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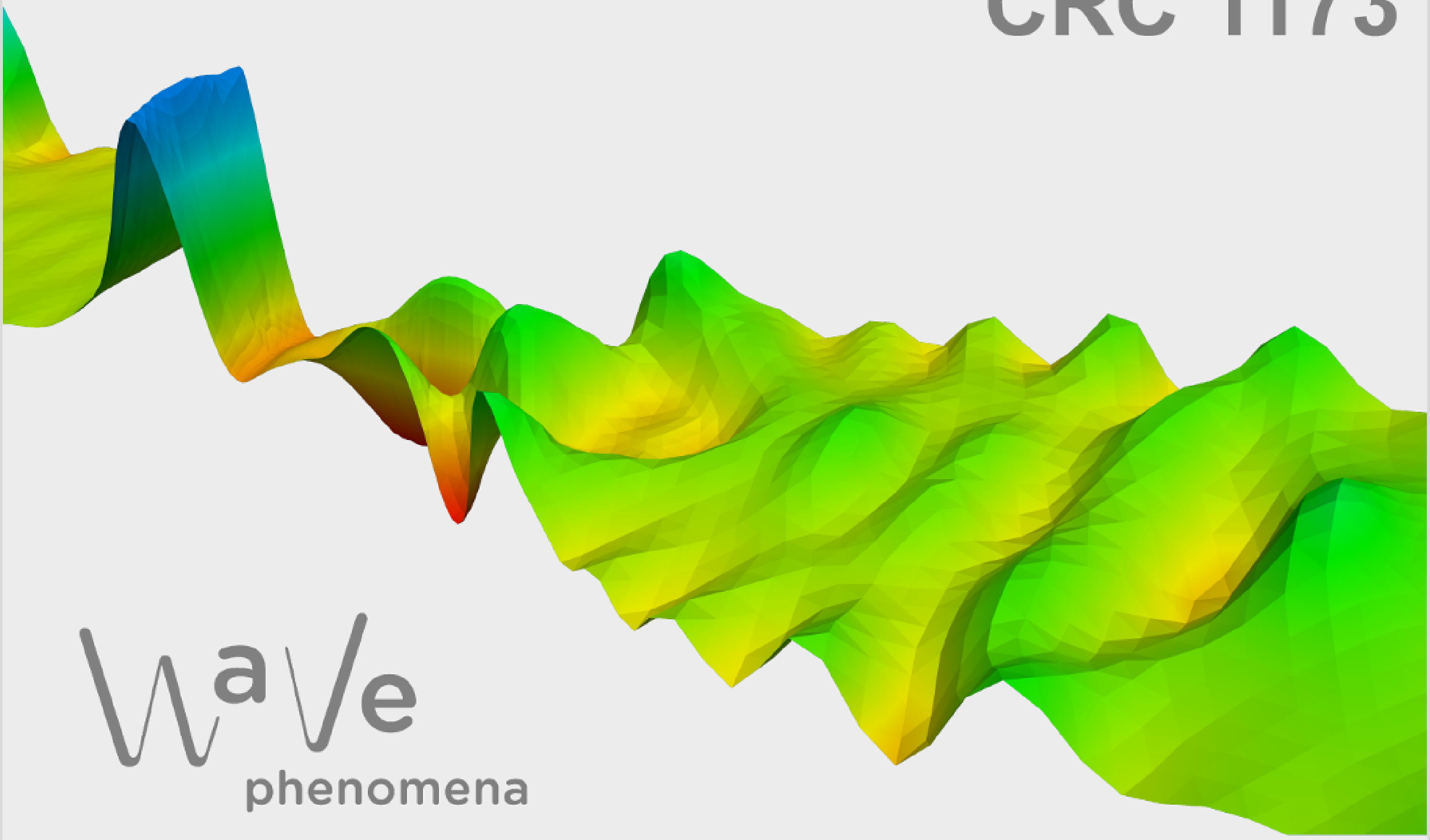

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A linear Schrödinger approximation for the KdV equation via IST beyond the natural NLS time scale

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Abstract

We are interested in improving validity results for the Nonlinear Schrödinger (NLS) approximation beyond the natural time scale for completely integrable systems. As a first step, we consider this approximation for the Korteweg-de Vries (KdV) equation with initial conditions for which the scattering data contains no eigenvalues. By performing a linear Schrödinger approximation for the scattering data the error made by this approximation has only to be estimated for a purely linear problem which gives estimates beyond the natural NLS time scale. The inverse scattering transform allows us to transfer these estimates to the original variables.

1 Introduction

The Nonlinear Schrödinger (NLS) equation describes slow modulations in time and space of oscillating wave packets in dispersive wave systems. It was derived through a multiple scaling perturbation ansatz in [Zak68] first. Various approximation results have been established in the mean-time, cf. [Kal88, KSM92, Sch05, TW12]. See [Dül21] which contains a recent overview. We are interested in the question whether it is possible to extend the validity of the NLS approximation for completely integrable systems beyond the natural time scale of the NLS approximation. As a first step in this direction, in this paper, we consider this question for the Korteweg-de Vries (KdV) equation which serves as a simple example of a completely integrable system.

Using the Miura transformation, cf. [DJ89], and Gronwall's inequality, in [Sch11] a simple proof was given that the NLS approximation

$$\varepsilon\Psi_u = \varepsilon\mathcal{A}(\varepsilon(x-c_u t), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_u x - \omega_u t)} + c.c. + \mathcal{O}(\varepsilon^2), \tag{1}$$

with $\mathcal{A}(X,T) \in \mathbb{C}$, $X = \varepsilon(x - c_u t)$, $T = \varepsilon^2 t$, $c_u = c_u(k_u)$ the linear group velocity, *c.c.* the complex conjugated terms and $k_u, \omega_u \in \mathbb{R}$ satisfying the linear dispersion relation, makes correct predictions about the dynamics of the KdV equation

$$\partial_t u - 6u\partial_x u + \partial_x^3 u = 0, \tag{2}$$

with $u(x,t) \in \mathbb{R}$, $x, t \in \mathbb{R}$, if $\mathcal{A}$ is chosen to be a solution of the defocusing NLS equation

$$\partial_T \mathcal{A} = -3\mathrm{i}k_u \partial_X^2 \mathcal{A} - 6\mathrm{i}k_u \mathcal{A}|\mathcal{A}|^2. \tag{3}$$

In detail, it was shown

Theorem 1.1 ([Sch11]). *Fix $s \geq 1$ and let $\mathcal{A} \in C([0,T_0], H^{s+4}(\mathbb{R},\mathbb{C}))$ be a solution of the NLS equation* (3). *Then there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there are solutions $u \in L^\infty([0,T], H^s(\mathbb{R},\mathbb{R}))$ of the KdV equation* (2) *with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|u(\cdot,t) - \varepsilon\Psi_u(\cdot,t)\|_{H^s} \leq C\varepsilon^{3/2}.$$

As already said, we are interested in improving such validity results for the NLS approximation beyond the natural $\mathcal{O}(1/\varepsilon^2)$-NLS time scale for completely integrable systems, here the KdV equation. We do so by restricting ourselves to initial conditions of the KdV equation for which the scattering data contains no eigenvalues and by performing an NLS approximation for the scattering variable b associated to the essential spectrum. Since the equation for b is linear, the NLS equation degenerates into a linear Schrödinger equation. On the level of the scattering variables the error made by this approximation has to be only estimated for a linear problem which gives estimates beyond the natural NLS time scale, cf Section 3. Hence, our approach allows us to extend the approximation time from $\mathcal{O}(1/\varepsilon^2)$ to $\mathcal{O}(1/\varepsilon^{3-\delta})$ with $\delta > 0$ arbitrarily small, but fixed. The inverse scattering transform finally allows us to transfer these results to the original variables, cf. Section 4 and Section 5. In particular for the last step, a number of functional analytic difficulties occur, cf. Remark 5.1. Our main result is formulated in Theorem 5.3. The paper is closed with some discussions where in particular we discuss the relation between the NLS approximation and the linear Schrödinger approximation for the KdV equation. See Remark 6.10.

For the KdV equation the subsequent Theorem 5.3 van be interpreted as a statement about long-time but transient dynamics. It makes no statement about the asymptotic behavior of the solutions of the KdV equation for $t \to \infty$. A more and more detailed description of the asymptotic behavior can be found in a number of papers such as [AS77, EvH81, DZ94, GT09].

Notation. Throughout this paper many possible different constants are denoted with the same symbol C if they can be chosen independently of the small perturbation parameter $0 < \varepsilon \ll 1$. Here and in the following $\int_{\mathbb{R}}$, respectively $\int_{-\infty}^{\infty}$, is abbreviated by $\int$.
Let $u(x) \in L^2(\mathbb{R})$, then its Fourier transform $\widehat{u}(\xi) \in L^2(\mathbb{R})$ is defined by

$$\widehat{u}(\xi) = \int \mathrm{e}^{-\mathrm{i}x\xi} u(x) dx.$$

The Sobolev space H^s, $s \geq 0$, of s times weakly differentiable functions is equipped with the norm

$$\|u\|_{H^s} = \left(\sum_{j=0}^{s} \int |\partial_x^j u(x)|^2 \mathrm{d}x \right)^{1/2}.$$

The weighted Lebesgue space L_s^2 is equipped with the norm

$$\|\widehat{u}\|_{L_s^2} = \left(\int |\widehat{u}(k)|^2 (1+k^2)^s \mathrm{d}k \right)^{1/2}.$$

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2 IST for the KdV equation

It is well known that the KdV equation

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u$$

can be solved with the help of the inverse scattering transform (IST). Since this theory plays a fundamental role in the following, we recall its basics for completeness. For more details we refer to [DJ89].

For a solution $u = u(x,t)$ of the KdV equation (2) we consider the associated (quantum mechanical) scattering problem, namely

$$L\psi = -\partial_x^2\psi - u\psi = \lambda\psi. \tag{4}$$

i) The scattering problem is to find the eigenvalues/spectral values $\lambda_k(t) \in \mathbb{R}$ and the associated (formal) eigenfunctions $\psi_k(\cdot,t)$ for a given $u = u(\cdot,t)$ where k is in some index set I.

ii) The inverse scattering problem is to reconstruct $u = u(\cdot,t)$ from the scattering data $\lambda_k(t)$ and $\psi_k(\cdot,t)$ for $k \in I$.

2.1 The scattering problem

The KdV equation is a completely integrable Hamiltonian system for which there exists a Lax pair formulation

$$\partial_t L = ML - LM,$$

with self-adjoint L defined in (4) and skew-symmetric

$$M\psi = -4\partial_x^3\psi - 3(u\partial_x\psi + (\partial_x u)\psi).$$

We recall that the Lax pair formulation has to be understood in the sense that $(\partial_t L)\psi = (\partial_t u)\psi$ and $(ML - LM)\psi = (-\partial_x^3 u + 6u\partial_x u)\psi$ are pure multiplication operators such that no ∂_x falls on ψ. The Lax pair representation implies that the eigenvalues/spectral values $\lambda_k(t)$ of the operator L are independent of time. The eigenfunctions $\psi_k(\cdot,t)$ satisfy

$$\partial_t\psi_k(\cdot,t) = M\psi_k(\cdot,t).$$

For spatially localized u the operator L possesses essential spectrum $[0,\infty)$ and a finite number, say N_0, of negative eigenvalues λ_n, with $n = 1,\ldots,N_0$.

i) The eigenfunctions to the negative eigenvalues decay with some exponential rate for $|x| \to \infty$, in particular we have

$$\psi_n(x,t) \sim c_n(t)\mathrm{e}^{-\kappa_n x},$$

for $x \to \infty$, where $\kappa_n^2 = -\lambda_n$ and $\kappa_n > 0$, which defines $c_n(t)$. It turns out that the coefficient $c_n(t)$ satisfies the simple evolution equation

$$\partial_t c_n = 4\kappa_n^3 c_n$$

which is solved by $c_n(t) = c_n(0)\mathrm{e}^{4\kappa_n^3 t}$.

ii) The eigenfunctions to the spectral values $\lambda_k = k^2 \in [0, \infty)$ for $k \in \mathbb{R}$ are of the form

$$\psi_k(x,t) \sim \mathrm{e}^{-\mathrm{i}kx} + \widehat{b}(k,t)\mathrm{e}^{\mathrm{i}kx}, \qquad \text{for } x \to \infty,$$

and

$$\psi_k(x,t) \sim \widehat{a}(k,t)\mathrm{e}^{-\mathrm{i}kx}, \qquad \text{for } x \to -\infty.$$

It turns out that because of

$$|\widehat{a}(k,t)|^2 + |\widehat{b}(k,t)|^2 = 1, \tag{5}$$

it is sufficient to control the coefficients $\widehat{b}(k,t)$ which satisfy the simple evolution equations

$$\partial_t \widehat{b}(k,t) = 8\mathrm{i}k^3 \widehat{b}(k,t). \tag{6}$$

2.2 The inverse scattering problem

The solution $u = u(x,t)$ can be reconstructed from the scattering data

$$\{\lambda_n, c_n(t), n = 1, \ldots, N_0; \widehat{b}(k,t), k \in \mathbb{R}\}$$

by solving the Gelfand-Levitan-Marchenko equation

$$K(x,y,t) + F(x+y,t) + \int_x^\infty K(x,z,t)F(y+z,t)\mathrm{d}z = 0 \tag{7}$$

for $K(x,y,t)$ with $y \geq x$ where

$$F(x,t) = \sum_{j=1}^{N_0} c_j^2(t)\mathrm{e}^{-\kappa_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\widehat{b}(k,t)\mathrm{d}k.$$

The solution is then given by

$$u(x,t) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x^+,t),$$

where x^+ indicates that the derivative is computed as right-hand limit in the second variable. The time t appears in these calculations only as a parameter. In the integral equation (7) also the variable x is a parameter.

3 The approximation for the scattering data

In this section we construct a Schrödinger approximation for the scattering variables $\widehat{b}(k,t)$, i.e., in the following we consider the case of no eigenvalues, i.e., we assume $N_0 = 0$ and comment on this assumption later on in Section 6.

The evolution equation (6) for the scattering variables $\widehat{b}(k,t)$ is solved by

$$\widehat{b}(k,t) = \mathrm{e}^{8\mathrm{i}k^3 t}\widehat{b}(k,0).$$

If k is interpreted as Fourier wave number and $\widehat{b}$ as Fourier transform of a function b, then b satisfies the so-called Airy equation

$$\partial_t b(x,t) = -8\partial_x^3 b(x,t). \tag{8}$$

For this equation we make the NLS ansatz

$$\varepsilon\Psi_{NLS}(x,t) = \varepsilon A(\varepsilon(x-c_0t),\varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0 t)} + c.c., \tag{9}$$

with some fixed $k_0 \in \mathbb{R}$. Plugging this ansatz into (8) and equating the coefficients of $\varepsilon^n \mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)}$ to zero gives the linear dispersion relation $\omega_0 = -8k_0^3$ at $\mathcal{O}(\varepsilon)$, the group velocity $c_0 = -24k_0^2$ at $\mathcal{O}(\varepsilon^2)$, and the linear Schrödinger equation

$$\partial_T A = -24\mathrm{i}k_0\partial_X^2 A \tag{10}$$

at $\mathcal{O}(\varepsilon^3)$. For this equation we have the global existence of solutions in every H^s for each $s \geq 0$ since the H^s-norm is conserved

$$\begin{aligned} \|A(\cdot,T)\|_{H^s} &= \|\mathrm{e}^{24\mathrm{i}k_0k^2T}\widehat{A}(K,0)(1+K^2)^{s/2}\|_{L^2_s(\mathrm{d}K)} \\ &= \|\widehat{A}(K,0)(1+K^2)^{s/2}\|_{L^2_s(\mathrm{d}K)} = \|A(\cdot,0)\|_{H^s}. \end{aligned} \tag{11}$$

We have the following approximation result:

Theorem 3.1 (**Approximation for** b**).** *For each $s \geq 0$ there exist $C > 0$ and $\varepsilon_0 > 0$ such that the following holds. Let $A \in C([0,\infty), H^{s+3})$ be a solution of the linear Schrödinger equation* (10). *Then for all $\varepsilon \in (0,\varepsilon_0)$ there is a unique solution b of the Airy equation* (8) *with initial data*

$$b(x,0) = \varepsilon A(\varepsilon x, 0)\mathrm{e}^{\mathrm{i}k_0x} + c.c.$$

such that

$$\sup_{t\in[0,t_0]} \|b(x,t)-(\varepsilon A(\varepsilon(x-c_0t),\varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)}+c.c.)\|_{H^s(\mathrm{d}x)} \leq C\varepsilon^{7/2}t_0\|A(\cdot,0)\|_{H^{s+3}},$$

for all $t_0 \geq 0$.

Proof. Let

$$R(x,t) = b(x,t) - (\varepsilon A(\varepsilon(x-c_0t), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)} + c.c.),$$

with $R(\cdot,0) = 0$. This error function satisfies

$$\partial_t R = -8\partial_x^3 R - \varepsilon^4(8\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)}\partial_X^3 A + c.c.).$$

Applying the variation of constant formula yields

$$R(\cdot,t) = -\int_0^t \mathrm{e}^{-8\partial_x^3(t-\tau)}\varepsilon^4(8\mathrm{e}^{\mathrm{i}(k_0x-\omega_0\tau)}\partial_X^3 A + c.c.)(\cdot,\tau)d\tau.$$

Taking care of the fact that we lose a factor $\varepsilon^{-1/2}$ due to the scaling properties of the $L^2(\mathbb{R})$-norm under $x \mapsto \varepsilon x$, we immediately find the estimate

$$\|R(\cdot,t)\|_{H^s} \leq C\varepsilon^4 t\varepsilon^{-1/2} \sup_{\tau\in[0,t]} \|A(\cdot,\tau)\|_{H^{s+3}} \leq C\varepsilon^{7/2} t\|A(\cdot,0)\|_{H^{s+3}}$$

due to (11). □

Corollary 3.2. *For each $s \geq 0$ and $\delta \in (0,1]$ there exist $C > 0$ and $\varepsilon_0 > 0$ such that the following holds. Let $A \in C([0,\infty), H^{s+3})$ be a solution of the linear Schrödinger equation* (10)*. Then for all $\varepsilon \in (0,\varepsilon_0)$ there is a unique solution b of the Airy equation* (8) *with $b(x,0) = \varepsilon A(\varepsilon x, 0)\mathrm{e}^{\mathrm{i}k_0x} + c.c.$ such that*

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]} \|b(x,t) - (\varepsilon A(\varepsilon(x-c_0t), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)} + c.c.)\|_{H^s(\mathrm{d}x)} \leq C\varepsilon^{1/2+\delta}.$$

Remark 3.3. The error of order $\mathcal{O}(\varepsilon^{1/2+\delta})$ is still smaller than the solution and the approximation which both are of order $\mathcal{O}(\varepsilon^{1/2})$ in H^s, $s \geq 0$. Thus, we improved the approximation time from $\mathcal{O}(1/\varepsilon^2)$ to $\mathcal{O}(1/\varepsilon^{3-\delta})$ with $\delta > 0$ arbitrarily small, but fixed.

Remark 3.4. The Schrödinger equation shows a decay rate like $T^{-1/2}$ for $T \to \infty$, whereas the Airy equation shows a decay rate like $t^{-1/3}$ for $t \to \infty$. The Fourier modes of the Schrödinger approximation are strongly concentrated at $k = k_0$, see Remark 3.9. Therefore, for $A \in H^s$ the part around $k = 0$, showing the slower decay rate $t^{-1/3}$, is ε^s initially. This part and the Schrödinger part at $k = k_0$ are of the same order if $\varepsilon^s t^{-1/3} = T^{-1/2} = (\varepsilon^2 t)^{-1/2}$, i.e., for $t = 1/\varepsilon^{6(s+1)} \gg 1/\varepsilon^3$. The faster decay rate of the Schrödinger equation is thus manifested outside its range of validity.

Higher order approximations can be computed, too.

Remark 3.5. The ansatz for the computation of higher order approximations is given by

$$\varepsilon\tilde{\Psi}_b(x,t) = \sum_{n=1}^{N} \varepsilon^n A_n(\varepsilon(x-c_0t),\varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)} + c.c.,$$

leading to the approximation equations

$$\partial_T A_1 = -24\mathrm{i}k_0\partial_X^2 A_1, \qquad \partial_T A_n = -24\mathrm{i}k_0\partial_X^2 A_n - 8\partial_X^3 A_{n-1},$$

for $n \in \{2,\ldots,N\}$, with a fixed $N \in \mathbb{N}$, and where $A_1 = A$ from above. These approximation equations for $n \geq 2$ can be solved with the variation of constant formula

$$A_n(\cdot,T) = -\int_0^T \mathrm{e}^{-24\mathrm{i}k_0\partial_X^2(T-\tau)} 8\partial_X^3 A_{n-1}(\cdot,\tau)\mathrm{d}\tau$$

where we have chosen vanishing initial conditions $A_n(\cdot,0) = 0$ for $n \in \{2,\ldots,N\}$. This immediately gives the estimate

$$\sup_{0\leq\tau\leq T} \|A_n(\cdot,\tau)\|_{H^s} \leq CT \sup_{0\leq\tau\leq T} \|A_{n-1}(\cdot,\tau)\|_{H^{s+3}}.$$

Therefore, we need

$$A_1 \in H^{s+3N}, \quad A_2 \in H^{s+3N-3}, \quad A_3 \in H^{s+3N-6}, \quad \ldots, \quad A_N \in H^{s+3}.$$

The error function then satisfies

$$\partial_t R = -8\partial_x^3 R - \varepsilon^{N+3}(8\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)}\partial_X^3 A_N + c.c.).$$

Remark 3.6. For obtaining estimates for the higher order approximation on the long $\mathcal{O}(1/\varepsilon^{3-\delta})$-time scale with $\delta > 0$ arbitrarily small, but fixed, we modify the ansatz into

$$\varepsilon\Psi_b(x,t) = \sum_{n=1}^{N} \varepsilon^{1+(n-1)\delta} A_n(\varepsilon(x-c_0t),\varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)} + c.c., \tag{12}$$

leading to the approximation equations

$$\partial_T A_1 = -24\mathrm{i}k_0\partial_X^2 A_1, \qquad \partial_T A_n = -24\mathrm{i}k_0\partial_X^2 A_n - 8\varepsilon^{1-\delta}\partial_X^3 A_{n-1},$$

with $n = 2, ..., N$. Since

$$\sup_{0\leq\tau\leq T} \|A_n(\cdot,\tau)\|_{H^s} \leq C\varepsilon^{1-\delta} T \sup_{0\leq\tau\leq T} \|A_{n-1}(\cdot,\tau)\|_{H^{s+3}},$$

all A_n remain $\mathcal{O}(1)$-bounded for $t \in [0, 1/\varepsilon^{3-\delta}]$. The error function then satisfies

$$\partial_t R = -8\partial_x^3 R - \varepsilon^{1+(N-1)\delta+3}(8\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} \partial_X^3 A_N + c.c.)$$

and so

$$\|R(\cdot,t)\|_{H^s} \leq C\varepsilon^{1+(N-1)\delta+3} t \varepsilon^{-1/2} \sup_{\tau\in[0,\varepsilon^2 t]} \|A_N(\cdot,\tau)\|_{H^{s+3}}.$$

Thus, we have proven

Theorem 3.7. *For each $N \in \mathbb{N}$, $s \geq 0$ and $\delta \in (0,1]$ there exist $C > 0$ and $\varepsilon_0 > 0$ such that the following holds. Let $A_1 \in C([0,\infty), H^{s+3N})$ be a solution of the linear Schrödinger equation (10) and let the A_n be solutions of*

$$\partial_T A_n = -24\mathrm{i}k_0 \partial_X^2 A_n - 8\varepsilon^{1-\delta} \partial_X^3 A_{n-1}, \qquad A_n|_{T=0} = 0$$

for $n = 2, \ldots, N$. Then for all $\varepsilon \in (0, \varepsilon_0)$ there is a unique solution b of the Airy equation (8) such that

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]} \|b(x,t) - \varepsilon\Psi_b(x,t)\|_{H^s(dx)} \leq C\varepsilon^{1/2+N\delta}$$

where

$$\varepsilon\Psi_b(x,t) = \sum_{n=1}^{N} \varepsilon^{1+(n-1)\delta} A_n(\varepsilon(x - c_0 t), \varepsilon^2 t) \mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} + c.c..$$

Remark 3.8. Sobolev's embedding theorem immediately yields

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]} \sup_{x\in\mathbb{R}} |b(x,t) - \varepsilon\Psi_b(x,t)| \leq C\varepsilon^{1/2+N\delta},$$

which is a non-void estimate if $1/2 + N\delta > 1$. Then, the error of order $\mathcal{O}(\varepsilon^{1/2+N\delta})$ is smaller than the solution and the approximation which both are of order $\mathcal{O}(\varepsilon)$ in C_b^0.

Remark 3.9. Since we are handling linear inhomogeneous equations, the above analysis holds in various other function spaces. For the subsequent analysis we need an L^∞-bound in Fourier space. Rewriting Remark 3.6 for obtaining estimates for the higher order approximation on the long $\mathcal{O}(1/\varepsilon^{3-\delta})$-time scale with $\delta > 0$ arbitrarily small, the modified ansatz in Fourier space is given by

$$\widehat{\varepsilon\Psi_b}(k,t) = \sum_{n=1}^{N} \varepsilon^{(n-1)\delta} \widehat{A}_n(\varepsilon^{-1}(k-k_0), \varepsilon^2 t) e^{i(-\omega_0 t) - ic_0(k-k_0)t} + c.c.f.,$$

leading to the approximation equations

$$\partial_T \widehat{A}_1 = 24ik_0 K^2 \widehat{A}_1, \qquad \partial_T \widehat{A}_n = 24ik_0 K^2 \widehat{A}_n + 8i\varepsilon^{1-\delta} K^3 \widehat{A}_{n-1}$$

where $c.c.f.$ corresponds to the complex conjugate in Fourier space. Since

$$\sup_{0\le\tau\le T} \|\widehat{A}_n(\cdot,\tau)\|_{L^\infty_s} \le C\varepsilon^{1-\delta} T \sup_{0\le\tau\le T} \|\widehat{A}_{n-1}(\cdot,\tau)\|_{L^\infty_{s+3}},$$

where

$$\|\widehat{A}(\cdot,\tau)\|_{L^\infty_s} = \sup_{K\in\mathbb{R}} |\widehat{A}(K,\tau)(1+K^2)^{s/2}|,$$

all $\widehat{A}_n$ remain $\mathcal{O}(1)$-bounded for $t \in [0, 1/\varepsilon^{3-\delta}]$. The error function then satisfies

$$\partial_t \widehat{R} = 8ik^3 \widehat{R} - \varepsilon^{(N-1)\delta+3}(8e^{i(-\omega_0 t) - ic_0(k-k_0)t}(iK)^3 \widehat{A}_N + c.c.f.),$$

and so

$$\begin{aligned} \|\widehat{R}(\cdot,t)\|_{L^\infty_s} &\le C\varepsilon^{(N-1)\delta+3} t \sup_{\tau\in[0,\varepsilon^2 t]} \|\widehat{A}_N(\cdot,\tau)\|_{L^\infty_{s+3}} \\ &\le C\varepsilon^{N\delta} \sup_{\tau\in[0,1/\varepsilon^{1-\delta}]} \|\widehat{A}_1(\cdot,\tau)\|_{L^\infty_{s+3N}}, \end{aligned}$$

for all $t \in [0, 1/\varepsilon^{3-\delta}]$.

4 The approximation of the KdV solutions via IST

In this section we use the Gelfand-Levitan-Marchenko equation to construct the approximation $\varepsilon\Psi_u$ for the KdV equation (2) associated to the linear

Schrödinger approximation $\varepsilon\Psi_b$ for a fixed N from Theorem 3.7. We compute

$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\widehat{b}(k,t)\mathrm{d}k = b(x,t)$$

for the solutions constructed in Section 3, i.e., for $b = \varepsilon\Psi_b$. Then, we set

$$\varepsilon\Psi_u(x,t) = -2\frac{\mathrm{d}}{\mathrm{d}x}(\varepsilon\Psi_K)(x,x^+,t) \tag{13}$$

where $\varepsilon\Psi_K$ is an approximate solution of the Gelfand-Levitan-Marchenko equation

$$\varepsilon\Psi_K(x,y,t) + \varepsilon\Psi_b(x+y,t) + \varepsilon^2\int_x^{\infty} \Psi_K(x,z,t)\Psi_b(y+z,t)\mathrm{d}z = 0, \tag{14}$$

with $y \geq x$. In the following we explain how to compute $\varepsilon\Psi_K$ iteratively. We have

$$\varepsilon\Psi_b(x,t) = \varepsilon A(\varepsilon(x-c_0t),\varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)} + c.c. + h.o.t..$$

In the subsequent computations **i)-iv)** we restrict ourselves to $\varepsilon\Psi_b$ without the h.o.t. for explaining the underlying approach. The general ansatz for $\varepsilon\Psi_K$ can be found in **v)**.

i) For approximately solving (14) we use perturbation theory. We make the ansatz

$$\varepsilon\Psi_K(x,y,t) = \varepsilon K_1(\varepsilon x,\varepsilon y,t)\mathrm{e}^{\mathrm{i}(k_0x+k_0y-\omega_0t)} + c.c.,$$

and compute

$$
\begin{aligned}
&\varepsilon^2 \int_x^\infty \Psi_K(x,z,t)\Psi_b(y+z,t)\mathrm{d}z \\
=\ & \int_x^\infty \varepsilon K_1(\varepsilon x, \varepsilon z, t)\varepsilon A(\varepsilon(y+z-c_0 t), \varepsilon^2 t)\mathrm{e}^{2\mathrm{i}k_0 z}\mathrm{d}z\mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} + c.c. \\
=\ & \varepsilon^2 K_1(\varepsilon x, \varepsilon z, t)A(\varepsilon(y+z-c_0 t), \varepsilon^2 t)\ \frac{\mathrm{e}^{2\mathrm{i}k_0 z}}{2\mathrm{i}k_0}\Big|_{z=x}^{\infty} \mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} \\
& - \int_x^\infty \varepsilon^3 \partial_Z (K_1(\varepsilon x, \varepsilon z, t)A(\varepsilon(y+z-c_0 t), \varepsilon^2 t))\frac{\mathrm{e}^{2\mathrm{i}k_0 z}}{2\mathrm{i}k_0}\mathrm{d}z\mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} + c.c. \\
=\ & -\varepsilon^2 K_1(\varepsilon x, \varepsilon x, t)A(\varepsilon(y+x-c_0 t), \varepsilon^2 t)\frac{\mathrm{e}^{2\mathrm{i}k_0 x}}{2\mathrm{i}k_0}\mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} \\
& + \varepsilon^3 \partial_Z (K_1(\varepsilon x, \varepsilon x, t)A(\varepsilon(y+x-c_0 t), \varepsilon^2 t))\frac{\mathrm{e}^{2\mathrm{i}k_0 x}}{(2\mathrm{i}k_0)^2}\mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} \\
& + \int_x^\infty \varepsilon^4 \partial_Z^2 (K_1(\varepsilon x, \varepsilon z, t)A(\varepsilon(y+z-c_0 t), \varepsilon^2 t))\frac{\mathrm{e}^{2\mathrm{i}k_0 z}}{(2\mathrm{i}k_0)^2}\mathrm{d}z\mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t} + c.c. \\
=\ & \ldots,
\end{aligned}
$$

such that equating the coefficient of $\varepsilon\mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-\mathrm{i}\omega_0 t}$ in (14) to zero yields

$$
K_1(\varepsilon x, \varepsilon y, t) = -A(\varepsilon(x+y-c_0 t), \varepsilon^2 t).
$$

The solution of the KdV equation is then given by

$$
u(x,t) = \varepsilon u_1(x,t) + h.o.t.,
$$

where

$$
\begin{aligned}
u_1(x,t) &= -2\frac{\mathrm{d}}{\mathrm{d}x}(K_1(\varepsilon x, \varepsilon x^+, t)\mathrm{e}^{\mathrm{i}(k_0 x + k_0 x - \omega_0 t)} + c.c.) \\
&= 2\frac{\mathrm{d}}{\mathrm{d}x}(A(\varepsilon(x+x-c_0 t), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(k_0 x + k_0 x - \omega_0 t)} + c.c.) \\
&= 4\mathrm{i}k_0 A(\varepsilon(2x-c_0 t), \varepsilon^2 t)\mathrm{e}^{2\mathrm{i}k_0 x - \mathrm{i}\omega_0 t} + c.c. \\
&\quad + 4\varepsilon(\partial_X A)(\varepsilon(2x-c_0 t), \varepsilon^2 t)\mathrm{e}^{2\mathrm{i}k_0 x - \mathrm{i}\omega_0 t} + c.c..
\end{aligned}
$$

ii) For getting rid of the terms of order $\mathcal{O}(\varepsilon^2)$ at $\mathrm{e}^{\mathrm{i}(3k_0 x + k_0 y - 2\omega_0 t)}$ we extend our ansatz to

$$
\begin{aligned}
\varepsilon\Psi_K(x,y,t) &= \varepsilon K_1(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(k_0 x + k_0 y - \omega_0 t)} + c.c. \\
&\quad + \varepsilon^2 K_2(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(3k_0 x + k_0 y - 2\omega_0 t)} + c.c. + h.o.t..
\end{aligned}
$$

Equating the coefficient of $\varepsilon^2 \mathrm{e}^{3\mathrm{i}k_0x+\mathrm{i}k_0y}\mathrm{e}^{-2\mathrm{i}\omega_0 t}$ in (14) to zero yields

$$\begin{aligned} K_2(\varepsilon x, \varepsilon y, t) &= \frac{1}{2\mathrm{i}k_0} K_1(\varepsilon x, \varepsilon y, t) A(\varepsilon(x+y-c_0t), \varepsilon^2 t) \\ &= -\frac{1}{2\mathrm{i}k_0} A(\varepsilon(x+y-c_0t), \varepsilon^2 t) A(\varepsilon(x+y-c_0t), \varepsilon^2 t). \end{aligned}$$

The next order solution of the KdV equation is then given by

$$u(x,t) = \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + h.o.t.,$$

where

$$\begin{aligned} u_2(x,t) &= -2\frac{\mathrm{d}}{\mathrm{d}x}(K_2(\varepsilon x, \varepsilon x^+, t)\mathrm{e}^{\mathrm{i}(3k_0x+k_0x-2\omega_0t)} + c.c.) \\ &= \frac{1}{\mathrm{i}k_0}\frac{\mathrm{d}}{\mathrm{d}x}(A^2(\varepsilon(2x-c_0t), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0t)} + c.c.) \\ &= 4A^2(\varepsilon(2x-c_0t), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0t)} + c.c. \\ &\quad + \frac{2}{\mathrm{i}k_0}\varepsilon(\partial_X(A^2))(\varepsilon(2x-c_0t), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0t)} + c.c.. \end{aligned}$$

iii) We use the same idea to get rid of the terms of order $\mathcal{O}(\varepsilon^3)$ at $\mathrm{e}^{\mathrm{i}(5k_0x+k_0y-3\omega_0t)}$. Again we extend our ansatz to

$$\begin{aligned} \varepsilon\Psi_K(x,y,t) &= \varepsilon K_1(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(k_0x+k_0y-\omega_0t)} + c.c. \\ &\quad + \varepsilon^2 K_2(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(3k_0x+k_0y-2\omega_0t)} + c.c. \\ &\quad + \varepsilon^3 K_3(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(5k_0x+k_0y-3\omega_0t)} + c.c. + h.o.t.. \end{aligned}$$

Equating the coefficient of $\varepsilon^3 \mathrm{e}^{5\mathrm{i}k_0x+\mathrm{i}k_0y}\mathrm{e}^{-3\mathrm{i}\omega_0 t}$ to zero in (14) yields

$$\begin{aligned} K_3(\varepsilon x, \varepsilon y, t) &= \frac{1}{2\mathrm{i}k_0} K_2(\varepsilon x, \varepsilon y, t) A(\varepsilon(x+y-c_0t), \varepsilon^2 t) \\ &= \frac{1}{4k_0^2} A^2(\varepsilon(x+y-c_0t), \varepsilon^2 t) A(\varepsilon(x+y-c_0t), \varepsilon^2 t). \end{aligned}$$

The next order solution of the KdV equation is then given by

$$u(x,t) = \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \varepsilon^3 u_3(x,t) + h.o.t.,$$

where

$$\begin{aligned}
u_3(x,t) &= -2\frac{\mathrm{d}}{\mathrm{d}x}(K_3(\varepsilon x, \varepsilon x^+, t)\mathrm{e}^{\mathrm{i}(5k_0x+k_0x-3\omega_0 t)} + c.c.) \\
&= -\frac{1}{2k_0^2}\frac{\mathrm{d}}{\mathrm{d}x}(A^3(\varepsilon(2x-c_0t), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(6k_0x-3\omega_0 t)} + c.c. \\
&= -\frac{3\mathrm{i}}{k_0}A^3(\varepsilon(2x-c_0t), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(6k_0x-3\omega_0 t)} + c.c. \\
&\quad -\frac{1}{k_0^2}\varepsilon(\partial_X(A^3))(\varepsilon(2x-c_0t), \varepsilon^2 t)\mathrm{e}^{\mathrm{i}(6k_0x-3\omega_0 t)} + c.c..
\end{aligned}$$

iv) As a last example we explain how to eliminate the terms of order $\mathcal{O}(\varepsilon^3)$ at $\mathrm{e}^{\mathrm{i}(3k_0x+k_0y-2\omega_0 t)}$. We extend our ansatz to

$$\begin{aligned}
\varepsilon\Psi_K(x,y,t) &= \varepsilon K_1(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(k_0x+k_0y-\omega_0 t)} + c.c. \\
&\quad +\varepsilon^2 K_2(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(3k_0x+k_0y-2\omega_0 t)} + c.c. \\
&\quad +\varepsilon^3 K_3(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(5k_0x+k_0y-3\omega_0 t)} + c.c. \\
&\quad +\varepsilon^3 K_{2,1}(\varepsilon x, \varepsilon y, t)\mathrm{e}^{\mathrm{i}(3k_0x+k_0y-2\omega_0 t)} + c.c. + h.o.t..
\end{aligned}$$

Equating the coefficient of $\varepsilon^3\mathrm{e}^{3\mathrm{i}k_0x+\mathrm{i}k_0y}\mathrm{e}^{-2\mathrm{i}\omega_0 t}$ in (14) to zero yields

$$\begin{aligned}
K_{2,1}(\varepsilon x, \varepsilon y, t) &= -\frac{1}{(2\mathrm{i}k_0)^2}\partial_X\left(K_1(\varepsilon x, \varepsilon y, t)A(\varepsilon(x+y-c_0t), \varepsilon^2 t)\right) \\
&= -\frac{1}{4k_0^2}\partial_X\left(A(\varepsilon(x+y-c_0t), \varepsilon^2 t)A(\varepsilon(x+y-c_0t), \varepsilon^2 t)\right).
\end{aligned}$$

The next order solution of the KdV equation is then given by

$$u(x,t) = \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \varepsilon^3 u_3(x,t) + \varepsilon^3 u_{2,1}(x,t) + h.o.t.,$$

where

$$\begin{aligned}
u_{2,1}(x,t) &= -2\frac{\mathrm{d}}{\mathrm{d}x}(K_{2,1}(\varepsilon x, \varepsilon x^+, t)\mathrm{e}^{\mathrm{i}(3k_0x+k_0x-2\omega_0 t)} + c.c.) \\
&= -\frac{1}{2k_0^2}\frac{\mathrm{d}}{\mathrm{d}x}(\partial_X\left(A^2(\varepsilon(2x-c_0t), \varepsilon^2 t)\right)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0 t)} + c.c. \\
&= -\frac{2\mathrm{i}}{k_0}(\partial_X(A^2))\left(\varepsilon(2x-c_0t), \varepsilon^2 t\right)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0 t)} + c.c. \\
&\quad -\frac{1}{k_0^2}\varepsilon(\partial_X^2(A^2))\left(\varepsilon(2x-c_0t), \varepsilon^2 t\right)\mathrm{e}^{\mathrm{i}(4k_0x-2\omega_0 t)} + c.c..
\end{aligned}$$

v) The inclusion of the h.o.t. does not change the calculations substantially.

Remark 4.1. Approximations $\varepsilon\Psi_K$ of the kernel can be computed up to arbitrary order. For solving (14) we then make the ansatz

$$\varepsilon\Psi_K(x,y,t) = \sum_{n\in I_{\tilde{N}}} \sum_{m=0}^{M_{\tilde{N},n}} \varepsilon^{\beta(n)+m} K_{n,m}(\varepsilon x, \varepsilon y, t) e^{i((2n-1)k_0 x + k_0 y - n\omega_0 t)},$$

with $\beta(n) = 1 + ||n| - 1|$, $I_{\tilde{N}} = \{-\tilde{N}, -\tilde{N}+1, \ldots, \tilde{N}-1, \tilde{N}\}$, and sufficiently large numbers $M_{\tilde{N},n} \in \mathbb{N}_0$. By shifting the integral term to higher orders again and again, as in **i)**, new terms occur which are balanced by extending the ansatz $\varepsilon\Psi_K$ with the new kernels $K_{n,m}$ where $K_{j,0} = K_j$ from above and $u_{j,0} = u_j$ for $j \in \mathbb{N}$.

Remark 4.2. The computation of the kernels seems to be rather complicated but once these calculations are made the formula for the solution

$$\begin{aligned}
u(x,t) &= \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \mathcal{O}(\varepsilon^3) \\
&= 4\varepsilon i k_0 A(\varepsilon(2x - c_0 t), \varepsilon^2 t) e^{2ik_0 x - i\omega_0 t} + c.c. \\
&\quad +4\varepsilon^2 (\partial_X A)(\varepsilon(2x - c_0 t), \varepsilon^2 t) e^{2ik_0 x - i\omega_0 t} + c.c. \\
&\quad +4\varepsilon^2 A^2(\varepsilon(2x - c_0 t), \varepsilon^2 t) e^{i(4k_0 x - 2\omega_0 t)} + c.c. + \mathcal{O}(\varepsilon^3)
\end{aligned}$$

in terms of A is rather simple.

Remark 4.3. In this section we performed the calculations to compute $\varepsilon\Psi_K$ and therefore $\varepsilon\Psi_u$ for the NLS ansatz Ψ_{NLS} in (9). In case $\delta = 1$ for the extended NLS ansatz $\varepsilon\Psi_b$ in (12) we can use the ansatz from Remark 4.1. Since the calculations for the extended ansatz do not differ from the previous ones we refrain from listing the formulas. In case $\delta \in (0,1)$ the ansatz from Remark 4.1 has to be modified in an obvious and straightforward way. Since the general form of $\varepsilon\Psi_K$ is then very lengthy we decided not list it here.

5 Error estimates via IST

Let

$$b(x,t) = \varepsilon\Psi_b(x,t) + \varepsilon^\beta R_b(x,t),$$

where $\beta = \frac{1}{2} + N\delta$ and $\varepsilon\Psi_b$ are given in Theorem 3.7. The kernel K is a sum of the approximation kernel $\varepsilon\Psi_K$ constructed in Section 4 and an error $\varepsilon^\beta R_K$. Plugging

$$K(x,y,t) = \varepsilon\Psi_K(x,y,t) + \varepsilon^\beta R_K(x,y,t)$$

into the Gelfand-Levitan-Marchenko equation (14) yields

$$\varepsilon\Psi_K(x,y,t)+\varepsilon^\beta R_K(x,y,t)+\varepsilon\Psi_b(x+y,t)+\varepsilon^\beta R_b(x+y,t) \\ +\int_x^\infty(\varepsilon\Psi_K(x,z,t)+\varepsilon^\beta R_K(x,z,t))(\varepsilon\Psi_b(y+z,t)+\varepsilon^\beta R_b(y+z,t))\mathrm{d}z=0,$$

and so

$$R_K+s_{inh}+s_{lin}+s_{non}+s_{res}=0, \tag{15}$$

with

$$\begin{aligned} s_{inh}(x,y,t) &= R_b(x+y,t)+\varepsilon\int_x^\infty \Psi_K(x,z,t)R_b(y+z,t)\mathrm{d}z, \\ s_{lin}(x,y,t) &= \varepsilon\int_x^\infty R_K(x,z,t)\Psi_b(y+z,t)\mathrm{d}z, \\ s_{non}(x,y,t) &= \varepsilon^\beta\int_x^\infty R_K(x,z,t)R_b(y+z,t)\mathrm{d}z, \\ s_{res}(x,y,t) &= \varepsilon^{-\beta}\Big(\varepsilon\Psi_K(x,y,t)+\varepsilon\Psi_b(x+y,t) \\ &\qquad +\varepsilon^2\int_x^\infty \Psi_K(x,z,t)\Psi_b(y+z,t)\mathrm{d}z\Big). \end{aligned}$$

The function $R_K(x,y,t)$ vanishes identically for $y<x$ since the Marchenko equation (7) is only valid for $y\geq x$.

The structure of (15) is as follows:

- The term s_{inh} is independent of R_K and does not contain residual terms.
- The term s_{lin} is linear in R_K. This term can be estimated with the help of energy estimates.
- The term s_{non} is nonlinear in R_K and will be of higher order due to the ε^β in front.
- The term s_{res} is the residual, i.e., it contains the terms which do not cancel after inserting the formal approximations Ψ_K and Ψ_b into the Gelfand-Levitan-Marchenko equation (14). The remaining terms can

be written as

$$\begin{aligned} s_{res} &= \varepsilon^{-\beta}(\varepsilon\Psi_K(x,y,t)+\varepsilon\Psi_b(x+y,t) \\ &\qquad\qquad\qquad +\varepsilon^2\int_x^\infty \Psi_K(x,z,t)\Psi_b(y+z,t)\mathrm{d}z) \\ &= \varepsilon^{-\beta}\left(\int_x^\infty \varepsilon^{N+1}K_{rest}(x,z,t)\Psi_b(y+z,t)\mathrm{d}z\right), \end{aligned}$$

where K_{rest} has an integral form similiar to

$$\int_x^\infty \varepsilon^3\partial_Z(K_1(\varepsilon x,\varepsilon z,t)A(\varepsilon(y+z-c_0t),\varepsilon^2 t))\frac{\mathrm{e}^{2\mathrm{i}k_0 z}}{2\mathrm{i}k_0}\mathrm{d}z\mathrm{e}^{\mathrm{i}k_0(x+y)}\mathrm{e}^{-2\mathrm{i}\omega_0 t}$$

from part **i)** of Section 4 and is finally a function of

$$A_1,...,A_N,\ldots,\partial_X^{s_A}A_1,...,\partial_X^{s_A}A_N,$$

where $A_1 = A$ from above, with s_A a number depending on $\tilde{N}$ from Lemma 4.1, cf. the construction in Section 4.

5.1 Outline

Equation (15) will be solved for every fixed t. Since (15) is formally of the form R_K plus some small perturbation in R_K plus some inhomogeneity, we will use Neumann's series to solve (15) w.r.t R_K.

Remark 5.1. In order to apply a fixed point argument to (15), we need suitable function spaces such that the above integrals $s_{inh}, s_{lin}, s_{non}$ and s_{res} are defined. However, the main difficulty is to find such spaces for K and thus the error R_K, since $K(\cdot,\cdot,t)\notin L^2(\mathbb{R}^2)$, as we will see. The key element in finding these spaces is an identity that can subsequently be found in Remark 5.2. This allows us to estimate $R_K(\cdot,\cdot,t)$ in the L^2-norm with respect to the second variable and in the supremum norm with respect to the first variable.

Multiplication of (15) with $R_K(x,y,t)$ and integration w.r.t. y yields

$$\int_{-\infty}^\infty |R_K(x,y,t)|^2\mathrm{d}y + r_{inh}(x,t)+r_{lin}(x,t)+r_{non}(x,t)+r_{res}(x,t)=0, \quad (16)$$

with

$$
\begin{aligned}
r_{inh}(x,t) &= \int_{-\infty}^{\infty} R_K(x,y,t)s_{inh}(x,y,t)\mathrm{d}y,\\
r_{lin}(x,t) &= \int_{-\infty}^{\infty} R_K(x,y,t)s_{lin}(x,y,t)\mathrm{d}y,\\
r_{non}(x,t) &= \int_{-\infty}^{\infty} R_K(x,y,t)s_{non}(x,y,t)\mathrm{d}y,\\
r_{res}(x,t) &= \int_{-\infty}^{\infty} R_K(x,y,t)s_{res}(x,y,t)\mathrm{d}y.
\end{aligned}
$$

Remark 5.2. In the following we use the fundamental identity, cf. [Seg73, p. 727],

$$
\begin{aligned}
& \int_{-\infty}^{\infty}\int_{x}^{\infty} \varepsilon R_1(x,z,t)\Psi_b(y+z,t)R_2(x,y,t)\mathrm{d}z\mathrm{d}y\\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon R_1(x,z,t)\Psi_b(y+z,t)R_2(x,y,t)\mathrm{d}z\mathrm{d}y\\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon R_1(x,z,t)\int_{-\infty}^{\infty}\widehat{\Psi}_b(k,t)\mathrm{e}^{\mathrm{i}k(y+z)}\mathrm{d}kR_2(x,y,t)\mathrm{d}z\mathrm{d}y\\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon R_1(x,z,t)\mathrm{e}^{\mathrm{i}kz}\widehat{\Psi}_b(k,t)R_2(x,y,t)\mathrm{e}^{\mathrm{i}ky}\mathrm{d}z\mathrm{d}y\mathrm{d}k\\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon\widehat{\Psi}_b(k,t)\widehat{R}_1(x,-k,t)\widehat{R}_2(x,-k,t)\mathrm{d}k\\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon\widehat{\Psi}_b(k,t)\overline{\widehat{R}_1(x,k,t)\widehat{R}_2(x,k,t)}\mathrm{d}k,
\end{aligned}
$$

where we used $R_K(x,z,t)=0$ for $x>z$, to obtain the second line from the first line, and the definition of Fourier transform.

5.2 Estimates for the inhomogeneous term r_{inh}

In this subsection we are going to estimate

$$
\begin{aligned}
r_{inh}(x,t) &= \int_{-\infty}^{\infty} R_K(x,y,t)R_b(x+y,t)\mathrm{d}y\\
&\quad +\varepsilon\int_{-\infty}^{\infty} R_K(x,y,t)\int_{x}^{\infty}\Psi_K(x,z,t)R_b(y+z,t)\mathrm{d}z\mathrm{d}y.
\end{aligned}
$$

With the Cauchy-Schwarz inequality we find

$$\left|\int_{-\infty}^{\infty} R_K(x,y,t) R_b(x+y,t) \mathrm{d}y\right| \leq \|R_K(x,\cdot,t)\|_{L^2} \|R_b(\cdot,t)\|_{L^2}$$

and with Remark 5.2, Plancherel's identity, and Young's inequality that

$$\begin{aligned} &\varepsilon \left|\int_{-\infty}^{\infty} R_K(x,y,t) \int_x^{\infty} \Psi_K(x,z,t) R_b(y+z,t) \mathrm{d}z \mathrm{d}y\right| \\ &\leq C\varepsilon \sup_{k\in\mathbb{R}} |\widehat{R}_b(\cdot,t)| \|R_K(x,\cdot,t)\|_{L^2} \|\Psi_K(x,\cdot,t)\|_{L^2}. \end{aligned}$$

Hence, we obtain the estimate

$$|r_{inh}(x,t)| \quad \leq \quad (C_{0,1} + C_{0,2}\varepsilon) \|R_K(x,\cdot,t)\|_{L^2} \leq 2C_{0,1} \|R_K(x,\cdot,t)\|_{L^2},$$

if $\varepsilon > 0$ is chosen so small that $C_{0,2}\varepsilon \leq C_{0,1}$, with constants

$$\begin{aligned} C_{0,1} &= \sup_{t\in[0,1/\varepsilon^{3-\delta}]} \|R_b(\cdot,t)\|_{L^2}, \\ C_{0,2}\varepsilon &= C\varepsilon \sup_{t\in[0,1/\varepsilon^{3-\delta}]} \sup_{k\in\mathbb{R}} |\widehat{R}_b(\cdot,t)| \sup_{t\in[0,1/\varepsilon^{3-\delta}]} \sup_{x\in\mathbb{R}} \|\Psi_K(x,\cdot,t)\|_{L^2}, \end{aligned}$$

where

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]} \sup_{k\in\mathbb{R}} |\widehat{R}_b(\cdot,t)| = C_B$$

is bounded by Remark 3.9 and the last factor is bounded by Remark 4.1.

5.3 Estimates for the linear term r_{lin}

With Remark 5.2 and Plancherel's identity we find

$$\begin{aligned} &\varepsilon \left|\int_{-\infty}^{\infty} R_K(x,y,t) \int_x^{\infty} R_K(x,z,t) \Psi_b(y+z,t) \mathrm{d}z \mathrm{d}y\right| \\ &\leq \varepsilon \|\widehat{\Psi}_b\|_{L^\infty} \|R_K(x,\cdot,t)\|_{L^2}^2. \end{aligned} \tag{17}$$

We assume

$$\sup_{K\in\mathbb{R}} |\widehat{A}(K,T)| \leq \sup_{K\in\mathbb{R}} |\widehat{A}(K,0)| \leq 1 - \delta' < 1 \tag{18}$$

for $T \geq 0$ and a $\delta' \in (0,1)$ which is a natural and necessary restriction due to the condition (5) which lead to (18). Then, the triangle inequality yields

$$\|\varepsilon\widehat{\Psi}_b\|_{L^\infty} \leq \|\widehat{A}\|_{L^\infty} + \mathcal{O}(\varepsilon) \leq 1 - \delta'/2 \tag{19}$$

for $\varepsilon > 0$ sufficiently small. Therefore, we achieve

$$|r_{lin}(x,t)| \leq C_1 \|R_K(x,\cdot,t)\|_{L^2}^2,$$

with a $C_1 < 1$.

5.4 Estimates for the nonlinear term r_{non}

In this subsection we are going to estimate

$$r_{non}(x,t) = \varepsilon^{\beta} \int_{-\infty}^{\infty} R_K(x,y,t) \int_x^{\infty} R_K(x,z,t) R_b(y+z,t) \mathrm{d}z \mathrm{d}y.$$

Again with Remark 5.2 and Plancherel's identity this can be estimated by

$$|r_{non}(x,t)| \leq C\varepsilon^{\beta} \|\widehat{R}_b(\cdot,t)\|_{L^\infty} \|R_K(x,\cdot,t)\|_{L^2}^2.$$

5.5 Estimates for the residual term r_{res}

In this subsection we are going to estimate

$$\begin{aligned} r_{res}(x,t) &= \varepsilon^{-\beta} \int_{-\infty}^{\infty} R_K(x,y,t) \big(\varepsilon \Psi_K(x,y,t) + \varepsilon \Psi_b(x+y,t) \\ &\qquad + \varepsilon^2 \int_x^{\infty} \Psi_K(x,z,t) \Psi_b(y+z,t) \mathrm{d}z \big) \mathrm{d}y \\ &= \varepsilon^{-\beta} \int_{-\infty}^{\infty} R_K(x,y,t) \left(\int_x^{\infty} \varepsilon^{\tilde{N}+1} K_{rest}(x,z,t) \Psi_b(y+z,t) \mathrm{d}z \right) \mathrm{d}y. \end{aligned}$$

With Remark 5.2 we find

$$r_{res}(x,t) = \varepsilon^{\tilde{N}+1-\beta} \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\Psi}_b(k,t) \overline{\widehat{R}_K(x,k,t) \widehat{K}_{rest}(x,k,t)} \mathrm{d}k.$$

The function K_{rest} can be expressed in terms of the functions $A_1, ..., A_N, ...,$ $\partial_X^{s_A} A_1, ..., \partial_X^{s_A} A_N$ with s_A a number depending on $\tilde{N}$. It contains terms which are at least quadratic in the argument. Moreover, there are no terms in K_{rest} w.r.t $A_1, ..., A_N$ and their derivatives of power bigger than $\tilde{N}+1$. Since multiplication becomes convolution under Fourier transform we have to estimate for instance

$$\|\widehat{A}^{*(\tilde{N}+1)}\|_{L^2}.$$

By Young's inequality for convolutions, the embedding $L^2_s \subset L^1$ for $s > 1/2$, we obtain for instance

$$\|\widehat{A}^{*(\tilde{N}+1)}\|_{L^2} \leq C\|\widehat{A}^{*(\tilde{N})}\|_{L^1}\|\widehat{A}\|_{L^2} \leq C\|\widehat{A}\|^{\tilde{N}}_{L^1}\|A\|_{H^s}$$

and analogously for the terms containing derivatives of A such that all terms in K_{rest} can be estimated in terms of $C\|A\|^j_{H^{s+s_A}}$ for $j = 2, \ldots, \tilde{N}+1$.

By the Cauchy-Schwarz inequality, Plancherel's identity and estimate (19) we obtain

$$\begin{aligned}
|r_{res}(x,t)| &= \left|\varepsilon^{N+1-\beta}\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\Psi}_b(k,t)\overline{\widehat{R}_K(x,k,t)\widehat{K}_{rest}(x,k,t)}\mathrm{d}k\right| \\
&\leq \varepsilon^{N-\beta}\varepsilon\|\widehat{\Psi}_b(\cdot,t)\|_{L^\infty}\|R_K(x,\cdot,t)\|_{L^2}\|K_{rest}(x,\cdot,t)\|_{L^2} \\
&\leq C\varepsilon^{N-\beta-1/2}\varepsilon\|\widehat{\Psi}_b(\cdot,t)\|_{L^\infty}\|R_K(x,\cdot,t)\|_{L^2}\|A(\cdot,t)\|_{H^{s+s_A}}\sum_{j=2}^{\tilde{N}+1}\|\widehat{A}(\cdot,t)\|^{j-1}_{L^1} \\
&\leq C\varepsilon^{N-\beta-1/2}C_1\|R_K(x,\cdot,t)\|_{L^2}\|A(\cdot,t)\|_{H^{s+s_A}}\sum_{j=2}^{\tilde{N}+1}\|\widehat{A}(\cdot,t)\|^{j-1}_{L^1} \\
&\leq C_2\varepsilon^{N-\beta-1/2}\|R_K(x,\cdot,t)\|_{L^2},
\end{aligned}$$

with the constant

$$C_2 = \sup_{t\in[0,1/\varepsilon^{3-\delta}]}\sup_{x\in\mathbb{R}}\left(CC_1\|A(\cdot,t)\|_{H^{s+s_A}}\sum_{j=2}^{\tilde{N}+1}\|\widehat{A}(\cdot,t)\|^{j-1}_{L^1}\right).$$

5.6 Final estimates

In the following we choose $\tilde{N}$ so large that $N - \beta \geq 1$. Recall that N and $\widetilde{N}$ are different numbers, cf. Remark 4.1, with $N \to \infty$ for $\widetilde{N} \to \infty$. From (16) we immediately find

$$\int_{-\infty}^{\infty}|R_K(x,y,t)|^2\mathrm{d}y \leq |r_{inh}(x,t)| + |r_{lin}(x,t)| + |r_{non}(x,t)| + |r_{res}(x,t)|.$$

First, Young's inequality gives

$$|r_{inh}(x,t)| \leq 2C_{0,1}\|R_K(x,\cdot,t)\|_{L^2} \leq \delta_1^2\|R_K(x,\cdot,t)\|^2_{L^2} + \frac{C_{0,1}}{\delta_1^2}$$

and

$$|r_{res}(x,t)| \leq C_2 \varepsilon^{N-\beta} \|R_K(x,\cdot,t)\|_{L^2} \leq C_2^2/4 + \varepsilon^{2(N-\beta)} \|R_K(x,\cdot,t)\|_{L^2}^2,$$

such that

$$\begin{aligned} \|R_K(x,\cdot,t)\|_{L^2}^2 \quad \leq \quad & \delta_1^2 \|R_K(x,\cdot,t)\|_{L^2}^2 + \frac{C_{0,1}}{\delta_1^2} \\ & + C_1 \|R_K(x,\cdot,t)\|_{L^2}^2 \\ & + C\varepsilon^\beta C_B \|R_K(x,\cdot,t)\|_{L^2}^2 \\ & + C_2^2/4 + \varepsilon^{2(N-\beta)} \|R_K(x,\cdot,t)\|_{L^2}^2 \end{aligned}$$

with $C_1 < 1$. Rearranging the terms we obtain

$$(1 - \delta_1^2 - C_1 - C\varepsilon^\beta C_B - \varepsilon^{2(N-\beta)}) \|R_K(x,\cdot,t)\|_{L^2}^2 \quad \leq \quad \frac{C_{0,1}}{\delta_1^2} + C_2^2/4.$$

Choosing $\delta_1 > 0$ and $\varepsilon > 0$ so small that

$$\delta_1^2 + C\varepsilon^\beta C_B + \varepsilon^{2(N-\beta)} \leq (1 - C_1)/2$$

gives

$$\sup_{t \in [0, 1/\varepsilon^{3-\delta}]} \sup_{x \in \mathbb{R}} \|R_K(x,\cdot,t)\|_{L^2} \leq 2(1 - C_1)^{-1} (\frac{C_{0,1}}{\delta_1^2} + C_2^2/4) =: C_R = \mathcal{O}(1),$$

and hence

$$\sup_{t \in [0, 1/\varepsilon^{3-\delta}]} \sup_{x \in \mathbb{R}} \|K(x,\cdot,t) - \varepsilon \Psi_K(x,\cdot,t)\|_{L^2} \leq C_R \varepsilon^\beta.$$

In exactly the same way, we prove

$$\sup_{t \in [0, 1/\varepsilon^{3-\delta}]} \sup_{x \in \mathbb{R}} \|\partial_x^{s_x} \partial_y^{s_y} (K(x,\cdot,t) - \varepsilon \Psi_K(x,\cdot,t))\|_{L^2} \leq C_R \varepsilon^\beta,$$

for $0 \leq s_x + s_y \leq s$. Therefore, we find

$$u(x,t) - \varepsilon \Psi_u(x,t) = -2 \frac{\mathrm{d}}{\mathrm{d}x} (K(x,x^+,t) - \varepsilon \Psi_K(x,x^+,t)).$$

and so

$$\sup_{t \in [0, 1/\varepsilon^{3-\delta}]} \|u(x,t) - \varepsilon \Psi_u(x,t)\|_{H^{s-1}} \leq C\varepsilon^\beta.$$

Hence, we have proven

Theorem 5.3. *For each $N \in \mathbb{N}$, $s \geq 1$ and $\delta \in (0,1]$ there exist $C > 0$ and $\varepsilon_0 > 0$ such that the following holds. Let $A_1 \in C([0,\infty), \mathcal{F}^{-1}L^\infty_{s+3N} \cap H^{s+3N})$ be a solution of the linear Schrödinger equation* (10) *with*

$$\sup_{K\in\mathbb{R}} |\widehat{A}(K,0)| < 1 \tag{20}$$

and let the A_n be solutions of

$$\partial_T A_n = 24\mathrm{i}k_0\partial_X^2 A_n - 8\varepsilon^{1-\delta}\partial_X^3 A_{n-1}, \qquad A_n|_{T=0} = 0,$$

for $n = 2,\dots,N$. Then for all $\varepsilon \in (0,\varepsilon_0)$ there are solutions u of the KdV equation (2) *such that*

$$\sup_{t\in[0,1/\varepsilon^{3-\delta}]} \|u(x,t) - \varepsilon\Psi_u(x,t)\|_{H^{s-1}(\mathrm{d}x)} \leq C\varepsilon^{1/2+N\delta},$$

with $\varepsilon\Psi_u$ as constructed in (13)*.*

Remark 5.4. As already said (20) is a natural and necessary restriction due to (5).

6 Discussion

In the previous sections we used the Gelfand-Levitan-Marchenko equation and the evolution of the scattering data to construct a linear Schrödinger approximation for the KdV equation. Although at a first view this detour only seems to be of theoretical use, the transfer of a nonlinear PDE problem into a pure integration problem allowed us to extend the approximation time beyond the natural NLS time scale.

Remark 6.1. Our result is what can be expected for completely integrable systems for which a representation in action and angle variables do exist. The action variables are conserved. The frequency of the angle variables are approximated up to order $\mathcal{O}(\varepsilon^2)$, i.e. with an error of order $\mathcal{O}(\varepsilon^3)$. The error for these variables then grows like $\mathcal{O}(\varepsilon^3)t$ which is of order $\mathcal{O}(\varepsilon^\delta)$ for a $\mathcal{O}(1/\varepsilon^{3-\delta})$-time scale.

Remark 6.2. The inverse scattering approach for the KdV equation and the NLS equation have been related in [ZK86]. Such correspondances have been analysed in a number of other papers, cf. [TLOB88, BCP02, KP03].

Remark 6.3. It is the goal of future research to describe the interaction of NLS scaled wave packets for completely integrable systems. Is it possible to extend the separation of internal and interaction dynamics of NLS scaled wave packets with different carrier waves for completely integrable systems even further than in the existing literature [PW95, CSU07, CCSU08, CS12, SC15] for general dispersive systems?

Remark 6.4. Due to the scaling properties of the slow spatial variable $X \sim \varepsilon x$ in the NLS ansatz and the scaling properties of the KdV solitons $\sim \varepsilon^2 A_{soliton}(\varepsilon(x-ct))$, we expect that the discrete eigenvalues in the scattering data are of order $\mathcal{O}(\varepsilon^2)$ for a general NLS ansatz in the original KdV equation. Rigorous estimates for the number and size of the eigenvalues can be found with the help of Lieb-Thirring inequalities, cf. [FLW23]. However, a detailed analysis with respect to this question is beyond the scope of this paper.

Remark 6.5. For the KdV equation, only the defocusing NLS equation can be derived. Unlike the focusing NLS equation, it has no pulse solutions and its dynamics for initial conditions vanishing for $|X| \to \infty$ is a bit annoying, since all solutions decay to zero for $T \to \infty$. However, the essential result is that the KdV equation can be approximated by a linear Schrödinger equation per se.

Remark 6.6. The Schrödinger equation shows a dispersive decay $\sim 1/\sqrt{T}$. Hence for $t = \mathcal{O}(1/\varepsilon^{3-\delta})$, respectively $T = \mathcal{O}(1/\varepsilon^{1-\delta})$ we have that the solutions are of order $\mathcal{O}(\varepsilon^{1/2-\delta/2})$ if they are initially of order $\mathcal{O}(1)$ or of order $\mathcal{O}(\varepsilon^{3/2-\delta/2})$ if they are initially of order $\mathcal{O}(\varepsilon)$ like for the NLS approximation. As explained in the introduction the KdV equation (2) can be transferred with the help of the Miura transformation

$$u = v^2 + \partial_x v \tag{21}$$

via direct substitution into the mKdV equation

$$\partial_t v - 6v^2 \partial_x v + \partial_x^3 v = 0. \tag{22}$$

For solutions of order $\mathcal{O}(\varepsilon^{3/2-\delta/2})$ of the mKdV equation Gronwall's inequality easily gives estimates on a time scale of order $\mathcal{O}(1/\varepsilon^{3-\delta})$ w.r.t. t. However, in our situation the solutions are initially of order $\mathcal{O}(\varepsilon)$ and much bigger than $\mathcal{O}(\varepsilon^{3/2-\delta/2})$ except at the end of the time interval $[0, 1/\varepsilon^{3-\delta}]$. Therefore, our result is non-trivial and gives a rather precise description of the decay of NLS scaled wave packets on the long time interval $[0, 1/\varepsilon^{3-\delta}]$.

Remark 6.7. Due to the decay of the solutions, cf Remark 6.6, we have a global-in-time approximation result with an error $\mathcal{O}(\varepsilon^{3/2-\delta/2})$.

Remark 6.8. It is the purpose of future research to transfer the presented analysis to other dispersive completely integrable systems, such as the sine-Gordon equation, the NLS equation, or the Toda-lattice.

Remark 6.9. There is a relation between the the scattering data of the KdV equation and of the NLS equation, cf. [ZK86, TLOB88]. It is the purpose of future research to replace the special initial conditions for the scattering data b by scattering data which allows us to handle all NLS approximations. In order to analyze the general NLS case, even with no discrete eigenvalues, scattering data concentrated at other integer multiples of the basic wave number have to be considered, too.

Remark 6.10. The question about the relation of the NLS approximation and the linear Schödinger approximation of the KdV question has not been discussed so far. At a first view it seems a little bit strange that the KdV equation can be approximated simultaneously by a NLS equation and a linear Schödinger equation. In the following we explain why this is not a contradiction.

We denote with x_u the space and with t_u the time variable used in the NLS approximation and with x the space and with t the time variable used in the linear Schödinger approximation.

On the one hand we have the NLS approximation

$$u(x_u, t_u) = \varepsilon \mathcal{A}(\varepsilon(x_u - c_u t_u), \varepsilon^2 t) \mathrm{e}^{\mathrm{i}(k_u x_u - \omega_u t_u)} + c.c. + \mathcal{O}(\varepsilon^2),$$

for the KdV equation

$$\partial_{t_u} u - 6u \partial_{x_u} u + \partial_{x_u}^3 u = 0,$$

with $\omega_u = -k_u^3$, $c_u = -3k_u^2$, and $\mathcal{A}$ satisfying the defocusing NLS equation

$$\partial_{T_u} \mathcal{A} = -3\mathrm{i}k_u \partial_{X_u}^2 \mathcal{A} - 6\mathrm{i}k_u \mathcal{A}|\mathcal{A}|^2.$$

On the other hand we have the linear Schödinger approximation

$$u(x, t) = 4\varepsilon \mathrm{i}k_0 A(\varepsilon(2x - c_0 t), \varepsilon^2 t) \mathrm{e}^{2\mathrm{i}k_0 x - \mathrm{i}\omega_0 t} + c.c. + \mathcal{O}(\varepsilon^2),$$

for the same KdV equation, with $\omega_0 = -8k_0^3$, $c_0 = -24k_0^2$, and A satisfying the linear Schödinger equation

$$\partial_T A = -24\mathrm{i}k_0 \partial_X^2 A.$$

A comparison of the NLS approximation and the linear Schödinger approximation gives $k_0 = k_u$, $2x = x_u$, $8t = t_u$, $c_0 = 8c_u$, $2X = X_u$, and $8T = T_u$.

For solutions of order $\mathcal{O}(\varepsilon)$ the nonlinear terms in the KdV equation will not affect the dynamics of the KdV equation in lowest order before $\mathcal{O}(1/\varepsilon)$-time scales but in general they play a role after this time scale. However, for initial conditions of the KdV equation which are $\mathcal{O}(\varepsilon^2)$ close to the approximation constructed by the linear Schrödinger equation at $t = 0$ there is a subset of initial conditions of the KdV equation for which the effect of the nonlinear terms stays small on the long $\mathcal{O}(1/\varepsilon^{3-\delta})$-time scale.

Remark 6.11. The previous analysis will work for all other members of the KdV-hierarchy, too.

7 Declarations

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