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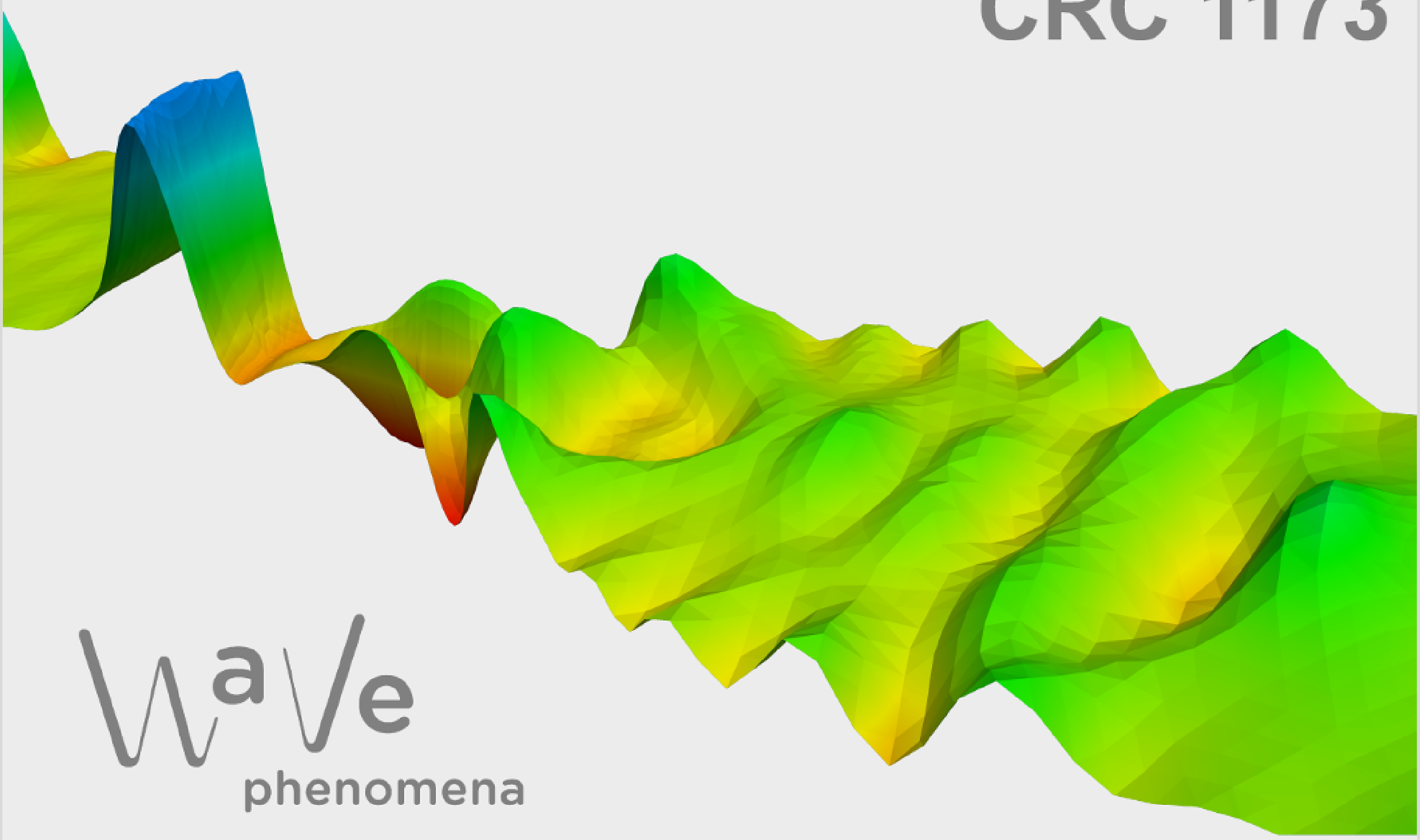

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From the Zakharov system to the NLS equation on the torus

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Abstract

The NLS equation can be derived from the Zakharov system in a singular limit. The goal of this paper is to add some new aspects to the existing analysis about this approximation. We outline the differences between the situation $x \in \mathbb{R}$ and $x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. We explain the difficulties occurring in the construction of higher order approximations and use these approximations to improve the approximation rate. Moreover, we point out that the standard validity proof requires a smallness condition which was not stated in the existing literature.

1 Introduction

We are interested in singular limits of the Klein-Gordon-Zakharov (KGZ) system or of systems related to the KGZ system. In this paper we consider the singular limit of the Zakharov system in which the NLS equation is obtained as a regular limit system. Our goal is to estimate the distance between the solutions obtained through the regular limit system and true solutions of the Zakharov system for small values of the perturbation parameter $0 < \varepsilon \ll 1$. In detail, we are interested in the Zakharov system in the form

$$2i\partial_t u = \partial_x^2 u - uv, \tag{1}$$

$$\varepsilon^2 \partial_t^2 v = \partial_x^2 v + \partial_x^2 |u|^2, \tag{2}$$

with $u = u(x,t) \in \mathbb{C}$, $v = v(x,t), x, t \in \mathbb{R}$ for spatially 2π-periodic boundary conditions in the singular limit $\varepsilon \to 0$. In this limit we first obtain $v = -|u|^2$ and then the NLS equation

$$2i\partial_t u = \partial_x^2 u + |u|^2 u, \tag{3}$$

with spatially 2π-periodic boundary conditions, i.e., $x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. It is the goal of this paper to give a proof of

Theorem 1.1. *There is a $C_{max} > 0$ such that for all $C_u \in [0, C_{max})$ the following holds. Let $u_0 \in C([0, T_0], H^6(\mathbb{T}, \mathbb{C}))$ be a solution of (3) satisfying*

$$\sup_{t\in[0,T_0]} \|u_0(\cdot, t)\|_{H^6} = C_u < \infty.$$

Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions $(u, v) \in C([0, T_0], H^1(\mathbb{T}, \mathbb{C}) \times L^2(\mathbb{T}, \mathbb{C}))$ of (1)-(2) satisfying

$$\sup_{t\in[0,T_0]} \|(u, v)(\cdot, t) - (u_0, -|u_0|^2)(\cdot, t)\|_{H^1\times L^2} \le C\varepsilon^2.$$

Such estimates have already been shown in [AA88, SW86] for $x \in \mathbb{R}^d$ with $d = 1, 2, 3$ with the help of energy estimates in order to study the asymptotic behavior of the solutions of the Zakharov system (1)-(2) when ε goes to zero. We remark that the approximation is only valid if the nonlinear part on the right-hand side of (1) has a negative sign. For a positive sign there exists a counter example which shows that the NLS approximation fails to make correct predictions about the dynamics of the Zakharov system, cf. [BSSZ20].

The goal of this paper is to add some new aspects to the existing analysis about the validity of this approximation. In detail, we outline the differences between the situation $x \in \mathbb{R}^d$ and $x \in \mathbb{T}^d$, explain the difficulties occurring in the construction of higher order approximations, and use these approximations to improve the approximation rate. Our energy estimates follow the ones of [AA88]. However, we point out that the standard validity proof requires a smallness condition which was not stated in the existing literature. It appears when estimating the Sobolev norms in terms of our energy, cf. Lemma 3.1.

Before we do so, we make a number of remarks about the occurrence and properties of the Zakharov system.

Remark 1.2. The Zakharov system was introduced by Zakharov ([Zak72]) to describe the propagation of Langmuir waves in an ionized plasma via the electric field u and the deviation v of the ions' equilibrium density. It can be derived directly from and justified for Maxwell's equation coupled with Euler's equation, cf. [Tex07]. For an overview about the significance of the Zakharov system we refer to the recent textbook [GGKZ16].

Remark 1.3. The Zakharov system can be rewritten as a semilinear evolutionary system for which local existence and uniqueness of solutions in Sobolov spaces can be established by using semigroup theory, cf. [OT92]. Going back to the original variables for the Zakharov system (1)-(2), there is local existence and uniqueness for $(u, v, \partial_t v) \in H^{s+2} \times H^{s+1} \times H^s$ for $s \geq 0$.

Remark 1.4. Estimating the solutions of (1)-(2) is a non-trivial task. This can be seen by writing (1)-(2) as

$$2i\partial_t u = \partial_x^2 u - uv, \qquad \partial_t v = \varepsilon^{-1} w, \qquad \partial_t w = \varepsilon^{-1}\partial_x^2 v + \varepsilon^{-1}\partial_x^2 |u|^2.$$

Hence the right-hand sides of the equations for v and w are of order $\mathcal{O}(\varepsilon^{-1})$ and so a direct application of Gronwall's inequality only gives estimates on a time interval of length $\mathcal{O}(\varepsilon)$. For details, see the discussion in [BSSZ20]. The error estimates on the time interval of length $\mathcal{O}(1)$ can only be achieved by a suitably chosen energy and require opposite signs in front of the nonlinear terms in the u and v equation (1)-(2). As already said above, in case of non-opposite signs in front of these nonlinear terms there exists a counter example which shows that the NLS approximation fails to make correct predictions about the dynamics of the Zakharov system, cf. [BSSZ20]. The fact that the signs of the nonlinear terms play a role for the approximation property immediately shows that the NLS approximation theory, started with [Kal88], developed for the NLS description of modulated wave packets cannot apply here.

Remark 1.5. The local existence and uniqueness of solutions $u \in H^s$, $s \geq 1$ of the NLS equation (3) is well known. It follows by using semigroup theory and a standard fixed point argument applied to the variation of constant formula.

The approximation theorem is proven by energy estimates and Gronwall's inequality. Before we do so, in the next section we bound the residual terms appearing for the Zakharov system. In Section 4 we discuss possible improvements of the result. In Section 5 we summarize our results and give an outlook to possible future applications.

Notation. Many possible different constants are denoted with the same symbol C if they can be chosen independently of the small perturbation parameter $0 < \varepsilon \ll 1$. In the following integration by parts all boundary terms vanish due to the periodic boundary conditions.

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2 Estimates for the residual

For the proof of Theorem 1.1 we use an improved approximation. Inserting the extended ansatz

$$\psi_u(x,t) = u_0(x,t), \qquad \psi_v(x,t) = v_0(x,t) + \varepsilon^2 v_2(x,t) \tag{4}$$

into the Zakharov system gives at ε^0 that

$$2i\partial_t u_0 = \partial_x^2 u_0 - u_0 v_0, \qquad 0 = \partial_x^2 v_0 + \partial_x^2(|u_0|^2),$$

and at ε^2 that

$$\partial_t^2 v_0 = \partial_x^2 v_2.$$

The residual of (1)-(2) is given by

$$\begin{aligned} \mathrm{Res}_u(u,v) &= -2i\partial_t u + \partial_x^2 u - uv, \\ \mathrm{Res}_v(u,v) &= -\varepsilon^2 \partial_t^2 v + \partial_x^2 v + \partial_x^2 |u|^2 \end{aligned}$$

and contains all terms which do not cancel after inserting the approximation into the Zakharov system. We choose $v_0 = -|u_0|^2$ and then u_0 to satisfy the NLS equation

$$2i\partial_t u_0 = \partial_x^2 u_0 + u_0|u_0|^2. \tag{5}$$

By this choice the residual will be of order $\mathcal{O}(\varepsilon^2)$ which turns out not to be sufficient for our proof of our approximation result stated in Theorem 1.1. Therefore, we additionally set

$$\widehat{v}_2(k,t) = -k^{-2}\partial_t^2 \widehat{v}_0(k,t) \tag{6}$$

for $k \in \mathbb{Z}\setminus\{0\}$ which finally brings the residual Res_v from $\mathcal{O}(\varepsilon^2)$ to $\mathcal{O}(\varepsilon^4)$. In order to have v_2 well-defined, due to the periodic boundary conditions, it is sufficient to show that the mean value of v_0 is conserved. This holds due to

$$\partial_t \int_{\mathbb{T}} v_0 \,\mathrm{d}x = -\partial_t \int_{\mathbb{T}} |u_0|^2 \,\mathrm{d}x = 0$$

which is the conservation of the L^2-norm for the solutions of the NLS equation. Therefore, we define $\widehat{v}_2(0,t) = \widehat{v}_2(0,0)$. Then by construction all $\partial_x^{-n}\partial_t^m v_2$ for $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ are well-defined and have a vanishing mean value.

Remark 2.1. Defining and removing the mean value of a function $v_2 \in L^2(\mathbb{R},\mathbb{R})$ is impossible. Nevertheless, the function v_2 can also be well defined if $x \in \mathbb{T}$ is replaced by $x \in \mathbb{R}$ since

$$\partial_t^2 v_0 = \frac{1}{4}\left(\partial_x^4(|u_0|^2) - 4\partial_x^2(\partial_x u_0 \partial_x \overline{u_0}) + \partial_x^2(|u_0|^4)\right)$$

respectively

$$v_2 = \frac{1}{4}\left(\partial_x^2(|u_0|^2) - 4|\partial_x u_0|^2 + |u_0|^4\right) = v_2^*(u_0).$$

If ψ_u and ψ_v are defined as in (4), we find for the residual that

$$\mathrm{Res}_u(\psi_u, \psi_v) = -\varepsilon^2 u_0 v_2, \qquad \mathrm{Res}_v(\psi_u, \psi_v) = -\varepsilon^4 \partial_t^2 v_2.$$

Thus, we can directly conclude the following lemma.

Lemma 2.2. *Let $s \geq 0$ and let $u_0 \in C([0,T_0], H^{s+6}(\mathbb{T}, \mathbb{C}))$ be a solution of the NLS equation* (3)*. Then there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0]} \|\mathrm{Res}_u(\psi_u, \psi_v)\|_{H^{s+4}} \leq C_{res}\varepsilon^2, \qquad \sup_{t\in[0,T_0]} \|\mathrm{Res}_v(\psi_u, \psi_v)\|_{H^s} \leq C_{res}\varepsilon^4.$$

Proof. In order to estimate $\partial_t^2 v_2$ in H^s we can use the representation of v_2 in terms of v_0 which can be found in Remark 2.1 and the NLS equation to express time derivatives of v_0 by space derivatives of u_0. Thus, the function has to be in H^{s+6}. The rest of the proof is straightforward. □

In the equations for the error not only the residual appears but also $\partial_x^{-1}\mathrm{Res}_v$. Hence we have to estimate the term $\varepsilon^4 \partial_x^{-1}\partial_t^2 v_2$, too. As above, due to the periodic boundary conditions, it would be sufficient to prove that the mean value of v_2 is conserved in order to have the term $\partial_x^{-1}\partial_t^2 v_2$ bounded in some Sobolev space on the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ but this we already proved above. Therefore, we have

Lemma 2.3. *Let $s \geq 0$ and let $u_0 \in C([0,T_0], H^{s+6}(\mathbb{T}, \mathbb{C}))$ be a solution of the NLS equation* (3)*. Then there exist $\varepsilon_0 > 0$ and $C_{res} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0]} \|\partial_x^{-1}\mathrm{Res}_v(\psi_u, \psi_v)\|_{L^2} = \sup_{t\in[0,T_0]} \varepsilon^4 \|\partial_x^{-1}\partial_t^2 v_2\|_{H^s} \leq C_{res}\varepsilon^4.$$

Remark 2.4. For $x \in \mathbb{R}$ a serious difficulty occurs at that point. In this case we have to choose $v_2 = v_2^*(u_0)$ from Remark 2.1 for which however we find

$$\partial_t \int_{\mathbb{R}} v_2 \,\mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}} \partial_x v_0 \ \mathrm{Im}(u_0 \partial_x \overline{u_0}) \,\mathrm{d}x \neq 0$$

after a straightforward calculation. However, in space dimensions $d \geq 3$ one may use that $\Delta^{-1} : L^2 \cap L^1 \to L^2$ is a bounded operator and that the nonlinear terms will be in L^1 due to the Cauchy-Schwarz inequality.

3 Estimates for the error

We introduce the error, made by the improved approximation (ψ_u, ψ_v) defined in (4), by

$$(u, v)(x, t) = (\psi_u, \psi_v)(x, t) + \varepsilon^2 (R_u, R_v)(x, t).$$

The error functions R_u and R_v satisfy

$$
\begin{aligned}
2i\partial_t R_u &= \partial_x^2 R_u - \psi_u R_v - \psi_v R_u - \varepsilon^2 R_u R_v + \varepsilon^{-2}\mathrm{Res}_u(\psi_u,\psi_v), &(7)\\
\varepsilon^2\partial_t^2 R_v &= \partial_x^2 R_v + \partial_x^2(\overline{\psi_u}R_u) + \partial_x^2(\psi_u\overline{R_u}) + \varepsilon^2\partial_x^2|R_u|^2 + \varepsilon^{-2}\mathrm{Res}_v(\psi_u,\psi_v). &(8)
\end{aligned}
$$

As already explained in Remark 1.4 bounds on the long $\mathcal{O}(1)$ time interval are a non-trivial task and require a suitable chosen energy. We follow [AA88] and multiply the first equation with $-i\overline{R_u}$ and integrate this equation w.r.t. x. Since ψ_v and R_v are real-valued we have

$$
\mathrm{Re}\int_{\mathbb{T}} i\overline{R_u}\psi_v R_u \,\mathrm{d}x = 0, \qquad \mathrm{Re}\int_{\mathbb{T}} i\overline{R_u}R_u R_v \,\mathrm{d}x = 0.
$$

Therefore, adding the complex conjugate gives

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|R_u\|_{L^2}^2 = \mathrm{Re}\int_{\mathbb{T}} i\overline{R_u}\psi_u R_v \,\mathrm{d}x - \mathrm{Re}\int_{\mathbb{T}} i\overline{R_u}\varepsilon^{-2}\mathrm{Res}_u(\psi_u,\psi_v)\,\mathrm{d}x.
$$

We multiply the first equation with $\partial_t\overline{R_u}$ and integrate this equation w.r.t. x. Adding the complex conjugate gives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x R_u\|_{L^2}^2 &= -\int_{\mathbb{T}}(\psi_u R_v\partial_t\overline{R_u} + \overline{\psi_u}R_v\partial_t R_u)\,\mathrm{d}x\\
&\quad -\int_{\mathbb{T}}(\psi_v R_u\partial_t\overline{R_u} + \psi_v\overline{R_u}\partial_t R_u)\,\mathrm{d}x\\
&\quad -\varepsilon^2\int_{\mathbb{T}}(R_v R_u\partial_t\overline{R_u} + R_v\overline{R_u}\partial_t R_u)\,\mathrm{d}x\\
&\quad +2\mathrm{Re}\int_{\mathbb{T}}\partial_t\overline{R_u}\varepsilon^{-2}\mathrm{Res}_u(\psi_u,\psi_v)\,\mathrm{d}x.
\end{aligned}
$$

Multiplying the second equation with $\partial_x^{-2}\partial_t R_v$ and integrating then w.r.t. x yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|R_v\|_{L^2}^2 + \frac{1}{2}\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{-1}\partial_t R_v\|_{L^2}^2\\
&= -\int_{\mathbb{T}}(\partial_t R_v)\overline{\psi_u}R_u\,\mathrm{d}x - \int_{\mathbb{T}}(\partial_t R_v)\psi_u\overline{R_u}\,\mathrm{d}x - \varepsilon^2\int_{\mathbb{T}}|R_u|^2\partial_t R_v\,\mathrm{d}x\\
&\quad + \int_{\mathbb{T}}(\partial_x^{-1}\partial_t R_v)\partial_x^{-1}(\varepsilon^{-2}\mathrm{Res}_v(\psi_u,\psi_v))\,\mathrm{d}x.
\end{aligned}
$$

Adding these equations gives

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}(\|R_u\|_{L^2}^2 + \|\partial_x R_u\|_{L^2}^2 + \frac{1}{2}\|R_v\|_{L^2}^2 + \frac{1}{2}\varepsilon^2\|\partial_x^{-1}\partial_t R_v\|_{L^2}^2) \\
= & \ \mathrm{Re}\int_{\mathbb{T}} i\overline{R_u}\psi_u R_v \,\mathrm{d}x - \mathrm{Re}\int_{\mathbb{T}} i\overline{R_u}\varepsilon^{-2}\mathrm{Res}_u(\psi_u,\psi_v)\,\mathrm{d}x + 2\mathrm{Re}\int_{\mathbb{T}} \partial_t\overline{R_u}\varepsilon^{-2}\mathrm{Res}_u(\psi_u,\psi_v)\,\mathrm{d}x \\
& + \int_{\mathbb{T}} (\partial_x^{-1}\partial_t R_v)(\varepsilon^{-2}\mathrm{Res}_v(\psi_u,\psi_v))\,\mathrm{d}x \\
& - \int_{\mathbb{T}} (\psi_u R_v \partial_t \overline{R_u} + \overline{\psi_u} R_v \partial_t R_u)\,\mathrm{d}x \\
& - \int_{\mathbb{T}} (\psi_v R_u \partial_t \overline{R_u} + \psi_v \overline{R_u} \partial_t R_u)\,\mathrm{d}x - \varepsilon^2\int_{\mathbb{T}} (R_v R_u \partial_t \overline{R_u} + R_v \overline{R_u}\partial_t R_u)\,\mathrm{d}x \\
& - \int_{\mathbb{T}} (\partial_t R_v)\overline{\psi_u} R_u \,\mathrm{d}x - \int_{\mathbb{T}} (\partial_t R_v)\psi_u \overline{R_u}\,\mathrm{d}x - \varepsilon^2\int_{\mathbb{T}} |R_u|^2 \partial_t R_v \,\mathrm{d}x = s.
\end{aligned}
$$

Keeping the first two lines and rewriting the following lines as total time derivative plus terms with the time derivative falling on approximation terms yields

$$
\begin{aligned}
s = & \ \mathrm{Re}\int_{\mathbb{T}} i\overline{R_u}\psi_u R_v \,\mathrm{d}x - \mathrm{Re}\int_{\mathbb{T}} i\overline{R_u}\varepsilon^{-2}\mathrm{Res}_u(\psi_u,\psi_v)\,\mathrm{d}x + 2\mathrm{Re}\int_{\mathbb{T}} \partial_t\overline{R_u}\varepsilon^{-2}\mathrm{Res}_u(\psi_u,\psi_v)\,\mathrm{d}x \\
& + \int_{\mathbb{T}} (\partial_x^{-1}\partial_t R_v)\partial_x^{-1}(\varepsilon^{-2}\mathrm{Res}_v(\psi_u,\psi_v))\,\mathrm{d}x - \varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} R_v|R_u|^2\,\mathrm{d}x \\
& - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \psi_v |R_u|^2\,\mathrm{d}x + \int_{\mathbb{T}} (\partial_t\psi_v)|R_u|^2\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \psi_u R_v \overline{R_u}\,\mathrm{d}x + \int_{\mathbb{T}} (\partial_t\psi_u) R_v\overline{R_u}\,\mathrm{d}x \\
& - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \overline{\psi_u} R_v R_u\,\mathrm{d}x + \int_{\mathbb{T}} (\partial_t\overline{\psi_u}) R_v R_u\,\mathrm{d}x.
\end{aligned}
$$

We collect all terms in a form of a time derivative in

$$
\begin{aligned}
E = & \ \|R_u\|_{L^2}^2 + \|\partial_x R_u\|_{L^2}^2 + \frac{1}{2}\|R_v\|_{L^2}^2 + \frac{1}{2}\varepsilon^2\|\partial_x^{-1}\partial_t R_v\|_{L^2}^2 + \int_{\mathbb{T}} \psi_v|R_u|^2\,\mathrm{d}x \\
& + \int_{\mathbb{T}} \psi_u R_v\overline{R_u}\,\mathrm{d}x + \int_{\mathbb{T}} \overline{\psi_u} R_v R_u\,\mathrm{d}x + \varepsilon^2\int_{\mathbb{T}} R_v|R_u|^2\,\mathrm{d}x.
\end{aligned}
$$

We have the following estimate

Lemma 3.1. *There is a $C_{max} > 0$ such that for all $C_u \in [0, C_{max})$ the following holds. Let $u_0 \in C([0,T_0], H^6(\mathbb{T},\mathbb{C}))$ be a solution of (3) satisfying*

$$\sup_{t\in[0,T_0]} \|u_0(\cdot,t)\|_{H^6} = C_u < \infty.$$

Then for all $C_R > 0$ there exist $C_2, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have

$$\|R_u\|_{H^1} + \|R_v\|_{L^2} \le C_2 E^{1/2},$$

if $\|R_u\|_{H^1} + \|R_v\|_{L^2} \le C_R$.

Remark 3.2. For rewriting the terms as time derivatives it is absolut fundamental that the nonlinear terms in (1)-(2) have opposite signs. In case that the nonlinear terms in (1)-(2) have the same sign such a rewriting is not possible and in fact there exists a counter example which shows that the NLS approximation fails to make correct predictions about the dynamics of the Zakharov system, cf. [BSSZ20]. However, it is not necessary to write the term

$$-\int_{\mathbb{T}} (\psi_v R_u \partial_t \overline{R_u} + \psi_v \overline{R_u} \partial_t R_u) \,\mathrm{d}x$$

as time derivative plus terms with less derivatives.

Rewriting the above calculations and estimating the non-time derivatives yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E \quad \leq \quad & \|\psi_u\|_{L^\infty} \|R_u\|_{L^2} \|R_v\|_{L^2} + \varepsilon^{-2} \|R_u\|_{L^2} \|\mathrm{Res}_u(\psi_u, \psi_v)\|_{L^2} \\ & + 2\mathrm{Re} \left| \int_{\mathbb{T}} \partial_t \overline{R_u} \varepsilon^{-2} \mathrm{Res}_u(\psi_u, \psi_v) \,\mathrm{d}x \right| + \|\partial_x^{-1} \partial_t R_v\|_{L^2} \|\partial_x^{-1} (\varepsilon^{-2} \mathrm{Res}_v(\psi_u, \psi_v))\|_{L^2} \\ & + 2\|\partial_t \psi_u\|_{L^\infty} \|R_v\|_{L^2} \|R_u\|_{L^2} + \|\partial_t \psi_v\|_{L^\infty} \|R_u\|_{L^2}^2. \end{aligned}$$

In order to estimate $\mathrm{Re} \left| \int_{\mathbb{T}} \partial_t \overline{R_u} \varepsilon^{-2} \mathrm{Res}_u \,\mathrm{d}x \right|$ we substitute $\partial_t \overline{R_u}$ by the right-hand side of (7). Integration by parts together with Lemma 3.1 finally gives

$$\begin{aligned} & \left| \mathrm{Re} \int_{\mathbb{T}} \partial_t R_u \varepsilon^{-2} \overline{\mathrm{Res}_u}(\psi_u, \psi_v) \,\mathrm{d}x \right| \\ & \quad = \left| \mathrm{Re} \int_{\mathbb{T}} (\partial_x^2 R_u - \psi_u R_v - \psi_v R_u - \varepsilon^2 R_u R_v + \varepsilon^{-2} \mathrm{Res}_u(\psi_u, \psi_v)) \varepsilon^{-2} \overline{\mathrm{Res}_u(\psi_u, \psi_v)} \,\mathrm{d}x \right| \\ & \quad \leq \varepsilon^{-2} \|R_u\|_{H^1} \|\mathrm{Res}_u(\psi_u, \psi_v)\|_{H^1} + \varepsilon^{-2} \|\psi_u\|_{L^\infty} \|R_v\|_{L^2} \|\mathrm{Res}_u(\psi_u, \psi_v)\|_{L^2} \\ & \qquad + \varepsilon^{-2} \|\psi_v\|_{L^\infty} \|R_u\|_{L^2} \|\mathrm{Res}_u(\psi_u, \psi_v)\|_{L^2} \\ & \qquad + \|R_u\|_{H^1} \|R_v\|_{L^2} \|\mathrm{Res}_u(\psi_u, \psi_v)\|_{L^2} + \varepsilon^{-4} \|\mathrm{Res}_u(\psi_u, \psi_v)\|_{L^2}^2 \\ & \quad \leq C_0 E^{1/2} + C_{res} \varepsilon^2 E + C_{res}^2. \end{aligned}$$

It is straightforward how to estimate $\|R_v\|_{L^2}$ and $\|R_u\|_{H^1}$ in terms of $E^{1/2}$. Less obvious is the estimate

$$\|\partial_x^{-1} \partial_t R_v\|_{L^2} \|\partial_x^{-1} (\varepsilon^{-2} \mathrm{Res}_v)\|_{L^2} \leq \varepsilon^2 \|\partial_x^{-1} \partial_t R_v\|_{L^2}^2 + \varepsilon^{-6} \|\partial_x^{-1} \mathrm{Res}_v\|_{L^2}^2 \leq E + C_{res} \varepsilon^2.$$

Hence, by using $E^{1/2} \leq 1 + E$ and Lemma 2.3 we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E & \leq C_0 + C_1 E + \varepsilon^2 \|\partial_x^{-1} \partial_t R_v\|_{L^2}^2 + \varepsilon^{-2} \|\partial_x^{-1} (\varepsilon^{-2} \mathrm{Res}_v(\psi_u, \psi_v))\|_{L^2}^2 \\ & \leq C_2 E + C_3. \end{aligned}$$

By Gronwall's inequality we have $E(t) \leq M$ for all $t \in [0, T_0]$ for a constant $M = \mathcal{O}(1)$. Hence, for u_0 sufficiently small, but still $\mathcal{O}(1)$, with Lemma 3.1, the bound for the error w.r.t. the $H^1 \times L^2$-norm, as stated in Theorem 1.1, follows.

4 Two improvements of the result

It is the purpose of this section to discuss two possible improvements of our approximation result stated in Theorem 1.1. We start with the following higher regularity result

Theorem 4.1. *Let $s \in \mathbb{N}_0$. There is a $C_{max} > 0$ such that for all $C_u \in [0, C_{max})$ the following holds. Let $u_0 \in C([0,T_0], H^{s+6}(\mathbb{T},\mathbb{C}))$ be a solution of (3) satisfying*

$$\sup_{t\in[0,T_0]} \|u_0(\cdot,t)\|_{H^{s+6}} = C_u < \infty.$$

Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have solutions $(u,v) \in C([0,T_0], H^{s+1}(\mathbb{T},\mathbb{C}) \times H^s(\mathbb{T},\mathbb{C}))$ of (1)-(2) satisfying

$$\sup_{t\in[0,T_0]} \|(u,v)(\cdot,t) - (u_0, -|u_0|^2)(\cdot,t)\|_{H^{s+1}\times H^s} \le C\varepsilon^2.$$

Proof. We proceed as in the proof of Theorem 1.1. The estimates for the residuals have already been stated in the required form. Thus, it remains to do the energy estimates. First we apply the operator ∂_x^s to the system (7)-(8). Then we multiply the first equation with $\partial_t\partial_x^s\overline{R_u}$, integrate w.r.t. x and add the complex conjugate. Further we multiply the second equation with $\partial_x^{s-2}\partial_t R_v$ and integrate w.r.t. x. Adding the resulting equations together gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{s+1}R_u\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^s R_v\|_{L^2}^2 + \frac{1}{2}\varepsilon^2\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^{s-1}\partial_t R_v\|_{L^2}^2 = I_1 + \ldots + I_5$$

where

$$\begin{aligned}
I_1 &= -\int_{\mathbb{T}} (\partial_x^s(\psi_u R_v)\partial_t\partial_x^s\overline{R_u} + \partial_x^s(\psi_u\overline{R_u})\partial_t\partial_x^s R_v)\,\mathrm{d}x \\
I_2 &= -\int_{\mathbb{T}} (\partial_x^s(\overline{\psi_u}R_v)\partial_t\partial_x^s R_u + \partial_x^s(\overline{\psi_u}R_u)\partial_t\partial_x^s R_v)\,\mathrm{d}x \\
I_3 &= -\int_{\mathbb{T}} (\partial_x^s(\psi_v R_u)\partial_t\partial_x^s\overline{R_u} + \partial_x^s(\psi_v\overline{R_u})\partial_t\partial_x^s R_u)\,\mathrm{d}x \\
I_4 &= -\varepsilon^2\int_{\mathbb{T}} (\partial_x^s(R_v R_u)\partial_t\partial_x^s\overline{R_u} + \partial_x^s(R_v\overline{R_u})\partial_t\partial_x^s R_u + \partial_x^s(|R_u|^2)\partial_t\partial_x^s R_v)\,\mathrm{d}x \\
I_5 &= 2\mathrm{Re}\int_{\mathbb{T}} \partial_t\partial_x^s\overline{R_u}\varepsilon^{-2}\partial_x^s\mathrm{Res}_u(\psi_u,\psi_v)\,\mathrm{d}x + \int_{\mathbb{T}} (\partial_x^{s-1}\partial_t R_v)\partial_x^{s-1}(\varepsilon^{-2}\mathrm{Res}_v(\psi_u,\psi_v))\,\mathrm{d}x.
\end{aligned}$$

Again it is essential that the R_u- and R_v-terms in the energy with the highest derivatives can be written as time derivative plus terms with less derivatives falling on R. As an example we consider the first two integrals I_1 and I_2. We use Leibniz's rule in order to get

$$\begin{aligned} I_1 = & -\int_{\mathbb{T}} (\psi_u \partial_x^s R_v \partial_t \partial_x^s \overline{R_u} + \psi_u \partial_x^s \overline{R_u} \partial_t \partial_x^s R_v) \,\mathrm{d}x \\ & - \sum_{k=1}^{s} \binom{s}{k} \int_{\mathbb{T}} \partial_x^k \psi_u \partial_x^{s-k} R_v \partial_t \partial_x^s \overline{R_u} \,\mathrm{d}x - \sum_{k=1}^{s} \binom{s}{k} \int_{\mathbb{T}} \partial_x^k \psi_u \partial_x^{s-k} \overline{R_u} \partial_t \partial_x^s R_v \,\mathrm{d}x. \end{aligned}$$

We apply the strategy from Section 3 and rewrite the terms with most derivatives falling on R as a total time derivative plus terms with the time derivative falling on approximation terms. This yields

$$\begin{aligned} I_1 = & -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} \psi_u \partial_x^s R_v \partial_x^s \overline{R_u} \,\mathrm{d}x + \int_{\mathbb{T}} \partial_t \psi_u \partial_x^s R_v \partial_x^s \overline{R_u} \,\mathrm{d}x \\ & - \sum_{k=1}^{s} \binom{s}{k} \int_{\mathbb{T}} \partial_x^k \psi_u \partial_x^{s-k} R_v \partial_t \partial_x^s \overline{R_u} \,\mathrm{d}x \\ & - \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=1}^{s} \binom{s}{k} \int_{\mathbb{T}} (\partial_x^k \psi_u \partial_x^{s-k} \overline{R_u} \partial_x^s R_v) \,\mathrm{d}x + \sum_{k=1}^{s} \binom{s}{k} \int_{\mathbb{T}} \partial_t (\partial_x^k \psi_u \partial_x^{s-k} \overline{R_u}) \partial_x^s R_v \,\mathrm{d}x \\ = & -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} \partial_x^s (\psi_u \overline{R_u}) \partial_x^s R_v \,\mathrm{d}x + \int_{\mathbb{T}} \partial_t \psi_u \partial_x^s R_v \partial_x^s \overline{R_u} \,\mathrm{d}x \\ & - \sum_{k=1}^{s} \binom{s}{k} \int_{\mathbb{T}} \partial_x^k \psi_u \partial_x^{s-k} R_v \partial_t \partial_x^s \overline{R_u} \,\mathrm{d}x + \sum_{k=1}^{s} \binom{s}{k} \int_{\mathbb{T}} \partial_t (\partial_x^k \psi_u \partial_x^{s-k} \overline{R_u}) \partial_x^s R_v \,\mathrm{d}x. \end{aligned}$$

We estimate

$$\begin{aligned} |I_{1,a} + I_{2,a}| &= 2 \left| \mathrm{Re} \int_{\mathbb{T}} \partial_t \psi_u \partial_x^s R_v \partial_x^s \overline{R_u} \,\mathrm{d}x \right| \leq C \|\partial_t \psi_u\|_{L^\infty} \|R_v\|_{H^s} \|R_u\|_{H^s}, \\ |I_{1,b} + I_{2,b}| &= 2 \left| \mathrm{Re} \sum_{k=1}^{s} \binom{s}{k} \int_{\mathbb{T}} \partial_x (\partial_x^k \psi_u \partial_x^{s-k} R_v) \partial_t \partial_x^{s-1} \overline{R_u} \,\mathrm{d}x \right| \\ &\leq C \|\psi_u\|_{H^{s+1}} \|R_v\|_{H^s} \|\partial_t R_u\|_{H^{s-1}}, \\ |I_{1,c} + I_{2,c}| &= 2 \left| \mathrm{Re} \sum_{k=1}^{s} \binom{s}{k} \int_{\mathbb{T}} \partial_t (\partial_x^k \psi_u \partial_x^{s-k} \overline{R_u}) \partial_x^s R_v \,\mathrm{d}x \right| \\ &\leq C (\|\partial_t \psi_u\|_{H^s} \|R_u\|_{H^{s-1}} + \|\psi_u\|_{H^s} \|\partial_t R_u\|_{H^{s-1}}) \|R_v\|_{H^s}, \end{aligned}$$

where $I_{2,j} = \overline{I_{1,j}}$. Here and in the following $\|\partial_t R_u\|_{H^{s-1}}$ occurs. This can be estimated by the right-hand side of the R_u-equation and so by

$$\begin{aligned} 2\|\partial_t R_u\|_{H^{s-1}} \leq \; & \|R_u\|_{H^{s+1}} + \|\psi_u\|_{H^{s-1}} \|R_v\|_{H^{s-1}} + \|\psi_v\|_{H^{s-1}} \|R_u\|_{H^{s-1}} \\ & + \varepsilon^2 \|R_u\|_{H^{s-1}} \|R_v\|_{H^{s-1}} + C_{res}. \end{aligned}$$

As above it is not necessary to write I_3 as time derivative plus terms with less derivatives. The terms in I_3 can be estimated directly. In detail, we find

$$|I_3| = 2\left|\mathrm{Re}\int_{\mathbb{T}} \partial_x^{s+1}(\psi_v R_u)\partial_t\partial_x^{s-1}\overline{R_u}\,\mathrm{d}x\right| \leq C\|\psi_u\|_{H^{s+1}}\|R_u\|_{H^{s+1}}\|\partial_t R_u\|_{H^{s-1}}.$$

The term I_4 can be written as

$$\begin{aligned}
&\int_{\mathbb{T}}(\partial_x^s(R_vR_u)\partial_t\partial_x^s\overline{R_u} + \partial_x^s(R_v\overline{R_u})\partial_t\partial_x^s R_u + \partial_x^s(|R_u|^2)\partial_t\partial_x^s R_v)\,\mathrm{d}x\\
&\quad= \int_{\mathbb{T}}(R_u\partial_x^s R_v\partial_t\partial_x^s\overline{R_u} + \partial_x^s R_v\overline{R_u}\partial_t\partial_x^s R_u + \partial_x^s R_u\overline{R_u}\partial_t\partial_x^s R_v + \partial_x^s\overline{R_u}R_u\partial_t\partial_x^s R_v)\,\mathrm{d}x\\
&\qquad+\sum_{k=1}^{s}\binom{s}{k}\int_{\mathbb{T}}\partial_x^k R_u\partial_x^{s-k}R_v\partial_t\partial_x^s\overline{R_u} + \partial_x^k\overline{R_u}\partial_x^{s-k}R_v\partial_t\partial_x^s R_u\,\mathrm{d}x\\
&\qquad+\sum_{k=1}^{s-1}\binom{s}{k}\int_{\mathbb{T}}\partial_x^k R_u\partial_x^{s-k}\overline{R_u}\partial_t\partial_x^s R_v\,\mathrm{d}x
\end{aligned}$$

Adapting the strategy for I_1 yields

$$\begin{aligned}
I_4 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}(R_u\partial_x^s R_v\partial_x^s\overline{R_u} + \partial_x^s R_v\overline{R_u}\partial_x^s R_u)\,\mathrm{d}x - \int_{\mathbb{T}}(\partial_t R_u\partial_x^s R_v\partial_x^s\overline{R_u} + \partial_x^s R_v\partial_t\overline{R_u}\partial_x^s R_u)\,\mathrm{d}x\\
&\quad+2\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int_{\mathbb{T}}\partial_x^k R_u\partial_x^{s-k}R_v\partial_t\partial_x^s\overline{R_u}\,\mathrm{d}x + \sum_{k=1}^{s-1}\binom{s}{k}\int_{\mathbb{T}}\partial_x^k R_u\partial_x^{s-k}\overline{R_u}\partial_t\partial_x^s R_v\,\mathrm{d}x.
\end{aligned}$$

After integration by parts in $I_{4,b}$ and $I_{4,c}$ we estimate

$$\begin{aligned}
|I_{4,a}| &= 2\varepsilon^2\left|\mathrm{Re}\int_{\mathbb{T}}\partial_t R_u\partial_x^s R_v\partial_x^s\overline{R_u}\,\mathrm{d}x\right| \leq C\varepsilon^2\|\partial_t R_u\|_{L^2}\|R_v\|_{H^s}\|R_u\|_{H^{s+1}},\\
|I_{4,b}| &= 2\varepsilon^2\left|\mathrm{Re}\sum_{k=1}^{s}\binom{s}{k}\int_{\mathbb{T}}\partial_x(\partial_x^k R_u\partial_x^{s-k}R_v)\partial_t\partial_x^{s-1}\overline{R_u}\,\mathrm{d}x\right|\\
&\leq C\varepsilon^2\|R_u\|_{H^{s+1}}\|R_v\|_{H^s}\|\partial_t R_u\|_{H^{s-1}},\\
|I_{4,c}| &= \varepsilon^2\left|\sum_{k=1}^{s-1}\binom{s}{k}\int_{\mathbb{T}}\partial_x(\partial_x^k R_u\partial_x^{s-k}\overline{R_u})\partial_t\partial_x^{s-1}R_v\,\mathrm{d}x\right|\\
&\leq C\varepsilon^2\|R_u\|_{H^{s+1}}\|R_u\|_{H^{s+1}}\|\partial_t\partial_x^{s-1}R_v\|_{L^2}.
\end{aligned}$$

The terms I_5 can be estimated through the Cauchy-Schwarz inequality, Lemma 2.2 and Lemma 2.3. We find

$$\begin{aligned}
|I_{5,a}| &= 2\left|\mathrm{Re}\int_{\mathbb{T}}\partial_t\partial_x^{s-1}\overline{R_u}\varepsilon^{-2}\partial_x^{s+1}\mathrm{Res}_u(\psi_u,\psi_v)\,\mathrm{d}x\right| \leq C\varepsilon^{-2}\|\partial_t R_u\|_{H^{s-1}}\|\mathrm{Res}_u(\psi_u,\psi_v)\|_{H^{s+1}},\\
|I_{5,b}| &= \left|\int_{\mathbb{T}}(\partial_x^{s-1}\partial_t R_v)\partial_x^{s-1}(\varepsilon^{-2}\mathrm{Res}_v(\psi_u,\psi_v))\,\mathrm{d}x\right| \leq C\varepsilon^{-2}\|\partial_t\partial_x^{s-1}R_v\|_{L^2}\|\mathrm{Res}_v(\psi_u,\psi_v)\|_{H^{s-1}}.
\end{aligned}$$

Again we collect all terms in form of a time derivative in our energy $\mathcal{E} = E + \widetilde{E}$, where

$$\begin{aligned}\widetilde{E} &= \|\partial_x^{s+1} R_u\|_{L^2}^2 + \frac{1}{2}\|\partial_x^s R_v\|_{L^2}^2 + \frac{1}{2}\varepsilon^2\|\partial_x^{s-1}\partial_t R_v\|_{L^2}^2 \\ &+ \int_{\mathbb{T}} (\partial_x^s(\psi_u \overline{R_u})\partial_x^s R_v + \partial_x^s(\overline{\psi_u} R_u)\partial_x^s R_v)\,\mathrm{d}x \\ &+ \varepsilon^2 \int_{\mathbb{T}} (R_u \partial_x^s R_v \partial_x^s \overline{R_u} + \partial_x^s R_v \overline{R_u} \partial_x^s R_u)\,\mathrm{d}x.\end{aligned}$$

Then $\mathcal{E}^{1/2}$ is equivalent to the $H^{s+1} \times H^s$-norm of (R_u, R_v) in the sense of Lemma 3.1.

Again we use $\mathcal{E}^{1/2} \leq 1 + \mathcal{E}$. Using Lemma 2.2 and the calculations from the previous section we finally come to an inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq C\mathcal{E} + C\varepsilon^2\mathcal{E}^{3/2} + C \leq (C+1)\mathcal{E} + C,$$

as long as $C\varepsilon\mathcal{E}^{1/2} \leq 1$. By Gronwall's inequality we have $\mathcal{E}(t) \leq M$ for all $t \in [0, T_0]$ for a constant $M = \mathcal{O}(1)$. We choose $\varepsilon_0 > 0$ so small that $C\varepsilon_0 M^{1/2} \leq 1$. Using the equivalence of $\mathcal{E}^{1/2}$ to the $H^{s+1} \times H^s$-norm we are done. $\square$

The second improvement is about the validity of the higher order approximations. For $u_0(\cdot, t)$ in a Sobolev space with sufficiently high regularity, the approximation rate in Theorem 1.1 and Theorem 4.1 can be increased. In the following, we outline how to achieve this. For computing higher order approximations in case $x \in \mathbb{T}$ we make the ansatz

$$\psi_{u,n}(x,t) = \sum_{k=0}^{n} \varepsilon^{2k} u_{2k}(x,t), \qquad \psi_{v,n}(x,t) = \sum_{k=0}^{n} \varepsilon^{2k} v_{2k}(x,t). \tag{9}$$

Then as before $v_0 = -|u_0|^2$ and u_0 solves the NLS equation (5). The functions u_{2k} solve inhomogeneous linear Schrödinger equations and the functions v_{2k} satisfy

$$\partial_t^2 v_{2(k-1)} = \partial_x^2 v_{2k} + \partial_x^2 G_{2k}(u_0, \ldots, u_{2k}), \tag{10}$$

where G_{2k} are quadratic nonlinearities. Suppose that $v_{2(k-1)}$ has a vanishing mean value. We look for v_{2k} having a vanishing mean value, too. Since in general G_{2k} will not have a vanishing mean value we add a constant $\beta_{2k} \in \mathbb{C}$ to v_{2k} to get rid of the non-vanishing mean value of G_{2k}. We can do this since the constant will cancel in (10). Then we set

$$v_{2k} = \partial_x^{-2}\partial_t^2 v_{2(k-1)} - G_{2k}(u_0, \ldots, u_{2k}) + \beta_{2k} \tag{11}$$

and

$$\beta_{2k} = \frac{1}{2\pi} \int_{\mathbb{T}} G_{2k}(u_0, \ldots, u_{2k})(x)\,\mathrm{d}x.$$

Under the condition that $u_0 \in H^{s+2n+5}(\mathbb{T}, \mathbb{C})$ we have that the residual is of order $\mathcal{O}(\varepsilon^{2n+2})$ for the higher order approximation.

Unlike in Section 3 we introduce the error by

$$(u, v)(x, t) = (\psi_{u,n}, \psi_{v,n})(x, t) + \varepsilon^{\beta}(R_u, R_v)(x, t).$$

The error functions R_u and R_v satisfy

$$\begin{aligned} 2i\partial_t R_u &= \partial_x^2 R_u - \psi_{u,n} R_v - \psi_{v,n} R_u - \varepsilon^{\beta} R_u R_v + \varepsilon^{-\beta}\mathrm{Res}_u(\psi_u, \psi_v), \quad (12)\\ \varepsilon^2 \partial_t^2 R_v &= \partial_x^2 R_v + \partial_x^2(\overline{\psi_{u,n}} R_u) + \partial_x^2(\psi_{u,n}\overline{R_u}) \\ &\quad + \varepsilon^{\beta}\partial_x^2 |R_u|^2 + \varepsilon^{-\beta}\mathrm{Res}_v(\psi_u, \psi_v). \quad (13) \end{aligned}$$

We set $\beta = 2n$ where n is the order of the approximation defined in (9). Then the energy estimates are analogous to those from above. In total, we have the following theorem.

Theorem 4.2. *Let $n \in \mathbb{N}$ and $s \in \mathbb{N}_0$. There is a $C_{max} > 0$ such that for all $C_u \in [0, C_{max})$ the following holds. Let $u_0 \in C([0, T_0], H^{s+2n+5}(\mathbb{T}, \mathbb{C}))$ be a solution of* (3) *satisfying*

$$\sup_{t \in [0, T_0]} \|u_0(\cdot, t)\|_{H^{s+2n+5}} = C_u < \infty.$$

Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions $(u, v) \in C([0, T_0], H^{s+1}(\mathbb{T}, \mathbb{C}) \times H^s(\mathbb{T}, \mathbb{C}))$ of (1)-(2) *satisfying*

$$\sup_{t \in [0, T_0]} \|(u, v)(\cdot, t) - (\psi_{u,n}, \psi_{v,n})(\cdot, t)\|_{H^{s+1} \times H^s} \leq C\varepsilon^{2n}.$$

5 Summary and Outlook

This paper investigates the singular limit of the Zakharov system, where the NLS equation appears as a regular limit equation. The paper explains the importance of opposite signs in the nonlinear terms of the Zakharov system for the validity of the NLS approximation, see Remark 1.4. The approximation results, Theorem 1.1, Theorem 4.1 and Theorem 4.2, are based on energy estimates and Gronwall's inequality. The analysis shows that the standard proof of validity requires an unstated smallness condition to relate the constructed energy to the usual Sobolev norm, cf. Lemma 3.1. The paper compares the cases where $x \in \mathbb{R}$ and

$x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. We explain in Remark 2.1 that in both cases the improved approximation (4) can be defined and the associated residual can be estimated. For the approximation proof we also need estimates for $\partial_x^{-1}\mathrm{Res}_v(\psi_u, \psi_v)$. These hold for $x \in \mathbb{T}$, cf. Lemma 2.3, but not for $x \in \mathbb{R}$, cf. Remark 2.4. We have outlined the challenges of properly defining the higher order approximations (9) by explaining the need to remove the mean in (11), which is only possible for $x \in \mathbb{T}$.

This paper is the starting point for some further investigations. The NLS equation also appears for the Klein-Gordon-Zakharov (KGZ) system.

$$\varepsilon^2\partial_t^2 u = \partial_x^2 u - \varepsilon^{-2}u - uv, \qquad \gamma^2\varepsilon^2\partial_t^2 v = \partial_x^2 v + \partial_x^2(|u|^2) \tag{14}$$

with a parameter $\gamma \in \mathbb{R} \setminus \{0\}$ in the limit $\varepsilon \to 0$. After eliminating all non-resonant terms by normal form transformations, we end up with a system that has the same properties as the Zakharov system. Therefore, we strongly expect that an NLS approximation result for the KGZ system can be obtained by combining normal form transformations with the energy estimates from this paper. This would provide an alternative to the approximation proof given in [MN05] and a first higher order NLS approximation result for the KGZ system on the one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. This will be the subject of future research.

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