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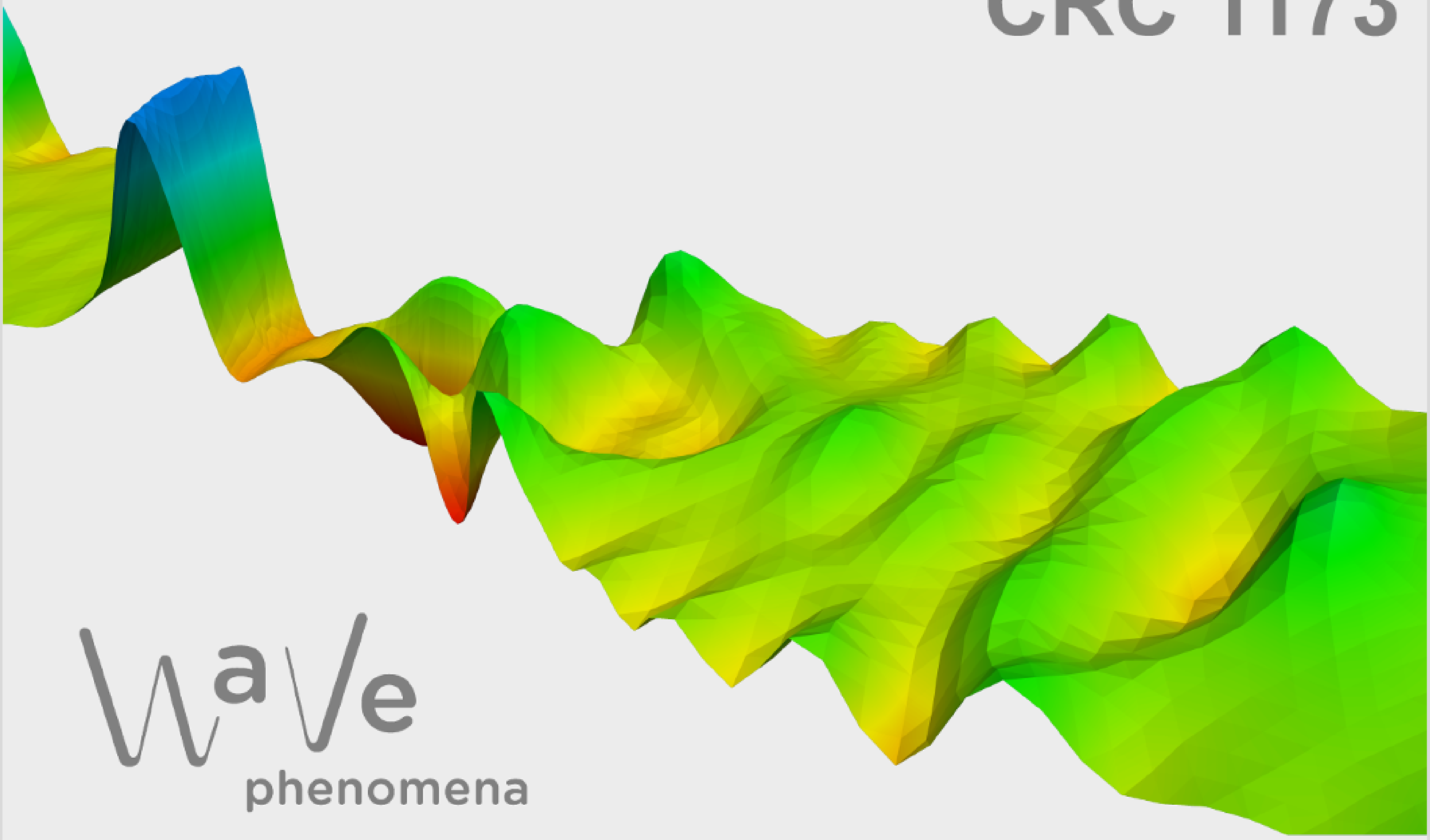

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APPROXIMATION OF MINIMIZERS OF THE GINZBURG–LANDAU ENERGY IN NON-CONVEX DOMAINS

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Abstract. In this work, we study the approximation of minimizers of the Ginzburg–Landau energy over non-convex polygonal and polyhedral domains. We discretize the order parameter with Lagrange finite elements and the vector potential with Nédélec elements. We show that under certain resolution conditions of the mesh sizes and the Ginzburg–Landau parameter, we obtain quasi-best approximation error bounds. In two dimensions, the order of convergence can be fully determined by the angle of the largest reentrant corner.

1. Introduction

We study the Ginzburg–Landau energy functional E given by

$$E(u, \mathbf{A}) = \frac{1}{2} \int_\Omega |\frac{\mathrm{i}}{\kappa} \nabla u + \mathbf{A} u|^2 + \frac{1}{2}(1 - |u|^2)^2 + |\operatorname{curl} \mathbf{A} - \mathbf{H}|^2 \, \mathrm{d}x \tag{1.1}$$

for the order parameter $u\colon \Omega \to \mathbb{C}$ and the vector potential $\mathbf{A}\colon \Omega \to \mathbb{R}^d$, which are global minimizers. of the energy E. Here, $\mathbf{H}$ is a given external magnetic field and $\kappa \in \mathbb{R}^+$ is a material parameter, often called the Ginzburg–Landau parameter. We minimize the energy under the side constraint $\operatorname{div} \mathbf{A} = 0$, where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a possibly non-convex polygonal or polyhedral domain Ω which for simplicity is assumed to be simply connected. Such minimizers describe the superconductivity in a material Ω, cf. [19, Sec. 3], where the physically relevant quantities are the density $|u(x)|^2$ as well as the effective magnetic field $\operatorname{curl} \mathbf{A}$. The present scaling of the model ensures $0 \leq |u|^2 \leq 1$. Here, the state $|u(x)|^2 = 1$ encodes local superconductivity whereas $|u(x)|^2 = 0$ means that the material is locally not superconducting. In these mixed normal-superconducting states, both phases can coexist in a so-called Abrikosov vortex lattice [1] with $|u(x)|^2 = 0$ in the vortex centers. The non-convex domains can for example model defects in the material.

In the recent works [8, 13, 14] the approximation of minimizers of (1.1) for large values of κ by different finite element methods was studied and best-approximation results have been shown. Further, resolution conditions of the meshes and the Ginzburg–Landau parameter where established. In the present work, we focus on a new aspect of the problem and consider polygonal and polyhedral domains which are not convex. This induces two major problems: First, the vector potential is now only slightly better than $H^{\frac{1}{2}}$ instead of H^2 (on a cube) and the embedding only yields slightly more than L^3, compared to L^∞. Second, for the discrete minimizers we have to enforce the divergence constraint in a weak sense, and thus end up with a non-conforming method which poses additional technical difficulties.

The first results for the numerical treatment of the Ginzburg–Landau equation were derived in the pioneering works by Du, Gunzburger and Peterson [19, 20]. The authors derived $H^1(\Omega)$-error estimates of optimal order in both variables, however without tracking the influence of the parameter κ. Later on, a covolume method [22] and a finite volume method [21] was used for the discretization. In [14] we considered a simplified version of (1.1) ($\mathbf{A}$ given) and we showed error bounds which are explicit in κ and the spatial parameter. In the same setting, a thorough investigation of the Localized Orthogonal Decomposition (LOD) applied to this problem was performed in [8]. The full problem (1.1) was then tackled in the work [13], where we used different meshes and ansatz spaces for the order parameter and the vector potential. With a detailed a-priori analysis on the

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regularity of the minimizers, we were ale to derive optimal a-priori bounds which turned out to be (almost) sharp in terms of κ and the spatial parameters.

In this work, we focus on the case of reduced regularity in both variables. Since one can in general not expect that the vector potential is in $\mathbf{H}^1(\Omega)$, we cannot use Lagrange element, but instead employ Nédélec elements. Hence, we have to enforce the divergence constraint in a discrete manner, which makes our ansatz non-confirming and thus several technical difficulties arise throughout the analysis. Further, the reduced regularity implies also a reduced integrability, and we have to be far more careful in the a-priori analysis of the minimizers and the bounds on the Fréchet derivatives of the energy E. In order to translate the abstract approximation results to specific domains, we keep the parameter $s = s(\Omega) > 0$ stemming from the elliptic regularity $H^{\frac{3}{2}+s}$ of the Poisson problem in non-convex polygonal and polyhedral domains, in all our estimates. For special geometries, such as the L-shaped domain in $\mathbb{R}^2$ where $s \sim \frac{1}{6}$, we can then directly read of the expected convergence rate.

This work is motivated by error analysis of the time-dependent Ginzburg–Landau equation by B. Li in [31]. Due to the time-dependence of the problem the low regularity prohibits the derivation of convergence rates, however abstract convergence could still be established. In contrast, any minimizer of (1.1) solves the critical point equation $E' = 0$, i.e. an elliptic problem, and thus additional regularity of the minimizers can be extracted which then allows also for convergence rates. Let us mention that there are many more results available on the time-dependent Ginzburg–Landau equation which has the same structure as a gradient flow applied to the presented energy above. Several results can be found in [5–7, 10, 11, 15–18, 23, 27, 28, 32–34] and the references therein, but most of them do not consider the precise dependence of the error bounds on the Ginzburg–Landau parameter κ.

The rest of the paper is organized as follows: In Section 2, we introduce the analytical framework and derive regularity estimates for the order parameter and the magnetic vector potential. Using a stabilized norm, we derive in Section 3 various estimate on the minimizers, the energy and its Fréchet derivatives. The space discretization is introduced in Section 4, where we also discuss the approximation by the Ritz projection in several norms. Finally, we conclude with our main results and the corresponding proofs in Section 5. Some technical results and a collection of regularity estimates is presented in the Appendices A, B, and C.

Notation. For a complex number $z \in \mathbb{C}$, we use z^* for the complex conjugate of z. In the whole paper we further denote by $L^2(\Omega) := L^2(\Omega, \mathbb{C})$ the Hilbert space of L^2-integrable complex functions, but equipped with the *real* scalar product $(u, v)_{L^2} := \mathrm{Re} \int_\Omega v\, w^*\, dx$ for $v, w \in L^2(\Omega)$. Hence, we interpret the space as a *real* Hilbert space. Analogously, we equip the space $H^1(\Omega) := H^1(\Omega, \mathbb{C})$, which will be the solution space for the order parameter, with the scalar product $(v, w)_{L^2} + (\nabla v, \nabla w)_{L^2}$. This interpretation is crucial so that the Fréchet derivatives of E are meaningful and exist on $H^1(\Omega)$. For any space X, we denote its dual space by X'. Note that this implies, that the elements of the dual space of $H^1(\Omega)$ consist of real-linear functionals, which are not necessarily complex-linear. For example, if $F(v) := (f, v)_{L^2}$ for some $f \in L^2(\Omega)$, then it holds $F(\alpha\, v) = \alpha\, F(v)$ if $\alpha \in \mathbb{R}$, but in general *not* if $\alpha \in \mathbb{C}$.

For the *real-valued* vector potentials, we use boldface letters and denote $\mathbf{L}^2(\Omega) := L^2(\Omega; \mathbb{R}^3)$ and $\mathbf{H}^1(\Omega) := H^1(\Omega; \mathbb{R}^3)$. Note that functions in $H^1(\Omega)$ are complex-valued, whereas functions in $\mathbf{H}^1(\Omega)$ are real-valued. Further, we use the standard spaces for the weak rotation $\mathbf{H}(\mathrm{curl})$ and divergence $\mathbf{H}(\mathrm{div})$, together with their closed subspaces $\mathbf{H}_0(\mathrm{div})$, i.e. $\mathbf{H} \cdot \nu = 0$ on $\partial\Omega$, and $\mathbf{H}_0(\mathrm{curl})$, i.e. $\mathbf{H} \times \nu = 0$ on $\partial\Omega$.

Throughout the paper C denotes a generic constant which is independent of κ and the spatial mesh parameters H and h, but might depend on numerical constants as well as Ω, $\mathbf{H}$. In particular, we write $\alpha \lesssim \beta$ if there is a constant C independent of κ, H and h such that $\alpha \leq C\, \beta$.

2. Solution spaces, gauges and first a-priori bounds on the minimizers

In this section, we present the functional analytic framework and the relevant properties of the spaces for the exact minimizers. For convenience of the reader, we recall also the known results and present, where necessary, the adaptions to the non-convex case. Further, throughout the paper we cover the cases $d = 2$ and $d = 3$. In the two-dimensional case, we assume without loss of generality that Ω is aligned in the x-y-plane and the magnetic field is given by $\mathbf{H}(x, y, z) = H_3(x, y)e_z$, i.e. it is orthogonal to Ω. The vector potential then has the form $\mathbf{A}(x, y, z) = (A_1(x, y), A_2(x, y), 0)$. Here, we make use of the standard 2d rotations by embedding $\mathbf{H}$ and $\mathbf{A}$ in their 3d representation. In particular, this implies that $\operatorname{div} \mathbf{H} = 0$ for any choice of H_3.

A crucial parameter for the quantification of the regularity of the minimizers is the real number $s > 0$, which is obtained from elliptic regularity results of the Poisson problem with solution in $H^{\frac{3}{2}+s}$. Since this s might be different for the Dirichlet and Neumann problem, we will work with the minimum of both numbers. Note that the formal limit $s \to \frac{1}{2}$ corresponds to the convex case, and we recover (at least formally) several results of the preceding works [13, 14].

2.1. **Spaces for the minimizers.** For the order parameter u, we employ the standard Sobolev space $H^1(\Omega)$, and for the vector potential, we employ the spaces

$$\mathbf{H}(\operatorname{curl}, \operatorname{div}) := \{B \in L^2(\Omega) \mid \operatorname{curl} B \in L^2(\Omega), \operatorname{div} B \in L^2(\Omega), B \cdot \nu = 0 \text{ on } \partial\Omega\}, \tag{2.1a}$$

$$\mathbf{H}_0(\operatorname{curl}, \operatorname{div}) := \{B \in \mathbf{H}(\operatorname{curl}, \operatorname{div}) \mid \operatorname{div} B = 0\}, \tag{2.1b}$$

which are in the literature often denoted by X_T and $X_{T,0}$, respectively, see [35, Section 3.8]. Both spaces can be equipped with the natural norm

$$||B||_{L^2}^2 + ||\operatorname{curl} B||_{L^2}^2 + ||\operatorname{div} B||_{L^2}^2. \tag{2.2}$$

Let us note that by [35, Corollary 3.49] the spaces $\mathbf{H}(\operatorname{curl}, \operatorname{div})$ and $\mathbf{H}_0(\operatorname{curl}, \operatorname{div})$ compactly embedd into L^2, and by [35, Corollary 3.51] the norm in (2.2) is equivalent to

$$||B||_{\mathbf{H}(\operatorname{curl},\operatorname{div})}^2 = ||\operatorname{curl} B||_{L^2}^2 + ||\operatorname{div} B||_{L^2}^2, \qquad ||B||_{\mathbf{H}_0(\operatorname{curl},\operatorname{div})} = ||\operatorname{curl} B||_{L^2},$$

respectively. For completness, we introduce the spaces of harmonics $K_N(\Omega)$ and $K_T(\Omega)$ by

$$\begin{aligned} K_T(\Omega) &= \{\mathbf{B} \in \mathbf{H}(\operatorname{curl}) \cap \mathbf{H}(\operatorname{div}) \mid \operatorname{curl} \mathbf{B} = \operatorname{div} \mathbf{B} = 0 \quad \text{and} \quad \mathbf{B} \cdot \nu|_{\partial\Omega} = 0\} \perp \nabla H^1(\Omega) \\ K_N(\Omega) &= \{\mathbf{B} \in \mathbf{H}(\operatorname{curl}) \cap \mathbf{H}(\operatorname{div}) \mid \operatorname{curl} \mathbf{B} = \operatorname{div} \mathbf{B} = 0 \quad \text{and} \quad \mathbf{B} \times \nu|_{\partial\Omega} = 0\} \perp \nabla H_0^1(\Omega). \end{aligned} \tag{2.3}$$

and recall that both spaces are finite dimensional, where $\dim K_T(\Omega)$ equals the first Betti number, which is zero if Ω is simply connected, and $\dim K_N(\Omega)$ equals the second Betti number, which counts the number of two-dimensional wholes.

We make frequent use of the following decomposition, using the spaces from (2.3), see for example [35, Theorem 3.45].

Theorem 2.1 (Helmholtz decomposition)**.** *Let* $\mathbf{B} \in L^2(\Omega)$.

(a) $\mathbf{B}$ *can also be decomposed as*

$$\mathbf{B} = \nabla p + \operatorname{curl} \mathbf{C} + f_T$$

with unique $p \in H^1(\Omega)$, $\mathbf{C} \in \mathbf{H}_0(\operatorname{curl})$ *and* $f_T \in K_T(\Omega)$.

(b) $\mathbf{B}$ *can be decomposed as*

$$\mathbf{B} = \nabla p + \operatorname{curl} \mathbf{C} + f_N$$

with unique $p \in H_0^1(\Omega)$, $f_N \in K_N(\Omega)$ *and* $\mathbf{C} \in \mathbf{H}_0(\operatorname{curl}, \operatorname{div})$.

We will often use the second decomposition in the form

$$\operatorname{curl} \mathbf{B} = \nabla p + \operatorname{curl} \mathbf{C} + f_N$$

and denote by π_{div} the map

$$\pi_{\operatorname{div}} \colon \mathbf{H}(\operatorname{curl}) \to \mathbf{H}_0(\operatorname{curl}, \operatorname{div}), \quad \mathbf{B} \mapsto \mathbf{C} \tag{2.4}$$

which lifts a function onto on the divergence free component of $\mathbf{B}$, see [35, Lemma 7.6]. A similar construction is found in [26, Sec. 44.2.1] called *curl-preserving lifting*. For the rest of this work, we assume that

$$\dim K_T(\Omega) = 0,$$

i.e. that Ω is simply connected.

In order to derive convergence rates, it is not sufficient to have regularity of the minimizers in $\mathbf{H}(\text{curl})$ and $\mathbf{H}(\text{div})$, but we need the following embeddings into standard Sobolev spaces. The first result can be found in [3, Prop. 3.7].

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz polyhedron. Then, there exists $s \in (0, \frac{1}{2}]$ such that*

$$||B||_{H^{\frac{1}{2}+s}} \leq C||B||_{\mathbf{H}(\text{curl},\text{div})}$$

for all $B \in \mathbf{H}(\text{curl}, \text{div})$. Further, we have

$$H^{\frac{1}{2}+s} \hookrightarrow L^{\frac{3}{1-s}}, \quad d = 3,$$
$$H^{\frac{1}{2}+s} \hookrightarrow L^{\frac{4}{1-2s}}, \quad d = 2.$$

Remark 2.3. *In order to compare the results from this work to the ones in the convex case, we note the following: For $d = 3$, (formally) setting $s = 1$ would give us $\mathbf{A} \in L^\infty$, which was used in the [13, 14], where obviously in the latter work, this was not done by the embedding from $\mathbf{H}(\text{curl}, \text{div})$, but from $\mathbf{A} \in H^2$. Let us already mention here, that setting $s = 1$ in the results obtained below allows us to recover the known results. Similarly, for $d = 2$ (formally) setting $s = \frac{1}{2}$ allows for the recovery of the known results. Here, we mainly focus on the case $s \to 0$.*

2.2. Explicit estimates in two dimensions.

In two dimensions we are able to quantify the exponent s by the angles of the corners. Thus, we present more detailed results in the following. In the next theorem, we require a special exponent defined via

$$p_{\max} = \begin{cases} \frac{\omega_{max}}{\omega_{max}-\pi/2}, & \frac{\pi}{2} < \omega_{max} < 2\pi, \\ \infty, & \frac{\pi}{2} \geq \omega_{max}, \end{cases} \tag{2.5}$$

and note that it always holds $p_{\max} > \frac{4}{3}$.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^2$ be a non-convex polygonal domain with maximal angle $\omega_{\max} \in (\pi, 2\pi)$. Then, we have the embedding*

$$||B||_{W^{1,p}} \leq C||B||_{\mathbf{H}(\text{curl},\text{div})}$$

for all $p < p_{\max} < 2$ given in (2.5). In addition, this implies

$$||B||_{H^{\frac{1}{2}+s}} \leq C||B||_{\mathbf{H}(\text{curl},\text{div})}, \qquad s < s_0 = \frac{2\pi - \omega_{\max}}{2\omega_{\max}},$$
$$||B||_{L^q} \leq C||B||_{\mathbf{H}(\text{curl},\text{div})}, \qquad q < q_0 = \frac{2\omega_{\max}}{\omega_{\max} - \pi},$$

where in all cases $s_0 > 0$ and $q_0 > 4$. If all angles ω satisfy $\omega \leq \pi$, the domain is convex, and we recover the cases $p = 2$, $s = 1$. Further, we note the relation $p_{\max} = \frac{4}{3-2s_0}$.

Let us note the following examples before we give the proof:

$$\omega_{\max} \sim \pi \implies p_{\max} \sim 2, \quad s_0 \sim 0, \quad q_0 \sim \infty$$
$$\omega_{\max} \sim 2\pi \implies p_{\max} \sim \frac{4}{3}, \quad s_0 \sim \frac{1}{2}, \quad q_0 \sim 4$$
$$\omega_{\max} = \frac{3\pi}{2} \implies p_{\max} = \frac{3}{2}, \quad s_0 = \frac{1}{6}, \quad q_0 = 6.$$

Proof of Theorem 2.4. Let $p < \min\{2, p_{\max}\}$. We follow the proof of [3, Prop. 3.7]. Here, we obtain some $B_2 \in H^1(\Omega)$ with $\operatorname{curl} B = \operatorname{curl} B_2$ and $\operatorname{div} B_2 = 0$. Since by construction $\operatorname{curl}(B - B_2) = 0$, there is $\chi \in H^1(\Omega)$ with $B - B_2 = \nabla\chi$ and χ solves the Neumann problem

$$\Delta\chi = \operatorname{div} B \text{ in } \Omega, \qquad \partial_\nu \chi = -B_2 \cdot \nu \text{ on } \partial\Omega,$$

and $B_2 \cdot \nu \in H^{\frac{1}{2}}(F)$ for all faces of $\partial\Omega$. In particular since $p < 2$, we have the trivial embedding $B_2 \cdot \nu \in W^{1-\frac{1}{p},p}(F)$ for all faces and, as there are no compatibility conditions (see [30, Theorem 5.1.2.3] which is trivial in the Neumann case for $d = 0$), it holds $B_2 \cdot \nu \in W^{1-\frac{1}{p},p}(\partial\Omega)$. With [30, Theorem 5.1.2.3] we find $\Psi \in W^{2,p}$ such that

$$\partial_\nu \Psi = -B_2 \cdot \nu \text{ on } \partial\Omega, \qquad \text{and} \qquad \Delta\Psi - \operatorname{div} B \perp 1,$$

and we are left to solve

$$\Delta(\chi - \Psi) = \operatorname{div} B - \Delta\Psi \in L^p \text{ in } \Omega, \qquad \partial_\nu(\chi - \Psi) = 0 \text{ on } \partial\Omega.$$

By [30, Cor. 4.4.3.4], there is a unique solution given by $\chi - \Psi \in W^{2,p}$, and thus $\nabla\chi \in W^{1,p}$. We have hence established the representation

$$B = B_2 + \nabla\chi \in W^{1,p} \hookrightarrow H^{\frac{2-2p}{p}} \cap L^{\frac{2p}{2-p}}.$$

Using the definition in (2.5), it holds

$$\frac{2 - 2p_{\max}}{p_{\max}} = \frac{\pi}{\omega_{\max}} = \frac{1}{2} + \frac{2\pi - \omega_{\max}}{2\omega_{\max}} \quad \text{and} \quad \frac{2p_{\max}}{2 - p_{\max}} = \frac{2\omega_{\max}}{\omega_{\max} - \pi},$$

and the claim follows. □

2.3. **Existence and first a-priori bounds on minimizers.** We first present the existence of (1.1) by adapting the proof in [19] to the non-convex case.

Theorem 2.5. *There exists at least one minimizer* $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}_0(\operatorname{curl}, \operatorname{div})$ *of* (1.1).

Proof. We follow the proof in [19, Prop. 3.4], and note that the energy E is nonnegative, continuous in the strong topology, and lower semicontinuous in the weak topology. We now consider a minimizing sequence and $(u_n, \mathbf{A}_n)$ and obtain without loss of generality $E(u_n, \mathbf{A}_n) \leq E(0, 0) \lesssim 1$. This directly gives us with Lemma 2.2

$$||\mathbf{A}_n||_{L^{\frac{3}{1-s}}} \lesssim ||\mathbf{A}_n||_{\mathbf{H}(\operatorname{curl},\operatorname{div})} \lesssim 1,$$

as well as

$$||\frac{\mathrm{i}}{\kappa}\nabla u_n||_{L^2} \lesssim 1 + ||\mathbf{A}_n u_n||_{L^2} \lesssim 1 + ||\mathbf{A}_n||_{L^{\frac{3}{1-s}}} ||u||_{L^{\frac{6}{1+2s}}} \leq 1 + \alpha||\nabla u_n||_{L^2} + C_\alpha||u_n||_{L^2},$$

where $\alpha > 0$ can be chosen arbitrarily small. Absorption, then gives a uniform bound (not in κ) on $||\nabla u_n||_{L^2}$ if $||u_n||_{L^2}$ is bounded. To this end, we observe that

$$||u_n||_{L^2}^2 = |||u_n|||_{L^2} \lesssim 1 + |||u_n| - 1||_{L^2} \lesssim 1 + ||(|u_n| - 1)(|u_n| + 1)||_{L^2} \lesssim 1,$$

and hence the uniform bound on the minimizing sequence, which yields a converging subsequence to some $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}_0(\operatorname{curl}, \operatorname{div})(\Omega)$. Let us now only focus on this subsequence. We observe that by the compact embeddings, we have strong convergence of u_n in L^4 as well as $\mathbf{A}_n$ in $L^{\frac{3}{1-s}-}$, see e.g. [24, Theorem 3.4], and u_n in $L^{\frac{6}{1+2s}+}$ and thus of $\mathbf{A}_n u_n$ in L^2. This then implies the weak convergence to a minimizer. □

Let us note that in principle one could also look for minimizers without the divergence constraint. However, the following result shows that this does not result in lower energy levels, but induces a larger kernel of the Fréchet derivative E''. We refer to [19, Lemma 3.1] for the corresponding result in convex domains.

Lemma 2.6 (Gauge invariance)**.** *Let* $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}(\operatorname{curl})$ *be a minimizer of* (1.1)*, then there is a* $\varphi \in H^1(\Omega)$ *such that*

$$\widetilde{\mathbf{A}} = \mathbf{A} + \nabla\varphi \in \mathbf{H}_0(\operatorname{curl}, \operatorname{div}),$$

and the pair

$$\left(u\, e^{\mathrm{i}\kappa\varphi}, \mathbf{A} + \nabla\varphi\right) \in H^1(\Omega) \times \mathbf{H}_0(\operatorname{curl}, \operatorname{div})$$

is also a minimizer of (1.1).

Proof. Let $\varphi \in H^1(\Omega)$ be a solution of

$$(\nabla\varphi, \nabla\psi) = -(\mathbf{A}, \nabla\psi) = f(\psi), \qquad \text{for all } \psi \in H^1(\Omega),$$

where $f \in H^{-1}$. We clearly have $\nabla\varphi \in \mathbf{H}(\mathrm{curl})$ and thus $\widetilde{\mathbf{A}} \in \mathbf{H}(\mathrm{curl})$, and further

$$0 = (\widetilde{\mathbf{A}}, \nabla\psi) \implies \widetilde{\mathbf{A}} \in \mathbf{H}(\mathrm{div}) \quad \text{with} \quad \mathrm{div}\, \widetilde{\mathbf{A}} = 0.$$

In addition, we have for all $\psi \in H^1(\Omega)$

$$0 = (\widetilde{\mathbf{A}}, \nabla\psi) = -(\mathrm{div}\, \widetilde{\mathbf{A}}, \psi) + \int_{\partial\Omega} \widetilde{\mathbf{A}} \cdot \nu\psi \,\mathrm{d}\sigma = \int_{\partial\Omega} \widetilde{\mathbf{A}} \cdot \nu\psi \,\mathrm{d}\sigma$$

and thus $\widetilde{\mathbf{A}} \cdot \nu = 0$ on $\partial\Omega$, which gives the claim. □

Next, we present the critical point equations for the minimizers of (1.1). Due to the divergence constraint we obtain additional terms compared to [19].

Lemma 2.7. *Let $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}(\mathrm{curl}, \mathrm{div})$ be a minimizer of (1.1). Then, there is Lagrange multiplier $\lambda \in H^1(\Omega)$ such that it holds*

$$\begin{aligned} \langle E'(u, \mathbf{A}), (\varphi, \mathbf{B})\rangle + (\nabla\lambda, \mathbf{B}) &= 0, \\ (\mathrm{div}\, \mathbf{A}, \mu) &= 0, \end{aligned}$$

for all $\varphi, \mu \in H^1(\Omega)$ and $\mathbf{B} \in \mathbf{H}(\mathrm{curl})$, with

$$\begin{aligned} \partial_u E(u, \mathbf{A})\varphi &= \mathrm{Re} \int_\Omega \big(\frac{\mathrm{i}}{\kappa}\nabla u + \mathbf{A}u\big) \cdot \big(\frac{\mathrm{i}}{\kappa}\nabla\varphi + \mathbf{A}\varphi\big)^* + \big(|u|^2 - 1\big)u\varphi^* \,\mathrm{d}x, \\ \partial_{\mathbf{A}} E(u, \mathbf{A})\mathbf{B} &= \int_\Omega |u|^2 \mathbf{A} \cdot \mathbf{B} + \frac{1}{\kappa} \mathrm{Re}\big(\mathrm{i}u^* \nabla u \cdot \mathbf{B}\big) + \mathrm{curl}\, \mathbf{A} \cdot \mathrm{curl}\, \mathbf{B} - \mathbf{H} \cdot \mathrm{curl}\, \mathbf{B} \,\mathrm{d}x \end{aligned}$$

for $\varphi \in H^1(\Omega)$ and $\mathbf{B} \in \mathbf{H}(\mathrm{curl})$.

For $\mathbf{B} \in \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$, we can neglect the Lagrange multiplier, and obtain the famous Ginzburg–Landau equations. Next, we present the immediate a-priori bounds on the minimizers.

Even though, the proof in [19, Prop. 3.11] for the following statement is presented in the convex case, it literally caries over to our setting with $\mathbf{B}, \mu = 0$ in Lemma 2.7.

Lemma 2.8. *Let $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$ be a minimizer of (1.1). Then, the following stability bounds hold*

$$|u| \le 1 \text{ a.e.}$$

The first a-priori bounds can by achieved by the fact that $E(u, \mathbf{A}) \le E(0, 0) \lesssim 1$ and again the critical point equation and is presented in [13].

Lemma 2.9. *Let $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$ be a minimizer of problem (1.1). Then, we have*

$$||\frac{1}{\kappa}\nabla u + \mathbf{A}u||_{L^2}^2 \le ||u||_{L^2}^2, \tag{2.6}$$

and the following stability bounds hold

$$||\tfrac{1}{\kappa}\nabla u||_{L^2} \lesssim ||u||_{L^2} + ||\mathbf{A}u||_{L^2}, \quad ||u||_{H^1_\kappa} \lesssim 1 + ||\mathbf{H}||_{L^2}, \quad ||\mathbf{A}||_{H^{\frac{1}{2}+s}} \lesssim ||\mathbf{A}||_{\mathbf{H}(\mathrm{curl},\mathrm{div})} \lesssim 1 + ||\mathbf{H}||_{L^2}.$$

To conlcude this section, we present a result on the regularity of the rotation of the vector potential $\mathbf{A}$. This is precisely, what is needed to show convergence for Nédélec elements, see [35, Theorem 5.41] and [2, Theorem B] and Lemma 4.2 below.

Lemma 2.10. *Let $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$ be a minimizer of problem (1.1).*

(a) Let $\mathbf{H} \in \mathbf{H}(\mathrm{curl}) \cap \mathbf{H}(\mathrm{div})$ and denote $\mathbf{H}_\Delta = \mathrm{curl}\, \mathbf{A} - \mathbf{H}$. Then, it holds

$$\mathbf{H}_\Delta \in \mathbf{H}(\mathrm{curl}) \cap \mathbf{H}(\mathrm{div}), \qquad \mathbf{H}_\Delta \times \nu\big|_{\partial\Omega} = 0,$$

and we have $\mathbf{H}_\Delta \in H^{\frac{1}{2}+s}(\Omega)$ with

$$||\mathbf{H}_\Delta||_{H^{\frac{1}{2}+s}} \lesssim 1.$$

(b) If in addition, $\mathbf{H} \in H^{\frac{1}{2}+s}(\Omega)$*, then*

$$||\operatorname{curl}\mathbf{A}||_{H^{\frac{1}{2}+s}} \lesssim 1.$$

Proof. We use Lemma 2.7 and rewrite $\partial_{\mathbf{A}} E(u,\mathbf{A})\mathbf{B} + (\nabla\lambda, \mathbf{B}) = 0$ as

$$\int_\Omega \mathbf{H}_\Delta \cdot \operatorname{curl}\mathbf{B}\,\mathrm{d}x = -\int_\Omega |u|^2\mathbf{A}\cdot\mathbf{B} + \frac{1}{\kappa}\operatorname{Re}\big(\mathrm{i}u^*\nabla u\cdot\mathbf{B}\big)\,\mathrm{d}x + \int_\Omega \nabla\lambda\cdot\mathbf{B}\,\mathrm{d}x$$

for all $\mathbf{B} \in \mathbf{H}(\operatorname{curl})$. We thus conclude that $\mathbf{H}_\Delta \in \mathbf{H}_0(\operatorname{curl})$ with

$$\operatorname{curl}\mathbf{H}_\Delta = -|u|^2\mathbf{A} - \frac{1}{\kappa}\operatorname{Re}\big(\mathrm{i}u^*\nabla u\big) + \nabla\lambda \in L^2$$

Inserting $\mathbf{B} = \nabla\lambda \in \mathbf{H}(\operatorname{curl})$ gives us $||\nabla\lambda||_{L^2} \lesssim 1$ and thus also $||\operatorname{curl}\mathbf{H}_\Delta||_{L^2} \lesssim 1$. Since by construction it holds $\operatorname{div}\mathbf{H}_\Delta \in L^2(\Omega)$, we obtain the claim form Lemma 2.2. □

2.4. **Refined regularity theory for the order parameter.** In this section, we derive additional properties of the minimizers u in terms of differentiability and integrability. Here, we heavily rely on the known and adapted results which we provide in Appendix B. Note that in the convex case it is not possible to derive more information on the vector potential $\mathbf{A}$ than done in Lemmas 2.9 and 2.10. Our aim is to understand how much of the derived results in [13, 14] is still valid in the non-smooth case. From now on, we work with the regularity obtained in Lemma 2.2, i.e.

$$\mathbf{A} \in \mathbf{H}_0(\operatorname{curl},\operatorname{div}) \hookrightarrow H^{\frac{1}{2}+s} \hookrightarrow L^{\frac{3}{1-s}},$$

and consider the fixed pair $(u,\mathbf{A})$.

Lemma 2.11. *Let* $(u,\mathbf{A}) \in H^1(\Omega)\times\mathbf{H}_0(\operatorname{curl},\operatorname{div})$ *be a minimizer of problem* (1.1)*. Further, let* $\varepsilon_2 > 0$ *be arbitrary, take* ε_0 *from Proposition B.3, and define the exponents*

$$p_{u,1} = 2p_{u,2} \quad \textit{and} \quad p_{u,2} = \min\{\frac{3}{2(1-s)} - \varepsilon_2, \frac{4}{3} + \varepsilon_0\} \geq \frac{4}{3}, \qquad d = 3,$$

$$p_{u,1} = 2p_{u,2} \quad \textit{and} \quad p_{u,2} = \min\{\frac{2}{1-2s} - \varepsilon_2, p_{\max}\} \geq \frac{4}{3}, \qquad d = 2,$$

with $p_{\max}$ *defined in* (2.5) *(with* $p_{\max} > 2$ *if and only if* Ω *is convex). Then, it holds* $u \in W^{2,p_{u,2}} \cap W^{1,p_{u,1}}$ *with*

$$\kappa^{-2}||\nabla^2 u||_{L^{p_{u,2}}} \lesssim 1 \qquad \textit{and} \qquad \kappa^{-1}||\nabla u||_{L^{p_{u,1}}} \lesssim 1.$$

In addition, by Sobolev embedding, we have

$$\kappa^{-2}||w||_{H^{\frac{7p_{u,2}-6}{2p_{u,2}}}} \lesssim 1, \quad d=3, \qquad \textit{and} \qquad \kappa^{-2}||w||_{H^{\frac{3p_{u,2}-2}{p_{u,2}}}} \lesssim 1, \quad d=2, \tag{2.7a}$$

and by interpolation between $W^{2,p_{u,2}}$ *and* L^∞

$$||u||_{H^{p_{u,2}}} \lesssim ||\nabla^2 u||_{L^{p_{u,2}}}^{\frac{p_{u,2}}{2}} \lesssim \kappa^{p_{u,2}}. \tag{2.7b}$$

Proof. In the following, we use the relation $\partial_u E(u,\mathbf{A})\varphi = 0$, i.e.

$$\begin{aligned} 0 &= \operatorname{Re}\int_\Omega \big(\frac{\mathrm{i}}{\kappa}\nabla u + \mathbf{A}u\big)\cdot\big(\frac{\mathrm{i}}{\kappa}\nabla\varphi + \mathbf{A}\varphi\big)^* + \big(|u|^2-1\big)u\varphi^*\,\mathrm{d}x \\ &= \operatorname{Re}\int_\Omega \frac{1}{\kappa}\nabla u\cdot\frac{1}{\kappa}\nabla\varphi^*\,\mathrm{d}x + \operatorname{Re}\int_\Omega \Big(2\frac{\mathrm{i}}{\kappa}\nabla u\cdot\mathbf{A} + |\mathbf{A}|^2 u + \big(|u|^2-1\big)u\Big)\varphi^*\,\mathrm{d}x, \end{aligned}$$

and extract the claimed regularity.

(a) We begin with the three-dimensional case.

(1) We have by Lemma 2.9 $\nabla u \in L^2$ and $\mathbf{A} \in L^{\frac{3}{1-s}} \subset L^3$ such that $|\mathbf{A}|^2 \in L^{\frac{3}{2-2s}} \subset L^{\frac{3}{2}}$ and

$$||2\frac{\mathrm{i}}{\kappa}\mathbf{A}\nabla u + |\mathbf{A}|^2 u + \big(|u|^2-1\big)u||_{L^{\frac{6}{5}}} \lesssim ||\frac{\mathrm{i}}{\kappa}\mathbf{A}\nabla u||_{L^{\frac{6}{5}}} + |||\mathbf{A}|^2||_{L^{\frac{6}{5}}} + 1 \lesssim 1$$

and hence with Proposition B.3

$$\kappa^{-2}||u||_{W^{2,\frac{6}{5}}} \lesssim 1.$$

(2) We now set up a bootstrap argument. Let $q_0 = \frac{6}{5}$ and use the interpolation estimate in Lemma B.5 to obtain that

$$||\frac{1}{\kappa}\nabla u||_{L^{2q_0}} \lesssim 1 \quad \text{and} \quad ||\frac{1}{\kappa}\mathbf{A}\nabla u||_{L^{q_1}} \lesssim 1, \quad \frac{1}{q_1} = \frac{1-s}{3} + \frac{1}{2q_0} \implies q_1 = \frac{6q_0}{3+2(1-s)q_0}$$

and gain with Proposition B.3

$$\kappa^{-2}||u||_{W^{2,q_1}} \lesssim 1.$$

This leads to a sequence of exponents $q_{k+1} = \frac{\alpha q_k}{\alpha+\beta q_k}$ (with $\alpha = 3, \beta = 2(1-s)$) which converges monotonically to $\frac{3}{2(1-s)}$ form below, see Lemma C.1. Thus, after finitely many steps, we have the established regularity with exponent $p_{u,1}$. For $p_{u,2}$ we again employ Lemma B.5.

(b) In two dimensions we make the following adaptions.

(1) We similarly use $|\mathbf{A}|^2 \in L^{\frac{2}{1-2s}} \subset L^2, \mathbf{A}\nabla u \in L^{\frac{4}{3-2s}} \subset L^{\frac{4}{3}}$ obtain thus with Proposition B.4

$$\kappa^{-2}||u||_{W^{2,\frac{4}{3}}} \lesssim 1.$$

(2) We use $q_0 = \frac{4}{3}$ and derive the recursion $q_{k+1} = \frac{4q_k}{2+(1-2s)q_k} \implies q = \frac{2}{1-2s}$ such that Lemma C.1 gives with with $\alpha = 2, \beta = 2(1-2s)$ the limit $\frac{3}{2(1-s)}$.

The interpolation estimate is obtained by Lemma B.5. □

In the next lemma, we derive norms in $W^{1,p}$, where p is larger than in Lemma 2.11. However, we have to pay with an additional power of κ.

Lemma 2.12. *Let* $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$ *be a minimizer of problem* (1.1).

(a) Let $\Omega \subset \mathbb{R}^3$. *For* $\widehat{p}_{u,1} = \min\{\frac{3}{1-s}, 3+\varepsilon_{-1}\}$ *we have* $u \in W^{1,\widehat{p}_{u,1}} \subset W^{1,3}$ *and the estimate*

$$||\frac{1}{\kappa}\nabla u||_{L^{p_{u,1}}} \lesssim \kappa.$$

(b) Let $\Omega \subset \mathbb{R}^2$. *For* $\widehat{p}_{u,1} = \min\{\frac{4}{1-2s}, 4+\varepsilon_{-1}\}$ *we have* $u \in W^{1,\widehat{p}_{u,1}} \subset W^{1,4}$ *and the estimate*

$$||\frac{1}{\kappa}\nabla u||_{L^{p_{u,1}}} \lesssim \kappa.$$

Proof. (a) We exploit $\operatorname{div}\mathbf{A} = 0$ to compute

$$|(\mathbf{A}\nabla u, \phi)| = |(u, \mathbf{A}\nabla\phi)| \leq ||\mathbf{A}||_{L^{\frac{3}{1-s}}} ||\nabla\phi||_{L^{\frac{3}{2+s}}}$$

and thus $\mathbf{A}\nabla u \in (W^{1,\frac{3}{2+s}})' = W^{-1,\frac{3}{1-s}}$. In addition, we have $|\mathbf{A}|^2 u \in L^{\frac{3}{2-2s}}$ and by Lemma B.1 also $|\mathbf{A}|^2 u \in W^{-1,\frac{3}{1-2s}} \subset W^{-1,\frac{3}{1-s}}$. With this, Proposition B.2 gives us $u \in W^{1,\widehat{p}_{u,1}}$ as well as

$$\kappa^{-1}||u||_{W^{1,\frac{3}{1-s}}} \lesssim ||2\frac{\mathrm{i}}{\kappa}\mathbf{A}\nabla u + |\mathbf{A}|^2 u + \big(|u|^2-1\big)u||_{W^{-1,\frac{3}{1-s}}} \lesssim \kappa,$$

and hence the assertion.

(b) For $d = 2$, we have $\mathbf{A}\nabla u (W^{1,\frac{4}{3+2s}})' = W^{-1,\frac{4}{1-2s}}$. In addition, by Lemma B.1 we conclude $|\mathbf{A}|^2 u \in L^{\frac{2}{1-2s}} \hookrightarrow W^{-1,\frac{4}{1-2s}}$ and thus obtain by Proposition B.2 $u \in W^{1,\widehat{p}_{u,1}}$. □

3. Adapted norm estimates

In the preceding papers, it was always assumed that $\mathbf{A}$ is in $L^\infty(\Omega)$ which enabled us to define a suitable norm for the treatment of the second Fréchet derivative of the energy with respect to u. As we have shown above, in the non-convex setting, one cannot expect such high integrability, and thus we have to modify this approach.

3.1. **A stabilized norm for the order parameter.** Let $(u, \mathbf{A}) \in H^1(\Omega)\times\mathbf{H}(\mathrm{curl})$ be a minimizer of (1.1), and let us recall the second Fréchet derivative of E with respect to u given by

$$\langle \partial_u^2 E(u,\mathbf{A})\psi, \varphi\rangle = \operatorname{Re}\int_\Omega \big(\frac{\mathrm{i}}{\kappa}\nabla\varphi + \mathbf{A}\varphi\big)\cdot\big(\frac{\mathrm{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\big)^* + \big(|u|^2-1\big)\varphi\psi^* + u^2\varphi^*\psi^* + |u|^2\varphi\psi^*\,\mathrm{d}x, \tag{3.1}$$

which motivates the use of the following norm

$$||\varphi||^2_{H^1_{\kappa,\mathbf{A}}} := ||\frac{\mathrm{i}}{\kappa}\nabla\varphi + \mathbf{A}\varphi||^2_{L^2} + C^2_{\mathrm{stab}}(\kappa)||\varphi||^2_{L^2} \tag{3.2}$$

with a stabilization constant $C_{\mathrm{stab}}(\kappa)$ determined below, which is necessary to ensure that $||\cdot||_{H^1_{\kappa,\mathbf{A}}}$ is a norm on $H^1(\Omega)$, see Lemmas 3.1 and 3.3. We associate to this the bilinear form

$$a_{\kappa,\mathbf{A}}(\varphi,\psi) = \operatorname{Re}\int_\Omega \big(\frac{\mathrm{i}}{\kappa}\nabla\varphi + \mathbf{A}\varphi\big)\cdot\big(\frac{\mathrm{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\big)^* + C^2_{\mathrm{stab}}(\kappa)\varphi\psi^*\,\mathrm{d}x.$$

Lemma 3.1 (Norm in 3d)**.** *Let $\Omega \subset \mathbb{R}^3$ and define*

$$C_{\mathrm{stab}}(\kappa) = c_3(1 + \kappa^{\frac{(1-s)}{s}}) \tag{3.3}$$

with a constant c_3 defined below, which is independent of κ and $\mathbf{A}$.

(a) For all $\varphi \in H^1(\Omega)$, it holds

$$||\varphi||_{L^{\frac{6}{1+2s}}} \lesssim ||\frac{1}{\kappa}\nabla\varphi||_{L^2} + C_{\mathrm{stab}}(\kappa)||\varphi||_{L^2}$$

independent of κ.

(b) Further, we have the norm equivalence

$$||\frac{1}{\kappa}\nabla\varphi||^2_{L^2} + C^2_{\mathrm{stab}}(\kappa)||\varphi||^2_{L^2} \lesssim ||\varphi||_{H^1_{\kappa,\mathbf{A}}} \lesssim ||\frac{1}{\kappa}\nabla\varphi||^2_{L^2} + C^2_{\mathrm{stab}}(\kappa)||\varphi||^2_{L^2}$$

with hidden constants which are independent of κ.

Remark 3.2. *(a) For fixed $\kappa \geq 1$, we have the limits*

$$\lim_{s\to 0} C_{\mathrm{stab}}(\kappa) = \infty \qquad \textit{and} \qquad \lim_{s\to 1} C_{\mathrm{stab}}(\kappa) = 1,$$

where the latter one precisely yields the (unscaled L^2)-norm employed in the convex case.

(b) In particular, we may estimate

$$||\mathbf{A}\varphi||_{L^2} \leq ||\frac{\mathrm{i}}{\kappa}\nabla\varphi + \mathbf{A}\varphi||_{L^2} + ||\frac{1}{\kappa}\nabla\varphi||_{L^2} \lesssim ||\varphi||_{H^1_{\kappa,\mathbf{A}}} \tag{3.4}$$

with constants independent of κ and $\mathbf{A}$.

Proof of Lemma 3.1. We compute with Young

$$||\frac{\mathrm{i}}{\kappa}\nabla\varphi + \mathbf{A}\varphi||^2_{L^2} = ||\frac{\mathrm{i}}{\kappa}\nabla\varphi||^2_{L^2} + ||\mathbf{A}\varphi||^2_{L^2} + 2\operatorname{Re}(\frac{\mathrm{i}}{\kappa}\nabla\varphi, \mathbf{A}\varphi) \geq (1-\gamma)||\frac{\mathrm{i}}{\kappa}\nabla\varphi||^2_{L^2} + (1-\frac{1}{\gamma})||\mathbf{A}\varphi||^2_{L^2}.$$

With $\mathbf{A} \in L^{\frac{3}{1-s}}$, we have $||\mathbf{A}\varphi||_{L^2} \leq ||\mathbf{A}||_{L^{\frac{3}{1-s}}}||\varphi||_{L^{\frac{6}{1+2s}}}$. Due to the relation for $\alpha = s \in (0,1)$

$$\frac{1+2s}{6} = \frac{\alpha}{2} + \frac{1-\alpha}{6},$$

we further estimate for any $\beta > 0$ by Young's inequality and Sobolev embedding

$$\begin{aligned} ||\varphi||^2_{L^{\frac{6}{1+2s}}} &\leq \beta^{-1}||\varphi||^{2\alpha}_{L^2}\,\beta||\varphi||^{2(1-\alpha)}_{L^6} \\ &\leq \alpha\beta^{\frac{-1}{\alpha}}||\varphi||^2_{L^2} + (1-\alpha)\beta^{\frac{1}{1-\alpha}}||\varphi||^2_{L^6} \\ &\lesssim \alpha\beta^{\frac{-1}{\alpha}}||\varphi||^2_{L^2} + \kappa^2(1-\alpha)\beta^{\frac{1}{1-\alpha}}||\varphi||^2_{H^1_\kappa}. \end{aligned}$$

The choice $\beta = c^{1-\alpha}\kappa^{2(s-1)}$ for some $c > 0$ then yields

$$||\varphi||^2_{L^{\frac{6}{1+2s}}} \lesssim s\kappa^{\frac{2(1-s)}{s}}||\varphi||^2_{L^2} + c(1-s)||\varphi||^2_{H^1_\kappa},$$

which gives the estimate in (a) as well as the upper bound in part (b). Further, we have shown for c sufficiently small

$$||\frac{\mathrm{i}}{\kappa}\nabla\varphi + \mathbf{A}\varphi||_{L^2}^2 \geq c_1||\frac{\mathrm{i}}{\kappa}\nabla\varphi||_{L^2}^2 - c_2\kappa^{\frac{2(1-s)}{s}}||\varphi||_{L^2}^2$$

which yields the lower bound in the norm equivalence for suitable $c_3 > 0$. □

In the two-dimensional case, we use a different scaling which requires fewer powers in κ in front of the L^2-term. This is due to the better integrability properties of the vector potential in lower dimensions.

Lemma 3.3 (Norm in 2d)**.** *Let $\Omega \subset \mathbb{R}^2$ and define*

$$C_{\mathrm{stab}}(\kappa) = c_2(1 + \kappa^{\frac{(1-2s)}{1+2s}}), \quad s \in (0, \frac{1}{2}),$$

with a constant c_2 defined below, which is independent of κ and $\mathbf{A}$.

(a) For all $\varphi \in H^1(\Omega)$, it holds

$$||\varphi||_{L^{\frac{4}{1+2s}}} \lesssim ||\frac{1}{\kappa}\nabla\varphi||_{L^2} + C_{\mathrm{stab}}(\kappa)||\varphi||_{L^2}$$

independent of κ.

(b) Further, we have the norm equivalence

$$||\frac{1}{\kappa}\nabla\varphi||_{L^2}^2 + C_{\mathrm{stab}}^2(\kappa)||\varphi||_{L^2}^2 \lesssim ||\varphi||_{H^1_{\kappa,\mathbf{A}}} \lesssim ||\frac{1}{\kappa}\nabla\varphi||_{L^2}^2 + C_{\mathrm{stab}}^2(\kappa)||\varphi||_{L^2}^2$$

with hidden constants which are independent of κ.

Remark 3.4. *For fixed $\kappa \geq 1$, we have the limits*

$$\lim_{s\to 0} C_{\mathrm{stab}}(\kappa) = 1 + \kappa \qquad \text{and} \qquad \lim_{s\to\frac{1}{2}} C_{\mathrm{stab}}(\kappa) = 1,$$

where the latter one precisely yields the (unscaled L^2)-norm employed in the convex case.

Proof of Lemma 3.3. We proceed as before with $\mathbf{A} \in L^{\frac{4}{1-2s}}$, we have $||\mathbf{A}\varphi||_{L^2} \leq ||\mathbf{A}||_{L^{\frac{4}{1-2s}}}||\varphi||_{L^{\frac{4}{1+2s}}}$ and hence estimate for some $q > 4$ (to be chosen later)

$$\frac{1+2s}{4} = \frac{\alpha}{2} + \frac{1-\alpha}{q} \qquad \Longrightarrow \qquad \alpha = \alpha(q) = \frac{1}{2}\frac{q(1+2s)-4}{q-2}, \quad s \in (0, \frac{1}{2}),$$

where we have $\alpha(q) \to \frac{1}{2}$ for $q \to \infty$. With this we estimate

$$\begin{aligned}
||\varphi||_{L^{\frac{4}{1+2s}}}^2 &\leq \beta^{-1}||\varphi||_{L^2}^{2\alpha}\,\beta||\varphi||_{L^q}^{2(1-\alpha)} \\
&\leq \alpha\beta^{\frac{-1}{\alpha}}||\varphi||_{L^2}^2 + (1-\alpha)\beta^{\frac{1}{1-\alpha}}||\varphi||_{L^q}^2 \\
&\lesssim \alpha\beta^{\frac{-1}{\alpha}}||\varphi||_{L^2}^2 + \kappa^2 q^2(1-\alpha)\beta^{\frac{1}{1-\alpha}}||\varphi||_{H^1_\kappa}^2
\end{aligned}$$

where we used $||u||_{L^q} \lesssim q||\varphi||_{H^1}$, and hence with $\beta = c^{1-\alpha}(q\,\kappa)^{2(\alpha-1)}$ we obtain

$$\begin{aligned}
||\varphi||_{L^{\frac{4}{1+2s}}}^2 &\leq \widetilde{c}\alpha(q\kappa)^{\frac{2(1-\alpha)}{\alpha}}||\varphi||_{L^2}^2 + c(1-\alpha)||u||_{H^1_\kappa}^2 \\
&= \widetilde{c}\alpha(q\kappa)^{\frac{2q(1-2s)}{q(1+2s)-4}}||\varphi||_{L^2}^2 + c(1-\alpha)||u||_{H^1_\kappa}^2 \\
&\sim \widetilde{c}\alpha\kappa^{\frac{2(1-2s)}{1+2s}}||\varphi||_{L^2}^2 + c(1-\alpha)||\varphi||_{H^1_\kappa}^2
\end{aligned}$$

for q sufficiently large (but independent of κ). This gives the desired estimates. □

In the following lemma, we derive a-priori estimates on solutions of the elliptic problems corresponding to the stabilized bilinear form $(\cdot,\cdot)_{H^1_{\kappa,\mathbf{A}}}$.

Lemma 3.5. *Let $f \in L^2$ and consider $w \in H^1(\Omega)$ to be the solution of*

$$(w, \varphi)_{H^1_{\kappa,\mathbf{A}}} = m(f, \varphi).$$

Then, it holds

$$||w||_{H^1_{\kappa,\mathbf{A}}} \lesssim C_{\text{stab}}^{-1}(\kappa)||f||_{L^2}.$$

(a) If $\Omega \subset \mathbb{R}^3$, then we have $w \in W^{2,\frac{6}{5-2s}}$ ($s < \frac{1}{4}$) with

$$\kappa^{-2}||w||_{H^{1+s}} \lesssim \kappa^{-2}||w||_{W^{2,\frac{6}{5-2s}}} \lesssim ||f||_{L^2}.$$

(b) If $\Omega \subset \mathbb{R}^2$, then we have $w \in W^{2,\frac{4}{3-2s}}$ with

$$\kappa^{-2}||w||_{H^{\frac{3}{2}+s}} \lesssim \kappa^{-2}||w||_{W^{2,\frac{4}{3-2s}}} \lesssim ||f||_{L^2}.$$

Proof. Simply inserting w in the elliptic problem and using Lemma 3.1 gives

$$||\frac{1}{\kappa}\nabla w||_{L^2}^2 + C_{\text{stab}}^2(\kappa)||w||_{L^2}^2 \lesssim (w, w)_{H^1_{\kappa,\mathbf{A}}} = m(f, w) \lesssim ||f||_{L^2}||w||_{L^2}$$

and hence with Young the estimate for $||w||_{H^1_{\kappa,\mathbf{A}}}$.

(a) We rewrite the critical point equation and define g as

$$m(\frac{1}{\kappa}\nabla w, \frac{1}{\kappa}\nabla\varphi) = 2m(\frac{\mathrm{i}}{\kappa}\nabla w, \mathbf{A}\varphi) + m(\mathbf{A}w, \mathbf{A}\varphi) + m(f, \varphi) = (g, \phi).$$

To show the claim, we have to provide a bound on the norm of g in $L^{\frac{6}{5-2s}}$. Since the dual space is isomorphic to $L^{\frac{6}{1+2s}}$, we may estimate

$$\begin{aligned}|(g, \phi)| &\lesssim ||w||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{A}||_{L^{\frac{3}{1-s}}}||\varphi||_{L^{\frac{6}{1+2s}}} + ||\mathbf{A}||^2_{L^{\frac{3}{1-s}}}||w||_{L^{\frac{6}{1+2s}}}||\varphi||_{L^{\frac{6}{1+2s}}} + ||f||_{L^2}||\varphi||_{L^{\frac{6}{1+2s}}} \\ &\lesssim \big(||w||_{H^1_{\kappa,\mathbf{A}}} + ||f||_{L^2}\big)||\varphi||_{L^{\frac{6}{1+2s}}} \\ &\lesssim (C_{\text{stab}}^{-1}(\kappa) + 1)||f||_{L^2}||\varphi||_{L^{\frac{6}{1+2s}}}.\end{aligned}$$

where we used Lemma 3.1 for the $L^{\frac{6}{1+2s}}$-norm of w. We thus established that

$$||g||_{L^{\frac{6}{5-2s}}} \lesssim ||f||_{L^2}$$

and we obtain with Proposition B.3 ($s < \frac{1}{4}$)

$$\kappa^{-2}||w||_{W^{2,\frac{6}{5-2s}}} \lesssim ||f||_{L^2}$$

as claimed. The first estimate follows from Sobolev's embedding.

(b) For $d = 2$, we modify the argument with the dual spaces $L^{\frac{4}{3-2s}}$ and $L^{\frac{4}{1+2s}}$ to

$$\begin{aligned}|(g, \phi)| &\lesssim ||w||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{A}||_{L^{\frac{4}{1-2s}}}||\varphi||_{L^{\frac{4}{1+2s}}} + ||\mathbf{A}||^2_{L^{\frac{4}{1-2s}}}||w||_{L^{\frac{4}{1+2s}}}||\varphi||_{L^{\frac{4}{1+2s}}} + ||f||_{L^2}||\varphi||_{L^{\frac{4}{1+2s}}} \\ &\lesssim (C_{\text{stab}}^{-1}(\kappa) + 1)||f||_{L^2}||\varphi||_{L^{\frac{4}{1+2s}}},\end{aligned}$$

where we used Lemma 3.3 for the $L^{\frac{4}{1+2s}}$-norm of w. The use of Lemma B.4 yields the claim. □

3.2. **Properties of the second Fréchet derivative of E.** In this section, we discuss upper and lower bounds of E''. To obtain results which are also applicable to study the energy (1.1) for a fixed vector potential $\mathbf{A}$ in the spirit of [8, 14], but under the reduced regularity assumptions. We therefore recall the representation of E'' from [13, Lemma 2.4].

Lemma 3.6. *Let* $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$ *be a minimizer of the energy defined in* (1.1). *We denote the second order partial Fréchet derivatives by*

$$\begin{aligned}
\langle \partial_u^2 E(u, \mathbf{A}) \cdot, \varphi \rangle &:= \frac{\partial}{\partial u}(\partial_u E(u, \mathbf{A})\varphi) : H^1(\Omega) \to \mathbb{R}, \\
\langle \partial_{\mathbf{A},u} E(u, \mathbf{A}) \cdot, \mathbf{B} \rangle &:= \frac{\partial}{\partial u}(\partial_{\mathbf{A}} E(u, \mathbf{A})\mathbf{B}) : H^1(\Omega) \to \mathbb{R}, \\
\langle \partial_{u,\mathbf{A}} E(u, \mathbf{A}) \cdot, \varphi \rangle &:= \frac{\partial}{\partial \mathbf{A}}(\partial_u E(u, \mathbf{A})\varphi) : \mathbf{H}(\mathrm{curl}) \to \mathbb{R}, \\
\langle \partial_{\mathbf{A}}^2 E(u, \mathbf{A}) \cdot, \mathbf{B} \rangle &:= \frac{\partial}{\partial \mathbf{A}}(\partial_{\mathbf{A}} E(u, \mathbf{A})\mathbf{B}) : \mathbf{H}(\mathrm{curl}) \to \mathbb{R},
\end{aligned}$$

where $\varphi \in H^1(\Omega)$ *and* $\mathbf{B} \in \mathbf{H}(\mathrm{curl})$. *The derivatives are given by*

$$\langle \partial_u^2 E(u, \mathbf{A})\psi, \varphi \rangle = \mathrm{Re} \int_\Omega \big(\frac{\mathrm{i}}{\kappa}\nabla\varphi + \mathbf{A}\varphi\big) \cdot \big(\frac{\mathrm{i}}{\kappa}\nabla\psi + \mathbf{A}\psi\big)^* + \big(|u|^2 - 1\big)\varphi\psi^* + u^2\varphi^*\psi^* + |u|^2\varphi\psi^* \,\mathrm{d}x,$$

$$\langle \partial_{\mathbf{A}}^2 E(u, \mathbf{A})\mathbf{B}, \mathbf{C} \rangle = \int_\Omega |u|^2 \mathbf{C} \cdot \mathbf{B} + \mathrm{curl}\,\mathbf{C} \cdot \mathrm{curl}\,\mathbf{B} \,\mathrm{d}x,$$

$$\langle \partial_{u,\mathbf{A}} E(u, \mathbf{A})\mathbf{B}, \psi \rangle = \int_\Omega 2\,\mathrm{Re}(u\psi^*)\mathbf{A} \cdot \mathbf{B} + \frac{1}{\kappa}\,\mathrm{Re}\big(\mathrm{i}u^*\nabla\psi + \mathrm{i}\psi^*\nabla u\big) \cdot \mathbf{B} \,\mathrm{d}x$$

and $\langle \partial_{u,\mathbf{A}} E(u, \mathbf{A})\mathbf{B}, \psi \rangle = \langle \partial_{\mathbf{A},u} E(u, \mathbf{A})\psi, \mathbf{B} \rangle$.

Fixing $\mathbf{A}$ then corresponds to setting $\mathbf{B} = \mathbf{C} = 0$.

3.3. **The derivative $\partial_u^2 E$ for fixed A.** Using the stabilized bilinear forms above, we can immediately give upper and lower bounds on $\partial_u^2 E$.

Lemma 3.7. *Let* $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$ *be a minimizer of* (1.1). *We have*

$$|\langle \partial_u^2 E(u, \mathbf{A})\psi, \varphi \rangle| \lesssim \big(||\frac{\mathrm{i}}{\kappa}\nabla\psi + \mathbf{A}\psi||_{L^2}^2 + ||\psi||_{L^2}^2\big)\big(||\frac{\mathrm{i}}{\kappa}\nabla\varphi + \mathbf{A}\varphi||_{L^2}^2 + ||\varphi||_{L^2}^2\big) \lesssim ||\varphi||_{H^1_{\kappa,\mathbf{A}}} ||\psi||_{H^1_{\kappa,\mathbf{A}}}$$

and there is a constant $\rho_{u,\mathbf{A}}(\kappa) \geq 1$ *such that*

$$|\langle \partial_u^2 E(u, \mathbf{A})\varphi, \varphi \rangle| \geq \rho_{u,\mathbf{A}}(\kappa)^{-1} C_{\mathrm{stab}}^{-1}(\kappa) ||\varphi||_{H^1_{\kappa,\mathbf{A}}}^2,$$

with $\rho_{u,\mathbf{A}}(\kappa)$ *bounded from above independently of* $C_{\mathrm{stab}}(\kappa)$, *i.e.* $\partial_u^2 E$ *is coercive on* $H^1_{\mathrm{i}u}$.

Proof. The first estimate is trivially satisfied, and as in [13] we have for the smallest non-zero eigenvalue λ_1 that

$$\langle \partial_u^2 E(u, \mathbf{A})\varphi, \varphi \rangle \geq \lambda_1 ||\varphi||_{L^2}^2.$$

By the proof of Lemma 3.1, we additionally observe that the Garding inequality in the form

$$\langle \partial_u^2 E(u, \mathbf{A})\varphi, \varphi \rangle \geq c_1 ||\varphi||_{H^1_{\kappa,\mathbf{A}}}^2 - c_2 C_{\mathrm{stab}}^2(\kappa) ||u||_{L^2}^2$$

is satisfied. We multiply the first equation by $c_2 C_{\mathrm{stab}}(\kappa)$ and the second by λ_1 and ad up to

$$(c_2 C_{\mathrm{stab}}(\kappa) + \lambda_1)\langle \partial_u^2 E(u, \mathbf{A})\varphi, \varphi \rangle \geq c_1\lambda_1 ||\varphi||_{H^1_{\kappa,\mathbf{A}}}^2$$

which gives the claim for

$$\rho_{u,\mathbf{A}}(\kappa) C_{\mathrm{stab}}(\kappa) = \frac{\lambda_1 + c_2 C_{\mathrm{stab}}(\kappa)}{c_1\lambda_1} \lesssim 1 + \frac{C_{\mathrm{stab}}(\kappa)}{\lambda_1} \lesssim (1 + \frac{1}{\lambda_1}) C_{\mathrm{stab}}(\kappa)$$

where we used that $C_{\mathrm{stab}}(\kappa) \geq 1$. □

Lemma 3.8. *Let* $f \in L^2(\Omega)$ *and consider* $w \in H^1_{\mathrm{i}u}$ *to be the solution of*

$$\langle \partial_u^2 E(u) w, \varphi \rangle = (f, \varphi).$$

Then, the solution satisfies

$$||w||_{H^1_{\kappa,\mathbf{A}}} \lesssim \rho_{u,\mathbf{A}}(\kappa) ||f||_{L^2}.$$

(a) If $\Omega \subset \mathbb{R}^3$, then we also have $w \in W^{2,\frac{6}{5-2s}}$ ($s < \frac{1}{4}$) with

$$\kappa^{-2}||w||_{H^{1+s}} \lesssim \kappa^{-2}||w||_{W^{2,\frac{6}{5-2s}}} \lesssim \rho_{u,\mathbf{A}}(\kappa)||f||_{L^2}.$$

(b) If $\Omega \subset \mathbb{R}^2$, then we also have $w \in W^{2,\frac{4}{3-2s}}$ with

$$\kappa^{-2}||w||_{H^{\frac{3}{2}+s}} \lesssim \kappa^{-2}||w||_{W^{2,\frac{4}{3-2s}}} \lesssim \rho_{u,\mathbf{A}}(\kappa)||f||_{L^2}.$$

Proof. The existence follows from Lemma 3.7, and we additionally conclude

$$||w||_{H^1_{\kappa,\mathbf{A}}} \lesssim \rho_{u,\mathbf{A}}(\kappa)C_{\text{stab}}(\kappa)||f||_{(H^1_{\kappa,\mathbf{A}})'} \lesssim \rho_{u,\mathbf{A}}(\kappa)||f||_{L^2},$$

since $C_{\text{stab}}(\kappa)$ cancels, and as before we rewrite the problem as

$$m(\frac{1}{\kappa}\nabla w, \frac{1}{\kappa}\nabla\varphi) = 2m(\frac{\mathrm{i}}{\kappa}\nabla w, \mathbf{A}\varphi) + m(\mathbf{A}w, \mathbf{A}\varphi) + m(f,\varphi) + \operatorname{Re}\int_\Omega \big(|u|^2-1\big)w\varphi^* + u^2 w^*\varphi^* + |u|^2 w\varphi^* \,\mathrm{d}x = (g,\phi),$$

where we only have to estimate g in the correct norm.

(a) We proceed as in Lemma 3.5 and obtain with Lemma 3.1

$$|(g,\phi)| \lesssim ||w||_{H^1_{\kappa,\mathbf{A}}}||\varphi||_{L^{\frac{6}{1+2s}}} + ||f||_{L^2}||\varphi||_{L^{\frac{6}{1+2s}}}, \lesssim (\rho_{u,\mathbf{A}}(\kappa)+1)||f||_{L^2}||\varphi||_{L^{\frac{6}{1+2s}}},$$

which gives the claim.

(b) The same reasoning gives the claim for $d = 2$ using $\mathbf{A} \in L^{\frac{4}{1-2s}}$. □

3.4. **Properties of E''.** As the last step, before we turn to the numerical discretization, we study the full energy (1.1), and thus have to consider both minimizers u, $\mathbf{A}$ as unknowns. To have better control over E'', we perform a similar composition to [13]. We define the inner product for $\mathbf{B}, \mathbf{C} \in \mathbf{H}(\text{curl})$ by

$$b(\mathbf{B},\mathbf{C}) = \int_\Omega \operatorname{curl}\mathbf{B}\cdot\operatorname{curl}\mathbf{C} + \mathbf{B}\cdot\mathbf{C}\,\mathrm{d}x$$

and with this the bilinear form r by

$$\langle E''(u,\mathbf{A})(\varphi,\mathbf{B}),(\psi,\mathbf{C})\rangle = a_{\kappa,\mathbf{A}}(\varphi,\psi) + b(\mathbf{B},\mathbf{C}) + r\big((\varphi,\mathbf{B}),(\psi,\mathbf{C})\big). \tag{3.5}$$

We further decompose the remainder r as

$$r\big((\varphi,\mathbf{B}),(\psi,\mathbf{C})\big) = r_1(\varphi,\psi)+r_2(\mathbf{B},\mathbf{C})+r_3(\mathbf{B},\psi)+r_3(\mathbf{C},\varphi)+r_4(\mathbf{B},\psi)+r_4(\mathbf{C},\varphi)+r_5(\mathbf{B},\psi)+r_5(\mathbf{C},\varphi),$$

where the different terms are defined by

$$r_1(\varphi,\psi) = \langle\partial_u^2 E(u,\mathbf{A})\psi,\varphi\rangle - a_{\kappa,\mathbf{A}}(\varphi,\psi), \qquad r_2(\mathbf{B},\mathbf{C}) = \langle\partial_{\mathbf{A}}^2 E(u,\mathbf{A})\mathbf{B},\mathbf{C}\rangle - b(\mathbf{B},\mathbf{C}),$$

$$r_3(\mathbf{B},\psi) = \operatorname{Re}\int_\Omega u\psi^*\mathbf{A}\cdot\mathbf{B}\,\mathrm{d}x, \quad r_4(\mathbf{B},\psi) = \operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}u^*\nabla\psi\cdot\mathbf{B}\,\mathrm{d}x, \quad r_5(\mathbf{B},\psi) = \operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}\psi^*\nabla u\cdot\mathbf{B}\,\mathrm{d}x.$$

We now have to bound the different contributions depending on the regularity and bonudary conditions of the arguments.

Lemma 3.9. *Let $\mathbf{B},\mathbf{C} \in \mathbf{L}^2(\Omega)$ and $\varphi,\psi \in L^2(\Omega)$. Then, it holds.*

$$|r_1(\varphi,\psi) + r_2(\mathbf{B},\mathbf{C})| \lesssim C^2_{\text{stab}}(\kappa)||\varphi||_{L^2}||\psi||_{L^2} + ||\mathbf{B}||_{L^2}||\mathbf{C}||_{L^2}.$$

Further, we have the following estimates:

(a) *For $\psi \in H^1(\Omega)$*

$$|r_3(\mathbf{B},\psi) + r_4(\mathbf{B},\psi)| \lesssim ||\psi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{B}||_{L^2},$$

and if $\mathbf{B} \in \mathbf{H}_0(\text{curl},\text{div})$

$$|r_5(\mathbf{B},\psi)| \lesssim ||\psi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{B}||_{L^2}.$$

(b) *For* $\mathbf{B} \in L^{\frac{3}{1-s}}$ *(or* $\mathbf{B} \in L^4$*, respectively)*

$$|r_3(\mathbf{B}, \psi)| \lesssim ||\psi||_{L^{\frac{3}{1+2s}}} ||\mathbf{B}||_{L^{\frac{3}{1-s}}}, \qquad d = 3,$$
$$|r_3(\mathbf{B}, \psi)| \lesssim ||\psi||_{L^2} ||\mathbf{B}||_{L^4}, \qquad d = 2.$$

(c) *If* $\psi \in H^1(\Omega)$ *and* $\mathbf{B} \in \mathbf{H}(\mathrm{curl})$

$$|r_4(\mathbf{B}, \psi)| \lesssim ||\psi||_{H^1_{\kappa,\mathbf{A}}} ||\pi_{\mathrm{div}}\mathbf{B} - \mathbf{B}||_{L^2} + \kappa ||\mathbf{B}||_{\mathbf{H}(\mathrm{curl})} ||\psi||_{L^{\frac{3}{1+s}}}, \qquad d = 3,$$
$$|r_4(\mathbf{B}, \psi)| \lesssim ||\psi||_{H^1_{\kappa,\mathbf{A}}} ||\pi_{\mathrm{div}}\mathbf{B} - \mathbf{B}||_{L^2} + \kappa ||\mathbf{B}||_{\mathbf{H}(\mathrm{curl})} ||\psi||_{L^2}, \qquad d = 2.$$

(d) *If* $\psi \in H^1(\Omega)$ *and* $\mathbf{B} \in \mathbf{H}(\mathrm{curl})$

$$|r_5(\mathbf{B}, \psi)| \lesssim ||\psi||_{H^1_{\kappa,\mathbf{A}}} ||\mathbf{B}||_{L^2} + \kappa^2 ||\psi||_{H^1_{\kappa,\mathbf{A}}} ||\pi_{\mathrm{div}}\mathbf{B} - \mathbf{B}||_{L^2}, \qquad d = 3,$$
$$|r_5(\mathbf{B}, \psi)| \lesssim ||\psi||_{H^1_{\kappa,\mathbf{A}}} ||\mathbf{B}||_{L^2} + \kappa ||\psi||_{H^1_{\kappa,\mathbf{A}}} ||\pi_{\mathrm{div}}\mathbf{B} - \mathbf{B}||_{L^2}, \qquad d = 2.$$

Remark 3.10. *We expect that several powers of* κ *can be removed, if one had a more precise knowledge on the range of exponents* p *in Proposition B.2.*

Proof of Lemma 3.9. We first note that by definition of the bilinear forms

$$|r_1(\varphi, \psi)| \lesssim C^2_{\mathrm{stab}}(\kappa) ||\varphi||_{L^2} ||\psi||_{L^2}, \qquad |r_2(\mathbf{B}, \mathbf{C})| \lesssim ||\mathbf{B}||_{L^2} ||\mathbf{C}||_{L^2}.$$

(a) With Lemma 2.8 and (3.4) we have for $\psi \in H^1(\Omega)$

$$|r_3(\mathbf{B}, \psi)| = |\operatorname{Re} \int_\Omega u\psi^* \mathbf{A} \cdot \mathbf{B} \,\mathrm{d}x| \le ||\mathbf{A}\psi||_{L^2} ||\mathbf{B}||_{L^2} \lesssim ||\psi||_{H^1_{\kappa,\mathbf{A}}} ||\mathbf{B}||_{L^2},$$
$$|r_4(\mathbf{B}, \psi)| = |\operatorname{Re} \int_\Omega \frac{\mathrm{i}}{\kappa} u^* \nabla\psi \cdot \mathbf{B} \,\mathrm{d}x| \le ||\frac{\mathrm{i}}{\kappa} \nabla\psi||_{L^2} ||\mathbf{B}||_{L^2} \lesssim ||\psi||_{H^1_\kappa} ||\mathbf{B}||_{L^2},$$
$$|r_5(\mathbf{B}, \psi)| = |\operatorname{Re} \int_\Omega \frac{\mathrm{i}}{\kappa} \psi^* \nabla u \cdot \mathbf{B} \,\mathrm{d}x| = |\operatorname{Re} \int_\Omega \frac{\mathrm{i}}{\kappa} \nabla\psi^* u \cdot \mathbf{B} \,\mathrm{d}x| \lesssim ||\psi||_{H^1_\kappa} ||\mathbf{B}||_{L^2},$$

where we used for the last estimate $\mathbf{B} \in \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$, integration by parts, and the fact that $\operatorname{div} \mathbf{B} = 0$.

(b) We estimate with Hölder depending on the dimension

$$|r_3(\mathbf{B}, \psi)| = |\operatorname{Re} \int_\Omega u\psi^* \mathbf{A} \cdot \mathbf{B} \,\mathrm{d}x| \le ||\mathbf{A}||_{L^{\frac{3}{1-s}}} ||\mathbf{B}||_{L^{\frac{3}{1-s}}} ||\psi||_{L^{\frac{3}{1+2s}}} \lesssim ||\psi||_{L^{\frac{3}{1+2s}}} ||\mathbf{B}||_{L^{\frac{3}{1-s}}},$$
$$|r_3(\mathbf{B}, \psi)| = |\operatorname{Re} \int_\Omega u\psi^* \mathbf{A} \cdot \mathbf{B} \,\mathrm{d}x| \le ||\mathbf{A}||_{L^4} ||\mathbf{B}||_{L^4} ||\psi||_{L^2} \lesssim ||\psi||_{L^2} ||\mathbf{B}||_{L^4}.$$

(c) For r_4 we compute with the lift $\pi_{\mathrm{div}}\mathbf{B}$

$$\begin{aligned}|r_4(\mathbf{B}, \psi)| &\le |\operatorname{Re} \int_\Omega \frac{\mathrm{i}}{\kappa} u^* \nabla\psi \cdot \pi_{\mathrm{div}}\mathbf{B} \,\mathrm{d}x| + |\operatorname{Re} \int_\Omega \frac{\mathrm{i}}{\kappa} u^* \nabla\psi \cdot (\pi_{\mathrm{div}}\mathbf{B} - \mathbf{B}) \,\mathrm{d}x| \\ &\lesssim |\operatorname{Re} \int_\Omega \frac{\mathrm{i}}{\kappa} u^* \nabla\psi \cdot \pi_{\mathrm{div}}\mathbf{B} \,\mathrm{d}x| + ||\psi||_{H^1_{\kappa,\mathbf{A}}} ||\pi_{\mathrm{div}}\mathbf{B} - \mathbf{B}||_{L^2} \\ &= |\operatorname{Re} \int_\Omega \frac{\mathrm{i}}{\kappa} \nabla u^* \psi \cdot \pi_{\mathrm{div}}\mathbf{B} \,\mathrm{d}x| + ||\psi||_{H^1_{\kappa,\mathbf{A}}} ||\pi_{\mathrm{div}}\mathbf{B} - \mathbf{B}||_{L^2},\end{aligned}$$

where we used integration by parts with the fact that $\pi_{\mathrm{div}}\mathbf{B} \in \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$. We use Lemma 2.12 for $d = 3$

$$|\operatorname{Re} \int_\Omega \frac{\mathrm{i}}{\kappa} \nabla u^* \psi \cdot \pi_{\mathrm{div}}\mathbf{B} \,\mathrm{d}x| \lesssim ||\frac{1}{\kappa} \nabla u||_{L^3} ||\pi_{\mathrm{div}}\mathbf{B}||_{L^{\frac{3}{1-s}}} ||\psi||_{L^{\frac{3}{1+s}}} \lesssim \kappa ||\mathbf{B}||_{\mathbf{H}(\mathrm{curl})} ||\psi||_{L^{\frac{3}{1+s}}},$$

and, for $d = 2$, we replace the last estimate by

$$|\operatorname{Re} \int_\Omega \frac{\mathrm{i}}{\kappa} \nabla u^* \psi \cdot \pi_{\mathrm{div}}\mathbf{B} \,\mathrm{d}x| \lesssim ||\frac{1}{\kappa} \nabla u||_{L^4} ||\pi_{\mathrm{div}}\mathbf{B}||_{L^2} ||\psi||_{L^2} \lesssim \kappa ||\mathbf{B}||_{\mathbf{H}(\mathrm{curl})} ||\psi||_{L^2}.$$

(d) Again Lemma 2.12 gives for $d = 3$

$$|r_5(\mathbf{B}, \psi)| = |\operatorname{Re} \int_\Omega \frac{\mathrm{i}}{\kappa} \psi^* \nabla u \cdot \mathbf{B} \,\mathrm{d}x| \lesssim ||\frac{1}{\kappa} \nabla u||_{L^3} ||\mathbf{B}||_{L^{\frac{3}{1-s}}} ||\psi||_{L^{\frac{3}{1+s}}} \lesssim \kappa ||\mathbf{B}||_{L^{\frac{3}{1-s}}} ||\psi||_{L^{\frac{3}{1+s}}},$$

and for $d = 2$

$$|r_5(\mathbf{B},\psi)| = |\operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}\psi^*\nabla u\cdot\mathbf{B}\,\mathrm{d}x| \lesssim ||\frac{1}{\kappa}\nabla u||_{L^4}||\mathbf{B}||_{L^4}||\psi||_{L^2} \lesssim \kappa||\mathbf{B}||_{L^4}||\psi||_{L^2}.$$

We insert the lift $\pi_{\mathrm{div}}\mathbf{B}$ to write

$$\begin{aligned}
|r_5(\mathbf{B},\psi)| &\le |\operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}\psi^*\nabla u\cdot\pi_{\mathrm{div}}\mathbf{B}\,\mathrm{d}x| + |\operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}\psi^*\nabla u\cdot(\pi_{\mathrm{div}}\mathbf{B}-\mathbf{B})| \\
&\le |\operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}\psi^*\nabla u\cdot\pi_{\mathrm{div}}\mathbf{B}\,\mathrm{d}x| + |\operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}\psi^*\nabla u\cdot(\pi_{\mathrm{div}}\mathbf{B}-\mathbf{B})| \\
&\le |\operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}\nabla\psi^* u\cdot\pi_{\mathrm{div}}\mathbf{B}\,\mathrm{d}x| + |\operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}\psi^*\nabla u\cdot(\pi_{\mathrm{div}}\mathbf{B}-\mathbf{B})| \\
&\le |\operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}\nabla\psi^* u\cdot\mathbf{B}\,\mathrm{d}x| + |\operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}\nabla\psi^* u\cdot(\pi_{\mathrm{div}}\mathbf{B}-\mathbf{B})\,\mathrm{d}x| \\
&\quad + |\operatorname{Re}\int_\Omega \frac{\mathrm{i}}{\kappa}\psi^*\nabla u\cdot(\pi_{\mathrm{div}}\mathbf{B}-\mathbf{B})|.
\end{aligned}$$

We now use the first estimate in Lemma 3.1 together with Lemma 4.3 for $d = 3$ as

$$\begin{aligned}
|r_5(\mathbf{B},\psi)| &\lesssim ||\psi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{B}||_{L^2} + ||\psi||_{H^1_{\kappa,\mathbf{A}}}||\pi_{\mathrm{div}}\mathbf{B}-\mathbf{B}||_{L^2} + ||\psi||_{L^6}||\frac{1}{\kappa}\nabla u||_{L^3}||\pi_{\mathrm{div}}\mathbf{B}-\mathbf{B}||_{L^2} \\
&\lesssim ||\psi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{B}||_{L^2} + \kappa^2||\psi||_{H^1_{\kappa,\mathbf{A}}}||\pi_{\mathrm{div}}\mathbf{B}-\mathbf{B}||_{L^2},
\end{aligned}$$

and for $d = 2$ as

$$\begin{aligned}
|r_5(\mathbf{B},\psi)| &\lesssim ||\psi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{B}||_{L^2} + ||\psi||_{H^1_{\kappa,\mathbf{A}}}||\pi_{\mathrm{div}}\mathbf{B}-\mathbf{B}||_{L^2} + ||\psi||_{L^4}||\frac{1}{\kappa}\nabla u||_{L^4}||\pi_{\mathrm{div}}\mathbf{B}-\mathbf{B}||_{L^2} \\
&\lesssim ||\psi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{B}||_{L^2} + \kappa||\psi||_{H^1_{\kappa,\mathbf{A}}}||\pi_{\mathrm{div}}\mathbf{B}-\mathbf{B}||_{L^2},
\end{aligned}$$

which gives the claim. □

Lemma 3.11. *Let $(u,\mathbf{A}) \in H^1(\Omega)\times\mathbf{H}_0(\mathrm{curl},\mathrm{div})$ be a minimizer of (1.1). The second Fréchet derivative is coercive on $H^1_{\kappa,\mathbf{A}}\times\mathbf{H}_0(\mathrm{curl},\mathrm{div})$ with*

$$\langle E''(u,\mathbf{A})(\varphi,\mathbf{B}),(\varphi,\mathbf{B})\rangle \ge \rho_{u,\mathbf{A}}(\kappa)^{-1}C_{\mathrm{stab}}^{-1}(\kappa)||\varphi,\mathbf{B}||^2_{H^1_{\kappa,\mathbf{A}}\times\mathbf{H}_0(\mathrm{curl},\mathrm{div})}$$

with $\rho_{u,\mathbf{A}}(\kappa)$ bounded from above independently of $C_{\mathrm{stab}}(\kappa)$.

Proof. As in Lemma 3.7, we have for the smallest non-zero eigenvalue λ_1 that

$$\langle E''(u,\mathbf{A})(\varphi,\mathbf{B}),(\varphi,\mathbf{B})\rangle \ge \lambda_1||(\varphi,\mathbf{B})||^2_{L^2\times L^2}.$$

With the estimates in Lemma 3.9, we obtain by a weighted Young's inequality

$$\langle E''(u,\mathbf{A})(\varphi,\mathbf{B}),(\varphi,\mathbf{B})\rangle \ge c_1||\varphi||^2_{H^1_{\kappa,\mathbf{A}}} + c_1||\operatorname{curl}\mathbf{B}||^2_{L^2} - c_2C^2_{\mathrm{stab}}(\kappa)||\varphi||^2_{L^2} - c_3||\mathbf{B}||^2_{L^2}.$$

Without loss of generality let $c_2C^2_{\mathrm{stab}}(\kappa) \ge c_3$, and multiply the first equation by $c_2C_{\mathrm{stab}}(\kappa)$ and the second by λ_1 and add up to

$$(c_2C_{\mathrm{stab}}(\kappa)+\lambda_1)\langle\partial_u^2E(u,\mathbf{A})\varphi,\varphi\rangle \ge c_1\lambda_1\big(||\varphi||^2_{H^1_{\kappa,\mathbf{A}}} + ||\operatorname{curl}\mathbf{B}||^2_{L^2}\big)$$

which gives the claim for

$$\rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa) = \frac{\lambda_1 + c_2C_{\mathrm{stab}}(\kappa)}{c_1\lambda_1}$$

the same reasoning as in Lemma 3.7 gives the claim. □

4. Space discretization

We now turn to the spatially discrete version of (1.1). Therefore, we introduce finite element spaces for the order parameter u and the vector potential $\mathbf{A}$. In addition, we have to discretize the divergence constraint or equivalently the Lagrange multiplier. In this section, we first present the spaces and then discuss the discrete minimization problem as well as the results on the discrete minimizers.

4.1. **Discrete spaces and properties.** To keep some flexibility in the framework, we consider the different meshes where the order parameter u is approximated with $\mathcal{T}_h$ and $\mathcal{T}_H$, where the parameters h and H indicate the largest diameter of each of the meshes. In the major part of the presentation, we work with the following assumption without mentioning it in every occasion.

Assumption 4.1 (Properties of the meshes)**.**

(a) *The meshes $\mathcal{T}_h, \mathcal{T}_H$ are shape regular.*
(b) *On the mesh $\mathcal{T}_h$ the L^2-projection is $H^1(\Omega)$-stable.*

We note that the second part is for example satisfied for quasi-uniform meshes, but also in the more general setting as for example discussed in [4]. Later on, we additionally require the uniform boundedness of a certain embedding for $\mathcal{T}_H$, see Assumption 4.4.

For the spaces we choose $V_h \subset H^1(\Omega)$ to be the space of (real) linear and continuous Lagrange elements and $\mathcal{R}_H \subset \mathbf{H}(\mathrm{curl})$ as Nédélec elements of the lowest order (of first or second kind). We then seek discrete minimizers

$$u_h \in V_{h,\mathbb{C}} = V_h + \mathrm{i}V_h, \qquad \mathbf{A}_H \in \mathcal{R}_H,$$

and enforce the divergence free condition on $\mathbf{A}_H$, cf. [35, Section 7.2.1], via

$$(\mathbf{A}_H, \nabla\varphi_H)_{L^2} = 0 \quad \text{ for all } \varphi_H \in V_H,$$

where V_H also consists of linear Lagrange finite element space but on the mesh $\mathcal{T}_H$. Note that this induces a discrete divergence operator $\mathrm{div_H}\colon \mathcal{R}_H \to V_H$ via

$$(\mathrm{div_H}\,\mathbf{A}_H, \varphi_H)_{L^2} = -(\mathbf{A}_H, \nabla\varphi_H)_{L^2},$$

such that we equivalently enforce

$$\mathrm{div_H}\,\mathbf{A}_H = 0.$$

We would like to have a discrete subspace of $\mathcal{R}_H$ which corresponds to $\mathbf{H}_0(\mathrm{curl}, \mathrm{div})$ from (2.1b). Thus, we define

$$\mathbf{V}_{H,0} \coloneqq \{\mathbf{B}_H \in \mathcal{R}_H \mid \mathrm{div_H}\,\mathbf{B}_H = 0\}, \tag{4.1}$$

and emphasize that the condition $\mathbf{B}_H \cdot \nu = 0$ is only enforced weakly. We endow the space with the norm

$$||\mathbf{B}_H||^2_{\mathbf{V}_{H,0}} \coloneqq ||\mathbf{B}_H||^2_{L^2} + ||\,\mathrm{curl}\,\mathbf{B}_H||^2_{L^2} + ||\,\mathrm{div_H}\,\mathbf{B}_H||^2_{L^2}.$$

Note that such subspaces are also used by Li in [31, Sec. 3.3] in the context of the time-dependent Ginzburg–Landau problem. In addition, we define the orthogonal projection $P_H\colon \mathbf{H}(\mathrm{curl}) \to \mathcal{R}_H$ for $\mathbf{B} \in \mathbf{H}(\mathrm{curl})$ by

$$(\mathrm{curl}(\mathbf{B} - P_H\mathbf{B}), \mathrm{curl}\, C_H) + (\mathbf{B} - P_H\mathbf{B}, C_H) = 0,$$

such that in particular it holds for C_H

$$(P_H\mathbf{B}, \nabla\varphi_H) = (\mathbf{B}, \nabla\varphi_H), \tag{4.2}$$

which implies that $P_H\mathbf{B} - \mathbf{B}$ is discrete divergence-free. Further, $\mathrm{div}\,\mathbf{B} = 0$ implies $\mathrm{div_H}\, P_H\mathbf{B} = 0$. We have in addition the following approximation result in Nédélec spaces, which only requires regularity of the function and its curl for optimal rates.

Lemma 4.2. *Let $\mathbf{B} \in \mathbf{H}^{\frac{1}{2}+s}(\Omega)$ and also $\mathrm{curl}\,\mathbf{B} \in \mathbf{H}^{\frac{1}{2}+s}(\Omega)$ for $\frac{1}{2} < s \leq 1$, then*

$$||\mathbf{B} - P_H\mathbf{B}||_{\mathbf{H}(\mathrm{curl})} \lesssim H^{\frac{1}{2}+s}\big(||\mathbf{B}||_{\mathbf{H}^{\frac{1}{2}+s}} + ||\,\mathrm{curl}\,\mathbf{B}||_{\mathbf{H}^{\frac{1}{2}+s}}\big).$$

Proof. See Section 7.2.1 and Theorem 5.41 in [35]. □

Since a function with vanishing discrete divergence is not automatically divergence free, we make use of the following construction in several parts of the numerical analysis. Here, we employ the projection defined in (2.4).

Lemma 4.3. *Let* $\mathbf{B}_H \in \mathbf{V}_{H,0}$ *and consider the projection* π_{div} *in* (2.4).

(a) Then $\pi_{\mathrm{div}}\mathbf{B}_H \in \mathbf{H}_0(\mathrm{curl},\mathrm{div})$ *and it holds*

$$||\pi_{\mathrm{div}}\mathbf{B}_H - \mathbf{B}_H||_{\mathbf{H}(\mathrm{curl})} = ||\pi_{\mathrm{div}}\mathbf{B}_H - \mathbf{B}_H||_{L^2} \lesssim H^{\frac{1}{2}+s}||\operatorname{curl}\mathbf{B}_H||_{L^2}.$$

(b) Further, it holds the embedding

$$||\mathbf{B}_H||_{L^2} \lesssim ||\operatorname{curl}\mathbf{B}_H||_{L^2}$$

with a constant independent of H*, i.e.* $||\operatorname{curl}\mathbf{B}_H||_{L^2}$ *is an equivalent norm on* $\mathbf{V}_{H,0}$.

Proof. See Appendix A or [35, Lemma 7.6]. □

When studying the discrete minimizers we would like to carry over as much of the techniques from the continuous case to the discrete setting. An essential tool in th continuous case was there to use the embedding stated in Lemma 2.2.

Assumption 4.4. *There is a constant* C *independent of* H *such that*

$$\begin{aligned} ||\mathbf{B}_H||_{L^{\frac{3}{1-s}}} &\lesssim ||\mathbf{B}_H||_{\mathbf{V}_{H,0}}, \quad d=3,\\ ||\mathbf{B}_H||_{L^{\frac{4}{1-2s}}} &\lesssim ||\mathbf{B}_H||_{\mathbf{V}_{H,0}}, \quad d=2, \end{aligned}$$

where $s>0$ *is such that* $\mathbf{H}_0(\mathrm{curl},\mathrm{div}) \hookrightarrow L^{\frac{3}{1-s}}$ *and* $\mathbf{H}_0(\mathrm{curl},\mathrm{div}) \hookrightarrow L^{\frac{4}{1-2s}}$*, respectively.*

We note that in the case of a quasi-uniform mesh $\mathcal{T}_H$ the estimate in Lemma 3.6 in [31] shows that this assumption is indeed satisfied. An interesting question would be to find criteria on the mesh such that Assumption 4.4 is satisfied for more general meshes.

Finally, we rely on the following quasi-interpolation operators that allow us to extract the optimal order of convergence in the following analysis. Let us emphasize that the properties only require the shape-regularity of $\mathcal{T}_h$.

Lemma 4.5. *There exists a quasi-interpolation operator* $I_h\colon L^1 \to V_h$ *such that for all* $p\in(1,\infty)$ *it holds*

$$||u - I_h u||_{L^p} \le C_p h^s ||u||_{W^{s,p}} \qquad \text{and} \qquad ||\nabla(u - I_h u)||_{L^p} \le C_p h^s ||u||_{W^{1+s,p}}$$

with a constant $C_p > 0$ *independent of* h.

Proof. This assertion is stated element-wise in [25, Theorem 22.6] and since the ansatz functions are $W^{1,\infty}$-conforming, we obtain the estimate on Ω. □

4.2. **Discrete minimization.** We are now in the position to state our discrete problems using the discrete spaces from the last section. The problem reads as follows: Seek $(u_h,\mathbf{A}_H)\in V_{h,\mathbb{C}}\times\mathcal{R}_H$ such that

$$E(u_h,\mathbf{A}_H) = \min_{v_h\in V_{h,\mathbb{C}},\mathbf{B}_H\in\mathcal{R}_H} \frac{1}{2}\int_\Omega |\frac{\mathrm{i}}{\kappa}\nabla v_h + \mathbf{B}_H v_h|^2 + \frac{1}{2}(1-|v_h|^2)^2 + |\operatorname{curl}\mathbf{B}_H - H|^2\,\mathrm{d}x, \tag{4.3}$$
$$\text{subject to} \quad \mathrm{div}_{\mathrm{H}}\,\mathbf{A}_H = 0.$$

Analogously to Lemma 2.7, we obtain that each minimizer is a solution to a saddle point problem, i.e. there exists $\lambda_H \in V_H$ such that for all $(\varphi_h,\mathbf{B}_H,\mu_H)\in V_{h,\mathbb{C}}\times\mathcal{R}_H\times V_H$ it holds

$$\begin{aligned} 0 &= \partial_u E(u_h,\mathbf{A}_H)\varphi_h + \partial_{\mathbf{A}}E(u,\mathbf{A}_H)\mathbf{B}_H + (\nabla\lambda_H,\mathbf{B}_H),\\ 0 &= (\mathrm{div}_{\mathrm{H}}\,\mathbf{A}_H,\mu_H), \end{aligned} \tag{4.4}$$

where the Lagrange multiplier terms vanishes for $\mathbf{B}_H\in\mathbf{V}_{H,0}$. Next, we state the existence result of discrete minimizers as well as the a-priori estimates.

Theorem 4.6 (Existence of discrete minmizers)**.** *For each $h > 0$, there exists at least one minimizer $(u_h, \mathbf{A}_H) \in V_{h,\mathbb{C}} \times \mathcal{R}_H$ of (4.3), and we have the a-priori bounds*

$$||\mathbf{A}_H||_{L^2} + ||\operatorname{curl} \mathbf{A}_H||_{L^2} \lesssim 1, \qquad ||u_h||_{L^4} \lesssim 1 \qquad ||\frac{\mathrm{i}}{\kappa}\nabla u_h + \mathbf{A}_H u_h||_{L^2} \lesssim 1.$$

In addition, under Assumption 4.4 it also holds $||\mathbf{A}_H||_{L^{\frac{3}{1-s}}} \lesssim 1$ $(d = 3)$ and $||\mathbf{A}_H||_{L^{\frac{4}{1-2s}}} \lesssim 1$ $(d = 2)$, respectively, and

$$||u_h||_{H^1_\kappa} + ||\mathbf{A}_H u_h||_{L^2} \lesssim 1 + \kappa^\alpha, \qquad \text{with} \qquad \begin{cases} \alpha = \max\{0, \frac{1-4s}{4s}\}, & d = 3, \\ \alpha = 0, & d = 2, \end{cases}$$

with hidden constants independent of h, H and κ.

Proof. Due to the finite dimensions in $V_{h,\mathbb{C}}$ and $\mathcal{R}_H$, the existence is clear. For the bounds note that $(0, 0) \in V_{h,\mathbb{C}} \times \mathbf{V}_{H,0}$ and thus $E(u_h, \mathbf{A}_H) \le E(0, 0) \lesssim 1$ and by construction also $\operatorname{div}_{\mathrm{H}} \mathbf{A}_H = 0$. The first three estimates are obtained as in [13] combined with Lemma 4.3. For the last one in $d = 3$, note that by Assumption 4.4

$$||\frac{\mathrm{i}}{\kappa}\nabla u_h||_{L^2} \lesssim ||\frac{\mathrm{i}}{\kappa}\nabla u_h + \mathbf{A}_H u_h||_{L^2} + ||\mathbf{A}_H u_h||_{L^2} \lesssim 1 + ||\mathbf{A}_H||_{\mathbf{V}_{H,0}} ||u_h||_{L^{\frac{6}{1+2s}}} \lesssim 1 + ||u_h||_{L^{\frac{6}{1+2s}}}.$$

For $s \ge \frac{1}{4}$, we obtain boundedness by a constant and thus only consider $s \in (0, \frac{1}{4})$. In this case we use the relation

$$\frac{1 + 2s}{6} = \frac{\alpha}{6} + \frac{1 - \alpha}{4}, \qquad \alpha = 1 - 4s \in (0, 1),$$

and estimate with $||u_h||_{L^2} + ||u_h||_{L^4} \lesssim 1$

$$||u_h||_{L^{\frac{6}{1+2s}}} \lesssim ||u_h||_{L^6}^{1-4s} ||u_h||_{L^4}^{4s} \lesssim ||u_h||_{L^6}^{1-4s} \lesssim \kappa^{1-4s} ||\frac{\mathrm{i}}{\kappa}\nabla u_h||_{L^2}^{1-4s} + 1 \lesssim C_\delta \kappa^{\frac{1-4s}{4s}} + \delta ||\frac{\mathrm{i}}{\kappa}\nabla u_h||_{L^2},$$

where we used Young's inequality in the last step for the pair $(\frac{1}{4s}, \frac{1}{1-4s})$ and some arbitrary $\delta > 0$. Hence, we obtain the last two estimates (for $d = 3$) by absorption. In two dimensions, we already have a uniform bound on $\mathbf{A}_H$ in L^4 and thus $||\mathbf{A}_H u_h||_{L^2} \lesssim 1$. □

Proposition 4.7. *Denote by $(u_h, \mathbf{A}_H) \in V_{h,\mathbb{C}} \times \mathcal{R}_H$ a family of minimizers of (4.3). Then, there exists an exact minimizer $(u_0, \mathbf{A}_0) \in H^1(\Omega) \times \mathbf{H}_0(\operatorname{curl}, \operatorname{div})$ of problem (1.1) such that there is a monotonically decreasing sequence $(h, H) \to 0$ with*

$$\lim_{(h,H)\to 0} ||u_0 - u_h||_{\mathbf{H}^1_n} + ||\mathbf{A}_0 - \mathbf{A}_H||_{\mathbf{H}(\operatorname{curl})} = 0.$$

In particular, we can require $u_h \perp \mathrm{i}u$.

Proof. We follow the strategy in [9]. By the a-priori bounds in Theorem 4.6, we obtain weakly convergent subsequences (with abuse of notation) as $(h, H) \to 0$

$$u_h \rightharpoonup u_0 \text{ in } H^1(\Omega), \qquad \mathbf{A}_H \rightharpoonup \mathbf{A}_0 \text{ in } \mathbf{H}(\operatorname{curl}),$$

which directly implies strong convergence of u_h in L^6 and of $\mathbf{A}_H$ in L^2. We now fix $\varphi \in H^1(\Omega)$ and write for $\varphi_H \in V_H$

$$\begin{aligned}
(\mathbf{A}_0, \nabla\varphi) &= (\mathbf{A}_0 - \mathbf{A}_H, \nabla\varphi) + (\mathbf{A}_H, \nabla\varphi) \\
&= (\mathbf{A}_0 - \mathbf{A}_H, \nabla\varphi) + (\mathbf{A}_H, \nabla\varphi_H) + (\mathbf{A}_H, \nabla\varphi - \nabla\varphi_H) \\
&= (\mathbf{A}_0 - \mathbf{A}_H, \nabla\varphi) + (\mathbf{A}_H, \nabla\varphi - \nabla\varphi_H),
\end{aligned}$$

where we used that $\operatorname{div}_{\mathrm{H}} \mathbf{A}_H = 0$. Now let $\lim_{H\to 0} ||\nabla(\varphi - \varphi_H)||_{L^2} = 0$, then the first term tends to zero due to the strong convergence in L^2 and the last term due to the uniform boundedness of $||\mathbf{A}_H||_{L^2}$. Thus, we have shown that $\mathbf{A}_0 \in \mathbf{H}_0(\operatorname{curl}, \operatorname{div})$.

By the weak semi lower continuity, we obtain

$$\liminf_{(h,H)\to 0} E(u_h, \mathbf{A}_H) \ge E(u_0, \mathbf{A}_0)$$

and by the density of $V_{h,\mathbb{C}}$ and $\mathcal{R}_H$ we obtain that $(u_h, \mathbf{A}_H)$ is a minimizing sequence. Hence, it holds equality

$$\lim_{(h,H)\to 0} E(u_h, \mathbf{A}_H) = E(u_0, \mathbf{A}_0) = \min_{\mathbf{H}^1_{\mathrm{n}} \times \mathbf{H}_0(\mathrm{curl},\mathrm{div})} E(u, \mathbf{A}).$$

In addition with the strong convergence of the lower order terms, the convergence of the energy implies

$$||\frac{1}{\kappa}\nabla u_h||^2_{L^2} + ||\operatorname{curl} \mathbf{A}_H||^2_{L^2} \to ||\frac{1}{\kappa}\nabla u_0||^2_{L^2} + ||\operatorname{curl} \mathbf{A}_0||^2_{L^2},$$

which finally yields the strong convergence as claimed. □

4.3. **The Ritz projection in $H^1_{\kappa,\mathbf{A}}$.** This section is devoted to a suitable projection for the order parameter u in the inner product corresponding to $H^1_{\kappa,\mathbf{A}}$ defined in (3.2). We present general best-approximation results as well as estimates for prescribed regularities. For a given minimizer u, we define the spaces $H^1_{\mathrm{i}u} = H^1(\Omega) \cap (\mathrm{i}u)^\perp$ and $V_{h,\mathrm{i}u} = V_{h,\mathbb{C}} \cap (\mathrm{i}u)^\perp$, where the orthogonality is with respect to the L^2-inner product. We define the Ritz projection $\mathrm{R}^\perp_{\kappa,\mathbf{A},h} \colon H^1_{\mathrm{i}u} \to V_{h,\mathrm{i}u}$ via

$$a_{\kappa,\mathbf{A}}(v - \mathrm{R}^\perp_{\kappa,\mathbf{A},h} v, \varphi_h) = 0 \quad \text{for all} \quad \varphi_h \in V_{h,\mathrm{i}u},$$

and in particular, we have

$$(\frac{\mathrm{i}}{\kappa}\nabla(v - \mathrm{R}^\perp_{\kappa,\mathbf{A},h}) + \mathbf{A}(v - \mathrm{R}^\perp_{\kappa,\mathbf{A},h} v), \frac{\mathrm{i}}{\kappa}\nabla\varphi_h + \mathbf{A}\varphi_h) = -C_{\mathrm{stab}}(\kappa)(v - \mathrm{R}^\perp_{\kappa,\mathbf{A},h} v, \varphi_h). \tag{4.5}$$

With this, we can conclude an adaption of the estimate in [14, Lemma 5.11].

Lemma 4.8. *Let $\varphi \in H^1_{\mathrm{i}u}$ and assume the resolution condition $C_{\mathrm{stab}}(\kappa)\kappa h \lesssim 1$, then*

$$||\varphi - \mathrm{R}^\perp_{\kappa,\mathbf{A},h}\varphi||_{H^1_{\kappa,\mathbf{A}}} \lesssim \inf_{\varphi_h \in V_h} \Big(\frac{1}{\kappa}||\nabla(\varphi - \varphi_h)||_{L^2} + C_{\mathrm{stab}}(\kappa)||\varphi - \varphi_h||_{L^2}\Big)$$

with a constant independent of κ and h.

Proof. We proceed analogously as in [14, Lemma 5.11] to get for any $\varphi \in H^1_{\mathrm{i}u}$

$$\begin{aligned}
||\varphi - \mathrm{R}^\perp_{\kappa,\mathbf{A},h}\varphi||_{H^1_{\kappa,\mathbf{A}}} &\le ||\varphi - \big(\pi_h\varphi - \frac{m(\pi_h\varphi - \varphi, \mathrm{i}u)}{m(\pi_h(\mathrm{i}u), \mathrm{i}u)}\pi_h(\mathrm{i}u)\big)||_{H^1_{\kappa,\mathbf{A}}} \\
&\le ||\varphi - \pi_h\varphi||_{H^1_{\kappa,\mathbf{A}}} + ||\varphi - \pi_h\varphi||_{L^2}\frac{||u||_{L^2}}{m(\pi_h(\mathrm{i}u) - \mathrm{i}u, \mathrm{i}u) + ||\mathrm{i}u||^2_{L^2}}||\pi_h(\mathrm{i}u)||_{H^1_{\kappa,\mathbf{A}}} \\
&\le ||\varphi - \pi_h\varphi||_{H^1_{\kappa,\mathbf{A}}} + ||\varphi - \pi_h\varphi||_{L^2}\frac{||\pi_h(\mathrm{i}u)||_{H^1_{\kappa,\mathbf{A}}}}{||u||_{L^2} - ||u - \pi_h u||_{L^2}}.
\end{aligned}$$

With the $H^1(\Omega)$-stability of π_h in Assumption 4.1 and the norm equivalence in Lemma 3.1, we have

$$||\pi_h(\mathrm{i}u)||_{H^1_{\kappa,\mathbf{A}}} \lesssim ||u||_{H^1_{\kappa,\mathbf{A}}} \lesssim C_{\mathrm{stab}}(\kappa)||u||_{L^2},$$

where we employed (2.6) for the last estimate. This leads to

$$||\varphi - \mathrm{R}^\perp_{\kappa,\mathbf{A},h}\varphi||_{H^1_{\kappa,\mathbf{A}}} \lesssim ||\varphi - \pi_h\varphi||_{H^1_{\kappa,\mathbf{A}}} + C_{\mathrm{stab}}(\kappa)||\varphi - \pi_h\varphi||_{L^2}\frac{||u||_{L^2}}{||u||_{L^2} - ||u - \pi_h u||_{L^2}}.$$

We further estimate the denominator again with the norm equivalence and (2.6) to obtain

$$||u||_{L^2} - c\kappa h||\frac{1}{\kappa}\nabla u||_{L^2} \ge ||u||_{L^2} - c\kappa h||u||_{H^1_{\kappa,\mathbf{A}}} \ge ||u||_{L^2}(1 - cC_{\mathrm{stab}}(\kappa)\kappa h)$$

to finally get with the assumed resolution condition

$$||\varphi - \mathrm{R}^\perp_{\kappa,\mathbf{A},h}\varphi||_{H^1_{\kappa,\mathbf{A}}} \lesssim ||\varphi - \pi_h\varphi||_{H^1_{\kappa,\mathbf{A}}} + C_{\mathrm{stab}}(\kappa)||\varphi - \pi_h\varphi||_{L^2} \lesssim ||\varphi - \pi_h\varphi||_{H^1_{\kappa,\mathbf{A}}}.$$

Again the stability of the L^2-projection in $H^1_{\kappa,\mathbf{A}}$ and the norm equivalence from Lemma 3.1 give the desired claim. □

Next, we present a series of lemmas which discuss the precise projection errors depending on the regularity of u.

Lemma 4.9. *Let $u \in H^{1+\sigma}(\Omega)$ for some $\sigma \in (0,1]$. Then,*

$$||u - \mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u||_{H^1_{\kappa,\mathbf{A}}} \lesssim h^{\sigma}\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)\frac{1}{\kappa}||u||_{H^{1+\sigma}}$$

with a constant independent of h and κ.

Proof. We employ Lemma 4.8 and choose the quasi-interpolation $\varphi_h = I_h u$ from Lemma 4.5 to obtain with $s = 1+\sigma$

$$\begin{aligned} ||u - \mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u||_{H^1_{\kappa,\mathbf{A}}} &\lesssim \frac{1}{\kappa}||\nabla(u - I_h u)||_{L^2} + C_{\mathrm{stab}}(\kappa)||u - I_h u||_{L^2} \\ &\lesssim \big(\frac{1}{\kappa}h^{s-1} + C_{\mathrm{stab}}(\kappa)h^{s}\big)||u||_{H^s} \\ &\lesssim h^{\sigma}\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)\frac{1}{\kappa}||u||_{H^{1+\sigma}}, \end{aligned}$$

which gives the claim. □

We can similarly show an increased convergence rate in the L^2-norm. As expected we do not obtain an additional full order of convergence, but only the available s from the smoothness of the domain.

Lemma 4.10. *(a) If $\Omega \subset \mathbb{R}^3$, we have*

$$||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u - u||_{L^2} \lesssim h^{s}\kappa\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u - u||_{H^1_{\kappa,\mathbf{A}}}.$$

(a) If $\Omega \subset \mathbb{R}^2$, we have

$$||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u - u||_{L^2} \lesssim h^{\frac{1}{2}+s}\kappa\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u - u||_{H^1_{\kappa,\mathbf{A}}}.$$

In both cases the hidden constant is independent of h and κ.

Proof. We use a dual argument, and define $e_\pi = \mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u - u$ and $w \in H^1(\Omega)$ as the solution of

$$a_{\kappa,\mathbf{A}}(w,\varphi) = m(e_\pi,\varphi),$$

and compute with the orthogonality of the Ritz projection

$$||e_\pi||^2_{L^2} = a_{\kappa,\mathbf{A}}(w,e_\pi) = a_{\kappa,\mathbf{A}}(w - \mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}w, \mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u - u) \le ||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}w - w||_{H^1_{\kappa,\mathbf{A}}}||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u - u||_{H^1_{\kappa,\mathbf{A}}}.$$

(a) With Lemmas 4.9 and 3.5 we conclude

$$||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}w - w||_{H^1_{\kappa,\mathbf{A}}} \lesssim h^{s}\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)\frac{1}{\kappa}||w||_{H^{1+s}} \lesssim h^{s}\kappa\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)||e_\pi||_{L^2}$$

and thus

$$||e_\pi||_{L^2} \lesssim h^{s}\kappa\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u - u||_{H^1_{\kappa,\mathbf{A}}},$$

as claimed.

(b) We simply use Lemma 4.9 and the increased regularity in part (b) of Lemma 3.5 to obtain

$$||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}w - w||_{H^1_{\kappa,\mathbf{A}}} \lesssim h^{\frac{1}{2}+s}\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)\frac{1}{\kappa}||w||_{H^{\frac{3}{2}+s}} \lesssim h^{\frac{1}{2}+s}\kappa\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)||e_\pi||_{L^2},$$

which gives the second assertion. □

Further, we show another approximation (for $d=3$) in the $L^{\frac{3}{1+s}}$-norm, which is between L^2 and $H^1 \hookrightarrow L^6$ for $s \in (0,\frac{1}{2}]$ and is useful in the later best-approximation estimates, cf. the proof of Lemma 5.7 below.

Lemma 4.11. *Let $\Omega \subset \mathbb{R}^3$ and $s \in (0,\frac{1}{2}]$. Then, it holds for*

$$||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u - u||_{L^{\frac{3}{1+s}}} \lesssim h^{s(\frac{1}{2}+s)}\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)^{\frac{1}{2}+s}\kappa||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u - u||_{H^1_{\kappa,\mathbf{A}}},$$

and for $s \ge \frac{1}{2}$ we are in the situation of Lemma 4.10.

Proof. We use interpolation between the estimates in L^2 and $H^1_{\kappa,\mathbf{A}}$. Therefore, note that

$$\frac{1+s}{3} = \frac{\alpha}{2} + \frac{1-\alpha}{6} \qquad \text{for} \qquad \alpha = \frac{1}{2} + s.$$

Hence, we obtain with $e_\pi = \mathrm{R}^\perp_{\kappa,\mathbf{A},h} u - u$

$$||e_\pi||_{L^{\frac{3}{1+s}}} \leq ||e_\pi||^{\frac{1}{2}+s}_{L^2} ||e_\pi||^{\frac{1}{2}-s}_{L^6} \leq \kappa^{\frac{1}{2}-s} ||e_\pi||^{\frac{1}{2}+s}_{L^2} ||e_\pi||^{\frac{1}{2}-s}_{H^1_{\kappa,\mathbf{A}}}.$$

Now with Lemma 4.10, we further estimate

$$||e_\pi||_{L^{\frac{3}{1+s}}} \lesssim (h^s\kappa)^{\frac{1}{2}+s}\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)^{\frac{1}{2}+s} \kappa^{\frac{1}{2}-s} ||e_\pi||_{H^1_{\kappa,\mathbf{A}}} = h^{s(\frac{1}{2}+s)}\kappa\big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big)^{\frac{1}{2}+s} ||e_\pi||_{H^1_{\kappa,\mathbf{A}}},$$

which gives the claim. □

4.4. **Inserting the minimizer.** We now combine the results on the convergence with the established regularity of the minimizers to derive convergence rates which are then independent of the minimizer itself and only depend on the regularity induced by the domain Ω. We first state the result in three dimensions.

Lemma 4.12 (3 dimensions)**.** *(a) Let $u \in W^{2,p_{u,2}}$ be a minimizer of (1.1), then*

$$||u - \mathrm{R}^\perp_{\kappa,\mathbf{A},h} u||_{H^1_{\kappa,\mathbf{A}}} \lesssim \min\{(\kappa h)^{p_{u,2}-1}, \kappa h^{\frac{5p_{u,2}-6}{2p_{u,2}}}\}(1 + \kappa^{\frac{1}{s}} h), \quad 0 < s < 1.$$

(b) Since $p_{u,2} \geq \frac{4}{3}$, we have at least

$$||u - \mathrm{R}^\perp_{\kappa,\mathbf{A},h} u||_{H^1_{\kappa,\mathbf{A}}} \lesssim (\kappa h)^{\frac{1}{3}}(1 + \kappa^{\frac{1}{s}} h), \quad 0 < s < 1.$$

Proof. We simply use Lemma 4.9 together with the estimates (2.7a) and (2.7b) and $C_{\mathrm{stab}}(\kappa)$ from (3.3) to conclude the assertion. □

Remark 4.13. *(a) Note that if Ω is convex, we obtain $s = \frac{1}{2}$ and hence $p_{u,2} = 2$. This implies*

$$||u - \mathrm{R}^\perp_{\kappa,\mathbf{A},h} u||_{H^1_{\kappa,\mathbf{A}}} \lesssim (\kappa h)(1 + \kappa^2 h).$$

(b) Further, formally inserting $s = 1$ yields $p_{u,2} \geq 2$ and hence the estimate

$$||u - \mathrm{R}^\perp_{\kappa,\mathbf{A},h} u||_{H^1_{\kappa,\mathbf{A}}} \lesssim (\kappa h)(1 + \kappa h).$$

Due to the better embedding in two dimensions, the error bounds improve in this case.

Lemma 4.14 (2 dimensions)**.** *(a) Let $u \in W^{2,p_{u,2}}$ be a minimizer of (1.1), then*

$$||u - \mathrm{R}^\perp_{\kappa,\mathbf{A},h} u||_{H^1_{\kappa,\mathbf{A}}} \lesssim \min\{(\kappa h)^{p_{u,2}-1}, \kappa h^{\frac{2p_{u,2}-2}{p_{u,2}}}\}(1 + \kappa^{\frac{2}{1+2s}} h), \quad s < \frac{1}{2}.$$

(b) Since $p_{u,2} \geq \frac{4}{3}$, we have at least

$$||u - \mathrm{R}^\perp_{\kappa,\mathbf{A},h} u||_{H^1_{\kappa,\mathbf{A}}} \lesssim \min\{(\kappa h)^{\frac{1}{3}}, \kappa h^{\frac{1}{2}}\}(1 + \kappa^{\frac{2}{1+2s}} h), \quad 0 < s < 1.$$

Remark 4.15. *If we write the largest angle $\omega_{max} = \alpha\pi$, $\alpha > \frac{1}{2}$, then*

$$p_{\max}(\alpha) = \frac{\alpha}{\alpha - \frac{1}{2}}, \implies p_{\max}(1) = 2, \quad p_{\max}(\frac{3}{2}) = \frac{3}{2}, \quad p_{\max}(2) = \frac{4}{3}.$$

(a) For example if $\alpha = \frac{3}{2}$, then $s = \frac{1}{6}$ and the convergence improves to

$$||u - \mathrm{R}^\perp_{\kappa,\mathbf{A},h} u||_{H^1_{\kappa,\mathbf{A}}} \lesssim \min\{(\kappa h)^{\frac{1}{2}}, \kappa h^{\frac{2}{3}}\}(1 + \kappa^{\frac{3}{2}} h), \quad 0 < s < 1,$$

which aligns with the expected regularity for an L-shaped domain where $u \in H^{\frac{5}{3}-}$.

(b) If Ω is convex, then $s = \frac{1}{2}$ and $p_{\max} = 2$ such that

$$||u - \mathrm{R}^\perp_{\kappa,\mathbf{A},h} u||_{H^1_{\kappa,\mathbf{A}}} \lesssim (\kappa h)(1 + \kappa h).$$

5. Quasi best-approximation results

We finally turn to the error bounds for the minimizers, where relate the error $e_h = u - u_h$ to the best approximation error e_π. We first study the case of a fixed vector potential, but only assume the regularity $\mathbf{A} \in H^{\frac{1}{2}+s}$, and in a second step derive the error bounds for the full problem. Let us note that in the reduced case, one can derive better estimates than in the coupled case.

5.1. **The case of fixed A.** Let u be a minimizer of (1.1) and $u_h \in V_{h,\mathrm{iu}}$ a minimizer of (4.3), and emphasize that we do not require Assumption 4.4 in this section. We use the standard decomposition

$$\langle \partial_u^2 E(u) \mathrm{R}_{\kappa,\mathbf{A},h}^\perp u - u_h, \varphi_h \rangle = \langle \partial_u^2 E(u) \mathrm{R}_{\kappa,\mathbf{A},h}^\perp u - u, \varphi_h \rangle + \langle \partial_u^2 E(u) u - u_h, \varphi_h \rangle =: \varepsilon_1(\varphi_h) + \varepsilon_2(\varphi_h),$$

and derive the following bounds.

Lemma 5.1. *Let u be a minimizer of (1.1) and $u_h \in V_{h,\mathrm{iu}}$ a minimizer of (4.3). Then, we have the estimates*

$$\varepsilon_1(\varphi_h) \lesssim C_{\mathrm{stab}}^2(\kappa) ||\mathrm{R}_{\kappa,\mathbf{A},h}^\perp u - u||_{L^2} ||\varphi_h||_{L^2},$$
$$\varepsilon_2(\varphi_h) \lesssim \big(||u - u_h||_{L^4}^2 + ||u - u_h||_{L^6}^3 \big) ||\varphi_h||_{L^2},$$

with constants independent of κ and h.

Proof. We follow the lines of [14, Lemma 5.5], and first consider using the representation in (3.1)

$$\begin{aligned} \varepsilon_1(\varphi_h) &= -C_{\mathrm{stab}}^2(\kappa) (\mathrm{R}_{\kappa,\mathbf{A},h}^\perp u - u, \varphi_h) \\ &\quad + \big((|u|^2 - 1)(\mathrm{R}_{\kappa,\mathbf{A},h}^\perp u - u) + u^2 (\mathrm{R}_{\kappa,\mathbf{A},h}^\perp u - u)^* + |u|^2 (\mathrm{R}_{\kappa,\mathbf{A},h}^\perp u - u), \varphi_h \big) \\ &\lesssim C_{\mathrm{stab}}^2(\kappa) ||\mathrm{R}_{\kappa,\mathbf{A},h}^\perp u - u||_{L^2} ||\varphi_h||_{L^2}, \end{aligned}$$

where we used (4.5). Further, by the estimate

$$\begin{aligned} |\varepsilon_2(\varphi_h)| &\leq \int_\Omega \big(2u|u - u_h|^2 + (u - u_h)^2 u^* - |u - u_h|^2 (u - u_h) \big) \phi h^* \,\mathrm{d}x \\ &\lesssim \big(||u - u_h||_{L^4}^2 + ||u - u_h||_{L^6}^3 \big) ||\varphi_h||_{L^2} \end{aligned}$$

we obtain the claim. □

This already yields us the best-approximation result in the $H_{\kappa,\mathbf{A}}^1$-norm.

Theorem 5.2. *Let Assumption 4.1 hold, and let u be a minimizer of (1.1) and $u_h \in V_{h,\mathrm{iu}}$ a minimizer of (4.3). For h sufficiently small, we obtain the following quasi best-approximation results.*

(a) If $\Omega \subset \mathbb{R}^3$, then it holds

$$||u - u_h||_{H_{\kappa,\mathbf{A}}^1} \lesssim \Big(1 + h^s \kappa \rho_{u,\mathbf{A}}(\kappa) \big(1 + \kappa^{\frac{1}{s}} h\big) \Big) ||\mathrm{R}_{\kappa,\mathbf{A},h}^\perp u - u||_{H_{\kappa,\mathbf{A}}^1}$$

with constants independent of κ and h.

(b) If $\Omega \subset \mathbb{R}^2$, then it holds

$$||u - u_h||_{H_{\kappa,\mathbf{A}}^1} \leq \Big(1 + h^{\frac{1}{2}+s} \kappa \rho_{u,\mathbf{A}}(\kappa) \big(1 + \kappa^{\frac{2}{1+2s}} h\big) \Big) ||\mathrm{R}_{\kappa,\mathbf{A},h}^\perp u - u||_{H_{\kappa,\mathbf{A}}^1}$$

with constants independent of κ and h.

In particular, we observe that for h is sufficiently small $||u - u_h||_{H_{\kappa,\mathbf{A}}^1} \lesssim ||\mathrm{R}_{\kappa,\mathbf{A},h}^\perp u - u||_{H_{\kappa,\mathbf{A}}^1}$.

Proof. We employ the lower bound in Lemma 3.7 and the estimates in Lemma 5.1

$$\begin{aligned}
&||\frac{1}{\kappa}\nabla(\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u_h)||^2_{L^2}+C^2_{\mathrm{stab}}(\kappa)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u_h||^2_{L^2}\\
&\lesssim \rho_{u,\mathbf{A}}(\kappa)\langle\partial^2_u E(u)\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u_h,\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u_h\rangle\\
&\leq \rho_{u,\mathbf{A}}(\kappa)\big(C^2_{\mathrm{stab}}(\kappa)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u||_{L^2}+||u-u_h||^2_{L^4}+||u-u_h||^3_{L^6}\big)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u_h||_{L^2}\\
&\leq \rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u||_{L^2}C_{\mathrm{stab}}(\kappa)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u_h||_{L^2}\\
&\qquad+\rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa)\big(||u-u_h||^2_{L^4}+||u-u_h||^3_{L^6}\big)C_{\mathrm{stab}}(\kappa)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u_h||_{L^2}
\end{aligned}$$

and thus with Young

$$\begin{aligned}
&||\frac{1}{\kappa}\nabla(\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u_h)||_{L^2}+C_{\mathrm{stab}}(\kappa)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u_h||_{L^2}\\
&\lesssim \rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u||_{L^2}+\rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa)\big(||u-u_h||^2_{L^4}+||u-u_h||^3_{L^6}\big)
\end{aligned}$$

Form this, we conclude for h sufficiently small

$$\begin{aligned}
&||\frac{1}{\kappa}\nabla(u-u_h)||_{L^2}+C_{\mathrm{stab}}(\kappa)||u-u_h||_{L^2}\\
&\lesssim ||\frac{1}{\kappa}\nabla(u-\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u)||_{L^2}+C_{\mathrm{stab}}(\kappa)||u-\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u||_{L^2}+\rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u||_{L^2}\\
&\lesssim ||\frac{1}{\kappa}\nabla(u-\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u)||_{L^2}+\rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u||_{L^2}
\end{aligned}$$

For $\Omega\subset\mathbb{R}^3$, we insert Lemma 4.10 (a) to eliminate the L^2-norm and obtain

$$||u-u_h||_{H^1_{\kappa,\mathbf{A}}}\leq\Big(1+h^s\kappa\rho_{u,\mathbf{A}}(\kappa)\big(1+C_{\mathrm{stab}}(\kappa)\kappa h\big)\Big)||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u||_{H^1_{\kappa,\mathbf{A}}}$$

and the definition of $C_{\mathrm{stab}}(\kappa)$ gives the desired bound. Using Lemma 4.10 (b) gives the claim in two dimensions. □

Using the estimate in $H^1_{\kappa,\mathbf{A}}$, we are now able to increase the convergence in the L^2-norm via an Aubin–Nitsche trick.

Theorem 5.3. *Let Assumption 4.1 hold, and let u be a minimizer of (1.1) and $u_h\in V_{h,\mathrm{iu}}$ a minimizer of (4.3).*

(a) If $\Omega\subset\mathbb{R}^3$, we have the estimate

$$||u-u_h||_{L^2}\lesssim\rho_{u,\mathbf{A}}(\kappa)\kappa h^s\big(1+C_{\mathrm{stab}}(\kappa)\kappa h\big)||u-u_h||_{H^1_{\kappa,\mathbf{A}}}$$

for h sufficiently small.

(b) If $\Omega\subset\mathbb{R}^2$, we have the estimate

$$||u-u_h||_{L^2}\leq\rho_{u,\mathbf{A}}(\kappa)\kappa h^{s+\frac{1}{2}}\big(1+C_{\mathrm{stab}}(\kappa)\kappa h\big)||u-u_h||_{H^1_{\kappa,\mathbf{A}}}$$

for h sufficiently small.

Proof. We use again a duality argument, but now with $\partial^2_u E$. Denote again $e_h=u-u_h$ and let $w\in H^1(\Omega)$ be the solution of

$$\langle\partial^2_u E(u)w,\varphi\rangle=(e_h,\varphi)$$

We then insert $\varphi=e_h$ to obtain

$$\begin{aligned}
||e_h||^2_{L^2}&=\langle\partial^2_u E(u)w,e_h\rangle=\langle\partial^2_u E(u)w-\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}w,e_h\rangle+\langle\partial^2_u E(u)\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}w,e_h\rangle\\
&\lesssim ||w-\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}w||_{H^1_{\kappa,\mathbf{A}}}||e_h||_{H^1_{\kappa,\mathbf{A}}}+|\varepsilon_2(\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}w)|\\
&\lesssim ||w-\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}w||_{H^1_{\kappa,\mathbf{A}}}||e_h||_{H^1_{\kappa,\mathbf{A}}}+||e_h||_{L^2}\big(||u-u_h||_{L^3}+||u-u_h||^2_{L^6}\big)\kappa||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}w||_{H^1_{\kappa,\mathbf{A}}}
\end{aligned}$$

where we used the representation of ε_2 in Lemma 5.1. We then use Lemma 3.8 to estimate

$$||\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}w||_{H^1_{\kappa,\mathbf{A}}}\leq||w||_{H^1_{\kappa,\mathbf{A}}}\lesssim\rho_{u,\mathbf{A}}(\kappa)||e_h||_{L^2}$$

as well as with Lemma 4.9 (a) (and part (b) in the case $d = 2$)

$$\|w - \mathrm{R}^{\perp}_{\kappa,\mathbf{A},h} w\|_{H^1_{\kappa,\mathbf{A}}} \lesssim h^s \big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big) \frac{1}{\kappa} \|w\|_{H^{1+s}} \lesssim h^s \kappa \big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big) \rho_{u,\mathbf{A}}(\kappa) \|e_h\|_{L^2}.$$

Combining all this, gives

$$\begin{aligned} \|u - u_h\|_{L^2} &\lesssim h^s \kappa \big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big) \rho_{u,\mathbf{A}}(\kappa) \|e_h\|_{H^1_{\kappa,\mathbf{A}}} \\ &\qquad + \rho_{u,\mathbf{A}}(\kappa) \big(\|u - u_h\|_{L^3} + \|u - u_h\|^2_{L^6}\big) \kappa \|u - u_h\|_{L^2} \\ &\lesssim \kappa h^s \rho_{u,\mathbf{A}}(\kappa) \big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big) \|e_h\|_{H^1_{\kappa,\mathbf{A}}} \\ &\qquad + \rho_{u,\mathbf{A}}(\kappa) \big(\|u - u_h\|_{L^3} + \|u - u_h\|^2_{L^6}\big) \kappa \|u - u_h\|_{L^2}. \end{aligned}$$

Thus, for h sufficiently small, we obtain by absorption

$$\|u - u_h\|_{L^2} \lesssim \rho_{u,\mathbf{A}}(\kappa) \kappa h^s \big(1 + C_{\mathrm{stab}}(\kappa)\kappa h\big) \|e_h\|_{H^1_{\kappa,\mathbf{A}}},$$

which gives the claim. □

5.2. **The case of full E''.** We now turn the coupled problem and treat the approximation on u and $\mathbf{A}$ together. A crucial ingredient in the latter analysis is the coercivity of the second Fréchet derivative to estimate the discrete error. However, in the magnetic part, the error satisfies by (4.2)

$$\operatorname{div}_{\mathrm{H}} \widehat{E}_H = 0 \quad \text{but} \quad \widehat{E}_H = P_H \mathbf{A} - \mathbf{A}_H \notin \mathbf{H}(\mathrm{div}).$$

Hence, we cannot apply Lemma 3.11 directly. However, the coercivity can be recovered under the resolution condition

$$\rho_{u,\mathbf{A}}(\kappa) C_{\mathrm{stab}}(\kappa) H^{\frac{1}{2}+s} \quad \text{sufficiently small,} \tag{5.1}$$

which is shown in the following lemma.

Proposition 5.4. *Let Assumption 4.1 hold, and let $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$ be a minimizer of (1.1) and $(u_h, \mathbf{A}_H) \in V_{h,\mathrm{iu}} \times \mathbf{V}_{H,0}$ a minimizer of (4.3). Then, there is a constant $c > 0$ independent of h, H, and κ such that*

$$\begin{aligned} &\big(1 - c\rho_{u,\mathbf{A}}(\kappa) C_{\mathrm{stab}}(\kappa) H^{\frac{1}{2}+s}\big) \|\varphi_h, \mathbf{B}_H\|^2_{H^1_{\kappa,\mathbf{A}} \times \mathbf{V}_{H,0}} \\ &\lesssim \rho_{u,\mathbf{A}}(\kappa) C_{\mathrm{stab}}(\kappa) \langle E''(u, \mathbf{A})(\varphi_h, \mathbf{B}_H), (\varphi_h, \mathbf{B}_H)\rangle \end{aligned}$$

for $(\varphi_h, \mathbf{B}_H) \in V_{h,\mathrm{iu}} \times \mathbf{V}_{H,0}$. In particular, E'' is coercive on $V_{h,\mathrm{iu}} \times \mathbf{V}_{H,0}$ under the resolution condition (5.1).

Proof. Let $\mathbf{B}_H$ be given and discrete divergence free. We associate $\widehat{\mathbf{B}} = \pi_{\mathrm{div}} \mathbf{B}_H \in \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$ such that $\operatorname{curl} \widehat{\mathbf{B}} = \operatorname{curl} \mathbf{B}_H$, with π_{div} from (2.4), and write $\mathbf{B}_H = \widehat{\mathbf{B}} + (\mathbf{B}_H - \widehat{\mathbf{B}})$ where the first summand is divergence free and the second one curl- and $\mathrm{div}_{\mathrm{H}}$-free.

We use the property of $\widehat{\mathbf{B}}$ to obtain

$$\|\varphi_h, \operatorname{curl} \mathbf{B}_H\|^2_{H^1_{\kappa,\mathbf{A}} \times L^2} = \|\varphi_h, \operatorname{curl} \widehat{\mathbf{B}}\|^2_{H^1_{\kappa,\mathbf{A}} \times L^2} \le \rho_{u,\mathbf{A}}(\kappa) C_{\mathrm{stab}}(\kappa) \langle E''(u, \mathbf{A})(\varphi_h, \widehat{\mathbf{B}}), (\varphi_h, \widehat{\mathbf{B}})\rangle$$

and further use the expansion $(\varphi_h, \widehat{\mathbf{B}}) = (\varphi_h, \mathbf{B}_H) + (0, \widehat{\mathbf{B}} - \mathbf{B}_H)$ to write

$$\begin{aligned} &\langle E''(u, \mathbf{A})(\varphi_h, \widehat{\mathbf{B}}), (\varphi_h, \widehat{\mathbf{B}})\rangle \\ &= \langle E''(u, \mathbf{A})(\varphi_h, \mathbf{B}_H), (\varphi_h, \mathbf{B}_H)\rangle + 2\langle E''(u, \mathbf{A})(\varphi_h, \widehat{\mathbf{B}}), (0, \widehat{\mathbf{B}} - \mathbf{B}_H)\rangle \\ &\quad + \langle E''(u, \mathbf{A})(0, \widehat{\mathbf{B}} - \mathbf{B}_H), (0, \widehat{\mathbf{B}} - \mathbf{B}_H)\rangle \\ &= \langle E''(u, \mathbf{A})(\varphi_h, \mathbf{B}_H), (\varphi_h, \mathbf{B}_H)\rangle + 4\langle \partial_{u,\mathbf{A}} E(u, \mathbf{A})(\widehat{\mathbf{B}} - \mathbf{B}_H), \varphi_h\rangle \\ &\quad + 2\langle \partial^2_{\mathbf{A}} E(u, \mathbf{A})(\widehat{\mathbf{B}} - \mathbf{B}_H), \mathbf{B}_H\rangle + \langle \partial^2_{\mathbf{A}} E(u, \mathbf{A})(\widehat{\mathbf{B}} - \mathbf{B}_H), \widehat{\mathbf{B}} - \mathbf{B}_H\rangle. \end{aligned}$$

The first term is precisely what we want, and for the second derivative in $\mathbf{A}$ we have

$$\langle \partial^2_{\mathbf{A}} E(u, \mathbf{A})(\widehat{\mathbf{B}} - \mathbf{B}_H), \widehat{\mathbf{B}} - \mathbf{B}_H\rangle \lesssim \|\widehat{\mathbf{B}} - \mathbf{B}_H\|^2_{\mathbf{H}(\mathrm{curl})} = \|\widehat{\mathbf{B}} - \mathbf{B}_H\|^2_{L^2} \lesssim \big(H^{\frac{1}{2}+s} \|\operatorname{curl} \mathbf{B}_H\|_{L^2}\big)^2,$$

where we used again Lemma 4.3. In the same manner we estimate

$$\langle \partial^2_{\mathbf{A}} E(u, \mathbf{A})(\widehat{\mathbf{B}} - \mathbf{B}_H), \mathbf{B}_H\rangle \lesssim H^{\frac{1}{2}+s} \|\operatorname{curl} \mathbf{B}_H\|^2_{L^2}.$$

Finally, with the representation of $\partial_{u,\mathbf{A}}E$ in Lemma 3.6,d the bounds derived in Section 3, and Assumption 4.4 we observe

$$\langle \partial_{u,\mathbf{A}}E(u,\mathbf{A})(\widehat{\mathbf{B}}-\mathbf{B}_H),\varphi_h\rangle \lesssim ||\varphi_h||_{H^1_{\kappa,\mathbf{A}}}||\widehat{\mathbf{B}}-\mathbf{B}_H||_{\mathbf{H}(\mathrm{curl})} \lesssim H^{\frac{1}{2}+s}\big(||\varphi_h||^2_{H^1_{\kappa,\mathbf{A}}}+||\operatorname{curl}\mathbf{B}_H||^2_{L^2}\big),$$

once again using Lemma 4.3. □

A crucial ingredient in the proofs of the error bounds is to exploit the critical point equation. However, due to the Lagrange multiplier in (4.4), it does not hold $\langle E'(u,\mathbf{A}),(\varphi_h,\mathbf{B}_H)\rangle = 0$ for $(\varphi_h,\mathbf{B}_H)\in V_{h,\mathrm{iu}}\times\mathbf{V}_{H,0}$. Nevertheless, we can quantify this error in the following result.

Lemma 5.5. *Let Assumption 4.1 hold, and let $(u,\mathbf{A})\in H^1(\Omega)\times\mathbf{H}_0(\mathrm{curl},\mathrm{div})$ be a minimizer of (1.1) and $(u_h,\mathbf{A}_H)\in V_{h,\mathrm{iu}}\times\mathbf{V}_{H,0}$ a minimizer of (4.3). Then, it holds*

$$\langle E'(u,\mathbf{A}),(\varphi_h,\mathbf{B}_H)\rangle \lesssim H^{\frac{1}{2}+s}||\operatorname{curl}\mathbf{B}_H||_{L^2}$$

for $(\varphi_h,\mathbf{B}_H)\in V_{h,\mathrm{iu}}\times\mathbf{V}_{H,0}$.

Proof. As above, we associate $\widehat{\mathbf{B}}\in\mathbf{H}_0(\mathrm{curl},\mathrm{div})$ such that $\operatorname{curl}\widehat{\mathbf{B}}=\operatorname{curl}\mathbf{B}_H$, and compute with Lemma 2.7

$$\langle E'(u,\mathbf{A}),(\varphi_h,\mathbf{B}_H)\rangle = \langle E'(u,\mathbf{A}),(0,\mathbf{B}_H-\widehat{\mathbf{B}})\rangle,$$

and Lemma 4.3 gives the claim. □

After these preparations, we now turn to the treatment of the error in the discrete minimizers. Defining the errors $\widehat{e}_h=\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u_h$ and $\widehat{E}_H=P_H\mathbf{A}-\mathbf{A}_H$, with Proposition 5.4 we can estimate

$$||\widehat{e}_h,\widehat{E}_H||^2_{H^1_{\kappa,\mathbf{A}}\times\mathbf{V}_{H,0}} \lesssim \rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa)\langle E''(u,\mathbf{A})(\widehat{e}_h,\widehat{E}_H),(\widehat{e}_h,\widehat{E}_H)\rangle,$$

and decompose as

$$\begin{aligned}\varepsilon_1(\psi_h,\mathbf{C}_H) &:= \langle E''(u,\mathbf{A})(\mathrm{R}^{\perp}_{\kappa,\mathbf{A},h}u-u,P_H\mathbf{A}-\mathbf{A}),(\psi_h,\mathbf{C}_H)\rangle,\\ \varepsilon_2(\psi_h,\mathbf{C}_H) &:= \langle E''(u,\mathbf{A})(u-u_h,\mathbf{A}-\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle.\end{aligned}$$

Thus, it remains to study the two different contributions. We begin with the second one.

Lemma 5.6. *Let Assumptions 4.1 and 4.4 hold, and let $(u,\mathbf{A})\in H^1(\Omega)\times\mathbf{H}_0(\mathrm{curl},\mathrm{div})$ be a minimizer of (1.1) and $(u_h,\mathbf{A}_H)\in V_{h,\mathrm{iu}}\times\mathbf{V}_{H,0}$ a minimizer of (4.3). Then, it holds*

$$\begin{aligned}|\varepsilon_2(\psi_h,\mathbf{C}_H)| \le C(\kappa)\big(&||u-u_h||^2_{L^6}+||u-u_h||^2_{H^1_{\kappa,\mathbf{A}}}+||\mathbf{A}-\mathbf{A}_H||^2_{L^3}\big)||\psi_h,\mathbf{C}_H||_{H^1_{\kappa,\mathbf{A}}\times\mathbf{H}(\mathrm{curl})}\\ &+\widetilde{C}H^{\frac{1}{2}+s}||\operatorname{curl}\mathbf{C}_H||_{L^2}\end{aligned}$$

with $\widetilde{C}$ independent of κ.

Proof. We perform the expansion

$$\begin{aligned}\varepsilon_2(\psi_h,\mathbf{C}_H) &= \langle E''(u,\mathbf{A})(u,\mathbf{A}),(\psi_h,\mathbf{C}_H)\rangle \pm \langle E'(u,\mathbf{A}),(\psi_h,\mathbf{C}_H)\rangle\\ &\qquad - \langle E''(u_h,\mathbf{A}_H)(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle \pm \langle E'(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle\\ &\qquad + \langle E''(u_h,\mathbf{A}_H)(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle - \langle E''(u,\mathbf{A})(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle\\ &= \big(\langle E''(u,\mathbf{A})(u,\mathbf{A}),(\psi_h,\mathbf{C}_H)\rangle - \langle E'(u,\mathbf{A}),(\psi_h,\mathbf{C}_H)\rangle\big)\\ &\qquad - \big(\langle E''(u_h,\mathbf{A}_H)(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle - \langle E'(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle\big)\\ &\qquad + \langle E''(u_h,\mathbf{A}_H)(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle - \langle E''(u,\mathbf{A})(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle\\ &\qquad + \langle E'(u,\mathbf{A}),(\psi_h,\mathbf{C}_H)\rangle\\ &\le \big|\big(\langle E''(u,\mathbf{A})(u,\mathbf{A}),(\psi_h,\mathbf{C}_H)\rangle - \langle E'(u,\mathbf{A}),(\psi_h,\mathbf{C}_H)\rangle\big)\\ &\qquad - \big(\langle E''(u_h,\mathbf{A}_H)(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle - \langle E'(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle\big)\\ &\qquad + \langle E''(u_h,\mathbf{A}_H)(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle - \langle E''(u,\mathbf{A})(u_h,\mathbf{A}_H),(\psi_h,\mathbf{C}_H)\rangle\big|\\ &\qquad + H^{\frac{1}{2}+s}||\operatorname{curl}\mathbf{C}_H||_{L^2},\end{aligned}$$

where we used that $\langle E'(u_h, \mathbf{A}_H), (\psi_h, \mathbf{C}_H)\rangle = 0$, and the estimate in Lemma 5.5. For the remaining parts, we modify the proof of [13, Lemma 5.5] such that only $||\mathbf{A}||_{L^3}, ||\mathbf{A}_H||_{L^3}$ are used instead of the $H^1(\Omega)$-norm. □

Lemma 5.7. *Let Assumptions 4.1 and 4.4 hold, and let* $(u, \mathbf{A}) \in H^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \mathrm{div})$ *be a minimizer of* (1.1) *and* $(u_h, \mathbf{A}_H) \in V_{h,\mathrm{iu}} \times \mathbf{V}_{H,0}$ *a minimizer of* (4.3)*.*

(a) If $\Omega \subset \mathbb{R}^3$*, we have the bound*

$$\frac{|\varepsilon_1(\psi_h, \mathbf{C}_H)|}{||\psi_h, \mathbf{C}_H||_{H^1_{\kappa,\mathbf{A}} \times \mathbf{H}(\mathrm{curl})}} \lesssim \big(h^s C_{\mathrm{stab}}(\kappa)\kappa(1 + C_{\mathrm{stab}}(\kappa)\kappa h)||e_\pi||_{H^1_{\kappa,\mathbf{A}}} + ||E_\pi||_{L^2} + H^{\frac{1}{2}+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}} + h^{s(\frac{1}{2}+s)}\kappa^2(1 + C_{\mathrm{stab}}(\kappa)\kappa h)^{\frac{1}{2}+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}} + \kappa^2 H^{\frac{1}{2}+s}||E_\pi||_{\mathbf{H}(\mathrm{curl})}\big).$$

(b) If $\Omega \subset \mathbb{R}^2$*, we have the bound*

$$\frac{|\varepsilon_1(\psi_h, \mathbf{C}_H)|}{||\psi_h, \mathbf{C}_H||_{H^1_{\kappa,\mathbf{A}} \times \mathbf{H}(\mathrm{curl})}} \lesssim \big(h^s C_{\mathrm{stab}}(\kappa)\kappa(1 + C_{\mathrm{stab}}(\kappa)\kappa h)||e_\pi||_{H^1_{\kappa,\mathbf{A}}} + ||E_\pi||_{L^2} + H^{\frac{1}{2}+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}} + h^{s(\frac{1}{2}+s)}\kappa^2(1 + C_{\mathrm{stab}}(\kappa)\kappa h)^{\frac{1}{2}+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}} + \kappa^2 H^{\frac{1}{2}+s}||E_\pi||_{\mathbf{H}(\mathrm{curl})}\big).$$

In both cases the hidden constants are independent of κ*,* h*, and* H*.*

Proof. By the definition of the two projections and r in (3.5)

$$\begin{aligned}\varepsilon_1(\psi_h, \mathbf{C}_H) &= r\big((e_\pi, E_\pi), (\psi_h, \mathbf{C}_H)\big) \\ &= r_1(e_\pi, \psi_h) + r_2(E_\pi, \mathbf{C}_H) \\ &\quad + r_3(E_\pi, \psi_h) + r_3(\mathbf{C}_H, e_\pi) + r_4(E_\pi, \psi_h) + r_4(\mathbf{C}_H, e_\pi) + r_5(E_\pi, \psi_h) + r_5(\mathbf{C}_H, e_\pi),\end{aligned}$$

Then, by Lemma 3.9 (a)

$$|r_1(e_\pi, \psi_h) + r_2(E_\pi, \mathbf{C}_H)| \lesssim C^2_{\mathrm{stab}}(\kappa)||e_\pi||_{L^2}||\psi_h||_{L^2} + ||E_\pi||_{L^2}||\mathbf{C}_H||_{L^2}.$$

This further is estimated by

$$\begin{aligned}||e_\pi||_{L^2} &\lesssim h^s\kappa(1 + C_{\mathrm{stab}}(\kappa)\kappa h)||e_\pi||_{H^1_{\kappa,\mathbf{A}}}, \qquad && d = 3, \\ ||e_\pi||_{L^2} &\lesssim h^{\frac{1}{2}+s}\kappa(1 + C_{\mathrm{stab}}(\kappa)\kappa h)||e_\pi||_{H^1_{\kappa,\mathbf{A}}}, && d = 2.\end{aligned}$$

We first consider the case $d = 3$. By Lemma 3.9 (a) and (b)

$$\begin{aligned}&|r_3(E_\pi, \psi_h) + r_3(\mathbf{C}_H, e_\pi)| \\ &\lesssim ||E_\pi||_{L^2}||\psi_h||_{H^1_{\kappa,\mathbf{A}}} + ||e_\pi||_{L^{\frac{3}{1+2s}}}||\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})} \\ &\lesssim ||E_\pi||_{L^2}||\psi_h||_{H^1_{\kappa,\mathbf{A}}} + h^{s(\frac{1}{2}+s)}\kappa(1 + C_{\mathrm{stab}}(\kappa)\kappa h)^{\frac{1}{2}+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})}.\end{aligned}$$

Further, by Lemma 3.9 (a) and (c)

$$\begin{aligned}|r_4(E_\pi, \psi_h) + r_4(\mathbf{C}_H, e_\pi)| &\lesssim ||E_\pi||_{L^2}||\psi_h||_{H^1_{\kappa,\mathbf{A}}} + ||e_\pi||_{H^1_{\kappa,\mathbf{A}}}||\pi_{\mathrm{div}}\mathbf{C}_H - \mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})} \\ &\quad + \kappa||e_\pi||_{L^{\frac{3}{1+s}}}||\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})} \\ &\lesssim ||E_\pi||_{L^2}||\psi_h||_{H^1_{\kappa,\mathbf{A}}} + H^{\frac{1}{2}+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})} \\ &\quad + h^{s(\frac{1}{2}+s)}\kappa^2(1 + C_{\mathrm{stab}}(\kappa)\kappa h)^{\frac{1}{2}+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})},\end{aligned}$$

where we used Lemmas 4.2 and 4.11 in the last step. Finally, we use Lemma 3.9 (d) and the same estimates to obtain

$$\begin{aligned}|r_5(E_\pi,\psi_h)+r_5(\mathbf{C}_H,e_\pi)| &\lesssim \kappa||\mathbf{C}_H||_{L^{\frac{3}{1-s}}}||e_\pi||_{L^{\frac{3}{1+s}}}+||E_\pi||_{L^2}||\psi_h||_{H^1_{\kappa,\mathbf{A}}}\\ &\quad+\kappa^2||\psi_h||_{H^1_{\kappa,\mathbf{A}}}||\pi_{\mathrm{div}}E_\pi-E_\pi||_{L^2}\\ &\lesssim h^{s(\frac12+s)}\kappa^2(1+C_{\mathrm{stab}}(\kappa)\kappa h)^{\frac12+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})}\\ &\quad+||E_\pi||_{L^2}||\psi_h||_{H^1_{\kappa,\mathbf{A}}}+\kappa^2H^{\frac12+s}||\psi_h||_{H^1_{\kappa,\mathbf{A}}}||E_\pi||_{\mathbf{H}(\mathrm{curl})}.\end{aligned}$$

Collecting all terms, then yields

$$\begin{aligned}\frac{|\varepsilon_1(\psi_h,\mathbf{C}_H)|}{||\psi_h,\mathbf{C}_H||_{H^1_{\kappa,\mathbf{A}}\times\mathbf{H}(\mathrm{curl})}} &\lesssim \big(h^sC_{\mathrm{stab}}(\kappa)\kappa(1+C_{\mathrm{stab}}(\kappa)\kappa h)||e_\pi||_{H^1_{\kappa,\mathbf{A}}}\\ &\quad+||E_\pi||_{L^2}+h^{s(\frac12+s)}\kappa(1+C_{\mathrm{stab}}(\kappa)\kappa h)^{\frac12+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}}\\ &\quad+H^{\frac12+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}}+h^{s(\frac12+s)}\kappa^2(1+C_{\mathrm{stab}}(\kappa)\kappa h)^{\frac12+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}}\\ &\quad+\kappa^2H^{\frac12+s}||E_\pi||_{\mathbf{H}(\mathrm{curl})}\big).\end{aligned}$$

Next, let $d=2$. By Lemma 3.9 (a) and (b)

$$|r_3(E_\pi,\psi_h)+r_3(\mathbf{C}_H,e_\pi)|\lesssim||E_\pi||_{L^2}||\psi_h||_{H^1_{\kappa,\mathbf{A}}}+||e_\pi||_{L^2}||\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})}$$

Further, by Lemma 3.9 (a) and (c)

$$\begin{aligned}|r_4(E_\pi,\psi_h)+r_4(\mathbf{C}_H,e_\pi)| &\lesssim ||E_\pi||_{L^2}||\psi_h||_{H^1_{\kappa,\mathbf{A}}}+||e_\pi||_{H^1_{\kappa,\mathbf{A}}}||\pi_{\mathrm{div}}\mathbf{C}_H-\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})}\\ &\quad+\kappa||e_\pi||_{L^2}||\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})}\\ &\lesssim ||E_\pi||_{L^2}||\psi_h||_{H^1_{\kappa,\mathbf{A}}}+H^{\frac12+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})}\\ &\quad+h^{\frac12+s}\kappa^2(1+C_{\mathrm{stab}}(\kappa)\kappa h)||e_\pi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})},\end{aligned}$$

where we used Lemmas 4.2 and 4.10 (b) in the last step. Finally, we use Lemma 3.9 (d) and the same estimates to obtain

$$\begin{aligned}|r_5(E_\pi,\psi_h)+r_5(\mathbf{C}_H,e_\pi)| &\lesssim \kappa||\mathbf{C}_H||_{L^4}||e_\pi||_{L^2}+||E_\pi||_{L^2}||\psi_h||_{H^1_{\kappa,\mathbf{A}}}\\ &\quad+\kappa||\psi_h||_{H^1_{\kappa,\mathbf{A}}}||\pi_{\mathrm{div}}E_\pi-E_\pi||_{L^2}\\ &\lesssim h^{\frac12+s}\kappa^2(1+C_{\mathrm{stab}}(\kappa)\kappa h)||e_\pi||_{H^1_{\kappa,\mathbf{A}}}||\mathbf{C}_H||_{\mathbf{H}(\mathrm{curl})}\\ &\quad+||E_\pi||_{L^2}||\psi_h||_{H^1_{\kappa,\mathbf{A}}}+\kappa H^{\frac12+s}||\psi_h||_{H^1_{\kappa,\mathbf{A}}}||E_\pi||_{\mathbf{H}(\mathrm{curl})}.\end{aligned}$$

Collecting all terms, then yields

$$\begin{aligned}\frac{|\varepsilon_1(\psi_h,\mathbf{C}_H)|}{||\psi_h,\mathbf{C}_H||_{H^1_{\kappa,\mathbf{A}}\times\mathbf{H}(\mathrm{curl})}} &\lesssim \big(h^{\frac12+s}C_{\mathrm{stab}}(\kappa)\kappa(1+C_{\mathrm{stab}}(\kappa)\kappa h)||e_\pi||_{H^1_{\kappa,\mathbf{A}}}\\ &\quad+||E_\pi||_{L^2}\\ &\quad+H^{\frac12+s}||e_\pi||_{H^1_{\kappa,\mathbf{A}}}+h^{\frac12+s}\kappa^2(1+C_{\mathrm{stab}}(\kappa)\kappa h)||e_\pi||_{H^1_{\kappa,\mathbf{A}}}\\ &\quad+\kappa H^{\frac12+s}||E_\pi||_{\mathbf{H}(\mathrm{curl})}\big).\end{aligned}$$

□

Finally, we can summarize our finding in our main result.

Theorem 5.8. *Let Assumptions 4.1 and 4.4 hold, and let* $(u,\mathbf{A})\in H^1(\Omega)\times\mathbf{H}_0(\mathrm{curl},\mathrm{div})$ *be a minimizer of* (1.1) *and* $(u_h,\mathbf{A}_H)\in V_{h,\mathrm{iu}}\times\mathbf{V}_{H,0}$ *a minimizer of* (4.3) *such that they satisfy the assertion of Proposition 4.7.*

(a) If $\Omega\subset\mathbb{R}^3$*, and* h,H *sufficiently small, in particular such that the resolution conditions*

$$\rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa)\big(h^sC_{\mathrm{stab}}(\kappa)\kappa+h^{s(\frac12+s)}\kappa^2+H^{\frac12+s}\kappa^2\big)\lesssim 1$$

are satisfied, we have

$$\begin{aligned}&\|u-u_h\|_{H^1_{\kappa,\mathbf{A}}}+\|\mathbf{A}-\mathbf{A}_H\|_{\mathbf{H}(\mathrm{curl})}\\&\lesssim \|\mathrm{R}^\perp_{\kappa,\mathbf{A},h}u-u\|_{H^1_{\kappa,\mathbf{A}}}+\|P_H\mathbf{A}-\mathbf{A}\|_{\mathbf{H}(\mathrm{curl})}+\rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa)\|P_H\mathbf{A}-\mathbf{A}\|_{L^2}\end{aligned}$$

with constants independent of κ*,* h *and* H*.*

(b) If $\Omega\subset\mathbb{R}^2$*, and* h,H *sufficiently small, in particular such that the resolution conditions*

$$\rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa)\big(h^{\frac12+s}C_{\mathrm{stab}}(\kappa)\kappa+h^{\frac12+s}\kappa^2+H^{\frac12+s}\kappa\big)\lesssim 1$$

are satisfied, we have

$$\begin{aligned}&\|u-u_h\|_{H^1_{\kappa,\mathbf{A}}}+\|\mathbf{A}-\mathbf{A}_H\|_{\mathbf{H}(\mathrm{curl})}\\&\lesssim \|\mathrm{R}^\perp_{\kappa,\mathbf{A},h}u-u\|_{H^1_{\kappa,\mathbf{A}}}+\|P_H\mathbf{A}-\mathbf{A}\|_{\mathbf{H}(\mathrm{curl})}+\rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa)\|P_H\mathbf{A}-\mathbf{A}\|_{L^2}\end{aligned}$$

with constants independent opf κ*,* h *and* H*.*

Proof. We split the error in the projection error and the discrete error and combine Proposition 5.4 with Lemmas 5.6 and 5.7. Absorption of the higher order terms gives the result. □

Remark 5.9. *Let us note that one would like to remove the* L^2*-norm together with the constants in front of it. However, from the regularity we derived so far for* $\mathbf{A}_H$*, we cannot hope to obtain an increased convergence in the* L^2*-norm using Nédélec of first kind. If we are able to show additional regularity and use elements of the second kind, one can hope for an additional* H *in front of the* L^2*-Norm, see for example* [35, Theorem 8.15]*, and then hide this term under a resolution condition.*

Conclusion 5.10. *Let the assumptions of Theorem 5.8 hold.*

(a) If $\Omega\subset\mathbb{R}^3$*, we have*

$$\|u-u_h\|_{H^1_{\kappa,\mathbf{A}}}+\|\mathbf{A}-\mathbf{A}_H\|_{\mathbf{H}(\mathrm{curl})}\lesssim \kappa h^s+(1+\rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa))H^{\frac12+s}$$

with constants independent of κ*,* h *and* H*.*

(b) If $\Omega\subset\mathbb{R}^2$*, we have*

$$\|u-u_h\|_{H^1_{\kappa,\mathbf{A}}}+\|\mathbf{A}-\mathbf{A}_H\|_{\mathbf{H}(\mathrm{curl})}\lesssim \kappa h^{\frac12+s}+(1+\rho_{u,\mathbf{A}}(\kappa)C_{\mathrm{stab}}(\kappa))H^{\frac12+s}$$

with constants independent opf κ*,* h *and* H*.*

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Appendix A. Proof of Lemma 4.3

A.1. **Discrete Helmholtz decomposition.** We follow the presentation in [35, Lemma 7.6] for the the Nédélec space which is $\mathbf{H}_0(\mathrm{curl})$-conforming. Here, the discrete divergence is defined using a discrete subspace of $H^1_0(\Omega)$. Recalling the definition of (4.1), we have for $\mathcal{R}_H\subset\mathbf{H}(\mathrm{curl})$ the discrete Helmholtz decomposition

$$\mathcal{R}_H=\mathbf{V}_{H,0}\oplus\nabla V_H$$

with the standard Lagrange finite element space V_H. We further introduce the div-conforming finite element space W_H, which satisfies $\mathrm{curl}\,\mathcal{R}_H\subset W_H$. Let us now define the discrete curl denoted by $\mathrm{curl_H}\colon W_H\to\mathcal{R}_H$ for $z_H\in W_H$ by

$$(\mathrm{curl_H}\,z_H,w_H)=(z_H,\mathrm{curl}\,w_H)\quad\text{for all}\quad w_H\in\mathcal{R}_H.$$

Lemma A.1. *Any function in* $\mathbf{B}_H \in \mathcal{R}_H$ *can be uniquely written as*

$$\mathbf{B}_H = \mathrm{curl_H}\, z_H + \nabla\varphi_H$$

for $z_H \in W_H$ *and* $\varphi_H \in V_H$*. In particular, if* $\mathbf{B}_H \in \mathbf{V}_{H,0}$*, then*

$$\mathbf{B}_H = \mathrm{curl_H}\, z_H.$$

Proof. Since ∇V_H is the kernel of curl restricted to $\mathcal{R}_H$, we obtain

$$\mathrm{ran}(\mathrm{curl_H}) = (\ker(\mathrm{curl}\,|_{\mathcal{R}_H}))^\perp = (\nabla V_H)^\perp$$

we have that $\mathbf{V}_{H,0} = (\nabla V_H)^\perp \oplus \nabla V_H = \mathrm{curl_H}\, W_H \oplus \nabla V_H$ and hence the assertion. □

A.2. **Commuting projections.** We provide now a proof following the presentation in [35, Lemma 7.6]. We therefore introduce the interpolations

$$r_H \colon \mathbf{H}(\mathrm{curl}) \to \mathcal{R}_H, \qquad w_H \colon \mathbf{H}(\mathrm{div}) \to W_H,$$

which satisfy the commuting property: for all $C \in \mathbf{H}(\mathrm{curl})$

$$\mathrm{curl}(r_H \mathbf{C}) = w_H(\mathrm{curl}\,\mathbf{C}). \tag{A.1}$$

A.3. **Proof of the properties of the projection** π_{div}**.**

Proof of Lemma 4.3. We reduce the claim in (b) to the estimate in (a). The idea is to expand as

$$||\mathbf{B}_H||_{L^2} \le ||\mathbf{C} - \mathbf{B}_H||_{L^2} + ||\mathbf{C}||_{L^2}$$

for some suitable C, where we follow the proof of [35, Lemma 7.6].

(b) We use $\mathbf{C} = \pi_{\mathrm{div}}\mathbf{B}_H$ from Theorem 2.1, which in particular satisfies by [35, Theorem 3.50 & Cor. 3.51]

$$||\mathbf{C}||_{H^{\frac{1}{2}+s}} \lesssim ||\,\mathrm{curl}\,\mathbf{C}||_{L^2}$$

as well as by (A.1)

$$\mathrm{curl}\, r_H\mathbf{C} = w_H(\mathrm{curl}\,\mathbf{C}) = w_H(\mathrm{curl}\,\mathbf{B}_H) = \mathrm{curl}\,\mathbf{B}_H \quad \Longrightarrow \quad \mathrm{curl}(r_H\mathbf{C} - \mathbf{B}_H) = 0. \tag{A.2}$$

This directly yields

$$||\mathbf{C}||_{L^2} \lesssim ||\,\mathrm{curl}\,\mathbf{C}||_{L^2} = ||\,\mathrm{curl}\,\mathbf{B}_H||_{L^2},$$

and thus we are left to estimate the difference $\mathbf{C} - \mathbf{B}_H$ as in (a).

(a) For this, we compute

$$\begin{aligned}
||\mathbf{C} - \mathbf{B}_H||^2_{L^2} &= (\mathbf{C} - \mathbf{B}_H, \mathbf{C} - \mathbf{B}_H) \\
&= (\mathbf{C} - \mathbf{B}_H, \mathbf{C} - r_H\mathbf{C}) + (\mathbf{C}, r_H\mathbf{C} - \mathbf{B}_H) - (\mathbf{B}_H, r_H\mathbf{C} - \mathbf{B}_H) \\
&\le ||\mathbf{C} - \mathbf{B}_H||_{L^2}||\mathbf{C} - r_H\mathbf{C}||_{L^2} + (\mathbf{C}, r_H\mathbf{C} - \mathbf{B}_H) - (\mathbf{B}_H, r_H\mathbf{C} - \mathbf{B}_H)
\end{aligned}$$

and show that the two inner products in fact vanish.

By Lemma A.1, it holds for some $z_H \in W_H$

$$(\mathbf{B}_H, r_H\mathbf{C} - \mathbf{B}_H) = (z_H, \mathrm{curl}(r_H\mathbf{C} - \mathbf{B}_H)) = 0,$$

By the Helmholtz decomposition in Theorem 2.1 (b) and the fact that $\mathrm{div}\,\mathbf{C} = 0$, there exists $\widetilde{\mathbf{C}} \in \mathbf{H}_0(\mathrm{curl})$ such that $\mathbf{C} = \mathrm{curl}\,\widetilde{\mathbf{C}}$ and hence

$$(\mathbf{C}, r_H\mathbf{C} - \mathbf{B}_H) = (\mathrm{curl}\,\widetilde{\mathbf{C}}, r_H\mathbf{C} - \mathbf{B}_H) = (\widetilde{\mathbf{C}}, \mathrm{curl}(r_H\mathbf{C} - \mathbf{B}_H)) = 0,$$

where we again used (A.2) in the last step. In the last step, we have to estimate the projection error of $\mathbf{C}$. We have by [35, Theorem 6.6]

$$||\mathbf{C} - r_H\mathbf{C}||_{L^2} \lesssim H^{\frac{1}{2}+s}(||\mathbf{C}||_{H^{\frac{1}{2}+s}} + ||\,\mathrm{curl}\,\mathbf{C}||_{L^2}) \lesssim H^{\frac{1}{2}+s}||\,\mathrm{curl}\,\mathbf{C}||_{L^2} = H^{\frac{1}{2}+s}||\,\mathrm{curl}\,\mathbf{B}_H||_{L^2},$$

and with this the assertion. □

Appendix B. Collection of regularity estimates

In this section of the appendix, we collect several useful regularity results which are used throughout the paper. Most of them are stated for convenience of the reader.

Lemma B.1. *Let$\Omega \subset \mathbb{R}^d$ and $f \in L^p$. Then $f \in W^{-1,q}$ with*

$$||f||_{L^p} \lesssim ||f||_{W^{-1,q}} = ||f||_{(W^{1,q'})'}, \quad \text{for} \quad 1 < p, q < \infty \text{ and } q = \frac{dp}{d-p}.$$

Further, it holds $W^{-1,p_1} \subset W^{-1,p_2}$ if $p_1 \geq p_2$.

(a) In particular, $L^{\frac{3}{2-s}} \hookrightarrow W^{-1,\frac{3}{1-s}}$, $d = 3$ and $L^{\frac{4}{3-2s}} \hookrightarrow W^{-1,\frac{4}{1-2s}}$, $d = 2$

Proof. We note that by the choice of p, q

$$W^{1,q'} \hookrightarrow L^{p'},$$

it holds for $\varphi \in W^{1,q'}$ that

$$(f, \varphi)_{L^2} \lesssim ||f||_{L^p} ||\varphi||_{L^{p'}} \lesssim ||f||_{L^p} ||\varphi||_{W^{1,q'}}.$$

Since $W^{-1,q} = \left(W^{1,q'}\right)'$ the first claim follows. Now let $p_1 \geq p_2$, then $p_1' \leq p_2'$ and $W^{1,p_2'} \subset W^{1,p_1'}$. This gives

$$W^{-1,p_1} = (W^{1,p_1'})' \subset (W^{1,p_2'})' = W^{-1,p_2},$$

and with this the second claim. □

B.1. **General results on regularity.** The next result can be found in a variant in [29, Theorem 1.2], and we only adapt the proof to our case.

Proposition B.2. *Let $w \in H^1(\Omega)$ be the solution of*

$$(\nabla w, \nabla \varphi) = f(\varphi) \quad \text{for all} \quad \varphi \in H^1(\Omega).$$

Then, there is a constant $\varepsilon_{-1} > 0$ such that for any p with

(a) $\frac{3}{2} - \varepsilon_{-1} < p < 3 + \varepsilon_{-1}$ *if* $d = 3$,
(b) $\frac{4}{3} - \varepsilon_{-1} < p < 4 + \varepsilon_{-1}$ *if* $d = 2$,

it holds that for $f \in W^{-1,p}$ the solution satisfies $w \in W^{1,p}$ with

$$||w||_{W^{1,p}} \lesssim ||f||_{W^{-1,p}}.$$

Proof. We proceed as in the proof of [29, Lemma 5.1] and let $g \in C_0^\infty(\Omega)$ be arbitrary and consider the dual solution v of

$$(\nabla v, \nabla \varphi) = (\operatorname{div} g, \varphi)$$

for all $\varphi \in H^1(\Omega)$. We then compute with the definition of v and w

$$|(\nabla w, g)| = |(w, \operatorname{div} g)| = |(\nabla w, \nabla v)| = |f(v)| \leq ||f||_{W^{-1,p}} ||v||_{W^{1,p'}} \lesssim ||f||_{W^{-1,p}} ||g||_{L^{p'}}$$

where we used [29, Theorem 1.2] in the last step, since p' is in the same range as p. From this we conclude by density

$$||\nabla w||_{L^p} = \sup_{||g||_{L^{p'}}=1} (\nabla w, g) \lesssim ||f||_{W^{-1,p}},$$

and thus the claim. □

The next result is on the $W^{2,p}$-regularity in three dimensions and is taken from [12, Cor. 3.10].

Proposition B.3. *Let $\Omega \subset \mathbb{R}^3$, and $w \in H^1(\Omega)$ be the solution of*

$$(\nabla w, \nabla \varphi) = (f, \varphi) \quad \text{for all} \quad \varphi \in H^1(\Omega). \tag{B.1}$$

Then, there is a constant $\varepsilon_0 \geq 0$ such that for $f \in L^p$ with $p \in [\frac{6}{5}, \frac{4}{3} + \varepsilon_0]$ the solution satisfies $w \in W^{2,p}$ with

$$||w||_{W^{2,p}} \lesssim ||f||_{L^p}.$$

In two spatial dimensions, we have the following result from [30, Cor. 4.4.3.4] and [30, Theorem 4.3.2.3].

Proposition B.4. *Let $\Omega \subset \mathbb{R}^2$, and $w \in H^1(\Omega)$ be the solution of (B.1). Further, let ω_{max} be the largest angle of Ω, and consider $p_{\max}$ defined in (2.5). Then, for $p < p_{\max}$ and $f \in L^p$ the solution satisfies $w \in W^{2,p}$ with*

$$||w||_{W^{2,p}} \lesssim ||f||_{L^p}.$$

Finally, we state some interpolation results, used several times in the analysis.

Lemma B.5. *Let $w \in W^{2,p} \cap L^\infty$ for some $p \leq 2$. Then, we have $w \in H^p = W^{p,2}$ with*

$$||w||_{H^p} \lesssim ||w||_{W^{2,p}}^{\frac{p}{2}} ||w||_{L^\infty}^{1-\frac{p}{2}}$$

$$||w||_{W^{1,2p}} \lesssim ||w||_{W^{2,p}}^{\frac{1}{2}} ||w||_{L^\infty}^{\frac{1}{2}}$$

In addition, we have by Sobolev embedding

$$||w||_{H^{\frac{7p-6}{2p}}} \lesssim ||w||_{W^{2,p}}, \quad d = 3,$$

$$||w||_{H^{\frac{3p-2}{p}}} \lesssim ||w||_{W^{2,p}}, \quad d = 2$$

Proof. Let $s_i \geq 0$ and $q_i \in (1, \infty)$ and define for $\alpha \in (0, 1)$

$$s = \alpha s_1 + (1 - \alpha) s_2, \quad \frac{1}{q} = \frac{\alpha}{q_1} + \frac{1-\alpha}{q_2}.$$

Then, for $w \in W^{s_1,q_1} \cap W^{s_2,q_2}$ we have $w \in W^{s,q}$, and it holds the estimate

$$||w||_{W^{s,q}} \lesssim ||w||_{W^{s_1,q_1}}^{\alpha} ||w||_{W^{s_2,q_2}}^{1-\alpha}$$

Then, our claim follows by applying the result with $s_1 = 2, s_2 = 0, p_1 = p, p_2 = \infty$ and $\alpha = \frac{p}{2}$, and $s_1 = 2, s_2 = 0, p_1 = p, p_2 = \infty$ and $\alpha = \frac{1}{2}$. □

Appendix C. Auxiliary result

Lemma C.1. *Let $\alpha > 0$ and consider the sequence*

$$q_{k+1} = \frac{2\alpha q_k}{\alpha + \beta q_k} \quad \textit{with} \quad q_0 \in (0, \frac{\alpha}{\beta}].$$

Then, the sequence converges monotonically to $\frac{\alpha}{\beta}$.

Proof. Define the function $f(x) = \frac{2\alpha x}{\alpha + \beta x}$, and note that

$$x = f(x) \quad \Longrightarrow \quad x = \frac{\alpha}{\beta},$$

and thus in the case of convergence the limit is $q = \frac{\alpha}{\beta}$. Further, we observe that f is positive and monotonically increasing for $x > 0$ such that if $q_k \leq \frac{\alpha}{\beta}$, it holds

$$q_{k+1} = f(q_k) \leq f(\frac{\alpha}{\beta}) = \frac{\alpha}{\beta},$$

and thus by induction the sequence is bounded form above by $\frac{\alpha}{\beta}$. Further, for $q_k \leq \frac{\alpha}{\beta}$ we have

$$\frac{q_{k+1}}{q_k} = \frac{2\alpha}{\alpha + \beta q_k} \geq \frac{2\alpha}{\alpha + \alpha} = 1,$$

and hence we have that the sequence is monotonically increasing. This gives the claim. □

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