} \frac{1}{|x|^{\mu_2(p^*-2)}}(\xi_\lambda^+)^2\varphi_R^\theta\,dx + \frac{\nu\alpha\beta pC}{2}\int_{\hat{B}_{2R}} \frac{1}{|x|^{\mu_2(p^*-2)}}(\zeta_\lambda^+)^2\varphi_R^\theta\,dx \\
\leqslant & \frac{\nu\alpha\beta pC}{|\lambda|^{\kappa^*}}\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta u^2\left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2}|\nabla \log u - \nabla \log u_\lambda|^2\,dx \\
& + \frac{\nu\alpha\beta pC}{|\lambda|^{\kappa^*}}\int_{\hat{B}_{2R}\cap\{v\geqslant v_\lambda\}} \varphi_R^\theta v^2\left(|\nabla v| + |\nabla v_\lambda|\right)^{p-2}|\nabla \log v - \nabla \log v_\lambda|^2\,dx + \frac{2\nu\alpha\beta pC}{|\lambda|^{\kappa^*}R^\kappa}.
\end{aligned}
\tag{3.22}
$$

Finally, collecting the estimates (3.5), (3.8), (3.10), (3.15), (3.17), (3.19), (3.22), and using it in (3.3), we are able to deduce that

$$
\begin{aligned}
&\left(c_1 - \frac{C + \nu\alpha(\bar{\bar{C}} + \beta pC)}{|\lambda|^{\kappa^*}}\right)\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta u^2\left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2}|\nabla \log u - \nabla \log u_\lambda|^2\,dx \\
&\leqslant \frac{\nu\alpha\beta p\hat{C}}{|\lambda|^{\kappa^*}}\int_{\hat{B}_{2R}\cap\{v\geqslant v_\lambda\}} \varphi_R^\theta v^2\left(|\nabla v| + |\nabla v_\lambda|\right)^{p-2}|\nabla \log v - \nabla \log v_\lambda|^2\,dx \\
&\quad + \frac{C}{R^\kappa} + \frac{C}{R^{\frac{\kappa}{p}}} + \frac{C}{|\lambda|^{\kappa^*}R^\kappa} + \frac{\nu\alpha\bar{\bar{C}}}{|\lambda|^{\kappa^*}R^\kappa} + \frac{2\nu\alpha\beta pC}{|\lambda|^{\kappa^*}R^\kappa}.
\end{aligned}
\tag{3.23}
$$

At this point we have to repeat the same arguments for the second equation (3.4) and hence we have to give an estimate for all the terms $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5$. Thus we get

$$
\begin{aligned}
&\left(c_2 - \frac{C + \nu\beta(\tilde{C} + \alpha pC)}{|\lambda|^{\kappa^*}}\right)\int_{\hat{B}_{2R}\cap\{v\geqslant v_\lambda\}} \varphi_R^\theta v^2\left(|\nabla v| + |\nabla v_\lambda|\right)^{p-2}|\nabla \log v - \nabla \log v_\lambda|^2\,dx \\
&\leqslant \frac{\nu\alpha\beta p\check{C}}{|\lambda|^{\kappa^*}}\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta v^2\left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2}|\nabla \log u - \nabla \log u_\lambda|^2\,dx \\
&\quad + \frac{C}{R^\kappa} + \frac{C}{R^{\frac{\kappa}{p}}} + \frac{C}{|\lambda|^{\kappa^*}R^\kappa} + \frac{\nu\beta\tilde{C}}{|\lambda|^{\kappa^*}R^\kappa} + \frac{2\nu\alpha\beta pC}{|\lambda|^{\kappa^*}R^\kappa}.
\end{aligned}
\tag{3.24}
$$

Summing the two contribution (3.23) and (3.24) we deduce

$$
\begin{aligned}
&\left(c_1 - \frac{C + \nu\alpha(\bar{\bar{C}} + \beta p(C + \check{C}))}{|\lambda|^{\kappa^*}}\right)\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta u^2\left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2}|\nabla \log u - \nabla \log u_\lambda|^2\,dx \\
&+\left(c_2 - \frac{C + \nu\beta(\tilde{C} + \alpha p(C + \hat{C}))}{|\lambda|^{\kappa^*}}\right)\int_{\hat{B}_{2R}\cap\{v\geqslant v_\lambda\}} \varphi_R^\theta v^2\left(|\nabla v| + |\nabla v_\lambda|\right)^{p-2}|\nabla \log v - \nabla \log v_\lambda|^2\,dx \\
&\leqslant \frac{C}{R^\kappa} + \frac{C}{R^{\frac{\kappa}{p}}} + \frac{C}{|\lambda|^{\kappa^*}R^\kappa} + \frac{\nu\beta\tilde{C}}{|\lambda|^{\kappa^*}R^\kappa} + \frac{2\nu\alpha\beta pC}{|\lambda|^{\kappa^*}R^\kappa}.
\end{aligned}
$$

For $|\lambda|$ sufficiently large, as R goes to $+\infty$, we deduce that

$$
\begin{aligned}
&\int_{\Sigma'_\lambda\cap\{u\geqslant u_\lambda\}} u^2\left(|\nabla u| + |\nabla u_\lambda|\right)^{p-2}|\nabla \log u - \nabla \log u_\lambda|^2\,dx \\
&\quad + \int_{\Sigma'_\lambda\cap\{v\geqslant v_\lambda\}} v^2\left(|\nabla v| + |\nabla v_\lambda|\right)^{p-2}|\nabla \log v - \nabla \log v_\lambda|^2\,dx \leqslant 0.
\end{aligned}
$$

Now we have to estimate the right hand side of (3.14), (3.18) and (3.21) in the case $1 < p < 2$. We first remark that $2\vartheta < 0$ (for $n > 2$) and, since $|x| \geqslant |x_\lambda|$, one has that $|x|^{2\vartheta} \leqslant |x_\lambda|^{2\vartheta}$. Then

$$
\begin{aligned}
\mathcal{I}_4 \leqslant & \frac{C}{|\lambda|^{\kappa^*}} \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} |x_\lambda|^{2\vartheta} \varphi_R^\theta |\nabla \log u - \nabla \log u_\lambda|^2 \, dx \\
& + \frac{C}{|\lambda|^{\kappa^*}} \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} |x|^{2\vartheta} \left(\log u - \log u_\lambda\right)^2 |\nabla \varphi_R|^2 \, dx.
\end{aligned} \tag{3.25}
$$

Let us consider $\check{R} = \max\{R_1, R_2\}$ and let $A_{\check{R},\check{R}_0} = \overline{B_{\check{R}}(0_\lambda) \setminus B_{\check{R}_0}(0_\lambda)}$. Then we get

$$
\hat{B}_{2R} = \hat{A}_{\check{R},\check{R}_0} \cup \left(\hat{B}_{2R} \setminus \hat{A}_{\check{R},\check{R}_0}\right).
$$

Thanks to Theorem 2.4 and in particular to (2.4), we deduce that

$$
\begin{aligned}
& \int_{\hat{B}_{2R}\setminus \hat{A}_{\check{R},\check{R}_0}} |x_\lambda|^{2\vartheta} \varphi_R^\theta |\nabla \log u - \nabla \log u_\lambda|^2 \, dx \\
=& C \int_{\hat{B}_{2R}\setminus \hat{A}_{\check{R},\check{R}_0}} |x_\lambda|^{(\mu_2+1)(2-p)} |x_\lambda|^{-2\mu_2} \varphi_R^\theta |\nabla \log u - \nabla \log u_\lambda|^2 \, dx \\
\leqslant & \, C \int_{\hat{B}_{2R}\setminus \hat{A}_{\check{R},\check{R}_0}} u_\lambda^2 \varphi_R^\theta \frac{|\nabla \log u - \nabla \log u_\lambda|^2}{(|\nabla u| + |\nabla u_\lambda|)^{2-p}} \, dx.
\end{aligned} \tag{3.26}
$$

In the annulus $A_{\check{R},\check{R}_0}$ it is true that $|x_\lambda| \geqslant \check{R}_0$ (thus $|x_\lambda|^{2\vartheta} \leqslant \check{R}_0^{2\vartheta}$) and, since we are far from the reflected origin 0_λ, we also obtain that $|\nabla u_\lambda|$ is bounded. Let $\mathcal{L}_u := \inf\limits_{B_{\check{R}}(0)\setminus B_{\check{R}_0}(0)} u$. Hence we get (by using (3.16) and the fact that $(|\nabla u| + |\nabla u_\lambda|)^{2-p} \leqslant C$ far from $\{0, 0_\lambda\}$)

$$
\begin{aligned}
\int_{A_{\check{R},\check{R}_0}} |x_\lambda|^{2\vartheta} \varphi_R^\theta |\nabla \log u - \nabla \log u_\lambda|^2 \, dx \leqslant & \, C \check{R}_0^{2\theta} \int_{A_{\check{R},\check{R}_0}} \varphi_R^\theta |\nabla \log u - \nabla \log u_\lambda|^2 \, dx \\
\leqslant & \, \frac{C \check{R}_0^{2\theta}}{\mathcal{L}_u^2} \int_{A_{\check{R},\check{R}_0}} u_\lambda^2 \varphi_R^\theta \frac{|\nabla \log u - \nabla \log u_\lambda|^2}{(|\nabla u| + |\nabla u_\lambda|)^{2-p}} \, dx.
\end{aligned} \tag{3.27}
$$

Combining the two estimate (3.26) and (3.27) in the first term of (3.25) and arguing as in (3.15) for the second term of (3.25) one yields to

$$
\mathcal{I}_4 \leqslant \frac{C}{|\lambda|^{\kappa^*}} \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} u_\lambda^2 \varphi_R^\theta \frac{|\nabla \log u - \nabla \log u_\lambda|^2}{(|\nabla u| + |\nabla u_\lambda|)^{2-p}} \, dx + \frac{C}{|\lambda|^{\kappa^*} R^\kappa}. \tag{3.28}
$$

Now, we go back to $\mathcal{I}_5$ and we want to estimate this term in the case $1 < p < 2$. Thanks to (3.18) and (3.21), we get

$$
\begin{aligned}
\mathcal{I}_5 = \mathcal{I}_{5,1} + \mathcal{I}_{5,2} \leqslant & \nu\alpha\check{\check{C}} \int_{\hat{B}_{2R}} u^{p^*-2} (\xi_\lambda^+)^2 \varphi_R^\theta \, dx + \frac{\nu\alpha\beta p C}{2} \int_{\hat{B}_{2R}} \frac{1}{|x|^{\mu_2(p^*-2)}} (\xi_\lambda^+)^2 \varphi_R^\theta \, dx \\
& + \frac{\nu\alpha\beta p C}{2} \int_{\hat{B}_{2R}} \frac{1}{|x|^{\mu_2(p^*-2)}} (\zeta_\lambda^+)^2 \varphi_R^\theta \, dx \\
\leqslant & K_1 \int_{\hat{B}_{2R}} \frac{1}{|x|^{\mu_2(p^*-2)}} (\xi_\lambda^+)^2 \varphi_R^\theta \, dx + K_2 \int_{\hat{B}_{2R}} \frac{1}{|x|^{\mu_2(p^*-2)}} (\zeta_\lambda^+)^2 \varphi_R^\theta \, dx.
\end{aligned}
$$

Repeating verbatim the argument used for $\mathcal{I}_4$, from (3.11) to (3.14), we get

$$
\begin{aligned}
\mathcal{I}_5 \leqslant & \frac{C}{|\lambda|^{\kappa^*}} \int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} |x|^{2\vartheta} |\nabla(\varphi_R^{\frac{\theta}{2}} (\log u - \log u_\lambda)^+)|^2 \, dx \\
& + \frac{C}{|\lambda|^{\kappa^*}} \int_{\hat{B}_{2R}\cap\{v\geqslant v_\lambda\}} |x|^{2\vartheta} |\nabla(\varphi_R^{\frac{\theta}{2}} (\log v - \log v_\lambda)^+)|^2 \, dx.
\end{aligned} \tag{3.29}
$$

Thus we can argue repeating verbatim the argument used for $\mathcal{I}_4$ in the case $1<p<2$, exactly from (3.25) to (3.28), for both the term in the right hand side of (3.29)

$$\begin{aligned}\mathcal{I}_5 \leqslant& \frac{\check{C}_1}{|\lambda|^{\kappa^*}}\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} u_\lambda^2\varphi_R^\theta \frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}\,dx\\ &+\frac{\check{C}_2}{|\lambda|^{\kappa^*}}\int_{\hat{B}_{2R}\cap\{v\geqslant v_\lambda\}} v_\lambda^2\varphi_R^\theta \frac{|\nabla\log v-\nabla\log v_\lambda|^2}{(|\nabla v|+|\nabla v_\lambda|)^{2-p}}\,dx+\frac{C}{|\lambda|^{\kappa^*}R^\kappa}.\end{aligned}\tag{3.30}$$

Hence, by collecting (3.6), (3.8), (3.10), (3.28) and (3.30) in (3.3), we get that for $1<p<2$ it holds

$$\begin{aligned}&\left(C_p-\frac{C+\check{C}_1K_1}{|\lambda|^{\kappa^*}}\right)\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} u_\lambda^2\varphi_R^\theta \frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}\\ &\leqslant \frac{\check{C}_2}{|\lambda|^{\kappa^*}}\int_{\hat{B}_{2R}\cap\{v\geqslant v_\lambda\}} v_\lambda^2\varphi_R^\theta \frac{|\nabla\log v-\nabla\log v_\lambda|^2}{(|\nabla v|+|\nabla v_\lambda|)^{2-p}}\,dx+\frac{C}{R^\kappa}+\frac{C}{R^{\frac{\kappa}{p}}}+\frac{C}{|\lambda|^{\kappa^*}R^\kappa}.\end{aligned}\tag{3.31}$$

As pointed out in the case $p>2$ we need to repeat the same arguments for the second equation (3.4) and hence we have to give an estimate for all the terms $\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3,\mathcal{E}_4,\mathcal{E}_5$. In particular, we are able to deduce that

$$\begin{aligned}&\left(C_p-\frac{C+\check{C}_2K_2}{|\lambda|^{\kappa^*}}\right)\int_{\hat{B}_{2R}\cap\{v\geqslant v_\lambda\}} \varphi_R^\theta v_\lambda^2 \frac{|\nabla\log v-\nabla\log v_\lambda|^2}{(|\nabla v|+|\nabla v_\lambda|)^{2-p}}\,dx\\ &\leqslant \frac{\check{C}_1}{|\lambda|^{\kappa^*}}\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} u_\lambda^2\varphi_R^\theta \frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}\,dx+\frac{C}{R^\kappa}+\frac{C}{R^{\frac{\kappa}{p}}}+\frac{C}{|\lambda|^{\kappa^*}R^\kappa}.\end{aligned}\tag{3.32}$$

Summing up the two contributions of (3.31) and (3.32), we obtain

$$\begin{aligned}&\left(C_p-\frac{\bar{C}_1}{|\lambda|^{\kappa^*}}\right)\int_{\hat{B}_{2R}\cap\{u\geqslant u_\lambda\}} \varphi_R^\theta u_\lambda^2 \frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}\\ &\qquad+\left(C_p-\frac{\bar{C}_2}{|\lambda|^{\kappa^*}}\right)\int_{\hat{B}_{2R}\cap\{v\geqslant v_\lambda\}} \varphi_R^\theta v_\lambda^2 \frac{|\nabla\log v-\nabla\log v_\lambda|^2}{(|\nabla v|+|\nabla v_\lambda|)^{2-p}}\\ &\qquad\leqslant \frac{\check{C}_1}{|\lambda|^{\kappa^*}}\frac{C}{R^\kappa}+\frac{C}{R^{\frac{\kappa}{p}}}+\frac{C}{|\lambda|^{\kappa^*}R^\kappa}.\end{aligned}$$

Once again we can choose $|\lambda|$ large enough so that, as R goes to $+\infty$, it yields

$$\int_{\Sigma'_\lambda\cap\{u\geqslant u_\lambda\}} u_\lambda^2\frac{|\nabla\log u-\nabla\log u_\lambda|^2}{(|\nabla u|+|\nabla u_\lambda|)^{2-p}}\,dx+\int_{\Sigma'_\lambda\cap\{v\geqslant v_\lambda\}} v_\lambda^2\frac{|\nabla\log v-\nabla\log v_\lambda|^2}{(|\nabla v|+|\nabla v_\lambda|)^{2-p}}\,dx\leqslant 0$$

Hence, in both cases, $\log u-\log u_\lambda$ and $\log v-\log v_\lambda$ are constants and, since $\log u-\log u_\lambda=0$ and $\log v-\log v_\lambda=0$ on T_λ, then $\log u-\log u_\lambda=0$ and $\log v-\log v_\lambda=0$ in $\Sigma'_\lambda\cap\{u\geqslant u_\lambda\}$. Therefore we deduce both $u\leqslant u_\lambda$ and $v\leqslant v_\lambda$ in Σ_λ. Thus we proved that the set Λ_0 is not empty and also that λ_0 exists and is finite.

Step 2: $\lambda_0=0$.
The second part of the moving plane procedure is based on a contradiction argument, i.e. assuming that $\lambda_0<0$. Arguing as in the proof of **Step 1** we will obtain a contradiction proving that, simultaneously $u\leqslant u_{\lambda_0+\varepsilon}$ and $v\leqslant v_{\lambda_0+\varepsilon}$ in $\Sigma_{\lambda_0+\varepsilon}$ for all $0<\varepsilon\leqslant\bar{\varepsilon}$ for some $\bar{\varepsilon}>0$.

A key ingredient that we will use in the rest of the proof is the well known classical *strong comparison principle*. In order to exploit this result, we notice that from **Step 1** and by continuity it holds that

$$u\leqslant u_{\lambda_0}\quad\text{and}\quad v\leqslant v_{\lambda_0}\quad\text{in}\quad\Sigma_{\lambda_0}\,.$$

Let us first define the critical sets

$$\mathcal{Z}_u := \{x \in \mathbb{R}^n \text{ s.t. } |\nabla u(x)| = 0\} \text{ and } \mathcal{Z}_v := \{x \in \mathbb{R}^n \text{ s.t. } |\nabla v(x)| = 0\}.$$

By [15, Theorem 1.4] we deduce that $u \equiv u_{\lambda_0}$ or $u < u_{\lambda_0}$ in any connected component $\mathcal{C}_u$ of $\Sigma_{\lambda_0} \setminus \mathcal{Z}_u$, and $v \equiv v_{\lambda_0}$ or $v < v_{\lambda_0}$ in any connected component $\mathcal{C}_v$ of $\Sigma_{\lambda_0} \setminus \mathcal{Z}_v$. We will frequently use the fact that $\mathcal{Z}_u$ and $\mathcal{Z}_v$ have zero Lebesgue measure [17, 22].

We first assume that $\Sigma_{\lambda_0} \setminus \mathcal{Z}_u$ and $\Sigma_{\lambda_0} \setminus \mathcal{Z}_v$ have only one connected component. We observe that the cases $u \equiv u_{\lambda_0}$ and $v \equiv v_{\lambda_0}$ are not possible since, by Theorem 2.4 and more precisely by (2.4), there exists $B_{\check{R}_0}(0_{\lambda_0})$ where $u < u_{\lambda_0}$ and $v < v_{\lambda_0}$; this fact immediately implies that $u < u_{\lambda_0}$ in $\Sigma_{\lambda_0} \setminus \mathcal{Z}_u$ and $v < v_{\lambda_0}$ in $\Sigma_{\lambda_0} \setminus \mathcal{Z}_v$.

Let us assume that there exist at least two connected components of $\Sigma_{\lambda_0} \setminus \mathcal{Z}_u$ and two connected components of $\Sigma_{\lambda_0} \setminus \mathcal{Z}_v$. Theorem 2.7 implies that $\mathcal{Z}_u$ and $\mathcal{Z}_v$ are bounded. As a consequence, we have that only one connected component of the set $\Sigma_{\lambda_0} \setminus \mathcal{Z}_u$ and one of the set $\Sigma_{\lambda_0} \setminus \mathcal{Z}_v$ can be unbounded. We refer to both these unbounded connected components as $\mathcal{C}_{1,u}$ and $\mathcal{C}_{1,v}$ and set

$$\mathcal{C}^u_\lambda := (\mathcal{C}^c_{1,u} \cap \Sigma_{\lambda_0}) \cup R_\lambda(\mathcal{C}^c_{1,u} \cap \Sigma_{\lambda_0}) \text{ and } \mathcal{C}^v_\lambda := (\mathcal{C}^c_{1,v} \cap \Sigma_{\lambda_0}) \cup R_\lambda(\mathcal{C}^c_{1,v} \cap \Sigma_{\lambda_0}).$$

If $u \equiv u_{\lambda_0}$ in $\mathcal{C}_{1,u}$ and $v \equiv v_{\lambda_0}$ in $\mathcal{C}_{1,v}$ it is easy to see that, by symmetry, $\mathcal{C}^u_\lambda$ contains at least one connected component of $\mathbb{R}^n \setminus \mathcal{Z}_u$ and $\mathcal{C}^v_\lambda$ contains at least one connected component of $\mathbb{R}^n \setminus \mathcal{Z}_v$. But this is impossible as proved in [17, Theorem 1.4], [31, Lemma 5] and [22, Proof of Theorem 1.1]. If else $u \equiv u_{\lambda_0}$ in $\mathcal{C}_{2,u}$ for some bounded component $\mathcal{C}_{2,u}$ and $v \equiv v_{\lambda_0}$ in $\mathcal{C}_{2,v}$ for some bounded component $\mathcal{C}_{2,v}$, then in this case we set

$$\mathcal{C}^u_\lambda := \mathcal{C}_{2,u} \cup R_\lambda(\mathcal{C}_{2,u}) \text{ and } \mathcal{C}^v_\lambda := \mathcal{C}_{2,v} \cup R_\lambda(\mathcal{C}_{2,v}),$$

and also in this case, by symmetry, $\mathcal{C}^u_\lambda$ would contain at least one connected component of $\mathbb{R}^n \setminus \mathcal{Z}_u$ and $\mathcal{C}^v_\lambda$ would contain at least one connected component of $\mathbb{R}^n \setminus \mathcal{Z}_v$ thus providing a contradiction. Resuming we just proved that

$$u < u_{\lambda_0} \quad \text{in} \quad \Sigma_{\lambda_0} \setminus \mathcal{Z}_u \quad \text{and} \quad v < v_{\lambda_0} \quad \text{in} \quad \Sigma_{\lambda_0} \setminus \mathcal{Z}_v\,.$$

Now, recalling that $\mathcal{Z}_u$ and $\mathcal{Z}_v$ are bounded by Theorem 2.7, we fix $\overline{R} > 0$ in such a way that

$$\mathcal{Z}_u, \mathcal{Z}_v \subset B_{\overline{R}}(0)\,,$$

and, for $\tau > 0$, we let $\mathcal{Z}^\tau_u$ be an open set containing $\mathcal{Z}_u$ such that $\mathcal{L}(\mathcal{Z}^\tau_u) < \tau$ and $\mathcal{Z}^\tau_v$ be an open set containing $\mathcal{Z}_v$ such that $\mathcal{L}(\mathcal{Z}^\tau_v) < \tau$ (that exists since $\mathcal{L}(\mathcal{Z}_u) = \mathcal{L}(\mathcal{Z}_v) = 0$). Then, for $\delta, \varepsilon, \overline{R}, \tau > 0$, we denote by

(i) $B_{\overline{R},\varepsilon} := B^c_{\overline{R}}(0) \cap \Sigma_{\lambda_0+\varepsilon}$;
(ii) $S_{\delta,\varepsilon} := \big((\Sigma_{\lambda_0+\varepsilon} \setminus \Sigma_{\lambda_0-\delta}) \cap B_{\overline{R}}(0)\big) \cup (\mathcal{Z}^\tau_u \cap \Sigma_{\lambda_0-\delta}) \cup (\mathcal{Z}^\tau_v \cap \Sigma_{\lambda_0-\delta})$;
(iii) $K_\delta := \overline{B_{\overline{R}}(0) \cap \Sigma_{\lambda_0-\delta} \cap (\mathcal{Z}^\tau_u \cup \mathcal{Z}^\tau_v)^c}$,

where $\delta \leqslant \bar{\delta}$ so that K_δ is nonempty. We remark that this construction gives

$$\Sigma_{\lambda_0+\varepsilon} = B_{\overline{R},\varepsilon} \cup S_{\delta,\varepsilon} \cup K_\delta.$$

We also remark that, since K_δ is compact, then by the uniform continuity of u, u_λ and v, v_λ, for $\bar{\varepsilon} > 0$ small enough one has that $u < u_{\lambda_0+\varepsilon}$ and $v < v_{\lambda_0+\varepsilon}$ in K_δ for every $\varepsilon \leqslant \bar{\varepsilon}$. Moreover, we point out the existence of $\tilde{R}_0$ such that $u < u_{\lambda_0+\varepsilon}$ and $v < v_{\lambda_0+\varepsilon}$ in $B_{\tilde{R}_0}(0_{\lambda_0+\varepsilon}) \subset \Sigma_{\lambda_0+\varepsilon}$ for every $\varepsilon \leqslant \bar{\varepsilon}$, and with $\tilde{R}_0$ independent of ε as done in **Step 1**.

In the sequel, for $R > \overline{R}$, we consider $\varphi_R \in \mathcal{C}^\infty_c(B_{2R}(0))$ a cut-off function with $0 \leqslant \varphi_R \leqslant 1$, $\varphi_R \equiv 1$ on $B_R(0)$ and $|\nabla \varphi_R| \leqslant \dfrac{2}{R}$.

Then, letting $\theta > \max\{2, p\}$, we consider the following test functions

$$\begin{aligned}
\Phi_{1,\lambda_0+\varepsilon} &= \varphi_R^\theta u^{1-p} \xi^+_{p,\lambda_0+\varepsilon} \chi_{\Sigma_{\lambda_0+\varepsilon}}, \qquad & \Phi_{2,\lambda_0+\varepsilon} &= \varphi_R^\theta u_{\lambda_0+\varepsilon}^{1-p} \xi^+_{p,\lambda_0+\varepsilon} \chi_{\Sigma_{\lambda_0+\varepsilon}},\\
\Psi_{1,\lambda_0+\varepsilon} &= \varphi_R^\theta v^{1-p} \zeta^+_{p,\lambda_0+\varepsilon} \chi_{\Sigma_{\lambda_0+\varepsilon}}, \qquad & \Psi_{2,\lambda_0+\varepsilon} &= \varphi_R^\theta v_{\lambda_0+\varepsilon}^{1-p} \zeta^+_{p,\lambda_0+\varepsilon} \chi_{\Sigma_{\lambda_0+\varepsilon}}.
\end{aligned}$$

Arguing similarly to **Step 1**, let us take $\Phi_{1,\lambda_0+\varepsilon}$ and $\Psi_{1,\lambda_0+\varepsilon}$ as test functions in both equation of ($\mathcal{S}^*$), and $\Phi_{2,\lambda_0+\varepsilon}$, $\Psi_{2,\lambda_0+\varepsilon}$ in ($\mathcal{S}^*_\lambda$) respectively and, reasoning as in **Step 1**, one yields to

$$\begin{aligned}
&\int_{\hat{B}_{2R}} \varphi_R^\theta \left(|\nabla u|^{p-2} \nabla u \cdot \nabla (u^{1-p} \xi^+_{p,\lambda_0+\varepsilon}) - |\nabla u_{\lambda_0+\varepsilon}|^{p-2} \nabla u_{\lambda_0+\varepsilon} \cdot \nabla (u_{\lambda_0+\varepsilon}^{1-p} \xi^+_{p,\lambda_0+\varepsilon}) \right) dx \\
&\quad \leqslant -\theta \int_{\hat{B}_{2R}} \varphi_R^{\theta-1} u^{1-p} \xi^+_{p,\lambda_0+\varepsilon} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi_R \, dx \\
&\qquad + \theta \int_{\hat{B}_{2R}} \varphi_R^{\theta-1} u_\lambda^{1-p} \xi^+_{p,\lambda_0+\varepsilon} |\nabla u_{\lambda_0+\varepsilon}|^{p-2} \nabla u_{\lambda_0+\varepsilon} \cdot \nabla \varphi_R \, dx \\
&\qquad + \int_{\hat{B}_{2R}} \left(u^{p^*-p} - u_{\lambda_0+\varepsilon}^{p^*-p} \, dx \right) \varphi_R^\theta \xi^+_{p,\lambda_0+\varepsilon} \, dx \\
&\qquad + \nu\alpha \int_{\hat{B}_{2R}} \left(u^{\alpha-p} v^\beta - u_{\lambda_0+\varepsilon}^{\alpha-p} v_{\lambda_0+\varepsilon}^\beta \right) \varphi_R^\theta \xi^+_{p,\lambda_0+\varepsilon} \, dx,
\end{aligned} \tag{3.33}$$

and to

$$\begin{aligned}
&\int_{\hat{B}_{2R}} \varphi_R^\theta \left(|\nabla v|^{p-2} \nabla v \cdot \nabla (v^{1-p} \zeta^+_{p,\lambda_0+\varepsilon}) - |\nabla v_{\lambda_0+\varepsilon}|^{p-2} \nabla v_{\lambda_0+\varepsilon} \cdot \nabla (v_{\lambda_0+\varepsilon}^{1-p} \zeta^+_{p,\lambda_0+\varepsilon}) \right) dx \\
&\quad \leqslant -\theta \int_{\hat{B}_{2R}} \varphi_R^{\theta-1} v^{1-p} \zeta^+_{p,\lambda_0+\varepsilon} |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi_R \, dx \\
&\qquad + \theta \int_{\hat{B}_{2R}} \varphi_R^{\theta-1} v_{\lambda_0+\varepsilon}^{1-p} \zeta^+_{p,\lambda_0+\varepsilon} |\nabla v_{\lambda_0+\varepsilon}|^{p-2} \nabla v_{\lambda_0+\varepsilon} \cdot \nabla \varphi_R \, dx \\
&\qquad + \int_{\hat{B}_{2R}} \left(v^{p^*-p} - v_{\lambda_0+\varepsilon}^{p^*-p} \right) \varphi_R^\theta \zeta^+_{p,\lambda_0+\varepsilon} \, dx \\
&\qquad + \nu\beta \int_{\hat{B}_{2R}} \left(u^{\alpha} v^{\beta-p} - u_{\lambda_0+\varepsilon}^{\alpha} v_{\lambda_0+\varepsilon}^{\beta-p} \right) \varphi_R^\theta \zeta^+_{p,\lambda_0+\varepsilon} \, dx.
\end{aligned} \tag{3.34}$$

We focus our attention only on (3.33). Arguing as in **Step 1**, in both cases $p > 2$ and $1 < p < 2$, we get

$$\begin{aligned}
&c_1 \int_{\hat{B}_{2R} \cap \{u \geqslant u_{\lambda_0+\varepsilon}\}} \varphi_R^\theta u^2 \left(|\nabla u| + |\nabla u_{\lambda_0+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0+\varepsilon}|^2 \, dx \\
&\quad \leqslant \int_{\hat{B}_{2R} \cap B_{\overline{R},\varepsilon}} (u^{p^*-p} - u_{\lambda_0+\varepsilon}^{p^*-p}) \varphi_R^\theta \xi^+_{p,\lambda_0+\varepsilon} \, dx + \int_{\hat{B}_{2R} \cap S_{\delta,\varepsilon}} (u^{p^*-p} - u_{\lambda_0+\varepsilon}^{p^*-p}) \varphi_R^\theta \xi^+_{p,\lambda_0+\varepsilon} \, dx \\
&\qquad + \nu\alpha \int_{\hat{B}_{2R} \cap B_{\overline{R},\varepsilon}} \left(u^{\alpha-p} v^\beta - u_{\lambda_0+\varepsilon}^{\alpha-p} v_{\lambda_0+\varepsilon}^\beta \right) \varphi_R^\theta \xi^+_{p,\lambda_0+\varepsilon} \, dx \\
&\qquad + \nu\alpha \int_{\hat{B}_{2R} \cap S_{\delta,\varepsilon}} \left(u^{\alpha-p} v^\beta - u_{\lambda_0+\varepsilon}^{\alpha-p} v_{\lambda_0+\varepsilon}^\beta \right) \varphi_R^\theta \xi^+_{p,\lambda_0+\varepsilon} \, dx + \frac{C}{R^{\frac{\kappa}{p}}} + \frac{C}{R^\kappa}.
\end{aligned} \tag{3.35}$$

We point out that we have used once again the fact that $\dfrac{u_{\lambda_0+\varepsilon}}{u} \geqslant \tilde{c}$ for every $0 \leqslant \varepsilon \leqslant \bar{\varepsilon}$ as to deduce (3.7).

In order to estimate the first term on the right hand side of (3.35), we argue exactly in the same way of estimating $\mathcal{I}_4$ in **Step 1** (taking into account Remark 2.2), where here $\overline{R}$ plays the role of λ in **Step 1**. Hence we get

$$\int_{\hat{B}_{2R}\cap B_{\overline{R},\varepsilon}} (u^{p^*-p} - u_{\lambda_0+\varepsilon}^{p^*-p})\varphi_R^\theta \xi_{p,\lambda_0+\varepsilon}^+ \, dx \leqslant \frac{C}{R^\kappa \bar{R}^{\kappa^*}}$$
$$+ \frac{C}{\overline{R}^{\kappa^*}} \int_{\hat{B}_{2R}\cap B_{\overline{R},\varepsilon}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}} \varphi_R^\theta u^2 \left(|\nabla u| + |\nabla u_{\lambda_0+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0+\varepsilon}|^2 \, dx.$$

For the second term on the right hand side of (3.35) we argue as in **Step 1**, obtaining

$$\int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}} (u^{p^*-p} - u_{\lambda_0+\varepsilon}^{p^*-p})\varphi_R^\theta \xi_{p,\lambda_0+\varepsilon}^+ \, dx \leqslant C_u \int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}} (\log u - \log u_{\lambda_0+\varepsilon})^2 \, dx, \tag{3.36}$$

where

$$C_u = \sup_{S_{\bar{\delta},\bar{\varepsilon}}} u^{p^*-2}.$$

Now we need to split the estimate in two cases; if $p > 2$ we apply a suitable weighted Poincaré inequality to the right hand side of (3.36) which can be found in [17, Theorem 3.2]. Hence in this case one has

$$\begin{aligned} &\int_{\hat{B}_{2R}\cap S_\delta^\varepsilon} (u^{p^*-p} - u_{\lambda_0+\varepsilon}^{p^*-p})\varphi_R^\theta \xi_{p,\lambda}^+ \, dx \\ &\leqslant C_p^2(S_{\delta,\varepsilon}) C_u \int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}} |\nabla u|^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0+\varepsilon}|^2 \, dx \\ &\leqslant \frac{C_p^2(S_{\delta,\varepsilon}) C_u}{\inf\limits_{S_{\bar{\delta},\bar{\varepsilon}}} u^2} \int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}} u^2 \left(|\nabla u| + |\nabla u_{\lambda_0+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0+\varepsilon}|^2 \, dx, \end{aligned} \tag{3.37}$$

where $C_p(\Omega)$ is the Poincaré constant tends to zero as $|\Omega| \to 0$. If $1 < p < 2$ one can apply the classical Poincaré inequality in order to have

$$\begin{aligned} &\int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}} (u^{p^*-p} - u_{\lambda_0+\varepsilon}^{p^*-p})\varphi_R^\theta \xi_{p,\lambda_0+\varepsilon}^+ \, dx \\ &\leqslant C_p^2(S_{\delta,\varepsilon}) C_u \int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}} |\nabla \log u - \nabla \log u_{\lambda_0+\varepsilon}|^2 \, dx \\ &\leqslant \frac{C C_p^2(S_{\delta,\varepsilon}) C_u}{\inf\limits_{S_{\bar{\delta},\bar{\varepsilon}}} u^2} \int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}} u^2 \left(|\nabla u| + |\nabla u_{\lambda_0+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0+\varepsilon}|^2 \, dx, \end{aligned} \tag{3.38}$$

which can be deduced since in $\Sigma_{\lambda_0+\varepsilon} \setminus B_{\tilde{R}_0}(0_{\lambda_0+\varepsilon})$ one has that

$$\left(|\nabla u| + |\nabla u_\lambda|\right)^{2-p} \leqslant C,$$

for some constant C that does not depend on $\varepsilon \leqslant \bar{\varepsilon}$.

Going back to (3.35) we have to give an estimate for the third and the fourth term on the right hand side. Starting by the third term we can argue in the same way of $\mathcal{I}_5$

$$\begin{aligned} &\int_{\hat{B}_{2R}\cap B_{\overline{R},\varepsilon}} \left(u^{\alpha-p} v^\beta - u_{\lambda_0+\varepsilon}^{\alpha-p} v_{\lambda_0+\varepsilon}^\beta\right) \varphi_R^\theta \xi_{p,\lambda_0+\varepsilon}^+ \, dx \\ \leqslant &\frac{C}{\overline{R}^{\kappa^*}} \int_{\hat{B}_{2R}\cap B_{\overline{R},\varepsilon}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}} \varphi_R^\theta u^2 \left(|\nabla u| + |\nabla u_{\lambda_0+\varepsilon}|\right)^{p-2} |\nabla \log u - \nabla \log u_{\lambda_0+\varepsilon}|^2 \, dx \\ &+ \frac{C}{\overline{R}^{\kappa^*}} \int_{\hat{B}_{2R}\cap B_{\overline{R},\varepsilon}\cap\{v\geqslant v_{\lambda_0+\varepsilon}\}} \varphi_R^\theta v^2 \left(|\nabla v| + |\nabla v_{\lambda_0+\varepsilon}|\right)^{p-2} |\nabla \log v - \nabla \log v_{\lambda_0+\varepsilon}|^2 \, dx + \frac{C}{R^\kappa \bar{R}^{\kappa^*}}. \end{aligned}$$

For the last term of (3.35), we argue as in **Step 1**, so that

$$\begin{aligned}&\int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}}\left(u^{\alpha-p}v^{\beta}-u^{\alpha-p}_{\lambda_0+\varepsilon}v^{\beta}_{\lambda_0+\varepsilon}\right)\varphi_R^{\theta}\xi^+_{p,\lambda_0+\varepsilon}\,dx\\&\leqslant C\int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}}u^{p^\star-2}(\xi_\lambda^+)^2\varphi_R^\theta\,dx+C\int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}}v^{p^*-2}(\zeta_\lambda^+)^2\varphi_R^\theta\,dx\\&\leqslant C_u\int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}}(\log u-\log u_{\lambda_0+\varepsilon})^2\,dx+C_v\int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}\cap\{v\geqslant v_{\lambda_0+\varepsilon}\}}(\log v-\log v_{\lambda_0+\varepsilon})^2\,dx,\end{aligned}$$

where

$$C_u:=\sup_{S_{\bar{\delta},\bar{\varepsilon}}}u^{p^*-2}\qquad\text{and}\qquad C_v:=\sup_{S_{\bar{\delta},\bar{\varepsilon}}}v^{p^*-2}.$$

At this point we can argue in two different cases $p>2$ as in (3.37) and $1<p<2$ as in (3.38).

$$\begin{aligned}&\int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}}\left(u^{\alpha-p}v^{\beta}-u^{\alpha-p}_{\lambda_0+\varepsilon}v^{\beta}_{\lambda_0+\varepsilon}\right)\varphi_R^{\theta}\xi^+_{p,\lambda_0+\varepsilon}\,dx\\&\leqslant\frac{CC_p^2(S_{\delta,\varepsilon})C_u}{\inf\limits_{S_{\bar{\delta},\bar{\varepsilon}}}u^2}\int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}}u^2\left(|\nabla u|+|\nabla u_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla\log u-\nabla\log u_{\lambda_0+\varepsilon}|^2\,dx\\&\quad+\frac{CC_p^2(S_{\delta,\varepsilon})C_v}{\inf\limits_{S_{\bar{\delta},\bar{\varepsilon}}}v^2}\int_{\hat{B}_{2R}\cap S_{\delta,\varepsilon}\cap\{v\geqslant v_{\lambda_0+\varepsilon}\}}v^2\left(|\nabla v|+|\nabla v_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla\log v-\nabla\log v_{\lambda_0+\varepsilon}|^2\,dx.\end{aligned}$$

Analogously, we can obtain the same estimate starting from (3.34) and repeating verbatim the above argument. At the end, we have to collect all the estimates in (3.35) and sum the contributions, in order to deduce that

$$\begin{aligned}&\check{c}\int_{\hat{B}_{2R}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}}\varphi_R^\theta u^2\left(|\nabla u|+|\nabla u_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla\log u-\nabla\log u_{\lambda_0+\varepsilon}|^2\,dx\\&\quad+\check{c}\int_{\hat{B}_{2R}\cap\{v\geqslant v_{\lambda_0+\varepsilon}\}}\varphi_R^\theta v^2\left(|\nabla v|+|\nabla v_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla\log v-\nabla\log v_{\lambda_0+\varepsilon}|^2\,dx\\&\leqslant\frac{C}{R^{\frac{\kappa}{p}}}+\frac{C}{R^{\kappa}}+\frac{C}{\bar{R}^{\kappa^*}}\int_{(\hat{B}_{2R}\cap B_{\bar{R},\varepsilon})\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}}\varphi_R^\theta u^2\left(|\nabla u|+|\nabla u_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla\log u-\nabla\log u_{\lambda_0+\varepsilon}|^2\,dx\\&\quad+\frac{\tilde{C}C_p^2(S_\delta^\varepsilon)C_u}{\inf\limits_{S_{\bar{\delta}}^{\bar{\varepsilon}}}u^2}\int_{S_\delta^\varepsilon\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}}u^2\left(|\nabla u|+|\nabla u_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla\log u-\nabla\log u_{\lambda_0+\varepsilon}|^2\,dx\\&\quad+\frac{C}{\bar{R}^{\kappa^*}}\int_{(\hat{B}_{2R}\cap B_{\bar{R},\varepsilon})\cap\{v\geqslant v_{\lambda_0+\varepsilon}\}}\varphi_R^\theta v^2\left(|\nabla v|+|\nabla v_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla\log v-\nabla\log v_{\lambda_0+\varepsilon}|^2\,dx\\&\quad+\frac{\bar{C}C_p^2(S_\delta^\varepsilon)C_v}{\inf\limits_{S_{\bar{\delta}}^{\bar{\varepsilon}}}v^2}\int_{S_\delta^\varepsilon\cap\{v\geqslant v_{\lambda_0+\varepsilon}\}}v^2\left(|\nabla v|+|\nabla v_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla\log v-\nabla\log v_{\lambda_0+\varepsilon}|^2\,dx,\end{aligned}$$

where $\check{c}=\min\{c_1,c_2\}$. Now we take care of the variable parameters $\bar{R},\delta,\bar{\varepsilon}$. First we fix $\bar{R}$ large such that

$$\frac{C}{\check{c}\bar{R}^{\kappa^*}}<1.$$

Then, since $C_p^2(\Omega)$ goes to zero if the Lebesgue measure of Ω goes to zero, we choose $\delta,\bar{\varepsilon},\tau$ small so that

$$\frac{CC_p^2(S_\delta^\varepsilon)C_u}{\check{c}\inf\limits_{S_{\bar{\delta}}^{\bar{\varepsilon}}}u^2}<1\quad\text{and}\quad\frac{CC_p^2(S_\delta^\varepsilon)C_v}{\check{c}\inf\limits_{S_{\bar{\delta}}^{\bar{\varepsilon}}}v^2}<1$$

for every $0 \leqslant \varepsilon \leqslant \bar{\varepsilon}$. Hence it follows that

$$\begin{aligned}&\int_{\hat{B}_{2R}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}} u^2\left(|\nabla u|+|\nabla u_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla \log u-\nabla \log u_{\lambda_0+\varepsilon}|^2\,dx\\&+\int_{\hat{B}_{2R}\cap\{v\geqslant v_{\lambda_0+\varepsilon}\}} v^2\left(|\nabla v|+|\nabla v_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla \log v-\nabla \log v_{\lambda_0+\varepsilon}|^2\,dx \leqslant \frac{C}{R^{\frac{\kappa}{p}}}+\frac{C}{R^{\kappa}},\end{aligned}$$

getting again (as $R \to +\infty$)

$$\begin{aligned}&\int_{\Sigma_{\lambda_0+\varepsilon}\cap\{u\geqslant u_{\lambda_0+\varepsilon}\}} u^2\left(|\nabla u|+|\nabla u_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla \log u-\nabla \log u_{\lambda_0+\varepsilon}|^2\,dx\\&+\int_{\hat{B}_{2R}\cap\{v\geqslant v_{\lambda_0+\varepsilon}\}} v^2\left(|\nabla v|+|\nabla v_{\lambda_0+\varepsilon}|\right)^{p-2}|\nabla \log v-\nabla \log v_{\lambda_0+\varepsilon}|^2\,dx = 0,\end{aligned}$$

which gives that $u \leqslant u_{\lambda_0+\varepsilon}$ and $v \leqslant v_{\lambda_0+\varepsilon}$ in $\Sigma_{\lambda_0+\varepsilon}$. This contradicts the definition of λ_0 and it proves that $\lambda_0 = 0$.

Step 3: Conclusion.
The symmetry of the solution (u, v) follows by performing the moving plane procedure in the opposite direction, and hence this gives the symmetry of u along the e_1-direction. Repeating the same arguments with respect to each direction $\eta \in \mathbb{S}^{n-1}_+$, then one deduces that (u, v) has radial symmetry with respect to the origin and that is a radially decreasing function. □

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