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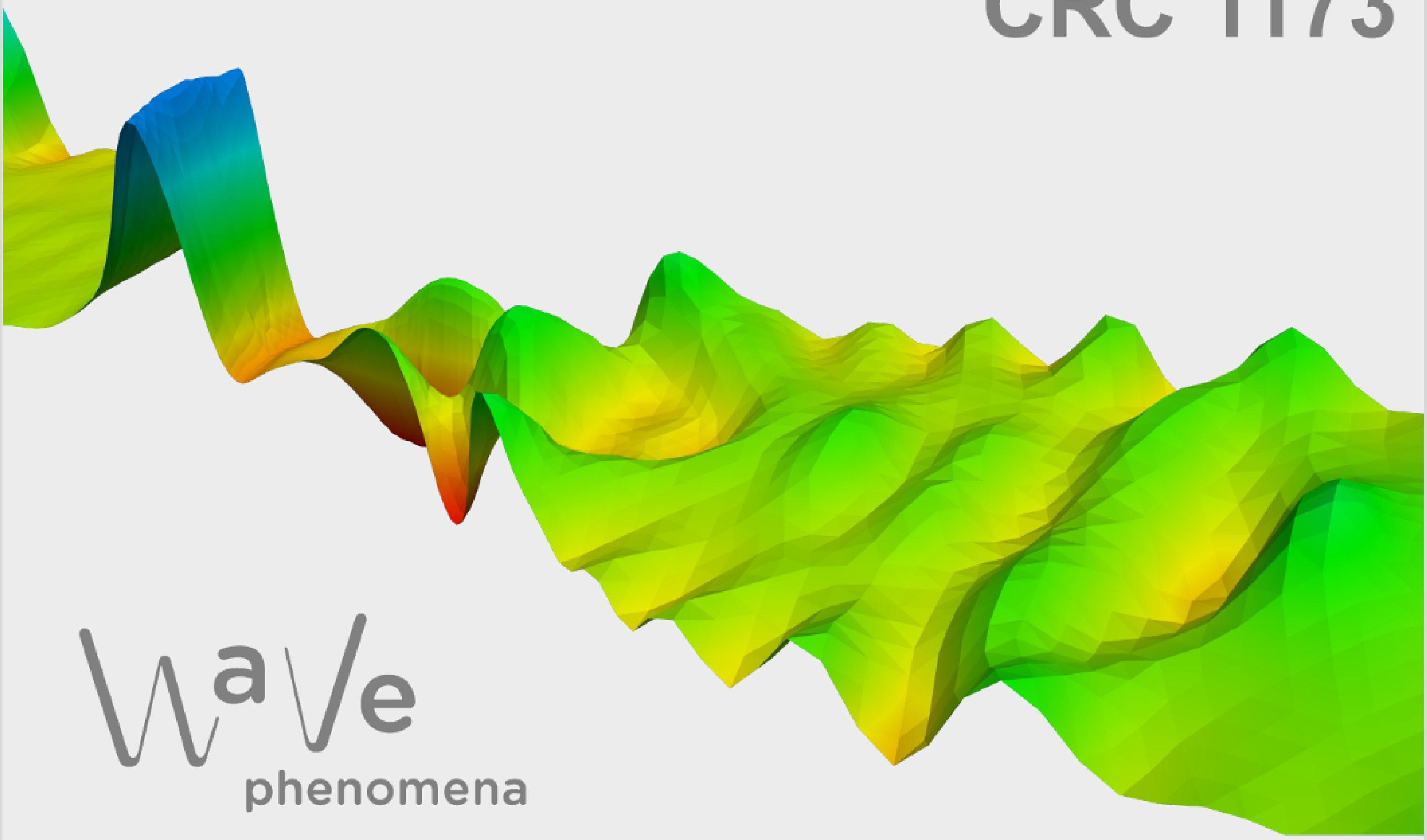

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An implicit-explicit time discretization scheme for second-order semilinear wave equations with a nonlocal material law and kinetic boundary conditions

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Abstract

In this paper we construct and analyze an implicit-explicit (IMEX) scheme for the semilinear viscoacoustic wave equation with a retarded material law. It contains a convolution term with exponential kernels and is thus nonlocal in time. Furthermore, the wave equation is equipped with kinetic boundary conditions. We treat the convolution term via additional variables. In order to make the problem well-posed, it is essential to perform an appropriate shift to derive auxiliary differential equations which are coupled to the first–order formulation of the wave equation. For the kinetic boundary conditions, we consider these equations in weighted bulk-surface Sobolev spaces. Second–order error bounds in time are proven for the IMEX scheme and are supported by numerical experiments, where the IMEX scheme is combined with an isoparametric finite element discretization in space.

Keywords: implicit-explicit time integration, IMEX, kinetic boundary condition, nonlocal material laws, auxiliary differential equation, semilinear wave equation

1 Introduction

In this paper, we consider the viscoacoustic wave equation with a convolution term of the form

$$\partial_{tt}u(t) - c^2\Delta u(t) + \int_{-\infty}^{t} b(t-\theta)\Delta u(\theta)\,\mathrm{d}\theta = f(t,u), \qquad u(0)=u_0, \quad u_t(0)=v_0, \tag{1.1a}$$

in a domain $\Omega \subset \mathbb{R}^d$ with smooth boundary Γ and we set $u(\theta) = u_0$ for $\theta < 0$. The viscoacoustic wave equation can be seen as a simplified model of the viscoelastic wave equation. The convolutionary memory term is determined by physical properties and describes the viscosity of the material. It makes the equation nonlocal in time, which is numerically challenging. Here, we study convolution kernels given by a linear combination of exponentials,

$$b(t) = \sum_{j=1}^{m} \beta_j \mathrm{e}^{-\lambda_j t}, \qquad \beta_j, \lambda_j > 0. \tag{1.1b}$$

This model that can be found in geophysical literature, e.g., [1, Chap. 2] and more specific in seismology [2]. The model problem using exponential kernels describes the case that most recent history has more influence on the materials reaction and diminishes in the past further away, cf. [1, Sec. 2.1.1]. We say, the material has a fading memory. In mechanical modeling this is used for the standard linear solid, where λ^{-1} is the relaxation time, cf. [3, p. 32]. We assume that the material properties do not change over time, cf. [1]. The kernels (1.1b) are of a special form, which enables us to derive additional auxiliary differential equations, which are coupled to the wave equation. For more general kernels, one can consider convolution quadrature, see e.g., [4–10].

To complement equation (1.1), we impose appropriate boundary conditions. In particular, we consider both Dirichlet and kinetic boundary conditions in a general framework. The latter are a special kind of dynamic boundary conditions, which take the form of another differential equation posed on the boundary. In two dimensions, kinetic boundary conditions admit the physical interpretation of a vibrating membrane with boundary mass density exposed to linear tension, see [11, Sec. 5&6], [12, p.56]; An example is the membrane of a bass drum which has a hole in the interior that has a thick border, cf. [7, Sec. 3.2]. In [11, 13], dynamic boundary conditions are considered for modeling heat conduction, where heat is created on the boundary.

For the time integration of (1.1) we employ a new implicit-explicit scheme in order to solve the stiff linear part implicitly and we avoid solving a nonlinear system in each step by treating the inhomogeneity explicitly. Due to the smooth boundary we combine this with a nonconforming space discretization with isoparametric finite elements.

In [14, 15] there are analytical results of wave equations with dynamic boundary conditions. In [16], the authors show wellposedness of a parabolic equation with dynamic boundary conditions. Other authors have investigated numerics for kinetic boundary conditions. Examples are [17–19] for linear and semilinear equations without nonlocal materials. A bulk-surface splitting method is proposed in [20].

The authors of [21] consider the viscoelastic wave equation and simulate its solution using a symmetric interior penalty discontinuous Galerkin method. In [22], space-time methods are used for the linear viscoacoustic equation. Several authors have investigated the viscoelastic or viscoacoustic wave equation as an inverse problem. Examples are [23], [24] with a finite difference method, and [25]. The goal of the last paper is a full wave form inversion of the viscoacoustic wave equation in a div-grad first-order formulation. The viscoelastic wave equation with standard boundary conditions is considered in [9] for several models. The authors show plots of the displacement function, using convolution quadrature and the finite element method. Furthermore, space-time plots are shown. However, this paper did not focus on the error analysis.

The literature on wave equations with auxiliary differential equations mostly deals with the analysis and stability including a different variable and possibly a delay term, cf. [26]. In [27], promising experiments for the viscoelastic wave equation with a nonlinear stress-strain relation are shown. For the numerical simulation, the ADE system of equations is discretized in time using a second order semi-implicit scheme. It is combined with a second-order time-stepping algorithm and a fourth-order staggered grid finite difference spatial discretization. However, the authors did not provide an error analysis.

There is a broad literature on implicit-explicit (IMEX) schemes, e.g., [28–31]. The authors of [32] propose an IMEX scheme combined with multiscale methods. A Crank-Nicolson-leapfrog IMEX scheme was constructed in [33, 34]. However, the schemes therein are not equivalent to our IMEX scheme.

The author of [35, 36] considers a related problem to (1.1), where the solution to a Maxwell system with exponential non-locality is approximated. However, in the setting there, the nonlocality is a bounded perturbation, which is not applicable in our situation.

The main contribution of this paper is the construction and the rigorous error analysis of a new IMEX scheme for a wave equation that has a nonlocality in time and is equipped with dynamic boundary conditions. In contrast to the setting in [37], the block structure of the operator in the first-order formulation has at least three components, which is why the calculations here are more involved than in [37]. In particular, we first derive a framework using weighted Sobolev spaces suitable for the wellposedness as well as for the error analysis. Moreover, defining the auxiliary variables is not straight forward, since an appropriate shift has to be included, cf. [26]. Finally, we show a uniform second-order error bound for the time discretization.

In this paper, for the sake of presentation, we do not carry out the full discretization, since it can be done along the lines of [37].

Outline of the paper

In Section 2, we state the problem, introduce auxiliary variables and the corresponding system of differential equations and investigate the wellposedness. In Section 3, we construct and analyze the numerical approximation by an implicit-explicit (IMEX) scheme in time. We conclude with numerical examples in Section 4.

Notation

For the partial derivative with respect to time we use the notation $\partial_t u = u' = u_t$. In this paper, we will consider (1.1a) with Dirichlet boundary conditions (D) or with kinetic boundary conditions (K). This will require different spaces, as defined next. We introduce the following bulk-surface Sobolev spaces,

$$H^k(\Omega,\Gamma) = \left\{ v \in H^k(\Omega) \,\middle|\, \gamma_D(v) \in H^k(\Gamma) \right\}, \quad k \geq 1,$$

of $H^k(\Omega)$-functions with $H^k(\Gamma)$-traces, where γ_D denotes the Dirichlet trace operator, cf. [13]. We further equip $H^k(\Omega,\Gamma)$ with the scalar product, which induces the norm

$$\|v\|^2_{H^k(\Omega,\Gamma)} = \|v\|^2_{H^k(\Omega)} + \|\gamma_D(v)\|^2_{H^k(\Gamma)} .$$

We denote the spaces

$$H = \begin{cases} L^2(\Omega), & \text{for (D)} \\ L^2(\Omega) \times L^2(\Gamma), & \text{for (K)} \end{cases} \qquad \text{and} \qquad V = \begin{cases} H^1_0(\Omega), & \text{for (D)} \\ H^1(\Omega,\Gamma), & \text{for (K)} \end{cases}. \tag{1.2a}$$

In [17, Cor. 6.7] it is shown, that $H^1(\Omega,\Gamma)$ is dense in $L^2(\Omega) \times L^2(\Gamma)$ and $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega,\Gamma)$.

We will make use of weighted spaces. Let V be a Hilbert space as defined in (1.2a) with scalar product $\langle \cdot, \cdot \rangle$, then we define the weighted Hilbert space V_α for some $\alpha > 0$ as the space V combined with the weighted scalar product, i.e.,

$$\big(V_\alpha, (\cdot,\cdot)\big), \qquad \text{where} \quad (x,y) \mapsto \alpha\langle x, y\rangle \quad \text{for} \quad x, y \in V. \tag{1.2b}$$

Let Γ be C^1-regular. For a function $v \in H^1(\Omega)$ and the outer unit normal vector $\boldsymbol{n}$, we define the surface gradient

$$\nabla_\Gamma v = (\partial_{j,\Gamma} v)_{j=1}^d = (I - \boldsymbol{n}\boldsymbol{n}^T)\nabla v \tag{1.3}$$

and the Laplace-Beltrami operator

$$\Delta_\Gamma v = \sum_{j=1}^d \partial^2_{j,\Gamma} v. \tag{1.4}$$

For the weak formulation of our PDEs we will make use of the well-known Gauss theorem, which also holds for the above surface operators, see also [13, p. 111],

$$-\int_\Omega (\Delta u)\varphi \,\mathrm{d}x = \int_\Omega \nabla u \nabla \varphi \,\mathrm{d}x - \int_\Gamma (\boldsymbol{n} \cdot \nabla u)\varphi \,\mathrm{d}x, \tag{1.5a}$$

$$-\int_\Gamma (\Delta_\Gamma u)\varphi \,\mathrm{d}x = \int_\Gamma \nabla_\Gamma u \nabla_\Gamma \varphi \,\mathrm{d}x. \tag{1.5b}$$

2 Analytical framework

Throughout this paper, we consider (1.1a) with Dirichlet boundary conditions as well as kinetic boundary conditions. In this section, we describe the framework for the two types of boundary conditions and how both cases will be handled in the further course of this paper. We will derive auxiliary differential equations for the treatment of the convolution and investigate its wellposedness by means of evolution equations in suitable spaces.

2.1 Problem statement

We assume Γ to be the smooth boundary of the domain $\Omega \subset \mathbb{R}^d$ and impose kinetic boundary conditions, i.e.,

$$\begin{aligned}
&\partial_{tt}u(t) - c^2\Delta u(t) + \int_{-\infty}^{t} b(t-s)\Delta u(s)\,\mathrm{d}s = f_\Omega(t), \\
&\partial_{tt}u(t) - c^2\Delta_\Gamma u(t) + c^2\boldsymbol{n}\cdot\nabla u + \int_{-\infty}^{t} b(t-s)\Delta_\Gamma u(s)\,\mathrm{d}s \\
&\qquad\qquad\qquad\qquad - \int_{-\infty}^{t} b(t-s)\boldsymbol{n}\cdot\nabla u(s)\,\mathrm{d}s = f_\Gamma(t) \\
&\quad u(0) = u_0, \quad \partial_t u(0) = v_0,
\end{aligned} \tag{2.1}$$

where we again set $u(s) = u_0$ for $s < 0$. In order to abbreviate the notation, we further define the extended Laplace operator

$$\Delta_{\Omega,\Gamma} = \begin{cases} \Delta, & \text{in} \quad \Omega, \\ \Delta_\Gamma - \boldsymbol{n}\cdot\nabla, & \text{on} \quad \Gamma. \end{cases} \tag{2.2}$$

Note that for $u \in H_0^1(\Omega)$, the extended Laplace operator only acts on the inner domain. Then, (1.1a) with Dirichlet or kinetic boundary conditions can be summarized as

$$\partial_{tt}u(t) - c^2\Delta_{\Omega,\Gamma}u(t) + \int_{-\infty}^{t} b(t-s)\Delta_{\Omega,\Gamma}u(s)\mathrm{d}s = f(t). \tag{2.3}$$

2.2 Auxiliary differential equations

In this subsection we derive auxiliary differential equations for the treatment of the convolution in (2.3) and investigate its wellposedness in the setting of an evolution equation. Following [26], we perform a shift of the variable in the convolution term and obtain

$$\int_{-\infty}^{t} \beta_j \mathrm{e}^{-\lambda_j(t-\theta)}\Delta_{\Omega,\Gamma}u(\theta)\,\mathrm{d}\theta = \frac{\beta_j}{\lambda_j}\Delta_{\Omega,\Gamma}u(t) - \beta_j\Delta_{\Omega,\Gamma}M_j(t), \quad j = 1,\dots,m,$$

with the auxiliary variables

$$M_j(t) = \int_{-\infty}^{t} \mathrm{e}^{-\lambda_j(t-\theta)} \big(u(t) - u(\theta)\big) \, \mathrm{d}\theta, \qquad t \geq 0.$$

We introduce the notation

$$M = (M_j)_{j=1}^m, \quad \beta = (\beta_j)_{j=1}^m, \quad \mathbf{1} = (1)_{j=1}^m, \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m). \tag{2.4}$$

Then, we obtain the following first-order in time coupled PDE system for (2.3)

$$\partial_t u = v, \qquad u(0) = u_0, \tag{2.5a}$$

$$\partial_t v = \alpha \Delta_{\Omega,\Gamma} u + \Delta_{\Omega,\Gamma} \beta^T M + f, \qquad v(0) = v_0, \qquad \alpha = c^2 - \sum_{j=1}^m \frac{\beta_j}{\lambda_j}, \tag{2.5b}$$

$$\partial_t M_j = -\lambda_j M_j + \frac{1}{\lambda_j} v, \qquad M_j(0) = 0, \qquad j = 1, \ldots, m. \tag{2.5c}$$

In a compact form, (2.5) can be written as

$$x' + \mathcal{A} x = F, \tag{2.6a}$$

where

$$x = \begin{pmatrix} u \\ v \\ M \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & -\mathrm{I} & 0 \\ -\alpha \Delta_{\Omega,\Gamma} & 0 & -\beta \otimes \Delta_{\Omega,\Gamma} \\ 0 & -\Lambda^{-1} \mathbf{1} & \Lambda \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}. \tag{2.6b}$$

Remark 1. *In applications, we often know the Laplace transformation B rather than the kernel b itself, see, e.g., [1, 2]. The Laplace transform of the differential equation (1.1) with frequency variable $s \in \mathbb{C}$ is given as*

$$\big(s^2 - (c^2 - B(s)) \Delta_{\Omega,\Gamma}\big) U(s) = F(s), \qquad B(s) = \sum_{j=1}^m \frac{\beta_j}{s + \lambda_j},$$

where $U(s), F(s)$ denote the Laplace transformations of u, f, respectively. Since $-\Delta_{\Omega,\Gamma}$ is a positive semidefinite operator, assuming

$$\alpha = c^2 - \sum_{j=1}^m \frac{\beta_j}{\lambda_j} > 0 \tag{2.7}$$

is sufficient to ensure that, for a given $F(s)$, this equation has a unique solution $U(s)$ for all $\mathrm{Re}\, s = \sigma > 0$. Hence we assume (2.7) in the remaining manuscript.

2.3 Wellposedness

Next we consider the wellposedness of the coupled PDE system (2.5) for Dirichlet (D) and for kinetic (K) boundary conditions. We will show that, in both cases, $\mathcal{A}$ generates a strongly continuous semigroup on the respective Hilbert space

$$X = V_\alpha \times H \times \bigtimes_{j=1}^{m} V_{\mu_j} \tag{2.8}$$

with V_α defined in (1.2) for $\alpha > 0$ and $\mu_j = \beta_j \lambda_j > 0$. In particular, $\mathcal{A}$ is monotone, i.e.,

$$(\mathcal{A}x, x)_X \geq -\mathrm{c}_m \|x\|_X^2 \qquad \text{for all} \quad x \in X, \tag{2.9}$$

cf. [12, Assumption 2.3]. For Dirichlet boundary conditions, we have $c_m = 0$, which means that $\mathcal{A}$ generates a contractive semigroup, while for kinetic boundary conditions, (2.9) holds for

$$\mathrm{c}_m > \frac{1}{2}, \qquad \mathrm{c}_m > \frac{\alpha + \mathbf{1}^T\beta}{2}, \qquad \lambda_j \mathrm{c}_m > \frac{1}{2} - \lambda_j^2, \qquad j = 1, \dots, m. \tag{2.10}$$

Theorem 1. *Let $\mathcal{A}$ and x be given as in (2.6b), describing the first-order formulation of (2.3).*

(D) The operator $-\mathcal{A}$ with domain

$$\mathcal{D}(\mathcal{A}) = \Big\{ x \in H_0^1(\Omega)^{m+2} \,\big|\, \alpha u + \beta^T M \in H^2(\Omega) \Big\} \tag{2.11}$$

generates a contractive C_0-semigroup on X defined in (2.8).

(K) For c_m chosen such that (2.10) holds, the shifted operator $-(\mathcal{A}+\mathrm{c}_m\mathrm{I})$ with domain

$$\mathcal{D}(\mathcal{A} + \mathrm{c}_m\mathrm{I}) = \Big\{ x \in H^1(\Omega,\Gamma)^{m+2} \,\big|\, \alpha u + \beta^T M \in H^2(\Omega,\Gamma) \Big\}$$

generates a contractive C_0-semigroup on X defined in (2.8).

In order to obtain local wellposedness of (2.6a) we make the following assumption on the inhomogeneity.

Assumption 2. *[37, Assumption 4.1] Let $\Theta \in \{\Omega, \Gamma\}$.*

(a) *The inhomogeneities $f = f_\Theta$, in (2.1) satisfy*

$$f_\Theta \in C^1([0,T] \times \overline{\Theta} \times \mathbb{R}; \mathbb{R})$$

and can be split into

$$f(t,\xi,u) = f_1(t,\xi) + f_2(\xi,u) \quad \textit{or} \quad f(t,\xi,u) = f_1(t,\xi) + f_2(t,u).$$

(b) *Furthermore, we assume the growth conditions, that there exist*

$$\zeta_\Omega \begin{cases} < \infty, & d = 2, \\ \leq \frac{d}{d-2} & d \geq 3, \end{cases} \quad \text{and} \quad \zeta_\Gamma \begin{cases} < \infty, & d = 2,3, \\ \leq \frac{d-1}{d-3} & d \geq 4, \end{cases}$$

such that for all $(t, \xi, u) \in [0,T] \times \Theta \times \mathbb{R}$ it holds that

$$|f_\Theta(t,\xi,u)| \leq C(1 + |u|^{\zeta_\Theta}), \qquad |\nabla f_\Theta(t,\xi,u)| \leq C(1 + |u|^{\zeta_\Theta - 1}).$$

Remark 2. *If Assumption 2 holds, then, (2.6a) is locally wellposed, i.e., for every initial value $x_0 \in X$ there exists $t^\star(x_0) > 0$ such that for all $T < t^\star(x_0)$, (2.6a) has a unique solution*

$$x \in C^1([0,T], X) \cap C([0,T], \mathcal{D}(\mathcal{A})).$$

The evolution equation (2.6a) fits into the framework of [12, 17, 38].

Proof of Theorem 1. For better readability, we only show the proof in the case $m = 1$, i.e., the case of one exponential kernel. We demonstrate the calculations for the wave equation with kinetic boundary conditions, the case of Dirichlet boundary conditions works analogously. Employing the Gauss theorems (1.5), we see that the kinetic boundary condition leads to solving (2.6b) on the space given in (2.8).

We will use the Lumer-Phillips theorem [39, Sec.1.3]. To this end, we show that $\mathcal{A} + \mathrm{c}_m \mathrm{I}$ is densely defined, monotone, and has full range.

For the monotonicity we calculate by using partial integration (1.5) and Young

$$\begin{aligned}
& (\mathcal{A}x + \mathrm{c}_m x, x)_X \\
= & -\alpha \langle v, u \rangle_{L^2(\Omega,\Gamma)} + \mathrm{c}_m \alpha \|\nabla u\|^2_{L^2(\Omega,\Gamma)} + \mathrm{c}_m \alpha \|u\|^2_{L^2(\Omega,\Gamma)} + \mathrm{c}_m \|v\|^2_{L^2(\Omega,\Gamma)} \\
& + \beta\lambda^2 \|\nabla M\|^2_{L^2(\Omega,\Gamma)} + \beta\lambda^2 \|M\|^2_{L^2(\Omega,\Gamma)} - \beta \langle v, M \rangle_{L^2(\Omega,\Gamma)} \\
& + \mathrm{c}_m \mu \|\nabla M\|^2_{L^2(\Omega,\Gamma)} + \mathrm{c}_m \mu \|M\|^2_{L^2(\Omega,\Gamma)} \\
\geq & -\frac{\alpha}{2} \left(\|v\|^2_{L^2(\Omega,\Gamma)} + \|u\|^2_{L^2(\Omega,\Gamma)} \right) - \frac{\beta}{2} \left(\|v\|^2_{L^2(\Omega,\Gamma)} + \|M\|^2_{L^2(\Omega,\Gamma)} \right) \\
& + \mathrm{c}_m \|v\|^2_{L^2(\Omega,\Gamma)} + \mathrm{c}_m \alpha \|u\|^2_{L^2(\Omega,\Gamma)} + \beta\lambda^2 \|M\|^2_{L^2(\Omega,\Gamma)} + \mathrm{c}_m \mu \|M\|^2_{L^2(\Omega,\Gamma)} \\
= & \, \alpha \|u\|^2_{L^2(\Omega,\Gamma)} \left(\mathrm{c}_m - \frac{1}{2} \right) + \|v\|^2_{L^2(\Omega,\Gamma)} \left(\mathrm{c}_m - \frac{\alpha + \beta}{2} \right) + \|M\|^2_{L^2(\Omega,\Gamma)} \left(\mathrm{c}_m \mu - \frac{\beta}{2} + \beta\lambda^2 \right) \\
\geq & \, 0.
\end{aligned}$$

To show that $\mathcal{A}$ has full range, let $\gamma > \mathrm{c}_m$, $y \coloneqq (f, g, h) \in X$. We claim that there exists $x \coloneqq (u, v, M) \in \mathcal{D}(\mathcal{A})$ such that $(\gamma + \mathcal{A})x = y$

$$\gamma u = v + f, \tag{2.12a}$$

$$\gamma v = \alpha \Delta_{\Omega,\Gamma} u + \beta \Delta_{\Omega,\Gamma} M + g, \tag{2.12b}$$

$$\gamma M = -\lambda M + \frac{1}{\lambda} v + h. \tag{2.12c}$$

We aim to insert

$$u = \frac{1}{\gamma}(v+f), \qquad M = \frac{1}{\gamma+\lambda}(\frac{1}{\lambda}v+h)$$

into (2.12b) and solve for v. To do so, we use the operator

$$\Delta_{\Omega,\Gamma} : H^1(\Omega,\Gamma) \to H^{-1}(\Omega,\Gamma)$$

for the moment. Then, we have to solve

$$\gamma v = \frac{\alpha}{\gamma}\Delta_{\Omega,\Gamma}(v+f) + \frac{\beta}{\gamma+\lambda}\Delta_{\Omega,\Gamma}(\frac{1}{\lambda}v+h) + g$$

which is equivalent to

$$v - \frac{1}{\gamma^2}(\alpha + \frac{\beta}{\gamma+\lambda})\Delta_{\Omega,\Gamma}v = \frac{\alpha}{\gamma^2}\Delta_{\Omega,\Gamma}f + \frac{\beta}{\gamma(\gamma+\lambda)}\Delta_{\Omega,\Gamma}h + \frac{1}{\gamma}g. \tag{2.13}$$

The weak formulation is given as: find $v \in H^1(\Omega,\Gamma)$ such that

$$\begin{aligned} &\langle v,\varphi\rangle_{L^2(\Omega,\Gamma)} + \frac{1}{\gamma^2}\big(\alpha + \tfrac{\beta}{\gamma+\lambda}\big)\langle\nabla v,\nabla\varphi\rangle_{L^2(\Omega,\Gamma)} \\ =& -\frac{\alpha}{\gamma^2}\langle\nabla f,\nabla\varphi\rangle_{L^2(\Omega,\Gamma)} - \frac{\beta}{\gamma(\gamma+\lambda)}\langle\nabla h,\nabla\varphi\rangle_{L^2(\Omega,\Gamma)} + \frac{1}{\gamma}\langle g,\varphi\rangle_{L^2(\Omega,\Gamma)}, \end{aligned}$$

for all $\varphi \in H^1(\Omega,\Gamma)$. By the Lax–Milgram theorem we obtain a unique solution $v \in H^1(\Omega,\Gamma)$ of (2.13). Inserting this solution into (2.12a) and (2.12c) gives us $u, M \in H^1(\Omega,\Gamma)$ such that

$$\alpha\Delta_{\Omega,\Gamma}u + \beta\Delta_{\Omega,\Gamma}M = \frac{\alpha}{\gamma}\Delta_{\Omega,\Gamma}(v+f) + \frac{\beta}{\gamma+\lambda}\Delta_{\Omega,\Gamma}(\frac{1}{\lambda}v+h)$$

In order to make sure, that $x = (u,v,M) \in \mathcal{D}(\mathcal{A})$, we test the above equation with $\varphi \in H^1(\Omega,\Gamma)$ which yields

$$\begin{aligned} \langle\nabla(\alpha u+\beta M),\nabla\varphi\rangle_{L^2(\Omega,\Gamma)} &= \frac{\alpha}{\gamma}\langle\nabla(v+f),\nabla\varphi\rangle_{L^2(\Omega,\Gamma)} + \frac{\beta}{\gamma+\lambda}\langle\nabla(\frac{1}{\lambda}v+h),\nabla\varphi\rangle_{L^2(\Omega,\Gamma)} \\ &= \langle g-\gamma v,\varphi\rangle_{L^2(\Omega,\Gamma)}, \end{aligned}$$

and using $g \in L^2(\Omega,\Gamma)$ and $v \in H^1(\Omega,\Gamma)$ shows $x = (u,v,M) \in \mathcal{D}(\mathcal{A})$.

Note that the full range of $\gamma + \mathcal{A}$ was shown for arbitrary $\gamma > \mathrm{c}_m$. The fact that $\mathcal{A}$ is densely defined therefore follows by [40, Prop. I.4.2]. $\square$

3 Implicit-explicit time integration

In this section, we introduce the IMEX scheme for the time discretization of the solution to (1.1a) and its properties.

3.1 Construction of the IMEX scheme

Following [37], we set up the IMEX scheme as a perturbation of the Crank-Nicolson scheme. The latter one computes $x^n \approx x(t_n)$ for $t_n = n\tau$, where $\tau > 0$ denotes the step size via

$$x^{n+1} = x^n + \frac{\tau}{2}\big(- \mathcal{A}(x^n + x^{n+1}) + F^n + F^{n+1}\big), \tag{3.1a}$$

where

$$F^n = \begin{pmatrix} 0 \\ f^n \\ 0 \end{pmatrix}, \qquad \text{and} \qquad f^n = f(t_n, u^n). \tag{3.1b}$$

Equivalently we can write this as

$$R_+ x^{n+1} = R_- x^n + \frac{\tau}{2}\big(F^n + F^{n+1}\big), \qquad R_\pm = \mathrm{I} \pm \frac{\tau}{2}\mathcal{A}. \tag{3.2}$$

With the parameters β_j and λ_j of the convolution kernel (1.1b), we define the following scalars, which will be used in the discretization of the differential equation of M,

$$\gamma_{j,\pm} = 1 \pm \frac{\tau\lambda_j}{2}, \qquad \gamma_j = \frac{\gamma_{j,-}}{\gamma_{j,+}}. \tag{3.3}$$

Furthermore, we introduce a modification of the matrix Λ as

$$\widetilde{\Lambda} = \operatorname{diag}(\lambda_1\gamma_{1,+}, \dots, \lambda_m\gamma_{m,+}), \tag{3.4}$$

and the modified scalars

$$\widetilde{\alpha} = \alpha + \beta^T \widetilde{\Lambda}^{-1}\mathbf{1} = \left(\alpha + \sum_{j=1}^m \frac{\beta_j}{\lambda_j \gamma_{j,+}} \right) > \alpha. \tag{3.5}$$

We further introduce the notation

$$\widetilde{\beta} = \frac{1}{2}\big(\beta_1(1+\gamma_1), \dots, \beta_m(1+\gamma_m)\big), \tag{3.6}$$

and note that $\beta_j(1+\gamma_j)/2 \in [0, \beta_j)$ for $j = 1, \dots, m$.

Similarly to [37, Lem. 2.5] it can be shown that with $j = 1, \dots, m$ and f^n defined in (3.1b), the Crank-Nicolson scheme (3.1) is equivalent to

$$\begin{aligned} u^{n+1} &= u^n + \tau v^{n+\frac{1}{2}}, \\ v^{n+\frac{1}{2}} &= v^n + \frac{\tau}{2}\alpha\Delta_{\Omega,\Gamma}u^n + \frac{\tau}{2}\Delta_{\Omega,\Gamma}\widetilde{\beta}^T M^n + \frac{\tau^2}{4}\widetilde{\alpha}\Delta_{\Omega,\Gamma}v^{n+\frac{1}{2}} + \frac{\tau}{4}(f^n + f^{n+1}), \end{aligned}$$

$$M_j^{n+1} = \gamma_j M_j^n + \gamma_{j,+}^{-1} \frac{\tau}{\lambda_j} v^{n+\frac{1}{2}},$$

$$v^{n+1} = v^{n+\frac{1}{2}} + \frac{\tau}{2}\alpha\Delta_{\Omega,\Gamma} u^n + \frac{\tau}{2}\Delta_{\Omega,\Gamma}\widetilde{\beta}^T M^n + \frac{\tau^2}{4}\widetilde{\alpha}\Delta_{\Omega,\Gamma} v^{n+\frac{1}{2}} + \frac{\tau}{4}(f^n + f^{n+1}).$$

If the right hand side f is nonlinear, we will have to solve a nonlinear system in each step of the Crank-Nicolson scheme, which is very expensive. To overcome this difficulty, we use the IMEX scheme instead. The idea of the IMEX scheme is to treat the stiff linear part implicitly and the non-stiff non-linear part explicitly, such that the solution of one linear system of equations in each time step is sufficient.

Following [37, Section 2.2.], we derive the scheme via a combination of the Crank-Nicolson and with the leapfrog scheme and, with $j = 1, \dots, m$, arrive at

$$v^{n+\frac{1}{2}} = v^n + \frac{\tau}{2}\alpha\Delta_{\Omega,\Gamma} u^n + \frac{\tau}{2}\Delta_{\Omega,\Gamma}\widetilde{\beta}^T M^n + \frac{\tau^2}{4}\widetilde{\alpha}\Delta_{\Omega,\Gamma} v^{n+\frac{1}{2}} + \frac{\tau}{2} f^n, \tag{3.7a}$$

$$u^{n+1} = u^n + \tau v^{n+\frac{1}{2}}, \tag{3.7b}$$

$$M_j^{n+1} = \gamma_j M_j^n + \gamma_{j,+}^{-1} \frac{\tau}{\lambda_j} v^{n+\frac{1}{2}}, \tag{3.7c}$$

$$v^{n+1} = v^{n+\frac{1}{2}} + \frac{\tau}{2}\alpha\Delta_{\Omega,\Gamma} u^n + \frac{\tau}{2}\Delta_{\Omega,\Gamma}\widetilde{\beta}^T M^n + \frac{\tau^2}{4}\widetilde{\alpha}\Delta_{\Omega,\Gamma} v^{n+\frac{1}{2}} + \frac{\tau}{2} f^{n+1}. \tag{3.7d}$$

An equivalent way to compute v^{n+1} is obtained by subtracting (3.7a) and (3.7d) as

$$v^{n+1} = -v^n + 2v^{n+1/2} + \frac{\tau}{2}(f^{n+1} - f^n), \tag{3.7e}$$

see also [37, Remark 2.6]. In the scheme (3.7), the nonlinearity f is treated explicitly.

3.2 Wellposedness and reformulation of the IMEX scheme

To prove wellposedness of the IMEX scheme, we define the operators

$$Q_\pm = I \pm \big(-\frac{\tau^2}{4}\widetilde{\alpha}\Delta_{\Omega,\Gamma}\big), \tag{3.8}$$

with weak formulation: for given $z \in V$ find $y \in H$ such that

$$\langle z, w\rangle_H \pm \langle \nabla z, \nabla w\rangle_H = \langle y, w\rangle_H \quad \text{for all} \quad w \in V,$$

we define $y = Q_\pm z$. With this notation, we can characterize the half step $v^{n+\frac{1}{2}}$ via

$$Q_+ v^{n+\frac{1}{2}} = v^n + \frac{\tau}{2}\Delta_{\Omega,\Gamma}\big(\alpha u^n + \widetilde{\beta}^T M^n\big) + \frac{\tau}{2} f^n.$$

For our error analysis we will use bounds on the operators $Q_\pm$ using the spaces defined in (1.2a). Up to the weighting constants the following lemma was given in [37, Lemma 2.7].

Lemma 3. *Let $Q_\pm$ be defined as in (3.8) with $\widetilde{\alpha}$ as in (3.5). Then $Q_+ : \mathcal{D}(\Delta_{\Omega,\Gamma}) \to H$ is invertible and we have*

$$\left\| \frac{\tau^2}{4} \widetilde{\alpha} \Delta_{\Omega,\Gamma} Q_+^{-1} \right\|_{H \leftarrow H} \leq 1, \tag{3.9a}$$

where $\|\cdot\|_{H\leftarrow H}$ denotes the operator norm. Furthermore, for $\tau > 0$ in case (D) and $0 < \tau < \sqrt{\frac{2}{\widetilde{\alpha}}}$ in case (K), we have the bounds

$$\|Q_+^{-1}\|_{V \leftarrow H} \leq \frac{1}{\tau}\sqrt{\frac{2}{\widetilde{\alpha}}}, \tag{3.9b}$$

$$\|Q_- Q_+^{-1}\|_{H \leftarrow H} \leq \mathrm{e}^{\frac{\tau^2}{2}}. \tag{3.9c}$$

Employing the invertibility of Q_+ we obtain wellposedness of the IMEX scheme as in [12, Cor. 4.7] and [37, Cor. 2.8].

Lemma 4. *The IMEX scheme is wellposed. We define $\widehat{\beta} = (\widehat{\beta}_j)_{j=1}^m$ with $\widehat{\beta}_j \in [-\beta_j, \beta_j]$. Then for initial values $u^0, v^0 \in V$ and $M^0 = \mathbf{0}_m$, we have for $j = 1, \dots, m$ that*

$$u^n, M_j^n \in V, \quad v^{n+\frac{1}{2}} \in \mathcal{D}(\Delta_{\Omega,\Gamma}), \quad v^{n+1} \in H, \quad \Delta_{\Omega,\Gamma}(\alpha u^n + \widehat{\beta}^T M^n) \in H.$$

Proof. As in [37, Cor. 2.8], the claim is proven by induction over $n \in \mathbb{N}_0$. Due to $M_j^0 = 0$, the statement is true for $n = 0$. Then, exploiting the fact that $\widetilde{\beta}_j \in [0, \beta_j)$ for each $j = 1, \dots, m$, such that $\alpha \Delta_{\Omega,\Gamma} u^n + \Delta_{\Omega,\Gamma} \widetilde{\beta}^T M^n \in H$, the claim follows. □

We derive an equivalent first–order formulation for the IMEX scheme, cf. [12, Lemma 4.8] and [37, Lemma 2.10], where we make use of the notation introduced in (3.4) and (3.6).

Lemma 5. *Let $\tau c_m < 2$ and if we have kinetic boundary conditions let $\tau < \sqrt{\frac{2}{\widetilde{\alpha}}}$. Then, the matrix R_+ from (3.2) is invertible and, with $S_+ = Q_+^{-1}\Delta_{\Omega,\Gamma}$ and $R = R_+^{-1} R_-$ we have*

$$R = \begin{pmatrix} 1 + \frac{\tau^2}{2}\alpha S_+ & \tau Q_+^{-1} & \frac{\tau^2}{2}\widetilde{\beta}^T \otimes S_+ \\ \tau \alpha S_+ & Q_- Q_+^{-1} & \tau\alpha\widetilde{\beta}^T \otimes S_+ \\ \frac{\tau^2}{2}\alpha\widetilde{\Lambda}^{-1} \otimes S_+ \mathbf{1} & \tau\widetilde{\Lambda}^{-1} \otimes Q_+^{-1}\mathbf{1} & \Lambda + \frac{\tau^2}{2}\big((\widetilde{\Lambda}^{-1}\widetilde{\beta})^T \otimes \mathbf{1}\big) \otimes S_+ \end{pmatrix}$$

and

$$R_+^{-1} = \begin{pmatrix} I + \frac{\tau^2}{4}\alpha S_+ & \frac{\tau}{2}Q_+^{-1} & \frac{\tau^2}{4}\beta^T\Lambda\widetilde{\Lambda}^{-1}\otimes S_+ \\ \frac{\tau}{2}\alpha S_+ & Q_+^{-1} & \frac{\tau}{2}\beta^T\Lambda\widetilde{\Lambda}^{-1}\otimes S_+ \\ \frac{\tau^2}{4}\alpha\widetilde{\Lambda}^{-1}\otimes S_+\mathbf{1} & \frac{\tau}{2}\widetilde{\Lambda}^{-1}\otimes Q_+^{-1}\mathbf{1} & \Lambda\widetilde{\Lambda}^{-1} + \frac{\tau^2}{4}\widetilde{\Lambda}^{-1}\mathbf{1}\beta^T\Lambda\widetilde{\Lambda}^{-1}\otimes S_+ \end{pmatrix}.$$

Furthermore, the matrix $R_+^{-1} : X \to \mathcal{D}(\mathcal{A})$ satisfies $\|R_+^{-1}\| \leq 1$ and R has a continuous extension on X and $\|R\| \leq \mathrm{e}^{\tau c_m}$.

The IMEX scheme is equivalent to the first-order formulation

$$x^{n+1} = Rx^n + \frac{\tau}{2}R_+^{-1}y^n + \frac{\tau^2}{4}R_+^{-1}z^n, \tag{3.10a}$$

where

$$y^n = \begin{pmatrix} 0 \\ 1 \\ \mathbf{0}_m \end{pmatrix} \otimes (f^n + f^{n+1}), \qquad z^n = \begin{pmatrix} 1 \\ 0 \\ \Lambda^{-1}\mathbf{1} \end{pmatrix} \otimes (f^n - f^{n+1}). \tag{3.10b}$$

Proof. The proof of the invertibility and the bounds follow along the lines of [37, Lemma 2.10] and [17, Lemma 2.14]. For the ease of presentation we only treat the case $m = 1$. A direct calculation shows that R and R_+^{-1} are given as above. Let $x = (u, v, M)^T \in X$ and we will first use $\Delta_{\Omega,\Gamma}$ mapping from V to its dual, this means that $S_+ : V \to V$. Then we have that the components of

$$w := R_+^{-1}x$$

are given by

$$\begin{aligned} w_1 &= \big(\mathrm{I} + \alpha\frac{\tau^2}{4}S_+\big)u + \frac{\tau}{2}Q_+^{-1}v + \frac{\tau^2}{4}\frac{\beta}{\gamma_+}S_+M, \\ w_2 &= \alpha\frac{\tau}{2}S_+u + Q_+^{-1}v + \alpha\frac{\tau}{2}\frac{\beta}{\gamma_+}S_+M, \\ w_3 &= \alpha\frac{\tau}{4\lambda\gamma_+}S_+u + \frac{\tau}{2\lambda\gamma_+}Q_+^{-1}v + \frac{1}{\gamma_+}\big(Q_+^{-1} - \alpha\frac{\tau^2}{4}S_+\big)M, \end{aligned}$$

and lie in V. We further obtain

$$\Delta_{\Omega,\Gamma}\big(\alpha w_1 + \beta w_3\big) = \alpha\Delta_{\Omega,\Gamma}Q_+^{-1}u + \big(\frac{\alpha\tau}{2} + \frac{\beta\tau}{2\lambda\gamma_+}\big)\Delta_{\Omega,\Gamma}Q_+^{-1}v + \frac{\beta}{\gamma_+}\Delta_{\Omega,\Gamma}Q_+^{-1}M \in H,$$

by Lemma 3, i.e., we verified that $R_+^{-1} : X \to \mathcal{D}(\mathcal{A})$.

The proof of the equivalence is done analogously to [12, Lemma 4.8]. □

3.3 Error analysis of the IMEX scheme

We now turn to the error estimation of the IMEX scheme. The main idea is, that the defect of the IMEX scheme can be written as the defect of the Crank-Nicolson scheme with an additional term. By Assumption 2, the inhomogeneity f is locally Lipschitz continuous, i.e., for $\|u\|_V, \|v\|_V \le \rho$ it holds

$$\|f(\cdot,\cdot,u) - f(\cdot,\cdot,v)\|_H \le L_\rho \|u-v\|_V, \qquad L_\rho = C(1+\rho^{\zeta_\Omega - 1} + \rho^{\zeta_\Gamma - 1}), \tag{3.11}$$

where C is a constant which is independent of ρ.

The following result can be found for the wave equation without retarded material laws in [37], for the more detailed version we reference to [41].

Theorem 6 (Error bound IMEX scheme)**.** *Assume that the solution $x = (u, v, M)$ of (2.6) satisfies $u \in C^4([0,T],H) \cap C^3([0,T],V)$ and $x \in C^2([0,T],\mathcal{D}(\mathcal{A}))$ and that $\tau > 0$ is sufficiently small. Then, the approximation $x^n \approx x(t_n)$, $t_n = n\tau$ given in (3.10) satisfies the error bound*

$$\|x^n - x(t_n)\|_X \le C\mathrm{e}^{Kt_n}\tau^2,$$

where $K = \mathrm{c}_m + \frac{L_\rho(1+\sqrt{2})}{\sqrt{\alpha} - L_\rho\tau(1+\sqrt{2})}$ and the constant C only depends on u and T, and L_ρ defined in (3.11).

Proof. For the ease of presentation we only present the case $m = 1$. During this proof we again use a $\sim$ for an exact evaluation, e.g., we write $\widetilde{x}^n = x(t_n)$.

(1) *Error recursion.* Denote the first-order error by

$$e^n = x^n - \widetilde{x}^n.$$

We insert the exact solution into the IMEX scheme (3.10) and obtain as in [12, (4.29)] that

$$\widetilde{x}^{n+1} = R\widetilde{x}^n + \frac{\tau}{2}R_+^{-1}\widetilde{y}^n + \frac{\tau^2}{4}R_+^{-1}\widetilde{z}^n - \delta_{\mathrm{IMEX}}^{n+1}, \tag{3.12}$$

with

$$\delta_{\mathrm{IMEX}}^{n+1} = R_+^{-1}\delta_{\mathrm{CN}}^{n+1} + \widetilde{\delta}^{n+1}, \qquad \widetilde{\delta}^{n+1} = \frac{\tau^2}{4}\begin{pmatrix} 1 \\ \frac{\tau}{2}\widetilde{\alpha}\Delta_{\Omega,\Gamma} \\ \frac{1}{\lambda\gamma_+} \end{pmatrix} \otimes Q_+^{-1}(\widetilde{f}^n - \widetilde{f}^{n+1}) \tag{3.13}$$

and $\delta_{\mathrm{CN}}^{n+1}$ is the defect from the Crank-Nicolson scheme (3.1). As in [12, Thm. 4.3], it can be seen that $\left\|\delta_{\mathrm{CN}}^{n+1}\right\| \le C\tau^3$. Substracting (3.12) from (3.10) we obtain the error recursion

$$e^{n+1} = Re^n + \frac{\tau}{2}R_+^{-1}(y^n - \widetilde{y}^n) + \frac{\tau^2}{4}R_+^{-1}(z^n - \widetilde{z}^n) + \delta_{\mathrm{IMEX}}^{n+1},$$

cf. [12, (4.31)]. With

$$\triangle f^n = f^n - \widetilde{f}^n - f^{n+1} + \widetilde{f}^{n+1} \tag{3.14}$$

we have

$$y^n - \widetilde{y}^n = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \triangle f^n \qquad \text{and} \qquad z^n - \widetilde{z}^n = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{\lambda} \end{pmatrix} \triangle f^n.$$

(2) *Stability.* For $e^0 = 0$ it holds that

$$e^n = \sum_{\ell=1}^{n} R^{n-\ell} \left(\frac{\tau}{2} R_+^{-1} (y^{\ell-1} - \widetilde{y}^{\ell-1}) + \frac{\tau^2}{4} R_+^{-1} (z^{\ell-1} - \widetilde{z}^{\ell-1}) + \delta_{\mathrm{IMEX}}^{\ell} \right).$$

Analogously to [12, (4.32)], taking the norm and using the triangle inequality, we obtain the estimate

$$\begin{aligned} \|e^n\|_X \leq \tau \sum_{\ell=1}^{n} \mathrm{e}^{(n-\ell)\tau c_m} \left(\frac{1}{2} \|y^{\ell-1} - \widetilde{y}^{\ell-1}\|_X + \frac{\tau}{4} \|R_+^{-1}(z^{\ell-1} - \widetilde{z}^{\ell-1})\|_X \right) \\ + \left\| \sum_{\ell=1}^{n} R^{n-\ell} \delta_{\mathrm{IMEX}}^{\ell} \right\|_X , \end{aligned} \tag{3.15}$$

Recalling the spaces from (1.2), we estimate similarly to [12, (4.33)]

$$\begin{aligned} \|y^\ell - \widetilde{y}^\ell\|_X = \|\triangle f^\ell\|_H &\leq L_\rho \big(\|u^\ell - \widetilde{u}^\ell\|_V + \|u^{\ell+1} - \widetilde{u}^{\ell+1}\|_V \big) \\ &\leq \frac{L_\rho}{\sqrt{\alpha}} \big(\|e^\ell\|_X + \|e^{\ell+1}\|_X \big). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \frac{1}{2} \sum_{\ell=1}^{n} \mathrm{e}^{(n-\ell)\tau c_m} \|y^{\ell-1} - \widetilde{y}^{\ell-1}\|_X &\leq \sum_{\ell=1}^{n} \mathrm{e}^{(n-\ell)\tau c_m} \frac{L_\rho}{2\sqrt{\alpha}} \big(\|e^\ell\|_X + \|e^{\ell-1}\|_X \big) \\ &\leq \sum_{\ell=0}^{n} \frac{L_\rho}{\sqrt{\alpha}} \mathrm{e}^{(n-\ell)\tau c_m} \|e^\ell\|_X . \end{aligned}$$

From

$$R_+^{-1} z^n = \begin{pmatrix} 1 \\ \frac{\widetilde{\alpha}\tau}{2} \Delta_{\Omega,\Gamma} \\ \frac{1}{\lambda\gamma_+} \end{pmatrix} \otimes Q_+^{-1} (f^n - f^{n+1}),$$

and the notation (3.14) we obtain

$$\begin{aligned} &\frac{\tau}{4} \|R_+^{-1}(z^\ell - \widetilde{z}^\ell)\|_X \\ &\quad = \frac{1}{2} \left(\alpha \left\| \frac{\tau}{2} Q_+^{-1} \triangle f^\ell \right\|_V^2 + \left\| \frac{\tau^2}{4} \widetilde{\alpha} \Delta_{\Omega,\Gamma} Q_+^{-1} \triangle f^\ell \right\|_H^2 + \mu \left\| \frac{\tau}{2} \frac{1}{\lambda\gamma_+} Q_+^{-1} \triangle f^\ell \right\|_V^2 \right)^{\frac{1}{2}} \end{aligned}$$

and from (3.9a) and (3.9b) we conclude

$$\alpha \left\| \frac{\tau}{2} Q_+^{-1} w \right\|_V^2 \le \frac{\alpha}{2\widetilde{\alpha}} \|w\|_H^2 ,$$
$$\mu \left\| \frac{\tau}{2} \frac{1}{\lambda\gamma_+} Q_+^{-1} w \right\|_V^2 \le \frac{\mu}{2(\lambda\gamma_+)^2 \widetilde{\alpha}} \|w\|_H^2 .$$

This yields

$$\begin{aligned}
&\sum_{\ell=1}^{n} \frac{\tau \mathrm{e}^{(n-\ell)\tau c_m}}{4} \left\| R_+^{-1} (z^{\ell-1} - \widetilde{z}^{\ell-1}) \right\|_X \\
&\qquad \le \sum_{\ell=1}^{n} \frac{L_\rho \mathrm{e}^{(n-\ell)\tau c_m}}{\sqrt{2\alpha}} \left(\left\| u^\ell - \widetilde{u}^\ell \right\|_{V_\alpha} + \left\| u^{\ell-1} - \widetilde{u}^{\ell-1} \right\|_{V_\alpha} \right) \\
&\qquad \le \sum_{\ell=0}^{n} \frac{L_\rho \sqrt{2}}{\sqrt{\alpha}} \mathrm{e}^{(n-\ell)\tau c_m} \left\| e^\ell \right\|_X .
\end{aligned}$$

Inserting these bounds yields

$$\|e^n\|_X \le \frac{\tau L_\rho}{\sqrt{\alpha}} (1 + \sqrt{2}) \sum_{\ell=0}^{n} \mathrm{e}^{(n-\ell)\tau c_m} \left\| e^\ell \right\|_H + \left\| \sum_{\ell=1}^{n} R^{n-\ell} \delta_{\mathrm{IMEX}}^\ell \right\|_X .$$

(3) *Bound of defects from* (3.13). Since we already know that the Crank-Nicolson defect is bounded by $C\tau^3$, it remains to bound $\widetilde{\delta}^{\ell+1}$. We follow [12, p. 41] and split it into two parts

$$\widetilde{\delta}^{\ell+1} = \widetilde{\delta}_1^{\ell+1} + \widetilde{\delta}_2^{\ell+1}$$

with

$$\widetilde{\delta}_1^{\ell+1} = \frac{\tau}{4} \begin{pmatrix} \tau Q_+^{-1} \\ Q_- Q_+^{-1} \\ \frac{\tau}{\lambda\gamma_+} Q_+^{-1} \end{pmatrix} \otimes (\widetilde{f}^\ell - \widetilde{f}^{\ell+1}), \qquad \widetilde{\delta}_2^{\ell+1} = \frac{\tau}{4} \begin{pmatrix} 0 \\ -(\widetilde{f}^\ell - \widetilde{f}^{\ell+1}) \\ 0 \end{pmatrix},$$

and then combine terms from two different steps to gain an extra order of τ. We observe that

$$\widetilde{\delta}_1^{\ell+1} + R\widetilde{\delta}_2^\ell = \frac{\tau}{2} \begin{pmatrix} \frac{\tau}{2} Q_+^{-1} \\ \frac{1}{2} Q_- Q_+^{-1} \\ \frac{\tau}{2\lambda\gamma_+} Q_+^{-1} \end{pmatrix} \otimes (-\widetilde{f}^{\ell-1} + 2\widetilde{f}^\ell - \widetilde{f}^{\ell+1}). \tag{3.16}$$

For the difference quotients we have the bounds

$$\left\| \widetilde{\delta}_2^{\ell+1} \right\|_X = \frac{\tau}{4} \left\| \widetilde{f}^\ell - \widetilde{f}^{\ell+1} \right\|_H \le C\tau^2, \qquad \tau \left\| \widetilde{f}^{\ell-1} - 2\widetilde{f}^\ell + \widetilde{f}^{\ell+1} \right\|_H \le C\tau^3. \tag{3.17}$$

To be more precise, we denote by $\mathcal{I}^\ell = [t_{\ell-1}, t_{\ell+1}]$ and as in [12, (4.36)] we obtain, due to the regularity assumptions on u, that

$$\begin{aligned}\tau \left\| \widetilde{f}^{\ell-1} - 2\widetilde{f}^{\ell} + \widetilde{f}^{\ell+1} \right\|_H &\leq C\tau^3 \left\| \partial_t^2 \big(u'' - \Delta_{\Omega,\Gamma}(\alpha u - \beta^T M)\big) \right\|_{L^\infty(\mathcal{I}^\ell, H)} \\ &\leq C\tau^3 \left(\left\| u^{(4)} \right\|_{L^\infty(\mathcal{I}^\ell, H)} + \left\| \partial_t^2 \big(\alpha u + \beta^T M\big) \right\|_{L^\infty(\mathcal{I}^\ell, H^2)} \right).\end{aligned}$$

where $H^2 = H^2(\Omega)$ for (D) and $H^2 = H^2(\Omega, \Gamma)$ in the case (K). Using the same arguments, we find that defects $\widetilde{\delta}_1^1$, $\widetilde{\delta}_2^n$ are of order τ^2. Therefore, we estimate

$$\begin{aligned}\tau \left\| \widetilde{f}^{\ell} - \widetilde{f}^{\ell+1} \right\|_H &\leq C\tau^2 \left\| \partial_t \big(u'' - \Delta_{\Omega,\Gamma}(\alpha u - \beta^T M)\big) \right\|_{L^\infty(\mathcal{I}^\ell, H)} \\ &\leq C'\tau^2 \left(\left\| u^{(3)} \right\|_{L^\infty(\mathcal{I}^\ell, H)} + \left\| \partial_t \big(\alpha u + \beta^T M\big) \right\|_{L^\infty(\mathcal{I}^\ell, H^2)} \right).\end{aligned}$$

Then, using (3.13) we split

$$\left\| \sum_{\ell=1}^n R^{n-\ell} \delta_{\mathrm{IMEX}}^\ell \right\|_X \leq \left\| \sum_{\ell=1}^n R^{n-\ell} \delta_{\mathrm{CN}}^\ell \right\|_X + \left\| \sum_{\ell=1}^n R^{n-\ell} (\widetilde{\delta}_1^\ell + \widetilde{\delta}_2^\ell) \right\|_X$$

and with an index shift and the previous calculations, we see

$$\begin{aligned}\left\| \sum_{\ell=1}^n R^{n-\ell} (\widetilde{\delta}_1^\ell + \widetilde{\delta}_2^\ell) \right\|_X &= \left\| \sum_{\ell=1}^n R^{n-\ell} \widetilde{\delta}_1^\ell + R^{n-\ell-1} R \widetilde{\delta}_2^\ell \right\|_X \\ &= \left\| R^{n-1} \widetilde{\delta}_1^1 + \widetilde{\delta}_1^n + \sum_{\ell=2}^n R^{n-\ell} (\widetilde{\delta}_1^\ell + R\widetilde{\delta}_2^\ell) \right\|_X \\ &\leq \mathrm{e}^{n\tau c_m} \left(\left\| \widetilde{\delta}_1^1 \right\|_X + \left\| \widetilde{\delta}_2^n \right\|_X + \sum_{\ell=2}^n \left\| \widetilde{\delta}_1^\ell + R\widetilde{\delta}_2^\ell \right\|_X \right) \\ &\leq C\mathrm{e}^{Tc_m} \tau^2.\end{aligned}$$

In the last estimate we used $t_n \leq T$, (3.16), (3.17), and that $\left\| \tau Q_+^{-1} \right\|_{V \leftarrow H} \leq C$ and $\left\| Q_- Q_+^{-1} \right\|_{H \leftarrow H} \leq C$ by Lemma 3.

With the stability bound (3.15), we see

$$\begin{aligned}\mathrm{e}^{-t_n c_m} \left\| e^n \right\|_X &\leq \frac{\tau L_\rho}{\sqrt{\alpha}} (1 + \sqrt{2}) \sum_{\ell=0}^n \mathrm{e}^{-\ell\tau c_m} \left\| e^\ell \right\|_X + \mathrm{e}^{-t_n c_m} \left\| \sum_{\ell=1}^n R^{n-\ell} \delta_{\mathrm{IMEX}}^\ell \right\|_X \\ &\leq \frac{\tau L_\rho}{\sqrt{\alpha}} (1 + \sqrt{2}) \sum_{\ell=0}^n \mathrm{e}^{-\ell\tau c_m} \left\| e^\ell \right\|_X + C\tau^2.\end{aligned}$$

Finally, the desired error bound follows from applying Grönwall's Lemma provided that x^n is uniformly bounded in X. The latter follows along the lines of [41, Thm. 4.9]. □

Remark 3. *The full discretization can be done analogously to [37] with isoparametric finite elements, see also [42]. Using the time discretization error analysis from the previous section, the full error estimate is a combination thereof with the one shown in [37]. When it comes to lift and reference operators for the error of space discretization, the additional variables M_j are treated in the same manner as u.*

4 Numerical experiments

In this section we show numerical examples to support our theoretical results. We consider the wave equation (1.1) with kinetic boundary conditions on the unit disc $B(0,1) \subset \mathbb{R}^2$. In order to measure exact numerical errors, we extend an example from [37] by including a convolution with $m = 4$ and the parameters $\alpha = 1, \beta_1 = 1, \beta_2 = 1.5, \beta_3 = 2, \beta_4 = 1.25$ and $\lambda_1 = 0.2, \lambda_2 = 1, \lambda_3 = 1.5, \lambda_4 = 2$. We choose the nonlinearities

$$\begin{aligned} f_\Omega(t,\xi,u) &= |u|\, u + \eta_\Omega(t,\xi), \\ f_\Gamma(t,\xi,u) &= |u|^2\, u + \eta_\Gamma(t,\xi), \end{aligned}$$

where

$$\begin{aligned} \eta_\Omega(t,\xi) &= -\big(4\pi^2 + |\sin(2\pi t)\xi_1\xi_2|\big)\sin(2\pi t)\xi_1\xi_2, \\ \eta_\Gamma(t,\xi) &= (6c^2 - 4\pi^2)\xi_1\xi_2\sin(2\pi t) - \big(\sin(2\pi t)\xi_1\xi_2\big)^3 \\ &\qquad - 12\pi\sum_{j=1}^{m}\frac{\beta_j}{4\pi^2+\lambda_j}\big(\mathrm{e}^{-\lambda_j t} - \cos(2\pi t) + \frac{\lambda_j}{2\pi}\sin(2\pi t)\big). \end{aligned}$$

The initial values are set to be

$$u(0,\xi) = 0, \qquad \partial_t u(0,\xi) = 2\pi\xi_1\xi_2,$$

and the exact solution to this example is given by

$$u(t,\xi) = \sin(2\pi t)\xi_1\xi_2.$$

The codes to reproduce our results are available at
https://github.com/MalikScheifinger/WaveKineticBC.git
The space discretization software is based on the FEM library `deal.II` [43] version 9.5.0, using quadrilateral mesh elements and isoparametric finite elements with polynomial degree $p = 2$ and maximal mesh width $h_{\max} \approx 0.014$. Our implementation follows [12, Chapter 6.2]. We shortly specify the notation we use and state our in space discretized IMEX scheme. We denote by $\boldsymbol{\mathcal{M}} \in \mathbb{R}^{N\times N}$ the mass matrix, by $\boldsymbol{\mathcal{S}} \in \mathbb{R}^{N\times N}$

the stiffness matrix and the load vector by $\boldsymbol{f}^n$ after choosing the standard nodal basis. Then the scheme reads

$$\begin{aligned}
\boldsymbol{\mathcal{M}}\boldsymbol{v}^{n+\frac{1}{2}} &= \boldsymbol{\mathcal{M}}\boldsymbol{v}^{n} - \frac{\tau}{2}\boldsymbol{\mathcal{S}}(\alpha\boldsymbol{u}^n + \widetilde{\beta}^T\boldsymbol{M}^n) - \frac{\tau^2}{4}\widetilde{\alpha}\boldsymbol{\mathcal{S}}\boldsymbol{v}^{n+\frac{1}{2}} + \frac{\tau}{2}\boldsymbol{f}^n, && \text{(4.1a)}\\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}^n + \tau\boldsymbol{v}^{n+\frac{1}{2}}, && \text{(4.1b)}\\
\boldsymbol{M}_j^{n+1} &= \gamma_j\boldsymbol{M}_j^n + \gamma_{j,+}^{-1}\frac{\tau}{\lambda_j}\boldsymbol{v}^{n+\frac{1}{2}}, && \text{(4.1c)}\\
\boldsymbol{\mathcal{M}}\boldsymbol{v}^{n+1} &= -\boldsymbol{\mathcal{M}}\boldsymbol{v}^{n} + 2\boldsymbol{\mathcal{M}}\boldsymbol{v}^{n+\frac{1}{2}} + \frac{\tau}{2}(\boldsymbol{f}^{n+1} - \boldsymbol{f}^n). && \text{(4.1d)}
\end{aligned}$$

The linear system (4.1a) is equivalent to

$$\boldsymbol{\mathcal{Q}}_+\boldsymbol{v}^{n+\frac{1}{2}} = \boldsymbol{\mathcal{M}}\boldsymbol{v}^{n} - \frac{\tau}{2}\boldsymbol{\mathcal{S}}(\alpha\boldsymbol{u}^n + \widetilde{\beta}^T\boldsymbol{M}^n) + \frac{\tau}{2}\boldsymbol{f}^n, \qquad \boldsymbol{\mathcal{Q}}_+ = \boldsymbol{\mathcal{M}} + \frac{\tau^2}{4}\widetilde{\alpha}\boldsymbol{\mathcal{S}}.$$

We solve this linear system with the conjugate gradient method and SSOR preconditioning.

In Figure 1 we show the numerical approximation of (1.1) using the data given in Section 4 at four different time stamps.

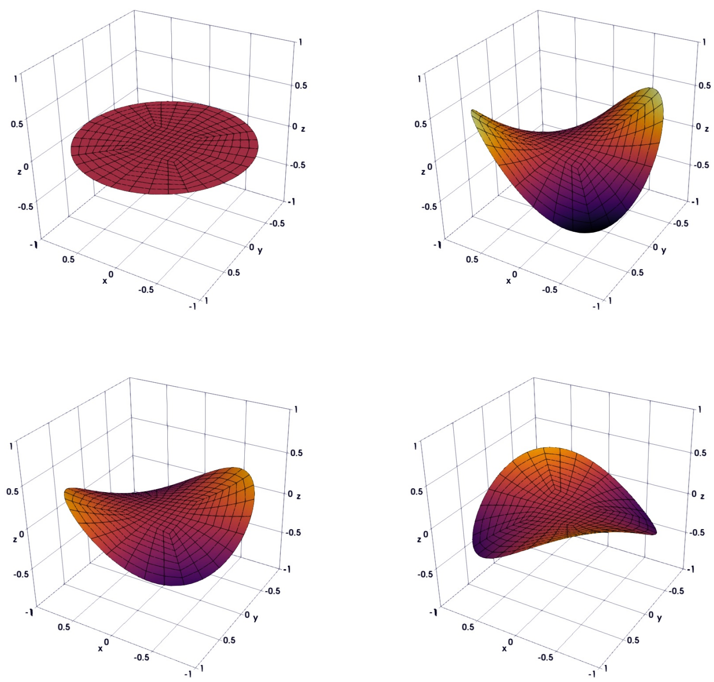

Fig. 1: Snapshots of the solution at times $t = 0, 0.2, 0.4, 0.6$ (from top left to bottom right) using the IMEX method with time stepsize $\tau = 0.1$.

In Figure 2 we illustrate the error of our method against the time stepsize τ. The error is measured at the endtime $T = 0.8$ in the $V_\alpha \times H$ norm. We observe second-order convergence in time.

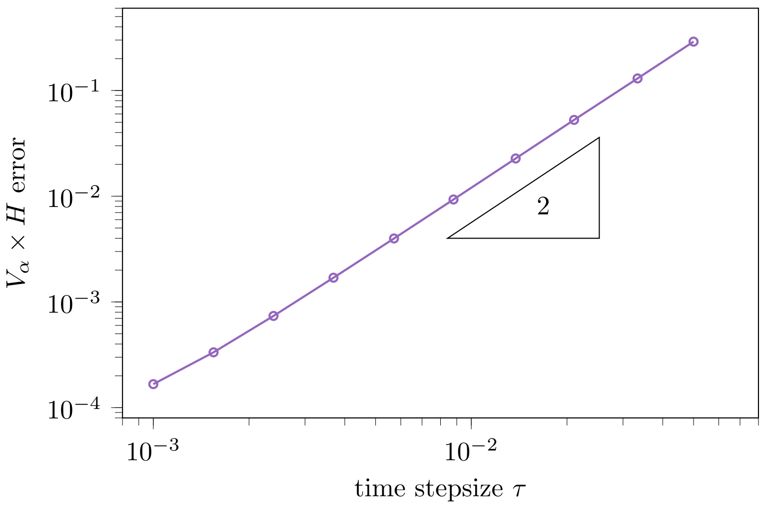

Fig. 2: Error evaluated at the endtime $T = 0.8$ in the $V_\alpha \times H$ norm plotted against time stepsize τ.

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