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Tim Buchholz, Marlis Hochbruck

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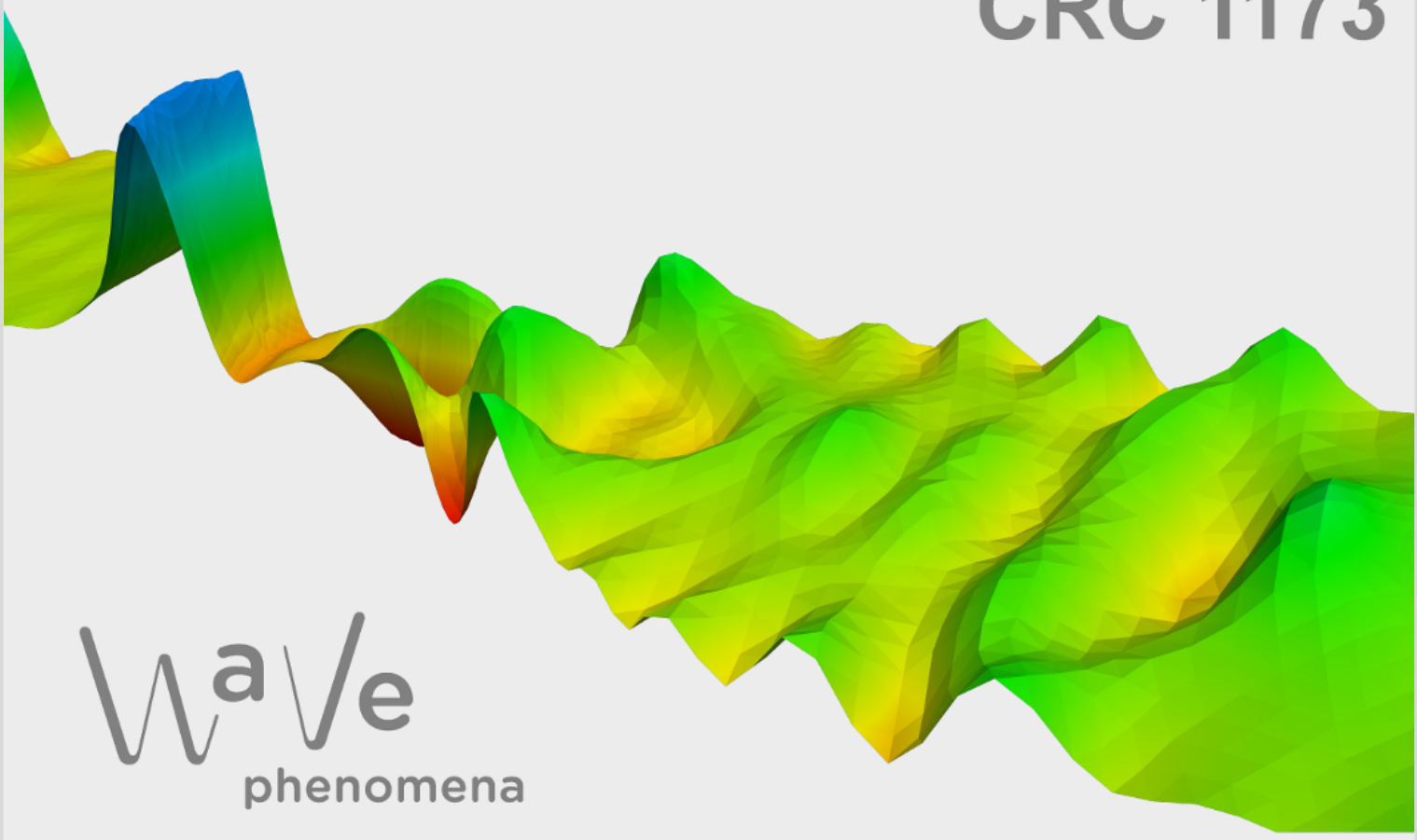

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A NON-ITERATIVE DOMAIN DECOMPOSITION TIME INTEGRATOR FOR LINEAR WAVE EQUATIONS

TIM BUCHHOLZ AND MARLIS HOCHBRUCK

Abstract. We propose and analyze a non-iterative domain decomposition integrator for the linear acoustic wave equation. The core idea is to combine an implicit Crank–Nicolson step on spatial subdomains with a local prediction step at the subdomain interfaces. This enables parallelization across space while advancing sequentially in time, without requiring iterations at each time step. The method is similar to the methods in [2, 7], which have been designed for parabolic problems. Our approach adapts them to the case of the wave equation in a fully discrete setting, using linear finite elements with mass lumping. Compared to explicit schemes, our method permits significantly larger time steps and retains high accuracy. We prove that the resulting method achieves second-order accuracy in time and global convergence of order $\mathcal{O}(h + \tau^2)$ under a CFL-type condition, which depends on the overlap width between subdomains. We conclude with numerical experiments which confirm the theoretical results.

1. Introduction

Wave equations are central to modeling a wide range of physical phenomena, including acoustic, electromagnetic, and elastic wave propagation, cf. [9, Chapter 1]. Numerical simulations of such problems pose significant challenges, particularly due to the computational cost associated with resolving wave dynamics accurately over large spatial and temporal domains.

A critical aspect of simulating wave propagation is the time integration process. Broadly, time-stepping methods fall into two categories: explicit and implicit methods. Explicit methods are only conditionally stable: Step size restrictions, often also called Courant-Friedrichs-Lewy (CFL) conditions, force the choice of the time step to directly depend (linearly or worse) on the space discretization width. Unfortunately, the need for numerous tiny steps can lead to a significant computational burden, slowing down the simulation process. Implicit methods, in contrast, are unconditionally stable in many cases and permit larger time steps. However, implicit methods incur a higher computational cost per step, as they require solving a system of equations at each step. This cost becomes particularly pronounced when dealing with three-dimensional or finely resolved two-dimensional problems.

To address these computational challenges, we propose a non-iterative domain decomposition (DD) method for time integration of a finite element discretization of the linear acoustic wave equation. Specifically, we consider the first-order formulation of the linear acoustic wave equation on an open, polygonal domain $\Omega \subset \mathbb{R}^m$,

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given by

$$
(1)\qquad \begin{cases} \partial_t u &= v, & \partial_t v &= Lu + f & \text{in } \Omega \times (0,T], \\ u(x,0) &= u^0(x), & v(x,0) &= v^0(x) & \text{in } \Omega, \\ u(\cdot,t)\big|_{\partial\Omega} &= 0 & & & \text{for } t \in (0,T], \end{cases}
$$

with $Lu = \nabla \cdot (\kappa^2 \nabla u)$, a constant wave propagation speed $\kappa > 0$ and final time T. The differential operator L is defined on its domain $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$. For the space discretization, we consider linear finite elements with mass lumping on a triangular mesh.

Domain decomposition methods are well-established tools for solving spatially discretized partial differential equations (PDEs) by splitting the spatial domain into subdomains, often enabling parallel computation. Classical Schwarz methods iteratively solve stationary problems by exchanging boundary data between subdomains. An overview of such methods can be found in [15] and [18], respectively.

For time-dependent problems, Schwarz waveform relaxation (SWR) methods extend this concept by integrating the PDE independently in each subdomain and exchanging interface data iteratively [14, 16, 17, 21]. However, their iterative nature introduces a computational overhead that conflicts with the goal of reducing the computational cost of the time integration.

We follow a non-iterative approach motivated by [2] for parabolic problems which is based on an overlapping decomposition of the spatial domain. Coupling between the subdomains is established via artificial boundary conditions obtained from local extrapolation at the interfaces in each time step. The authors of [2] show that this *domain splitting* method is stable and exhibits optimal order of convergence if the overlap regions between the subdomains are sufficiently large. Related non-iterative methods are discussed in [7, 8].

Another non-iterative DD approach for hyperbolic problems is tent pitching [10, 19, 20] in which space-time domains are constructed incrementally with tent-shaped subdomains, adhering to causality constraints. Yet, this can be complex for non-constant coefficients and lacks straightforward parallelization due to subdomain solution dependencies. A different approach relying on finite propagation speed and superposition principles was recently proposed in [13]. Similar to our convergence analysis, the authors of [13] show convergence against the Crank–Nicolson scheme and utilize exponential decay in space. However, their localized implicit time stepping method requires significant larger overlap regions than the scheme presented here.

In this work, we adapt the domain splitting method from [2] to the setting of the linear acoustic wave equation in first order formulation. In this method, which we call *domain splitting* method hereafter, we divide the spatial domain into overlapping subdomains on which the Crank–Nicolson method is applied independently. We use a local variant of the leapfrog scheme for the prediction of boundary data on the subdomain interfaces.

We prove that the proposed domain splitting method achieves second-order accuracy in time and overall convergence of order $\mathcal{O}(h+\tau^2)$ under a CFL-type condition that depends on the wave propagation speed, the spatial mesh size, and the width of the overlap between subdomains. Compared to the explicit leapfrog scheme, the proposed method allows $\mathcal{O}(\ell)$ bigger time steps, where ℓ describes the number of mesh layers within the overlap. At the same time the domain splitting method retains convergence of order $\mathcal{O}(h+\tau^2)$, given sufficient regularity of the exact solution, the initial data, and the source term.

The paper is organized as follows. In Section 2 we first introduce some notation and discuss standard space and second-order time discretizations on the full spatial domain. We then construct the domain splitting scheme in Section 3 and present the main result in Section 4. The subsequent error analysis spans Sections 5 to 8. In Section 5, we bound the error introduced by the predictions at the boundaries of the subdomains. In Section 6, we show that the local error can be interpreted as a solution of a stationary problem, from which we deduce an exponential decay in space. Section 7 gathers some properties of the averaging process, which is necessary to obtain a globally continuous approximation. In Section 8, we derive an error recursion and conclude the error analysis with the proof of the main result. Finally, we present numerical experiments in Section 9.

2. Discretization and notation

2.1. **Discretization in space.** Let $\mathcal{T}_h = \mathcal{T}_h(\Omega)$ be a (shape- and contact-regular) matching simplicial mesh of Ω, see, e.g. [11, Definition 8.11]. The parameter h denotes the minimal diameter of the elements in $\mathcal{T}_h$. We use continuous linear finite elements and define the approximation spaces

$$W_h(\Omega) = \{q \in H^1(\Omega) : \forall K \in \mathcal{T}_h : q\big|_K \in P_1\} \quad \text{and} \quad W_{h,0}(\Omega) = W_h(\Omega) \cap H_0^1(\Omega),$$

where P_1 denotes the set of all linear polynomials in m variables.

Let $\Theta \subseteq \Omega$ such that

$$K \cap \Theta = K \quad \text{or} \quad K \cap \Theta = \emptyset \quad \text{for all } K \in \mathcal{T}_h. \tag{2}$$

We denote the submesh containing all elements in Θ by $\mathcal{T}_h(\Theta)$.

Moreover, we define the set of all nodal points of the triangulation $\mathcal{T}_h$ by

$$\mathcal{N}_{\overline{\Omega}} \coloneqq \bigcup_{K \in \mathcal{T}_h} \{\boldsymbol{x}_{K,j}\}_{j=1}^{m+1},$$

where $\{\boldsymbol{x}_{K,j}\}_{j=1}^{m+1}$ are the vertices of the element $K \in \mathcal{T}_h$. The set of interior nodal points of $\Theta \subseteq \Omega$ satisfying (2) is given by $\mathcal{N}_\Theta = \mathcal{N}_{\overline{\Omega}} \cap \Theta$ and for $\Theta = \Omega$ we write $\mathcal{N} = \mathcal{N}_\Omega$.

2.2. **Norms and bilinear forms.** For $\Theta \subseteq \Omega$ with (2) we denote the standard $L^2(\Theta)$ inner product by $(\cdot,\cdot)_\Theta = (\cdot,\cdot)_{L^2(\Theta)}$ and define the bilinear form

$$a_\Theta(q,p) \coloneqq \int_\Theta \kappa^2 \nabla q \cdot \nabla p \,\mathrm{d}x, \qquad \text{for } q,p \in H^1(\Theta). \tag{3}$$

We formulate (1) weakly by seeking $\big(u,v\big)^\top \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\partial_t u, \varphi)_\Omega &= (v,\varphi)_\Omega\,, && \text{for all } \varphi \in H_0^1(\Omega),\\ (\partial_t v, \psi)_\Omega &= a_\Omega\big(u,\psi\big) + (f,\psi)_\Omega\,, && \text{for all } \psi \in H_0^1(\Omega). \end{aligned} \tag{4}$$

Moreover, we use mass-lumping, i.e., we approximate the L^2 inner product cell-wise by a (Lobatto-)quadrature approximation, see, e.g., [6]. If we denote the nodal interpolation on a single cell K by $\mathcal{I}_h^K$, we can express the mass-lumped inner product by

$$\big(\!\big(q_h, p_h\big)\!\big)_\Theta = \sum_{K \in \mathcal{T}_h(\Theta)} \int_K \mathcal{I}_h^K(q_h p_h)\,\mathrm{d}x, \qquad q_h, p_h \in W_h(\Theta).$$

On a mesh consisting of intervals, triangles, or tetrahedrons we can give the mass-lumped bilinear form $\big(\!\big(\cdot,\cdot\big)\!\big)_\Theta$ explicitly by the following representation

$$\big(\!\big(q_h, p_h\big)\!\big)_\Theta = \sum_{K \in \mathcal{T}_h(\Theta)} \sum_{j=1}^{m+1} \frac{|K|}{m+1} q_h(\boldsymbol{x}_{K,j}) p_h(\boldsymbol{x}_{K,j}), \tag{5}$$

see, e.g., [5, Chapter 4]. The mass-lumped inner product induces a norm on $W_h(\Omega)$, which is equivalent to the standard $L^2(\Omega)$-norm, i.e.

$$C_{\mathrm{ML}}^{-1}\big(q_h,q_h\big)_\Omega^{1/2} \le \|q_h\|_{L^2(\Omega)} \le C_{\mathrm{ML}}\big(q_h,q_h\big)_\Omega^{1/2}, \qquad q_h \in W_h(\Omega), \tag{6}$$

see, e.g., [29, Chapter 15] and [27, Section 3 and 4].

We then introduce the finite-dimensional Hilbert spaces $V_h(\Theta)$, $H_h(\Theta)$ by

$$V_h(\Theta) = \big(W_h(\Theta), a_\Theta(\cdot,\cdot)\big), \qquad H_h(\Theta) = \big(W_h(\Theta), \big(\!\big(\cdot,\cdot\big)\!\big)_\Theta\big)$$

with norms

$$\|\cdot\|_{V_h(\Theta)}^2 = a_\Theta(\cdot,\cdot) \quad \text{and} \quad \|\cdot\|_{H_h(\Theta)}^2 = \big(\!\big(\cdot,\cdot\big)\!\big)_\Theta,$$

respectively. The subspaces with a zero trace are denoted as $V_{h,0}(\Theta)$ and $H_{h,0}(\Theta)$. The linear operator $L_h : V_{h,0}(\Omega) \to V_{h,0}(\Omega)$ associated to $a_\Omega(\cdot,\cdot)$ is given by

$$\big(\!\big(L_h q_h, p_h\big)\!\big)_\Omega = a_\Omega\big(q_h,p_h\big), \quad q_h,p_h \in V_{h,0}(\Omega). \tag{7}$$

For $X(\Theta) = H^1(\Theta) \times L^2(\Theta)$ and $x = \big(u,v\big)^\top \in X(\Theta)$ we define

$$\big\|x\big\|_{X(\Theta)}^2 = a_\Theta\big(u,u\big) + (v,v)_\Theta\,.$$

Similarly, for $X_h(\Theta) = V_h(\Theta) \times H_h(\Theta)$ and $x_h = \big(u_h,v_h\big)^\top \in X_h(\Theta)$ we denote the discrete variant using the mass-lumped scalar product by

$$\big\|x_h\big\|_{X_h(\Theta)}^2 = \|u_h\|_{V_h(\Theta)}^2 + \|v_h\|_{H_h(\Theta)}^2.$$

If $\Theta = \Omega$, we drop the set Θ and write, e.g., $\|\cdot\|_X = \|\cdot\|_{X(\Omega)}$ and $\|\cdot\|_{X_h} = \|\cdot\|_{X_h(\Omega)}$. We use the L^2-projection with respect to $\big(\!\big(\cdot,\cdot\big)\!\big)_\Omega$ to approximate the right-hand side $f_h(t) \approx f(t)$. Note, that this projection coincides with the nodal interpolation $\mathcal{I}_h$, as

$$\big(\!\big(q - \mathcal{I}_h q, w_h\big)\!\big)_\Omega = 0, \quad \text{for } q \in C(\Omega), w_h \in H_h. \tag{8}$$

For $q \in H^2(K)$ on $K \in \mathcal{T}_h$, the standard interpolation estimate

$$|q - \mathcal{I}_h q|_{1,K} \le C_{\mathcal{I}_h} h\, |q|_{2,K} \tag{9}$$

holds with $|\cdot|_{\alpha,K}$ denoting the standard H^α-seminorms on K, see, e.g. [3, Theorem 4.4.4]. We also use the nodal interpolation $\mathcal{I}_h$ for the initial values, so that we have

$$u_h^0 = \mathcal{I}_h u^0, \qquad v_h^0 = \mathcal{I}_h v^0, \qquad f_h^n = \mathcal{I}_h f(t_n). \tag{10}$$

In X_h the discrete counterpart of (4) becomes finally

$$\partial_t \begin{pmatrix} u_h \\ v_h \end{pmatrix} = \begin{pmatrix} 0 & I \\ -L_h & 0 \end{pmatrix} \begin{pmatrix} u_h \\ v_h \end{pmatrix} + \begin{pmatrix} 0 \\ f_h(t) \end{pmatrix}. \tag{11}$$

2.3. **Classical second-order time integration.** Let $\tau > 0$ be a given time-step size and $t_n = n\tau$ for $n = 0,1,\dots,n_T$ with final time $T = n_T\tau$. The implicit Crank–Nicolson scheme for the semi-discrete system (11) yields approximations $x_{\mathrm{CN}}^n = \big(u_{\mathrm{CN}}^n, v_{\mathrm{CN}}^n\big)^\top \in X_h$ given by

$$u_{\mathrm{CN}}^n = u_{\mathrm{CN}}^{n-1} + \frac{\tau}{2}\big(v_{\mathrm{CN}}^n + v_{\mathrm{CN}}^{n-1}\big), \tag{12a}$$

$$v_{\mathrm{CN}}^n = v_{\mathrm{CN}}^{n-1} - \frac{\tau}{2} L_h\big(u_{\mathrm{CN}}^n + u_{\mathrm{CN}}^{n-1}\big) + \tau \overline{f}_h^{\,n-1/2}, \qquad \overline{f}_h^{\,n-1/2} = \frac{1}{2}(f_h^n + f_h^{n-1}) \tag{12b}$$

where we denote $\overline{f}_h^{\,n-1/2} = \frac{1}{2}(f_h^n + f_h^{n-1})$. For the implementation it is advantageous to solve

$$\big(I + \frac{\tau^2}{4} L_h\big) u_{\mathrm{CN}}^n = \big(I - \frac{\tau^2}{4} L_h\big) u_{\mathrm{CN}}^{n-1} + \tau v_{\mathrm{CN}}^{n-1} + \frac{\tau^2}{2} \overline{f}_h^{\,n-1/2} \tag{13}$$

for u_{CN}^n instead of solving the coupled system (12).

The one-step formulation of the explicit leapfrog scheme applied to (11) yields an approximation $x_{\mathrm{lf}}^n = \left(u_{\mathrm{lf}}^n, v_{\mathrm{lf}}^n\right)^\top \in X_h$ given by

$$u_{\mathrm{lf}}^{n-1/2} = u_{\mathrm{lf}}^{n-1} + \frac{\tau}{2} v_{\mathrm{lf}}^{n-1} \tag{14a}$$

$$v_{\mathrm{lf}}^{n} = v_{\mathrm{lf}}^{n-1} - \tau L_h u_{\mathrm{lf}}^{n-1/2} + \tau \overline{f}_h^{n-1/2} \tag{14b}$$

$$u_{\mathrm{lf}}^{n} = u_{\mathrm{lf}}^{n-1/2} + \frac{\tau}{2} v_{\mathrm{lf}}^{n} \tag{14c}$$

If we choose a nodal basis of $W_h(\Omega)$ and assemble the mass and stiffness matrices for (14), we get one linear system with a diagonal mass matrix in each step. We would like to stress that this makes the method local in space. It is well known that the leapfrog scheme suffers from a strong CFL condition, namely

$$\tau^2 \left\| L_h \right\|_{H_h \leftarrow H_h} \leq 4, \tag{15}$$

see, e.g., [25, Theorem 1].

3. Domain splitting scheme

The construction of our new scheme is inspired by the non-iterative DD scheme from [2] proposed for finite element discretizations of parabolic problems. We start by decomposing the spatial domain into non-overlapping subdomains Ω_i, $i = 1, \ldots, N$, such that

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega_i}.$$

Next, we define an overlapping decomposition with overlapping subdomains Ω_i^δ by extending Ω_i with ℓ layers of elements. Here, δ denotes the minimal width of the layers extending Ω_i, see Figure 1. Obviously, we have $\delta \sim \ell h$ for a regular mesh.

Solving the discrete wave equation (11) on each subdomain Ω_i^δ requires appropriate boundary conditions. Hence, one time step of the domain splitting method consists of the following three steps:

1) **Prediction:** We generate boundary values at all artificial boundaries $\partial\Omega_i^\delta \cap \Omega$ by a leapfrog step, which is local in space.
2) **Crank–Nicolson on subdomains:** Using the predicted boundary values, we perform one time step of the Crank–Nicolson method (12) on each of the overlapping subdomains Ω_i^δ.
3) **Averaging:** In this step, we restrict the subdomain solution on Ω_i^δ to the non–overlapping subdomain Ω_i, $i = 1, \ldots, N$. On the interfaces $\partial\Omega_i \cap \partial\Omega_j$, we apply averaging to obtain a unique continuous solution on Ω.

We now present the three steps in detail.

3.1. Prediction.

To generate boundary conditions at each of the interfaces $\Gamma_i^\delta = \partial\Omega_i^\delta \cap \Omega$ we perform a leapfrog step. This step is fully explicit and only uses the degrees of freedom surrounding the interface Γ_i^δ since mass lumping leads to a diagonal matrix in (14).

In contrast to our approach, in [2] extrapolation in time was used locally at the desired degrees of freedom. Using a leapfrog step for the prediction is advantageous here, since it leads to a one-step scheme which simplifies the implementation as well as the analysis.

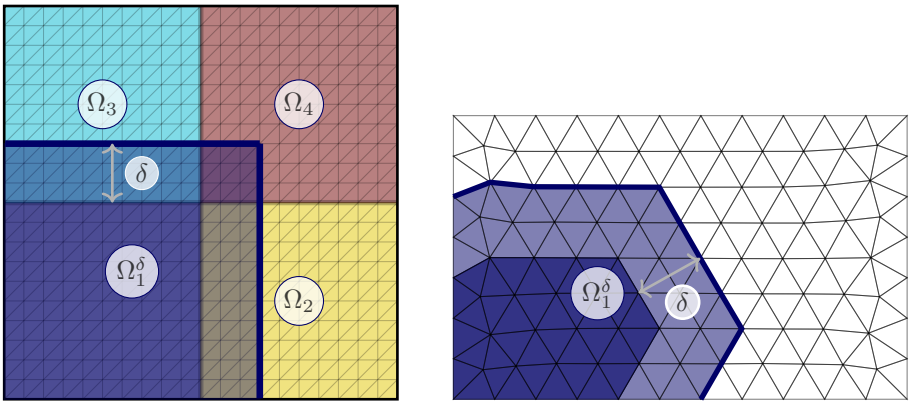

FIGURE 1. Non-overlapping and overlapping decomposition for $\ell = 3$ on an equidistant mesh and for $\ell = 2$ on a non-equidistant mesh.

3.2. **Crank–Nicolson step on subdomains.** The prediction step provides inhomogeneous Dirichlet boundary conditions on all interfaces Γ_i^δ, $i = 1, \dots, N$. This enables us to apply a Crank–Nicolson step on each subdomain Ω_i^δ. Using the Crank–Nicolson method on subdomains is considerably cheaper than on the full domain Ω, since the linear systems of equations are smaller and can be solved in parallel.

3.3. **Averaging.** Recall that $\mathcal{N}_{\Omega_i}$ is the set of all interior nodes in Ω_i, i.e., $\mathcal{N}_{\Omega_i} = \mathcal{N} \cap \Omega_i$. We define the averaging operator

$$\zeta : \bigtimes_{i=1}^{N} X_h(\Omega_i^\delta) \to X_h, \quad \text{or} \quad \zeta : \bigtimes_{i=1}^{N} V_h(\Omega_i^\delta) \to V_h,$$

mapping a set of subdomain functions $q_i \in X_h(\Omega_i^\delta)$ or $q_i \in V_h(\Omega_i^\delta)$, $i = 1, \dots, N$, to a global approximation by its nodal values

(16)
$$\zeta\big(\{q_i\}_{i=1,\dots,N}\big)(\boldsymbol{x}_j) = \begin{cases} q_i(\boldsymbol{x}_j), & \boldsymbol{x}_j \in \mathcal{N}_{\Omega_i} \text{ for exactly one } i \in \{1, \dots, N\}, \\ \sum\limits_{k \in J_j} \frac{q_k(\boldsymbol{x}_j)}{|J_j|}, & \boldsymbol{x}_j \in \mathcal{N} \setminus \big(\bigcup_{i=1,\dots,N} \mathcal{N}_{\Omega_i}\big), \\ & \text{with } J_j = \big\{i \in \{1, \dots, N\} : \boldsymbol{x}_j \in \overline{\Omega}_i\big\}. \end{cases}$$

Note that $\boldsymbol{x}_j$ is contained in at most one set of interior nodal points $\mathcal{N}_{\Omega_i}$ as the subdomains $\Omega_i, i = 1, \dots, N$, are non-overlapping. Moreover, for $q \in V_h(\Omega)$ we immediately see

$$\zeta\big(\{q|_{\Omega_i^\delta}\}_{i=1,\dots,N}\big) = q, \tag{17}$$

i.e., first taking restrictions to the overlapping subdomains and then applying ζ keeps functions in $V_h(\Omega)$ invariant.

3.4. **Full algorithm.** We summarize the domain splitting method in Algorithm 1. For the initial values we assume $u^0, v^0 \in H^2(\Omega)$ to give the point-wise evaluations within the interpolation a proper meaning.

3.5. **Further notation.** For the subsequent error analysis, we introduce some additional notation. By x_{DS}^n and x_i^n we denote the approximations of Algorithm 1. In addition, the Crank–Nicolson solution after n steps started from $x^0 = \big(u^0, v^0\big)$ is denoted by x_{CN}^n. Moreover, we will make use of *local in time* approximations, which are obtained by doing one time step with the Crank–Nicolson or the leapfrog scheme

Algorithm 1: Domain splitting scheme

input: initial data $u_{\mathrm{DS}}^0 = \mathcal{I}_h u^0 \in V_{h,0}$, $v_{\mathrm{DS}}^0 = \mathcal{I}_h v^0 \in H_h$,
overlapping decomposition $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i^\delta$,
time-step size τ, final time $T = n_T \tau$

$n = 1$
while $n \le n_T$ **do**
  prediction step: prediction by $\widetilde{u}_{\mathrm{lf}}^n$ (leapfrog step with i.c. x_{DS}^{n-1})
  for *i = 1,. . . ,* N **do**
    Crank–Nicolson on subdomains:
    calculate x_i^n on Ω_i^δ with $\widetilde{u}_{\mathrm{lf}}^n|_{\Gamma_i^\delta}$ as b.c. on Γ_i^δ
  end
  averaging step: construct global $x_{\mathrm{DS}}^n = \zeta\left(\{x_i^n\}_{i=1,\dots,N}\right)$ according to (16)
  update $n \leftarrow n+1$
end
result: $x_{\mathrm{DS}}^{n_T} \approx x(\cdot, T)$.

starting from the domain splitting approximation x_{DS}^{n-1}. These approximations are denoted by $\widetilde{x}_{\mathrm{CN}}^n$ and $\widetilde{x}_{\mathrm{lf}}^n$, respectively.

We summarize the notation in Table 1.

domain splitting	subdomain approx. in nth step	Crank–Nicolson	*local in time* Crank–Nicolson	*local in time* leapfrog
$x_{\mathrm{DS}}^n = \begin{pmatrix} u_{\mathrm{DS}}^n \\ v_{\mathrm{DS}}^n \end{pmatrix}$	$x_i^n = \begin{pmatrix} u_i^n \\ v_i^n \end{pmatrix}$	$x_{\mathrm{CN}}^n = \begin{pmatrix} u_{\mathrm{CN}}^n \\ v_{\mathrm{CN}}^n \end{pmatrix}$	$\widetilde{x}_{\mathrm{CN}}^n = \begin{pmatrix} \widetilde{u}_{\mathrm{CN}}^n \\ \widetilde{v}_{\mathrm{CN}}^n \end{pmatrix}$	$\widetilde{x}_{\mathrm{lf}}^n = \begin{pmatrix} \widetilde{u}_{\mathrm{lf}}^n \\ \widetilde{v}_{\mathrm{lf}}^n \end{pmatrix}$

TABLE 1. Notation for different approximations

4. Main result

A main contribution of this work is to show that the domain splitting method for approximating the solution of (11) satisfies an error bound of the form

$$\left\| x_{\mathrm{DS}}^n - x(t_n) \right\|_X \lesssim \tau^2 + h$$

under suitable conditions on the exact solution x of (1), the right-hand side f, and the parameters h, τ and ℓ.

A key idea is to start with comparing the domain splitting method and the Crank–Nicolson method, since for the latter, rigorous error bounds of the full discretization exist, see [23] and [22, Example 5.3, Corollary 5.9]. More precisely, we will show

$$E^n = \left\| x_{\mathrm{DS}}^n - x_{\mathrm{CN}}^n \right\|_{X_h} \lesssim \tau^2$$

in the following lemma.

Lemma 4.1. *Let $u^0, v^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f \in L^\infty\big([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\big)$ and consider the domain splitting method of the previous section with a sufficiently large overlap $\delta \sim \ell h$. Assume that the CFL condition*

$$\tau^2 \big\|L_h\big\|_{H_h \leftarrow H_h} \leq 4\ell^2 \tag{18}$$

holds for a time step $\tau \in (0,1]$. Then there exist constants $0 \leq \sigma \leq 1$ and $C_{\mathrm{data}} > 0$ such that

$$E^n = \big\|x_{\mathrm{DS}}^n - x_{\mathrm{CN}}^n\big\|_{X_h} \leq \tau^2 C_{\mathrm{data}} \min\{e^{\sigma t_n} - 1, \sigma t_n \mathrm{e}^{\sigma\tau t_n}\}$$

The constant C_{data} is independent of h and τ, for details see Lemma 5.2 below.

The proof of this result is quite involved and postponed to Section 8.1.

Theorem 4.2 (Error bound against the exact solution)**.** *Let the assumptions of Lemma 4.1 be fulfilled and let the solution of (1) suffice*

$$u \in C^4\big([0,T]; L^2(\Omega)\big) \cap C^3\big([0,T]; H_0^1(\Omega)\big) \cap C^2\big([0,T]; H^2(\Omega)\big) \cap C\big([0,T]; D(L^2)\big).$$

Then, the error of the domain splitting scheme satisfies

$$\big\|x_{\mathrm{DS}}^n - x(t_n)\big\|_X \lesssim \tau^2 + h.$$

Proof. Let $\mathcal{J}_h$ be the projection from X to X_h obtained by taking a Ritz projection $\mathcal{R}_h$ (see, e.g., [12, Section 32.4]) in both components. Then, by the triangle inequality, we can estimate

$$\big\|x(t_n) - x_{\mathrm{DS}}^n\big\|_X \leq \big\|x(t_n) - \mathcal{J}_h x(t_n)\big\|_X + C_{\mathrm{ML}}\big\|\mathcal{J}_h x(t_n) - x_{\mathrm{CN}}^n\big\|_{X_h} + C_{\mathrm{ML}}\big\|x_{\mathrm{CN}}^n - x_{\mathrm{DS}}^n\big\|_{X_h},$$

with $C_{\mathrm{ML}} > 0$ from (6). For the first term we can apply standard approximation results to obtain

$$\begin{aligned}\big\|x(t_n) - \mathcal{J}_h x(t_n)\big\|_X &= \Big(\big|u(t_n) - \mathcal{R}_h u(t_n)\big|_{H^1(\Omega)}^2 + \big\|\partial_t u(t_n) - \mathcal{R}_h(\partial_t u(t_n))\big\|_{L^2(\Omega)}^2\Big)^{1/2} \\ &\lesssim h\Big(\big\|u(t_n)\big\|_{H^2(\Omega)}^2 + \big\|\partial_t u(t_n)\big\|_{H^1(\Omega)}^2\Big)^{1/2},\end{aligned}$$

see, e.g., [12, Theorem 32.15 and 33.2]. For the second term, we have

$$\begin{aligned}\big\|x_{\mathrm{CN}}^n - \mathcal{J}_h x(t_n)\big\|_{X_h} \lesssim \sqrt{2}\Big(&h\|u^0\|_{H^2(\Omega)} + h\|v^0\|_{H^1(\Omega)} + \frac{\tau^2}{8}\int_0^{t_{n+1}} \|\partial_t^3 x(s)\|_X \,\mathrm{d}s \\ &+ h\tau \sum_{j=1}^n \max_{s\in[t_j,t_{j+1}]} \|\partial_t^2 u(s)\|_{H^1(\Omega)} \\ &+ h\tau \sum_{j=1}^n \|Lu(t_j)\|_{H^2(\Omega)} + \|Lu(t_{j+1})\|_{H^2(\Omega)}\Big),\end{aligned}$$

from [23, Section 4.8] and [22, Section 5.3 and Corollary 5.9].

Finally, the bound for the third term follows from Lemma 4.1. □

5. Prediction error

A key part of the error analysis is estimating the prediction error at each step, which is challenging because each step begins with a solution that has been cut and reassembled along the interface, offering little regularity. To address this, we aim to analyze the prediction error globally in space, but only locally in time. This is done by using equivalent formulations of the leapfrog and Crank–Nicolson methods on X_h, which can be found in [9, Sections 11.1 and 11.2].

Lemma 5.1. *a) The Crank–Nicolson method (12) is equivalent to*

$$R_- \begin{pmatrix} u_{\mathrm{CN}}^n \\ v_{\mathrm{CN}}^n \end{pmatrix} = R_+ \begin{pmatrix} u_{\mathrm{CN}}^{n-1} \\ v_{\mathrm{CN}}^{n-1} \end{pmatrix} + \tau \begin{pmatrix} 0 \\ \overline{f}_h^{n-1/2} \end{pmatrix}, \tag{19}$$

where

$$R_- = \begin{pmatrix} \mathrm{I} & -\frac{\tau}{2}\mathrm{I} \\ \frac{\tau}{2}L_h & \mathrm{I} \end{pmatrix}, \qquad R_+ = \begin{pmatrix} \mathrm{I} & \frac{\tau}{2}\mathrm{I} \\ -\frac{\tau}{2}L_h & \mathrm{I} \end{pmatrix}.$$

b) The leapfrog method (14) is equivalent to

$$\widehat{R}_- \begin{pmatrix} u_{\mathrm{lf}}^n \\ v_{\mathrm{lf}}^n \end{pmatrix} = \widehat{R}_+ \begin{pmatrix} u_{\mathrm{lf}}^{n-1} \\ v_{\mathrm{lf}}^{n-1} \end{pmatrix} + \tau \begin{pmatrix} 0 \\ \overline{f}_h^{n-1/2} \end{pmatrix}, \tag{20}$$

where

$$\widehat{R}_- = \begin{pmatrix} \mathrm{I} & -\frac{\tau}{2}\mathrm{I} \\ \frac{\tau}{2}L_h & \mathrm{I} - \frac{\tau^2}{4}L_h \end{pmatrix} \qquad \widehat{R}_+ = \begin{pmatrix} \mathrm{I} & \frac{\tau}{2}\mathrm{I} \\ -\frac{\tau}{2}L_h & \mathrm{I} - \frac{\tau^2}{4}L_h \end{pmatrix}$$

Moreover, we define

$$R = R_-^{-1}R_+ \quad \text{and} \quad \widehat{R} = \widehat{R}_-^{-1}\widehat{R}_+. \tag{21}$$

We start with a stability result for the Crank–Nicolson method, which makes use of the inverse inequality

$$\|L_h q_h\|_{H_h} \le C_{\mathrm{inv}}\kappa h^{-1}\|q_h\|_{V_h} \le C_{\mathrm{inv}}^2\kappa^2 h^{-2}\|q_h\|_{H_h}, \tag{22}$$

see, e.g., [11, Lemma 12.1]. For a function $q \in H^2(\Omega)$ we moreover use, that

$$\big\|(\mathcal{I}_h - \mathcal{R}_h)q\big\|_{V_h} \le C_{\mathcal{R}_h} h \|q\|_{H^2(\Omega)}, \tag{23}$$

see, e.g., [3, Theorem 4.4.20 and 8.5.3]. Lastly, for all $p \in H^2(\Omega)$ it holds that $p_h = \mathcal{I}_h p$ satisfies

$$\|L_h^{1/2} p_h\|_{H_h} = \|\mathcal{I}_h p\|_{V_h} \le C_{\mathrm{Int}}\|p\|_{H^2(\Omega)}, \tag{24}$$

see, e.g., [11, Section 11.5.1]. Note here that we could also require $p \in W^{1,s}(\Omega)$ with $s > m$, since we only need an embedding into $C^0(\Omega)$ to make $\mathcal{I}_h p$ well defined, see, e.g. [3, Theorem 4.4.4].

Lemma 5.2. *Let $u^0, v^0 \in D(L)$ and $f \in L^\infty\big([0,T]; D(L)\big)$. With $u_{\mathrm{CN}}^0 = u_h^0$ and $v_{\mathrm{CN}}^0 = v_h^0$ from (10), the Crank–Nicolson approximations (19) satisfy*

$$\Big(\big\|L_h^{1/2}u_{\mathrm{CN}}^{n-1}\big\|_{V_h}^2 + \big\|L_h^{1/2}v_{\mathrm{CN}}^{n-1}\big\|_{H_h}^2\Big)^{1/2} + \Big\|\frac{\tau}{2}L_h^{1/2}\big(\overline{f}_h^{n-1/2}\big)\Big\|_{H_h} \le C_{\mathrm{data}}, \tag{25}$$

with $C_{\mathrm{data}} = C_{\mathrm{data}}(u^0, v^0, \kappa, f, t_n)$ independent of h.

Proof. By the discrete variation-of-constants formula, we have

$$\begin{pmatrix} u_{\mathrm{CN}}^{n-1} \\ v_{\mathrm{CN}}^{n-1} \end{pmatrix} = R^{n-1} \begin{pmatrix} u_{\mathrm{CN}}^0 \\ v_{\mathrm{CN}}^0 \end{pmatrix} + \tau \sum_{j=1}^{n-1} R^{n-j} R_-^{-1} \begin{pmatrix} 0 \\ \overline{f}_h^{j-1/2} \end{pmatrix}.$$

It is well known that

$$\big\|R\big\|_{X_h \leftarrow X_h} = 1 \quad \text{and} \quad \big\|R_-\big\|_{X_h \leftarrow X_h} \le 1$$

so that we obtain

$$\left\| \begin{pmatrix} L_h^{1/2}u_{\mathrm{CN}}^{n-1} \\ L_h^{1/2}v_{\mathrm{CN}}^{n-1} \end{pmatrix} \right\|_{X_h} \le \left\| \begin{pmatrix} L_h^{1/2}u_{\mathrm{CN}}^{0} \\ L_h^{1/2}v_{\mathrm{CN}}^{0} \end{pmatrix} \right\|_{X_h} + \tau \sum_{j=0}^{n-1} \big\|L_h^{1/2} f_h^j\big\|_{H_h}.$$

It remains to bound $\|L_h^{1/2} u_{\mathrm{CN}}^0\|_{V_h}$, $\|L_h^{1/2} v_{\mathrm{CN}}^0\|_{H_h}$, and $\|L_h^{1/2} f_h^j\|_{H_h}$ in terms of the initial values and the inhomogeneity. Recall that the Ritz projection $\mathcal{R}_h$ satisfies

$$\big\|L_h \mathcal{R}_h \phi\big\|_{H_h} = \big\|L\phi\big\|_{L^2(\Omega)}, \tag{26}$$

for all $\phi \in D(L)$, since

$$\big(\!\big(L_h \mathcal{R}_h \phi, \psi_h\big)\!\big)_\Omega = a_\Omega\big(\mathcal{R}_h\phi, \psi_h\big) = a_\Omega\big(\phi, \psi_h\big) = \big(L\phi, \psi_h\big)_\Omega\,, \quad \text{for all } \psi_h \in H_h.$$

Hence, we have from (22) that

$$\begin{aligned}\big\|L_h^{1/2} u_{\mathrm{CN}}^0\big\|_{V_h} = \big\|L_h \mathcal{I}_h u^0\big\|_{H_h} &\le \big\|L_h(\mathcal{I}_h - \mathcal{R}_h)u^0\big\|_{H_h} + \big\|L_h \mathcal{R}_h u^0\big\|_{H_h} \\ &\le C_{\mathrm{inv}} h^{-1} \kappa \big\|(\mathcal{I}_h - \mathcal{R}_h)u^0\big\|_{V_h} + \big\|Lu^0\big\|_{L^2(\Omega)} \\ &\le (C_{\mathrm{Int}} C_{\mathcal{R}_h} + 1)\kappa \big\|u^0\big\|_{H^2(\Omega)},\end{aligned}$$

since $u^0 \in D(L) \subset H^2(\Omega)$ by assumption and (23). The remaining bounds in the H_h-norm follow directly from (24) and using that $f \in L^\infty([0,T]; D(L))$. Setting all these bounds together yields an upper bound for (25) given by

$$C_{\mathrm{data}} = \big(C_{\mathrm{inv}} C_{\mathcal{R}_h} + 1\big)\kappa \|u^0\|_{H^2(\Omega)} + C_{\mathrm{Int}} \kappa \|v^0\|_{H^2(\Omega)} + t_n C_{\mathrm{Int}} \kappa \max_{j=0,\dots,n} \|f(t_j)\|_{H^2(\Omega)}\,. \tag{27}$$

□

Based on this stability result, we can bound the prediction error $\big\|\widetilde{x}_{\mathrm{CN}}^n - \widetilde{x}_{\mathrm{lf}}^n\big\|_{X_h}^2$ which is *global in space* but *local in time* (we only consider one time step of the Crank–Nicolson and the leapfrog method starting from the domain splitting approximation x_{DS}^{n-1}).

Lemma 5.3. *Let the assumptions of Lemma 5.2 and in addition the CFL condition (18) be satisfied. Then it holds*

$$\big\|\widetilde{x}_{\mathrm{CN}}^n - \widetilde{x}_{\mathrm{lf}}^n\big\|_{X_h} \le M_\ell \left(\big\|x_{\mathrm{DS}}^{n-1} - x_{\mathrm{CN}}^{n-1}\big\|_{X_h} + \tau C_{\mathrm{data}}\right),$$

with a constant M_ℓ only depending on ℓ.

Proof. We rewrite the difference we are interested in with Lemma 5.1 by

$$\begin{aligned}\widetilde{x}_{\mathrm{CN}}^n - \widetilde{x}_{\mathrm{lf}}^n &= \big(R - \widehat{R}\big) x_{\mathrm{DS}}^{n-1} + \tau\big(R_-^{-1} - \widehat{R}_-^{-1}\big)\overline{g}_h^{n-1/2} \\ &= \big(R - \widehat{R}\big)\big(x_{\mathrm{DS}}^{n-1} - x_{\mathrm{CN}}^{n-1}\big) + \big(R - \widehat{R}\big) x_{\mathrm{CN}}^{n-1} + \tau\big(R_-^{-1} - \widehat{R}_-^{-1}\big)\overline{g}_h^{n-1/2}\end{aligned} \tag{28}$$

with $\overline{g}_h^{n-1/2} = \begin{pmatrix} 0 \\ \overline{f}_h^{n-1/2} \end{pmatrix}$. A simple calculation shows

$$R_-^{-1} - \widehat{R}_-^{-1} = \frac{\tau^2}{4} \begin{pmatrix} A_h^{-1} L_h & \\ & A_h^{-1} L_h \end{pmatrix} \begin{pmatrix} \frac{\tau^2}{4} L_h & -\frac{\tau}{2}\mathrm{I} \\ \frac{\tau}{2} L_h & -\mathrm{I} \end{pmatrix},$$

and

$$R - \widehat{R} = \frac{\tau^2}{4} \begin{pmatrix} A_h^{-1} L_h^2 & \\ & A_h^{-1} L_h^2 \end{pmatrix} \begin{pmatrix} \frac{\tau^2}{2}\mathrm{I} & \frac{\tau^3}{4}\mathrm{I} \\ -\tau\mathrm{I} & \frac{\tau^2}{2}\mathrm{I} \end{pmatrix},$$

where we denote $A_h = \mathrm{I} + \frac{\tau^2}{4} L_h$. Since for $\alpha \in [0,2]$, there holds

$$0 \le \frac{\xi^{\alpha/2}}{1+\xi} \le 1, \qquad \text{for all } \xi \ge 0,$$

it follows that

$$\tau^2 \big\|A_h^{-1} L_h\big\|_{H_h \leftarrow H_h} \le 4 \quad \text{and} \quad \tau^2 \big\|A_h^{-1} L_h\big\|_{V_h \leftarrow V_h} \le 4.$$

From this and (18) we can derive for $x = \big(u, v\big)^\top$

(29)
$$\begin{aligned}\big\|(R-\widehat{R})x\big\|_{X_h}^2 &= \Big\|\frac{\tau^2}{4}A_h^{-1}L_h^2\big(\frac{\tau^2}{2}u+\frac{\tau^3}{4}v\big)\Big\|_{V_h}^2 + \Big\|\frac{\tau^2}{4}A_h^{-1}L_h^2\big(-\tau u+\frac{\tau^2}{2}v\big)\Big\|_{H_h}^2\\ &\le 2\Big(\big\|\frac{\tau^2}{2}L_h u\big\|_{V_h}^2 + \big\|\tau L_h^{1/2}u\big\|_{V_h}^2 + \big\|\tau L_h^{1/2}v\big\|_{H_h}^2 + \big\|\frac{\tau^2}{2}L_h v\big\|_{H_h}^2\Big)\\ &\le M_\ell^2\Big(\big\|u\big\|_{V_h}^2 + \big\|v\big\|_{H_h}^2\Big)\\ &= M_\ell^2\big\|x\big\|_{X_h}^2\end{aligned}$$

with a generic constant M_ℓ only depending on ℓ. Alternatively, we can also keep a prefactor τ by using

$$\big\|(R-\widehat{R})x\big\|_{X_h} \le M_\ell\tau\Big(\big\|L_h^{1/2}u\big\|_{V_h}^2 + \big\|L_h^{1/2}v\big\|_{H_h}^2\Big)^{1/2}. \tag{30}$$

Analogously, we derive for $g = \big(0, f\big)^\top$

(31)
$$\big\|(R_-^{-1}-\widehat{R}_-^{-1})g\big\|_{X_h}^2 \le \big\|\frac{\tau^3}{8}A_h^{-1}L_h^{3/2}f\big\|_{H_h}^2 + \big\|\frac{\tau^2}{4}A_h^{-1}L_h f\big\|_{H_h}^2 \le M_\ell^2\big\|\frac{\tau}{2}L_h^{1/2}f\big\|_{H_h}^2.$$

From (28) we deduce

$$\begin{aligned}\big\|\widetilde{x}_{\mathrm{CN}}^n - \widetilde{x}_{\mathrm{lf}}^n\big\|_{X_h} \le{}& \big\|(R-\widehat{R})\big(x_{\mathrm{DS}}^{n-1} - x_{\mathrm{CN}}^{n-1}\big)\big\|_{X_h} + \big\|(R-\widehat{R})x_{\mathrm{CN}}^{n-1}\big\|_{X_h}\\ &+ \tau\big\|(R_-^{-1}-\widehat{R}_-^{-1})\overline{g}_h^{n-1/2}\big\|_{X_h}.\end{aligned} \tag{32}$$

For the first term we use (29) to get

$$\big\|(R-\widehat{R})\big(x_{\mathrm{DS}}^{n-1} - x_{\mathrm{CN}}^{n-1}\big)\big\|_{X_h} \le M_\ell\big\|x_{\mathrm{DS}}^{n-1} - x_{\mathrm{CN}}^{n-1}\big\|_{X_h},$$

while we take (30) for the second term to obtain

$$\big\|(R-\widehat{R})x_{\mathrm{CN}}^{n-1}\big\|_{X_h} \le M_\ell\tau\Big(\big\|L_h^{1/2}u_{\mathrm{CN}}^{n-1}\big\|_{V_h}^2 + \big\|L_h^{1/2}v_{\mathrm{CN}}^{n-1}\big\|_{H_h}^2\Big)^{1/2},$$

and (31) to bound

$$\big\|(R_-^{-1}-\widehat{R}_-^{-1})\overline{g}_h^{n-1/2}\big\|_{X_h} \le M_\ell\big\|\frac{\tau}{2}L_h^{1/2}\overline{f}_h^{n-1/2}\big\|_{H_h}.$$

Putting these estimates together in (32) and using Lemma 5.2 proves the claim. □

6. Local deviation from global Crank–Nicolson approximation

Next, we investigate the local Crank–Nicolson approximations on all subdomains. After the for loop in Algorithm 1 we end up with subdomain approximations x_i^n on each of the overlapping subdomains Ω_i^δ, which we now compare to the global in space Crank–Nicolson approximation $\widetilde{x}_{\mathrm{CN}}^n$ given in (12) on Ω. We refer to Table 1 for an overview on the notation.

6.1. Variational problem for local error on subdomains.

Recall that the subdomain approximation x_i^n and the Crank–Nicolson approximation $\widetilde{x}_{\mathrm{CN}}^n$ both start from the same approximation x_{DS}^{n-1}.

Lemma 6.1. *The difference $z_{u,i}^n = u_i^n - \widetilde{u}_{\mathrm{CN}}^n\big|_{\Omega_i^\delta}$ satisfies*

$$\begin{aligned}&\big(\!\big(z_{u,i}^n, \varphi\big)\!\big)_{\Omega_i^\delta} + \frac{\tau^2}{4}a_{\Omega_i^\delta}\big(z_{u,i}^n, \varphi\big) = 0, && \text{for all } \varphi \in V_{h,0}(\Omega_i^\delta),\\ &\qquad z_{u,i}^n = \widetilde{u}_{\mathrm{lf}}^n - \widetilde{u}_{\mathrm{CN}}^n, && \text{on } \partial\Omega_i^\delta.\end{aligned} \tag{33a}$$

Moreover, for $z_{v,i}^n = v_i^n - \widetilde{v}_{\mathrm{CN}}^n\big|_{\Omega_i^\delta}$ *it holds, that*

$$z_{u,i}^n = \frac{\tau}{2} z_{v,i}^n \quad \text{in } H_h(\Omega_i^\delta). \tag{33b}$$

The problem (33a) is well posed, cf. [12, Lemma 33.1].

Proof. The proof is based on comparing two weak forms, one on the overlapping subdomains Ω_i^δ and one on the full domain Ω.

On the one hand, $x_i^n \in \big(V_h(\Omega_i^\delta) \cap H_0^1(\Omega)\big) \times H_h(\Omega_i^\delta)$ is the solution of

$$\begin{aligned}
\big(\!\big(u_i^n, \varphi_i\big)\!\big)_{\Omega_i^\delta} - \frac{\tau}{2}\big(\!\big(v_i^n, \varphi_i\big)\!\big)_{\Omega_i^\delta} &= \big(\!\big(u_{\mathrm{DS}}^{n-1}, \varphi_i\big)\!\big)_{\Omega_i^\delta} + \frac{\tau}{2}\big(\!\big(v_{\mathrm{DS}}^{n-1}, \varphi_i\big)\!\big)_{\Omega_i^\delta}, \\
\big(\!\big(v_i^n, \psi_i\big)\!\big)_{\Omega_i^\delta} + \frac{\tau}{2} a_{\Omega_i^\delta}\big(u_i^n, \psi_i\big) &= \big(\!\big(v_{\mathrm{DS}}^{n-1}, \psi_i\big)\!\big)_{\Omega_i^\delta} - \frac{\tau}{2} a_{\Omega_i^\delta}\big(u_{\mathrm{DS}}^{n-1}, \psi_i\big) + \tau\big(\!\big(\overline{f}_h^{\,n-1/2}, \psi_i\big)\!\big)_{\Omega_i^\delta}, \\
u_i^n &= \widetilde{u}_{\mathrm{lf}}^n, \qquad \text{on } \Gamma_i^\delta.
\end{aligned}$$

for all $\varphi_i, \psi_i \in V_{h,0}(\Omega_i^\delta)$. On the other hand, $\widetilde{x}_{\mathrm{CN}}^n \in V_{h,0}(\Omega) \times H_{h,0}(\Omega)$ satisfies

$$\begin{aligned}
\big(\!\big(\widetilde{u}_{\mathrm{CN}}^n, \varphi\big)\!\big)_{\Omega} - \frac{\tau}{2}\big(\!\big(\widetilde{v}_{\mathrm{CN}}^n, \varphi\big)\!\big)_{\Omega} &= \big(\!\big(u_{\mathrm{DS}}^{n-1}, \varphi\big)\!\big)_{\Omega} + \frac{\tau}{2}\big(\!\big(v_{\mathrm{DS}}^{n-1}, \varphi\big)\!\big)_{\Omega}, \\
\big(\!\big(\widetilde{v}_{\mathrm{CN}}^n, \psi\big)\!\big)_{\Omega} + \frac{\tau}{2} a_{\Omega}\big(\widetilde{u}_{\mathrm{CN}}^n, \psi\big) &= \big(\!\big(v_{\mathrm{DS}}^{n-1}, \psi\big)\!\big)_{\Omega} - \frac{\tau}{2} a_{\Omega}\big(u_{\mathrm{DS}}^{n-1}, \psi\big) + \tau\big(\!\big(\overline{f}_h^{\,n-1/2}, \psi\big)\!\big)_{\Omega},
\end{aligned}$$

for all $\varphi, \psi \in V_{h,0}(\Omega)$.

In order to compare these two approximations, we extend the test functions φ_i, ψ_i by zero onto Ω. By this extension we get valid test functions for the weak formulation of the global Crank–Nicolson on Ω, which vanish outside of Ω_i^δ. The difference

$$x_i^n - \widetilde{x}_{\mathrm{CN}}^n\big|_{\Omega_i^\delta} = \begin{pmatrix} u_i^n - \widetilde{u}_{\mathrm{CN}}^n\big|_{\Omega_i^\delta} \\ v_i^n - \widetilde{v}_{\mathrm{CN}}^n\big|_{\Omega_i^\delta} \end{pmatrix} = \begin{pmatrix} z_{u,i}^n \\ z_{v,i}^n \end{pmatrix} \in V_h(\Omega_i^\delta) \times H_h(\Omega_i^\delta)$$

satisfies

$$\big(\!\big(z_{u,i}^n, \varphi_i\big)\!\big)_{\Omega_i^\delta} - \frac{\tau}{2}\big(\!\big(z_{v,i}^n, \varphi_i\big)\!\big)_{\Omega_i^\delta} = 0, \qquad \varphi_i \in V_{h,0}(\Omega_i^\delta), \tag{34a}$$

$$\big(\!\big(z_{v,i}^n, \psi_i\big)\!\big)_{\Omega_i^\delta} + \frac{\tau}{2} a_{\Omega_i^\delta}\big(z_{u,i}^n, \psi_i\big) = 0, \qquad \psi_i \in V_{h,0}(\Omega_i^\delta), \tag{34b}$$

and

$$\begin{aligned}
(u_i^n - \widetilde{u}_{\mathrm{CN}}^n)\big|_{\Gamma_i^\delta} &= \widetilde{u}_{\mathrm{lf}}^n\big|_{\Gamma_i^\delta} - \widetilde{u}_{\mathrm{CN}}^n\big|_{\Gamma_i^\delta}, \\
(u_i^n - \widetilde{u}_{\mathrm{CN}}^n)\big|_{\partial\Omega \cap \partial\Omega_i^\delta} &= 0.
\end{aligned} \tag{34c}$$

(34a) is equivalent to (33b). Inserting this relation into (34b) and using the boundary conditions (34c) shows that $z_{u,i}^n$ solves (33a). □

6.2. **Exponential decay of errors in boundary data.** The goal of this subsection is to prove a discrete exponential decay result for the solution of (34a). For that we additionally need to introduce weighted bilinear forms.

Definition 6.2. *Let* $\lambda > 0$ *be arbitrarily chosen and* $q_h, p_h \in V_h$. *For a positive and continuous weight function* $\omega : \Omega \to \mathbb{R}_+$, *we introduce the weighted bilinear forms*

$$a_{\Theta,\omega}(q_h, p_h) = \int_\Theta \omega(x) \kappa^2 \nabla q_h(x) \cdot \nabla p_h(x)\, \mathrm{d}x, \tag{35a}$$

$$b_{\Theta,\omega}(q_h, p_h) = \frac{1}{\lambda^2}\big(\!\big(\omega q_h, p_h\big)\!\big)_\Theta + a_{\Theta,\omega}(q_h, p_h)\ . \tag{35b}$$

For $\omega \equiv 1$ *we abbreviate this via*

$$b_\Theta(q_h, p_h) = b_{\Theta,1}(q_h, p_h) = \frac{1}{\lambda^2}\big(\!\big(q_h, p_h\big)\!\big)_\Theta + a_\Theta(q_h, p_h). \tag{35c}$$

Next, we show a discrete decay estimate, which serves as key ingredient of our error analysis. This estimate is strongly motivated by [2, Lemma 1], which, however, does not apply directly to our setting for wave equations. For instance, we have to include mass lumping.

Theorem 6.3. *Let $\lambda > 0$, and $g \in V_h(\Omega_i^\delta)$. If $z \in V_h(\Omega_i^\delta)$ solves*

$$\begin{aligned} \langle\!\langle z, \varphi \rangle\!\rangle_{\Omega_i^\delta} + \lambda^2 a_{\Omega_i^\delta}(z, \varphi) &= 0, \qquad \forall \varphi \in V_{h,0}(\Omega_i^\delta), \\ z &= g, \qquad \text{on } \partial\Omega_i^\delta, \end{aligned} \tag{36}$$

then it holds

$$\frac{1}{\lambda^2}\langle\!\langle z, z \rangle\!\rangle_{\Omega_i} + a_{\Omega_i}(z, z) \leq \beta \exp\Big(-\frac{\gamma\delta}{\max\{\lambda, h\}}\Big)\Big(\frac{1}{\lambda^2}\langle\!\langle g, g \rangle\!\rangle_{\Omega_i^\delta} + a_{\Omega_i^\delta}(g, g)\Big)$$

with $\gamma \in (0, 1]$ and $\beta > 0$, which are independent of λ, h, and the size of the subdomain Ω_i.

Proof. We choose some $\gamma \in (0, 1]$, which will be fixed later and define a weight function

$$\omega(x) \coloneqq \exp\left(\frac{\gamma x}{\max\{\lambda, h\}}\right), \quad x \in \Omega_i^\delta. \tag{37}$$

The distance of a point $x \in \Omega_i^\delta$ to the prediction interface is denoted by $\operatorname{dist}(x, \Gamma_i^\delta)$. Since the distance function is not contained in $V_h(\Omega_i^\delta)$, we define its nodal interpolation $\mathcal{I}_h$ on the grid points in Ω_i^δ as

$$d_h \coloneqq \mathcal{I}_h\big(x \mapsto \operatorname{dist}(x, \Gamma_i^\delta)\big) \in V_h(\Omega_i^\delta). \tag{38}$$

For the sake of presentation, we suppress the space dependencies in the rest of the proof, i.e., we write $\omega(d_h) = \omega(d_h(x))$. Let us first gather some properties of ω and d_h

$$\omega(\xi) \geq 1 \geq \omega(-\xi) \geq 0, \qquad \text{for } \xi \geq 0, \tag{39a}$$

$$\omega(\xi + \nu) = \omega(\xi)\omega(\nu), \qquad \text{for } \xi, \nu \in \mathbb{R}, \tag{39b}$$

$$d_h(x) - \delta \geq 0, \qquad \text{for } x \in \overline{\Omega}_i. \tag{39c}$$

By definitions (35c) for $\Theta = \Omega_i$, (37), (39a), and (39c), we have

$$\begin{aligned} b_{\Omega_i}(z, z) &= \frac{1}{\lambda^2}\langle\!\langle z, z \rangle\!\rangle_{\Omega_i} + a_{\Omega_i}(z, z) \\ &\leq \frac{1}{\lambda^2}\langle\!\langle \omega(d_h - \delta)z, z \rangle\!\rangle_{\Omega_i} + a_{\Omega_i, \omega(d_h - \delta)}(z, z) \\ &= \omega(-\delta) b_{\Omega_i, \omega(d_h)}(z, z) \\ &\leq \omega(-\delta) b_{\Omega_i^\delta, \omega(d_h)}(z, z). \end{aligned} \tag{40}$$

Thus, it remains to bound $b_{\Omega_i^\delta, \omega(d_h)}(z, z)$ in terms of the boundary data g. To shorten the notation, we write

$$\omega_{dz} = \omega(d_h) z. \tag{41}$$

Since $\mathcal{I}_h \omega_{dz} - g \in V_{h,0}(\Omega_i^\delta)$ is a valid test function for (36), we conclude that

$$\frac{1}{\lambda^2}\langle\!\langle z, \mathcal{I}_h \omega_{dz} \rangle\!\rangle_{\Omega_i^\delta} + a_{\Omega_i^\delta}(z, \mathcal{I}_h \omega_{dz}) = \frac{1}{\lambda^2}\langle\!\langle z, g \rangle\!\rangle_{\Omega_i^\delta} + a_{\Omega_i^\delta}(z, g) = b_{\Omega_i^\delta}(z, g). \tag{42}$$

Using this together with the product rule

$$\omega(d_h)\nabla z = \nabla \omega_{dz} - z \nabla \omega(d_h)\,.$$

yields

$$
\begin{aligned}
b_{\Omega_i^\delta,\omega(d_h)}(z,z) &= \frac{1}{\lambda^2}\langle\!\langle \omega_{dz}, z\rangle\!\rangle_{\Omega_i^\delta} + a_{\Omega_i^\delta}(\omega_{dz}, z) - \int_{\Omega_i^\delta} z\kappa^2 \nabla\omega(d_h)\cdot\nabla z \,\mathrm{d}x \\
&= \frac{1}{\lambda^2}\langle\!\langle \omega_{dz} - \mathcal{I}_h\omega_{dz}, z\rangle\!\rangle_{\Omega_i^\delta} + a_{\Omega_i^\delta}(\omega_{dz} - \mathcal{I}_h\omega_{dz}, z) + b_{\Omega_i^\delta}(g,z) \\
&\quad - \int_{\Omega_i^\delta} z\kappa^2 \nabla\omega(d_h)\cdot\nabla z \,\mathrm{d}x
\end{aligned} \tag{43}
$$

The first term vanishes because of (8). The third term can be estimated with Cauchy–Schwarz inequality by

$$
\left|b_{\Omega_i^\delta}(g,z)\right| \le b_{\Omega_i^\delta}(g,g)^{1/2} b_{\Omega_i^\delta}(z,z)^{1/2} \le b_{\Omega_i^\delta}(g,g)^{1/2} b_{\Omega_i^\delta,\omega(d_h)}(z,z)^{1/2},
$$

where we also used that $\omega(d_h) \ge 1$ on Ω_i^δ. The bounds on the remaining two terms are given in Lemmas A.1 and A.2 which we postponed to Appendix A because the proofs are rather technical. Plugging these bounds into (43) yields

$$
b_{\Omega_i^\delta,\omega(d_h)}(z,z)^{1/2} \le \gamma(C_1+C_2) b_{\Omega_i^\delta,\omega(d_h)}(z,z)^{1/2} + b_{\Omega_i^\delta}(g,g)^{1/2} \tag{44}
$$

with constants C_1, C_2 independent of γ, h and λ. Thus, by possibly reducing $\gamma \in (0,1]$, such that

$$
\beta^{-1/2} := 1 - \gamma\Big(C_1 + C_2\Big) > 0.
$$

This allows us to rearrange (44), which gives

$$
b_{\Omega_i^\delta,\omega(d_h)}(z,z)^{1/2} \le \beta^{1/2} b_{\Omega_i^\delta}(g,g)^{1/2} \tag{45}
$$

By setting together (40) and (45) we finally obtain

$$
b_{\Omega_i}(z,z) \le \beta\omega(-\delta) b_{\Omega_i^\delta}(g,g),
$$

which concludes the proof. □

We now set together Lemma 6.1 and Theorem 6.3 to apply the decay result to our setting.

Lemma 6.4. *Let the conditions from Lemma 6.1 and Theorem 6.3 be satisfied. Then,*

$$
\left\|x_i^n - \widetilde{x}_{\mathrm{CN}}^n\right\|_{X_h(\Omega_i)}^2 \le \beta \exp\Big(-\frac{\gamma\delta}{\max\{\frac{\tau}{2}, h\}}\Big)\left\|\widetilde{x}_{\mathrm{lf}}^n - \widetilde{x}_{\mathrm{CN}}^n\right\|_{X_h(\Omega_i^\delta)}^2.
$$

Proof. As in Lemma 6.1 we write

$$
z_i^n = x_i^n - \widetilde{x}_{\mathrm{CN}}^n\big|_{\Omega_i^\delta} = \begin{pmatrix} u_i^n - \widetilde{u}_{\mathrm{CN}}^n\big|_{\Omega_i^\delta} \\ v_i^n - \widetilde{v}_{\mathrm{CN}}^n\big|_{\Omega_i^\delta} \end{pmatrix} = \begin{pmatrix} z_{u,i}^n \\ z_{v,i}^n \end{pmatrix}.
$$

From (33b) it holds

$$
\left\|z_i^n\right\|_{X_h(\Omega_i)}^2 = \left\|z_{u,i}^n\right\|_{V_h(\Omega_i)}^2 + \left\|z_{v,i}^n\right\|_{H_h(\Omega_i)}^2 = \left\|z_{u,i}^n\right\|_{V_h(\Omega_i)}^2 + \frac{4}{\tau^2}\left\|z_{u,i}^n\right\|_{H_h(\Omega_i)}^2. \tag{46}
$$

The leapfrog steps (14a) and (14c) started from x_{DS}^{n-1} yield

$$
\widetilde{u}_{\mathrm{lf}}^n = u_{\mathrm{DS}}^{n-1} + \frac{\tau}{2}\big(v_{\mathrm{DS}}^{n-1} + \widetilde{v}_{\mathrm{lf}}^n\big)
$$

On the other hand, we have for the Crank–Nicolson method by (12a) that

$$
\widetilde{u}_{\mathrm{CN}}^n = u_{\mathrm{DS}}^{n-1} + \frac{\tau}{2}\big(v_{\mathrm{DS}}^{n-1} + \widetilde{v}_{\mathrm{CN}}^n\big).
$$

It follows

$$
\widetilde{u}_{\mathrm{lf}}^n - \widetilde{u}_{\mathrm{CN}}^n = \frac{\tau}{2}\big(\widetilde{v}_{\mathrm{lf}}^n - \widetilde{v}_{\mathrm{CN}}^n\big),
$$

and thus we have as in (46)

$$\left\|\widetilde{x}^n_{\mathrm{lf}} - \widetilde{x}^n_{\mathrm{CN}}\right\|^2_{X_h(\Omega_i^\delta)} = \left\|\widetilde{u}^n_{\mathrm{lf}} - \widetilde{u}^n_{\mathrm{CN}}\right\|^2_{V_h(\Omega_i^\delta)} + \frac{4}{\tau^2}\left\|\widetilde{u}^n_{\mathrm{lf}} - \widetilde{u}^n_{\mathrm{CN}}\right\|^2_{H_h(\Omega_i^\delta)}.$$

The stated bound then follows immediately by Theorem 6.3, where we set $g = \widetilde{u}^n_{\mathrm{lf}} - \widetilde{u}^n_{\mathrm{CN}}$ and $\lambda = \frac{\tau}{2}$. □

7. Averaging

In this section we investigate the stability of the averaging operator ζ defined in (16) and deduce some bound for the specific use of ζ within a single domain splitting step. The statement of the following lemma relies on the notation summarized in Table 1.

Lemma 7.1. *The averaging operator ζ satisfies the following stability estimate in the mass lumped L^2-norm*

(47a)
$$\left\|\zeta\left(\{v_i\}_{i=1,\dots,N}\right)\right\|^2_{H_h(\Omega)} \le c_{\mathrm{avg}} \sum_{i=1}^N \left\|v_i\right\|^2_{H_h(\Omega_i)}, \quad \text{for all } v_i \in H_h(\Omega_i^\delta),\ i = 1,\dots N,$$

with a constant $c_{\mathrm{avg}} > 0$ independent of h, κ, ℓ.

If x^n_{DS} is the CFL condition (18) is satisfied, it further holds

(47b)
$$\begin{aligned}\left\|x^n_{\mathrm{DS}} - \widetilde{x}^n_{\mathrm{CN}}\right\|^2_{X_h} &= \left\|\zeta\Big(\{x_i^n - \widetilde{x}^n_{\mathrm{CN}}|_{\Omega_i^\delta}\}_{i=1,\dots,N}\Big)\right\|^2_{X_h} \\ &\le C_{\mathrm{avg}} \sum_{i=1}^N \left\|x_i^n - \widetilde{x}^n_{\mathrm{CN}}\right\|^2_{X_h(\Omega_i)}.\end{aligned}$$

There, the constant C_{avg} may depend on ℓ, but it does not depend on h and τ.

Note that (47b) relies heavily on the specific structure of $x_i^n - \widetilde{x}^n_{\mathrm{CN}}|_{\Omega_i^\delta}$ from Lemma 6.1.

Proof. Note that, with $v_\Omega = \zeta\left(\{v_i\}_{i=1,\dots,N}\right)$ it holds, that

$$\left\|v_\Omega\right\|^2_{H_h(\Omega)} = \sum_{i=1}^N \left\|v_\Omega\right\|^2_{H_h(\Omega_i)} = \sum_{i=1}^N \sum_{K\in\mathcal{T}_h(\Omega_i)} \sum_{j=1}^{m+1} \frac{|K|}{m+1} v_\Omega(\boldsymbol{x}_{K,j})^2.$$

The stability bound (47a) follows directly by

$$v_\Omega(\boldsymbol{x}_{K,j})^2 = \left(\sum_{k\in J_{K,j}} \frac{v_k(\boldsymbol{x}_{K,j})}{|J_{K,j}|}\right)^2 \le \sum_{k\in J_{K,j}} \frac{v_k(\boldsymbol{x}_{K,j})^2}{|J_{K,j}|},$$

with $J_{K,j} = \left\{k \in \{1,\dots,N\} : \boldsymbol{x}_{K,j} \in \overline{\Omega}_k\right\}$ and the fact that the sizes of neighboring cells are uniformly equivalent w.r.t. h, see [11, Proposition 11.6].

To show (47b) we note that

$$\left\|\zeta\Big(\{x_i^n - \widetilde{x}^n_{\mathrm{CN}}|_{\Omega_i^\delta}\}_{i=1,\dots,N}\Big)\right\|^2_{X_h} = \sum_{i=1}^N \left\|\zeta\Big(\{x_i^n - \widetilde{x}^n_{\mathrm{CN}}|_{\Omega_i^\delta}\}_{i=1,\dots,N}\Big)\right\|^2_{X_h(\Omega_i)}.$$

Let $K \in \mathcal{T}_h$ be a cell of which all vertices $\{\boldsymbol{x}_{K,j}\}_{j=1}^{m+1}$ lie inside of one of these subdomains, say in Ω_k. For this cell the averaged solution coincides with the local solution, i.e.,

$$\zeta\Big(\{x_i^n - \widetilde{x}^n_{\mathrm{CN}}|_{\Omega_i^\delta}\}_{i=1,\dots,N}\Big)\Big|_K = \Big(x_k^n - \widetilde{x}^n_{\mathrm{CN}}|_{\Omega_k}\Big)\Big|_K. \tag{48}$$

This means we only have to focus on the remaining cells for which one of the vertices, say $\boldsymbol{x}_{K,r}$, lies on an interface, i.e. in

$$\boldsymbol{x}_{K,r} \in \Gamma = \bigcup_{i=1,\dots,N} \Gamma_i.$$

Furthermore, we define the set of all cells having at least one vertex on the interface by

(49)
$$\mathcal{T}_h(\Gamma) \coloneqq \big\{K \in \mathcal{T}_h \mid \boldsymbol{x}_{K,j} \in \mathcal{N} \setminus \big(\bigcup_{i=1,\dots,N} \mathcal{N}_{\Omega_i}\big) \text{ for at least one } j \in 1,\dots,m+1\big\}.$$

By Lemma 6.1 we have $u_i^n - \widetilde{u}_{\mathrm{CN}}^n = \frac{\tau}{2}\,(v_i^n - \widetilde{v}_{\mathrm{CN}}^n)$ in $H_h(\Omega_i^\delta)$, see (33b), and thus

$$(u_{\mathrm{DS}}^n - \widetilde{u}_{\mathrm{CN}}^n)\,(\boldsymbol{x}_{K,j}) = \frac{\tau}{2}\,(v_{\mathrm{DS}}^n - \widetilde{v}_{\mathrm{CN}}^n)\,(\boldsymbol{x}_{K,j}) \quad \text{for all } \boldsymbol{x}_{K,j} \in \mathcal{N}.,$$

For $K \in \mathcal{T}_h(\Gamma)$, we derive with the CFL condition (18) that

$$\begin{aligned}
\big\|x_{\mathrm{DS}}^n - \widetilde{x}_{\mathrm{CN}}^n\big\|_{X_h(K)}^2 &= \big\|u_{\mathrm{DS}}^n - \widetilde{u}_{\mathrm{CN}}^n\big\|_{V_h(K)}^2 + \frac{4}{\tau^2}\big\|u_{\mathrm{DS}}^n - \widetilde{u}_{\mathrm{CN}}^n\big\|_{H_h(K)}^2 \\
&\le \big\|L_h\big\|_{H_h \leftarrow H_h}\big\|u_{\mathrm{DS}}^n - \widetilde{u}_{\mathrm{CN}}^n\big\|_{H_h(K)}^2 + \frac{4}{\tau^2}\big\|u_{\mathrm{DS}}^n - \widetilde{u}_{\mathrm{CN}}^n\big\|_{H_h(K)}^2 \\
&\le (\ell^2+1)\frac{4}{\tau^2}\big\|u_{\mathrm{DS}}^n - \widetilde{u}_{\mathrm{CN}}^n\big\|_{H_h(K)}^2 \\
&= (\ell^2+1)\big\|v_{\mathrm{DS}}^n - \widetilde{v}_{\mathrm{CN}}^n\big\|_{H_h(K)}^2 .
\end{aligned}$$

We continue by using (47a) which implies that

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h(\Gamma)} \big\|v_{\mathrm{DS}}^n - \widetilde{v}_{\mathrm{CN}}^n\big\|_{H_h(K)}^2 &\le c_{\mathrm{avg}} \sum_{i=1}^N \sum_{\substack{K\in\mathcal{T}_h(\Gamma)\\ K\subset\overline{\Omega}_i}} \big\|v_i^n - \widetilde{v}_{\mathrm{CN}}^n\big\|_{H_h(K)}^2 \\
&\le c_{\mathrm{avg}} \sum_{i=1}^N \sum_{\substack{K\in\mathcal{T}_h(\Gamma)\\ K\subset\overline{\Omega}_i}} \big\|x_i^n - \widetilde{x}_{\mathrm{CN}}^n\big\|_{X_h(K)}^2 .
\end{aligned}$$

Putting these derivations together we obtain finally

$$\sum_{K\in\mathcal{T}_h(\Gamma)} \big\|x_{\mathrm{DS}}^n - \widetilde{x}_{\mathrm{CN}}^n\big\|_{X_h(K)}^2 \le C_{\mathrm{avg}} \sum_{i=1}^N \sum_{\substack{K\in\mathcal{T}_h(\Gamma)\\ K\subset\overline{\Omega}_i}} \big\|x_i^n - \widetilde{x}_{\mathrm{CN}}^n\big\|_{X_h(K)}^2 \tag{50}$$

with $C_{\mathrm{avg}} = c_{\mathrm{avg}}\big(\ell^2+1\big)$. However, by (48) we also know that

$$\sum_{K\in\mathcal{T}_h\setminus\mathcal{T}_h(\Gamma)} \big\|x_{\mathrm{DS}}^n - \widetilde{x}_{\mathrm{CN}}^n\big\|_{X_h(K)}^2 = \sum_{i=1}^N \sum_{\substack{K\in\mathcal{T}_h\setminus\mathcal{T}_h(\Gamma)\\ K\subset\overline{\Omega}_i}} \big\|x_i^n - \widetilde{x}_{\mathrm{CN}}^n\big\|_{X_h(K)}^2 .$$

Together with (50) this yields (47b). □

8. Global error

Recall the definitions of $x_{\mathrm{DS}}^n, x_i^n, x_{\mathrm{CN}}^n, \widetilde{x}_{\mathrm{CN}}^n$ and $\widetilde{x}_{\mathrm{lf}}^n$, which are summarized in Table 1. We now derive a bound for the difference E^n of the domain splitting approximation and the Crank–Nicolson approximation after n time steps, i.e.

$$E^n = \big\|x_{\mathrm{DS}}^n - x_{\mathrm{CN}}^n\big\|_{X_h} . \tag{51}$$

For that we will use the results of the previous sections, so recall the constants C_{data} from Lemma 5.2, M_ℓ from Lemma 5.3, β, γ from Theorem 6.3, and C_{avg} from Lemma 7.1.

To set everything together, we denote the maximum number of mutually overlapping subdomains by

$$\operatorname{card}\{i \mid x \in \Omega_i^\delta\} \leq C_{\text{glob}}, \qquad \text{for all } x \in \Omega. \tag{52}$$

Note that C_{glob} does not depend on the total number of subdomains N. This property is known as *bounded local overlap property*, see for instance [24].

Lemma 8.1. *Let $u^0, v^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f \in L^\infty([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$. Assume that the CFL condition (18) holds. Then, E^n defined in (51) satisfies*

$$E^n \leq \tau C_{\text{data}} D_{\tau,h,\delta} \sum_{j=0}^{n-1} (1 + D_{\tau,h,\delta})^j \tag{53}$$

with

$$D_{\tau,h,\delta}^2 = C_{\text{avg}} C_{\text{glob}} \beta \exp\Big(- \frac{\gamma\delta}{\max\{\frac{\tau}{2}, h\}} \Big) M_\ell^2. \tag{54}$$

Proof. Firstly, we use the triangle inequality to obtain

$$E^n = \big\| x_{\text{DS}}^n - x_{\text{CN}}^n \big\|_{X_h} \leq \big\| x_{\text{DS}}^n - \widetilde{x}_{\text{CN}}^n \big\|_{X_h} + \big\| \widetilde{x}_{\text{CN}}^n - x_{\text{CN}}^n \big\|_{X_h} = E_A^n + E_B^n.$$

Secondly, we use (47b) and then exploit the exponential decay in Lemma 6.4 and (52)

$$\begin{aligned}
(E_A^n)^2 = \big\| x_{\text{DS}}^n - \widetilde{x}_{\text{CN}}^n \big\|_{X_h}^2 &\leq C_{\text{avg}} \sum_{i=1}^N \big\| x_i^n - \widetilde{x}_{\text{CN}}^n \big\|_{X_h(\Omega_i)}^2 \\
&\leq C_{\text{avg}} \beta \exp\Big(- \frac{\gamma\delta}{\max\{\frac{\tau}{2}, h\}} \Big) \sum_{i=1}^N \big\| \widetilde{x}_{\text{lf}}^n - \widetilde{x}_{\text{CN}}^n \big\|_{X_h(\Omega_i^\delta)}^2 \\
&\leq C_{\text{glob}} C_{\text{avg}} \beta \exp\Big(- \frac{\gamma\delta}{\max\{\frac{\tau}{2}, h\}} \Big) \big\| \widetilde{x}_{\text{lf}}^n - \widetilde{x}_{\text{CN}}^n \big\|_{X_h}^2
\end{aligned}$$

Next, Lemma 5.3 yields

$$\big\| \widetilde{x}_{\text{lf}}^n - \widetilde{x}_{\text{CN}}^n \big\|_{X_h} \leq M_\ell \left(E^{n-1} + \tau C_{\text{data}} \right),$$

so that we obtain

$$(E_A^n)^2 \leq C_{\text{avg}} C_{\text{glob}} \beta \exp\Big(- \frac{\gamma\delta}{\max\{\frac{\tau}{2}, h\}} \Big) M_\ell^2 \left(E^{n-1} + \tau C_{\text{data}} \right)^2.$$

For term E_B^n we use that the Crank–Nicolson operator R defined in (21) is unitary, which implies

$$E_B^n = \big\| \widetilde{x}_{\text{CN}}^n - x_{\text{CN}}^n \big\|_{X_h} = \big\| R \left(x_{\text{DS}}^{n-1} - x_{\text{CN}}^{n-1} \right) \big\|_{X_h} = \big\| x_{\text{DS}}^{n-1} - x_{\text{CN}}^{n-1} \big\|_{X_h} = E^{n-1}.$$

Combining the estimates of E_A^n and E_B^n and obtain

$$\begin{aligned}
E^n &\leq D_{\tau,h,\delta} \left(E^{n-1} + \tau C_{\text{data}} \right) + E^{n-1} \\
&= (1 + D_{\tau,h,\delta}) E^{n-1} + \tau D_{\tau,h,\delta} C_{\text{data}}.
\end{aligned}$$

Resolving this inequality and using that $E^0 = \big\| x_{\text{DS}}^0 - x_{\text{CN}}^0 \big\|_{X_h} = 0$ proves the claim. ∎

8.1. **Proof of Lemma 4.1.** In this section, we complete the proof of the main result Theorem 4.2 which relies on Lemma 4.1.

Proof. By assumption, the overlap $\delta \sim \ell h$ is sufficiently large and hence there exists a constant $0 < \sigma \leq 1$ such that

$$D_{\tau,h,\delta} = \left(C_{\text{avg}} C_{\text{glob}} \beta \exp\Big(- \frac{\gamma \delta}{\max\{\frac{\tau}{2}, h\}} \Big) M_\ell^2 \right)^{1/2} \leq \sigma \tau^2, \tag{55}$$

where γ was introduced in Theorem 6.3. From (53) we obtain

$$E^n \leq \tau^2 C_{\text{data}} \sigma \tau \sum_{j=0}^{n-1} \big(1 + \sigma \tau^2\big)^j . \tag{56}$$

This allows us to derive two error bounds. Since $\tau \leq 1$, there holds

$$\sigma \tau \sum_{j=0}^{n-1} \big(1 + \sigma \tau^2\big)^j \leq \sigma \tau \sum_{j=0}^{n-1} (1 + \sigma \tau)^j \leq \mathrm{e}^{\sigma t_n} - 1$$

and

$$\sigma \tau \sum_{j=0}^{n-1} \big(1 + \sigma \tau^2\big)^j \leq \frac{1}{\tau} \big(\mathrm{e}^{\sigma \tau t_n} - 1\big) \leq \sigma t_n \mathrm{e}^{\sigma \tau t_n}.$$

This proves the lemma. □

8.2. **Discussion on the CFL condition.** Next, let us discuss the CFL condition (18) required for Theorem 4.2. Obviously, (18) becomes weaker if ℓ is chosen larger.

The second condition is that the overlap $\delta \sim \ell h$ has to be chosen sufficiently large such that (55) holds for a constant $\sigma \in (0, 1)$. This condition ensures, that the damping of the prediction error is strong enough. The condition can be satisfied for fixed τ, h by taking ℓ sufficiently large such that the condition on $\delta \sim h\ell$ above is satisfied.

9. Numerical experiments

After proving the theoretical results in the previous sections, we confirm and illustrate these with some numerical experiments. All experiments were implemented within the FEniCSx framework [1,28]. The code corresponding to the experiments in this section is made publicly available at

https://gitlab.kit.edu/tim.buchholz/ds-acoustic-wave.

In all graphs, solid lines refer to errors against the analytical solution, dashed lines refer to errors against the Crank–Nicolson solution.

9.1. **One-dimensional experiments.** Inspired by [26] we construct an analytical solution to the homogeneous wave equation on a one-dimensional domain $(0, 1) \subseteq \mathbb{R}$. We choose a function

$$\mu_{\xi,s}(z) = \mathbb{1}_{\{|z-\xi|<s\}} \sin\Big(\frac{z - (\xi + s)}{2s} \pi\Big)^3, \tag{57}$$

where $\mathbb{1}_S$ denotes the indicator function corresponding to a set S. Then, we define the initial conditions for (1) as

$$u^0(x) = \mu(x), \qquad v^0(x) = \partial_t \mu(x - t)\big|_{t=0} \tag{58}$$

with

$$\mu(z) = \mu_{\xi_1,s}(z) - \mu_{\xi_2,s}(z), \qquad \xi_1 = 0.55, \quad \xi_2 = 0.45, \quad s = 0.2.$$

In order to avoid over approximation effects, we perturbed a uniform mesh randomly, which resulted in a non-equidistant mesh with minimum mesh width

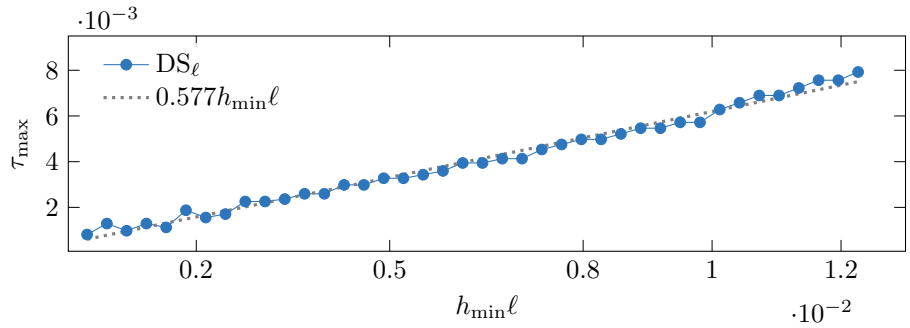

FIGURE 2. Maximal stable step sizes $\tau_{\max}$ of the domain splitting (DS_ℓ) depending on the overlap parameter ℓ.

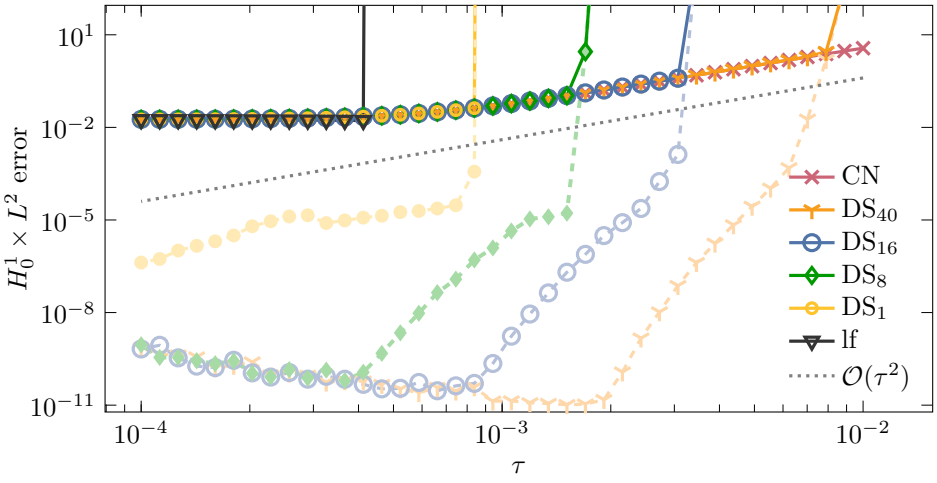

FIGURE 3. Errors of the domain splitting approximation (DS_ℓ) for the 1d example of Section 9.1 for various numbers of ℓ measured at the final time $T = 5.0$ against the exact solution (solid) and against the Crank–Nicolson approximation (dashed). For comparison, we also show the error of the Crank–Nicolson (CN) and the leapfrog (lf) methods on the full domain.

$h_{\min} = 3 \cdot 10^{-4}$ and maximum mesh width $h_{\max} = 6.9 \cdot 10^{-4}$ with 2000 subintervals). All domain splitting approximations use two subdomains and the index indicates the choice of the overlap parameter ℓ.

We start by illustrating the dependence of the CFL condition (18) on the overlap parameter ℓ in Figure 2. One can clearly see the linear dependence, confirming the theoretical result.

In Figure 3 we show the $H_0^1(\Omega) \times L^2(\Omega)$ errors measured at the final time $T = 5.0$ against the exact solution (which can be found in [26]). For comparison, we also show the error of Crank–Nicolson (CN) and the leapfrog (lf) methods on the full domain. Here, we observe that the error of DS_ℓ roughly coincides with the Crank–Nicolson error, if the time step satisfies the CFL condition. This is confirmed in Figure 3, where we illustrate Lemma 4.1 by plotting not only the error against the exact solution but also the error of the domain splitting method (DS_8) against the Crank–Nicolson solution on the full domain.

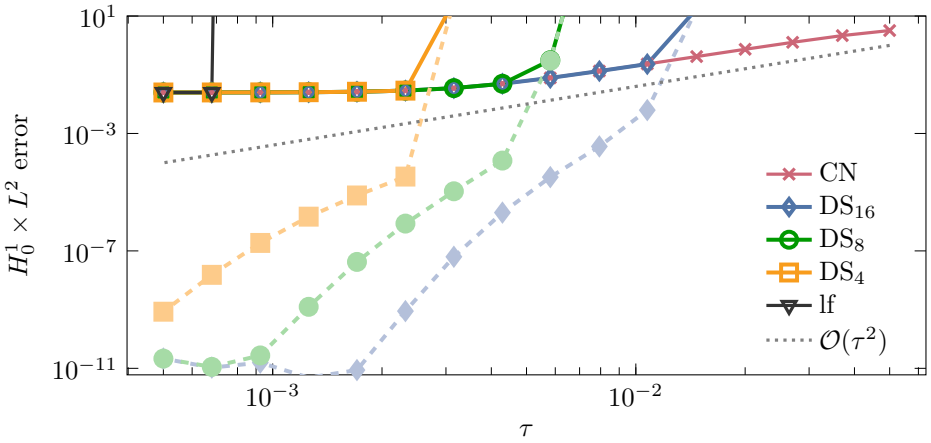

FIGURE 4. Errors of the domain splitting approximation (DS_ℓ) with 4×4 subdomains and $\ell = 8$ for the example from Section 9.2 measured at the final time $T = 1.0$ against the exact solution (solid) and against the Crank–Nicolson approximation (dashed). For comparison, we also show the error of the Crank–Nicolson (CN) and the leapfrog (lf) methods on the full domain.

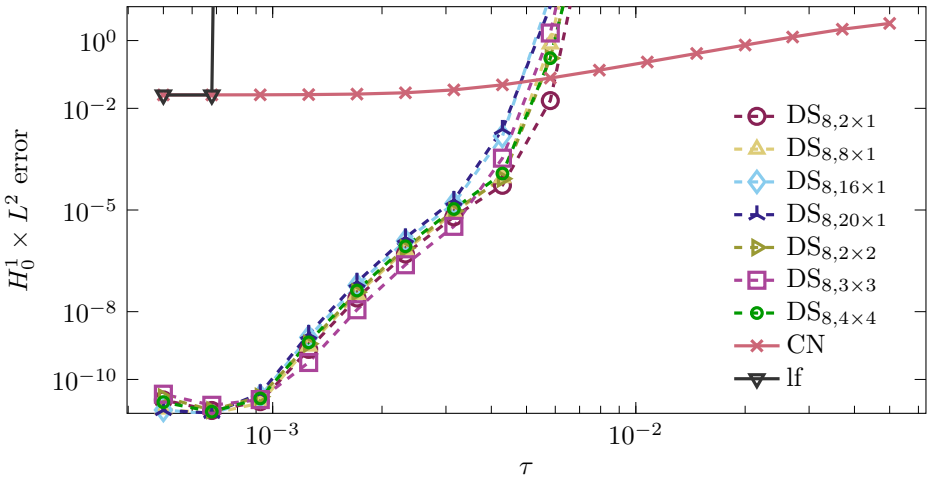

FIGURE 5. Errors of the domain splitting approximation against the Crank–Nicolson approximation on different, rectangular subdomain configurations. All runs were made with overlap parameter $\ell = 8$.

9.2. **Two-dimensional experiments.** Let $\Omega = (0,1)^2$, $\xi = 0.5$, $s = 0.2$, and Using (57), we prescribe the exact solution of (1) as

$$u(x,y,t) = u_{1\mathrm{D}}(x,t)\mu_{\xi,s}(y) + u_{1\mathrm{D}}(y,t)\mu_{\xi,s}(x),$$

where $u_{1\mathrm{D}}$ solves the homogeneous one-dimensional wave equation on $(0,1)$ with initial conditions (58) with $\mu = \mu_{\xi,s}$. Inserting this solution into the wave operator defines the inhomogeneity f in (1).

We discretize Ω by a regular, triangular mesh containing $2 \cdot 10^6$ cells with mesh width $h = 10^{-3}$. The error is measured at the final time $T = 1.0$. In Figure 4

the convergence against the exact solution is depicted for the leapfrog scheme, the Crank–Nicolson method and the domain splitting method with different choices of the overlap parameter ℓ. Additionally, the norm of the difference between the final domain splitting approximation and the Crank–Nicolson solution is measured. The results are in line with the one-dimensional example in Figure 3.

In Figure 5 compare the error of the domain splitting approximation against the Crank–Nicolson approximation on different, rectangular subdomain configurations comprising $N_x \times N_y$ subdomains in x and y-direction, respectively, for a fixed overlap parameter $\ell = 8$. The approximations are denoted by $\mathrm{DS}_{\ell,N_x\times N_y}$. While the error constants depend slightly on the number of neighboring subdomains, the stability is not affected by the number or configuration of subdomains as also stated by our error analysis.

Appendix A. Bounds of Theorem 6.3

With the notation introduced in Definition 6.2 and in the proof of Theorem 6.3 we now prove two Lemmas used there. Recall the definition of the constants C_{ML} and $C_{\mathcal{I}_h}$ from (6) and (9), respectively, and the definition of the weight function ω in (37). Let $|\cdot|_{2,K,\omega(-d_h)}$ denote the H^2 semi-norm, weighted by $\omega(-d_h)$, in the same manner as the bilinear forms from Definition 6.2. Moreover, $|\cdot|$ denotes the Euclidean norm when applied to a vector.

Lemma A.1. *Let $\gamma \in (0,1]$, $\lambda > 0$, $z \in V_h(\Omega_i^\delta)$. For $\omega_{dz} = \omega(d_h)z$ defined in (41) the interpolation error $\varepsilon_{dz} = \omega_{dz} - \mathcal{I}_h\omega_{dz}$ satisfies*

$$\left|a_{\Omega_i^\delta}(\varepsilon_{dz}, z)\right| \leq \gamma C_1 b_{\Omega_i^\delta,\omega(d_h)}(z,z),$$

with $C_1 = \kappa C_{\mathcal{I}_h}\sqrt{3m\mathrm{e}\cdot\max\{\mathrm{e}C_{\mathrm{ML}},2\}}$.

Proof. By definition of $a_{\Omega_i^\delta}(\cdot,\cdot)$ in (3), $a_{\Omega_i^\delta,\omega}(\cdot,\cdot)$ in (35a), the Cauchy–Schwarz inequality, and (39b) we obtain

$$\begin{aligned}\left|a_{\Omega_i^\delta}(\varepsilon_{dz}, z)\right| &= \left|\left(\kappa^2\nabla(\varepsilon_{dz}), \nabla z\right)_{\Omega_i^\delta}\right| \\ &= \left|\left(\kappa^2\omega(-d_h)^{1/2}\nabla(\varepsilon_{dz}), \omega(d_h)^{1/2}\nabla z\right)_{\Omega_i^\delta}\right| \\ &\leq a_{\Omega_i^\delta,\omega(-d_h)}(\varepsilon_{dz},\varepsilon_{dz})^{1/2} a_{\Omega_i^\delta,\omega(d_h)}(z,z)^{1/2}.\end{aligned} \tag{59}$$

We estimate the first factor via

$$\begin{aligned}a_{\Omega_i^\delta,\omega(-d_h)}(\varepsilon_{dz},\varepsilon_{dz}) &= \sum_{K\in\mathcal{T}_h(\Omega_i^\delta)} a_{K,\omega(-d_h)}(\varepsilon_{dz},\varepsilon_{dz}) \\ &\leq \sum_{K\in\mathcal{T}_h(\Omega_i^\delta)} \max_{x\in K}\omega(-d_h)\kappa^2 |\varepsilon_{dz}|_{1,K}^2 .\end{aligned} \tag{60}$$

Since $z\big|_K \in P_1$, for $K \in \mathcal{T}_h(\Omega_i^\delta)$ the function $\omega_{dz}\big|_K$ is smooth, so that the interpolation estimate (9) yields

$$|\varepsilon_{dz}|_{1,K}^2 \leq C_{\mathcal{I}_h}^2 h^2 |\omega_{dz}|_{2,K}^2 \leq \frac{C_{\mathcal{I}_h}^2 h^2}{\min_{x\in K}\omega(-d_h)} |\omega_{dz}|_{2,K,\omega(-d_h)}^2 . \tag{61}$$

Moreover, since $\omega(x)$ is monotonically increasing, there holds on $K \in \mathcal{T}_h(\Omega_i^\delta)$

$$\frac{\max_{x\in K}\omega(-d_h)}{\min_{x\in K}\omega(-d_h)} = \frac{\omega(-\min_{x\in K} d_h)}{\omega(-\max_{x\in K} d_h)} = \omega(\max_{x\in K} d_h - \min_{x\in K} d_h) \leq \omega(\mathrm{diam}K) \leq \omega(h). \tag{62}$$

By combining this with (60) and (61) we get

$$a_{\Omega_i^\delta,\omega(-d_h)}(\varepsilon_{dz},\varepsilon_{dz}) \le \kappa^2 C_{\mathcal{I}_h}^2 h^2 \omega(h) \sum_{K\in\mathcal{T}_h(\Omega_i^\delta)} |\omega_{dz}|_{2,K,\omega(-d_h)}^2 . \tag{63}$$

Since $\partial_r(\partial_j v_h\big|_K) = 0$ for $v_h \in P_1$ and $r,j = 1,\dots,m$, we get

$$\frac{\partial^2}{\partial_r\partial_j}\omega_{dz} = \frac{\gamma}{\max\lambda,h}\Big(\partial_r z\partial_j d_h + z\frac{\gamma}{\max\{\lambda,h\}}\partial_r d_h\partial_j d_h + \partial_j z\partial_r d_h\Big)\omega(d_h). \tag{64}$$

Note, that $d : x \mapsto \operatorname{dist}(x,\Gamma_i^\delta)$ satisfies

$$|\nabla d(\widehat{x})| = 1, \quad \text{for all } \widehat{x}\in\Omega_i^\delta,$$

cf. [4]. By definition of d_h in (38), we deduce with a geometric argument

$$|\partial_j d_h(\widehat{x})| \le |\nabla d_h(\widehat{x})| \le |\nabla d(\widehat{x})| = 1, \quad \text{for } j = 1,\dots,m \tag{65}$$

for all $\widehat{x}\in\Omega_i^\delta$, see also [2] after equation (24).

By combining (64) and (65) and using the linearity of the integral we have

$$\begin{aligned}\sum_{K\in\mathcal{T}_h(\Omega_i^\delta)} |\omega_{dz}|_{2,K,\omega(-d_h)}^2 &= \sum_{K\in\mathcal{T}_h(\Omega_i^\delta)} \int_K \omega(-d_h)\sum_{r,j=1}^m \left|\frac{\partial^2\omega_{dz}}{\partial_r\partial_j}\right|^2 \mathrm{d}x \\ &\le \frac{3m\gamma^2}{\max\{\lambda,h\}^4}\left(\gamma^2\,(\omega(d_h)z,z)_{\Omega_i^\delta} + 2\max\{\lambda,h\}^2 a_{\Omega_i^\delta,\omega(d_h)}(z,z)\right).\end{aligned} \tag{66}$$

Using (6), one can easily show, that

$$(\omega(d_h)u,u)_{\Omega_i^\delta} \le C_{\mathrm{ML}}\omega(h)\big(\omega(d_h)u,u\big)_{\Omega_i^\delta}, \qquad u\in V_h(\Omega_i^\delta). \tag{67}$$

Combining this with (66) yields

$$\sum_{K\in\mathcal{T}_h(\Omega_i^\delta)} |\omega_{dz}|_{2,K,\omega(-d_h)}^2 \le \frac{3m\gamma^2}{\max\{\lambda,h\}^2}\max\{\gamma^2 C_{\mathrm{ML}}\omega(h),2\} b_{\Omega_i^\delta,\omega(d_h)}(z,z). \tag{68}$$

Putting (63), (68) together yields an estimate for the first factor of (59) of the form

$$a_{\Omega_i^\delta,\omega(-d_h)}(\varepsilon_{dz},\varepsilon_{dz}) \le \kappa^2 C_{\mathcal{I}_h}^2 h^2\omega(h)\frac{3m\gamma^2}{\max\{\lambda,h\}^2}\max\{\gamma^2 C_{\mathrm{ML}}\omega(h),2\} b_{\Omega_i^\delta,\omega(d_h)}(z,z).$$

By (35b) and since $\frac{1}{\lambda^2}\big(\omega(d_h)z,z\big)_{\Omega_i^\delta} \ge 0$, we bound the second factor in (59) by

$$a_{\Omega_i^\delta,\omega(d_h)}(z,z) \le b_{\Omega_i^\delta,\omega(d_h)}(z,z).$$

In total, we receive the estimate

$$\left|a_{\Omega_i^\delta}(\varepsilon_{dz},z)\right| \le \kappa C_{\mathcal{I}_h}\omega(\frac{h}{2})\frac{\sqrt{3m}\gamma h}{\max\{\lambda,h\}}\max\{\gamma C_{\mathrm{ML}}^{1/2}\omega(\frac{h}{2}),\sqrt{2}\} b_{\Omega_i^\delta,\omega(d_h)}(z,z).$$

The claim follows by using that $\frac{h}{\max\{\lambda,h\}} \le 1$, $\gamma\le 1$, and thus $\omega(h/2) \le \mathrm{e}^{\gamma/2} \le \sqrt{\mathrm{e}}$. □

By similar arguments as in the proof of Lemma A.1 we can also derive a bound for the last remaining term in (43).

Lemma A.2. *Let the assumptions of Lemma A.1 hold. Then, we have*

$$\left|\int_{\Omega_i^\delta} z\kappa^2\nabla\omega(d_h)\cdot\nabla z\,\mathrm{d}x\right| \le \gamma C_2 b_{\Omega_i^\delta,\omega(d_h)}(z,z),$$

with $C_2 = \frac{1}{2}\kappa(C_{\mathrm{ML}}\mathrm{e})^{1/2}$.

Proof. By (65) we have $|\nabla d_h| \le 1$. Thus, with definition (37) of $\omega(\cdot)$ we derive

$$|\nabla\omega(d_h)| = \left|\nabla \exp\left(\frac{\gamma d_h}{\max\{\lambda,h\}}\right)\right| = \left|\frac{\gamma}{\max\{\lambda,h\}}\exp\left(\frac{\gamma d_h}{\max\{\lambda,h\}}\right)\nabla d_h\right| \le \frac{\gamma}{\max\{\lambda,h\}}\omega(d_h).$$

The desired bound then follows by using again the Cauchy–Schwarz inequality, which yields

$$\begin{aligned}\left|\int_{\Omega_i^\delta} z\kappa^2\nabla\omega(d_h)\cdot\nabla z\,\mathrm{d}x\right| &\le \kappa\int_{\Omega_i^\delta}|z|\,\frac{\gamma}{\max\{\lambda,h\}}\omega(d_h)\kappa\,|\nabla z|\,\mathrm{d}x\\ &\le \frac{\kappa\gamma}{\max\{\lambda,h\}}\,(\omega(d_h)z,z)_{\Omega_i^\delta}^{1/2}\,a_{\Omega_i^\delta,\omega(d_h)}(z,z)^{1/2}\\ &\le \gamma\frac{\kappa(C_{\mathrm{ML}}\mathrm{e}^\gamma)^{1/2}}{2}b_{\Omega_i^\delta,\omega(d_h)}(z,z).\end{aligned}$$

The claim follows then as in Lemma A.1 from $\mathrm{e}^\gamma \le \mathrm{e}$. □

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References

[1] Igor A. Baratta, Joseph P. Dean, Jørgen S. Dokken, Michal Habera, Jack S. Hale, Chris N. Richardson, Marie E. Rognes, Matthew W. Scroggs, Nathan Sime, and Garth N. Wells, *DOLFINx: the next generation FEniCS problem solving environment*, 2023.

[2] H. Blum, S. Lisky, and R. Rannacher, *A domain splitting algorithm for parabolic problems*, Computing **49** (1992), no. 1, 11–23. MR1182439

[3] Susanne C. Brenner and L. Ridgway Scott, *The mathematical theory of finite element methods*, Third, Texts in Applied Mathematics, vol. 15, Springer, New York, 2008. MR2373954

[4] Luis A. Caffarelli and Michael G. Crandall, *Distance functions and almost global solutions of eikonal equations*, Comm. Partial Differential Equations **35** (2010), no. 3, 391–414. MR2748630

[5] Philippe G. Ciarlet, *The finite element method for elliptic problems*, Studies in Mathematics and its Applications, vol. Vol. 4, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978. MR520174

[6] Gary Cohen and Sébastien Pernet, *Finite element and discontinuous Galerkin methods for transient wave equations*, Scientific Computation, Springer, Dordrecht, 2017. With a foreword by Patrick Joly. MR3526765

[7] Clint N. Dawson and Todd F. Dupont, *Explicit/implicit conservative Galerkin domain decomposition procedures for parabolic problems*, Math. Comp. **58** (1992), no. 197, 21–34. MR1106964

[8] ———, *Noniterative domain decomposition for second order hyperbolic problems*, Domain decomposition methods in science and engineering (Como, 1992), 1994, pp. 45–52. MR1262604

[9] Willy Dörfler, Marlis Hochbruck, Jonas Köhler, Andreas Rieder, Roland Schnaubelt, and Christian Wieners, *Wave phenomena: Mathematical analysis and numerical approximation*, Vol. 49, Springer Nature, 2023.

[10] Dow Drake, Jay Gopalakrishnan, Joachim Schöberl, and Christoph Wintersteiger, *Convergence analysis of some tent-based schemes for linear hyperbolic systems*, Math. Comp. **91** (2022), no. 334, 699–733. MR4379973

[11] Alexandre Ern and Jean-Luc Guermond, *Finite elements I—Approximation and interpolation*, Texts in Applied Mathematics, vol. 72, Springer, Cham, [2021] ©2021. MR4242224

[12] ———, *Finite elements II—Galerkin approximation, elliptic and mixed PDEs*, Texts in Applied Mathematics, vol. 73, Springer, Cham, [2021] ©2021. MR4269305

[13] Dietmar Gallistl and Roland Maier, *Localized implicit time stepping for the wave equation*, SIAM J. Numer. Anal. **62** (2024), no. 4, 1589–1608. MR4772556

[14] M. J. Gander, L. Halpern, and F. Nataf, *Optimal convergence for overlapping and non-overlapping Schwarz waveform relaxation*, Eleventh International Conference on Domain Decomposition Methods (London, 1998), 1999, pp. 27–36. MR1827406
[15] Martin J Gander and Laurence Halpern, *Techniques for locally adaptive time stepping developed over the last two decades*, Domain decomposition methods in science and engineering xx, 2013, pp. 377–385.
[16] Martin J. Gander, Laurence Halpern, and Frédéric Nataf, *Optimal Schwarz waveform relaxation for the one dimensional wave equation*, SIAM J. Numer. Anal. **41** (2003), no. 5, 1643–1681. MR2035001
[17] Martin J. Gander, Felix Kwok, and Bankim C. Mandal, *Dirichlet-Neumann waveform relaxation methods for parabolic and hyperbolic problems in multiple subdomains*, BIT **61** (2021), no. 1, 173–207. MR4235305
[18] Martin J. Gander and Hui Zhang, *Schwarz methods by domain truncation*, Acta Numer. **31** (2022), 1–134. MR4436585
[19] J. Gopalakrishnan, J. Schöberl, and C. Wintersteiger, *Mapped tent pitching schemes for hyperbolic systems*, SIAM J. Sci. Comput. **39** (2017), no. 6, B1043–B1063. MR3725284
[20] Jay Gopalakrishnan, Peter Monk, and Paulina Sepúlveda, *A tent pitching scheme motivated by Friedrichs theory*, Comput. Math. Appl. **70** (2015), no. 5, 1114–1135. MR3378991
[21] Laurence Halpern and Jérémie Szeftel, *Nonlinear nonoverlapping Schwarz waveform relaxation for semilinear wave propgation*, Math. Comp. **78** (2009), no. 266, 865–889. MR2476563
[22] David Hipp, *A unified error analysis for spatial discretizations of wave-type equations with applications to dynamic boundary conditions*, Dissertation, 2017.
[23] David Hipp, Marlis Hochbruck, and Christian Stohrer, *Unified error analysis for nonconforming space discretizations of wave-type equations*, IMA J. Numer. Anal. **39** (2019), no. 3, 1206–1245. MR3984056
[24] M. Holst, *Application of domain decomposition and partition of unity methods in physics and geometry*, Domain decomposition methods in science and engineering, 2003, pp. 63–78. MR2093735
[25] Patrick Joly, *Variational methods for time-dependent wave propagation problems*, Topics in computational wave propagation, 2003, pp. 201–264. MR2032871
[26] Qiong-Gui Lin, *Yet another approach to solutions of one-dimensional wave equations with inhomogeneous boundary conditions*, American Journal of Physics **90** (January 2022), no. 1, 31–36.
[27] P.-A. Raviart, *The use of numerical integration in finite element methods for solving parabolic equations*, Topics in numerical analysis (Proc. Roy. Irish Acad. Conf., University Coll., Dublin, 1972), 1973, pp. 233–264. MR345428
[28] Matthew W. Scroggs, Jørgen S. Dokken, Chris N. Richardson, and Garth N. Wells, *Construction of arbitrary order finite element degree-of-freedom maps on polygonal and polyhedral cell meshes*, ACM Transactions on Mathematical Software **48** (2022), no. 2, 18:1–18:23.
[29] Vidar Thomée, *Galerkin finite element methods for parabolic problems*, Springer Series in Computational Mathematics, vol. 25, Springer-Verlag, Berlin, 1997. MR1479170

Institute for Applied and Numerical Mathematics, Karlsruhe Institute of Technology, Englerstr. 2, 76131 Karlsruhe, Germany
Email address: {tim.buchholz,marlis.hochbruck}@kit.edu