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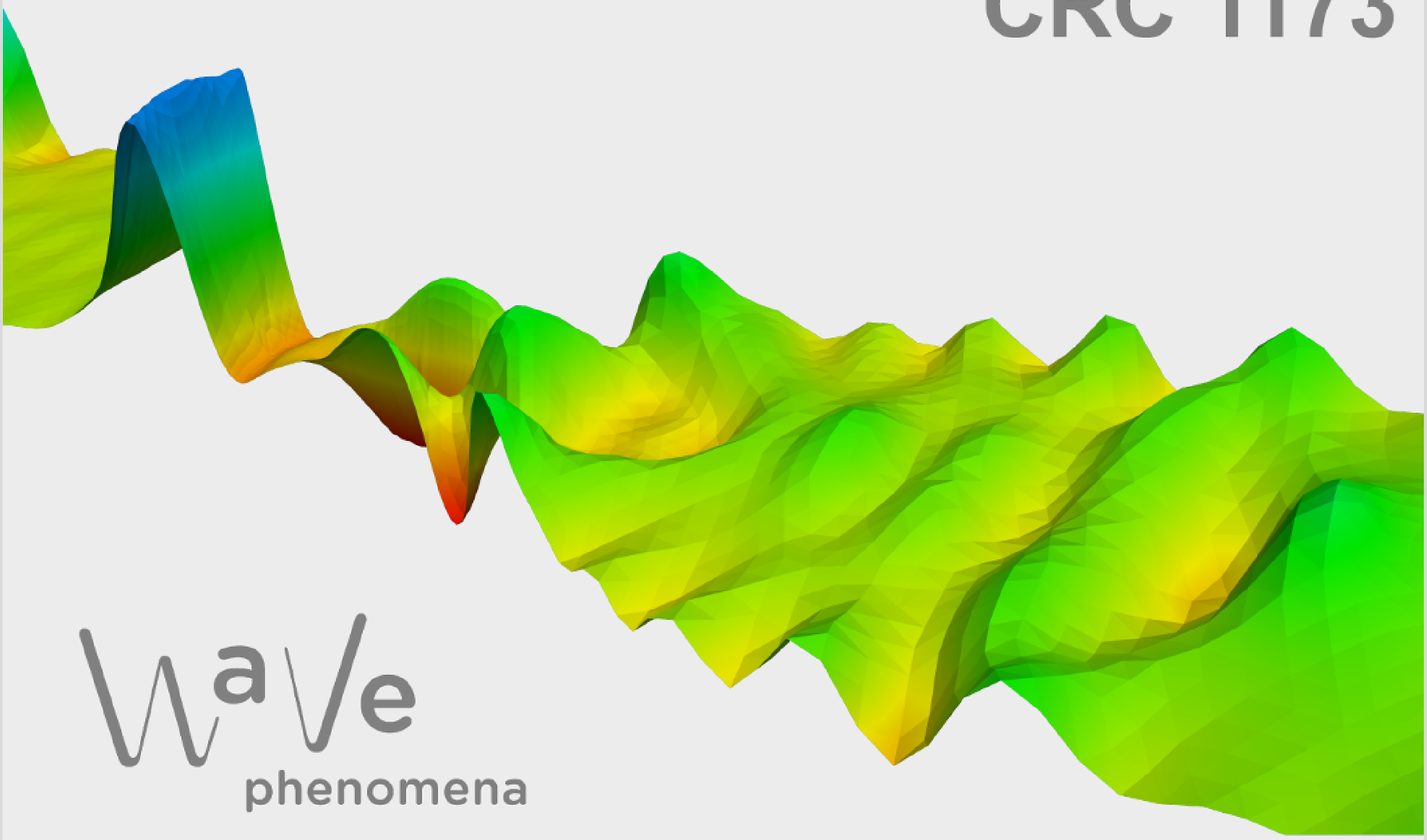

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Approximate Peregrine solitons in dispersive nonlinear wave equations

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Abstract

The purpose of this short note is to explain how the existing results on the validity of the NLS approximation can be extended from Sobolev spaces $H^s(\mathbb{R})$ to the spaces of functions $u = v + w$ where $v \in H^s_{per}$ and $w \in H^s(\mathbb{R})$. This allows us to use the Peregrine solution of the NLS equation to find freak or rogue wave dynamics in more complicated systems.

1 Introduction

The NLS equation

$$i\partial_T A = \nu_1 \partial_X^2 A + \nu_2 A|A|^2, \qquad (\nu_1, \nu_2 \in \mathbb{R})$$

can be derived by multiple scaling perturbation analysis for the description of slow modulations in time and space of the envelope of spatially and temporarily oscillating wave packets, as they appear in nonlinear optics, water wave theory, plasma physics, waves in DNA, Bose-Einstein condensates, etc.

As an example we consider the cubic Klein-Gordon equation

$$\partial_t^2 u = \partial_x^2 u - u + u^3, \tag{1}$$

for $x, t, u(x,t) \in \mathbb{R}$. We make the ansatz

$$u(x,t) = \varepsilon \Psi_{NLS}(x,t) = \varepsilon A(\varepsilon(x - c_g t), \varepsilon^2 t) e^{i(k_0 x - \omega_0 t)} + c.c. \qquad (2)$$

with spatial and temporal wave number $k_0 > 0$ and $\omega_0 \in \mathbb{R}$, with linear group velocity $c_g \in \mathbb{R}$, with envelope function $A = A(X,T) \in \mathbb{C}$, scaled variables $X = \varepsilon(x - c_g t)$, $T = \varepsilon^2 t$, and small perturbation parameter $0 < \varepsilon \ll 1$. By inserting the ansatz in (1) and by equating the coefficients in front of $\varepsilon^j e^{i(k_0 x - \omega_0 t)}$ to zero for $j = 1, 2, 3$, we find the linear dispersion relation $\omega_0^2 = k_0^2 + 1$, the linear group velocity, here $c_g = k_0/\omega_0$, and finally the NLS equation

$$2i\omega_0 \partial_T A = (c_g^2 - 1)\partial_X^2 A - 3A|A|^2. \qquad (3)$$

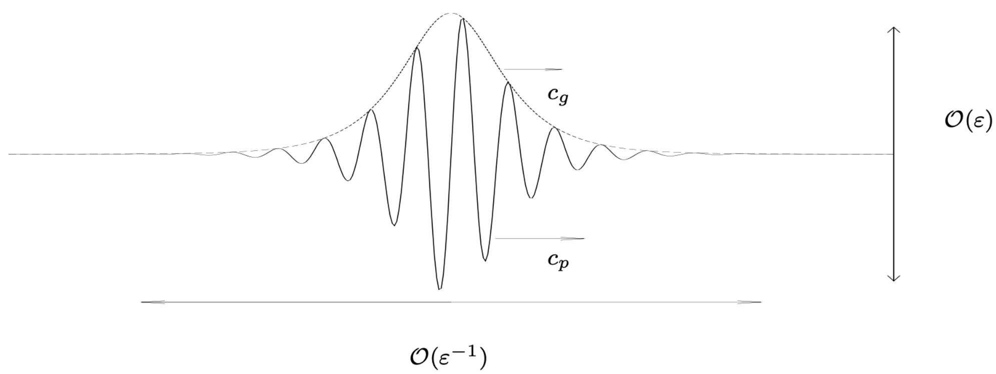

Figure 1: A modulating pulse described by the NLS equation. The envelope, advancing with group velocity c_g in the laboratory frame, modulates the underlying carrier wave $e^{i(k_0 x - \omega_0 t)}$, advancing with phase velocity c_p. The envelope evolves approximately as a solution of the NLS equation.

Several NLS approximation results have been established in the last decades. For the above cubic Klein-Gordon equation we have for instance [KSM92]:

Theorem. Let $A \in C([0, T_0], H^4)$ be a solution of the NLS equation (3). Then there exists an $\varepsilon_0 > 0$ and a $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there are solutions u of the original system (1) which can be approximated by $\varepsilon \Psi_{NLS}$ with

$$\sup_{t \in [0, T_0/\varepsilon^2]} \|u(t) - \varepsilon \Psi_{NLS}(t)\|_{H^1} < C\varepsilon^{3/2}.$$

The NLS equation was first derived by Zakharov in 1968 for the water wave problem [Zak68]. In general, error estimates are non-trivial for quadratic nonlinearities since solutions of order $\mathcal{O}(\varepsilon)$ have to be controlled on a time scale of order $\mathcal{O}(1/\varepsilon^2)$. The results are based on near identity change of variables using the oscillatory character of the $\mathcal{O}(\varepsilon)$-terms in the error equations, cf. [Kal88]. Resonant terms can however lead to non-approximation results. There are rigorous counter examples that the NLS approximation fails to make correct predictions, see [SSZ15]. An NLS approximation result for the water wave problem can be found for instance in [Dül21].

It is the purpose of this short note to explain how the existing approximation results can be extended from Sobolev spaces $H^s(\mathbb{R})$ to the spaces of functions $u = v + w$ where $v \in H^s_{per}$ and $w \in H^s(\mathbb{R})$. This allows us to use the Peregrine solution of the NLS equation to find rogue wave dynamics in more complicated systems, too. Rogue waves, also known as freak waves or killer waves, are large and unpredictable surface waves that can be extremely dangerous to ships. Their height is significantly greater than that of average waves, and they seem to appear out of nowhere. These phenomena are not confined to water waves; they also occur in liquid helium, non-linear optics, and microwave cavities, see [GTY⁺17].

The focusing NLS equation in its normalized form

$$i\partial_\tau \psi + \frac{1}{2}\partial_\xi^2 \psi + |\psi|^2 \psi = 0$$

possesses so called Peregrine solutions

$$A(\xi, \tau) = \left[1 - \frac{4(1 + 2i\tau)}{1 + 4\xi^2 + 4\tau^2}\right] e^{i\tau} \tag{4}$$

which were discovered by Peregrine [Per83] and are prototype of a rogue wave. See Figure 2.

The existing NLS approximation results do not apply since this solution does not decay to zero for $|\xi| \to \infty$. However, it can be written as

$$A(\xi, \tau) = e^{i\tau} + B(\xi, \tau),$$

with $B(\cdot, \tau) \in W^{s,p}(\mathbb{R})$ for all $p \in [1, \infty]$ and $s \in \mathbb{N}_0$. In a similar manner we split the ansatz, defined in (2), into

$$\varepsilon \Psi_{NLS}(x, t) = \varepsilon \Psi_v(x, t) + \varepsilon \Psi_w(x, t),$$

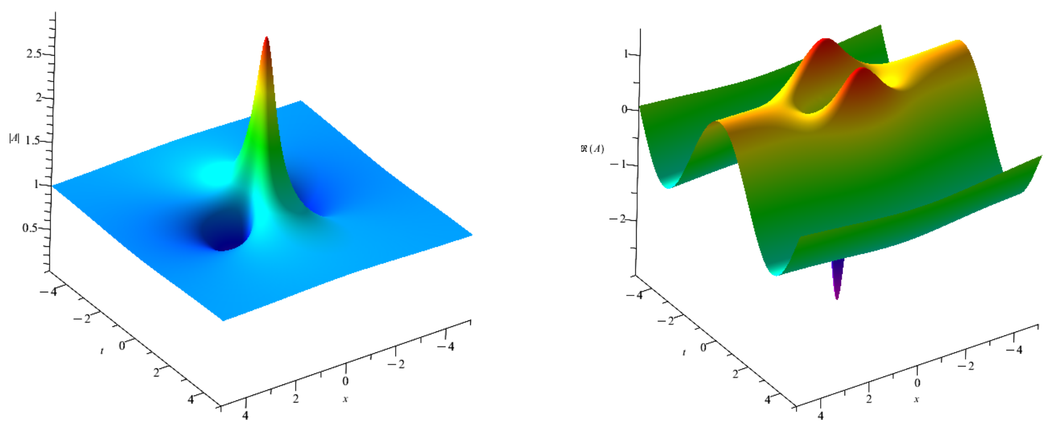

Figure 2: Absolute value (left) and real part (right) of the Peregrine soliton plotted as a function of space and time.

with

$$\varepsilon\Psi_v(x,t) = \varepsilon\Psi_v(x+2\pi/k_0,t), \quad \varepsilon\Psi_v(\cdot,t) \in W^{s,p}_{per},$$

and

$$\varepsilon\Psi_w(\cdot,t) \in W^{s,p}(\mathbb{R}).$$

In a similar manner we split the solution u of the cubic Klein-Gordon equation into

$$u = v + w,$$

with

$$v(x,t) = v(x+2\pi/k_0,t), \quad v(\cdot,t) \in W^{s,p}_{per}, \quad \text{and} \quad w(\cdot,t) \in W^{s,p}(\mathbb{R}).$$

The new variables v and w satisfy

$$\partial_t^2 v = \partial_x^2 v - v + v^3, \tag{5}$$
$$\partial_t^2 w = \partial_x^2 w - w + 3v^2 w + 3vw^2 + w^3. \tag{6}$$

Then we have the following approximation result

Theorem 1.1. *Fix $s > 1/2$ and let $A = A_v + A_w$, with*

$$(A_v, A_w) \in C([0,T_0], \mathbb{C} \times H^{s+4}(\mathbb{R})),$$

be a solution of the NLS equation (3). *Then there exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *there are solutions*

$$(v, w) \in C([0, T], H^s_{per} \times H^s(\mathbb{R}))$$

of the KG equation (1) *with*

$$\sup_{t \in [0, T_0/\varepsilon^2]} \|(v, w)(\cdot, t) - (\varepsilon \Psi_v, \varepsilon \Psi_w)(\cdot, t)\|_{H^s_{per} \times H^s} \leq C \varepsilon^{3/2}.$$

Since we have a cubic nonlinearity the application of Gronwall's inequality to the variation of constant formula is sufficient for getting an $\mathcal{O}(1)$-bound for R_v and R_w on the long $\mathcal{O}(1/\varepsilon^2)$-time scale. We expect that most of the existing NLS approximation theory (the handling of quadratic terms by normal form transformations, cf. [Kal88], the handling of resonances, cf. [Sch05], and quasilinear systems, cf. [Dül17, WC17, DH18]) can be transferred to these function spaces. There are other interesting solutions of the NLS equation having the same analytic properties as the Peregrine solutions. Hence the above approximation result also applies to so called Kuznetsov-Ma solitons and higher order Peregrine solitons. See Section 6.

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2 The functional analytic set-up

In this section we introduce the functional analytic set-up which we need to prove Theorem 1.1.

A) In order to solve (5) we use the space

$$H^s_{per,k_0} = \{v : \mathbb{R} \to \mathbb{R} : v(x) = v(x + 2\pi/k_0), \|v\|_{H^s_{per}} < \infty\}$$

for a $k_0 > 0$, with

$$\|v\|_{H^s_{per}} = \|\widehat{v}\|_{\ell^2_s} = \|(\langle k \rangle^{s/2} \widehat{v}_k)_{k \in \mathbb{Z}}\|_{\ell^2} = \left(\sum_{k \in \mathbb{Z}} |\widehat{v}_k|^2 (1 + k^2) \right)^{1/2},$$

where $v(x) = \frac{k_0}{2\pi} \sum_{k \in \mathbb{Z}} \widehat{v}_k e^{ikk_0 x}$, i.e., $(\widehat{v}_k)_{k \in \mathbb{Z}}$ is the discrete Fourier transform of v. In the following we write H^s_{per} instead of H^s_{per,k_0} if no confusion is possible. The space H^s_{per} is closed under pointwise multiplication if $s > 1/2$ and can be embedded in $C^0_{b,unif}$ if $s > 1/2$. In detail, for each $s > 1/2$ there exists a $C > 0$ such that for all $v_1, v_2 \in H^s_{per}$ we have

$$\|v_1 v_2\|_{H^s_{per}} \leq C\|v_1\|_{H^s_{per}}\|v_2\|_{H^s_{per}}.$$

B) In order to solve (6) we use the space

$$H^s = H^s(\mathbb{R}) = \{w : \mathbb{R} \to \mathbb{R} : \|w\|_{H^s} < \infty\},$$

with

$$\|w\|_{H^s} = \|\widehat{w}\|_{L^2_s} = \| \langle\cdot\rangle^{s/2} \, \widehat{w}(\cdot)\|_{L^2} = \left(\int (1+k^2)^s |\widehat{w}(k)|^2 dk\right)^{1/2},$$

i.e., $\widehat{w}$ is the continuous Fourier transform of w. The space $H^s(\mathbb{R})$ is closed under pointwise multiplication if $s > 1/2$ and can be embedded in $C^0_{b,unif}$ if $s > 1/2$. In detail, for each $s > 1/2$ there exists a $C > 0$ such that for all $w_1, w_2 \in H^s(\mathbb{R})$ we have

$$\|w_1 w_2\|_{H^s} \leq C\|w_1\|_{H^s}\|w_2\|_{H^s}.$$

C) In (6) there are multiplications between $v \in H^s_{per}$ and $w \in H^s$, too. For $s > 1/2$ these can be estimated with

$$\begin{aligned}
\|vw\|_{H^s} &= \|\langle k\rangle^{s/2}\widehat{vw}(k)\|_{L^2(dk)} \leq \left\|\int \langle k\rangle^{s/2}|\widehat{v}(k-\ell)\widehat{w}(\ell)|d\ell\right\|_{L^2(dk)} \\
&\leq C\left\|\int \langle k-\ell\rangle^{s/2}|\widehat{v}(k-\ell)\widehat{w}(\ell)|d\ell + \int \langle \ell\rangle^{s/2}|\widehat{v}(k-\ell)\widehat{w}(\ell)|d\ell\right\|_{L^2} \\
&\leq C\||\langle\cdot\rangle^{s/2}\widehat{v}| * |\widehat{w}| + |\widehat{v}| * |\langle\cdot\rangle^{s/2}\widehat{w}|\|_{L^2} \\
&\leq C\left(\|\langle\cdot\rangle^{s/2}\widehat{v}\|_{\ell^2}\|\widehat{w}\|_{L^1} + \|\widehat{v}\|_{\ell^1}\|\langle\cdot\rangle^{s/2}\widehat{w}\|_{L^2}\right) \\
&= C(\|\widehat{v}\|_{\ell^2_s}\|\widehat{w}\|_{L^1} + \|\widehat{v}_i\|_{\ell^1_0}\|\widehat{w}\|_{L^2_s}) \\
&\leq 2C\|v\|_{H^s_{per}}\|w\|_{H^s},
\end{aligned}$$

where we used the identity

$$\langle k\rangle^s \leq C(\langle k-\ell\rangle^s + \langle\ell\rangle^s),$$

in the second inequality, Young's inequality in the fourth inequality, and Sobolev's embedding theorem in the last inequality.

Remark 2.1. The goal of this paper can also be reformulated as follows. It is the purpose of this short note is to explain how the existing results about the validity of the NLS approximation can be extended from Sobolev spaces to spaces formally defined by

$$M^s = \{u = v + w : v \in H^s_{per}, w \in H^s\},$$

with the corresponding norm

$$\|u\|_{M^s} = \|v\|_{H^s_{per}} + \|w\|_{H^s}.$$

As a direct consequence of the above estimates the spaces M^s are algebras for $s > 1/2$, i.e., for each $s > 1/2$ there exists a $C > 0$ such that for all $u_1, u_2 \in M^s$ we have

$$\|u_1 u_2\|_{M^s} \leq C \|u_1\|_{M^s} \|u_2\|_{M^s}.$$

3 Construction of an improved approximation

We follow the lines of [KSM92] and consider the improved approximation

$$\begin{aligned} u(x,t) = \varepsilon\Psi(x,t) &= \varepsilon A(\varepsilon(x - c_g t), \varepsilon^2 t) e^{i(k_0 x - \omega_0 t)} \\ &\quad + \varepsilon^3 A_3(\varepsilon(x - c_g t), \varepsilon^2 t) e^{3i(k_0 x - \omega_0 t)} + c.c.. \end{aligned}$$

By this choice we have that the so called residual

$$\mathrm{Res} = -\partial_t^2 u + \partial_x^2 u - u + u^3,$$

i.e., the terms which do not cancel after inserting the approximation into the equation, is formally of order $\mathcal{O}(\varepsilon^4)$. For estimating the approximation and the residual we split $A = A_v + A_w$ into a spatially constant part A_v and into a spatially decaying part A_w. Similarly, we split $A_3 = A_{3,v} + A_{3,w}$. Thus the improved approximation is then of the form

$$u(x,t) = \varepsilon\Psi(x,t) = v(x,t) + w(x,t) = \varepsilon\Psi_v(x,t) + \varepsilon\Psi_w(x,t),$$

where

$$\begin{aligned} \varepsilon\Psi_v(x,t) &= \varepsilon A_v(\varepsilon(x - c_g t), \varepsilon^2 t) e^{i(k_0 x - \omega_0 t)} \\ &\quad + \varepsilon^3 A_{3,v}(\varepsilon(x - c_g t), \varepsilon^2 t) e^{3i(k_0 x - \omega_0 t)} + c.c., \\ \varepsilon\Psi_w(x,t) &= \varepsilon A_w(\varepsilon(x - c_g t), \varepsilon^2 t) e^{i(k_0 x - \omega_0 t)} \\ &\quad + \varepsilon^3 A_{3,w}(\varepsilon(x - c_g t), \varepsilon^2 t) e^{3i(k_0 x - \omega_0 t)} + c.c.. \end{aligned}$$

By inserting this ansatz into (1) and by equating the coefficients in front of $\varepsilon^j e^{i(k_0x-\omega_0 t)}$ to zero, for $j = 1, 2, 3$, we find, as in the introduction, the linear dispersion relation $\omega_0^2 = k_0^2 + 1$, the linear group velocity $c_g = k_0/\omega_0$, and the NLS equation

$$\begin{aligned} 2i\omega_0\partial_T A_v &= -3A_v|A_v|^2, &(7)\\ 2i\omega_0\partial_T A_w &= (c_g^2-1)\partial_X^2 A_w - 3A_v^2\overline{A}_w - 6A_vA_w\overline{A}_v &(8)\\ &\quad -3A_w^2\overline{A}_v - 6A_wA_v\overline{A}_w - 3A_w^2\overline{A}_w. \end{aligned}$$

By equating the coefficient in front of $\varepsilon^3 e^{3i(k_0x-\omega_0t)}$ to zero we finally find

$$\begin{aligned} 0 &= (9k_2 - 9\omega_0^2 + 1)A_{3,v} - A_v^3, &(9)\\ 0 &= (9k_2 - 9\omega_0^2 + 1)A_{3,w} - 3A_v^2A_w - 3A_vA_w^2 - A_w^3. &(10) \end{aligned}$$

Similarly to the splitting of $u = v+w$ we split the residual $\mathrm{Res} = \mathrm{Res}_v + \mathrm{Res}_w$, into

$$\begin{aligned} \mathrm{Res}_v(v) &= -\partial_t^2 v + \partial_x^2 v - v + v^3, &(11)\\ \mathrm{Res}_w(v,w) &= -\partial_t^2 w + \partial_x^2 w - w + 3v^2w + 3vw^2 + w^3. &(12) \end{aligned}$$

4 Estimates for the residual

We have

Lemma 4.1. *Assume $s_A - 3 \geq s \geq 1$. Let $(A_v, A_w) \in C([0,T_0], \mathbb{C}\times H^{s_A})$ be a solution of the NLS equation (7)-(8) and let $(A_{3,v}, A_{3,w})$ be defined as solution of (9)-(10). Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{Res}_v(\varepsilon\Psi_v)\|_{H^s_{per}} \leq C\varepsilon^4$$

and

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{Res}_w(\varepsilon\Psi_v, \varepsilon\Psi_w)\|_{H^s} \leq C\varepsilon^{7/2}.$$

Proof. By construction we have that the remaining terms in the residuals are formally of order $\mathcal{O}(\varepsilon^4)$. The loss of $\varepsilon^{-1/2}$ for $\mathrm{Res}_w(\varepsilon\Psi_v, \varepsilon\Psi_w)$ comes from the scaling properties of the $L^2(\mathbb{R})$-norm w.r.t. the scaling $X = \varepsilon x$. We refrain from giving the detailed estimates which are completely straightforward, cf. [SU17, Section 11.2]. □

5 The error estimates

The subsequent error estimates adapt the proof in [SU17, Section 11.2]. We introduce error functions $\varepsilon^\beta R_v$ and $\varepsilon^\beta R_w$ with $\beta = 3/2$ by:

$$v = \varepsilon\Psi_v + \varepsilon^\beta R_v, \qquad w = \varepsilon\Psi_w + \varepsilon^\beta R_w,$$

where

$$R_v(x,t) = R_v(x + 2\pi/k_0, t), \qquad R_w(\cdot, t) \in L^2(\mathbb{R}).$$

The error functions R_v and R_w satisfy

$$\partial_t^2 R_v = \partial_x^2 R_v - R_v + \varepsilon^2 f_v, \tag{13}$$

$$\partial_t^2 R_w = \partial_x^2 R_w - R_w + \varepsilon^2 f_w, \tag{14}$$

where

$$\begin{aligned} \varepsilon^2 f_v &= 3\varepsilon^2 \Psi_v^2 R_v + 3\varepsilon^{1+\beta}\Psi_v R_v^2 + \varepsilon^{2\beta} R_v^3 + \varepsilon^{-\beta}\mathrm{Res}_v(\varepsilon\Psi_v), \\ \varepsilon^2 f_w &= 3\varepsilon^2(\Psi_v + \Psi_w)^2 R_w + 3\varepsilon^2\Psi_w^2 R_v + 6\varepsilon^2\Psi_v\Psi_w R_v \\ &\quad +3\varepsilon^{1+\beta}(\Psi_v + \Psi_w)R_w^2 + 3\varepsilon^{1+\beta}\Psi_w R_v^2 + 6\varepsilon^{1+\beta}\Psi_w R_v R_w \\ &\quad +3\varepsilon^{2\beta} R_v^2 R_w + 3\varepsilon^{2\beta} R_v R_w^2 + \varepsilon^{2\beta} R_w^3 + \varepsilon^{-\beta}\mathrm{Res}_w(\varepsilon\Psi_v, \varepsilon\Psi_w). \end{aligned}$$

The associated equations in Fourier space, namely

$$\partial_t^2 \widehat{R}_q = -\omega^2 \widehat{R}_q + \varepsilon^2 \widehat{f}_q,$$

with $q = v, w$ and $\omega(k) = \sqrt{k^2+1}$, are written as a first order system

$$\begin{aligned} \partial_t \widehat{R}_{q,1} &= i\omega \widehat{R}_{q,2}, \\ \partial_t \widehat{R}_{q,2} &= i\omega \widehat{R}_{q,1} + \varepsilon^2 \frac{1}{i\omega}\widehat{f}_q. \end{aligned}$$

This system is abbreviated as

$$\partial_t \widehat{\mathcal{R}}_q(k,t) = \widehat{\Lambda}(k)\widehat{\mathcal{R}}_q(k,t) + \varepsilon^2 \widehat{F}_q(k,t),$$

with

$$\widehat{\Lambda}(k) = \begin{pmatrix} 0 & i\omega(k) \\ i\omega(k) & 0 \end{pmatrix}, \qquad \widehat{F}_q(k,t) = \begin{pmatrix} 0 \\ \frac{1}{i\omega}\widehat{f}_q(k,t) \end{pmatrix}.$$

We use the variation of constant formula

$$\widehat{\mathcal{R}}_q(k,t) = e^{t\widehat{\Lambda}(k)}\widehat{\mathcal{R}}_q(k,0) + \varepsilon^2 \int_0^t e^{(t-\tau)\widehat{\Lambda}(k)}\widehat{F}_q(k,\tau)d\tau$$

to estimate the solutions of this system. We start with the estimate for the linear semigroup.

Lemma 5.1. *The semigroup* $(e^{t\widehat{\Lambda}(k)})_{t\geq 0}$ *is uniformly bounded in every* L^2_s *and every* ℓ^2_s*, i.e., for every* $s \geq 0$ *there exists a* $C > 0$ *such that*

$$\sup_{t\in\mathbb{R}} \|e^{t\widehat{\Lambda}(k)}\|_{L^2_s \to L^2_s} + \sup_{t\in\mathbb{R}} \|e^{t\widehat{\Lambda}(k)}\|_{\ell^2_s \to \ell^2_s} \leq C.$$

Proof. In the following X^s denotes L^2_s or ℓ^2_s. We have $\widehat{\Lambda}(k) = S\widehat{D}(k)S^{-1}$ and as a consequence $e^{t\widehat{\Lambda}(k)} = Se^{t\widehat{D}(k)}S^{-1}$ where

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{and} \qquad \widehat{D}(k) = \begin{pmatrix} i\omega(k) & 0 \\ 0 & -i\omega(k) \end{pmatrix}.$$

The estimate follows from

$$\|e^{t\widehat{\Lambda}(k)}\widehat{u}\|_{X^s} \leq \sup_{k\in\mathbb{R}} \|e^{t\widehat{\Lambda}(k)}\|_{\mathbb{C}^2\to\mathbb{C}^2} \|\widehat{u}\|_{X^s},$$

and

$$\begin{aligned} \sup_{k\in\mathbb{R}} \|e^{t\widehat{\Lambda}(k)}\|_{\mathbb{C}^2\to\mathbb{C}^2} &\leq \|S\|_{\mathbb{C}^2\to\mathbb{C}^2} \sup_{k\in\mathbb{R}} \|e^{t\widehat{D}(k)}\|_{\mathbb{C}^2\to\mathbb{C}^2} \cdot \|S^{-1}\|_{\mathbb{C}^2\to\mathbb{C}^2} \\ &\leq \|S\|_{\mathbb{C}^2\to\mathbb{C}^2} \|S^{-1}\|_{\mathbb{C}^2\to\mathbb{C}^2} < \infty. \end{aligned}$$

□

For the nonlinear terms we have

Lemma 5.2. *For every* $s > 1/2$ *there is a* $C > 0$ *such that for all* $\varepsilon \in (0, 1]$ *we have*

$$\begin{aligned} \|\widehat{F}_v\|_{\ell^2_s} &\leq C\left(\|\widehat{\mathcal{R}}_v\|_{\ell^2_s} + \varepsilon^{\beta-1}\|\widehat{\mathcal{R}}_v\|^2_{\ell^2_s} + \varepsilon^{2\beta-2}\|\widehat{\mathcal{R}}_v|^3_{\ell^2_s} + 1\right), \\ \|\widehat{F}_w\|_{L^2_s} &\leq C\Big(\|\widehat{\mathcal{R}}_v\|_{\ell^2_s} + \|\widehat{\mathcal{R}}_w\|_{L^2_s} + \varepsilon^{\beta-1}(\|\widehat{\mathcal{R}}_v\|_{\ell^2_s} + \|\widehat{\mathcal{R}}_w\|_{L^2_s})^2 \\ &\qquad + \varepsilon^{2\beta-2}(\|\widehat{\mathcal{R}}_v\|_{\ell^2_s} + \|\widehat{\mathcal{R}}_w\|_{L^2_s})^3 + 1\Big). \end{aligned}$$

Proof. Let $q \in \{v, w\}$, and X^s_q, Y^s_q be ℓ^2_s, ℓ^1_s for $q = v$ and L^2_s, L^1_s for $q = w$. Then the estimates follow from

$$\left\|\frac{1}{\omega}\widehat{u}\right\|_{X^s_q} \leq \|\widehat{u}\|_{X^s_q}, \quad \|\varepsilon^{-\beta}\widehat{\mathrm{Res}_q}\|_{X^s_q} \leq C,$$

as well as from

$$
\begin{aligned}
&\Big\|\widehat{\Psi}_{q_1} * \widehat{\Psi}_{q_2} * \widehat{R}_{q_3}\Big\|_{X_q^s} \leq C\|\widehat{\Psi}_{q_1}\|_{Y^s_{q_1}}\|\widehat{\Psi}_{q_2}\|_{Y^s_{q_2}}\|\widehat{R}_{q_3}\|_{X^s_{q_3}},\\
&\|\widehat{\Psi}_{q_1} * \widehat{R}_{q_2} * \widehat{R}_{q_3}\|_{X_q^s} \leq C\|\widehat{\Psi}_{q_1}\|_{Y^s_{q_1}}\|\widehat{R}_{q_2}\|_{X^s_{q_2}}\|\widehat{R}_{q_3}\|_{X^s_{q_3}},\\
&\|\widehat{R}_{q_1} * \widehat{R}_{q_2} * \widehat{R}_{q_3}\|_{X_q^s} \leq C\|\widehat{R}_{q_1}\|_{X^s_{q_1}}\|\widehat{R}_{q_2}\|_{X^s_{q_2}}\|\widehat{R}_{q_3}\|_{X^s_{q_3}},
\end{aligned}
$$

where for $q = v$ we have $q_1 = q_2 = q_3 = v$ and where for $q = w$ at least one of the indices q_j equals w. Finally for estimating $\|\widehat{\Psi}_q\|_{Y_q^s}$ we use

$$
\begin{aligned}
&2\Big\|\frac{1}{\varepsilon}\widehat{A}_q\Big(\frac{\cdot - k_0}{\varepsilon}\Big) + \text{h.o.t.}\Big\|_{Y_q^s} \leq C\Big\|\frac{1}{\varepsilon}\widehat{A}_q\Big(\frac{\cdot}{\varepsilon}\Big)\Big\|_{Y_q^s} + \text{h.o.t.}\\
&\leq C\|\widehat{A}_q\|_{Y_q^s} + \text{h.o.t.} \leq C\|\widehat{A}_q\|_{X_q^{s+1}} + \text{h.o.t.}.
\end{aligned}
$$

Note that we estimated $\widehat{\Psi}_q$ in Y_q^s and not in X_q^s since $\|\widehat{\Psi}_q\|_{X_q^s} = \mathcal{O}(\varepsilon^{-1/2})$ for $q = w$ which is too large to derive estimates on the natural time scale $\mathcal{O}(1/\varepsilon^2)$ with respect to t. □

Using the previous lemmas shows that

$$
Z(t) = \|\widehat{\mathcal{R}}_v(t)\|_{\ell^2_s} + \|\widehat{\mathcal{R}}_w(t)\|_{L^2_s}
$$

satisfies

$$
\begin{aligned}
Z(t) &\leq C\varepsilon^2\int_0^t \big(Z(\tau) + \varepsilon^{\beta-1}Z(\tau)^2 + \varepsilon^{2\beta-2}Z(\tau)^3 + 1\big)\,d\tau\\
&\leq C\varepsilon^2\int_0^t (Z(\tau)+2)\,d\tau \leq 2CT_0 + C\varepsilon^2\int_0^t Z(\tau)d\tau
\end{aligned}
$$

which holds as long as

$$
\varepsilon^{\beta-1}Z(\tau)^2 + \varepsilon^{2\beta-2}Z(\tau)^3 \leq 1. \tag{15}
$$

Applying Gronwall's inequality yields

$$
Z(t) = \|\widehat{\mathcal{R}}_v(t)\|_{\ell^2_s} + \|\widehat{\mathcal{R}}_w(t)\|_{L^2_s} \leq 2CT_0e^{C\varepsilon^2 t} \leq 2CT_0e^{CT_0} = M
$$

for all $t \in [0, T_0/\varepsilon^2]$. Choosing $\varepsilon_0 > 0$ such that $\varepsilon_0^{\beta-1}M^2 + \varepsilon_0^{2\beta-2}M^3 \leq 1$ ensures that condition (15) is satisfied. This completes the proof of our approximation result.

Remark 5.3. Local existence and uniqueness of solutions to the nonlinear wave equation (1), as well as to the error equations (13)-(14), hold in the function spaces in which the error estimates were derived.

6 Discussion

We strongly expect that the existing theory about the validity of NLS approximations, i.e., the handling of quadratic nonlinearities by normal form transformations, cf. [Kal88], the handling of resonances, cf. [Sch05], or the handling of quasilinear systems, cf. [Dül17], can be transferred in the same way. Therefore, we strongly expect that an approximation result, similar to Theorem 1.1, holds for the water wave problem, too. This will be the subject of future research. The present paper goes beyond some formal arguments and is a strong indication for the occurrence of freak or rogue wave behavior in almost all dispersive wave system where the defocusing NLS equation occurs as an amplitude equation in the above sense.

The Peregrine soliton is not the only interesting solution where our theory applies, too. For illustration we recall some of these solutions from the existing literature. The Peregrine soliton can be obtained from the family of solutions

$$A(X,T) = e^{iT}\left(1 + \frac{2(1-2a)\cosh(RT) + iR\sinh(RT)}{\sqrt{2a}\cos(\Omega X) - \cosh(RT)}\right)$$

in the limit $a \to 1/2$, cf. [AAT09], where

$$R = \sqrt{8a(1-2a)}, \qquad \Omega = 2\sqrt{1-2a}.$$

For $0 < a < 1/2$ the members of this family are called Akhmediev Breathers. They are spatially periodic and localized in time above a time-periodic spatially constant background state. See Figure 3. For $a > 1/2$ the members of this family are called Kuznetsov-Ma solitons. They are time-periodic and spatially localized above a time-periodic spatially constant background state. See Figure 4. Our approximation result, Theorem 1.1, easily applies for all members of the Kuznetsov-Ma family, too.

Finally, the Peregrine soliton can also be seen as a member of a family of solutions of the form

$$A(X,T) = e^{iT}\left((-1)^j + \frac{G_j + iH_j}{D_j}\right)$$

where the G_j and H_j are suitable chosen complex polynomials and the D_j are suitable chosen positive real polynomials, cf. [AASC09]. For the Peregrine soliton we have $j = 1$, $G_1 = 4$, $H_1 = 8x$, and $D_1 = 1 + 4x^2 + 4t^2$. Other members of this family are plotted in Figure 5. Our approximation result, Theorem 1.1, applies to all members of this family.

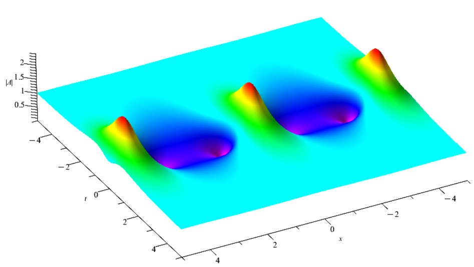

Figure 3: Absolute value of an Akhmediev breather (periodic in space) for $a = \frac{1}{4}$.

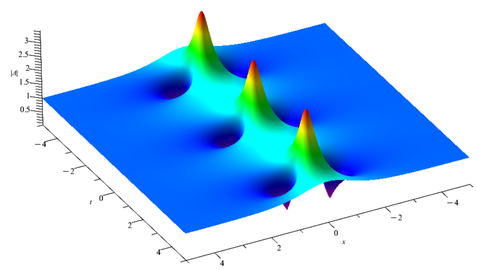

Figure 4: Absolute value of a Kuznetsov-Ma soliton (periodic in time) for $a = \frac{3}{4}$.

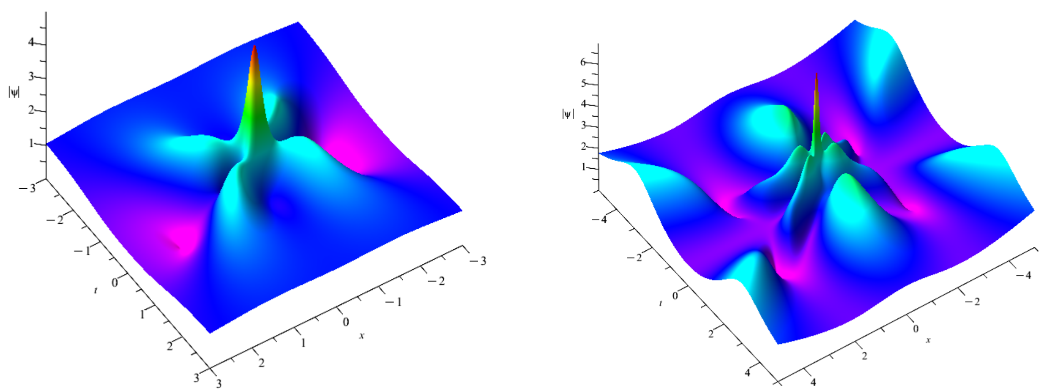

Figure 5: Absolute value of the higher order Peregrine soliton for $j = 2, 3$.

References

[AASC09] Nail Akhmediev, Adrian Ankiewicz, and J. M. Soto-Crespo. Rogue waves and rational solutions of the nonlinear schrödinger equation. *Phys. Rev. E*, 80:026601, Aug 2009.

[AAT09] N. Akhmediev, A. Ankiewicz, and M. Taki. Waves that appear from nowhere and disappear without a trace. *Physics Letters A*, 373(6):675–678, 2009.

[DH18] Wolf-Patrick Düll and Max Heß. Existence of long time solutions and validity of the nonlinear Schrödinger approximation for a quasilinear dispersive equation. *J. Differ. Equations*, 264(4):2598–2632, 2018.

[Dül17] Wolf-Patrick Düll. Justification of the nonlinear Schrödinger approximation for a quasilinear Klein-Gordon equation. *Commun. Math. Phys.*, 355(3):1189–1207, 2017.

[Dül21] Wolf-Patrick Düll. Validity of the nonlinear Schrödinger approximation for the two-dimensional water wave problem with and without surface tension in the arc length formulation. *Arch. Ration. Mech. Anal.*, 239(2):831–914, 2021.

[GTY+17] Boling Guo, Lixin Tian, Zhenya Yan, Liming Ling, and Yu-Feng Wang. *Rogue waves. Mathematical theory and applications in physics*. Berlin: De Gruyter, 2017.

[Kal88] L. A. Kalyakin. Asymptotic decay of a one-dimensional wavepacket in a nonlinear dispersive medium. *Math. USSR, Sb.*, 60(2):457–483, 1988.

[KSM92] Pius Kirrmann, Guido Schneider, and Alexander Mielke. The validity of modulation equations for extended systems with cubic nonlinearities. *Proc. R. Soc. Edinb., Sect. A, Math.*, 122(1-2):85–91, 1992.

[Per83] D. H. Peregrine. Water waves, nonlinear schrödinger equations and their solutions. *The Journal of the Australian Mathematical Society. Series B. Applied Mathematics*, 25(1):16–43, 1983.

[Sch05] Guido Schneider. Justification and failure of the nonlinear Schrödinger equation in case of non-trivial quadratic resonances. *J. Differ. Equations*, 216(2):354–386, 2005.

[SSZ15] Guido Schneider, Danish Ali Sunny, and Dominik Zimmermann. The NLS approximation makes wrong predictions for the water wave problem in case of small surface tension and spatially periodic boundary conditions. *J. Dyn. Differ. Equations*, 27(3-4):1077–1099, 2015.

[SU17] Guido Schneider and Hannes Uecker. *Nonlinear PDEs. A dynamical systems approach*, volume 182 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2017.

[WC17] C. Eugene Wayne and Patrick Cummings. Modified energy functionals and the NLS approximation. *Discrete Contin. Dyn. Syst.*, 37(3):1295–1321, 2017.

[Zak68] V.E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Sov. Phys. J. Appl. Mech. Tech. Phys*, 4:190–194, 1968.