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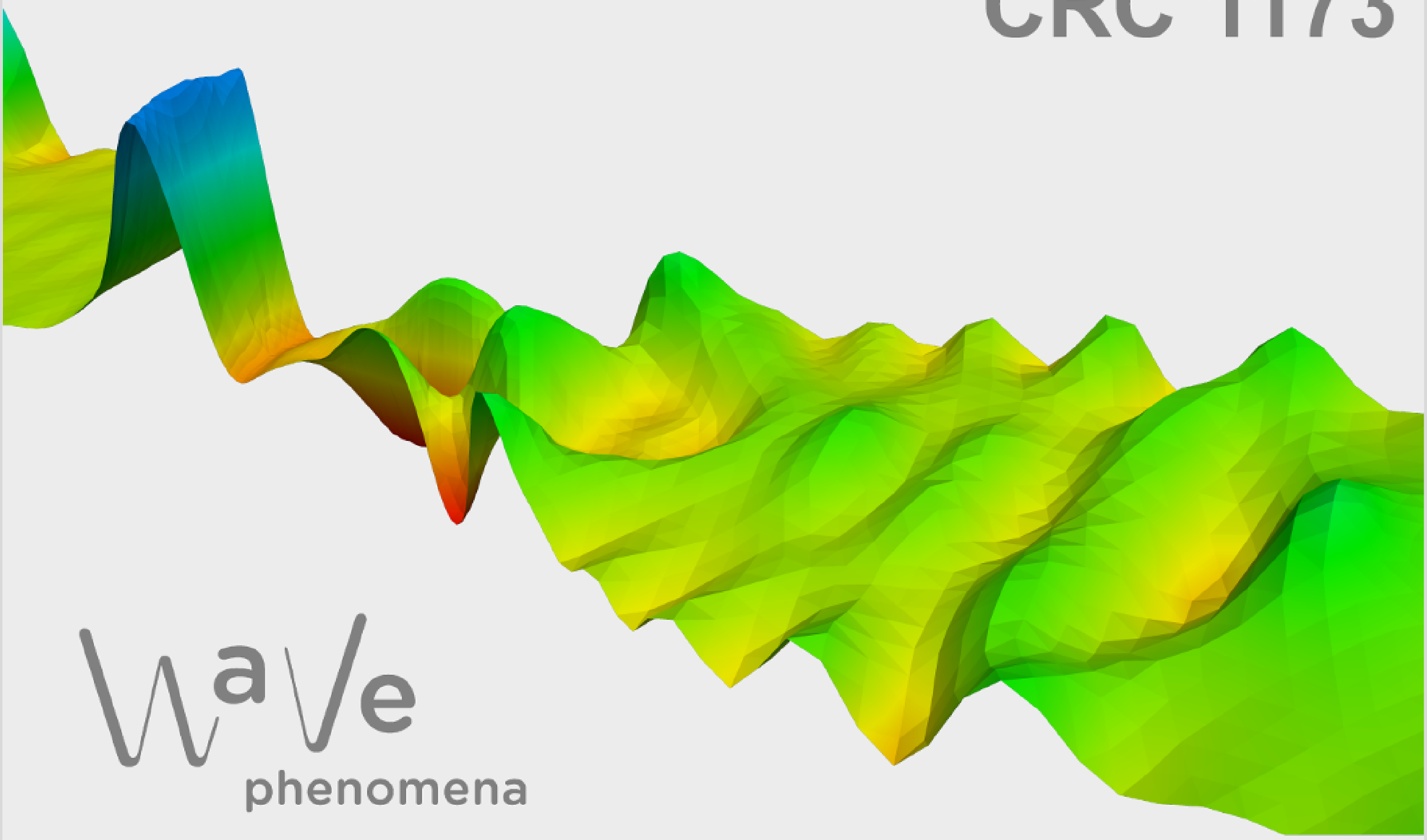

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Some remarks about an effective description of high-frequency wave-packet propagation

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Abstract

We consider systems of the form

$$\partial_\tau \mathcal{U} + \mathcal{A}(\partial_\xi)\mathcal{U} + \frac{1}{\varepsilon}\mathcal{E}\mathcal{U} = \mathcal{T}_2(\mathcal{U},\mathcal{U}) + \varepsilon\mathcal{T}_3(\mathcal{U},\mathcal{U},\mathcal{U}),$$

with $0 < \varepsilon \ll 1$ a small perturbation parameter. We are interested in an effective description of high-frequency wave-packet propagation associated to highly oscillatory initial conditions

$$\mathcal{U}(\xi,0) = \mathcal{U}_*(\xi)e^{ik_0\xi/\varepsilon} + c.c..$$

By classical perturbation analysis for polarized initial conditions NLS approximations up to an arbitrary order and for non-polarized initial conditions a system of decoupled NLS equations can be derived for the approximate description of the associated solutions. Under the validity of a number of non-resonance conditions we prove error estimates between these formal approximations and true solutions of the original system. The result improves results from the existing literature in at least two directions, firstly, the handling of higher order approximations in case of quadratic nonlinearities $\mathcal{T}_2(\mathcal{U},\mathcal{U})$ and secondly, the handling of non-polarized initial conditions.

1 Introduction

We consider

$$\partial_\tau \mathcal{U} + \mathcal{A}(\partial_\xi)\mathcal{U} + \frac{1}{\varepsilon}\mathcal{E}\mathcal{U} = \mathcal{T}_2(\mathcal{U},\mathcal{U}) + \varepsilon \mathcal{T}_3(\mathcal{U},\mathcal{U},\mathcal{U}), \tag{1}$$

with $\mathcal{U}(\xi,\tau) \in \mathbb{R}^N$, $\xi \in \mathbb{R}^d$, $N, d \in \mathbb{N}$, $\tau \in [0, \tau_0/\varepsilon]$ for a $\tau_0 > 0$, $\mathcal{A}(\partial_\xi) = \sum_{j=1}^d \mathcal{A}_j \partial_{\xi_j}$, $\mathcal{A}_j = \mathcal{A}_j^T \in \mathbb{R}^{N\times N}$, $\mathcal{E} = -\mathcal{E}^T \in \mathbb{R}^{N\times N}$, $\mathcal{T}_2 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ a bilinear mapping and $\mathcal{T}_3 : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ a trilinear mapping. For most of the paper we assume $d = 1$ due to the possible application in nonlinear optics and due to the purpose of this paper. The initial conditions for (1) are given by

$$\mathcal{U}(\xi, 0) = \mathcal{U}_*(\xi) e^{ik_0\xi/\varepsilon} + c.c., \tag{2}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter and $k_0 > 0$ is a fixed spatial wave number. This class of problems (1) include, e.g., the Maxwell-Lorentz system and Klein-Gordon systems, cf. [Col02, CL09, Section 2.1]. This class of problems has been considered for instance in [CL09, Rau12] or recently in [BJL24] where in case $\mathcal{T}_2 = 0$ higher order NLS approximations have been derived and justified by establishing approximation results. The results presented in the following improve these papers in at least two directions, firstly, the handling of higher order NLS approximations for quadratic nonlinearities $\mathcal{T}_2(\mathcal{U},\mathcal{U})$ and secondly the handling of more than one NLS-scaled wave-packet which is called the non-polarized situation in the following. We obtain these improved results by relating these questions to questions which have already been solved in the existing literature or which can easily be generalized to the present situation, namely first the justification of higher order NLS approximations for systems with quadratic nonlinearities and secondly the separation of internal and interaction dynamics of NLS-scaled wave-packets.

As will be clear in the following, higher order nonlinear terms

$$\ldots + \varepsilon^2 \mathcal{T}_4(\mathcal{U},\mathcal{U},\mathcal{U},\mathcal{U}) + \varepsilon^3 \mathcal{T}_5(\mathcal{U},\mathcal{U},\mathcal{U},\mathcal{U},\mathcal{U}) + \ldots$$

can easily be incorporated in the subsequent analysis. However, for notational simplicity we restrict ourselves to (1).

Error estimates for the NLS approximation of dispersive systems without small perturbation parameter in the underlying equations have already been known for a few decades. The NLS equation as envelope equation has been derived first in [Zak68]. For systems without quadratic terms a simple application of Gronwall's inequality is sufficient to obtain such estimates [KSM92]. A very general NLS approximation result including quadratic nonlinearities has been shown in [Kal87]. This result was improved in a number of papers by weakening the non-resonance conditions which are necessary to eliminate the quadratic terms in order to apply Gronwall's inequality again, e.g. [Sch98, Sch05, DHSZ16]. An application of the theory to the water wave problem can be found in [TW12, DSW16, Dül21], and quasilinear wave equations are considered in [CW17, Dül17, DH18]. Another example of an NLS approximation result for dispersive systems with a small perturbation parameter in the underlying equations exists for instance for the Klein-Gordon-Zakharov system from plasma physics, e.g. [MN02]. Our approach to handle higher order NLS approximations for quadratic nonlinearities $\mathcal{T}_2(\mathcal{U},\mathcal{U})$ follows [Sch98].

Our approach to handle more than one NLS-scaled wave-packet is based on the literature about the separation of internal and interaction dynamics of NLS-scaled wave-packets, in particular on [CBCSU08, CBS12]. For dispersive systems it has first been observed in [PW95] that for spatially localized NLS-scaled wave-packets no interaction appears in lowest order w.r.t. the small perturbation parameter $0 < \varepsilon \ll 1$. This result has been improved in a number of papers, e.g. [CBSU07, CBCSU08, CBS12, CBS15], now allowing to separate the internal from the interaction dynamics of the wave-packets up to high order.

The plan of the paper is as follows. In Section 2 we present two examples which show how the existing theory for NLS approximations can be made applicable for systems of the form (1). In Section 3 we rescale (1) in such a way that the existing theory for dispersive systems without small perturbation parameter in the equations becomes applicable. In Section 4 we explain the handling of quadratic nonlinearities $\mathcal{T}_2(\mathcal{U},\mathcal{U})$ and in Section 5 we explain our main result, namely the handling of more than one NLS-scaled wave-packet. Finally, in Section 6 we explain how the results change if $x \in \mathbb{R}^d$ with $d \geq 2$ is considered.

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2 Two examples

The first example motivates our subsequent approach.

Example 2.1. We consider the nonlinear Klein-Gordon equation

$$\partial_t^2 u = \partial_x^2 u - u + u^2 + u^3, \tag{3}$$

with $t \in \mathbb{R}$, $x \in \mathbb{R}$, and $u(x,t) \in \mathbb{R}$. By introducing $v = \partial_t u$ and $w = \partial_x u$ we obtain the system

$$\begin{aligned} \partial_t u &= v, \\ \partial_t v &= \partial_x w - u + u^2 + u^3, \\ \partial_t w &= \partial_x v. \end{aligned}$$

This system can be written in the form

$$\partial_t U + \mathcal{A}\partial_x U + \mathcal{E}U = \mathcal{T}_2(U,U) + \mathcal{T}_3(U,U,U), \tag{4}$$

with

$$U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathcal{T}_2(U,U) = \begin{pmatrix} 0 \\ u^2 \\ 0 \end{pmatrix}, \quad \mathcal{T}_3(U,U,U) = \begin{pmatrix} 0 \\ u^3 \\ 0 \end{pmatrix}.$$

By setting $\tau = \varepsilon t$, $\xi = \varepsilon x$, and $U = \varepsilon \mathcal{U}$ this system transforms in our starting system (1).

For (3) a NLS equation

$$\partial_T A = i\nu_1 \partial_X^2 A + i\nu_2 A|A|^2,$$

with coefficients ν_1, $\nu_2 \in \mathbb{R}$, can be derived with the ansatz

$$u(x,t) = \varepsilon A(X,T) e^{i(k_0 x - \omega_0 t)} + c.c. + \mathcal{O}(\varepsilon^2),$$

where $X = \varepsilon(x - ct)$ and $T = \varepsilon^2 t$. Herein, k_0 and ω_0 satisfy the linear dispersion relation $\omega_0^2 = k_0^2 + 1$, and $c = \frac{d\omega_0}{dk_0}$ is the linear group velocity. By definition of v and w the associated ansatz for the solution U of (4) is given by

$$U(x,t) = \varepsilon A(X,T) \begin{pmatrix} 1 \\ -i\omega_0 \\ ik_0 \end{pmatrix} e^{i(k_0 x - \omega_0 t)} + c.c. + \mathcal{O}(\varepsilon^2).$$

Hence, the associated ansatz for the solution $\mathcal{U}$ of (1) is given by

$$\mathcal{U}(\xi,\tau) = A(X,T) \begin{pmatrix} 1 \\ -i\omega_0 \\ ik_0 \end{pmatrix} e^{i\varepsilon^{-1}(k_0 \xi - \omega_0 \tau)} + c.c. + \mathcal{O}(\varepsilon),$$

where $X = \xi - c\tau$, $T = \varepsilon\tau$. As a consequence we have for the initial condition of (1) that

$$\mathcal{U}(\xi,0) = A(\xi,0) \begin{pmatrix} 1 \\ -i\omega_0 \\ ik_0 \end{pmatrix} e^{ik_0\xi/\varepsilon} + c.c..$$

In order to have an example for which all assumptions, used in the subsequent validity proofs, can be checked easily, we also consider

Example 2.2. We consider

$$\partial_t U + \mathcal{A}\partial_x U + \mathcal{E} U = \mathcal{T}_2(U,U) + \mathcal{T}_3(U,U,U), \tag{5}$$

with

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$\mathcal{T}_2(U,U) = \mathcal{O}(u^2 + v^2), \quad \mathcal{T}_3(U,U,U) = \mathcal{O}(|u|^3 + |v|^3).$$

By setting $\tau = \varepsilon t$, $\xi = \varepsilon x$, and $U = \varepsilon\mathcal{U}$ this system transforms in our starting system (1). The linear operator $\mathcal{M} = \mathcal{A}\partial_x + \mathcal{E}$ is given in Fourier space by

$$\widehat{\mathcal{M}}(k) = \begin{pmatrix} 0 & 1+ik \\ -1+ik & 0 \end{pmatrix},$$

The matrix $\widehat{\mathcal{M}}(k)$ has two eigenvalues, namely

$$\omega_1(k) = -i\sqrt{1+k^2}, \qquad \omega_2(k) = i\sqrt{1+k^2}.$$

3 Rescaling and transforming the system

In this section we rescale and transform (1) in such a way that the existing theory for dispersive systems without small perturbation parameter in the underlying equations becomes applicable. Following Example 2.1 of the last section we introduce t, x, U by $\tau = \varepsilon t$, $\xi = \varepsilon x$, and $U = \varepsilon \mathcal{U}$, and consider then

$$\partial_t U + \mathcal{A}\partial_x U + \mathcal{E}U = \mathcal{T}_2(U,U) + \mathcal{T}_3(U,U,U), \tag{6}$$

with initial conditions

$$U(x,0) = \varepsilon U_*(\varepsilon x)e^{ik_0 x} + c.c.$$

This is a special form of a dispersive wave system. For dispersive wave systems a complete theory exists how to handle the validity question of NLS approximations and of generalizations of NLS approximations. For proving the validity of NLS approximations for systems with quadratic nonlinearities it is essential to consider the Fourier transformed systems.

i) In Fourier space we have

$$\partial_t \widehat{U}(k,t) + ik\mathcal{A}\widehat{U}(k,t) + \mathcal{E}\widehat{U}(k,t) = \widehat{\mathcal{T}}_2(\widehat{U},\widehat{U})(k,t) + \widehat{\mathcal{T}}_3(\widehat{U},\widehat{U},\widehat{U})(k,t).$$

ii) We diagonalize the linear part in Fourier space with $\widehat{U}(k,t) = \widehat{S}(k)\widehat{V}(k,t)$, with $\widehat{S}(k) \in \mathbb{C}^{N\times N}$, for $k \in \mathbb{R}$. We assume for a moment that such a diagonalization is possible for all $k \in \mathbb{R}$. After the diagonalization we find

$$\partial_t \widehat{V}(k,t) = \widehat{D}(k)\widehat{V}(k,t) + \widehat{N}(\widehat{V})(k,t), \tag{7}$$

where

$$\begin{aligned} \widehat{D}(k) &= (i\omega_n(k))_{n\in\mathbb{N}} = -(\widehat{S}(k))^{-1}(ik\mathcal{A} + \mathcal{E})\widehat{S}(k), \\ \widehat{N}(\widehat{V})(k,t) &= \widehat{G}_2(\widehat{V},\widehat{V})(k,t) + \widehat{G}_3(\widehat{V},\widehat{V},\widehat{V})(k,t), \end{aligned}$$

with

$$\begin{aligned} &(\widehat{G}_2(\widehat{V},\widehat{V}))_j(k,t) \\ &= \sum_{j_1,j_2=1}^{N} \int \widehat{g}^j_{j_1 j_2}(k,k-k_1,k_1)\widehat{V}_{j_1}(k-k_1,t)\widehat{V}_{j_2}(k_1,t)dk_1 \end{aligned}$$

and

$$
\begin{aligned}
&(\widehat{G}_3(\widehat{V},\widehat{V},\widehat{V}))_j(k,t) \\
&= \sum_{j_1,j_2,j_3=1}^{N} \int\int \widehat{g}^{j}_{j_1j_2j_3}(k,k-k_1,k_1-k_2,k_2) \\
&\qquad \times \widehat{V}_{j_1}(k-k_1,t)\widehat{V}_{j_2}(k_1-k_2,t)\widehat{V}_{j_3}(k_2,t)dk_2dk_1.
\end{aligned}
$$

Without loss of generality we assume that $\widehat{g}^{j}_{j_1j_2}$ is symmetric in j_1 and j_2, as well as $\widehat{g}^{j}_{j_1j_2j_3}$ is symmetric in j_1, j_2 and j_3. These are exactly the dispersive wave systems for which the analysis for the validity of the NLS approximation has been carried out. In the following we transfer the higher order validity results for systems with quadratic nonlinearities and the validity results for solutions with more than one NLS-scaled wave-packet to (1).

For (1) we have

$$
\begin{aligned}
\widehat{G}_2(\widehat{V},\widehat{V})(k,t) &= (\widehat{S}(k))^{-1}\widehat{\mathcal{T}}_2(\widehat{S}\widehat{V},\widehat{S}\widehat{V})(k,t), \\
\widehat{G}_3(\widehat{V},\widehat{V},\widehat{V})(k,t) &= (\widehat{S}(k))^{-1}\widehat{\mathcal{T}}_3(\widehat{S}\widehat{V},\widehat{S}\widehat{V},\widehat{S}\widehat{V})(k,t)
\end{aligned}
$$

If

$$
(\widehat{\mathcal{T}}_2(\widehat{U},\widehat{U}))_j(k,t) = \sum_{j_1,j_2=1}^{N} \int \widehat{b}^{j}_{j_1j_2}\widehat{U}_{j_1}(k-k_1,t)\widehat{U}_{j_2}(k_1,t)dk_1
$$

with $\widehat{b}^{j}_{j_1j_2} \in \mathbb{R}$ we have

$$
\widehat{g}^{j}_{j_1j_2}(k,k-k_1,k_1) = \sum_{j_3,j_4,j_5=1}^{N} ((\widehat{S}(k))^{-1})_{j,j_3}\widehat{b}^{j_3}_{j_4j_5}\widehat{S}_{j_4,j_1}(k-k_1)\widehat{S}_{j_5,j_2}(k_1)
$$

and similar for $\widehat{\mathcal{T}}_3$ and $\widehat{g}^{j}_{j_1j_2j_3}(k,k-k_1,k_1-k_2,k_2)$.

iii) Since the derivation of the approximation equations is notationally more simple in physical space we transfer the diagonlized system (7) back to physical space where for the components we have

$$
\partial_t V_n(x,t) = i\omega_n(-i\partial_x)V_n(x,t) + N_n(V)(x,t) \tag{8}
$$

for $n = 1,\ldots,N$ which is the starting point of the subsequent analysis.

4 Higher order NLS approximation

In this section first we recall the results from the existing literature about the validity of higher order NLS approximations for (7), see for instance [Kal87, SU17]. Secondly, we use these results to extend the results from [BJL24] with a pure cubic nonlinearity to the situation with quadratic nonlinearities, i.e., $\mathcal{T}_2 \neq 0$. We consider so called polarized initial conditions which are of order $\mathcal{O}(\varepsilon)$ only in one component, say, we make the NLS ansatz for V_{n_0} for an $n_0 \in \{1, \ldots, N\}$. The ansatz for the derivation of a higher order NLS approximation is then given by

$$V_{n_0}(x,t) = \sum_{m=-m_*}^{m_*} \sum_{j=0}^{j_*(m)} \varepsilon^{\beta_{n_0}(m)+j} A_{n_0,m,j}(X,T) e^{im(k_0 x - \omega_0 t)} \tag{9}$$

and by

$$V_n(x,t) = \sum_{m=-m_*}^{m_*} \sum_{j=0}^{j_*(m)} \varepsilon^{\beta_n(m)+j} A_{n,m,j}(X,T) e^{im(k_0 x - \omega_0 t)} \tag{10}$$

for $n \in \{1, \ldots, N\}$ with $n \neq n_0$, where $X = \varepsilon(x - ct)$ and $T = \varepsilon^2 t$, and where m^* is a fixed chosen number and where $j_*(m)$ is defined below. Herein, we have the basic temporal wave number $\omega_0 = -\omega_{n_0}(k_0)$ and $c = \frac{d}{dk}\omega_{n_0}(k_0)$ the linear group velocity. The appearing numbers are given by

$$\beta_{n_0}(m) = 1 + ||m| - 1|, \qquad \beta_n(m) = 1 + ||m| - 1| + 2\delta_{|m|1},$$

and

$$j_*(m) = m_* - |m| - 2\delta_{|m|1},$$

with δ_{mn} the Kronecker delta. Plugging this ansatz into (8) and equating equal powers of ε and of $e^{im(k_0 x - \omega_0 t)}$ to zero gives that $A_{n_0,1,0}$ has to satisfy an NLS equation

$$\partial_T A_{n_0,1,0} = i\nu_1 \partial_X^2 A_{n_0,1,0} + i\nu_2 A_{n_0,1,0} |A_{n_0,1,0}|^2, \tag{11}$$

with coefficients ν_1, $\nu_2 \in \mathbb{R}$, where $A_{n_0,-1,0} = \overline{A_{n_0,1,0}}$. We find that the $A_{n_0,1,j}$ and $A_{n_0,-1,j}$ for $j \geq 1$ have to satisfy linear inhomogeneous Schrödinger equations and that all other $A_{n,m,j}$ have to satisfy algebraic equations which can be solved w.r.t. $A_{n,m,j}$ if the non-resonance conditions

$$\omega_n(mk_0) - m\omega_{n_0}(k_0) \neq 0 \tag{12}$$

are satisfied for all $n \in \{1, \dots, N\}$ and $m \in \{-m_*, \dots, m_*\}$ except if $(m, n) = (1, n_0)$. The approximation constructed in this way is called in the following $\varepsilon\Psi_{n_0}$.

In case of no quadratic terms, i.e. $\mathcal{T}_2 = 0$, this approximation can be justified with an approximation theorem by a simple application of Gronwall's inequality, cf. [KSM92]. In case of $\mathcal{T}_2 \neq 0$ the quadratic terms have to be eliminated by some normal form transformation. This requires the validity of additional non-resonance conditions, namely

$$\inf_{n_1, n_2 \in \{1, \dots, N\}} \inf_{k \in \mathbb{R}} |\omega_{n_1}(k) - \omega_{n_0}(k_0) - \omega_{n_2}(k - k_0)| > 0. \tag{13}$$

This condition can be weakened to

$$\sup_{n_1, n_2 \in \{1, \dots, N\}} \sup_{k \in \mathbb{R}} \left| \frac{\widehat{g}^{n_1}_{n_0 n_2}(k, k_0, k - k_0)}{\omega_{n_1}(k) - \omega_{n_0}(k_0) - \omega_{n_2}(k - k_0)} \right| < \infty. \tag{14}$$

Moreover, we assume that

$$\sup_{k \in \mathbb{R}} \|\widehat{S}(k)\| + \sup_{k \in \mathbb{R}} \|(\widehat{S}(k))^{-1}\| < \infty. \tag{15}$$

Then we have the following approximation theorem, cf. [Kal87, DLP$^+$11, SU17].

Theorem 4.1. *For all $m \in \mathbb{N}$ with $m \geq 4$ the following holds. Assume the validity of* (15) *and of the non-resonance conditions* (12) *and* (14)*. Let $A_{n_0,1,0} \in C([0, T_0], H^{3(m-3)+2})$ be a solution of the NLS equation* (11) *and let $\varepsilon\Psi_{n_0}$ be the approximation defined above with $m^* = m - 1$. Then there exist $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions V of* (8) *with*

$$\sup_{t \in [0, T_0/\varepsilon^2]} \sup_{x \in \mathbb{R}} |V(x, t) - \varepsilon\Psi_{n_0}(x, t)| \leq C\varepsilon^{m - 5/2}.$$

Remark 4.2. In case of $\mathcal{T}_2 = 0$ from the non-resonance condition (12) the cases $m \in \{-2, 0, 2\}$ only play a role for the higher order terms, i.e. $j \geq 1$, and (14) is no longer necessary.

For the non-diagonalized system we obtain

Corollary 4.3. *Under the assumptions of Theorem 4.1 there exist $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions U of* (6) *with*

$$\sup_{t \in [0, T_0/\varepsilon^2]} \sup_{x \in \mathbb{R}} |U(x, t) - \varepsilon S\Psi_{n_0}(x, t)| \leq C\varepsilon^{m - 5/2}.$$

For the original system (1) the higher order approximation result is as follows.

Corollary 4.4. *Under the assumptions of Theorem 4.1 there exist $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions $\mathcal{U}$ of (1) with*

$$\sup_{\tau \in [0, T_0/\varepsilon]} \sup_{\xi \in \mathbb{R}} |\mathcal{U}(\xi, \tau) - S\Psi_{n_0}(\xi, \tau)| \leq C\varepsilon^{m-7/2}.$$

Remark 4.5. Theorem 4.1 should not be taken for granted since solutions of order $\mathcal{O}(\varepsilon)$ have to be controlled on an $\mathcal{O}(1/\varepsilon^2)$-time-scale.

Remark 4.6. The previous approximation results, in particular Corollary 4.4 guarantee that the NLS approximation can be used for an effective simulation of solutions of (1) to polarized initial conditions of the form (2). As already said an initial condition is called polarized if in the diagonalized system (8) the initial condition is $\mathcal{O}(\varepsilon)$ in only one component at $k = k_0$. The situation of initial conditions being $\mathcal{O}(\varepsilon)$ in all components at $k = k_0$ is handled in the following Section 5.

Remark 4.7. For having an approximation of the form (9)-(10), i.e., that $A_{n_0,-1,0}$ also belongs to the V_{n_0}-component we need that $\omega_{n_0}(k) = -\omega_{n_0}(-k)$ around $k = k_0$. In general this is only possible if ω_{n_0} is defined with at least one jump. For the derivation of the amplitude equations this jump should be chosen at a small but order $\mathcal{O}(1)$ wave number $k \neq 0$.

Remark 4.8. The assumption on the diagonalization can be weakened strongly. Only a separation of the NLS modes near $k = \pm k_0$ is necessary. However, since this requires a complete rewriting of all non-resonance conditions, cf. [Kal87, Col02], and gives less insight we prefer to stay at the chosen presentation.

Remark 4.9. The eigenvalues for Example 2.2 are the same as for the Klein-Gordon model (3). It is well known that the non-resonance condition (14) is satisfied for the Klein-Gordon model (3). The matrix S can be defined in Fourier space by

$$\widehat{S}(k) = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} 1+ik & 1+ik \\ -i\sqrt{1+k^2} & i\sqrt{1+k^2} \end{pmatrix},$$

for which the validity of the assumption (15) is obvious since the limits exist for $|k| \to \infty$.

5 The handling of more than one wave-packet

It is the purpose of this section to explain how to handle more than one wave-packet, i.e, how to handle the case of non-polarized initial conditions. These are of order $\mathcal{O}(\varepsilon)$ in all components. In order to handle this situation we make an NLS approximation not only for V_{n_0} alone, but for all V_n with $n \in \{1, \ldots, N\}$. We restrict ourselves again to the case $x \in \mathbb{R}$. Due to the fact that in general the group velocities $\frac{d}{dk}\omega_n$ of the wave-packets are different, no consistent ansatz of the above form (9)-(10) is possible. However, by a slight modification of this ansatz a consistent ansatz up to an error of order $\mathcal{O}(\varepsilon^3)$ is possible. In detail, the analysis made in [CBS12] for the interaction of two NLS-scaled wave-packets for dispersive wave systems on the one hand can be specialized and on the other hand can be generalized to handle the present situation.

The ansatz of [CBS12] specialized and generalized to the present situation is given by

$$\begin{aligned} V_n(x,t) &= \sum_{r=0}^{2} \varepsilon^{1+r} A_{n,1,r}(X_n,T) e^{iY_n} \\ &\quad + \sum_{r=0}^{2} \varepsilon^{1+r} A_{n,-1,r}(X_n,T) e^{-iY_n} + M_{mixed,n}, \\ X_n &= X + \varepsilon\omega_n'(k_0)t + \varepsilon^2 \sum_{j \neq n} \Psi_{n,j}^{(1)}(X + \varepsilon\omega_j'(k_0)t, T), \\ Y_n &= k_0 x - \omega_n(k_0)t + \sum_{l=1,2} \varepsilon^l \sum_{j \neq n} \Omega_{n,j}^{(l)}(X + \varepsilon\omega_j'(k_0)t, T), \\ X &= \varepsilon x, \quad T = \varepsilon^2 t, \end{aligned}$$

with $M_{mixed,n} = \mathcal{O}(\varepsilon^2)$ determined below. It is a specialization in the sense that all spatial wave numbers of the wave-packets are the same. It is generalization in the sense that more than two wave-packets are considered.

In case of $\mathcal{T}_2 = 0$ in (1) we recall the explicit formulas which determine the amplitudes $A_{n,1,r}$, the phase shifts $\Omega_{n,j}^{(1)}$, the envelope shifts $\Psi_{n,j}^{(1)}$, and the second order corrections of the phase shifts and amplitudes $\Omega_{n,j}^{(2)}$.

Similar to [CBS12], at $\mathcal{O}(\varepsilon^3)$ we find that $A_{n,1,0}$ has to satisfy the NLS equation

$$\begin{aligned}\partial_2 A_{n,1,0}(X_n,T) &= -i(\omega_n''(k_0)/2)\partial_1^2 A_{n,1,0}(X_n,T) \\ &\quad +3g^n_{nnn}(k_0,k_0,k_0,-k_0)|A_{n,1,0}(X_n,T)|^2 A_{n,1,0}(X_n,T),\end{aligned} \tag{16}$$

and that $\Omega^{(1)}_{n,j}$ has to satisfy the phase shift formula

$$\Omega^{(1)}_{n,j}(X_j,T) = \frac{6g^n_{njj}(k_0,k_0,k_0,-k_0)}{i(\omega_n'(k_0)-\omega_j'(k_0))}\int_{-\infty}^{X_j} |A_{j,1,0}(\zeta,T)|^2 d\zeta.$$

In particular, both the NLS equation and the phase shift formula are analogous to (3.10) and (3.11) in [CBS12] respectively. We have that $\Omega^{(1)}_{n,j}$ is a real quantity because of $g^n_{njj}(k_0,k_0,k_0,-k_0) \in i\mathbb{R}$. Therefore, it is a pure phase correction. Moreover, we have a number of mixed terms which can be eliminated by setting $M_{mixed,n}$ equal to

$$\sum_{r_1,r_2,r_3=\pm 1}\ \sum_{j_1,j_2,j_3,r_1j_1+r_2j_2+r_3j_3\neq 1}^{N} \varepsilon^3 M^{r_1,r_2,r_3}_{n,j_1,j_2,j_3}(X,T)e^{i(r_1Y_{j_1}+r_2Y_{j_2}+r_3Y_{j_3})} + c.c.,$$

where

$$\begin{aligned}M^{r_1,r_2,r_3}_{n,j_1,j_2,j_3}(X,T) &= ((\omega_{j_1}(r_1k_0)+\omega_{j_2}(r_2k_0)+\omega_{j_3}(r_3k_0)-\omega_n((r_1+r_2+r_3)k_0))^{-1} \\ &\quad \times g^n_{j_1,j_2,j_3}((r_1+r_2+r_3)k_0,r_1k_0,r_2k_0,r_3k_0) \\ &\quad \times A_{j_1,r_1,0}(X_{j_1},T)A_{j_2,r_2,0}(X_{j_2},T)A_{j_3,r_3,0}(X_{j_3},T).\end{aligned}$$

Comparable to (3.12) in [CBS12], at $\mathcal{O}(\varepsilon^4)$ we find that the $A_{n,1,1}$ solve linear inhomogeneous evolution equations

$$\partial_2 A_{n,1,1}(X_1,T) = i(\omega_1''(k_0)/2)\partial_1^2 A_{n,1,1}(X_1,T) + t_{n,1,1},$$

where $t_{n,1,1}$ is a function of $A_{n,1,0}$ and $A_{n,-1,0}$. Thus, $A_{n,1,1}$ describes internal dynamics of a single pulse.

At $\mathcal{O}(\varepsilon^4)$ we also find the envelope shift formula

$$\Psi^{(1)}_{n,j}(\underline{X}_j,T) = C_{n,j}\int_{-\infty}^{\underline{X}_j} |A_{j,1,0}(\zeta,T)|^2 d\zeta,$$

with an explicitly computable prefactor $C_{n,j}$, equivalent to (3.13) in [CBS12].

The quantities $\Omega_{n,j}^{(2)}$ are determined at $\mathcal{O}(\varepsilon^4)$. The real part is a second order correction to the phase shift, whereas its imaginary part gives a correction to the amplitude. We refrain from explicitly displaying the rather lengthy expression for $\Omega_{n,j}^{(2)}$ and only note that it is pure integration of spatially localized terms similar to the expressions for determining $\Omega_{n,j}^{(1)}$ and $\Psi_{n,j}^{(1)}$.

Finally, there are even more mixed terms which we also do not display explicitly.

For the computation of the mixed terms again a number of non-resonance conditions have to be satisfied. For the computation of the quadratic mixed terms we need

$$\omega_n((r_1 + r_2)k_0) - \omega_{j_1}(r_1 k_0) - \omega_{j_2}(r_2 k_0) \neq 0 \tag{17}$$

for all $n, j_1, j_2 \in \{1, \ldots, N\}$ and $r_1, r_2 \in \{-1, 1\}$. For the computation of the cubic mixed terms we need

$$\omega_n((r_1 + r_2 + r_3)k_0) - \omega_{j_1}(r_1 k_0) - \omega_{j_2}(r_2 k_0) - \omega_{j_3}(r_3 k_0) \neq 0 \tag{18}$$

for all $n, j_1, j_2, j_3 \in \{1, \ldots, N\}$ and $r_1, r_2, r_3 \in \{-1, 1\}$ with $r_1 + r_2 + r_3 \not\in \{-1, 1\}$. For the computation of the quartic mixed terms we need

$$\omega_n((r_1+r_2+r_3+r_4)k_0) - \omega_{j_1}(r_1 k_0) - \omega_{j_2}(r_2 k_0) - \omega_{j_3}(r_3 k_0) - \omega_{j_4}(r_4 k_0) \neq 0 \tag{19}$$

for all $n, j_1, j_2, j_3, j_4 \in \{1, \ldots, N\}$ and $r_1, r_2, r_3, r_4 \in \{-1, 1\}$. For the computation of the phase shifts $\Omega_{n,j}^{(r)}$ and the envelope shifts $\Psi_{n,j}^{(1)}$ the additional condition

$$\omega_n'(k_0) - \omega_j'(k_0) \neq 0, \tag{20}$$

for all $n, j \in \{1, \ldots, N\}$, with $n \neq j$, on the group velocities is necessary.

In case of no quadratic terms, i.e. $\mathcal{T}_2 = 0$, again this approximation can be justified with a simple application of Gronwall's inequality, cf. [KSM92]. In case of $\mathcal{T}_2 \neq 0$ the quadratic terms have to be eliminated by some normal form transformation. This requires additional non-resonance conditions, cf. [SU17, §11.5], namely

$$\inf_{n_1,n_2,n_3 \in \{1,\ldots,N\}} \inf_{k \in \mathbb{R}} |\omega_{n_1}(k) - \omega_{n_2}(k_0) - \omega_{n_3}(k - k_0)| > 0. \tag{21}$$

This condition again can be weakened, namely to

$$\sup_{n_1,n_2,n_3 \in \{1,\ldots,N\}} \sup_{k \in \mathbb{R}} \left| \frac{\widehat{g}^{n_1}_{n_2 n_3}(k, k_0, k - k_0)}{\omega_{n_1}(k) - \omega_{n_2}(k_0) - \omega_{n_3}(k - k_0)} \right| < \infty. \tag{22}$$

Remark 5.1. For the computation of the phase shifts $\Omega_{n,j}^{(r)}$ and the envelope shift $\Psi_{n,j}^{(1)}$ some integral has to be computed and so a certain spatial localization of the solutions of the NLS equations is necessary. Therefore, we define the space H_m^s for $s, m \in \mathbb{N}_0$ as a subspace of H^s for which the norm

$$\|A\|_{H_m^s} = \|A\rho\|_{H^s}$$

is finite, where $\rho(x) = (1 + x^2)^{m/2}$. Local existence and uniqueness for the solutions of the NLS equations (16) in spaces $H_0^{s+m} \cap H_m^s$ is well known if $s \geq 1$ and follows by application of the variation of constant formula and using the fact that $i\partial_X^2$ is the generator of a strongly continuous semigroup in $H_0^{s+m} \cap H_m^s$, cf. [CKS95].

By the above ansatz the residual terms formally are of order $\mathcal{O}(\varepsilon^5)$. Therefore, we have the following approximation theorem, cf. [SU17, Theorem 11.2.6].

Theorem 5.2. *Assume the validity of* (15) *and of the non-resonance conditions* (17), (18), (19), (20), *and* (22). *Let the* $A_{n,1,0} \in C([0, T_0], H_2^{12} \cap H_0^{14})$ *be solutions of the NLS equations* (16) *and let* $\varepsilon\Psi$ *be the approximation to these solutions defined above in Section 5. Then there exist* $C > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *we have solutions* V *of* (8) *with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |V(x, t) - \varepsilon\Psi(x, t)| \leq C\varepsilon^{5/2}.$$

For the non-diagonalized system we obtain

Corollary 5.3. *Under the assumptions of Theorem 5.2 there exist* $C > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *we have solutions* U *of* (6) *with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |U(x, t) - \varepsilon S\Psi(x, t)| \leq C\varepsilon^{5/2}.$$

For the original system (1) the approximation result is as follows.

Corollary 5.4. *Under the assumptions of Theorem 5.2 there exist* $C > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *we have solutions* $\mathcal{U}$ *of* (1) *with*

$$\sup_{\tau\in[0,T_0/\varepsilon]} \sup_{\xi\in\mathbb{R}} |\mathcal{U}(\xi, \tau) - S\Psi(\xi, \tau)| \leq C\varepsilon^{3/2}.$$

Remark 5.5. As before, Theorem 5.2 should not be taken for granted since solutions of order $\mathcal{O}(\varepsilon)$ have to be controlled on an $\mathcal{O}(1/\varepsilon^2)$-time-scale.

Remark 5.6. The previous approximation results, in particular Corollary 5.4 guarantee that the NLS approximation can be used for an effective simulation of solutions of (1) to general initial conditions of the form (2). The polarization is no longer needed.

6 The higher dimensional situation

In this section we explain how the results change if $x \in \mathbb{R}^d$ with $d \geq 2$ is considered.

The results from Section 4 transfer in a straightforward way from $x \in \mathbb{R}$ to $x \in \mathbb{R}^d$ with $d \geq 2$. The group velocity in (9) and (10) is then given by $c = \nabla_{k_0}\omega_0 \in \mathbb{R}^d$ where $k \in \mathbb{R}^d$. The NLS equation is then given by

$$\partial_T A_{n_0,1,0} = i \sum_{j_1,j_2}^{d} \nu_{j_1 j_2} \partial_{X_{j_1}} \partial_{X_{j_2}} A_{n_0,1,0} + i\nu_2 A_{n_0,1,0} |A_{n_0,1,0}|^2, \tag{23}$$

where $\nu_{j_1 j_2} = \frac{1}{2}\partial_{k_{j_1}}\partial_{k_{j_2}}\omega_0|_{k=k_0}$. The proof of the approximation result is line for line the same. In a similar way the results from Section 5 can be modified.

However, in general the non-resonance condition to eliminate the quadratic terms (13) will not be valid in higher space dimensions. In [DHSZ16] it has been pointed out that this problem can be solved by working in modulational Gevrey spaces.

We would like to close the paper with the remark that the results from Section 5 are less relevant in higher space dimensions due to the fact that for $x \in \mathbb{R}^d$ the wave-packets have more space and so in general spatially localized solutions will miss each other. In case that the wave-packets are not at the same place at the same time, there is a decoupling up to high order. The ansatz is then given by

$$V(x,t) = \sum_{n_0=1}^{N} \varepsilon \Psi_{n_0}(x,t)$$

where $\varepsilon\Psi_{n_0}$ is the approximation defined in (9)-(10) but now with

$$X = X_{n_0} = \varepsilon(x - \nabla\omega_{n_0}|_{k=k_0} t) + \varepsilon^{-1} X_{n_0,0} = \widetilde{X} + \varepsilon^{-1}(X_{n_0,0} - \nabla\omega_{n_0}|_{k=k_0} T),$$

with $\widetilde{X} = \varepsilon x$. Suppose now that

(ASS) the lines $(X,T) = (X_{n_0,0} - \nabla\omega_{n_0}|_{k=k_0}T, T) \in \mathbb{R}^{d+1}$ have no intersection points, i.e., assume that the wave-packets are not at the same place at the same time.

Then we have

Theorem 6.1. *For all $m \in \mathbb{N}$ there exists a $s \in \mathbb{N}$ such that the following holds. Assume the validity of* (15)*, of the non-resonance conditions* (12) *and* (14)*, and of the non-interaction assumption* **(ASS)***. For all $n_0 \in \{1, \ldots, N\}$ let $A_{n_0,1,0} \in C([0,T_0], H_m^{s+3m_*} \cap H_0^{s+3m_*+m})$ be solution of the NLS equations* (11) *and let the $\varepsilon\Psi_{n_0}$ be the approximations defined in* (9)-(10) *with the above modification. Then there exist $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have solutions V of* (8) *with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |V(x,t) - \sum_{n_0=1}^{N} \varepsilon\Psi_{n_0}(x,t)| \leq C\varepsilon^{m-1/2}.$$

By the localization $A_{n_0,1,0} \in C([0,T_0], H_m^{s+3m_*})$ the interaction terms are of sufficiently high order w.r.t. ε, cf. [PW95, CBCSU08].

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