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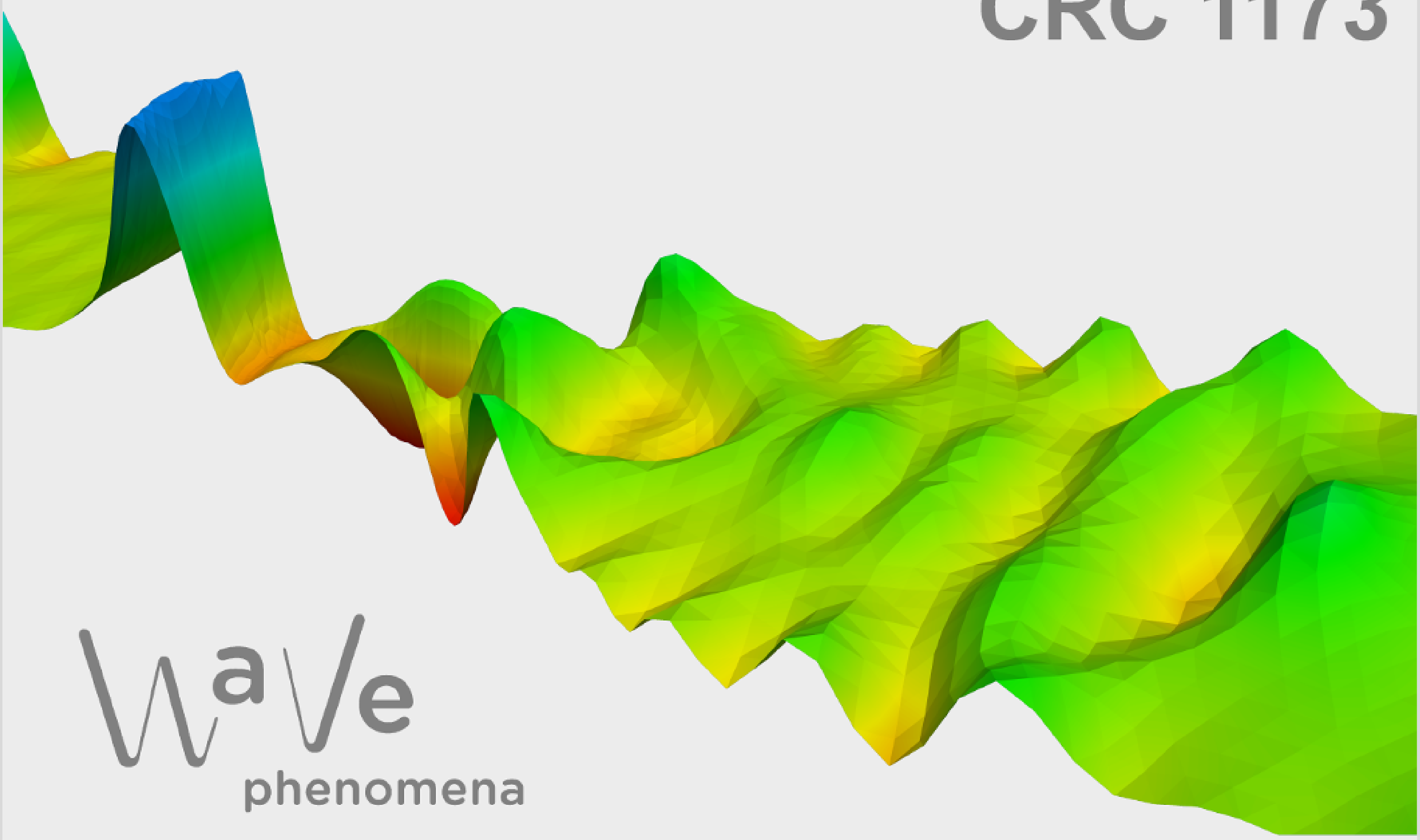

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HYBRID DISCONTINUOUS GALERKIN DISCRETIZATIONS FOR THE DAMPED TIME-HARMONIC GALBRUN'S EQUATION

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Abstract. In this article, we study the damped time-harmonic Galbrun's equation which models solar and stellar oscillations. We introduce and analyze hybrid discontinuous Galerkin discretizations (HDG) that are stable and optimally convergent for all polynomial degrees greater than or equal to one. The proposed methods are robust with respect to the drastic changes in the magnitude of the coefficients that naturally occur in stars. Our analysis is based on the concept of discrete approximation schemes and weak T-compatibility, which exploits the weakly T-coercive structure of the equation. Compared to the H^1-conforming discretization of [Halla, Lehrenfeld, Stocker, 2022], our method offers improved stability and robustness. Furthermore, it significantly reduces the computational costs compared to the $H(\mathrm{div})$-conforming DG discretization of [Halla, 2023], which has similar stability properties. These advantages make the proposed HDG methods well-suited for astrophysical simulations.

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1. Introduction

Helioseismology studies the interior of the Sun through acoustic oscillations measured at the surface [21]. Reconstructing physical quantities in the interior, such as the density, the sound speed, or subsurface flows, requires solving a passive imagining problem. To tackle this problem, approaches such as *helioseismic holography* [37, 39] rely on an accurate and computationally efficient solution of the forward problem. To model solar and stellar oscillations, we consider the damped time-harmonic *Galbrun's equation*: Find $\mathbf{u} : \mathcal{O} \to \mathbb{C}^d$ such that

$$\begin{aligned} -\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)^2\mathbf{u} - \nabla\left(\rho c_s^2 \operatorname{div}\mathbf{u}\right) + (\operatorname{div}\mathbf{u})\nabla p - \nabla(\nabla p \cdot \mathbf{u}) \\ + (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\mathbf{u} + \gamma\rho(-i\omega)\mathbf{u} = \mathbf{f} \quad \text{in } \mathcal{O}, \end{aligned} \tag{1a}$$

$$\boldsymbol{\nu} \cdot \mathbf{u} = 0 \quad \text{on } \partial\mathcal{O}, \tag{1b}$$

where $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded Lipschitz domain. We denote by ρ the density, by c_s the sound speed, by p the pressure, by ϕ the gravitational potential, and by $\mathbf{f}$ the source term. Furthermore, $\Omega \in \mathbb{R}^d$ is the angular velocity of the frame of reference, ω is the frequency and $\mathbf{b}$ is the background velocity. The operator

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$\partial_{\mathbf{b}} \coloneqq \sum_{l=1}^{d} \mathbf{b}_l \partial_{x_l}$ is the directional derivative in the direction of $\mathbf{b}$, $\mathrm{Hess}(\cdot)$ is the Hessian, and by $\boldsymbol{\nu}$, we denote the exterior unit normal vector on $\partial\mathcal{O}$. The damping is modeled with the term $-i\gamma\rho\omega\mathbf{u}$, where γ is a scalar damping coefficient.

Galbrun's equation was first derived in [20] and is a linearization of the nonlinear Euler equations with the Lagrangian perturbation of displacement as unknown. Without the additional rotational terms as in (1), it is commonly applied in aeroacoustics [38]. For the well-posed analysis in the time domain, we refer to [24].

Assuming that the Mach number of the background flow $\mathbf{b}$ is bounded suitably, the well-posedness of problem (1) has been shown in [27]. The main ingredient of the proof is a generalized Helmholtz decomposition and a weak T-coercivity argument. Here, we call a problem weakly T-coercive if it is a compact perturbation of a T-coercive problem. The T-coercivity technique [6, 9, 12] relies on the explicit construction of an operator that realizes the inf-sup condition. This approach has been successfully applied to a variety of problems, including Helmholtz-like problems [14, 19, 25, 31, 46] and problems with sign-changing coefficients [4–6, 28].

The key to develop stable discretizations of (weakly) T-coercive problems is to transfer the construction of the T-operator to the discrete level in a stable manner. In particular, the stability of the discretization is obtained when the constructions fulfill a *T-compatibility* condition [25, 29].

The construction and analysis of reliable finite element schemes for (1) was initiated in [29], where suitable H^1-conforming discretizations were considered, primarily to circumvent the challenges associated with analyzing non-conforming methods. Since the stability of the discrete divergence operator is essential for stable discretizations of (1) [1, 29], H^1-conforming discretizations of (1) face similar restrictions as those encountered for finite element discretizations for the Stokes problem. As with the Scott-Vogelius [**?**] pair, the polynomial degree has to be sufficiently large (e.g., $k \geq 4$ in 2d and $k \geq 8$ in 3d) and/or special meshes (e.g., barycentric refinements) have to be used. In addition, the assumed bound on the Mach number lacked robustness with respect to changes in magnitude of the physical parameters.

The challenges posed by non-conforming discretizations were then overcome in [26] for $H(\mathrm{div})$-conforming and in [45] for fully discontinuous Galerkin (DG) finite elements. Notably, these schemes are stable for all polynomial degrees $k \geq 1$, and the assumed bound on the Mach number remains robust even in the presence of highly heterogeneous physical parameters. To achieve this, the directional derivative $\partial_{\mathbf{b}}$ is stabilized through a *lifting operator* [2, 8], which ensures stability without the need to choose a suitable penalty parameter.

However, these developments were primarily motivated by theoretical considerations, and they lack computational efficiency, in particular because the lifting operator drastically increases the computational costs in a DG setting (see Remark 28). Thus, we propose hybrid discontinuous Galerkin (HDG) discretizations of (1) in the current work. The key idea of *hybridization* [15, 35] is to introduce additional facet unknowns, which increases the total number of degrees of freedom but reduces the number of global couplings. Due to the resulting structure of the linear system, static condensation can be applied to eliminate the volume unknowns, leading to a significant reduction in the computational costs. Furthermore, in the hybrid setting, relying on a lifting operator to stabilize $\partial_{\mathbf{b}}$ is feasible, since it is a local operator.

Our analysis extends the work of [26, 45] to the hybrid setting and covers both, the fully non-conforming and the $H(\mathrm{div})$-conforming case. We show stability and quasi-optimality for all polynomial degrees $k \geq 1$, and the required boundedness assumption on the Mach number is robust with respect to the physical parameters. Moreover, the proposed methods significantly reduce the computational costs, making them well-suited for large-scale, efficient, and accurate simulations of solar oscillations.

Structure of paper. In Section 2, we repeat the abstract framework which we use to analyze the proposed discretizations of (1). In particular, we recall the concepts of weak T-coercivity, discrete approximation schemes, and weak T-compatibility which provide sufficient criteria for the convergence of approximations. In Section 3, we introduce hybrid discontinuous Galerkin methods for (1) and show that the discretizations are discrete approximation schemes which allows us to apply the framework introduced in Section 2. Afterwards, we utilize the weak T-compatibility criteria to prove the stability and convergence of the proposed discretization in Section 4 and conclude with numerical experiments in Section 5.

2. Abstract framework

This section recalls the abstract tools which we will use to analyze the proposed discretizations of (1). For more details and proofs we refer to [26, 29, 45]. In Section 2.1, we discuss the concept of weak T-coercivity which essentially asks for an operator to be a compact perturbation of a bijective operator, cf. Definition 1 for a precise definition. Afterwards, we study the approximation of weakly T-coercive operators in Section 2.2. In particular, we introduce the (much broader) framework of *discrete approximation schemes* and discuss sufficient conditions for the convergence of discrete approximations of weakly T-coercive operators.

2.1. Weak T-coercivity

For two Hilbert spaces $(X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \langle\cdot,\cdot\rangle_Y)$, we denote by $L(X, Y)$ the space of bounded linear operators from X to Y. In particular, we set $L(X) := L(X, X)$. Through the Riesz-isomorphism, there exists a one-to-one relation between bounded sesquilinear forms $a(\cdot,\cdot)$ on $X \times X$ and bounded linear operators $A \in L(X)$ via $\langle Au, u'\rangle_X := a(u, u')$ for all $u, u' \in X$. Thus, we discuss the following concepts for linear operators $A \in L(X)$, but also associate them with the corresponding sesquilinear form.

Recall that an operator $A \in L(X)$ is called *coercive* if it holds that

$$\inf_{u\in X\setminus\{0\}} \frac{|\langle Au, u\rangle_X|}{\|u\|_X^2} > 0.$$

This condition is equivalent, cf. [18, Lem. C.58] to the existence of $\xi \in \mathbb{C}$, $|\xi| = 1$, such that

$$\inf_{u\in X\setminus\{0\}} \frac{\operatorname{Re}(\xi\langle Au, u\rangle_X)}{\|u\|_X^2} > 0. \tag{2}$$

The well-known Lax-Milgram lemma states that bounded coercive operators are bijective. More generally, a bounded operator $A \in L(X)$ is bijective if and only if the adjoint operator $A^* \in L(X)$ is injective and the *inf-sup condition* holds:

$$\inf_{u\in X\setminus\{0\}} \sup_{v\in X\setminus\{0\}} \frac{|\langle Au, v\rangle_X|}{\|u\|_X\|v\|_X} > 0.$$

Equivalently, we can prove *T-coercivity* [14], which asks for the existence of a bijective operator $T \in L(X)$ such that T^*A (or AT) is coercive. We recall the following generalization of T-coercivity.

Definition 1 (Weak T-coercivity)**.** We call an operator $A \in L(X)$ *weakly T-coercive* if there exists a bijective operator $T \in L(X)$ and a compact operator $K \in L(X)$ such that $AT + K$ is coercive.

In other words, an operator is weakly T-coercive if it is a compact perturbation of a T-coercive, operator. Thus, weakly T-coercive operators are Fredholm with index zero and therefore bijective if and only if they are injective.

2.2. Discrete approximation schemes and weak T-compatibility

We want to study the approximation of weakly T-coercive operators in a general setting. To this end, we discuss the notion of *weak T-compatibility* [25, 29] which is build upon the framework of *discrete approximation schemes* [43, 44]. For a more extensive review of these concepts, we refer to [45, Chap. 2]. In the following, let X be a Hilbert space and $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite dimensional Hilbert spaces, which are not necessarily subspaces of X. Instead, we assume that there exists a sequence of bounded linear operators $(p_n)_{n\in\mathbb{N}}$, $p_n \in L(X, X_n)$, such that $\lim_{n\to\infty} \|p_n u\|_{X_n} = \|u\|_X$ for all $u \in X$. Finally, let $A \in L(X)$ be a bounded linear operator and $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n)$, be a sequence of bounded linear operators.

Definition 2. In the setting from above, we define the following concepts:

(i) A sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$, is said to *converge* to $u \in X$, if $\lim_{n\to\infty} \|p_n u - u_n\|_{X_n} = 0$.

(ii) A sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$, is called *compact*, if for every subsequence $\mathbb{N}' \subset \mathbb{N}$ there exists a subsubsequence $\mathbb{N}'' \subset \mathbb{N}'$ and $u \in X$ such that $(u_n)_{n\in\mathbb{N}''}$ converges to u.

(iii) A sequence of operators $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n)$, *approximates* (also called *asymptotic consistency*) an operator $A \in L(X)$, if $\lim_{n\to\infty} \|A_n p_n u - p_n A u\|_{X_n} = 0$ for all $u \in X$.

(iv) A sequence of operators $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n)$, is called *compact*, if for every bounded sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$, $\|u_n\|_{X_n} \leq C$, the sequence $(A_n u_n)_{n\in\mathbb{N}}$ is compact.

(v) A sequence of operators $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n)$, is called *stable*, if there exist constants $C > 0$, $n_0 > 0$, such that A_n is invertible and $\|A_n^{-1}\|_{L(X_n)} \leq C$ for all $n > n_0$.

(vi) A sequence of operators $(A_n)_{n\in\mathbb{N}}$, $A_n \in L(X_n)$, is said to be *regular*, if $\|u_n\|_{X_n} \leq C$ and the compactness of $(A_n u_n)_{n\in\mathbb{N}}$ imply the compactness of $(u_n)_{n\in\mathbb{N}}$ itself.

We call the triple (X_n, p_n, A_n) a *discrete approximation scheme* (DAS) of (X, A) if we have that $\lim_{n\to\infty} \|p_n u\|_{X_n} = \|u\|_X$ for all $u \in X$ and A_n approximates A.

A conforming Galerkin scheme (X_n, p_n, A_n), where $X_n \subset X$ fulfills an approximation property, $p_n \in L(X, X_n)$ is the orthogonal projection onto X_n, and $A_n := p_n A|_{X_n}$, is always a DAS of (X, A). Our main goal is to show that the sequence of approximations $(u_n)_{n\in\mathbb{N}}$, $u_n \in X_n$, converges to the continuous solution $u \in X$, so we are interested in the stability of the sequence $(A_n)_{n\in\mathbb{N}}$. The following result shows that we can focus on the regularity of the sequence $(A_n)_{n\in\mathbb{N}}$ instead.

Lemma 3 (Lem. 1 & 2 of [29])**.** *Let $A \in L(X)$ be bijective and (X_n, p_n, A_n) be a DAS of (X, A). If $(A_n)_{n\in\mathbb{N}}$ is regular, then $(A_n)_{n\in\mathbb{N}}$ is stable. Further, if $u \in X$ solves $Au = f$ and $u_n \in X_n$ are solutions to $A_n u_n = f_n$ where $\lim_{n\to\infty} \|p_n f - f_n\|_{X_n} = 0$, then $(u_n)_{n\in\mathbb{N}}$ converges to u.*

The following theorem gives sufficient conditions for the regularity of approximations of weakly T-coercive operators and therefore the stability of the approximation. It is the key motivation for the analysis presented in Section 4. We note that if $A \in L(X)$ is weakly T-coercive , then there exists a bijective operator $B \in L(X)$ and a compact operator $K \in L(X)$ such that $AT = B + K$.

Theorem 4 (Thm. 3 of [29])**.** *Assume that there exists a constant $C > 0$, sequences $(A_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(K_n)_{n\in\mathbb{N}}$ and $B, T \in L(X)$ such that for each $n \in \mathbb{N}$ it holds that $A_n, T_n, B_n, K_n \in L(X_n)$, $\|T_n\|_{L(X_n)}$, $\|T_n^{-1}\|_{L(X_n)}$, $\|B_n\|_{L(X_n)}$, $\|B_n^{-1}\|_{L(X_n)} \leq C$, B is bijective, $(K_n)_{n\in\mathbb{N}}$ is compact and*

$$\lim_{n\to\infty} \|T_n p_n u - p_n T u\|_{X_n} = 0, \qquad \lim_{n\to\infty} \|B_n p_n u - p_n B u\|_{X_n} = 0, \qquad \forall u \in X,$$

$$A_n T_n = B_n + K_n.$$

Then the sequence $(A_n)_{n\in\mathbb{N}}$ is regular.

To summarize, Theorem 4 yields the stability of a discrete approximation scheme, provided that we can transfer the weakly T-coercive structure of the continuous operator $A \in L(X)$ to the discrete level in a stable manner.

3. Hybrid discontinuous Galerkin discretizations

In this section, we introduce the considered discretizations of (1). After discussing preliminaries and the continuous weak formulation in Section 3.1, we introduce the HDG discretizations of Galbrun's equation in Section 3.2. To conclude, we show in Section 3.3 that the proposed discretization is indeed a DAS allowing us apply the framework from Section 2.

3.1. **Preliminaries and weak formulation**

For simplicity, we assume that $\mathcal{O} \subset \mathbb{R}^d$ is a convex Lipschitz polyhedron and consider $\mathcal{O}$ to be the default domain of all function spaces. Thus, we write for example $L^2 := L^2(\mathcal{O})$. Further, we denote by $\langle\cdot,\cdot\rangle$ the standard L^2-scalar product. For any space X of scalar valued functions, we denote its vectorial version $\boldsymbol{X} := [X]^d$ using the boldface notation. With abuse of notation, we use also use the notation $\langle\cdot,\cdot\rangle$ for the vector valued $\mathbf{L}^2$-scalar product. For any $X \subset L^2$ we denote $X_* := \{u \in X : \langle u, 1\rangle = 0\}$ with the special case $L^2_0 := L^2_*$. Furthermore, we define the space

$$\mathbf{H}^1_{\boldsymbol{\nu}0} := \{\mathbf{v} \in \mathbf{H}^1 : \mathbf{v}\cdot\boldsymbol{\nu} = 0 \text{ on } \partial\mathcal{O}\}. \tag{3}$$

Let $\omega \in \mathbb{R}\setminus\{0\}$, $\Omega \in \mathbb{R}^d$, and $c_s, \rho \in W^{1,\infty}(\mathcal{O},\mathbb{R})$, $\gamma \in L^\infty(\mathcal{O},\mathbb{R})$ be measurable and bounded from above and below. For any function, we denote by $\underline{\cdot}$ and $\bar{\cdot}$ its minimal and maximal value in the domain under consideration. Thus, the boundedness assumptions on the coefficients translate to

$$\underline{c_s} \le c_s(\boldsymbol{x}) \le \overline{c_s}, \quad \underline{\rho} \le \rho(\boldsymbol{x}) \le \overline{\rho}, \quad \underline{\gamma} \le \gamma(\boldsymbol{x}) \le \overline{\gamma} \quad \text{for all } \boldsymbol{x} \in \mathcal{O}, \tag{4}$$

for constants $\underline{c_s}, \overline{c_s}, \underline{\rho}, \overline{\rho}, \underline{\gamma}, \overline{\gamma} > 0$. Finally, let the background flow $\mathbf{b} \in \boldsymbol{W}^{1,\infty}(\mathcal{O},\mathbb{R}^d)$ be compactly supported in $\mathcal{O}$. We assume that the background flow conserves mass in the sense that $\operatorname{div}(\rho\mathbf{b}) = 0$. In particular, the former assumptions imply that $\operatorname{div}(\rho\mathbf{b}) \in L^2$ and $\mathbf{b}\cdot\boldsymbol{\nu} = 0$ on $\partial\mathcal{O}$. Let the pressure and gravitational potential $p, \phi \in W^{2,\infty}(\mathcal{O},\mathbb{R})$.

Throughout the manuscript, we use the notation $A \lesssim B$, if there exists a constant $C > 0$ such that $A \le CB$, where the constant C may be different at each occurrence. The constant C may depend on the domain $\mathcal{O}$ and the physical parameters, but not on the index $n \in \mathbb{N}$ and functions involved in A and B. In particular, the constant C is not allowed to depend on the ratio $\frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}}$.

To introduce the weak formulation of (1), we define the space $\mathbb{X}$ through

$$\mathbb{X} := \{\mathbf{u} \in \mathbf{L}^2 : \operatorname{div}\mathbf{u} \in L^2, \partial_{\mathbf{b}}\mathbf{u} \in \mathbf{L}^2, \boldsymbol{\nu}\cdot\mathbf{u} = 0 \text{ on } \partial\mathcal{O}\}. \tag{5}$$

In contrast to the original definitions from [27], we consider the associated inner product on $\mathbb{X}$ to be weighted:

$$\langle \mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}} := \langle c_s^2\rho \operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{u}'\rangle + \langle \mathbf{u}, \mathbf{u}'\rangle + \langle \rho\partial_{\mathbf{b}}\mathbf{u}, \partial_{\mathbf{b}}\mathbf{u}'\rangle$$

Due to the smoothness assumptions $c_s, \rho \in W^{1,\infty}(\mathcal{O},\mathbb{R})$ and the boundedness assumptions (4), the weighted inner product is equivalent to the canonical inner product on $\mathbb{X}$ and the proof that $\mathbb{X}$ is a Hilbert space follows with the same argumentation as in [27, Lem. 2.1]. The smoothness assumption $\mathbf{b} \in \boldsymbol{W}^{1,\infty}$ and the compactness of $\operatorname{supp}\mathbf{b} \subset \mathcal{O}$ ensure that the embedding $\boldsymbol{C}_0^\infty \subset \mathbb{X}$ is dense [29, Thm. 6].

For $\mathbf{u}, \mathbf{u}' \in \mathbb{X}$, we define the following sesquilinear forms

$$a^{\operatorname{div}}(\mathbf{u}, \mathbf{u}') := \langle c_s^2\rho \operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{u}'\rangle_{L^2} + \langle \operatorname{div}\mathbf{u}, \nabla p\cdot\mathbf{u}'\rangle_{L^2} + \langle \nabla p\cdot\mathbf{u}, \operatorname{div}\mathbf{u}'\rangle_{L^2}, \tag{6a}$$

$$a^{\partial_{\mathbf{b}}}(\mathbf{u}, \mathbf{u}') := \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}'\rangle_{\mathbf{L}^2}, \tag{6b}$$

$$a^{r}(\mathbf{u}, \mathbf{u}') := \langle (\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\mathbf{u}, \mathbf{u}'\rangle_{\mathbf{L}^2} - i\omega\langle \gamma\rho\mathbf{u}, \mathbf{u}'\rangle_{\mathbf{L}^2}. \tag{6c}$$

Then, we define the sesquilinear form $a\colon \mathbb{X}\times\mathbb{X} \to \mathbb{C}$ by

$$a(\mathbf{u}, \mathbf{u}') := a^{\operatorname{div}}(\mathbf{u}, \mathbf{u}') - a^{\partial_{\mathbf{b}}}(\mathbf{u}, \mathbf{u}') + a^{r}(\mathbf{u}, \mathbf{u}') \tag{7}$$

and denote by $A \in L(\mathbb{X})$ the associated operator. Assuming mass conservation $\operatorname{div}(\rho\mathbf{b}) = 0$, the variational formulation of (1) is given by

$$\text{find } \mathbf{u} \in \mathbb{X} \text{ such that } a(\mathbf{u}, \mathbf{u}') = \langle \mathbf{f}, \mathbf{u}'\rangle \text{ for all } \mathbf{u}' \in \mathbb{X}, \tag{8}$$

cf. [27, Sec. 2.3]. If the Mach number $\|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}$ is bounded suitably, it can be shown that the operator A is weakly T-coercive and injective such that problem (8) is well-posed [27, Thm. 3.11]. Defining $\mathbf{q} := c_s^{-2}\rho^{-1}\nabla p$

the sesquilinear form $a(\cdot,\cdot)$ can be written as

$$\begin{aligned} a(\mathbf{u},\mathbf{u}') =& \langle c_s^2\rho(\operatorname{div} + \mathbf{q}\cdot)\mathbf{u}, (\operatorname{div} + \mathbf{q}\cdot)\mathbf{u}'\rangle - \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}'\rangle \\ &+ \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi) - c_s^2\rho\mathbf{q}\otimes\mathbf{q})\mathbf{u}, \mathbf{u}'\rangle - i\omega\langle\gamma\rho\mathbf{u},\mathbf{u}'\rangle. \end{aligned} \tag{9}$$

This representation will be useful for the discussion of the well-posedness of the continuous and the discrete problem in Section 4. Similar to (6), we define

$$a^{(\operatorname{div} + \mathbf{q}\cdot)}(\mathbf{u},\mathbf{u}') := \langle c_s^2\rho(\operatorname{div} + \mathbf{q}\cdot)\mathbf{u}, (\operatorname{div} + \mathbf{q}\cdot)\mathbf{u}'\rangle \tag{10a}$$

$$a^{(r-\mathbf{q}\cdot)}(\mathbf{u},\mathbf{u}') := \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi) - c_s^2\rho\mathbf{q}\otimes\mathbf{q})\mathbf{u}, \mathbf{u}'\rangle - i\omega\langle\gamma\rho\mathbf{u},\mathbf{u}'\rangle, \tag{10b}$$

such that $a(\mathbf{u},\mathbf{u}') = a^{(\operatorname{div} + \mathbf{q}\cdot)}(\mathbf{u},\mathbf{u}') - a^{\partial_{\mathbf{b}}}(\mathbf{u},\mathbf{u}') + a^{(r-\mathbf{q}\cdot)}(\mathbf{u},\mathbf{u}')$. In particular, considering $a(\mathbf{u},\mathbf{u})$ for $\mathbf{u} \in \mathbb{X} \cap \ker\{\operatorname{div} + \mathbf{q}\}$ reveals that the sesquilinear form $a(\cdot,\cdot)$ is not coercive.

3.2. HDG-Discretization

Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of shape regular, simplicial triangulations of the domain $\mathcal{O}$. Let $\mathcal{F}_n$ be the collection of all faces of the triangulation $\mathcal{T}_n$, and let $\partial\mathcal{T}_n$ be the collection of all element boundaries $\partial\tau$ of elements $\tau \in \mathcal{T}_n$. Notice the subtle difference between $\mathcal{F}_n$ and $\partial\mathcal{T}_n$; for instance, summing over all element boundaries counts each interior facet *twice*.

For an element $\tau \in \mathcal{T}_n$ or a face $F \in \mathcal{F}_n$, we denote by h_τ and h_F their diameters, respectively, and we set $h_{\partial\tau} = \max_{F\in\partial\tau} h_F$. For a unified presentation, we define a function $\mathfrak{h}|_\sigma := h_\sigma$, $\sigma \in \mathcal{S}_n$, where $\mathcal{S}_n \in \{\mathcal{T}_n, \partial\mathcal{T}_n, \mathcal{F}_n\}$. Finally, let $h_n := \max_{\tau\in\mathcal{T}_n} h_\tau$ be the maximal mesh size.

For a generic Hilbert space $\mathbb{S}$, we denote by $\mathbb{S}(\mathcal{T}_n)$ its broken version on $\mathcal{T}_n$. In particular, we denote by $\mathbb{P}^k(\mathcal{T}_n)$ and $\mathbb{P}^k(\mathcal{F}_n)$ the spaces of piecewise polynomials up to degree k on $\mathcal{T}_n$ and $\mathcal{F}_n$.

On broken spaces $\mathbb{S}(\mathcal{S}_n)$, where $\mathcal{S}_n \in \{\mathcal{T}_n, \partial\mathcal{T}_n, \mathcal{F}_n\}$, we use the abbreviations:

$$\langle\cdot,\cdot\rangle_{\mathbb{S}(\mathcal{S}_n)} := \sum_{\sigma\in\mathcal{S}_n}\langle\cdot,\cdot\rangle_{\mathbb{S}(\sigma)}, \qquad \|\cdot\|^2_{\mathbb{S}(\mathcal{S}_n)} := \sum_{\sigma\in\mathcal{S}_n}\|\cdot\|^2_{\mathbb{S}(\sigma)}.$$

In particular, we set $\langle\cdot,\cdot\rangle_{\mathcal{S}_n} := \langle\cdot,\cdot\rangle_{L^2(\mathcal{S}_n)}$. With abuse of notation, we will also use this notation for the respective broken vector-valued scalar products, i.e. with L^2 replaced by $\mathbf{L}^2$. We introduce the discrete space

$$\mathbb{X}_n := \mathbb{X}_{\mathcal{T}_n} \times \mathbb{X}_{\mathcal{F}_n},$$

where $\mathbb{X}_{\mathcal{T}_n}$ and $\mathbb{X}_{\mathcal{F}_n}$ are discrete polynomial spaces defined on $\mathcal{T}_n$ and $\mathcal{F}_n$, respectively. The default choices are $\mathbb{X}_{\mathcal{T}_n} = [\mathbb{P}^k(\mathcal{T}_n)]^d$ and $\mathbb{X}_{\mathcal{F}_n} = [\mathbb{P}^k(\mathcal{F}_n)]^d$, $k \in \mathbb{N}$, yielding a fully non-conforming HDG discretization. However, the forthcoming analysis also covers different choices, for example $H(\operatorname{div})$-conforming spaces, cf. Remark 5.

For functions $\mathbf{u}_n \in \mathbb{X}_n$, we write $\mathbf{u}_n = (\mathbf{u}_\tau, \mathbf{u}_F)$, where $\mathbf{u}_\tau \in \mathbb{X}_{\mathcal{T}_n}$ is the volume and $\mathbf{u}_F \in \mathbb{X}_{\mathcal{F}_n}$ is the facet component of $\mathbf{u}_n$. On occasion, we make use of the projection operators onto the volume or facet components defined by $(\cdot)_\tau\colon \mathbb{X}_n \to \mathbb{X}_{\mathcal{T}_n}, (\mathbf{u}_\tau,\mathbf{u}_F) \mapsto \mathbf{u}_\tau$ and $(\cdot)_F\colon \mathbb{X}_n \to \mathbb{X}_{\mathcal{F}_n}, (\mathbf{u}_\tau,\mathbf{u}_F) \mapsto \mathbf{u}_F$.

We define the following HDG-jump operators element-wise on $\tau \in \mathcal{T}_n$

$$[\![\underline{\mathbf{u}_n}]\!] := \mathbf{u}_\tau - \mathbf{u}_F, \quad [\![\underline{\mathbf{u}_n}]\!]_{\boldsymbol{\nu}} := \boldsymbol{\nu}\cdot[\![\underline{\mathbf{u}_n}]\!], \quad [\![\underline{\mathbf{u}_n}]\!]_{\mathbf{b}} := (\mathbf{b}\cdot\boldsymbol{\nu})[\![\underline{\mathbf{u}_n}]\!], \tag{11}$$

where we interpret $\mathbf{u}_\tau$ in a trace sense. Further, we define $[\![\mathbf{u}_\tau]\!]_F := \mathbf{u}_\tau|_{\tau_1} - \mathbf{u}_\tau|_{\tau_2}$, $F \in \mathcal{F}_n$, $\tau_1,\tau_2 \in \mathcal{T}_n$, $\tau_1 \cap \tau_2 = F$, to be the usual DG-jump operator (distinguished by the absence of the underline) on $\mathbb{X}_{\mathcal{T}_n}$. Here, we assume a unique numbering of the aligned elements for each facet to fix the sign of the jump.

Let $l \in \mathbb{N}$. For all $\mathbf{u}_n \in \mathbb{X}_n$, we define the weighted lifting operators of degree l as $\mathbf{R}^l\mathbf{u}_n \in [\mathbb{P}^l(\mathcal{T}_n)]^d$ and $R^l\mathbf{u}_n \in \mathbb{P}^l(\mathcal{T}_n)$ solving

$$\langle \rho \mathbf{R}^l \mathbf{u}_n, \boldsymbol{\psi}_n \rangle_{\mathcal{T}_n} = -\langle \rho [\![\underline{\mathbf{u}_n}]\!]_{\mathbf{b}}, \boldsymbol{\psi}_n \rangle_{\mathbf{L}^2(\partial \mathcal{T}_n)} \quad \text{for all } \boldsymbol{\psi}_n \in [\mathbb{P}^l(\mathcal{T}_n)]^d, \tag{12a}$$

$$\langle c_s^2 \rho R^l \mathbf{u}_n, \psi_n \rangle_{\mathcal{T}_n} = -\langle c_s^2 \rho [\![\underline{\mathbf{u}_n}]\!]_{\boldsymbol{\nu}}, \psi_n \rangle_{L^2(\partial \mathcal{T}_n)} \quad \text{for all } \psi_n \in \mathbb{P}^l(\mathcal{T}_n). \tag{12b}$$

Due to the Cauchy-Schwarz and the discrete trace inequality, we have that

$$\begin{aligned} \|\rho^{1/2} \mathbf{R}^l \mathbf{u}_n\|_{\mathbf{L}^2}^2 &\leq C_{dt}^2 \|\rho^{1/2} \mathfrak{h}^{-1/2} [\![\underline{\mathbf{u}_n}]\!]_{\mathbf{b}}\|_{\mathbf{L}^2(\partial \mathcal{T}_n)}^2, \\ \|(c_s^2 \rho)^{1/2} R^l \mathbf{u}_n\|_{L^2}^2 &\leq C_{dt}^2 \|(c_s^2 \rho)^{1/2} \mathfrak{h}^{-1/2} [\![\underline{\mathbf{u}_n}]\!]_{\boldsymbol{\nu}}\|_{L^2(\partial \mathcal{T}_n)}^2. \end{aligned} \tag{13}$$

The lifting operators allow us to define the following discrete versions of the differential operators $\mathbf{D}_{\mathbf{b}}^n$ and $\operatorname{div}_{\boldsymbol{\nu}}^n$ on $\mathbb{X}_n$. For $\mathbf{u}_n \in \mathbb{X}_n$ and $\tau \in \mathcal{T}_n$, we define

$$(\mathbf{D}_{\mathbf{b}}^n \mathbf{u}_n)|_\tau \coloneqq (\partial_{\mathbf{b}} \mathbf{u}_\tau) + \mathbf{R}^l \mathbf{u}_n \quad \text{and} \quad (\operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n)|_\tau \coloneqq (\operatorname{div} \mathbf{u}_\tau) + R^l \mathbf{u}_n. \tag{14}$$

These operators can be interpreted as distributional versions of their continuous counterpart on the broken polynomial space $\mathbb{X}_n$, cf. [10]. To stabilize $\partial_{\mathbf{b}}$ on the discrete level, we replace $\partial_{\mathbf{b}}$ by $\mathbf{D}_{\mathbf{b}}^n$ in the discrete sesquilinear form. This treatment stems from a Bassi-Rebay lifting technique [2, 8] and enables us to obtain a stable method without an additional stabilization term. In particular, this technique allows us to avoid further assumptions on the magnitude of the Mach number. In contrast, the terms involving the divergence operator do not depend on the background flow $\mathbf{b}$. As such, we use a classical symmetric interior penalty (SIP) technique for those terms and note that the discrete divergence operator is mainly introduced to obtain a unified notation with $\mathbf{D}_{\mathbf{b}}^n$, for which the lifting technique is essential. To be precise, we introduce the following discrete sesquilinear forms:

$$\begin{aligned} a_n^{\mathrm{div}}(\mathbf{u}_n, \mathbf{u}_n') &\coloneqq \langle c_s^2 \rho \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n' \rangle_{\mathcal{T}_n} + s_n(\mathbf{u}_n, \mathbf{u}_n') \\ &\quad + \langle \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n, \nabla p \cdot \mathbf{u}_\tau' \rangle_{\mathcal{T}_n} + \langle \nabla p \cdot \mathbf{u}_\tau, \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n' \rangle_{\mathcal{T}_n} \end{aligned} \tag{15a}$$

$$a_n^{\partial_{\mathbf{b}}}(\mathbf{u}_n, \mathbf{u}_n') \coloneqq \langle \rho(\omega + i\mathbf{D}_{\mathbf{b}}^n + i\Omega\times)\mathbf{u}_n, (\omega + i\mathbf{D}_{\mathbf{b}}^n + i\Omega\times)\mathbf{u}_n' \rangle_{\mathcal{T}_n} \tag{15b}$$

$$a_n^{r}(\mathbf{u}_n, \mathbf{u}_n') \coloneqq \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\mathbf{u}_\tau, \mathbf{u}_\tau' \rangle_{\mathcal{T}_n} - i\omega \langle \gamma \rho \mathbf{u}_\tau, \mathbf{u}_\tau' \rangle_{\mathcal{T}_n} \tag{15c}$$

where the stabilization term $s_n(\cdot,\cdot)$ is defined for $\alpha > 0$ as

$$s_n(\mathbf{u}_n, \mathbf{u}_n') \coloneqq -\langle c_s^2 \rho \alpha \mathfrak{h}^{-1} [\![\underline{\mathbf{u}_n}]\!]_{\boldsymbol{\nu}}, [\![\underline{\mathbf{u}_n'}]\!]_{\boldsymbol{\nu}} \rangle_{\partial \mathcal{T}_n} - \langle c_s^2 \rho R^l \mathbf{u}_n, R^l \mathbf{u}_n' \rangle_{\mathcal{T}_n}.$$

The unusual minus in front of the first term is required to show stability in Theorem 23. In particular, the construction of the T_n in Section 4.1 flips the sign in front of the normal jump, which makes the terms stemming from $s_n(\cdot,\cdot)$ positive.

Altogether, we define the discrete version of (7) through

$$a_n(\mathbf{u}_n, \mathbf{u}_n') \coloneqq a_n^{\mathrm{div}}(\mathbf{u}_n, \mathbf{u}_n') - a_n^{\partial_{\mathbf{b}}}(\mathbf{u}_n, \mathbf{u}_n') + a_n^{r}(\mathbf{u}_n, \mathbf{u}_n'). \tag{16}$$

We denote by $A_n \in L(\mathbb{X}_n)$ the operator associated to the sesquilinear form $a_n(\cdot,\cdot)$. The use of the discrete divergence operator $\operatorname{div}_{\boldsymbol{\nu}}^n$ in combination with the stabilization term $s_n(\cdot,\cdot)$ indeed yields a SIP formulation for $a_n^{\mathrm{div}}(\cdot,\cdot)$, since

$$\begin{aligned} &\langle c_s^2 \rho \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n' \rangle_{\mathcal{T}_n} + s_n(\mathbf{u}_n, \mathbf{u}_n') \\ &\quad = \langle c_s^2 \rho \operatorname{div} \mathbf{u}_\tau, \operatorname{div} \mathbf{u}_\tau \rangle_{\mathcal{T}_n} - \langle c_s^2 \rho [\![\underline{\mathbf{u}_n}]\!]_{\boldsymbol{\nu}}, \operatorname{div} \mathbf{u}_\tau' \rangle_{\partial \mathcal{T}_n} \\ &\quad - \langle c_s^2 \rho \operatorname{div} \mathbf{u}_\tau, [\![\underline{\mathbf{u}_n'}]\!]_{\boldsymbol{\nu}} \rangle_{\partial \mathcal{T}_n} + \langle c_s^2 \rho \alpha \mathfrak{h}^{-1} [\![\underline{\mathbf{u}_n}]\!]_{\boldsymbol{\nu}}, [\![\underline{\mathbf{u}_n'}]\!]_{\boldsymbol{\nu}} \rangle_{\partial \mathcal{T}_n}. \end{aligned}$$

Altogether, we consider the discrete problem

$$\text{find } \mathbf{u}_n \in \mathbb{X}_n \text{ such that } a_n(\mathbf{u}_n, \mathbf{u}_n') = \langle \mathbf{f}, \mathbf{u}_n' \rangle \text{ for all } \mathbf{u}_n' \in \mathbb{X}_n. \tag{17}$$

For functions $\mathbf{u}_n, \mathbf{u}_n' \in \mathbb{X}_n$, we define the following scalar product

$$\begin{aligned}\langle \mathbf{u}_n, \mathbf{u}_n' \rangle_{\mathbb{X}_n} :=& \langle c_s^2 \rho \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n' \rangle_{\mathcal{T}_n} + \langle \mathbf{u}_\tau, \mathbf{u}_\tau' \rangle_{\mathcal{T}_n} \\ &+ \langle \rho \mathbf{D}_{\mathbf{b}}^n \mathbf{u}_n, \mathbf{D}_{\mathbf{b}}^n \mathbf{u}_n' \rangle_{\mathcal{T}_n} + \langle c_s^2 \rho \mathfrak{h}^{-1} [\![\underline{\mathbf{u}_n}]\!]_{\boldsymbol{\nu}}, [\![\underline{\mathbf{u}_n'}]\!]_{\boldsymbol{\nu}} \rangle_{\partial \mathcal{T}_n},\end{aligned} \tag{18}$$

and denote by $\|\cdot\|_{\mathbb{X}_n} = \sqrt{\langle \cdot, \cdot \rangle_{\mathbb{X}_n}}$ the induced norm. The terms involving the normal jump $[\![\underline{\cdot}]\!]_{\boldsymbol{\nu}}$ are added to control the terms of $a_n(\cdot,\cdot)$ arising from the SIP stabilization of the $\langle \operatorname{div} \cdot, \operatorname{div} \cdot \rangle_{\mathcal{T}_n}$-term.

In preparation for the forthcoming analysis, we define the following bounded interpolation operators. For $s > 1/2$, let

- $\pi_n^d : \mathbf{H}^s \to [\mathbb{P}^k(\mathcal{T}_n)]^d \cap H(\operatorname{div})$ be an $H(\operatorname{div})$-conforming interpolation operator,
- $\pi_n^l : L^2 \to \mathbb{P}^k(\mathcal{T}_n)$ be the (scalar) L^2-interpolation operator,
- $\pi_n : \mathbf{H}^s \to \mathbb{X}_{\mathcal{T}_n}$ be an interpolation operator into $\mathbb{X}_{\mathcal{T}_n}$, e.g. the standard $\mathbf{L}^2$-interpolation operator if $\mathbb{X}_{\mathcal{T}_n} = [\mathbb{P}^k(\mathcal{T}_n)]^d$ or an $H(\operatorname{div})$-conforming interpolation operator if $\mathbb{X}_{\mathcal{T}_n} = [\mathbb{P}^k(\mathcal{T}_n)]^d \cap H(\operatorname{div})$.

We assume that the interpolation operators π_n^d and π_n^l fulfill the commutation property $\operatorname{div} \pi_n^d = \pi_n^l \operatorname{div}$ and that the following estimates hold: for all $\mathbf{v} \in \mathbf{H}^r(\tau)$, $r \in [1, k+1]$, $m \in [0, r]$, $\tau \in \mathcal{T}_n$, we have that

$$|\mathbf{v} - \tilde{\pi}_n \mathbf{v}|_{\mathbf{H}^m(\tau)} \le C_{\mathrm{apr}} h_\tau^{r-m} |\mathbf{v}|_{\mathbf{H}^r(\tau)}, \tag{19a}$$

$$\|\mathbf{v} - \tilde{\pi}_n \mathbf{v}\|_{\mathbf{L}^2(\partial\tau)} \le C_{\mathrm{ab}} h_\tau^{r-1/2} |\mathbf{v}|_{\mathbf{H}^r(\tau)}, \tag{19b}$$

where $\tilde{\pi}_n \in \{\pi_n^d, \pi_n\}$. For standard constructions satisfying these assumptions, we refer to [17].

We extend the interpolation operator π_n from the DG space $\mathbb{X}_{\mathcal{T}_n}$ to the HDG space $\mathbb{X}_n$ through

$$\underline{\pi_n} : \mathbf{H}^s \to \mathbb{X}_n, \mathbf{u}_\tau \mapsto (\pi_n \mathbf{u}_\tau, \operatorname{tr}|_{\mathcal{F}_\mathrm{n}}(\pi_n \mathbf{u}_\tau)),$$

where the trace operator $\operatorname{tr}|_{\mathcal{F}_\mathrm{n}}$ of a discontinuous function is defined through averaging. For $\mathbf{u}_\tau \in \mathbb{X}_{\mathcal{T}_n}$, $F \in \mathcal{F}_n$ and $\tau_1, \tau_2 \in \mathcal{T}_n$ such that $\tau_1 \cap \tau_2 = F$, we set

$$(\operatorname{tr}|_{\mathcal{F}_\mathrm{n}} \mathbf{u}_\tau)|_F := \frac{1}{2}(\operatorname{tr} \mathbf{u}_\tau|_{\tau_1} + \operatorname{tr} \mathbf{u}_\tau|_{\tau_2}).$$

We define a specific extension of the $H(\operatorname{div})$-conforming interpolation operator π_n^d to $\mathbb{X}_n$ for future use. For any vector-valued function $\mathbf{u}$, let $P_{\boldsymbol{\nu}} \mathbf{u} := \boldsymbol{\nu}(\boldsymbol{\nu} \cdot \mathbf{u})$ denote the normal projection and $P_{\boldsymbol{\nu}}^{\perp} := \operatorname{id} - P_{\boldsymbol{\nu}}$ be the tangential projection. For $s \ge 0$, we set

$$\underline{\tilde{\pi}}_n^d : \mathbf{H}^{1+s} \to \mathbb{X}_{\mathcal{T}_n} \cap H(\operatorname{div}) \times \mathbb{X}_{\mathcal{F}_n}, \mathbf{u} \mapsto (\pi_n^d \mathbf{u}, P_{\boldsymbol{\nu}}(\pi_n^d \mathbf{u}) + P_{\boldsymbol{\nu}}^{\perp}(\pi_n^{\mathcal{F}_n} \mathbf{u})), \tag{20}$$

where $\pi_n^{\mathcal{F}_n}$ is the L^2-projection onto $\mathbb{X}_{\mathcal{F}_n}$. The properties of the interpolation operator $\underline{\tilde{\pi}}_n^d$ are studied in Lemma 6.

Remark 5 (Choices for $\mathbb{X}_{\mathcal{T}_n}$ and $\mathbb{X}_{\mathcal{F}_n}$)**.** As discussed above, we may choose $\mathbb{X}_{\mathcal{T}_n}$ as the Brezzi-Douglas-Marini (BDM) space [7] defined as

$$\mathbb{X}_{\mathcal{T}_n} := \mathbb{BDM}_k := \{\mathbf{u}_\tau \in [\mathbb{P}^k(\mathcal{T}_n)]^d : [\![\mathbf{u}_\tau \cdot \boldsymbol{\nu}]\!]_F = 0 \ \forall F \in \mathcal{F}_n\} \subset H(\operatorname{div}, \mathcal{O}),$$

where we recall that $[\![\cdot]\!]_F$, $F \in \mathcal{F}_n$, denotes the usual DG-jump operator. To obtain an $H(\operatorname{div})$-conforming HDG discretization [36], we set $\mathbb{X}_{\mathcal{F}_n}$ to be the space of tangential polynomials on the skeleton $\mathcal{F}_n$:

$$\mathbb{X}_{\mathcal{F}_n} := \mathbb{P}^{k,\mathrm{tang}}(\mathcal{F}_n) = \{\mathbf{u}_F \in [\mathbb{P}^k(\mathcal{F}_n)]^d : P_{\boldsymbol{\nu}} \mathbf{u}_F = 0\}.$$

For these choices, we redefine the HDG-jump operators as $[\![\underline{\mathbf{u}_n}]\!] := P_{\boldsymbol{\nu}}^{\perp}\mathbf{u}_\tau - \mathbf{u}_F$ such that $[\![\underline{\mathbf{u}_n}]\!]_{\boldsymbol{\nu}} = 0$. Consequently, we obtain that $R^l\mathbf{u}_n = 0$ for all $\mathbf{u}_n \in \mathbb{X}_n$ and therefore we have that $\operatorname{div}_{\boldsymbol{\nu}}^n = \operatorname{div}$ and $s_n(\cdot,\cdot) = 0$.

To optimize the computational efficiency, we can use relaxed $H(\operatorname{div})$-conforming spaces as introduced in [33, 34]. We define

$$\mathbb{BDM}_k^- := \{\mathbf{u}_\tau \in [\mathbb{P}^k(\mathcal{T}_n)]^d : \Pi_F^{k-1}[\![\mathbf{u}_\tau \cdot \boldsymbol{\nu}]\!]_F = 0 \ \forall F \in \mathcal{F}_n\},$$

where Π_F^{k-1} is the L^2-projection onto $[\mathcal{P}^{k-1}(F)]^d$. While functions in $\mathbb{BDM}_k^-$ are not normal-continuous in the highest order, this relaxation reduces the number of coupling degrees of freedoms improving the sparsity pattern of the system matrices. The choice of $\mathbb{X}_{\mathcal{T}_n} = \mathbb{BDM}_k^-$ and $\mathbb{X}_{\mathcal{F}_n} = \mathbb{P}^{k,\mathrm{tang}}(\mathcal{F}_n)$ still yields $R^l\mathbf{u}_n = 0$ if the lifting degree satisfies $l \leq k-1$.

3.3. **Interpretation as DAS**

To apply the abstract framework discussed in Section 2, we have to show that the proposed discretization can be interpreted as a DAS. As a first step, we have to define a suitable quasi projection $p_n \in L(\mathbb{X}, \mathbb{X}_n)$. Since the trace of functions in $\mathbb{X}$ is not necessarily well-defined, the evaluation of the discrete operator $\mathbf{D}_{\mathbf{b}}^n$ and therefore the evaluation of $\langle\cdot,\cdot\rangle_{\mathbb{X}_n}$ is not well-defined for functions in $\mathbb{X}$. Nevertheless, we want to define the quasi projection p_n in the spirit of an orthogonal projection. Thus, for $\mathbf{u} \in \mathbb{X}$, we define $p_n\mathbf{u} \in \mathbb{X}_n$ to be the solution to

$$\langle p_n\mathbf{u}, \mathbf{u}_n'\rangle_{\mathbb{X}_n} = \langle c_s^2\rho \operatorname{div}\mathbf{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n'\rangle_{L^2} + \langle \mathbf{u}, \mathbf{u}_\tau'\rangle_{\mathbf{L}^2} + \langle \rho\partial_{\mathbf{b}}\mathbf{u}, \mathbf{D}_{\mathbf{b}}^n\mathbf{u}_n'\rangle_{\mathbf{L}^2} \quad \forall \mathbf{u}_n' \in \mathbb{X}_n. \tag{21}$$

If the function $\mathbf{u}$ has enough regularity to allow for a well-defined trace, for instance if $\mathbf{u} \in \mathbb{X}\cap\mathbf{H}^1$, the discrete operators $\operatorname{div}_{\boldsymbol{\nu}}^n$ and $\mathbf{D}_{\mathbf{b}}^n$ can be applied to the pair $\underline{\mathbf{u}} := (\mathbf{u}, \operatorname{tr}\mathbf{u})$ and thus $\underline{\mathbf{u}}$ may be plugged into $\langle\cdot,\cdot\rangle_{\mathbb{X}_n}$, cf. (18). Then, (21) can be written as $\langle p_n\mathbf{u}, \mathbf{u}_n'\rangle_{\mathbb{X}_n} = \langle \underline{\mathbf{u}}, \mathbf{u}_n'\rangle_{\mathbb{X}_n}$.

The Cauchy-Schwarz inequality yields that $p_n \in L(\mathbb{X}, \mathbb{X}_n)$ with $\|p_n\|_{L(\mathbb{X},\mathbb{X}_n)} \leq 1$. Furthermore, for all $\mathbf{u}_n' \in \mathbb{X}_n$ the following Galerkin orthogonality property holds:

$$\begin{aligned} 0 =& \langle c_s^2\rho(\operatorname{div}\mathbf{u} - \operatorname{div}_{\boldsymbol{\nu}}^n p_n\mathbf{u}), \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n'\rangle_{\mathcal{T}_n} + \langle \mathbf{u} - (p_n\mathbf{u})_\tau, \mathbf{u}_n'\rangle_{\mathcal{T}_n} \\ &+ \langle \rho(\partial_{\mathbf{b}}\mathbf{u} - \mathbf{D}_{\mathbf{b}}^n p_n\mathbf{u}), \mathbf{D}_{\mathbf{b}}^n\mathbf{u}_n'\rangle_{\mathcal{T}_n} - \langle c_s^2\rho\mathfrak{h}^{-1}[\![\underline{p_n\mathbf{u}}]\!]_{\boldsymbol{\nu}}, [\![\mathbf{u}_n']\!]_{\boldsymbol{\nu}}\rangle_{\partial\mathcal{T}_n} \end{aligned} \tag{22}$$

As discussed above, the norm $\|\cdot\|_{\mathbb{X}_n}$ cannot be evaluated for functions in $\mathbb{X}$. To circumvent this issue, we introduce a distance function $d_n : \mathbb{X}\times\mathbb{X}_n \to \mathbb{R}_0^+$,

$$\begin{aligned} d_n(\mathbf{u}, \mathbf{u}_n')^2 :=& \|(c_s^2\rho)^{1/2}(\operatorname{div}\mathbf{u} - \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n)\|_{L^2}^2 + \|\mathbf{u} - \mathbf{u}_\tau\|_{\mathbf{L}^2}^2 \\ &+ \|\rho^{1/2}(\partial_{\mathbf{b}}\mathbf{u} - \mathbf{D}_{\mathbf{b}}^n\mathbf{u}_n)\|_{\mathbf{L}^2}^2 + [\![\underline{\mathbf{u}_h}]\!]_{\partial\mathcal{T}_n,1/2,\boldsymbol{\nu}}^2, \end{aligned} \tag{23}$$

where we define the following jump norm on $\mathbb{X}_n$:

$$[\![\underline{\mathbf{u}_h}]\!]_{\partial\mathcal{T}_n,1/2,\boldsymbol{\nu}}^2 := \|(c_s^2\rho)^{1/2}\mathfrak{h}^{-1/2}[\![\underline{\mathbf{u}_n}]\!]_{\boldsymbol{\nu}}\|_{L^2(\partial\mathcal{T}_n)}^2. \tag{24}$$

If the trace is well-defined, e.g. for $\mathbf{u} \in \mathbb{X}\cap\mathbf{H}^1$, then we obtain $d_n(\mathbf{u}, \mathbf{u}_n) = \|\underline{\mathbf{u}} - \mathbf{u}_n\|_{\mathbb{X}_n}$ for $\underline{\mathbf{u}} = (\mathbf{u}, \operatorname{tr}\mathbf{u})$. Furthermore, the distance function $d_n(\cdot,\cdot)$ fulfills the following triangle inequalities

$$d_n(\mathbf{u}, \mathbf{u}_n) \leq d_n(\tilde{\mathbf{u}}, \mathbf{u}_n) + \|\tilde{\mathbf{u}} - \mathbf{u}\|_{\mathbb{X}}, \quad d_n(\mathbf{u}, \mathbf{u}_n) \leq d_n(\mathbf{u}, \tilde{\mathbf{u}}_n) + \|\tilde{\mathbf{u}}_n - \mathbf{u}_n\|_{\mathbb{X}_n}, \tag{25}$$

for all $\mathbf{u}, \tilde{\mathbf{u}} \in \mathbb{X}$ and $\mathbf{u}_n, \tilde{\mathbf{u}}_n \in \mathbb{X}_n$.

To show that the triple $(\mathbb{X}_n, p_n, A_n)$ is a DAS of $(\mathbb{X}, A)$, we will show in Lemma 11 below that $\lim_{n\to\infty}\|p_n\mathbf{u}\|_{\mathbb{X}_n} = \|\mathbf{u}\|_{\mathbb{X}}$ for all $\mathbf{u} \in \mathbb{X}$ and in Theorem 13 that A_n approximates A. As a preparation, we proceed to analyze the projection operators defined in the previous section. We remark that most results follow with the same argumentation as in the $H(\operatorname{div})$-conforming DG [26] and the fully discontinuous DG [45] case.

Lemma 6. *Let* $\mathbf{u} \in \mathbf{H}^{1+s}$, $0 \le s \le k$, *and* $\underline{\tilde{\pi}}_n^d$ *be defined by* (20). *Then*

$$[\![\underline{\tilde{\pi}}_n^d \mathbf{u}]\!]_{\boldsymbol{\nu}} = 0 \qquad \textit{and} \qquad [\![\underline{\tilde{\pi}}_n^d \mathbf{u}]\!] = P_{\boldsymbol{\nu}}^{\perp}(\pi_n^d \mathbf{u} - \pi_n^{\mathcal{F}_n} \mathbf{u}). \tag{26}$$

Furthermore, there exists a constant $C > 0$ *such that*

$$\|\underline{\mathbf{u}} - \underline{\tilde{\pi}}_n^d \mathbf{u}\|_{\mathbb{X}_n} \le C h_n^s \|\mathbf{u}\|_{\mathbf{H}^{1+s}}, \tag{27}$$

where $\underline{\mathbf{u}} = (\mathbf{u}, \operatorname{tr} \mathbf{u})$.

Proof. The properties (26) hold by definition of $\underline{\tilde{\pi}}_n^d$. Since $[\![\underline{\tilde{\pi}}_n^d \mathbf{u}]\!]_{\boldsymbol{\nu}} = 0$, it follows that $R^l \underline{\tilde{\pi}}_n^d \mathbf{u} = 0$ and therefore by definition of the discrete divergence operator $\operatorname{div}_{\boldsymbol{\nu}}^n$, we have that $\operatorname{div}_{\boldsymbol{\nu}}^n \underline{\tilde{\pi}}_n^d \mathbf{u} = \operatorname{div}(\underline{\tilde{\pi}}_n^d \mathbf{u})_\tau$. The triangle inequality yields that

$$\|\underline{\mathbf{u}} - \underline{\tilde{\pi}}_n^d \mathbf{u}\|_{\mathbb{X}_n} \le \|\mathbf{u} - (\underline{\tilde{\pi}}_n^d \mathbf{u})_\tau\|_{\mathbb{X}(\mathcal{T}_n)} + \|\rho^{1/2} \mathbf{R}^l (\underline{\mathbf{u}} - \underline{\tilde{\pi}}_n^d \mathbf{u})\|_{\mathcal{T}_n}.$$

For the first term, the commutation property $\operatorname{div} \pi_n^d = \pi_n^l \operatorname{div}$ and the approximation properties of π_n^d and π_n^l imply that $\|\mathbf{u} - (\underline{\tilde{\pi}}_n^d \mathbf{u})_\tau\|_{\mathbb{X}(\mathcal{T}_n)} \lesssim h_n^s \|\mathbf{u}\|_{\mathbf{H}^{1+s}}$.

For the second term, we calculate with (13) and $[\![\underline{\mathbf{u}}]\!] = 0$ that

$$\begin{aligned}
\|\rho^{1/2} \mathbf{R}^l (\underline{\mathbf{u}} - \underline{\tilde{\pi}}_n^d \mathbf{u})\|_{\mathcal{T}_n} &\lesssim \|\rho^{1/2} \mathfrak{h}^{-1/2} [\![\underline{\tilde{\pi}}_n^d \mathbf{u}]\!]_{\mathbf{b}}\|_{\mathbf{L}^2(\partial \mathcal{T}_n)} \\
&\lesssim \|\rho^{1/2} \mathfrak{h}^{-1/2} (\mathbf{b} \cdot \boldsymbol{\nu}) P_{\boldsymbol{\nu}}^{\perp} (\pi_n^d \mathbf{u} - \pi_n^{\mathcal{F}_n} \mathbf{u})\|_{\mathbf{L}^2(\partial \mathcal{T}_n)} \\
&\lesssim \|\rho^{1/2} \mathfrak{h}^{-1/2} (\mathbf{b} \cdot \boldsymbol{\nu}) (\pi_n^d \mathbf{u} - \mathbf{u})\|_{\mathbf{L}^2(\partial \mathcal{T}_n)} \\
&\quad + \|\rho^{1/2} \mathfrak{h}^{-1/2} (\mathbf{b} \cdot \boldsymbol{\nu}) (\pi_n^{\mathcal{F}_n} \mathbf{u} - \mathbf{u})\|_{\mathbf{L}^2(\partial \mathcal{T}_n)} \\
&\lesssim h_n^s \|\mathbf{u}\|_{\mathbf{H}^{1+s}},
\end{aligned} \tag{28}$$

where we use the definition of $\underline{\tilde{\pi}}_n^d$ in the second line, the boundedness of $P_{\boldsymbol{\nu}}^{\perp}$ in the third and the approximation properties of π_n^d and $\pi_n^{\mathcal{F}_n}$ in the last line. Altogether, we conclude that there exists a constant $C > 0$ such that $\|\underline{\mathbf{u}} - \underline{\tilde{\pi}}_n^d \mathbf{u}\|_{\mathbb{X}_n} \le C \|\mathbf{u}\|_{\mathbf{H}^{1+s}}$ which proves (27). ☐

Lemma 7. *For all* $\mathbf{u} \in \mathbf{H}^1_{\boldsymbol{\nu}0}$, *it holds that* $d_n(\mathbf{u}, p_n \mathbf{u}) \le d_n(\mathbf{u}, \underline{\pi}_n \mathbf{u})$.

Proof. Follows from an application of (22) and the Cauchy-Schwarz inequality. ☐

Lemma 8. *For all* $\mathbf{u} \in \mathbf{H}^1_{\boldsymbol{\nu}0} \cap \mathbf{H}^{1+s}$, $0 < s \le k$, *there exists a constant* $C > 0$ *independent of* h *such that* $d_n(\mathbf{u}, \underline{\pi}_n \mathbf{u}) \le C h_n^s \|\mathbf{u}\|_{\mathbf{H}^{1+s}}$.

Proof. Follows from the boundedness of the lifting operators, cf. (13) and the approximation properties of π_n, cf. (19). ☐

Lemma 9. *For each* $\mathbf{u} \in \mathbb{X}$, *it holds that* $\lim_{n\to\infty} d_n(\mathbf{u}, p_n \mathbf{u}) = 0$.

Proof. Due to the density of $\boldsymbol{C}_0^\infty$ in $\mathbb{X}$ [29, Thm. 6], we can choose $\tilde{\mathbf{u}} \in \boldsymbol{C}_0^\infty$ such that $\|\mathbf{u} - \tilde{\mathbf{u}}\|_{\mathbb{X}} < \epsilon$ for any $\epsilon > 0$. Then, we can estimate with (25) that

$$d_n(\mathbf{u}, p_n \mathbf{u}) \le d_n(\tilde{\mathbf{u}}, p_n \tilde{\mathbf{u}}) + \|\tilde{\mathbf{u}} - \mathbf{u}\|_{\mathbb{X}} + \|p_n(\tilde{\mathbf{u}} - \mathbf{u})\|_{\mathbb{X}_n} \le d_n(\tilde{\mathbf{u}}, p_n \tilde{\mathbf{u}}) + 2\epsilon.$$

Thus, the claim follows from the previous Lemma 7 and Lemma 8. ☐

Lemma 10. *For all* $\mathbf{u} \in \mathbf{H}^1_{\boldsymbol{\nu}0}$, *it holds that*

$$\lim_{n\to\infty} d_n(\mathbf{u}, \underline{\pi}_n \mathbf{u}) = 0 \quad \textit{and} \quad \lim_{n\to\infty} d_n(\mathbf{u}, \underline{\tilde{\pi}}_n^d \mathbf{u}) = 0. \tag{29}$$

Proof. By construction of $\tilde{\underline{\pi}}_n^d$ and the approximation properties of π_n^d, we have for $\mathbf{u} \in \mathbf{H}^{1+s}$ that

$$d_n(\mathbf{u}, \tilde{\underline{\pi}}_n^d \mathbf{u}) \lesssim \|\mathbf{u} - \pi_n^d \mathbf{u}\|_{\mathbb{X}(\mathcal{T}_n)} + \|\rho^{1/2} \mathbf{R}^l \tilde{\underline{\pi}}_n^d \mathbf{u}\|_{\mathcal{T}_n} \lesssim h^s \|\mathbf{u}\|_{\mathbf{H}^{1+s}}.$$

Lemma 8 yields the same estimate for $\underline{\pi}_n$. The proof of the claim then follows with similar density arguments as in the proof of Lemma 9 with the additional technicality of constructing a smooth approximation that respects the boundary condition $\boldsymbol{\nu} \cdot \tilde{\mathbf{u}} = 0$ on $\partial\mathcal{O}$. For technical details, we refer to the proof of [26, Lem. 6]. □

Lemma 11. *For all* $\mathbf{u} \in \mathbb{X}$ *it holds that* $\lim_{n\to\infty} \|p_n \mathbf{u}\|_{\mathbb{X}_n} = \|\mathbf{u}\|_{\mathbb{X}}$.

Proof. With (21), we compute

$$\begin{aligned}
\|p_n \mathbf{u}\|_{\mathbb{X}_n}^2 &= \langle p_n \mathbf{u}, p_n \mathbf{u} \rangle_{\mathbb{X}_n} \\
&= \langle c_s^2 \rho \operatorname{div} \mathbf{u}, \operatorname{div}_{\boldsymbol{\nu}}^n p_n \mathbf{u} \rangle_{L^2} + \langle \mathbf{u}, (p_n \mathbf{u})_\tau \rangle_{\mathbf{L}^2} + \langle \rho \partial_{\mathbf{b}} \mathbf{u}, \mathbf{D}_{\mathbf{b}}^n p_n \mathbf{u} \rangle_{\mathbf{L}^2} \\
&= \|\mathbf{u}\|_{\mathbb{X}} + \langle c_s^2 \rho \operatorname{div} \mathbf{u}, \operatorname{div}_{\boldsymbol{\nu}}^n p_n \mathbf{u} - \operatorname{div} \mathbf{u} \rangle_{L^2} + \langle \mathbf{u}, (p_n \mathbf{u})_\tau - \mathbf{u} \rangle_{\mathbf{L}^2} \\
&\quad + \langle \rho \partial_{\mathbf{b}} \mathbf{u}, \mathbf{D}_{\mathbf{b}}^n p_n \mathbf{u} - \partial_{\mathbf{b}} \mathbf{u} \rangle_{\mathbf{L}^2}.
\end{aligned}$$

Since $\lim_{n\to\infty} d_n(\mathbf{u}, p_n \mathbf{u}) = 0$ by Lemma 9, the claim follows from the estimate

$$\begin{aligned}
&|\langle c_s^2 \rho \operatorname{div} \mathbf{u}, \operatorname{div}_{\boldsymbol{\nu}}^n p_n \mathbf{u} - \operatorname{div} \mathbf{u} \rangle_{L^2} + \langle \mathbf{u}, (p_n \mathbf{u})_\tau - \mathbf{u} \rangle_{\mathbf{L}^2} + \langle \rho \partial_{\mathbf{b}} \mathbf{u}, \mathbf{D}_{\mathbf{b}}^n p_n \mathbf{u} - \partial_{\mathbf{b}} \mathbf{u} \rangle_{\mathbf{L}^2}| \\
\leq &\|\mathbf{u}\|_{\mathbb{X}} d_n(\mathbf{u}, p_n \mathbf{u}).
\end{aligned}$$

□

Recall that $A_n \in L(\mathbb{X}_n)$ and $A \in L(\mathbb{X})$ are the linear operators associated with the sesquilinear forms $a_n(\cdot,\cdot)$ defined by (16) and $a(\cdot,\cdot)$ defined by (7). In preparation to show that A_n approximates A, we prove the following compactness result.

Lemma 12. *Let* $(\mathbf{u}_n)_{n\in\mathbb{N}} \subset \mathbb{X}_n$ *be such that* $\sup_{n\in\mathbb{N}} \|\mathbf{u}_n\|_{\mathbb{X}_n} < \infty$. *Then there exists* $\mathbf{u} \in \mathbb{X}$ *and a subsequence* $\mathbb{N}' \subset \mathbb{N}$ *such that* $(\mathbf{u}_n)_\tau \xrightarrow{\mathbf{L}^2} \mathbf{u}$, $c_s^2 \rho \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n \xrightarrow{L^2} c_s^2 \rho \operatorname{div} \mathbf{u}$ *and* $\rho \mathbf{D}_{\mathbf{b}}^n \mathbf{u}_n \xrightarrow{\mathbf{L}^2} \rho \partial_{\mathbf{b}} \mathbf{u}$, $n \in \mathbb{N}'$.

Proof. We modify standard arguments from the DG-case, see e.g. [10] and [26], to the HDG setting, see also [32]. By assumption, the sequences $(\mathbf{u}_n)_\tau$, $\rho \mathbf{D}_{\mathbf{b}}^n \mathbf{u}_n$ and $c_s^2 \rho \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n$ are bounded in $\mathbf{L}^2$ and L^2, respectively. Thus, there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and elements $\mathbf{u}, \boldsymbol{g} \in \mathbf{L}^2$, $q \in L^2$ such that $(\mathbf{u}_n)_\tau \xrightarrow{\mathbf{L}^2} \mathbf{u}$, $c_s^2 \rho \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n \xrightarrow{L^2} q$ and $\rho \mathbf{D}_{\mathbf{b}}^n \mathbf{u}_n \xrightarrow{\mathbf{L}^2} \boldsymbol{g}$. It remains to show that $\boldsymbol{g} = \rho \partial_{\mathbf{b}} \mathbf{u}$ and $q = c_s^2 \rho \operatorname{div} \mathbf{u}$. We only show the former, as the latter follows with a similar argumentation for the scalar lifting operator R^l, see also [45, Lem. 6.7]. Let $\boldsymbol{\psi} \in \boldsymbol{C}_0^\infty$ and $\boldsymbol{\psi}_n$ be the lowest order standard $\mathbf{H}^1$-interpolant of $\boldsymbol{\psi}$ on $\mathcal{T}_n$. Then, we compute with element-wise partial integration

$$\begin{aligned}
&-\langle (\mathbf{u}_n)_\tau, \rho \partial_{\mathbf{b}} \boldsymbol{\psi} \rangle_{\mathcal{T}_n} \\
&\quad = \langle \rho \partial_{\mathbf{b}} (\mathbf{u}_n)_\tau, \boldsymbol{\psi} \rangle_{\mathcal{T}_n} - \langle \rho (\mathbf{b}\cdot\boldsymbol{\nu}) (\mathbf{u}_n)_\tau, \boldsymbol{\psi} \rangle_{\partial\mathcal{T}_n} + \underbrace{\langle \rho (\mathbf{b}\cdot\boldsymbol{\nu}) (\mathbf{u}_n)_F, \boldsymbol{\psi} \rangle_{\partial\mathcal{T}_n}}_{=0} \\
&\quad = \langle \rho \partial_{\mathbf{b}} (\mathbf{u}_n)_\tau, \boldsymbol{\psi} \rangle_{\mathcal{T}_n} - \langle \rho [\![\underline{\mathbf{u}_n}]\!]_{\mathbf{b}}, \boldsymbol{\psi} \rangle_{\partial\mathcal{T}_n} \\
&\quad \overset{(12a)}{=} \langle \rho \partial_{\mathbf{b}} (\mathbf{u}_n)_\tau, \boldsymbol{\psi} \rangle_{\mathcal{T}_n} - \langle \rho [\![\underline{\mathbf{u}_n}]\!]_{\mathbf{b}}, \boldsymbol{\psi} - \boldsymbol{\psi}_n \rangle_{\partial\mathcal{T}_n} + \langle \rho \mathbf{R}^l \underline{\mathbf{u}}_n, \boldsymbol{\psi}_n \rangle_{\mathcal{T}_n} \\
&\quad \overset{(14)}{=} \langle \rho \partial_{\mathbf{b}} (\mathbf{u}_n)_\tau, \boldsymbol{\psi} - \boldsymbol{\psi}_n \rangle_{\mathcal{T}_n} - \langle \rho [\![\underline{\mathbf{u}_n}]\!]_{\mathbf{b}}, \boldsymbol{\psi} - \boldsymbol{\psi}_n \rangle_{\partial\mathcal{T}_n} + \langle \rho \mathbf{D}_{\mathbf{b}}^n \underline{\mathbf{u}}_n, \boldsymbol{\psi}_n \rangle_{\mathcal{T}_n} \\
&\quad = -\langle (\mathbf{u}_n)_\tau, \rho \partial_{\mathbf{b}} (\boldsymbol{\psi} - \boldsymbol{\psi}_n) \rangle_{\mathcal{T}_n} + \langle \rho \mathbf{D}_{\mathbf{b}}^n \underline{\mathbf{u}}_n, \boldsymbol{\psi}_n \rangle_{\mathcal{T}_n},
\end{aligned} \tag{30}$$

where we recall that $\operatorname{div}(\rho\mathbf{b}) = 0$ by assumption. Since $\|\boldsymbol{\psi} - \boldsymbol{\psi}_n\|_{\mathbf{H}^1} \lesssim h_n\|\boldsymbol{\psi}\|_{\mathbf{H}^2}$ and $\|\mathbf{u}_n\|_{\mathbb{X}_n} \lesssim 1$, it follows that $\lim_{n\to\infty}\langle\rho\mathbf{D}^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi}_n\rangle_{\mathcal{T}_n} = \lim_{n\to\infty} -\langle(\mathbf{u}_n)_\tau, \rho\partial_{\mathbf{b}}\boldsymbol{\psi}_n\rangle_{\mathcal{T}_n}$. Thus, we obtain

$$\begin{aligned}\langle \boldsymbol{g}, \boldsymbol{\psi}\rangle &= \lim_{n\to\infty}\langle\rho\mathbf{D}^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi}\rangle = \lim_{n\to\infty}\left(\langle\rho\mathbf{D}^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi} - \boldsymbol{\psi}_n\rangle_{\mathcal{T}_n} + \langle\rho\mathbf{D}^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi}_n\rangle_{\mathcal{T}_n}\right) \\ &\overset{(30)}{=} \lim_{n\to\infty} -\langle(\mathbf{u}_n)_\tau, \rho\partial_{\mathbf{b}}\boldsymbol{\psi}\rangle_{\mathcal{T}_n} = -\langle \mathbf{u}, \rho\partial_{\mathbf{b}}\boldsymbol{\psi}\rangle_{\mathcal{T}_n}.\end{aligned}$$

Consequently, it holds that $\boldsymbol{g} = \rho\partial_{\mathbf{b}}\mathbf{u}$ and with similar arguments $q = c_s^2\rho \operatorname{div}\mathbf{u}$. ☐

Theorem 13. *The operator $A_n \in L(\mathbb{X}_n)$ approximates the operator $A \in L(\mathbb{X})$, i.e. for each $\mathbf{u} \in \mathbb{X}$, it holds that*

$$\lim_{n\to\infty}\|(A_n p_n - p_n A)\mathbf{u}\|_{\mathbb{X}_n} = 0.$$

Proof. As $\|\mathbf{u}_n\|_{\mathbb{X}_n} = \sup_{\mathbf{u}_n\in\mathbb{X}_n, \|\mathbf{u}'_n\|_{\mathbb{X}_n}=1}|\langle\mathbf{u}_n, \mathbf{u}'_n\rangle_{\mathbb{X}_n}|$, we can choose for any $\mathbf{u} \in \mathbb{X}$ a sequence $(\mathbf{u}_n)_{n\in\mathbb{N}} \subset \mathbb{X}_n$ such that $\|\mathbf{u}_n\|_{\mathbb{X}_n} = 1$ and

$$\|(A_n p_n - p_n A)\mathbf{u}\|_{\mathbb{X}_n} \le |\langle(A_n p_n - p_n A)\mathbf{u}, \mathbf{u}_n\rangle_{\mathbb{X}_n}| + 1/n.$$

For any subsequence $\mathbb{N}' \subset \mathbb{N}$, we can choose a subsubsequence $\mathbb{N}'' \subset \mathbb{N}'$ as in Lemma 12 such that for a $\mathbf{u}' \in \mathbb{X}$ it holds that

$$\begin{aligned}\lim_{n\in\mathbb{N}''}\langle p_n A\mathbf{u}, \mathbf{u}_n\rangle_{\mathbb{X}_n} &\overset{(21)}{=} \lim_{n\in\mathbb{N}''}\left(\langle\operatorname{div} A\mathbf{u}, c_s^2\rho\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{u}_n\rangle_{L^2} + \langle A\mathbf{u}, \mathbf{u}_n\rangle_{\mathbf{L}^2} + \langle\partial_{\mathbf{b}}A\mathbf{u}, \rho\mathbf{D}^n_{\mathbf{b}}\mathbf{u}_n\rangle_{\mathbf{L}^2}\right) \\ &= \langle\operatorname{div} A\mathbf{u}, c_s^2\rho\operatorname{div}\mathbf{u}'\rangle_{L^2} + \langle A\mathbf{u}, \mathbf{u}'\rangle_{\mathbf{L}^2} + \langle\partial_{\mathbf{b}}A\mathbf{u}, \rho\partial_{\mathbf{b}}\mathbf{u}'\rangle_{\mathbf{L}^2} = \langle A\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}}.\end{aligned}$$

Furthermore, recalling (15) and (16), we have that

$$\langle A_n p_n\mathbf{u}, \mathbf{u}_n\rangle_{\mathbb{X}_n} = a_n(p_n\mathbf{u}, \mathbf{u}_n) = a_n^{\operatorname{div}}(p_n\mathbf{u}, \mathbf{u}_n) - a_n^{\partial_{\mathbf{b}}}(p_n\mathbf{u}, \mathbf{u}_n) + a_n^r(p_n\mathbf{u}, \mathbf{u}_n)$$

As discussed before, the trace of functions in $\mathbb{X}$ is not well-defined, but the normal trace is. Thus, we can plug $\mathbf{u} \in \mathbb{X}$ into the forms $a_n^y(\cdot,\cdot)$, $y \in \{\operatorname{div}, r\}$, but not into $a_n^{\partial_{\mathbf{b}}}(\cdot,\cdot)$. For $y \in \{\operatorname{div}, r\}$, we recall the definitions from (6) and calculate

$$\lim_{n\in\mathbb{N}''} a_n^y(p_n\mathbf{u}, \mathbf{u}_n) = \lim_{n\in\mathbb{N}''}\left(a_n^y(\mathbf{u}, \mathbf{u}_n) + a_n^y(p_n\mathbf{u} - \mathbf{u}, \mathbf{u}_n)\right) = a^y(\mathbf{u}, \mathbf{u}'),$$

where the last equality follows from $|a_n^y(p_n\mathbf{u} - \mathbf{u}, \mathbf{u}_n)| \lesssim d_n(\mathbf{u}, p_n\mathbf{u})$, Lemma 9 and Lemma 12. For the remaining term $a_n^{\partial_{\mathbf{b}}}(p_n\mathbf{u}, \mathbf{u}_n)$, we calculate

$$\begin{aligned}a_n^{\partial_{\mathbf{b}}}(p_n\mathbf{u}, \mathbf{u}_n) &= \langle\rho(\omega + i\mathbf{D}^n_{\mathbf{b}} + i\Omega\times)p_n\mathbf{u}, (\omega + i\mathbf{D}^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}_n\rangle_{\mathcal{T}_n} \\ &= \langle\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, (\omega + i\mathbf{D}^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}_n\rangle_{\mathcal{T}_n} \\ &\quad + \langle\rho(\omega + i\Omega\times)((p_n\mathbf{u})_\tau - \mathbf{u}), (\omega + i\mathbf{D}^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}_n\rangle_{\mathcal{T}_n} \\ &\quad + \langle\rho(\mathbf{D}^n_{\mathbf{b}}p_n\mathbf{u} - \partial_{\mathbf{b}}\mathbf{u}), (\omega + i\mathbf{D}^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}_n\rangle_{\mathcal{T}_n}\end{aligned}$$

While the first term converges to $a^{\partial_{\mathbf{b}}}(\mathbf{u}, \mathbf{u}')$ due to Lemma 12, the second and third terms are again bounded by $d_n(\mathbf{u}, p_n\mathbf{u})$ and thus converge to zero by Lemma 9. Altogether, we obtain that

$$\lim_{n\in\mathbb{N}''}\langle A_n p_n\mathbf{u}, \mathbf{u}_n\rangle_{\mathbb{X}_n} = a^{\operatorname{div}}(\mathbf{u}, \mathbf{u}') - a^{\partial_{\mathbf{b}}}(\mathbf{u}, \mathbf{u}') + a^r(\mathbf{u}, \mathbf{u}') = \langle A\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}},$$

and therefore $\lim_{n\in\mathbb{N}''}\|(A_n p_n - p_n A)\mathbf{u}\|_{\mathbb{X}_n} = 0$, which completes the proof. ☐

Thus, we have shown in Lemma 11 that $\lim_{n\to\infty}\|p_n\mathbf{u}\|_{\mathbb{X}_n} = \|\mathbf{u}\|_{\mathbb{X}}$ and in Theorem 13 that A_n approximates A. Consequently, we conclude that the triple $(\mathbb{X}_n, p_n, A_n)$ is a DAS of $(\mathbb{X}, A)$ which allows us to apply the results from Theorem 4 to analyze the discrete problem (17).

4. Convergence Analysis

The main goal of this section is to show that the sequence of approximations $(A_n)_{n\in\mathbb{N}}$ is stable and that the sequence of discrete solutions $(\mathbf{u}_n)_{n\in\mathbb{N}}$ converges to the solution of the continuous problem with optimal order. To this end, we want to use Theorem 4 to show that the sequence $(A_n)_{n\in\mathbb{N}}$ is regular and apply Lemma 3 to obtain stability and convergence. In Section 4.1, we introduce T-operators T and T_n on the continuous and the discrete level. Afterwards, we show in Section 4.2 that T_n satisfies the assumptions from Theorem 4. In Section 4.3 we show that the remaining requirements from Theorem 4 are satisfied and in Section 4.4 we conclude the analysis of the discrete problem (17). The roadmap for the analysis is shown in Fig. 1.

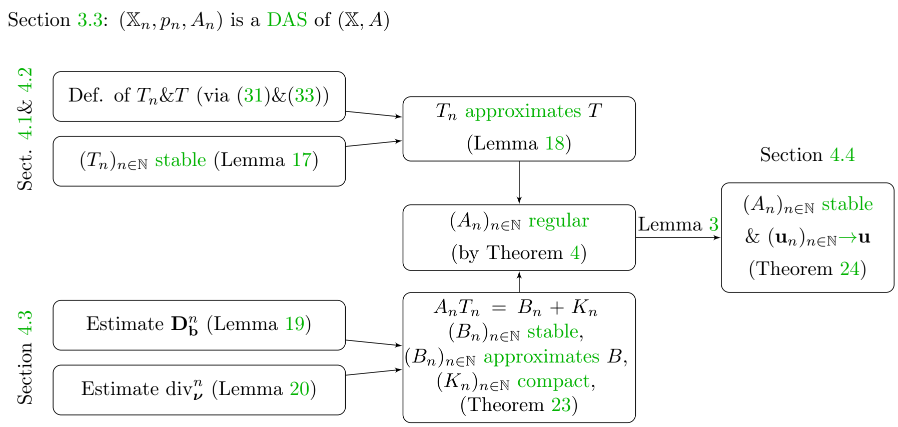

Figure 1. Roadmap for the analysis of the discrete problem (17).

4.1. Construction of T and T_n

Let us recall the construction of T on the continuous level as considered in [26]. In Section 3.1, we introduced $\mathbf{q} := c_s^{-2}\rho^{-1}\nabla p$, yielding the reformulation (9) of the sesquilinear form $a(\cdot,\cdot)$; recall also the definition of $a^{(\operatorname{div}+\mathbf{q}\cdot)}(\cdot,\cdot)$ in (10). Intuitively, the strategy to show that sesquilinear form $a(\cdot,\cdot)$ is weakly T-coercive is to construct the operator T to flip the sign in front of $a^{\partial_{\mathbf{b}}}(\cdot,\cdot)$ for elements in $\ker\{a^{(\operatorname{div}+\mathbf{q}\cdot)}(\cdot,\cdot)\}$.

To this end, we want to decompose the space $\mathbb{X}$ into a subspace associated with the perturbed divergence operator $(\operatorname{div}+\mathbf{q}\cdot)$ and its orthogonal complement. In essence, the following construction is a generalized Helmholtz decomposition, where we want to identify the kernel of $(\operatorname{div}+\mathbf{q}\cdot)$ instead of the kernel of div. In particular, if the pressure p is constant, we have that $\mathbf{q}=0$ and we recover the classical Helmholtz decomposition. A similar (though less involved) argument is applied in [40, Sec. 15.1] to the Helmholtz equation.

For $\mathbf{u}\in H_0(\operatorname{div}) := \{\mathbf{u}\in H(\operatorname{div}) : \mathbf{u}\cdot\boldsymbol{\nu} = 0 \text{ on } \partial\mathcal{O}\}$, let $v\in H^2_*$ solve

$$\begin{aligned}(\operatorname{div}+P_{L^2_0}\mathbf{q}\cdot+M)\nabla v &= (\operatorname{div}+P_{L^2_0}\mathbf{q}\cdot+M)\mathbf{u} \text{ in } \mathcal{O},\\ \boldsymbol{\nu}\cdot\nabla v &= 0 \text{ on } \partial\mathcal{O},\end{aligned} \tag{31}$$

where $P_{L^2_0}$ is the L^2-projection onto L^2_0 and M is a suitable finite rank operator constructed below. The operator M is necessary to ensure the well-posedness of the problem, since $(\operatorname{div}+P_{L^2_0}\mathbf{q}\cdot)\nabla$ might not be bijective. It is, however, a compact perturbation of a bijective operator and therefore Fredholm with index zero. For any Fredholm operator with index zero, there exists a finite rank operator such that the sum of both operators is bijective, cf. [22, Thm. 5.3].

Since we exploit the specific structure of M later on, we discuss an explicit construction of M. We set

$$H^2_{*,\mathrm{Neu}} := \{\phi\in H^2_*, \boldsymbol{\nu}\cdot\nabla\phi = 0 \text{ on } \partial\mathcal{O}\},\quad \mathcal{N} := \ker\left\{(\operatorname{div}+P_{L^2_0}\mathbf{q}\cdot)\nabla\right\}\subset H^2_{*,\mathrm{Neu}}.$$

Let $L := \dim \mathcal{N}$ and $(\phi_l)_{1\le l\le L} \subset H^2_{*,\mathrm{Neu}}$ be an orthonormal basis of $\mathcal{N}$ with respect to the inner product $\langle \operatorname{div}\nabla\cdot, \operatorname{div}\nabla\cdot\rangle$, which is equivalent to the canonical $H^2_{*,\mathrm{Neu}}$-inner product since $\boldsymbol{\nu}\cdot\nabla\phi = 0$ and $\langle \phi, 1\rangle = 0$ for all $\phi \in H^2_{*,\mathrm{Neu}}$.

Let $(\psi_l)_{1\le l\le L} \subset H^2_{*,\mathrm{Neu}}$ be an orthonormal basis of the L^2_0-orthogonal complement $\Big((\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot)\nabla H^2_{*,\mathrm{Neu}}\Big)^{\perp}$. Then, we set

$$M := \sum_{l=1}^{L} \psi_l \langle \operatorname{div}\cdot, \operatorname{div}\nabla\phi_l\rangle. \tag{32}$$

By construction, M can be applied to $H(\operatorname{div})$-functions and is compact. Thus, the operator $(\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot + M)$ is a Fredholm operator with index zero, and hence it is bijective if and only if it is injective. However, the construction of M ensures the injectivity of the operator and therefore the well-posedness of the problem (31). Thus, for any $\mathbf{u} \in \mathbb{X} \subset H_0(\operatorname{div})$ there exists a unique $v \in H^2_*$ solving (31).

Thus, we can define a unique decomposition of $\mathbf{v}+\mathbf{w} = \mathbf{u} \in \mathbb{X}$ by setting $\mathbf{v} := P_V\mathbf{u} := \nabla v$ and $\mathbf{w} := \mathbf{u}-\mathbf{v}$. In particular, this construction yields that $(\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot)\mathbf{w} = -M\mathbf{w}$. The construction of $P_V : H_0(\operatorname{div}) \to \mathbf{H}^1, \mathbf{u} \mapsto \nabla v$ allows us to use the compactness of the embedding $\mathbf{H}^1 \hookrightarrow \mathbf{L}^2$.

If $\mathbf{q} = 0$, then $\operatorname{div}\nabla = \Delta$ is bijective and $M = 0$, so that we recover the standard Helmholtz decomposition into a gradient potential and a divergence-free part.

Further, we define a bijective operator $T \in L(\mathbb{X})$ through $T\mathbf{u} := \mathbf{v} - \mathbf{w}$. That $a(\cdot,\cdot)$ is indeed weakly T-coercive with respect to this construction will be shown in Theorem 23.

Now, we want to construct a similar decomposition of the discrete space $\mathbb{X}_n$. To account for the discontinuity of the discrete functions, we have to modify the right-hand side of (31). In particular, we replace the divergence operator and the operator M with corresponding discrete counterparts.

For $\mathbf{u}_n \in \mathbb{X}_n$, let $\tilde{v} \in H^2_*$ be the solution to

$$\begin{aligned} (\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot + M)\nabla\tilde{v} &= (\operatorname{div}^n_{\boldsymbol{\nu}} + \pi^l_n P_{L^2_0}\mathbf{q}\cdot + M_n)\mathbf{u}_n \text{ in } \mathcal{O}, \\ \boldsymbol{\nu}\cdot\nabla\tilde{v} &= 0 \text{ on } \partial\mathcal{O}, \end{aligned} \tag{33}$$

where we interpret $\pi^l_n P_{L^2_0}\mathbf{q}\cdot\mathbf{u}_n = \pi^l_n P_{L^2_0}\mathbf{q}\cdot(\mathbf{u}_n)_\tau$ and define the operator M_n similarly to (32) as

$$M_n := \sum_{l=1}^{L} \psi_l \langle \operatorname{div}^n_{\boldsymbol{\nu}}\cdot, \operatorname{div}\nabla\phi_l\rangle. \tag{34}$$

Since we only changed the right-hand side of the problem, the well-posedness of the problem is not affected. Then, we define the decomposition $\mathbf{u}_n = \mathbf{v}_n + \mathbf{w}_n$ where we choose $\mathbf{v}_n$ as the $H(\operatorname{div})$-conforming HDG interpolation of $\nabla\tilde{v}$. To be precise, we recall the definition (20) of the projection operator $\underline{\tilde{\pi}}^d_n$ and its properties studied in Lemma 6 and define

$$\mathbf{v}_n := P_{V_n}\mathbf{u}_n := \underline{\tilde{\pi}}^d_n\nabla\tilde{v} = (\pi^d_n\nabla\tilde{v}, P_{\boldsymbol{\nu}}(\pi^d_n\nabla\tilde{v}) + P^{\perp}_{\boldsymbol{\nu}}(\pi^{\mathcal{F}_n}_n\nabla\tilde{v})), \quad \mathbf{w}_n := \mathbf{u}_n - \mathbf{v}_n. \tag{35}$$

For later use, let $S_n : \mathbb{X}_n \to \mathbf{H}^1$, $\mathbf{u}_n \mapsto \nabla\tilde{v}$ be the solution operator of (33) composed with ∇. Then, we have that $P_{V_n}\mathbf{u}_n = \underline{\tilde{\pi}}^d_n S_n\mathbf{u}_n$. Finally, we define the operator $T_n : \mathbb{X}_n \to \mathbb{X}_n$ through

$$T_n\mathbf{u}_n := \mathbf{v}_n - \mathbf{w}_n, \quad \text{i.e., } T_n = 2P_{V_n} - \mathrm{id}_{\mathbb{X}_n}. \tag{36}$$

Since (33) is well-posed, we have the stability estimate

$$\|S_n\mathbf{u}_n\|_{\mathbf{H}^1} \lesssim \|(\operatorname{div}^n_{\boldsymbol{\nu}} + \pi^l_n P_{L^2_0}\mathbf{q}\cdot + M_n)\mathbf{u}_n\|_{L^2} \lesssim \|\mathbf{u}_n\|_{\mathbb{X}_n} \quad \text{for all } \mathbf{u}_n \in \mathbb{X}_n. \tag{37}$$

Furthermore, since $\operatorname{ran}(S_n) \subset \mathbf{H}^1$, we can utilize the compact embedding $\mathbf{H}^1 \hookrightarrow \mathbf{L}^2$ to expose the weakly T-coercive structure of A_n in Theorem 23 below.

Remark 14 (Alternative decomposition of $\mathbb{X}_n$)**.** In the construction above, the normal jump is attributed to $\mathbf{w}_n$ and the T_n-operator flips its sign. Alternatively, we can isolate the normal jump through a suitably defined lifting operator, cf. [1,45]. In this case, we would decompose $\mathbf{u}_n = \mathbf{v}_n + \mathbf{w}_n + \mathbf{z}_n$ with $\mathbf{v}_n$ as above and $[\![\underline{\mathbf{u}_n}]\!]_{\boldsymbol{\nu}} = [\![\underline{\mathbf{z}_n}]\!]_{\boldsymbol{\nu}}$. This construction is more natural, because since we associate $\mathbf{w}_n$ with the *divergence free* part of the Helmholtz decomposition, we would expect the normal jump to be zero. When defining T_n, we now have explicit control over the sign of the normal jump. The previous construction corresponds to $T_n\mathbf{u}_n \coloneqq \mathbf{v}_n - \mathbf{w}_n - \mathbf{z}_n$, but we could also define $T_n\mathbf{u}_n \coloneqq \mathbf{v}_n - \mathbf{w}_n + \mathbf{z}_n$. Note that for the latter construction, the stabilization term $s_n(\cdot,\cdot)$ would have to be redefined to have a positive sign in front of the normal contribution and in the forthcoming analysis, the stabilization parameter α would have to be chosen sufficiently large to ensure that $s_n(\cdot,\cdot)$ is positive definite. To avoid further technicalities, we do not further consider this alternative decomposition.

4.2. Analysis of T_n

We want to show that the sequence $(T_n)_{n\in\mathbb{N}}$ is bounded, stable, and approximates the operator T. By definition of T_n, we have that $T_n = 2P_{V_n} - \mathrm{id}_{\mathbb{X}_n}$ and therefore we mainly have to focus on the properties of P_{V_n}.

Lemma 15. *There exists a constant $C > 0$ such that $\|T_n\|_{L(\mathbb{X}_n)} \le C$ for all $n \in \mathbb{N}$.*

Proof. It suffices to show the statement for P_{V_n}. Since $P_{V_n} = \underline{\tilde{\pi}}_n^d S_n$ and $S_n\mathbf{u}_n \in \mathbf{H}^1$, we obtain with Lemma 6 and (37) that

$$\|P_{V_n}\mathbf{u}_n\|_{\mathbb{X}_n} = \|\underline{\tilde{\pi}}_n^d S_n\mathbf{u}_n\|_{\mathbb{X}_n} \lesssim \|S_n\mathbf{u}_n\|_{\mathbf{H}^1} \lesssim \|\mathbf{u}_n\|_{\mathbb{X}_n}. \tag{38}$$

Thus, there exists a constant $C > 0$ such that $\|P_{V_n}\|_{L(\mathbb{X}_n)} \le C$ for all $n \in \mathbb{N}$. ☐

For $\mathbf{q} = 0$, the projection P_{V_n} is idempotent, that is $P_{V_n}^2 = P_{V_n}$. In the case where $\mathbf{q} \neq 0$, we can still show that P_{V_n} is asymptotically idempotent.

Lemma 16. *Let $O_n \coloneqq P_{V_n}P_{V_n} - P_{V_n}$. Then $\lim_{n\to\infty} \|O_n\|_{L(\mathbb{X}_n)} = 0$.*

Proof. Let $\mathbf{u}_n \in \mathbb{X}_n$ and $\mathbf{w}_n \coloneqq (\mathrm{id}_{\mathbb{X}_n} - P_{V_n})\mathbf{u}_n$. Since $P_{V_n} = \underline{\tilde{\pi}}_n^d S_n$, Lemma 6 implies that $\|(P_{V_n}P_{V_n} - P_{V_n})\mathbf{u}_n\|_{\mathbb{X}_n} \lesssim \|S_n\mathbf{w}_n\|_{\mathbf{H}^1}$. By construction of P_{V_n}, we have that $[\![\underline{P_{V_n}\mathbf{u}_n}]\!]_{\boldsymbol{\nu}} = 0$ and $\mathrm{div}_{\boldsymbol{\nu}}^n P_{V_n}\mathbf{u}_n = \mathrm{div}(P_{V_n}\mathbf{u}_n)_\tau = \mathrm{div}\,\pi_n^d S_n\mathbf{u}_n$, and therefore $S_n(P_{V_n}\mathbf{u}_n) \in \mathbf{H}^1_{\boldsymbol{\nu}0}$ solves

$$(\mathrm{div} + P_{L^2_0}\mathbf{q}\cdot + M)S_n(P_{V_n}\mathbf{u}_n) \overset{(33)}{=} (\mathrm{div} + \pi_n^l P_{L^2_0}\mathbf{q}\cdot + M)\pi_n^d(S_n\mathbf{u}_n).$$

Thus, we calculate that

$$\begin{aligned}
&(\mathrm{div} + P_{L^2_0}\mathbf{q}\cdot + M)S_n\mathbf{w}_n \\
&\overset{(33)}{=} (\mathrm{div} + P_{L^2_0}\mathbf{q}\cdot + M)S_n\mathbf{u}_n - (\mathrm{div} + \pi_n^l P_{L^2_0}\mathbf{q}\cdot + M)\pi_n^d(S_n\mathbf{u}_n) \\
&= \mathrm{div}\left(\mathrm{id}_{\mathbb{X}} - \pi_n^d\right) S_n\mathbf{u}_n + (P_{L^2_0}\mathbf{q}\cdot - \pi_n^l P_{L^2_0}\mathbf{q}\cdot\pi_n^d)S_n\mathbf{u}_n + M\left(\mathrm{id}_{\mathbb{X}} - \pi_n^d\right) S_n\mathbf{u}_n \\
&= \mathrm{div}\left(\mathrm{id}_{\mathbb{X}} - \pi_n^d\right) S_n\mathbf{u}_n + \left(\mathrm{id}_{L^2_0} - \pi_n^l\right) P_{L^2_0}\mathbf{q}\cdot S_n\mathbf{u}_n \\
&\qquad + \pi_n^l P_{L^2_0}\mathbf{q}\cdot\left(\mathrm{id}_{\mathbb{X}} - \pi_n^d\right) S_n\mathbf{u}_n + M\left(\mathrm{id}_{\mathbb{X}} - \pi_n^d\right) S_n\mathbf{u}_n \\
&= (\mathrm{id}_{L^2_0} - \pi_n^l)(\mathrm{div} + P_{L^2_0}\mathbf{q}\cdot)S_n\mathbf{u}_n + \pi_n^l P_{L^2_0}\mathbf{q}\cdot\left(\mathrm{id}_{\mathbb{X}} - \pi_n^d\right) S_n\mathbf{u}_n \\
&\qquad + M\left(\mathrm{id}_{\mathbb{X}} - \pi_n^d\right) S_n\mathbf{u}_n \eqqcolon -\tilde{O}_n\mathbf{u}_n,
\end{aligned} \tag{39}$$

where we use the commutation property $\mathrm{div}\,\pi_n^d = \pi_n^l\,\mathrm{div}$ in the last step. Consequently, $S_n\mathbf{w}_n$ solves (33) with right-hand side $-\tilde{O}_n\mathbf{u}_n$ and the stability estimate (37) implies that $\|S_n\mathbf{w}_n\|_{\mathbf{H}^1} \lesssim \|\tilde{O}_n\mathbf{u}_n\|_{\mathbb{X}_n}$. We note that the minus in front of $\tilde{O}_n$ is purely for notational convenience in later calculations.

It remains to show that $\lim_{n\to\infty} \|\tilde{O}_n\|_{L(\mathbb{X}_n, L^2_0)} = 0$. Due to (33), we have that

$$\begin{aligned}(\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot)S_n\mathbf{u}_n &= (\operatorname{div}^n_{\boldsymbol{\nu}} + \pi^l_n P_{L^2_0}\mathbf{q}\cdot + M_n)\mathbf{u}_n - MS_n\mathbf{u}_n,\\ \pi^l_n(\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot)S_n\mathbf{u}_n &= (\operatorname{div}^n_{\boldsymbol{\nu}} + \pi^l_n P_{L^2_0}\mathbf{q}\cdot + M_n)\mathbf{u}_n - \pi^l_n MS_n\mathbf{u}_n\\ &\quad + (\pi^l_n - \operatorname{id}_{L^2_0})M_nS_n\mathbf{u}_n.\end{aligned} \tag{40}$$

Because π^l_n converges to $\operatorname{id}_{L^2_0}$ pointwise and the operators MS_n and M_nS_n are compact, it follows that

$$\begin{aligned}&\|(\operatorname{id}_{L^2_0} - \pi^l_n)(\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot)S_n\|_{L(\mathbb{X}_n, L^2_0)}\\ &\lesssim \|(\operatorname{id}_{L^2_0} - \pi^l_n)MS_n\|_{L(\mathbb{X}_n, L^2_0)} + \|(\pi^l_n - \operatorname{id}_{L^2_0})M_nS_n\|_{L(\mathbb{X}_n, L^2_0)} \overset{n\to\infty}{\to} 0.\end{aligned}$$

Furthermore, by construction of the operator M, the commutation property $\operatorname{div}\pi^d_n = \pi^l_n \operatorname{div}$, and (40) we have that

$$\begin{aligned}\|M(\operatorname{id}_{\mathbb{X}} - \pi^d_n)S_n\|_{L^2(\mathbb{X}_n, L^2_0)} &\lesssim \|\operatorname{div}\left(\operatorname{id}_{\mathbb{X}} - \pi^d_n\right)S_n\|_{L^2(\mathbb{X}_n, L^2_0)}\\ &\lesssim \|(\operatorname{id}_{L^2_0} - \pi^l_n)(\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot)S_n\|_{L(\mathbb{X}_n, L^2_0)} + \|(\operatorname{id}_{L^2_0} - \pi^l_n)P_{L^2_0}\mathbf{q}\cdot S_n\|_{L(\mathbb{X}_n, L^2_0)},\end{aligned}$$

where the second term converges to zero as well, since π^l_n converges to $\operatorname{id}_{L^2_0}$ pointwise and $P_{L^2_0}\mathbf{q}\cdot S_n$ is compact. Finally, we estimate with (19)

$$\|\pi^l_n P_{L^2_0}\mathbf{q}\cdot(\operatorname{id}_{\mathbb{X}} - \pi^d_n)S_n\|_{L(\mathbb{X}_n, L^2_0)} \lesssim \|(\operatorname{id}_{\mathbb{X}} - \pi^d_n)S_n\|_{L(\mathbb{X}_n, \mathbf{L}^2)} \lesssim h_n\|S_n\|_{L(\mathbb{X}_n, \mathbf{H}^1)} \overset{n\to\infty}{\to} 0.$$

Altogether, we obtain that $\lim_{n\to\infty} \|\tilde{O}_n\|_{L(\mathbb{X}_n, L^2_0)} = 0$ and thus

$$\|(P_{V_n}P_{V_n} - P_{V_n})\mathbf{u}_n\|_{\mathbb{X}_n} \lesssim \|S_n\mathbf{w}_n\|_{\mathbf{H}^1} \overset{(37)}{\lesssim} \|\tilde{O}_n\mathbf{u}_n\|_{\mathbb{X}_n} \lesssim \|\tilde{O}_n\|_{L(\mathbb{X}_n, L^2_0)}\|\mathbf{u}_n\|_{\mathbb{X}_n} \overset{n\to\infty}{\to} 0.$$

□

The calculations in (39) yield

$$(\operatorname{div}^n_{\boldsymbol{\nu}} + \pi^l_n P_{L^2_0}\mathbf{q}\cdot)\mathbf{w}_n \overset{(33)}{=} (\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot + M)S_n\mathbf{w}_n - M_n\mathbf{w}_n \overset{(39)}{=} -M_n\mathbf{w}_n - \tilde{O}_n\mathbf{u}_n \tag{41}$$

In particular, if $\mathbf{q} = 0$, then $M_n = 0$ and $\tilde{O}_n = 0$, so $\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}_n = 0$ and we recover the standard Helmholtz decomposition on the discrete level. From (41), we observe that even in the case where $\mathbf{q} \neq 0$ the discrete perturbed divergence of $\mathbf{w}_n$ consists of the compact operator M_n and $\tilde{O}_n$ which can be absorbed in the compact part of the weakly T-coercive structure.

Lemma 17. *There exists an index $n_0 > 0$ and $C > 0$ such that $\|T_n^{-1}\|_{L(\mathbb{X}_n)} \leq C$ for all $n > n_0$.*

Proof. We have that $T_nT_n = 4P_{V_n}P_{V_n} - 4P_{V_n} + \operatorname{id}_{\mathbb{X}} = \operatorname{id}_{\mathbb{X}} + 4O_n$ with O_n as defined in Lemma 16. Since $\lim_{n\to\infty} \|O_n\|_{L(\mathbb{X}_n)} = 0$, there exists $n_0 > 0$ such that $\|4O_n\|_{L(\mathbb{X}_n)} < 1$ for all $n > n_0$ and thus there exists $C > 0$ such that $\|(\operatorname{id}_{\mathbb{X}} + 4O_n)^{-1}\|_{L(\mathbb{X}_n)} \leq C$ for all $n > n_0$. Writing

$$(\operatorname{id}_{\mathbb{X}} + 4O_n)^{-1}T_n = (T_nT_n)^{-1}T_n = T_n^{-1},$$

we conclude that $\|T_n^{-1}\|_{L(\mathbb{X}_n)} \leq C\|T_n\|_{L(\mathbb{X}_n)}$ for all $n > n_0$, which proves the claim. □

The next lemma shows that T_n indeed approximates the operator T.

Lemma 18. *For each $\mathbf{u} \in \mathbb{X}$, it holds that $\lim_{n\to\infty} \|(T_n p_n - p_n T)\mathbf{u}\|_{\mathbb{X}_n} = 0$.*

Proof. As before, we only have to show the statement for P_{V_n}. First, we estimate

$$\|(P_{V_n} p_n - p_n P_V)\mathbf{u}\|_{\mathbb{X}_n} \leq d_n(P_V \mathbf{u}, p_n P_V \mathbf{u}) + d_n(P_V \mathbf{u}, P_{V_n} p_n \mathbf{u}),$$

and note that the first term converges to zero by Lemma 9. By definition, we have that $P_{V_n} p_n \mathbf{u} = \underline{\tilde{\pi}}_n^d S_n p_n \mathbf{u}$ and estimate for the second term

$$d_n(P_V \mathbf{u}, P_{V_n} p_n \mathbf{u}) \overset{(25)}{\leq} d_n(P_V \mathbf{u}, \underline{\tilde{\pi}}_n^d P_V \mathbf{u}) + \|\underline{\tilde{\pi}}_n^d (P_V \mathbf{u} - S_n p_n \mathbf{u})\|_{\mathbb{X}_n}. \tag{42}$$

Since $P_V \mathbf{u} \in \mathbf{H}^1_{\boldsymbol{\nu}0}$, the first term converges to zero by Lemma 10. For the second term, we estimate

$$\begin{aligned}\|\underline{\tilde{\pi}}_n^d (P_V \mathbf{u} - S_n p_n \mathbf{u})\|_{\mathbb{X}_n} &\lesssim \|P_V \mathbf{u} - S_n p_n \mathbf{u}\|_{\mathbf{H}^1} \\ &\lesssim \underbrace{\|\operatorname{div} \mathbf{u} - \operatorname{div}_{\boldsymbol{\nu}}^n p_n \mathbf{u}\|_{\mathcal{T}_n}}_{\text{(I)}} + \underbrace{\|P_{L^2_0} \mathbf{q} \cdot \mathbf{u} - \pi_n^l P_{L^2_0} \mathbf{q} \cdot (p_n \mathbf{u})_\tau \|_{\mathcal{T}_n}}_{\text{(II)}} + \underbrace{\|M\mathbf{u} - M_n p_n \mathbf{u}\|_{\mathcal{T}_n}}_{\text{(III)}}.\end{aligned}$$

By definition of $d_n(\cdot,\cdot)$, we have that (I) $\lesssim d_n(\mathbf{u}, p_n \mathbf{u})$. We further estimate

$$\begin{aligned}\text{(II)} &\lesssim \|(P_{L^2_0} \mathbf{q} \cdot - \pi_n^l P_{L^2_0} \mathbf{q} \cdot)\mathbf{u}\|_{L^2} + \|\pi_n^l P_{L^2_0} \mathbf{q} \cdot (\mathbf{u} - (p_n \mathbf{u})_\tau)\|_{\mathcal{T}_n} \\ &\lesssim \|(P_{L^2_0} \mathbf{q} \cdot - \pi_n^l P_{L^2_0} \mathbf{q} \cdot)\mathbf{u}\|_{L^2} + d_n(\mathbf{u}, p_n \mathbf{u}).\end{aligned}$$

By construction of M and M_n it holds that (III) $\lesssim \|\operatorname{div} \mathbf{u} - \operatorname{div}_{\boldsymbol{\nu}}^n p_n \mathbf{u}\|_{\mathcal{T}_n} \lesssim d_n(\mathbf{u}, p_n \mathbf{u})$.

Altogether, we obtain

$$\begin{aligned}\|\underline{\tilde{\pi}}_n^d (P_V \mathbf{u} - S_n p_n \mathbf{u})\|_{\mathbb{X}_n} &\lesssim \|(P_V \mathbf{u} - S_n p_n \mathbf{u})\|_{\mathbb{X}_n} \\ &\lesssim d_n(\mathbf{u}, p_n \mathbf{u}) + \|(P_{L^2_0} \mathbf{q} \cdot - \pi_n^l P_{L^2_0} \mathbf{q} \cdot)\mathbf{u}\|_{L^2} \overset{n\to\infty}{\to} 0,\end{aligned} \tag{43}$$

where the first term converges to zero by Lemma 10 and the second term converges to zero due to the pointwise convergence of π_n^l to $\mathrm{id}_{L^2_0}$, which proves the claim. ☐

4.3. Weak T-compatibility

In the previous section, we have defined and analyzed the operators T and T_n. To prepare for the application of Theorem 4 in Section 4.4, we have to construct a characterization $A_n T_n = B_n + K_n$ that satisfies the conditions from Theorem 4. Before we do so in Theorem 23, we introduce the following notation and prove some auxiliary results.

For $\mathbf{u} \in \mathbf{H}^1_{\boldsymbol{\nu}0}$, we define the weighted $\mathbf{H}^1$-seminorm through

$$|\mathbf{u}|^2_{\mathbf{H}^1_{c_s^2\rho}} := \|(c_s^2 \rho)^{1/2} \nabla \mathbf{u}\|^2_{(L^2)^{d\times d}}. \tag{44}$$

We show that the construction of $P_{V_n} \mathbf{u}_n := \underline{\tilde{\pi}}_n^d \nabla \tilde{v}$, where $\tilde{v} \in H^2_{*,\mathrm{Neu}}$ solves (33), allows us to bound the norm of the differential operators $\mathbf{D}^n_{\mathbf{b}}$ and $\operatorname{div}^n_{\boldsymbol{\nu}}$ suitably. These estimates are crucial to show that the conditions from Theorem 4 are satisfied.

Lemma 19. *There exists constant $C_{(19)} > 0$ and $n_0 \in \mathbb{N}$ such that for all $\mathbf{u}_n \in \mathbb{X}_n$ and $n > n_0$, it holds that*

$$\|\rho^{1/2} \mathbf{D}^n_{\mathbf{b}} (P_{V_n} \mathbf{u}_n)\|^2_{\mathcal{T}_n} \leq C_{(19)} \|c_s^{-1} \mathbf{b}\|^2_{\mathbf{L}^\infty} |S_n \mathbf{u}_n|^2_{\mathbf{H}^1_{c_s^2\rho}}. \tag{45}$$

In particular, $C_{(19)} = 1 + O(h_n^2)$.

Proof. For $\mathbf{u}_n \in \mathbb{X}_n$, we set $\mathbf{v}_n := P_{V_n}\mathbf{u}_n$ and denote by $\tilde{v}$ the solution to (33) such that $S_n\mathbf{u}_n = \nabla\tilde{v}$. For an element $\tau \in \mathcal{T}_n$ and $\eta \in W^{1,\infty}$ we use the notation $\eta_\tau = \eta|_\tau$ and estimate

$$\|\rho^{1/2}\mathbf{D}^n_{\mathbf{b}}\mathbf{v}_n\|_{\mathbf{L}^2(\tau)} \le \|\rho^{1/2}\partial_{\mathbf{b}}(\mathbf{v}_n)_\tau\|_{\mathbf{L}^2(\tau)} + \|\rho^{1/2}\mathbf{R}^l\mathbf{v}_n\|_{\mathbf{L}^2(\tau)}. \tag{46}$$

By definition, $(\mathbf{v}_n)_\tau = \pi^d_n\nabla\tilde{v}$, and thus we can estimate the first term by

$$\begin{aligned}\|\rho^{1/2}\partial_{\mathbf{b}}(\pi^d_n\nabla\tilde{v})\|^2_{\mathbf{L}^2(\tau)} &\le \|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}\overline{c_{s_\tau}}^2\overline{\rho_\tau}|\pi^d_n\nabla\tilde{v}|^2_{\mathbf{H}^1(\tau)} \lesssim \|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}\overline{c_{s_\tau}}^2\overline{\rho_\tau}|\nabla\tilde{v}|^2_{\mathbf{H}^1(\tau)}\\ &\lesssim \|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}\overline{c_{s_\tau}}^2\overline{\rho_\tau}(\underline{c^2_{s_\tau}}\underline{\rho_\tau})^{-1}|\nabla\tilde{v}|^2_{\mathbf{H}^1_{c_s^2\rho}(\tau)}\\ &\lesssim \|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}\underbrace{\left(1 + h_n^2\frac{1}{\underline{c_{s_\tau}}^2\underline{\rho_\tau}}(C^L_{c_s\rho^{1/2}})^2\right)}_{=:C^L_n(\tau)}|\nabla\tilde{v}|^2_{\mathbf{H}^1_{c_s^2\rho}(\tau)}\end{aligned}$$

where we use the Lipschitz continuity of $c_s\rho^{1/2} \in W^{1,\infty}$ with constant $C^L_{c_s\rho^{1/2}}$ in the last step. Since $h_n \to 0$ for $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $C^L_n(\tau) \le 2$ for all $n > n_0$ and all $\tau \in \mathcal{T}_n$. For the second term in (46), we use (13) and same argumentation as in the proof of Lemma 6 to obtain

$$\begin{aligned}\|\rho^{1/2}\mathbf{R}^l\mathbf{v}_n\|^2_{\mathbf{L}^2(\tau)} &\le C^2_{\mathrm{dt}}\overline{\rho_\tau}\|\mathfrak{h}^{-1/2}[\![\underline{\mathbf{v}_n}]\!]_{\mathbf{b}}\|^2_{\mathbf{L}^2(\partial\tau)} \le \|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}C^2_{\mathrm{dt}}\overline{c_{s_\tau}}^2\overline{\rho_\tau}\|\mathfrak{h}^{-1/2}[\![\underline{\mathbf{v}_n}]\!]\|^2_{\mathbf{L}^2(\partial\tau)}\\ &\overset{(28)}{\lesssim} \|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}C^2_{\mathrm{dt}}C^L_n(\tau)|\nabla\tilde{v}|_{\mathbf{H}^1_{c_s^2\rho}(\tau)},\end{aligned}$$

Inserting both estimates into (46) and summing over all elements $\tau \in \mathcal{T}_n$ yields

$$\|\rho^{1/2}\mathbf{D}^n_{\mathbf{b}}\mathbf{v}_n\|^2_{\mathcal{T}_n} \le C_{(19)}\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}|\nabla\tilde{v}|^2_{\mathbf{H}^1_{c_s^2\rho}},$$

where $C_{(19)} > 0$ is independent of $n > n_0$. Since $\mathbf{v}_n = P_{V_n}\mathbf{u}_n$ and $S_n\mathbf{u}_n = \nabla\tilde{v}$, the proof is finished. □

Let us stress that the constant $C_{(19)} > 0$ only depends locally on the coefficients and their Lipschitz constant, and is independent of the ratio $(\underline{c_s}^2\underline{\rho})/(\overline{c_s}^2\overline{\rho})$. In particular, the quadratic factor h_n^2 mitigates the effects of large Lipschitz constants and asymptotically, the constant tends to one with order h_n^2.

In the following lemma, we show that the decomposition (33) allows us to bound the norm of the discrete divergence operator from below.

Lemma 20. *For any $\delta \in (0,1)$, there exist $C_\delta > 0$, so that*

$$\|c_s\rho^{1/2}\mathrm{div}^n_{\boldsymbol{\nu}}P_{V_n}\mathbf{u}_n\|^2_{\mathcal{T}_n} \ge \left((1-\delta)^2|S_n\mathbf{u}_n|^2_{\mathbf{H}^1_{c_s^2\rho}} - C_\delta\|S_n\mathbf{u}_n\|^2_{\mathbf{L}^2}\right) + \langle\check{O}_n\mathbf{u}_n,\mathbf{u}_n\rangle_{\mathbb{X}_n}, \tag{47}$$

for all $\mathbf{u}_n \in \mathbb{X}_n$ and $\lim_{n\to\infty}\|\check{O}_n\|_{L(\mathbb{X}_n)} = 0$.

Proof. For $\mathbf{u}_n \in \mathbb{X}_n$, let $\tilde{v} \in H^2_{*,\mathrm{Neu}}$ be the solution to (33) such that $S_n\mathbf{u}_n = \nabla\tilde{v}$. Due to the properties of $\underline{\tilde{\pi}}^d_n$ discussed in Lemma 6, we obtain for $P_{V_n} = \underline{\tilde{\pi}}^d_nS_n$ that $\mathrm{div}^n_{\boldsymbol{\nu}}P_{V_n}\mathbf{u}_n = \mathrm{div}(P_{V_n}\mathbf{u}_n)_\tau = \mathrm{div}\,\pi^d_n\nabla\tilde{v}$.

Using the commutation property $\pi_n^l \operatorname{div} = \operatorname{div} \pi_n^d$, we calculate

$$\begin{aligned}
\operatorname{div} \pi_n^d \nabla \tilde{v} &= \pi_n^l \operatorname{div} \nabla \tilde{v} \\
&\overset{(33)}{=} \pi_n^l \Big(-(P_{L_0^2} \mathbf{q} \cdot + M) \nabla \tilde{v} + (\operatorname{div}_{\boldsymbol{\nu}}^n + \pi_n^l P_{L_0^2} \mathbf{q} \cdot + M_n) \mathbf{u}_n \Big) \\
&= -(P_{L_0^2} \mathbf{q} \cdot + M) \nabla \tilde{v} + (\operatorname{div}_{\boldsymbol{\nu}}^n + \pi_n^l P_{L_0^2} \mathbf{q} \cdot + M_n) \mathbf{u}_n \\
&\quad + (\operatorname{id} - \pi_n^l)(P_{L_0^2} \mathbf{q} \cdot + M) \nabla \tilde{v} + (\pi_n^l - \operatorname{id}) M_n \mathbf{u}_n \\
&\overset{(33)}{=} \operatorname{div} \nabla \tilde{v} + (\operatorname{id} - \pi_n^l)(P_{L_0^2} \mathbf{q} \cdot + M) S_n \mathbf{u}_n + (\pi_n^l - \operatorname{id}) M_n \mathbf{u}_n \\
&=: \operatorname{div} \nabla \tilde{v} + \hat{O}_n \mathbf{u}_n.
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
\| c_s \rho^{1/2} \operatorname{div}_{\boldsymbol{\nu}}^n P_{V_n} \mathbf{u}_n \|_{\mathcal{T}_n}^2 &= \langle c_s^2 \rho \operatorname{div} \pi_n^d \nabla \tilde{v}, \operatorname{div} \pi_n^d \nabla \tilde{v} \rangle \\
&= \langle c_s^2 \rho \operatorname{div} \nabla \tilde{v}, \operatorname{div} \nabla \tilde{v} \rangle + \langle \check{O}_n \mathbf{u}_n, \mathbf{u}_n \rangle_{\mathbb{X}_n},
\end{aligned} \tag{48}$$

where we define the operator $\check{O}_n$ through

$$\begin{aligned}
\langle \check{O}_n \mathbf{u}_n, \mathbf{u}_n' \rangle_{\mathbb{X}_n} :=& \langle c_s^2 \rho \operatorname{div} \pi_n^d \nabla \tilde{v}, \hat{O}_n \mathbf{u}_n' \rangle + \langle c_s^2 \rho \hat{O}_n \mathbf{u}_n, \operatorname{div} (\pi_n^d \nabla \tilde{v})' \rangle \\
&+ \langle c_s^2 \rho \hat{O}_n \mathbf{u}_n, \hat{O}_n \mathbf{u}_n' \rangle.
\end{aligned}$$

With similar arguments as in Lemma 16, we can show $\lim_{n\to\infty} \|\hat{O}_n\|_{L(\mathbb{X}_n, L_0^2)} = 0$ and thus $\lim_{n\to\infty} \|\check{O}_n\|_{L(\mathbb{X}_n)} = 0$.

In the following, we use similar techniques as in the proof of [27, Thm. 3.5] to show that the first term can be estimated suitably by a weighted $\mathbf{H}^1$-seminorm. By assumption, $\mathcal{O}$ is a convex Lipschitz polyhedron and thus, we can apply [23, Thm. 3.1.1.2] to estimate for any $\mathbf{v} \in \mathbf{H}^1_{\boldsymbol{\nu}0}$

$$\| \operatorname{div}(\mathbf{v}) \|_{L^2}^2 = \sum_{i,j=1}^d \langle \partial_{x_j} \mathbf{v}_i, \partial_{x_i} \mathbf{v}_j \rangle_{\mathbf{L}^2} - \int_{\partial\mathcal{O}} \mathcal{B}(P_{\boldsymbol{\nu}}^\perp \mathbf{v}, P_{\boldsymbol{\nu}}^\perp \mathbf{v}) \mathrm{d}s \geq \sum_{i,j=1}^d \langle \partial_{x_j} \mathbf{v}_i, \partial_{x_i} \mathbf{v}_j \rangle_{\mathbf{L}^2}. \tag{49}$$

where $\mathcal{B}(\boldsymbol{\tau}, \boldsymbol{\tau}') := -\partial_{\boldsymbol{\tau}} \boldsymbol{\nu} \cdot \overline{\boldsymbol{\tau}'}$ is the second fundamental quadratic form of $\partial\mathcal{O}$ applied to tangential vector fields $\boldsymbol{\tau}, \boldsymbol{\tau}'$ and $\partial_{\boldsymbol{\tau}}$ is the tangential derivative. The last estimate follows since the form $\mathcal{B}$ is non-positive for convex domains [23, Sec. 3.1.1].

For any $\eta \in W^{1,\infty}(\mathcal{O})$, the product rule gives that $\eta \operatorname{div} \mathbf{v} = \operatorname{div}(\eta \mathbf{v}) - \mathbf{v} \cdot \nabla \eta$ and thus we estimate with the weighted Young's inequality for any $\delta \in (0,1)$ that

$$\| \eta \operatorname{div} \mathbf{v} \|_{L^2}^2 \geq (1-\delta) \| \operatorname{div}(\eta \mathbf{v}) \|_{\mathbf{L}^2}^2 + (1 - \frac{1}{\delta}) \| \mathbf{v} \cdot \nabla \eta \|_{\mathbf{L}^2}^2. \tag{50}$$

Since $\eta \mathbf{v} \in \mathbf{H}^1_{\boldsymbol{\nu}0}$, we can apply (49) to the first term to obtain

$$\begin{aligned}
\| \operatorname{div}(\eta \mathbf{v}) \|_{L^2}^2 \geq& \sum_{i,j=1}^d \langle \partial_{x_j} (\eta \mathbf{v}_i), \partial_{x_i} (\eta \mathbf{v}_j) \rangle_{\mathbf{L}^2} \\
\geq& \sum_{i,j=1}^d \big(\langle \eta \partial_{x_j} \mathbf{v}_i, \eta \partial_{x_i} \mathbf{v}_j \rangle_{\mathbf{L}^2} + \langle \mathbf{v}_i \partial_{x_j} \eta, \mathbf{v}_j \partial_{x_i} \eta \rangle_{\mathbf{L}^2} + \langle \mathbf{v}_i \partial_{x_j} \eta, \eta \partial_{x_i} \mathbf{v}_j \rangle_{\mathbf{L}^2} \\
&+ \langle \eta \partial_{x_j} \mathbf{v}_i, \mathbf{v}_j \partial_{x_i} \eta \rangle_{\mathbf{L}^2} \big).
\end{aligned}$$

Applying this estimate to $\mathbf{v} = \nabla\tilde{v}$, we notice that $\partial_{x_j}\mathbf{v}_i = \partial_{x_i}\mathbf{v}_j$ and consequently

$$\begin{aligned}\|\operatorname{div}(\eta\mathbf{v})\|_{L^2}^2 &\geq \|\eta\nabla\mathbf{v}\|_{\mathbf{L}^2}^2 + \sum_{i,j=1}^{d} (\langle \mathbf{v}_i\partial_{x_j}\eta, \mathbf{v}_j\partial_{x_i}\eta\rangle_{\mathbf{L}^2} + \langle \mathbf{v}_i\partial_{x_j}\eta, \eta\partial_{x_i}\mathbf{v}_j\rangle_{\mathbf{L}^2} \\ &\quad + \langle \eta\partial_{x_j}\mathbf{v}_i, \mathbf{v}_j\partial_{x_i}\eta\rangle_{\mathbf{L}^2}) \\ &\geq \|\eta\nabla\mathbf{v}\|_{\mathbf{L}^2}^2 - C\left(\|\mathbf{v}\cdot\nabla\eta\|_{\mathbf{L}^2}^2 + \|\eta\nabla\mathbf{v}\|_{\mathbf{L}^2}\|\mathbf{v}\cdot\nabla\eta\|_{\mathbf{L}^2}\right) \\ &\geq (1-\delta)\|\eta\nabla\mathbf{v}\|_{\mathbf{L}^2}^2 - (C + \frac{1}{4\delta})\|\mathbf{v}\cdot\nabla\eta\|_{\mathbf{L}^2}^2,\end{aligned}$$

where we use the Cauchy-Schwarz and the weighted Young's inequality in the last two lines. Inserting this into the estimate (50) yields

$$\|\eta \operatorname{div}\mathbf{v}\|_{L^2}^2 \geq (1-\delta)^2\|\eta\nabla\mathbf{v}\|_{\mathbf{L}^2}^2 + \underbrace{\left(C(\delta-1) - \frac{5}{4\delta} + \frac{5}{4}\right)}_{=:-\tilde{C}_\delta}\|\mathbf{v}\cdot\nabla\eta\|_{\mathbf{L}^2}^2,$$

where we note that $\tilde{C}_\delta \geq 0$ for all $\delta \in (0,1)$. Since $\eta \in W^{1,\infty}$ is Lipschitz continuous, $\nabla\eta$ is bounded such that $\|\mathbf{v}\cdot\nabla\eta\|_{\mathbf{L}^2}^2 \leq C_{\nabla\eta}\|\mathbf{v}\|_{\mathbf{L}^2}$. Inserting $\mathbf{v} = \nabla\tilde{v} = S_n\mathbf{u}_n$ and $\eta = c_s\rho^{1/2}$ we obtain together with (48) that

$$\|c_s\rho^{1/2}\operatorname{div}_{\boldsymbol{\nu}}^n P_{V_n}\mathbf{u}_n\|_{\mathcal{T}_n}^2 \geq (1-\delta)^2|S_n\mathbf{u}_n|_{\mathbf{H}^1_{c_s^2\rho}}^2 - C_\delta\|S_n\mathbf{u}_n\|_{\mathbf{L}^2}^2 + \langle \check{O}_n\mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}, \tag{51}$$

where $C_\delta := \tilde{C}_\delta C_{\nabla\eta}$. Thus, the claim is proven. □

To prove our main result, we have to assume that the Mach number of the background flow is bounded suitably. To be precise, we define the matrix $\underline{\underline{m}} := -\rho^{-1}\operatorname{Hess}(p) + \operatorname{Hess}(\phi)$ and denote by $\lambda_-(\underline{\underline{m}}) \in L^\infty$ its smallest eigenvalue[1]. Then, we set for $\omega \neq 0$

$$C_{\underline{\underline{m}}} := \max\left\{0, \sup_{x\in\mathcal{O}} \frac{\lambda_-(\underline{\underline{m}}(x))}{\gamma(x)}\right\}, \qquad \theta := \arctan(C_{\underline{\underline{m}}}/|\omega|) \in [0, \pi/2). \tag{52}$$

This definition of θ allows us to estimate

$$\langle \rho\underline{\underline{m}}\mathbf{u}_\tau, \mathbf{u}_\tau\rangle \geq -|\omega|\tan(\theta)\|(\gamma\rho)^{1/2}\mathbf{u}_\tau\|_{\mathbf{L}^2} \qquad \forall \mathbf{u}_\tau \in \mathbb{X}_{\mathcal{T}_n}, \tag{53}$$

which we will use in the proof of Theorem 23. In preparation of Theorem 23, we make the following assumption.

Assumption 21. The background flow $\mathbf{b} \in \mathbf{W}^{1,\infty}$ satisfies

$$\|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}^2 < \frac{1}{C_{(19)}(1 + C_{\underline{\underline{m}}}^2/|\omega|^2)} = (C_{(19)}(1+\tan^2(\theta)))^{-1}, \tag{54}$$

where $C_{(19)} = 1 + O(h_n^2) > 0$ is the constant appearing in Lemma 19.

Remark 22. The strict inequality in Assumption 21 implies that the inequality holds even for a slightly smaller r.h.s., i.e. there is $\delta_0 \in (0,1)$ so that

$$\|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}^2 < \frac{(1-\delta_0)^2}{C_{(19)}(1+(1+\delta_0)^2\tan^2(\theta)/|\omega|^2)} \Leftrightarrow \frac{(1-\delta_0)^2}{C_{(19)}\|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}^2} > 1 + (1+\delta_0)^2\tan^2(\theta)$$

[1]Note that $\underline{\underline{m}}$ is symmetric.

where we made use of the definition of θ. Similarly, we can bound $\tan^2(\theta)$ from below by $\kappa^{-1}\tan^2(\theta+\tau)$ with $\kappa > 1$ close to 1 for $\tau > 0$ sufficiently small. To be precise, there is $\tau_0 \in (0, \pi/2 - \theta)$ and $\epsilon_0 \in (0, 1/2)$ so that for all $\tau \in (0, \tau_0)$ and $\epsilon \in (0, \epsilon_0)$ we have that

$$\frac{(1-\delta_0)^2}{C_{(19)}\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}} - 1 > (1+\delta_0)^2\tan^2(\theta) > \tan^2(\theta+\tau)(1-\epsilon)^{-1}(1-2\epsilon)^{-1}. \tag{55}$$

Multiplying with $(1-\epsilon)$ and rearranging the terms, we obtain that for $\epsilon \in (0, \epsilon_0)$, $\delta \in (0, \delta_0)$, and $\tau \in (0, \tau_0)$, we have that

$$C_{\theta,\tau,\epsilon,\delta} := (1-\epsilon)(1-\delta)^2 - C_{(19)}\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}\big(1+\tan^2(\theta+\tau)(1-2\epsilon)^{-1} - \epsilon\big) > 0. \tag{56}$$

This constant appears in the proof of Theorem 23 below and its positivity is essential to obtain stability. In [29], where an $\mathbf{H}^1$-conforming discretization of (8) was analyzed, the following smallness assumption was assumed:

$$\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} < \beta_h^2 \frac{\underline{c_s}^2\underline{\rho}}{\overline{c_s}^2\overline{\rho}}(1+\tan^2(\theta))^{-1}. \tag{57}$$

Here β_h is the discrete stability constant of the divergence operator. This assumption is much more restrictive than (54) because it depends on the ratio $\underline{c_s}^2\underline{\rho}/\overline{c_s}^2\overline{\rho}$, whereas the constant $C_{(19)} > 0$ is independent of this ratio. In particular, the constant $C_{(19)}$ tends to one asymptotically, and thus Assumption 21 tends to the boundedness assumption from the continuous analysis, cf. [27, Thm. 3.11].

To avoid the dependence on the ratio, we used the weighted $\mathbf{H}^1$-seminorm $|\cdot|_{\mathbf{H}^1_{c_s^2\rho}}$ in Lemma 19 and Lemma 20, and the constants $C_{(19)}$ can be interpreted as the continuity of the operator $\mathbf{D}^n_{\mathbf{b}}$ with respect to $|\cdot|_{\mathbf{H}^1_{c_s^2\rho}}$. We compare both assumptions numerically in Section 5.4.

Theorem 23. *Assume that Assumption 21 is satisfied. Then, there exist sequences $(B_n)_{n\in\mathbb{N}}, (K_n)_{n\in\mathbb{N}}$, $B_n \in L(\mathbb{X}_n)$, $K_n \in L(\mathbb{X}_n)$, $n \in \mathbb{N}$ such that $A_nT_n = B_n + K_n$ with $(B_n)_{n\in\mathbb{N}}$ being uniformly bounded. $(B_n)_{n\in\mathbb{N}}$ is stable, $(K_n)_{n\in\mathbb{N}}$ is compact and there exists a bijective operator $B \in L(\mathbb{X}_n)$ such that B_n approximates B.*

Proof. We split the proof of the statement into four steps. In the first step, we define the sequences $(B_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$ and argue that indeed $A_nT_n = B_n + K_n$. Afterwards, we show in the second and third step that the sequence $(B_n)_{n\in\mathbb{N}}$ is stable. In the last step, we show that there exists a bijective operator $B \in L(\mathbb{X})$ and a compact operator $K \in L(\mathbb{X})$ such that $AT = B + K$ and B_n approximates B. In the following, we denote $\mathbf{v}_n := P_{V_n}\mathbf{u}_n$, $\mathbf{w}_n := \mathbf{u}_n - \mathbf{v}_n$ for an element $\mathbf{u}_n \in \mathbb{X}_n$ and $\mathbf{v}'_n, \mathbf{w}'_n$ defined analogously for an element $\mathbf{u}'_n \in \mathbb{X}_n$.
Step 1: Definition of B_n and K_n. We want to define $B_n, K_n \in L(\mathbb{X}_n)$ such that $A_nT_n = B_n + K_n$, where $(B_n)_{n\in\mathbb{N}}$ is uniformly coercive and $(K_n)_{n\in\mathbb{N}}$ is compact. In particular, we define $B_n = B_n^{(1)} + B_n^{(2)}$, where $B_n^{(1)}$ is constructed to yield the essential control of the $\|\cdot\|_{\mathbb{X}_n}$-norm and $B_n^{(2)}$ contains the remaining terms, which we will estimate in *Step 3*. We add compact terms $K_n^{(1)}$ and $K_n^{(2)}$ to both operators which are subtracted again through $K_n = -K_n^{(1)} - K_n^{(2)}$.

We will consider a splitting $a_n(\cdot,\cdot) = a_n^{(1)}(\cdot,\cdot) + a_n^{(2)}(\cdot,\cdot)$ so that terms that directly help for (T_n-)coercivity will be collected in $a_n^{(1)}(\cdot,\cdot)$ and the remainder will be collected in $a_n^{(2)}(\cdot,\cdot)$. By construction, T_n swaps the sign in front of $\mathbf{w}_n$ such that $\langle A_nT_n\mathbf{u}_n, \mathbf{u}'_n\rangle = a_n(\mathbf{v}_n - \mathbf{w}_n, \mathbf{v}'_n + \mathbf{w}'_n)$.

We recall $\mathbf{q} := c_s^{-2}\rho^{-1}\nabla p$, the associated rewriting of the sesquilinear form (9), and note that we can use a corresponding split on the discrete level: $a_n(\cdot,\cdot) = a_n^{(\operatorname{div}+\mathbf{q}\cdot)}(\cdot,\cdot) + s_n(\cdot,\cdot) - a_n^{\partial_{\mathbf{b}}}(\cdot,\cdot) + a_n^{(r-\mathbf{q}\cdot)}(\cdot,\cdot)$.

We have that $\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{v}_n = \operatorname{div}\mathbf{v}_\tau$ and thus

$$\begin{aligned} a_n^{(\operatorname{div}+\mathbf{q}\cdot)}(T_n\mathbf{u}_n, \mathbf{u}'_n) &= \langle c_s^2\rho(\operatorname{div}^n_{\boldsymbol{\nu}}+\mathbf{q}\cdot)\mathbf{u}_n, (\operatorname{div}^n_{\boldsymbol{\nu}}+\mathbf{q}\cdot)\mathbf{u}'_n\rangle = \textstyle\sum_{J=I,II,III,IV}(58_{\mathrm{J}}) \\ &= \langle c_s^2\rho(\operatorname{div}+\mathbf{q}\cdot)\mathbf{v}_\tau, (\operatorname{div}+\mathbf{q}\cdot)\mathbf{v}'_\tau\rangle + \langle c_s^2\rho(\operatorname{div}+\mathbf{q}\cdot)\mathbf{v}_\tau, (\operatorname{div}^n_{\boldsymbol{\nu}}+\mathbf{q}\cdot)\mathbf{w}'_n\rangle \\ &\quad - \langle c_s^2\rho(\operatorname{div}^n_{\boldsymbol{\nu}}+\mathbf{q}\cdot)\mathbf{w}_n, (\operatorname{div}+\mathbf{q}\cdot)\mathbf{v}'_\tau\rangle - \langle c_s^2\rho(\operatorname{div}^n_{\boldsymbol{\nu}}+\mathbf{q}\cdot)\mathbf{w}_n, (\operatorname{div}^n_{\boldsymbol{\nu}}+\mathbf{q}\cdot)\mathbf{w}'_n\rangle \end{aligned} \tag{58}$$

The div-parts of the first term in (58_{I}) will directly be used in $a_n^{(1)}(\cdot,\cdot)$ to control the divergence, below. To rewrite (58_{IV}), we want to use that $(\operatorname{div}^n_{\boldsymbol{\nu}} + \pi_n^l P_{L^2_0}\mathbf{q}\cdot)\mathbf{w}_n \overset{(41)}{=} -M_n\mathbf{w}_n - \tilde{O}_n\mathbf{u}_n$ for both arguments, so we shift in the terms involving the projection $\pi_n^l P_{L^2_0}$. Then, we obtain[2] that

$$\begin{aligned}(58_{\mathrm{IV}}) = &-\langle c_s^2\rho(M_n\mathbf{w}_n+\tilde{O}_n\mathbf{u}_n), M_n\mathbf{w}'_n+\tilde{O}_n\mathbf{u}'_n\rangle - \langle c_s^2\rho \operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}_n, (\mathrm{id}-\pi_n^l P_{L^2_0})\mathbf{q}\cdot\mathbf{w}'_n\rangle \\ &- \langle c_s^2\rho(\mathrm{id}-\pi_n^l P_{L^2_0})\mathbf{q}\cdot\mathbf{w}_n, \operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}'_n\rangle + \langle c_s^2\rho\pi_n^l P_{L^2_0}\mathbf{q}\cdot\mathbf{w}_n, \pi_n^l P_{L^2_0}\mathbf{q}\cdot\mathbf{w}'_n\rangle \\ &- \langle c_s^2\rho\mathbf{q}\otimes\mathbf{q}\cdot\mathbf{w}_\tau, \mathbf{w}'_\tau\rangle\end{aligned} \tag{59}$$

We apply the same argumentation to the mixed terms (58_{II}) and (58_{III}), and note that the terms $\langle c_s^2\rho\mathbf{q}\otimes\mathbf{q}\cdot\mathbf{z}_\tau, \mathbf{z}'_\tau\rangle$, $\mathbf{z}_\tau \in \{\mathbf{v}_\tau, \mathbf{w}_\tau\}$, $\mathbf{z}'_\tau \in \{\mathbf{v}'_\tau, \mathbf{w}'_\tau\}$, cancel out with the $\mathbf{q}$-terms in $a_n^{(r-\mathbf{q}\cdot)}(T_n\mathbf{u}_n, \mathbf{u}'_n)$. Thus, these terms do not appear in the following, and we are left with $a_n^r(\cdot,\cdot)$ instead of $a_n^{(r-\mathbf{q}\cdot)}(\cdot,\cdot)$.

For $a_n^{(1)}(\cdot,\cdot)$, we will only use the fourth term of (59) and shift the remaining terms into $a_n^{(2)}(\cdot,\cdot)$. Hence, we define

$$\begin{aligned}&a_n^{(1)}(\mathbf{u}_n, \mathbf{u}'_n) := && (60|a_n^{(1)})\\ &\quad \langle c_s^2\rho\operatorname{div}\mathbf{v}_\tau, \operatorname{div}\mathbf{v}'_\tau\rangle + \langle c_s^2\rho\pi_n^l P_{L^2_0}\mathbf{q}\cdot\mathbf{w}_\tau, \pi_n^l P_{L^2_0}\mathbf{q}\cdot\mathbf{w}'_\tau\rangle - s_n(\mathbf{w}_n, \mathbf{w}'_n) && (60a|a_n^{(1)})\\ &\quad - \langle\rho i\mathbf{D}^n_{\mathbf{b}}\mathbf{v}_n, i\mathbf{D}^n_{\mathbf{b}}\mathbf{v}'_n\rangle + \langle\rho(\omega+i\mathbf{D}^n_{\mathbf{b}}+i\Omega\times)\mathbf{w}_n, (\omega+i\mathbf{D}^n_{\mathbf{b}}+i\Omega\times)\mathbf{w}'_n\rangle && (60b|a_n^{(1)})\\ &\quad + \langle\rho(\omega+i\mathbf{D}^n_{\mathbf{b}}+i\Omega\times)\mathbf{w}_n, i\mathbf{D}^n_{\mathbf{b}}\mathbf{v}'_n\rangle - \langle\rho i\mathbf{D}^n_{\mathbf{b}}\mathbf{v}_n, (\omega+i\mathbf{D}^n_{\mathbf{b}}+i\Omega\times)\mathbf{w}'_n\rangle && (60c|a_n^{(1)})\\ &\quad + \langle\rho(\underline{\underline{m}}+i\omega\gamma)\mathbf{w}_\tau, \mathbf{w}'_\tau\rangle. && (60d|a_n^{(1)})\end{aligned}$$

The div-terms of (58_{I}) and the fourth term in (59) from $a_n^{(\operatorname{div}+\mathbf{q}\cdot)}(T_n\mathbf{u}_n, \mathbf{u}'_n)$ together with $s_n(T_n\mathbf{u}_n, \mathbf{u}'_n)$ form line ($60a|a_n^{(1)}$), where we note that $s_n(T_n\mathbf{u}_n, \mathbf{u}'_n) = -s_n(\mathbf{w}_n, \mathbf{w}'_n)$ since $[\![\underline{P_{V_n}\mathbf{u}_n}]\!]_{\boldsymbol{\nu}} = 0$ for all $\mathbf{u}_n \in \mathbb{X}_n$ by construction of P_{V_n}. The terms in the lines ($60b|a_n^{(1)}$) and ($60c|a_n^{(1)}$) arise naturally from a selection of terms from $a_n^{\partial_{\mathbf{b}}}(T_n\mathbf{u}_n, \mathbf{u}'_n)$. By definition of $\underline{\underline{m}} = -\rho^{-1}\operatorname{Hess}(p) + \operatorname{Hess}(\phi)$, we can write $a_n^r(\mathbf{u}_n, \mathbf{u}'_n) = -\langle\rho(\underline{\underline{m}}+i\omega\gamma)\mathbf{u}_n, \mathbf{u}'_n\rangle$, which is where the line ($60d|a_n^{(1)}$) originates from.

Below, in *Step 2*, we want to estimate these terms from below by applying Lemmas 19 and 20. To this end, we add suitable compact terms to $B_n^{(1)}$ which we simultaneously subtract from K_n. For $C_1 > 0$ to be specified later on, we set

$$\langle K_n^{(1)}\mathbf{u}_n, \mathbf{u}'_n\rangle_{\mathbb{X}_n} := \langle\mathbf{v}_\tau, \mathbf{v}'_\tau\rangle + C_1\langle S_n\mathbf{u}_n, S_n\mathbf{u}'_n\rangle + \langle c_s^2\rho M_n\mathbf{w}_n, M_n\mathbf{w}'_n\rangle + \langle c_s^2\rho\tilde{O}_n\mathbf{u}_n, \tilde{O}_n\mathbf{u}'_n\rangle. \qquad (60|K_n^{(1)})$$

Then, we define $\langle B_n^{(1)}\mathbf{u}_n, \mathbf{u}'_n\rangle_{\mathbb{X}_n} := a_n^{(1)}(\mathbf{u}_n, \mathbf{u}'_n) + \langle K_n^{(1)}\mathbf{u}_n, \mathbf{u}'_n\rangle_{\mathbb{X}_n}$. To account for the remaining terms in $a_n(T_n\mathbf{u}_n, \mathbf{u}'_n)$, we define

$$a_n^{(2)}(\mathbf{u}_n, \mathbf{u}'_n) := a_n(T_n\mathbf{u}_n, \mathbf{u}'_n) - a_n^{(1)}(\mathbf{u}_n, \mathbf{u}'_n). \qquad (60|a_n^{(2)})$$

To treat the terms in $a_n^{(2)}(\mathbf{u}_n, \mathbf{u}'_n)$ we add suitable compact terms which together with $a_n^{(2)}(\mathbf{u}_n, \mathbf{u}'_n)$ can be bounded from below. To this end, we define, for $C_2 > 0$ to be specified later on,

$$\begin{aligned}\langle K_n^{(2)}\mathbf{u}_n, \mathbf{u}'_n\rangle_{\mathbb{X}_n} := C_2(&\langle\mathbf{v}_\tau, \mathbf{v}'_\tau\rangle + \langle S_n\mathbf{u}_n, S_n\mathbf{u}'_n\rangle + \langle c_s^2\rho\tilde{O}_n\mathbf{u}_n, \tilde{O}_n\mathbf{u}'_n\rangle && (61a|K_n^{(2)})\\ &+ \langle c_s^2\rho M_n\mathbf{w}_n, M_n\mathbf{w}'_n\rangle + \langle\operatorname{mean}(\mathbf{q}\cdot\mathbf{w}_\tau), \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}'_\tau)\rangle) && (61b|K_n^{(2)})\end{aligned}$$

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[2] We use that $\langle a+b, a'+b'\rangle = \langle a+c, a'+c'\rangle + \langle a, b'-c'\rangle + \langle b-c, a'\rangle + \langle b, b'\rangle - \langle c, c'\rangle$ with $a = \operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}_n$, $b = \mathbf{q}\cdot\mathbf{w}_n$, and $c = \pi_n^l P_{L^2_0}\mathbf{q}\cdot\mathbf{w}_n$; a', b', c' analogously with $\mathbf{w}'_n$.

where mean denotes the mean value operator $\frac{1}{\mathcal{O}}\int_{\mathcal{O}}\cdot d\mathbf{x}$, and set $\langle B_n^{(2)}\mathbf{u}_n,\mathbf{u}_n'\rangle_{\mathbb{X}_n} := a_n^{(2)}(\mathbf{u}_n,\mathbf{u}_n')+\langle K_n^{(2)}\mathbf{u}_n,\mathbf{u}_n'\rangle_{\mathbb{X}_n}$, which we analyze in *Step 3*.

With $K_n := -K_n^{(1)} - K_n^{(2)}$ and $B_n := B_n^{(1)} + B_n^{(2)}$, we then obtain that $A_nT_n = B_n + K_n$. The explicit expressions for the operators $B_n^{(1)}$ and especially the lenghty one for $B_n^{(2)}$ are written out in Appendix A. We note that the uniform boundedness of B_n, $n\in\mathbb{N}$, follows straightforwardly. Furthermore, it can be shown that the sequence $(K_n)_{n\in\mathbb{N}}$ is indeed compact with the same argumentation as in [26, Lem. 17]. In particular, we note that $\lim_{n\to\infty}\|\tilde{O}_n\|_{L(\mathbb{X}_n,L^2)} = 0$, that the operators M_n and mean$(\cdot)$ give rise to compact terms, and that the construction of S_n allows us to use the compact embedding $\mathbf{H}^1 \hookrightarrow \mathbf{L}^2$.

Step 2: Uniform coercivity of $B_n^{(1)}$. First of all, we show that there exists an index $n_0 > 0$ such that $B_n^{(1)}$ is uniformly coercive for all $n > n_0$.

Let $\mathbf{u}_n \in \mathbb{X}_n$ be arbitrary and $\delta_0,\epsilon_0 \in (0,1)$, $\tau_0 \in (0,\pi/2-\theta)$ be such that the constant $C_{\theta,\tau,\epsilon,\delta}$ defined by (56) is positive for all $\delta\in(0,\delta_0)$, $\epsilon\in(0,\epsilon_0)$ and $\tau\in(0,\tau_0)$. We recall that this is possible due to Assumption 21, as detailed in Remark 22. We further recall that by definition of θ, the estimate (53) holds.

Targeting at coercivity of $B_n^{(1)}$ in the sense of (2), we set $\xi := e^{-i(\theta+\tau)\mathrm{sgn}\omega}$, $|\xi| = 1$, so that $\mathrm{Re}(\xi(a+ib))/\cos(\theta+\tau) = a + \mathrm{sgn}(\omega)\tan(\theta+\tau)b$ for $a,b\in\mathbb{R}$. Using (60$|a_n^{(1)}$) and (60$|K_n^{(1)}$), we note that (60c$|a_n^{(1)}$) becomes purely imaginary for $(\mathbf{v}_n',\mathbf{w}_n') = (\mathbf{v}_n,\mathbf{w}_n)$ and obtain

$$\begin{aligned}
&\mathrm{Re}\left(\xi\langle B_n^{(1)}\mathbf{u}_n,\mathbf{u}_n\rangle_{\mathbb{X}_n}\right)/\cos(\theta+\tau)\\
&= \|c_s\rho^{1/2}\operatorname{div}\mathbf{v}_\tau\|_{L^2}^2 - \|\rho^{1/2}\mathbf{D}_\mathbf{b}^n\mathbf{v}_n\|_{\mathbf{L}^2}^2 + \|\mathbf{v}_\tau\|_{\mathbf{L}^2} + C_1\|S_n\mathbf{u}_n\|_{\mathbf{L}^2}^2 + \|c_s\rho^{1/2}M_n\mathbf{w}_n\|_{L^2}^2\\
&\quad + \|c_s\rho^{1/2}\tilde{O}_n\mathbf{u}_n\|_{L^2}^2 + \|c_s\rho^{1/2}\pi_n^l P_{L_0^2}\mathbf{q}\cdot\mathbf{w}_\tau\|_{L^2}^2 + \|\rho^{1/2}(\omega+i\mathbf{D}_\mathbf{b}^n+i\Omega\times)\mathbf{w}_n\|_{\mathbf{L}^2}^2\\
&\quad + 2\tan(\theta+\tau)\mathrm{sgn}(\omega)\,\mathrm{Im}\left(\langle\rho(\omega+i\mathbf{D}_\mathbf{b}^n+i\Omega\times)\mathbf{w}_n, i\mathbf{D}_\mathbf{b}^n\mathbf{v}_n\rangle\right) - s_n(\mathbf{w}_n,\mathbf{w}_n)\\
&\quad + \langle\rho\underline{\underline{m}}\mathbf{w}_\tau,\mathbf{w}_\tau\rangle_{\mathbf{L}^2} + |\omega|\tan(\theta+\tau)\|(\gamma\rho)^{1/2}\mathbf{w}_\tau\|_{\mathbf{L}^2} \geq \dots
\end{aligned}$$

Regrouping and applying a weighted Young's inequality $(2ab \leq (1-2\epsilon)^{-1}a^2 + (1-2\epsilon)b^2)$ on the mixed term (involving $\mathbf{w}_n$ and $\mathbf{v}_n$) and using (53) yields

$$\begin{aligned}
\dots\geq\ & \|c_s\rho^{1/2}\operatorname{div}\mathbf{v}_\tau\|_{L^2}^2 - \|\rho^{1/2}\mathbf{D}_\mathbf{b}^n\mathbf{v}_n\|_{\mathbf{L}^2}^2(1+\underbrace{\tan^2(\theta+\tau)(1-2\epsilon)^{-1}}_{=:C_{\theta,\tau,\epsilon}})+\|\mathbf{v}_\tau\|_{\mathbf{L}^2}^2 && \text{(I)}\\
&+ C_1\|S_n\mathbf{u}_n\|_{\mathbf{L}^2}^2 && \text{(II)}\\
&+ \|c_s\rho^{1/2}M_n\mathbf{w}_n\|_{L^2}^2 + \|c_s\rho^{1/2}\tilde{O}_n\mathbf{u}_n\|_{L^2}^2 + \|c_s\rho^{1/2}\pi_n^l P_{L_0^2}\mathbf{q}\cdot\mathbf{w}_\tau\|_{L^2}^2 && \text{(III)}\\
&\left.\begin{aligned}&+2\epsilon\|\rho^{1/2}(\omega+i\mathbf{D}_\mathbf{b}^n+i\Omega\times)\mathbf{w}_n\|_{\mathbf{L}^2}^2\\ &+|\omega|\left(\tan(\theta+\tau)-\tan(\theta)\right)\|(\gamma\rho)^{1/2}\mathbf{w}_\tau\|_{\mathbf{L}^2} - s_n(\mathbf{w}_n,\mathbf{w}_n)\end{aligned}\right\} && \text{(IV)}
\end{aligned}$$

(I) and (II) allow us to control $\mathbf{v}_n$ while (III) and (IV) are responsible for $\mathbf{w}_n$. We start with estimating (I) from below. Splitting off an ϵ-scaled $\mathbb{X}_n$-norm of $\mathbf{v}_n$ (note that $[\![\underline{\mathbf{v}_n}]\!]_{\partial\mathcal{T}_n,1/2,\boldsymbol{\nu}} = 0$), and using Lemmas 19 and 20 and (56) for the remainder, we obtain

$$\begin{aligned}
\text{(I)} &= \epsilon\|\mathbf{v}_n\|_{\mathbb{X}_n}^2 + (1-\epsilon)\|c_s\rho^{1/2}\operatorname{div}\mathbf{v}_\tau\|_{L^2}^2 - (1+C_{\theta,\tau,\epsilon}-\epsilon)\|\rho^{1/2}\mathbf{D}_\mathbf{b}^n\mathbf{v}_n\|_{\mathbf{L}^2}^2 + \overbrace{(1-\epsilon)\|\mathbf{v}_\tau\|_{\mathbf{L}^2}^2}^{\geq 0}\\
&\geq \epsilon\|\mathbf{v}_n\|_{\mathbb{X}_n}^2 + \underbrace{C_{\theta,\tau,\epsilon,\delta}|S_n\mathbf{u}_n|_{\mathbf{H}^1_{c_s^2\rho}}^2}_{\geq 0} + (1-\epsilon)\left(\langle\check{O}_n\mathbf{u}_n,\mathbf{u}_n\rangle - C_\delta\|S_n\mathbf{u}_n\|_{\mathbf{L}^2}^2\right)
\end{aligned}$$

We then have

$$\text{(I)}+\text{(II)} \geq \epsilon\|\mathbf{v}_n\|_{\mathbb{X}_n}^2 + (C_1-(1-\epsilon)C_\delta)\|S_n\mathbf{u}_n\|_{\mathbf{L}^2}^2 - (1-\epsilon)\|\check{O}_n\|_{L(\mathbb{X}_n)}\|\mathbf{u}_n\|_{\mathbb{X}_n}^2.$$

Furthermore, recalling (41), i.e. $(\operatorname{div}^n_{\boldsymbol{\nu}} + \pi^l_n P_{L^2_0}\mathbf{q}\cdot)\mathbf{w}_n = -M_n\mathbf{w}_n - \tilde{O}_n\mathbf{u}_n$, we see

$$\begin{aligned}(\mathrm{III}) &= \|c_s\rho^{1/2}M_n\mathbf{w}_n\|^2_{L^2} + \|c_s\rho^{1/2}\tilde{O}_n\mathbf{u}_n\|^2_{L^2} + \|c_s\rho^{1/2}\pi^l_n P_{L^2_0}\mathbf{q}\cdot\mathbf{w}_\tau\|^2_{L^2}\\ &\geq \frac{1}{4}\|c_s\rho^{1/2}\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}_n\|^2_{\mathcal{T}_n}.\end{aligned}$$

For the first term in (IV) we obtain from a weighted Young's inequality (as in [27])

$$2\epsilon\|\rho^{1/2}(\omega + i\mathbf{D}^n_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n\|^2_{\mathbf{L}^2} \geq \epsilon\|\rho^{1/2}\mathbf{D}^n_{\mathbf{b}}\mathbf{w}_n\|^2_{\mathbf{L}^2} - C\epsilon\|\rho^{1/2}\mathbf{w}_\tau\|^2_{\mathbf{L}^2}$$

for $C \in \mathbb{R}$ only depending on ω and Ω. For sufficiently small ϵ we can dominate the latter part by the second line of (IV). Exploiting $-s_n(\mathbf{w}_n, \mathbf{w}_n) \geq \alpha[\![\underline{\mathbf{w}_n}]\!]^2_{\partial\mathcal{T}_n,1/2,\boldsymbol{\nu}}$ and choosing ϵ small enough (compared to α, $\frac{1}{4}$ and especially τ), we get control of the $\mathbb{X}_n$-norm of $\mathbf{v}_n$:

$$(\mathrm{III}) + (\mathrm{IV}) \geq \epsilon\|\mathbf{w}_n\|^2_{\mathbb{X}_n}.$$

Combining all estimates and $\|\mathbf{u}_n\|^2_{\mathbb{X}_n} \leq 2\|\mathbf{v}_n\|^2_{\mathbb{X}_n} + 2\|\mathbf{w}_n\|^2_{\mathbb{X}_n}$, we obtain

$$\begin{aligned}&\operatorname{Re}\left(\xi\langle B^{(1)}_n\mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}\right)/\cos(\theta+\tau) \geq (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV})\\ &\geq \frac{\epsilon}{2}\|\mathbf{u}_n\|^2_{\mathbb{X}_n} + (C_1 - (1-\epsilon)C_\delta)\|S_n\mathbf{u}_n\|^2_{\mathbf{L}^2} - (1-\epsilon)\|\check{O}_n\|_{L(\mathbb{X}_n)}\|\mathbf{u}_n\|^2_{\mathbb{X}_n}.\end{aligned}$$

Since $\lim_{n\to\infty}\|\check{O}_n\|_{L(\mathbb{X}_n)} = 0$, we can choose $n_1 > n_0$ large enough such that $(1-\epsilon)\|\check{O}_n\|_{L(\mathbb{X}_n)} < \epsilon/4$ and thus with $C_1 > (1-\epsilon)C_\delta$

$$\operatorname{Re}\left(\xi\langle B^{(1)}_n\mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}\right)/\cos(\theta+\tau) \geq \frac{\epsilon}{4}\|\mathbf{u}_n\|^2_{\mathbb{X}_n}$$

for $n > n_1$. Thus, $B^{(1)}_n$ is uniformly coercive (in the sense of (2)) for all $n > n_1$.

Step 3: Uniform coercivity of B_n. We have shown that $B^{(1)}_n$ is uniformly coercive in *Step 2*, so it remains to show that B_n (after addition of $B^{(2)}_n$) is uniformly coercive as well. To this end, we want to derive a bound for $B^{(2)}_n$ of the form

$$\operatorname{Re}(\xi\langle B^{(2)}_n\mathbf{u}_n,\mathbf{u}_n\rangle_{\mathbb{X}_n})/\cos(\theta+\tau) \geq -(\epsilon/8+\zeta_{1,n})\|\mathbf{u}_n\|^2_{\mathbb{X}_n} + (C_2-\zeta_2)|\mathbf{u}_n|^2_{K^{(2)}_n} \tag{62}$$

for $(\zeta_{1,n})_{n\in\mathbb{N}}$ and $\zeta_{1,n}, \zeta_2 \in \mathbb{R}_+$ with the semi-norm

$$\begin{aligned}|\mathbf{u}_n|^2_{K^{(2)}_n} &:= \|\mathbf{v}_\tau\|^2_{\mathbf{L}^2} + \|S_n\mathbf{u}_n\|^2_{\mathbf{L}^2} + \|c_s\rho^{1/2}\tilde{O}_n\mathbf{u}_n\|^2_{L^2} + \|c_s\rho^{1/2}M_n\mathbf{w}_n\|^2_{L^2} + \|\mathrm{mean}(\mathbf{q}\cdot\mathbf{w}_\tau)\|^2_{L^2}\\ &= C_2^{-1}K^{(2)}_n(\mathbf{u}_n, \mathbf{u}_n).\end{aligned}$$

We will show that $\zeta_{1,n} \to 0$ for $n \to \infty$ so that $-(\epsilon/8+\zeta_{1,n})\|\mathbf{u}_n\|^2_{\mathbb{X}_n}$ can be dominated by the $B^{(1)}_n$-contribution for sufficiently large n. The bound (62) allows us to choose C_2 sufficiently large to compensate for the $-\zeta_2|\mathbf{u}_n|^2_{K^{(2)}_n}$ term and obtain the uniform coercivity of B_n.

To prove (62), it suffices to show the boundedness of $a^{(2)}_n$ in the following form

$$|\langle a^{(2)}_n\mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}| \leq \eta_{1,n}\|\mathbf{u}_n\|^2_{\mathbb{X}_n} + \eta_2\|\mathbf{u}_n\|_{\mathbb{X}_n}|\mathbf{u}_n|_{K^{(2)}_n}. \tag{63}$$

for $(\eta_{1,n})_{n\in\mathbb{N}} \to 0$ for $n \to \infty$ and η_2 bounded. Then, a weighted Young's inequality of the form $\eta_2 ab \leq \frac{1}{2}\frac{\epsilon}{4\cos(\theta+\tau)}a^2 + \frac{1}{2}\frac{4\cos(\theta+\tau)\eta_2^2}{\epsilon}b^2$ yields

$$|\langle a_n^{(2)}\mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}| \leq \frac{\epsilon/8+\zeta_{1,n}}{\cos(\theta+\tau)}\|\mathbf{u}_n\|^2_{\mathbb{X}_n} + \frac{\zeta_2}{\cos(\theta+\tau)}|\mathbf{u}_n|^2_{K_n^{(2)}}. \tag{64}$$

with $\zeta_{1,n} = \eta_{1,n}\cos(\theta+\tau)$ and $\zeta_2 = 2/\epsilon\cos^2(\theta+\tau)\eta_2^2$ which implies (62). It hence remains to show (63).

$a_n^{(2)}(\cdot,\cdot)$ effectively contains all the terms of $a_n(\cdot,\cdot)$ that are not considered in $a_n^{(1)}(\cdot,\cdot)$. Most terms are of the form that they pair a term that can be bounded by $|\cdot|_{K_n^{(2)}}$ with another term that can be bounded by $|\cdot|_{\mathbb{X}_n}$. Those terms are thence directly suitable for (64) and in the following, we only discuss the terms that do not match this pattern; for completeness we state the full expression $a_n^{(2)}(\cdot,\cdot)$ in Appendix A.

The terms of (64) that do not match the pattern described above stem from contributions of $\operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{w}_n + \mathbf{q}\cdot\mathbf{w}_n$. We then shift in $\pi_n^l P_{L_0^2}(\mathbf{q}\cdot\mathbf{w}_n)$, cf. (59), and make use of (41) to bound $\operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{w}_n + \pi_n^l P_{L_0^2}\mathbf{q}\cdot\mathbf{w}_n$ by $|\mathbf{u}_n|_{K_n^{(2)}}$. The remaining terms are then of the form $\langle c_s^2\rho(\operatorname{id}-\pi_n^l P_{L_0^2})(\mathbf{q}\cdot\mathbf{w}_\tau), \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{z}_n\rangle$ for $\mathbf{z}_n \in \{\mathbf{v}_n', \mathbf{w}_n'\}$. To exploit the approximation properties of π_n^l here, we first shift in another $\operatorname{mean}(\mathbf{q}\cdot\mathbf{w}_n)$ term (which is itself bounded by constant times $\|\mathbf{u}_n\|_{K_n^{(2)}}$) and only have to deal with the following expression where we abbreviate $\Pi^* = \operatorname{id}-\operatorname{mean}-\pi_n^l P_{L_0^2}$:

$$\begin{aligned}|\langle c_s^2\rho\Pi^*(\mathbf{q}\cdot\mathbf{w}_\tau), \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{z}_n\rangle| &= |\langle \mathbf{q}\cdot\mathbf{w}_\tau, \Pi^*(c_s^2\rho\operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{z}_n)\rangle| \\ &\leq \|\mathbf{q}\|_{\mathbf{L}^\infty}\|\mathbf{w}_\tau\|_{\mathbf{L}^2}\|\Pi^*(c_s^2\rho\operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{z}_n)\|_{L^2} \lesssim \|\mathbf{u}_n\|_{\mathbb{X}_n}\|\Pi^*(c_s^2\rho\operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{z}_n)\|_{L^2},\end{aligned}$$

We estimate the last term using a discrete commutator technique [3] to obtain

$$\begin{aligned}\|\Pi^*(c_s^2\rho\operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{z}_n)\|^2_{L^2} &= \sum_{\tau\in\mathcal{T}_n}\|\Pi^*(c_s^2\rho\operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{z}_n)\|^2_{L^2(\tau)} = \sum_{\tau\in\mathcal{T}_n}\|\Pi^*((c_s^2\rho - c_\tau)\operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{z}_n)\|^2_{L^2(\tau)} \\ &\leq \sum_{\tau\in\mathcal{T}_n}\|c_s^2\rho - c_\tau\|^2_{L^\infty(\tau)}\|\operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{z}_n\|^2_{L^2(\tau)} \leq (C^L_{c_s^2\rho})^2 h_n^2\|\mathbf{z}_n\|^2_{\mathbb{X}_n} \lesssim h_n^2\|\mathbf{u}_n\|^2_{\mathbb{X}_n}.\end{aligned}$$

Here, we used that $\Pi^* r = 0$ for any piecewise polynomial r of degree $k-1$ and hence $\Pi^* c_\tau \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{z}_n = 0$ for any piecewise constant c_τ.

This reveals that all terms in $a_n^{(2)}$ are suitable for (63) and hence (62) holds. Together with *Step 2*, C_2 and n sufficiently large we hence obtain

$$\operatorname{Re}(\xi\langle B_n\mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}) \geq 3\epsilon/16\cos(\theta+\tau)\|\mathbf{u}_n\|^2_{\mathbb{X}_n}. \tag{65}$$

Step 4: Asymptotic consistency of B_n. In the last step, we want to show that there exists a bijective operator $B \in L(\mathbb{X})$ such that B_n approximates B.

For $\mathbf{u}, \mathbf{u}' \in \mathbb{X}$, we define

$$\begin{aligned}\langle K\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}} \coloneqq &-(1+C_2)\langle \mathbf{v}, \mathbf{v}'\rangle - (C_1+C_2)\langle \mathbf{v}, \mathbf{v}'\rangle &\quad (K\text{-a})\\ &-(1+C_2)\langle c_s^2\rho M\mathbf{w}, M\mathbf{w}'\rangle - C_2\langle \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}), \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}')\rangle &\quad (K\text{-b})\end{aligned}$$

and set $B \coloneqq AT - K$. Since M is of finite rank and $\mathbf{v} = P_V\mathbf{u} \in \mathbf{H}^1$, the compactness of the operator K follows from the compact embedding $\mathbf{H}^1 \hookrightarrow \mathbf{L}^2$. Rewriting the operator B with similar arguments as in *Step 1*, cf. (79), we can show that B is coercive using the same arguments as in *Step 2* and *Step 3*. Thus, B is bijective. It remains to show that B_n approximates B, that is $\lim_{n\to\infty}\|(B_n p_n - p_n B)\mathbf{u}\|_{\mathbb{X}_n} = 0$.

To this end, we note that it suffices to show that K_n approximates K, since we can use that $B_n = A_nT_n - K_n$, $B = AT - K$ and estimate

$$\begin{aligned}\|(B_np_n - p_nB)\mathbf{u}\|_{\mathbb{X}_n} &\leq \|(K_np_n - p_nK)\mathbf{u}\|_{\mathbb{X}_n} + \|(A_nT_np_n - p_nAT)\mathbf{u}\|_{\mathbb{X}_n}\\ &\leq \|(K_np_n - p_nK)\mathbf{u}\|_{\mathbb{X}_n} + \|(A_np_n - p_nA)T\mathbf{u}\|_{\mathbb{X}_n} + \|A_n\|_{L(\mathbb{X}_n)}\|(T_np_n - p_nT)\mathbf{u}\|_{\mathbb{X}}.\end{aligned}$$

By Theorem 13 and Lemma 18, A_n approximates A and T_n approximates T so the last two terms converge to zero. Thus, it indeed suffices to show that K_n approximates K to conclude that B_n approximates B.

Similar to the proof of Theorem 13, we choose $\mathbf{u}'_n \in \mathbb{X}_n$, $n \in \mathbb{N}$, $\|\mathbf{u}'_n\|_{\mathbb{X}_n} = 1$, such that $\|(K_np_n - p_nK)\mathbf{u}\|_{\mathbb{X}_n} \leq |\langle (K_np_n - p_nK)\mathbf{u}, \mathbf{u}'_n\rangle_{\mathbb{X}_n}| + 1/n$. For an arbitrary subsequence $\mathbb{N}' \subset \mathbb{N}$, we can choose $\mathbf{u}' \in \mathbb{X}$ and a subsequence $\mathbb{N}'' \subset \mathbb{N}$ such that $\mathbf{u}'_n \overset{\mathbf{L}^2}{\rightharpoonup} \mathbf{u}'$, $c_s^2\rho \operatorname{div}^n_{\boldsymbol{\nu}} \mathbf{u}'_n \overset{L^2}{\rightharpoonup} c_s^2\rho \operatorname{div} \mathbf{u}'$ and $\rho \mathbf{D}^n_{\mathbf{b}}\mathbf{u}'_n \overset{\mathbf{L}^2}{\rightharpoonup} \rho\partial_{\mathbf{b}}\mathbf{u}'$ due to Lemma 12.

We compute that

$$\begin{aligned}\langle p_nK\mathbf{u}, \mathbf{u}'_n\rangle_{\mathbb{X}_n} &\overset{(21)}{=} \langle \operatorname{div} K\mathbf{u}, c_s^2\rho \operatorname{div}^n_{\boldsymbol{\nu}} \mathbf{u}'_n\rangle + \langle K\mathbf{u}, \mathbf{u}'_n\rangle + \langle \partial_{\mathbf{b}}K\mathbf{u}, \rho\mathbf{D}^n_{\mathbf{b}}\mathbf{u}'_n\rangle\\ &\overset{n\in\mathbb{N}''}{\to} \langle c_s^2\rho \operatorname{div} K\mathbf{u}, \mathbf{u}'\rangle + \langle K\mathbf{u}, \mathbf{u}'\rangle + \langle \rho\partial_{\mathbf{b}}K\mathbf{u}, \partial_{\mathbf{b}}\mathbf{u}'\rangle = \langle K\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}}.\end{aligned}$$

Thus, we have to show that $\lim_{n\in\mathbb{N}''}\langle K_np_n\mathbf{u}, \mathbf{u}'_n\rangle_{\mathbb{X}_n} = -\langle K\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}}$. We recall that $K_n := -K_n^{(1)} - K_n^{(2)}$ and therefore

$$\begin{aligned}&\langle K_np_n\mathbf{u}, \mathbf{u}'_n\rangle_{\mathbb{X}_n}\\ &\quad= -(1 + C_2)\langle (P_{V_n}p_n\mathbf{u})_\tau, \mathbf{v}'_\tau\rangle - (C_1 + C_2)\langle S_np_n\mathbf{u}, S_n\mathbf{u}'_n\rangle \qquad (K_n\text{-a})\\ &\qquad - (1 + C_2)\langle c_s^2\rho\tilde{O}_np_n\mathbf{u}, \tilde{O}_n\mathbf{u}'_n\rangle - (1 + C_2)\langle c_s^2\rho M_n(\operatorname{id}_{\mathbb{X}_n} - P_{V_n})p_n\mathbf{u}, \mathbf{w}'_n\rangle \qquad (K_n\text{-b})\\ &\qquad - (1 + C_2)\langle \operatorname{mean}(\mathbf{q}\cdot(\operatorname{id}_{\mathbb{X}_n} - P_{V_n})(p_n\mathbf{u})_\tau), \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}'_\tau)\rangle. \qquad (K_n\text{-c})\end{aligned}$$

In the following, we show that (K_n-a) converges to (K-a) and (K_n-b) + (K_n-c) converges to (K-b).

Step 4a: Convergence of (K_n-a). To show that (K_n-a) converges to (K-a), we exploit the convergence of $p_n\mathbf{u}$. Using the approximation properties of π_n^d and the same argumentation as in Lemma 18, cf. especially (43), we have

$$\begin{aligned}&|\langle \mathbf{v}, \mathbf{v}'_\tau\rangle - \langle (P_{V_n}p_n\mathbf{u})_\tau, \mathbf{v}'_\tau\rangle| = |\langle (P_V - \pi_n^dS_np_n)\mathbf{u}, \mathbf{v}'_\tau\rangle|\\ &\quad\lesssim |\langle (P_V - \pi_n^dP_V)\mathbf{u}, \mathbf{v}'_\tau\rangle| + |\langle \pi_n^d(P_V - S_np_n)\mathbf{u}, \mathbf{v}'_\tau\rangle|\\ &\quad\lesssim h_n\|P_V\mathbf{u}\|_{\mathbf{H}^1} + d_n(\mathbf{u}, p_n\mathbf{u}) + \|(\operatorname{id}_{L^2_0} - \pi_n^l)P_{L^2_0}\mathbf{q}\cdot\mathbf{u}\|_{L^2} \overset{n\in\mathbb{N}''}{\to} 0,\\ \text{and}\quad &|\langle \mathbf{v}, S_n\mathbf{u}'_n\rangle - \langle S_np_n\mathbf{u}, S_n\mathbf{u}'_n\rangle| = |\langle (P_V - S_np_n)\mathbf{u}, S_n\mathbf{u}'_n\rangle|\\ &\quad\lesssim \|(P_V - S_np_n)\mathbf{u}\|_{\mathbf{L}^2} \lesssim d_n(\mathbf{u}, p_n\mathbf{u}) + \|(\operatorname{id}_{L^2_0} - \pi_n^l)P_{L^2_0}\mathbf{q}\cdot\mathbf{u}\|_{L^2} \overset{n\in\mathbb{N}''}{\to} 0,\end{aligned}$$

where we used Lemma 9 and the pointwise convergence of π_n^l to $\operatorname{id}_{L^2_0}$. Hence,

$$\lim_{n\in\mathbb{N}''}(K_n\text{-a}) = \lim_{n\in\mathbb{N}''}\left(-(1 + C_2)\langle \mathbf{v}, \mathbf{v}'_\tau\rangle - (C_1 + C_2)\langle \mathbf{v}, S_n\mathbf{u}'_n\rangle\right).$$

It remains to show that the right-hand side converges to (K-a), i.e. $\langle \mathbf{v}, S_n \mathbf{u}'_n\rangle \overset{n\in\mathbb{N}''}{\to} \langle \mathbf{v}, \mathbf{v}'\rangle$. Let $S := \nabla((\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot + M)\nabla)^{-1}$ such that

$$\begin{aligned}
&\langle \mathbf{v}, S_n \mathbf{u}'_n\rangle - \langle \mathbf{v}, \mathbf{v}'\rangle = \langle \mathbf{v}, S_n \mathbf{u}'_n - P_V \mathbf{u}'\rangle \\
&= \langle \mathbf{v}, S((\operatorname{div}^n_{\boldsymbol{\nu}} + \pi^l_n P_{L^2_0}\mathbf{q}\cdot + M_n)\mathbf{u}'_n - (\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot + M)\mathbf{u}')\rangle \\
&= \langle S^*\mathbf{v}, \operatorname{div}^n_{\boldsymbol{\nu}} \mathbf{u}'_n - \operatorname{div}\mathbf{u}'\rangle + \langle S^*\mathbf{v}, \pi^l_n P_{L^2_0}\mathbf{q}\cdot(\mathbf{u}'_n)_\tau - P_{L^2_0}\mathbf{q}\cdot\mathbf{u}'\rangle + \langle S^*\mathbf{v}, M_n\mathbf{u}'_n - M\mathbf{u}'\rangle.
\end{aligned}$$

By choice of the subsequence $\mathbb{N}'' \subset \mathbb{N}$ and Lemma 12, we have that $c_s^2\rho \operatorname{div}^n_{\boldsymbol{\nu}} \mathbf{u}'_n \overset{L^2}{\rightharpoonup} c_s^2\rho\operatorname{div}\mathbf{u}'$ (and hence $\operatorname{div}^n_{\boldsymbol{\nu}} \mathbf{u}'_n \overset{L^2}{\rightharpoonup} \operatorname{div}\mathbf{u}'$) and therefore $M_n\mathbf{u}'_n \to M\mathbf{u}'$ by construction of M_n and M. Therefore, the first and the last term converge to zero as $S^*\mathbf{v} \in L^2$. Furthermore, we have that

$$\begin{aligned}
&\langle S^*\mathbf{v}, \pi^l_n P_{L^2_0}\mathbf{q}\cdot(\mathbf{u}'_n)_\tau - P_{L^2_0}\mathbf{q}\cdot\mathbf{u}'\rangle \\
=&\langle S^*\mathbf{v}, (\pi^l_n - \operatorname{id})P_{L^2_0}\mathbf{q}\cdot\mathbf{u}\rangle + \langle S^*\mathbf{v}, \pi^l_n P_{L^2_0}\mathbf{q}\cdot((\mathbf{u}'_n)_\tau - \mathbf{u}')\rangle
\end{aligned}$$

The first term converges to zero due to the pointwise convergence of π^l_n to id. For the second term, we first notice that π^l_n is uniformly bounded and by Lemma 12 $(\mathbf{u}'_n)_\tau \overset{\mathbf{L}^2}{\rightharpoonup} \mathbf{u}'$. Therefore $P_{L^2_0}\mathbf{q}\cdot\mathbf{u}'_n \overset{\mathbf{L}^2}{\rightharpoonup} P_{L^2_0}\mathbf{q}\cdot\mathbf{u}'$ because the compact operator $P_{L^2_0}\mathbf{q}\cdot$ maps weakly convergent sequences onto weakly convergent sequences. We conclude $\langle \mathbf{v}, S_n\mathbf{u}'_n\rangle \overset{n\in\mathbb{N}''}{\to} \langle \mathbf{v}, \mathbf{v}'\rangle$ and hence $\lim_{n\in\mathbb{N}''}$ (K_n-a) $=$ (K-a).

Step 4b: Convergence of (K_n-b) *&* (K_n-c) *to* (K-b). For (K_n-b), we first note that $\lim_{n\to\infty} \|\tilde{O}_n\|_{L(\mathbb{X}_n, L^2_0)} = 0$ due to the arguments in Lemma 16 so that we only have to consider the second term. With similar arguments as in the proof of Lemma 18 we have (recall that $\mathbf{w} = (\operatorname{id} - P_V)\mathbf{u}$)

$$\begin{aligned}
&|\langle M\mathbf{w}, M_n\mathbf{w}'_n\rangle - \langle M_n(\operatorname{id} - P_{V_n})(p_n\mathbf{u}), M_n\mathbf{w}'_n\rangle| \\
&\lesssim \|\operatorname{div}(\mathbf{u} - P_V\mathbf{u}) - \operatorname{div}^n_{\boldsymbol{\nu}}(p_n\mathbf{u} - P_{V_n}p_n\mathbf{u})\|_{L^2} \lesssim d_n(\mathbf{u}, p_n\mathbf{u}) + d_n(P_V\mathbf{u}, P_{V_n}p_n\mathbf{u}),
\end{aligned}$$

where the first term converges to zero due to Lemma 9 and the second term converges to zero with the same argumentation as in the proof of Lemma 18, cf. (42). Applying Lemma 12 yields $\lim_{n\in\mathbb{N}''}$ (K_n-b) $= \lim_{n\in\mathbb{N}''}(1 + C_2)\langle M\mathbf{w}, M_n\mathbf{w}'_n\rangle = (1 + C_2)\langle M\mathbf{w}, M\mathbf{w}'\rangle$.

Finally, we consider (K_n-c) and calculate that

$$\begin{aligned}
&|\langle \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}), \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}'_\tau)\rangle - \langle \operatorname{mean}(\mathbf{q}\cdot((\operatorname{id} - P_{V_n})(p_n\mathbf{u}))_\tau), \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}'_\tau)\rangle| \\
&\lesssim \|\mathbf{u} - P_V\mathbf{u} - (p_n\mathbf{u} - P_{V_n}p_n\mathbf{u})_\tau\|_{\mathbf{L}^2} \\
&\lesssim \|p_nP_V\mathbf{u} - (P_{V_n}p_n\mathbf{u})_\tau\|_{\mathbf{L}^2} + d_n(\mathbf{u}, p_n\mathbf{u}) + d_n(P_V\mathbf{u}, p_nP_{V_n}\mathbf{u}) \overset{n\in\mathbb{N}''}{\to} 0,
\end{aligned}$$

where we again apply Lemma 9 and Lemma 18.

Altogether, we obtain that $\lim_{n\in\mathbb{N}''}$ (K_n-a) $=$ (K-a) and $\lim_{n\in\mathbb{N}''}$((K_n-b) $+$ (K_n-c)) $=$ (K-b) and thus K_n approximates K. Due to the argumentation above, we conclude that B_n approximates B which finishes the proof.

To summarize, in *Step 1* we have defined the decomposition $A_nT_n = B_n + K_n$, where the sequence $(K_n)_{n\in\mathbb{N}}$ is compact. In *Step 2* and *Step 3*, we have shown that the sequence $(B_n)_{n\in\mathbb{N}}$ is uniformly coercive and therefore stable. Finally, in *Step 4*, we have shown that there exists a bijective operator $B \in L(\mathbb{X})$ such that B_n approximates B. □

4.4. **Convergence estimates**

To conclude the analysis of the discrete problem, we show that the sequence of discrete solutions $(\mathbf{u}_n)_{n\in\mathbb{N}}$ converges to the solution of the continuous problem. Further, if we assume additional regularity for the continuous solution, we obtain convergence with optimal order.

Theorem 24. *Assume that Assumption 21 holds. For $\mathbf{f} \in \mathbf{L}^2$, let $\mathbf{u} \in \mathbb{X}$ be the solution to (8). Then, there exists an index $n_0 > 0$ such that for all $n > n_0$ the problem (17) has a unique solution $\mathbf{u}_n \in \mathbb{X}_n$ and* $\lim_{n\to\infty} d_n(\mathbf{u}, \mathbf{u}_n) = 0$.

Proof. In Theorem 13, Lemma 18, and Theorem 23, we have shown that the operators A, $(A_n)_{n\in\mathbb{N}}$, T, and $(T_n)_{n\in\mathbb{N}}$ fulfill the necessary conditions for the application of Theorem 4 to conclude that the sequence $(A_n)_{n\in\mathbb{N}}$ is regular. To be able to apply Lemma 3, which yields stability and convergence, we still have to show that the continuous right-hand side converges to the discrete right-hand side in the sense of discrete approximation schemes. Let $\boldsymbol{g} \in \mathbb{X}$ be such that $\langle \boldsymbol{g}, \mathbf{u}\rangle_{\mathbb{X}} = \langle \mathbf{f}, \mathbf{u}\rangle_{\mathbf{L}^2}$ for all $\mathbf{u} \in \mathbb{X}$ and $\boldsymbol{g}_n \in \mathbb{X}_n$ be such that $\langle \boldsymbol{g}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n} = \langle \mathbf{f}, \mathbf{u}_n\rangle_{\mathbf{L}^2}$ for all $\mathbf{u}_n \in \mathbb{X}_n$. Take $\mathbf{u}'_n \in \mathbb{X}_n$, $\|\mathbf{u}'_n\|_{\mathbb{X}_n} = 1$ such that $\|p_n\boldsymbol{g} - \boldsymbol{g}_n\|_{\mathbb{X}_n} \leq |\langle p_n\boldsymbol{g} - \boldsymbol{g}_n, \mathbf{u}'_n\rangle| + 1/n$ and for an arbitrary subsequence $\mathbb{N}' \subset \mathbb{N}$ choose $\mathbb{N}'' \subset \mathbb{N}'$ according to Lemma 12. Then, we have that with (21)

$$\begin{aligned}
&\langle p_n\boldsymbol{g} - \boldsymbol{g}_n, \mathbf{u}'_n\rangle_{\mathbb{X}_n} = \langle p_n\boldsymbol{g}, \mathbf{u}'_n\rangle_{\mathbb{X}_n} - \langle \mathbf{f}, \mathbf{u}'_n\rangle_{\mathbf{L}^2} \\
=&\langle c_s^2\rho \operatorname{div} \boldsymbol{g}, \operatorname{div}^n_{\boldsymbol{\nu}} \mathbf{u}'_n\rangle - \langle \boldsymbol{g}, \mathbf{u}'_n\rangle + \langle \rho\partial_{\mathbf{b}}\boldsymbol{g}, \mathbf{D}^n_{\mathbf{b}}\mathbf{u}'_n\rangle - \langle \mathbf{f}, \mathbf{u}'_n\rangle_{\mathbf{L}^2} \overset{n\in\mathbb{N}''}{\to} \langle \boldsymbol{g}, \mathbf{u}'\rangle_{\mathbb{X}} - \langle \mathbf{f}, \mathbf{u}'\rangle_{\mathbf{L}^2} = 0.
\end{aligned}$$

Thus, we can apply Lemma 3 to conclude that the sequence $(A_n)_{n\in\mathbb{N}}$ is stable, i.e. there exists an index n_0 such that A_n^{-1} exists and is bounded for all $n > n_0$ and problem (17) has a unique solution for all $n > n_0$. Furthermore, it holds that $\lim_{n\to\infty} \|p_n\mathbf{u} - \mathbf{u}_n\|_{\mathbb{X}_n} = 0$. We estimate with Lemma 7

$$d_n(\mathbf{u}, \mathbf{u}_n) \leq d_n(\mathbf{u}, p_n\mathbf{u}) + \|p_n\mathbf{u} - \mathbf{u}_n\|_{\mathbb{X}_n} \leq d_n(\mathbf{u}, \underline{\pi}_n\mathbf{u}) + \|p_n\mathbf{u} - \mathbf{u}_n\|_{\mathbb{X}_n}, \tag{68}$$

and apply Lemma 10 to conclude that $\lim_{n\to\infty} d_n(\mathbf{u}, p_n\mathbf{u}) = 0$. □

Theorem 25. *Let the assumptions from Theorem 24 be satisfied. If additionally $\mathbf{u} \in \mathbb{X} \cap \mathbf{H}^{2+s}$, $s > 0$, $\rho \in W^{1+s,\infty}$, and $\mathbf{b} \in \mathbf{W}^{1+s,\infty}$, then there exists a constant $C > 0$ independent of n such that*

$$d_n(\mathbf{u}, \mathbf{u}_n) \leq Ch_n^{\min\{s,k,l\}}\|\mathbf{u}\|_{\mathbf{H}^{2+s}}.$$

Proof. To show the convergence rate, we continue to estimate (68). For the first term, Lemma 8 yields that $d_n(\mathbf{u}, \underline{\pi}_n\mathbf{u}) \lesssim h_n^{\min\{1+s,k\}}$. For the second term, we note that

$$\|p_n\mathbf{u} - \mathbf{u}_n\|_{\mathbb{X}_n} \leq (\sup_{n>n_0} \|A_n^{-1}\|_{L(\mathbb{X}_n)})\|A_n(p_n\mathbf{u} - \mathbf{u}_n)\|_{\mathbb{X}_n}$$

and compute with similar arguments as in Theorem 13 that

$$\begin{aligned}
\|A_n(p_n\mathbf{u} - \mathbf{u}_n)\|_{\mathbb{X}_n} &= \sup_{\mathbf{u}'_n\in\mathbb{X}_n\|\mathbf{u}'_n\|_{\mathbb{X}_n}=1} |a_n(p_n\mathbf{u} - \mathbf{u}_n, \mathbf{u}'_n)| \\
&\leq Cd_n(\mathbf{u}, p_n\mathbf{u}) + \sup_{\mathbf{u}'_n\in\mathbb{X}_n\|\mathbf{u}'_n\|_{\mathbb{X}_n}=1} |\langle c_s^2\rho \operatorname{div} \mathbf{u}, \operatorname{div}^n_{\boldsymbol{\nu}} \mathbf{u}'_n\rangle \\
&\quad - \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, (\omega + i\mathbf{D}^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}'_n\rangle \\
&\quad + \langle \operatorname{div} \mathbf{u}, \nabla p \cdot \mathbf{u}'_n\rangle + \langle \nabla p \cdot \mathbf{u}, \operatorname{div}^n_{\boldsymbol{\nu}} \mathbf{u}'_n\rangle + \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\mathbf{u}, \mathbf{u}'_n\rangle \\
&\quad - i\omega\langle \gamma\rho\mathbf{u}, \mathbf{u}'_n\rangle - \langle \mathbf{f}, \mathbf{u}'_n\rangle|.
\end{aligned}$$

For the first term, we again use the estimates from Lemma 7 and Lemma 8. For the remainder, we want to integrate by parts and use the fact that $\mathbf{u}$ solves (1). This requires that $\mathbf{u} \in \mathbf{H}^2$ is regular enough, because a right hand-side $\mathbf{f} \in \mathbf{L}^2$ only grants $-\nabla(c_s^2\rho \operatorname{div} \mathbf{u}) + \rho\partial_{\mathbf{b}}\partial_{\mathbf{b}}\mathbf{u}$ which however does not imply that $-\nabla(c_s^2\rho \operatorname{div} \mathbf{u}) \in \mathbf{L}^2$ or $\rho\partial_{\mathbf{b}}\partial_{\mathbf{b}}\mathbf{u} \in \mathbf{L}^2$.

Let $\boldsymbol{\psi}_n \in [\mathbb{P}^l(\mathcal{T}_n)]^d$ be a suitable H^1-projection of $(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}$, for example as in [16]. Then, we have that

$$\langle(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, \rho\mathbf{D}_{\mathbf{b}}^n\mathbf{u}_n'\rangle = \langle\boldsymbol{\psi}_n, \rho\mathbf{D}_{\mathbf{b}}^n\mathbf{u}_n'\rangle + \langle(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u} - \boldsymbol{\psi}_n, \rho\mathbf{D}_{\mathbf{b}}^n\mathbf{u}_n'\rangle,$$

and similar to (30) we calculate with the definition (12a) and $\operatorname{div}(\rho\mathbf{b}) = 0$ that

$$\begin{aligned}
\langle\boldsymbol{\psi}_{\boldsymbol{n}}, \rho\mathbf{D}_{\mathbf{b}}^n\mathbf{u}_n'\rangle &= \langle\boldsymbol{\psi}_n, \rho\partial_{\mathbf{b}}\mathbf{u}_\tau'\rangle_{\mathcal{T}_n} + \langle\boldsymbol{\psi}_n, \rho\mathbf{R}^l\mathbf{u}_n'\rangle_{\mathcal{T}_n} \\
&= -\langle\rho\partial_{\mathbf{b}}\boldsymbol{\psi}_n, \mathbf{u}_\tau'\rangle_{\mathcal{T}_n} + \langle\boldsymbol{\psi}_n, \rho[\![\underline{\mathbf{u}_n'}]\!]_{\mathbf{b}}\rangle_{\partial\mathcal{T}_n} + \langle\boldsymbol{\psi}_n, \rho\mathbf{R}^l\mathbf{u}_n'\rangle_{\mathcal{T}_n} \\
&= -\langle\rho\partial_{\mathbf{b}}(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, \mathbf{u}_\tau'\rangle_{\mathcal{T}_n} + \langle\rho\partial_{\mathbf{b}}((\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u} - \boldsymbol{\psi}_n), \mathbf{u}_\tau'\rangle_{\mathcal{T}_n}.
\end{aligned}$$

With similar techniques, cf. [45, Thm. 6.26], we obtain for $\psi_n, \tilde{\psi}_n \in \mathbb{P}^l(\mathcal{T}_n)$ being suitable H^1-projections of $c_s^2\rho \operatorname{div} \mathbf{u}$ and $\nabla p \cdot \mathbf{u}$ that

$$\begin{aligned}
\langle c_s^2\rho \operatorname{div} \mathbf{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n'\rangle = &- \langle\nabla(c_s^2\rho \operatorname{div} \mathbf{u}), \mathbf{u}_n'\rangle + \langle\nabla(c_s^2\rho \operatorname{div} \mathbf{u} - \psi_n), \mathbf{u}_n'\rangle \\
&+ \langle c_s^2\rho \operatorname{div} \mathbf{u} - \psi_n, \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n'\rangle, \\
\langle\nabla p \cdot \mathbf{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n'\rangle = &- \langle\nabla(\nabla p \cdot \mathbf{u}), \mathbf{u}_n'\rangle + \langle\nabla(\nabla p \cdot \mathbf{u} - \tilde{\psi}_n), \mathbf{u}_n'\rangle \\
&+ \langle\nabla p \cdot \mathbf{u} - \tilde{\psi}_n, \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n'\rangle.
\end{aligned}$$

Altogether, we obtain that

$$\begin{aligned}
\sup_{\mathbf{u}_n' \in \mathbb{X}_n \|\mathbf{u}_n'\|_{\mathbb{X}_n} = 1} &|\langle c_s^2\rho \operatorname{div} \mathbf{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n'\rangle - \langle\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, (\omega + i\mathbf{D}_{\mathbf{b}}^n + i\Omega\times)\mathbf{u}_n'\rangle \\
&+ \langle\operatorname{div} \mathbf{u}, \nabla p \cdot \mathbf{u}_n'\rangle + \langle\nabla p \cdot \mathbf{u}, \operatorname{div}_{\boldsymbol{\nu}}^n \mathbf{u}_n'\rangle + \langle(\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\mathbf{u}, \mathbf{u}_n'\rangle \\
&- i\omega\langle\gamma\rho\mathbf{u}, \mathbf{u}_n'\rangle - \langle\mathbf{f}, \mathbf{u}_n'\rangle| \\
&\lesssim \|\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u} - \boldsymbol{\psi}_{\boldsymbol{n}}\|_{\mathbf{H}^1} + \|c_s^2\rho \operatorname{div} \mathbf{u} - \psi_n\|_{H^1} \\
&+ \|\nabla p \cdot \mathbf{u} - \tilde{\psi}_n\|_{H^1} \\
&\lesssim h^{\min\{l,s\}}.
\end{aligned}$$

Combining all estimates, we obtain the desired result. □

Remark 26 (The case $l = k - 1$)**.** While the convergence estimate in Theorem 25 suggest the choice $l = k$ for the degree of the lifting operator $\mathbf{R}^l$, the numerical experiments in Section 5.3 suggest that the choice $l = k - 1$ (together with a reduced facet space $\mathbb{X}_{\mathcal{F}_n} = \mathbb{P}^{k-1}(\mathcal{F}_n)$) might be sufficient to obtain optimal convergence rates. Thus, it might be possible to improve the results presented in Theorem 25 to obtain the estimate $d_n(\mathbf{u}, \mathbf{u}_n) \lesssim h_n^{\min\{s,k,l+1\}}$.

5. Numerical Experiments

In this section, we study the discretization of Galbrun's equation with HDG methods numerically. After some preliminary discussions in Section 5.1 and implementational aspects in Section 5.2, we study the convergence of different HDG methods towards an exact solution in Section 5.3. The main goal is to verify the convergence rates obtained in Theorem 25 numerically. In Section 5.4, we investigate the influence of the Mach number on the discretization error. Afterwards, we compare the proposed discretization of the convection term through lifting operators with a naive SIP discretization in Section 5.5. To conclude, we consider numerical examples with physically relevant coefficients from the Sun in Section 5.6. For the implementation we use the finite element software `NGSolve` [41, 42]. Replication data is available in [30].

5.1. **Preliminaries**

In the numerical examples presented below, we want to compare different HDG discretizations. The most natural choices are a *fully nonconforming HDG* method or an $H(\mathrm{div})$*-conforming HDG* method, cf. Remark 5. However, choosing polynomials of order k for the facet unknowns might not be optimal in terms of computational efficiency. For different problems, optimal convergence rates have been obtained with facet unknowns of only order $k-1$, i.e. one order reduced compared to the volume unknowns. This can be achieved by involving only the L^2-projection on polynomials of degree $k-1$ for the hybrid DG jump operator on the facets. For a SIP discretization this has been achieved through the *projected jumps* modification in [36].

For our proposed discretizations, reducing the lifting order to $l = k-1$ implicitly includes a projection of the facet jumps onto the desired space as well and we can therefore also reduce the order of the facet unknowns to $k-1$ without further modifications. In the following, we call the simultaneous reduction of the facet and lifting degree to $k-1$ a *reduced* method, e.g. a *reduced fully non-conforming HDG* and a *reduced* $H(\mathrm{div})$*-conforming HDG* method. To obtain this improved efficiency also for the normal component, we also consider an *optimized HDG* method. Here the finite element space, which we denote by $\mathbb{BDM}_k^-$ as discussed in Remark 5, has so-called *relaxed* $H(\mathrm{div})$*-conformity.* In this case, we set $s_n(\cdot,\cdot) = 0$ to avoid an additional penalty on the highest order normal jump.

An overview of the different discretizations together with their associated costs is given in Table 1. The analysis from Section 4.4 yields optimal converges rates for the fully-nonconforming and the $H(\mathrm{div})$-conforming HDG method, but not for the reduced methods where we choose the lifting degree $l = k-1$.

HDG method	discrete spaces			associated costs		
	$\mathbb{X}_{\mathcal{T}_n}$	$\mathbb{X}_{\mathcal{F}_n}$	lifting	ndofs	ncdofs	nze
full	$[\mathbb{P}^k(\mathcal{T}_n)]^d$	$[\mathbb{P}^k(\mathcal{F}_n)]^d$	$[\mathbb{P}^k(\mathcal{T}_n)]^d$	124	20	784
red. full	$[\mathbb{P}^k(\mathcal{T}_n)]^d$	$[\mathbb{P}^{k-1}(\mathcal{F}_n)]^d$	$[\mathbb{P}^{k-1}(\mathcal{T}_n)]^d$	74	10	**196**
$H(\mathrm{div})$	$\mathbb{BDM}^k(\mathcal{T}_n)$	$[\mathbb{P}^{k,\mathrm{tang}}(\mathcal{F}_n)]^d$	$[\mathbb{P}^k(\mathcal{T}_n)]^d$	88	20	784
red. $H(\mathrm{div})$	$\mathbb{BDM}^k(\mathcal{T}_n)$	$[\mathbb{P}^{k-1,\mathrm{tang}}(\mathcal{F}_n)]^d$	$[\mathbb{P}^{k-1}(\mathcal{T}_n)]^d$	51	15	441
optimized	$\mathbb{BDM}_k^-(\mathcal{T}_n)$	$[\mathbb{P}^{k-1,\mathrm{tang}}(\mathcal{F}_n)]^d$	$[\mathbb{P}^{k-1}(\mathcal{T}_n)]^d$	56	10	**196**

TABLE 1. We compare different HDG methods in terms of the associated computational costs measured through the number of degrees of freedom (ndofs), the number of coupling degrees of freedom (ncdofs), and the number of non-zero entries in the system matrices (nze) for polynomial degree $k = 1$ and a mesh with 6 elements. The *red. full* and the optimzed method have significantly fewer nzes than the *full HDG* method. This reduction becomes less pronounced for higher polynomial degree.

For the experiments carried out below, we consider background flows of the following form

$$\mathbf{b}_\eta \coloneqq \eta c_{\mathbf{b}} \begin{pmatrix} -y \\ x \end{pmatrix}, \tag{69}$$

where $\eta \in W^{1,\infty}$ is chosen in the specific experiment and the parameter $c_{\mathbf{b}} \in \mathbb{R}$ controls the Mach number of the flow. For all experiments, we restrict ourselves to the case where the gravitational potential ϕ is constant.

5.2. **Computational aspects**

In the next two remarks, we briefly address computational aspects of the implementation of the lifting.

Remark 27 (Implementation of the lifting operator)**.** In practice, we implement the lifting operator $\mathbf{R}^l$ through a mixed formulation adding an auxiliary equation and variable. For all $\mathbf{u}_n, \mathbf{u}'_n \in \mathbb{X}_n$, we need to form

$$\begin{aligned}\langle \rho \mathbf{D}^n_{\mathbf{b}} \mathbf{u}_n, \mathbf{D}^n_{\mathbf{b}} \mathbf{u}'_n \rangle_{\mathcal{T}_n} =& \langle \rho \partial_{\mathbf{b}} \mathbf{u}_n, \partial_{\mathbf{b}} \mathbf{u}'_n \rangle_{\mathcal{T}_n} + \langle \rho \mathbf{R}^l \mathbf{u}_n, \partial_{\mathbf{b}} \mathbf{u}'_n \rangle_{\mathcal{T}_n} + \langle \rho \partial_{\mathbf{b}} \mathbf{u}_n, \mathbf{R}^l \mathbf{u}'_n \rangle_{\mathcal{T}_n} \\ &+ \langle \rho \mathbf{R}^l \mathbf{u}_n, \mathbf{R}^l \mathbf{u}'_n \rangle_{\mathcal{T}_n}.\end{aligned} \tag{70}$$

For the mixed terms, we obtain by definition (12a) that

$$\langle \rho \mathbf{R}^l \mathbf{u}_n, \partial_{\mathbf{b}} \mathbf{u}'_n \rangle_{\mathcal{T}_n} + \langle \rho \partial_{\mathbf{b}} \mathbf{u}_n, \mathbf{R}^l \mathbf{u}'_n \rangle_{\mathcal{T}_n} = -\langle \rho [\![\underline{\mathbf{u}_n}]\!]_{\mathbf{b}}, \partial_{\mathbf{b}} \mathbf{u}'_n \rangle_{\partial \mathcal{T}_n} - \langle \rho \partial_{\mathbf{b}} \mathbf{u}_n, [\![\underline{\mathbf{u}'_n}]\!]_{\mathbf{b}} \rangle_{\partial \mathcal{T}_n}. \tag{71}$$

For the remaining term, we introduce an auxiliary variable $\boldsymbol{r} = \mathbf{R}^l \mathbf{u}_n \in [\mathbb{P}^l(\mathcal{T}_n)]^d$ with the defining that $\boldsymbol{r}$ fulfills $\langle \rho \boldsymbol{r}, \boldsymbol{s} \rangle_{\mathcal{T}_n} = -\langle \rho [\![\underline{\mathbf{u}_n}]\!]_{\mathbf{b}}, \boldsymbol{s} \rangle_{\partial \mathcal{T}_n}$ for all $\boldsymbol{s} \in [\mathbb{P}^l(\mathcal{T}_n)]^d$. Then, we have

$$\langle \rho \mathbf{R}^l \mathbf{u}_n, \mathbf{R}^l \mathbf{u}'_n \rangle_{\mathcal{T}_n} = \langle \rho \boldsymbol{r}, \mathbf{R}^l \mathbf{u}'_n \rangle_{\mathcal{T}_n} = \overline{\langle \rho \mathbf{R}^l \mathbf{u}'_n, \boldsymbol{r} \rangle_{\mathcal{T}_n}} = -\langle \rho \boldsymbol{r}, [\![\underline{\mathbf{u}'_n}]\!]_{\mathbf{b}} \rangle_{\partial \mathcal{T}_n}, \tag{72}$$

Thus, we can implement the term (70) through a mixed formulation with the auxiliary variable $\boldsymbol{r}$ (and corresponding test function $\boldsymbol{s}$). As discussed in Section 3.2 the scalar lifting operator R^l is only introduced for notational convenience *in the analysis* and by definition of the stabilization term s_n, we are considering a SIP method for the diffusion term. Thus, we do not have to implement the scalar lifting operator R^l explicitly.

Remark 28 (Computational costs associated with the lifting operator)**.** In a DG setting, the implementation of lifting operators is usually associated with higher computational costs in the resulting linear systems because the lifting operator introduces new, less local couplings compared to classical SIP operators. In contrast, the support of the HDG-lifting operator is local since the volume unknowns only couple through the facet unknowns indirectly. Note that especially $\boldsymbol{r}$ and $\boldsymbol{s}$ in Remark 27 only occur locally on each element and can be eliminated locally. Thus, in an HDG setting, the implementation of the lifting operators leads to similar computational costs than the implementation of a corresponding SIP variant. To visualize the associated computational costs, we consider the sparsity pattern of the respective system matrices in Fig. 2.

5.3. Convergence studies

We consider the unit disk $\mathcal{O} = \{x \in \mathbb{R}^2 : \|x\| < 1\}$ and choose the parameters

$$\rho = \sqrt{10/\pi} \exp(-10(x^2 + y^2)), \quad c_s^2 = 1.44 + 0.25\rho, \quad \omega = 0.78 \times 2\pi, \tag{73a}$$

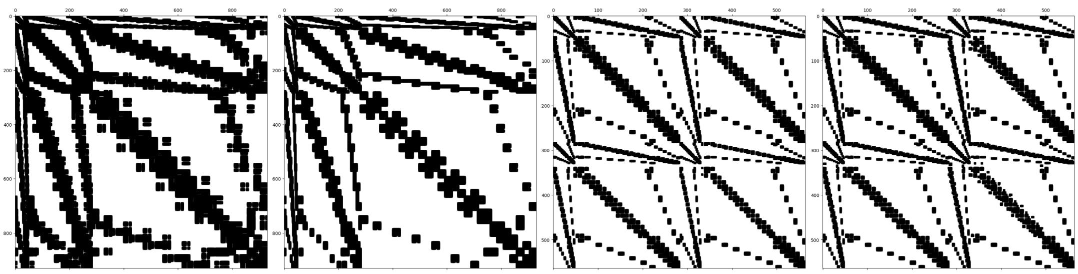
$$\gamma = 0.1, \quad \Omega = (0, 0), \quad p = 1.44\rho + 0.08\rho^2. \tag{73b}$$

While these parameters are artificially chosen, the density and the sound speed are modeled to behave similarly (though less extreme) than the respective parameters in the Sun. We consider the background flow $\mathbf{b}_{c_s}$ given by (69) and note that $\operatorname{div}(\rho \mathbf{b}_{c_s}) = 0$, $\mathbf{b}_{c_s} \cdot \boldsymbol{\nu} = 0$ and $\|c_s^{-1} \mathbf{b}_{c_s}\|_{\mathbf{L}^\infty} = c_{\mathbf{b}}$. The source term $\mathbf{f}$ is calculated such that the exact solution is given by

$$\mathbf{u}_{\mathrm{ex}} = \frac{1}{\rho} \sin(x^2 + y^2) \sin((x^2 + y^2) - 1) (+(1 + i)g - (1 + i)g \tag{74}$$

where $g = \sqrt{\alpha/\pi} \exp(-\alpha(x^2 + y^2))$, $\alpha = \log(10^9)$, is a Gaussian.

In Fig. 3, we compare the discretization error in the $\|\cdot\|_{\mathbb{X}(\mathcal{T}_n)}$-norm of the methods from Table 1. As expected by Theorem 25, the fully non-conforming and $H(\mathrm{div})$-conforming HDG methods converge with optimal order. Additionally, the reduced and the optimized methods converge with optimal order as well, even though these cases are not covered by Theorem 25. The absolute error of the optimized HDG method is larger than the error of the other methods by a constant factor while the degrees of freedom are reduced significantly[3]. For the fully non-conforming methods, we observe that the stabilization parameter $\alpha = 100$ seems to be more robust than



$H(\text{div})$-DG+Lift | $H(\text{div})$-DG+SIP | $H(\text{div})$-HDG+Lift | $H(\text{div})$-HDG+SIP

`nzes:` $235,036$ | `nzes:` $133,652$ | `nzes:` $30,096$ | `nzes:` $30,096$

Figure 2. We compare the sparsity pattern of the stiffness matrix obtained with the following four methods: $H(\text{div})$-conforming DG with lifting stabilization (left), $H(\text{div})$-conforming DG with SIP (middle-left), $H(\text{div})$-conforming HDG with lifting stabilization (middle-right), and $H(\text{div})$-conforming HDG with SIP (right). For the HDG methods, we use static condensation and for the DG method with the lifting operator, we apply the Schur complement to eliminate the unknowns associated with the lifting operator. In the HDG setting, both methods lead to the same number of non-zero entries (`nzes`) (even though the couplings differ slightly), whereas in the DG setting, the lifting operator almost doubles the number of non-zero entries. For the computations, we chose a mesh with 27 elements and the polynomial degree $k = 5$.

$\alpha = 10$, where we observe a longer preasymptotic phase. Overall, the numerical results confirm the theoretical results from Theorem 25, and suggest that the dependence of the convergence order on the lifting degree l might be improved, cf. Remark 26.

5.4. Mach number robustness

As formalized in Assumption 21, the stability of the method depends on the Mach number of the background flow. Here, we want to study the influence of the Mach number on the error of the discretization. In particular, we want to compare the methods considered in this manuscript with the $\mathbf{H}^1$-conforming discretization introduced in [29], and therefore the assumptions on the Mach number from (54) and (57).

We consider the parameters given by (73), but choose the right-hand side independent of the background flow:

$$\mathbf{f}(x, y) := \frac{1}{2}\sqrt{55/\pi}\,\exp(-55((x - 0.35)^2 + (y - 0.35)^2)) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{75}$$

To measure the discretization error, we calculate a reference solution on a fine mesh ($h \sim 0.5^5$) with high polynomial degree ($k = 7$) with the $H(\text{div})$-conforming HDG method, cf. Table 1. Then, we solve the problem on a coarser mesh ($h \sim 0.5^4$) with polynomial degree $k = 5$ using the $H(\text{div})$-conforming HDG and the $\mathbf{H}^1$-conforming discretization. To compare the resulting discretization error with the approximation quality of the discrete space, we calculate the best-approximation of the reference solution with respect to the $\mathbb{X}(\mathcal{T}_n)$-inner product, i.e. we compute $\Pi^{\mathbb{X}_n}\mathbf{u}_{\text{ref}} \in \mathbb{X}_n$ such that $\langle \Pi^{\mathbb{X}_n}\mathbf{u}_{\text{ref}}, \mathbf{v}_n\rangle_{\mathbb{X}(\mathcal{T}_n)} = \langle \mathbf{u}, \mathbf{v}_n\rangle_{\mathbb{X}(\mathcal{T}_n)}$ for all $\mathbf{v}_n \in \mathbb{X}_n$.

The results for three different background flows $\mathbf{b}_\eta$, $\eta \in \{1, c_s, c_s/\rho\}$, modeled by (69) are displayed in Fig. 4. For the flows $\mathbf{b}_1$ and $\mathbf{b}_{c_s}$, we observe that the discretization error of the $H(\text{div})$-conforming HDG method is close to the best-approximation error until $\|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty} \approx 1.0$. In contrast, the discretization error of the $\mathbf{H}^1$-conforming method starts to deviate from the best-approximation error as soon as the Mach number approaches

[3] For $k = 4$ on the finest mesh level, we have the following number of non-zero matrix entries for the system matrix: $\sim 5 \cdot 10^6$ (full HDG & $H(\text{div})$-conforming HDG), $\sim 4 \cdot 10^6$ (red. $H(\text{div})$-conforming HDG), $\sim 3 \cdot 10^6$ (optimized HDG).

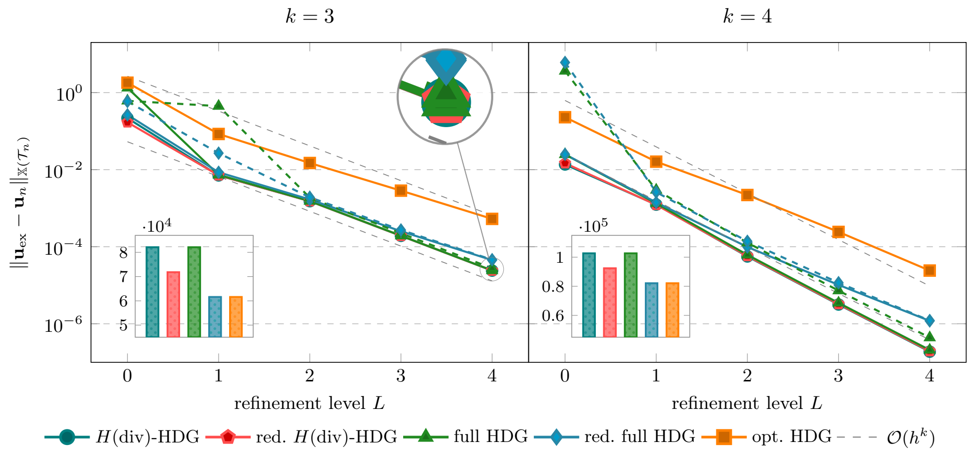


FIGURE 3. Convergence of the methods listed in Table 1 against (74) for polynomial degrees $k = 3$ and $k = 4$ with Mach number $\|c_s^{-1}\mathbf{b}_{c_s}\|^2_{\mathbf{L}^\infty} = 0.25$. For the fully non-conforming methods, we consider the choices $\alpha \in \{10k^2, 100k^2\}$ (dashed, solid). The error is measured in the $\mathbb{X}(\mathcal{T}_n)$-norm. In the embedded bar charts we show the number of coupled degrees of freedom for each method at the last refinement level $L = 4$.

0.3. Thus, the $H(\text{div})$-conforming method seems to be more robust with respect to the Mach number than the $\mathbf{H}^1$-conforming method.

For the background flow $\mathbf{b}_{c_s/\rho}$, the methods produce much more similar results and in particular, the error does not increase drastically once the Mach number exceeds 1.0.

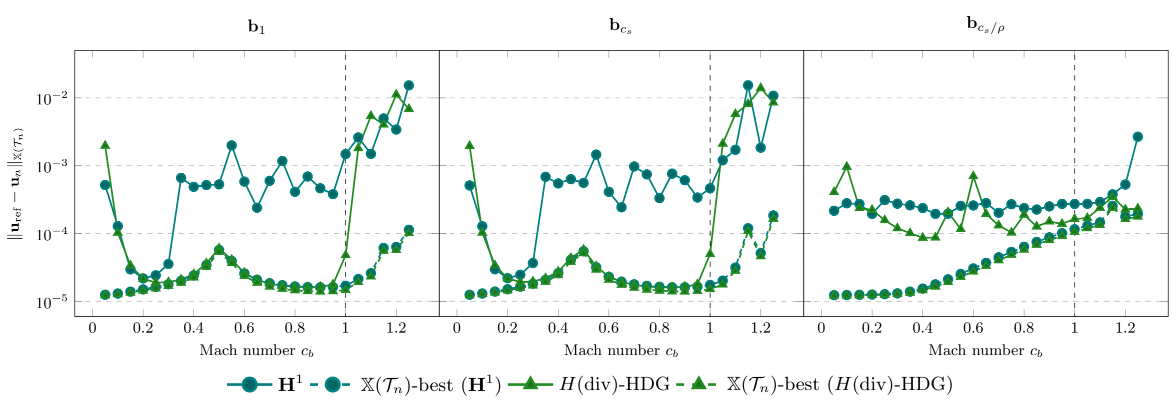


FIGURE 4. $\mathbb{X}(\mathcal{T}_n)$-error of the $\mathbf{H}^1$-conforming discretization, the $H(\text{div})$-conforming HDG discretization and the respective best-approximation error for $\mathbf{b}_\eta$, $\eta \in \{1, c_s, c_s/\rho\}$, for increasing Mach numbers from 0.05 to 1.25.

5.5. Comparison with SIP

The lifting operator $\mathbf{R}^l$ allows us to stabilize the directional derivative $\partial_{\mathbf{b}}$ without balancing a stabilization parameter against the Mach number of the background flow $\mathbf{b}$. In this section, we compare the proposed method with a SIP version to assess its practical relevance. To avoid balancing two stabilization parameters against each other, we only consider the $H(\mathrm{div})$-conforming HDG method. For the SIP version, we simply replace the term

$$-\langle \rho(\omega + i\mathbf{D}^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}_n, (\omega + i\mathbf{D}^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}'_n\rangle_{\mathcal{T}_n}$$

by

$$\begin{aligned} -\langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}_\tau, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}'_\tau\rangle_{\mathcal{T}_n} - i\langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}_\tau, [\![\underline{\mathbf{u}'_n}]\!]_{\mathbf{b}}\rangle_{\partial\mathcal{T}_n} \\ - i\langle [\![\underline{\mathbf{u}_n}]\!]_{\mathbf{b}}, \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}'_\tau\rangle_{\partial\mathcal{T}_n} + \langle \rho\lambda\mathfrak{h}^{-1}[\![\underline{\mathbf{u}_n}]\!]_{\mathbf{b}}, [\![\underline{\mathbf{u}'_n}]\!]_{\mathbf{b}}\rangle_{\partial\mathcal{T}_n}, \end{aligned}$$

where $\lambda > 0$ is a stabilization parameter that has to be chosen sufficiently large to ensure stability. We choose the same examples as considered in Section 5.3, where the parameters are given by (73) and the reference solution by (74). In Fig. 5, we compare the discretization error of the lifting stabilized method with the SIP method for stabilization parameters $\lambda \in \{1k^2, 10k^2, 100k^2\}$ and polynomial degree $k = 5$. We choose the background flow $\mathbf{b}_{c_s}$ and consider the Mach numbers $\|c_s\mathbf{b}\|_{\mathbf{L}^\infty} \in \{0.01, 0.45\}$.

The lifting stabilized discretization seems to be more stable and the error is (significantly) smaller. In particular, the choice of a suitable SIP stabilization parameter λ seems to depend on the Mach number. It's also worth mentioning that the condition number of the system matrix grows with the stabilization parameter λ in the SIP version, which is not the case for the lifting stabilized version. Altogether, we conclude that the lifting stabilized version is more robust and reliable than the SIP version, in particular because the computational costs are not significantly higher, cf. Remark 28.

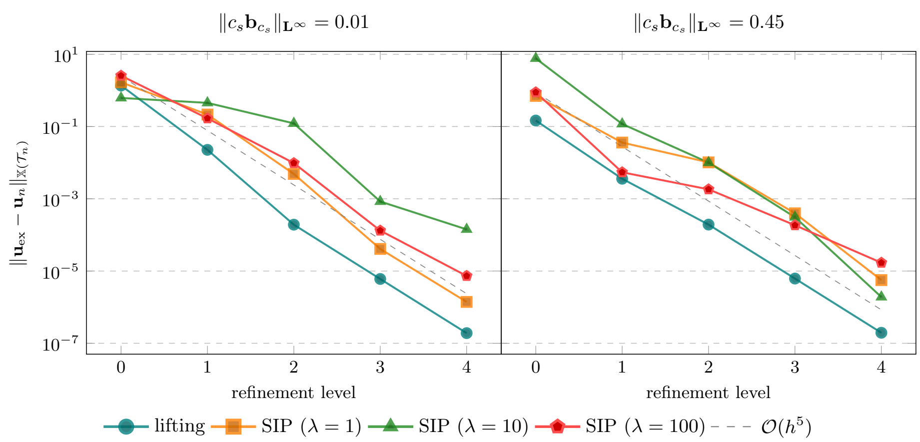


FIGURE 5. Discretization error measured in the $\|\cdot\|_{\mathbb{X}(\mathcal{T}_n)}$-norm for the lifting stabilized method and the SIP variant with $\lambda \in \{1k^2, 10k^2, 100k^2\}$ for polynomial degree $k = 5$ and Mach numbers $\|c_s\mathbf{b}_{c_s}\|_{\mathbf{L}^\infty} \in \{0.01, 0.45\}$. The choice of a suitable penalty parameters λ seems to depend on the Mach number and the error of the lifting stabilized method is smaller.

5.6. **Sun parameters**

Finally, let us consider a numerical example using the density, sound speed and pressure provided by the `modelS` [13] for the Sun. Due to the extreme variation of these coefficients towards the boundary of the domain, we use special meshes that are finer towards the boundary but more coarse in the interior, see Fig. 6. In addition to the parameters given by the `modelS`, we follow [11] and set

$$\omega = 0.003 \cdot 2\pi \cdot R_\circ, \quad \gamma = \omega/100, \quad \Omega = (0, 0),$$

where $R_\circ \approx 1.0007126$ is the radius of the sun. We choose the right-hand side

$$\mathbf{f} = 10^7 \begin{pmatrix} g \\ 0 \end{pmatrix},$$

where $g(x,y) = \sqrt{(\log(10^6)/0.1^2)/\pi}\exp((-\log(10^6)/0.1^2)((x-0.5)^2+(y-0.5)^2))$ is a Gaussian. We consider the case of a uniform and a non-uniform background flow. Specifically, we consider the flows $\mathbf{b}_{1/R_\circ}$ and $\mathbf{b}_{c_s/R_\circ}$ which are of the form (69) such that we can use the parameter $c_\mathbf{b}$ to control the Mach number. We note that in both cases, we have that $\operatorname{div}(\rho\mathbf{b}) = 0$.

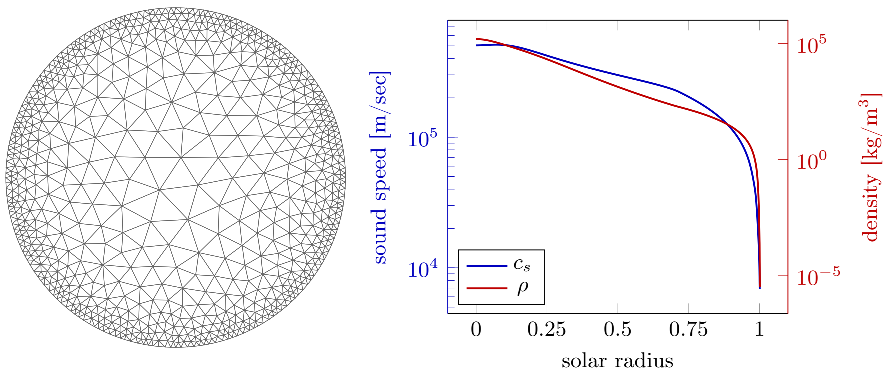


FIGURE 6. The density and sound speed provided by the `modelS` (right) and an example mesh adapted to these coefficients (left).

To improve the computational efficiency, we use the *optimized HDG* method as described in Table 1. In Figs. 7 and 8, we display the real part of the x-components of the computed solutions for the two backgrounds flows. For the background flow $\mathbf{b}_{1/R_\circ}$, the computed solution seems to be stable, even when the Mach number exceed 1.0. In contrast, we observe instabilities in the solutions computed for the second background flow $\mathbf{b}_{c_s/R_\circ}$ once the Mach number grows large, which is however still in agreement with the results from Theorem 23.

Acknowledgements

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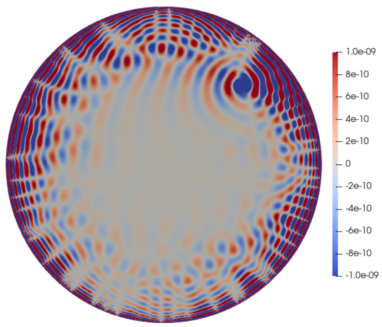
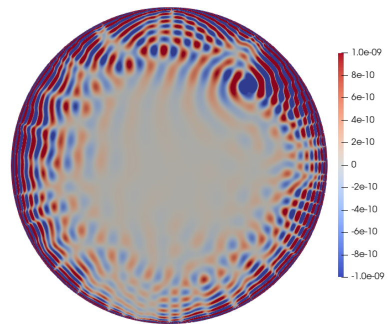
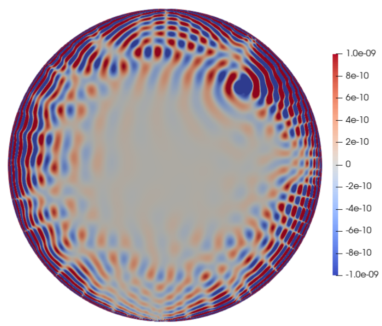


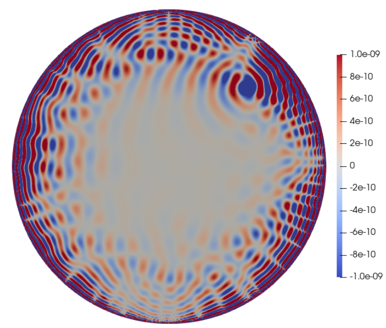
$\|c_s^{-1}\mathbf{b}_{1/R_\circ}\|^2_{\mathbf{L}^\infty} \approx 0.05$



$\|c_s^{-1}\mathbf{b}_{1/R_\circ}\|^2_{\mathbf{L}^\infty} \approx 0.50$



$\|c_s^{-1}\mathbf{b}_{1/R_\circ}\|^2_{\mathbf{L}^\infty} \approx 1.00$



$\|c_s^{-1}\mathbf{b}_{1/R_\circ}\|^2_{\mathbf{L}^\infty} \approx 1.50$

FIGURE 7. Real part of the x-component of the solution computed with the *optimized HDG* method for polynomial degree $k = 7$ for the flow $\mathbf{b}_{1/R_\circ}$. We consider the Mach numbers $\|c_s^{-1}\mathbf{b}_{1/R_\circ}\|^2_{\mathbf{L}^\infty} \in \{0.05, 0.5, 1.0, 1.5\}$.

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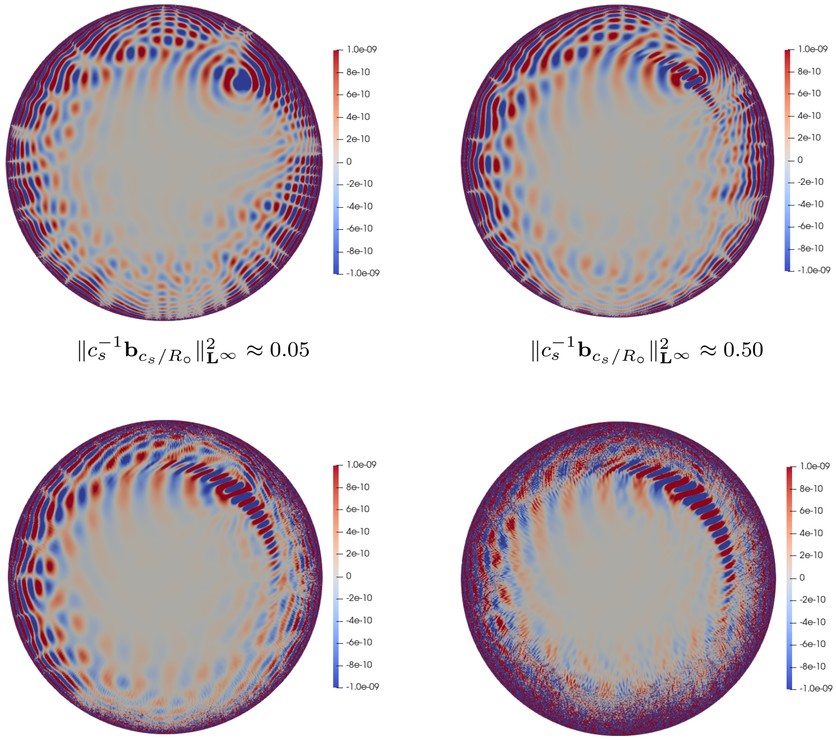

$\|c_s^{-1}\mathbf{b}_{c_s/R_\circ}\|^2_{\mathbf{L}^\infty} \approx 0.05$ $\|c_s^{-1}\mathbf{b}_{c_s/R_\circ}\|^2_{\mathbf{L}^\infty} \approx 0.50$

$\|c_s^{-1}\mathbf{b}_{c_s/R_\circ}\|^2_{\mathbf{L}^\infty} \approx 1.00$ $\|c_s^{-1}\mathbf{b}_{c_s/R_\circ}\|^2_{\mathbf{L}^\infty} \approx 1.50$

Figure 8. Real part of the x-component of the solution computed with the *optimized HDG* method for polynomial degree $k = 7$ for the flow $\mathbf{b}_{c_s/R_\circ}$. We consider the Mach numbers $\|c_s^{-1}\mathbf{b}_{c_s/R_\circ}\|^2_{\mathbf{L}^\infty} \in \{0.05, 0.5, 1.0, 1.5\}$.

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Appendix A. Supplementary material to Theorem 23

This section contains the operators defined in the proof of Theorem 23. In *Step 1*, we defined the operator $B_n^{(1)}$ as the sum of $a_n^{(1)}$ and $K_n^{(1)}$, i.e.

$$\begin{aligned}
\langle B_n^{(1)}\mathbf{u}_n, \mathbf{u}'_n\rangle_{\mathbb{X}_n} :=& \\
&\langle c_s^2\rho \operatorname{div}\mathbf{v}_\tau, \operatorname{div}\mathbf{v}'_\tau\rangle + \langle c_s^2\rho\pi_n^l P_{L_0^2}\mathbf{q}\cdot\mathbf{w}_\tau, \pi_n^l P_{L_0^2}\mathbf{q}\cdot\mathbf{w}'_\tau\rangle - s_n(\mathbf{w}_n, \mathbf{w}'_n) && \text{(76a)}\\
&- \langle \rho i D_{\mathbf{b}}^n\mathbf{v}_n, iD_{\mathbf{b}}^n\mathbf{v}'_n\rangle + \langle \rho(\omega + i\mathbf{D}_{\mathbf{b}}^n + i\Omega\times)\mathbf{w}_n, (\omega + i\mathbf{D}_{\mathbf{b}}^n + i\Omega\times)\mathbf{w}'_n\rangle && \text{(76b)}\\
&+ \langle \rho(\omega + i\mathbf{D}_{\mathbf{b}}^n + i\Omega\times)\mathbf{w}_n, iD_{\mathbf{b}}^n\mathbf{v}'_n\rangle - \langle \rho i D_{\mathbf{b}}^n\mathbf{v}_n, (\omega + i\mathbf{D}_{\mathbf{b}}^n + i\Omega\times)\mathbf{w}'_n\rangle && \text{(76c)}\\
&+ \langle \mathbf{v}_\tau, \mathbf{v}'_\tau\rangle + C_1\langle S_n\mathbf{u}_n, S_n\mathbf{u}'_n\rangle + \langle c_s^2\rho M_n\mathbf{w}_n, M_n\mathbf{w}'_n\rangle + \langle c_s^2\rho\tilde{O}_n\mathbf{u}_n, \tilde{O}_n\mathbf{u}'_n\rangle && \text{(76d)}\\
&+ \langle \rho(\underline{\underline{m}} + i\omega\gamma)\mathbf{w}_\tau, \mathbf{w}'_\tau\rangle. && \text{(76e)}
\end{aligned}$$

and the operator $B_n^{(2)}$ as the sum of $a_n^{(2)}$ and $K_n^{(2)}$. With $a_n^{(2)}$ as follows

$$\begin{aligned}
a_n^{(2)}(\mathbf{u}_n, \mathbf{u}'_n) &:= a_n(T_n\mathbf{u}_n, \mathbf{u}'_n) - a_n^{(1)}(\mathbf{u}_n, \mathbf{u}'_n)\\
&= a_n^{(2)}(\mathbf{v}_n, \mathbf{v}'_n) + a_n^{(2)}(\mathbf{v}_n, \mathbf{w}'_n) + a_n^{(2)}(\mathbf{w}_n, \mathbf{v}'_n) + a_n^{(2)}(\mathbf{w}_n, \mathbf{w}'_n) \text{ with}
\end{aligned}$$

$$\begin{aligned}
a_n^{(2)}(\mathbf{v}_n, \mathbf{v}'_n) =& \langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_\tau, \operatorname{div}\mathbf{v}'_\tau\rangle + \langle c_s^2\rho \operatorname{div}\mathbf{v}_\tau, \mathbf{q}\cdot\mathbf{v}'_\tau\rangle + \langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_\tau, \mathbf{q}\cdot\mathbf{v}'_\tau\rangle\\
&- \langle \rho(\omega + i\Omega\times)\mathbf{v}_\tau, (\omega + i\Omega\times)\mathbf{v}'_\tau\rangle - \langle \rho(\underline{\underline{m}} + i\omega\gamma)\mathbf{v}_\tau, \mathbf{v}'_\tau\rangle\\
&- \langle \rho(\omega + i\Omega\times)\mathbf{v}_\tau, i\mathbf{D}_{\mathbf{b}}^n\mathbf{v}'_n\rangle - \langle \rho i\mathbf{D}_{\mathbf{b}}^n\mathbf{v}_n, (\omega + i\Omega\times)\mathbf{v}'_\tau\rangle
\end{aligned}$$

$$
\begin{aligned}
a_n^{(2)}(\mathbf{v}_n,\mathbf{w}'_n) = &-\langle\rho(\omega+i\Omega\times)\mathbf{v}_\tau,(\omega+i\mathbf{D}^n_{\mathbf{b}}+i\Omega\times)\mathbf{w}'_n\rangle-\langle\rho(\underline{\underline{m}}+i\omega\gamma)\mathbf{v}_\tau,\mathbf{w}'_\tau\rangle+\langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_\tau,\mathbf{q}\cdot\mathbf{w}'_\tau\rangle\\
&-\langle c_s^2\rho(\operatorname{div}+\pi^l_n P_{L^2_0}\mathbf{q}\cdot)\mathbf{v}_\tau,M_n\mathbf{w}'_n+\tilde{O}_n\mathbf{u}'_n\rangle+\langle c_s^2\rho(\operatorname{id}-\pi^l_n P_{L^2_0})(\mathbf{q}\cdot\mathbf{v}_\tau),\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}'_n\rangle\\
&+\langle c_s^2\rho\operatorname{div}\mathbf{v}_\tau,(\operatorname{id}-\pi^l_n P_{L^2_0})\mathbf{q}\cdot\mathbf{w}'_n\rangle-\langle c_s^2\rho\pi^l_n P_{L^2_0}(\mathbf{q}\cdot\mathbf{v}_\tau),\pi^l_n P_{L^2_0}(\mathbf{q}\cdot\mathbf{w}'_\tau)\rangle
\end{aligned}
$$

$$
\begin{aligned}
a_n^{(2)}(\mathbf{w}_n,\mathbf{v}'_n) = &\langle\rho(\omega+i\mathbf{D}^n_{\mathbf{b}}+i\Omega\times)\mathbf{w}_n,(\omega+i\Omega\times)\mathbf{v}'_\tau\rangle+\langle\rho(\underline{\underline{m}}+i\omega\gamma)\mathbf{w}_\tau,\mathbf{v}'_\tau\rangle-\langle c_s^2\rho\mathbf{q}\cdot\mathbf{w}_\tau,\mathbf{q}\cdot\mathbf{v}'_\tau\rangle\\
&+\langle c_s^2\rho(M_n\mathbf{w}_n+\tilde{O}_n\mathbf{u}_n),(\operatorname{div}+\pi^l_n P_{L^2_0}\mathbf{q}\cdot)\mathbf{v}'_\tau\rangle-\langle c_s^2\rho\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}_n,(\operatorname{id}-\pi^l_n P_{L^2_0})(\mathbf{q}\cdot\mathbf{v}'_\tau)\rangle\\
&-\langle c_s^2\rho(\operatorname{id}-\pi^l_n P_{L^2_0})(\mathbf{q}\cdot\mathbf{w}_\tau),\operatorname{div}\mathbf{v}'_\tau\rangle+\langle c_s^2\rho\pi^l_n P_{L^2_0}(\mathbf{q}\cdot\mathbf{w}_\tau),\pi^l_n P_{L^2_0}(\mathbf{q}\cdot\mathbf{v}'_\tau)\rangle
\end{aligned}
$$

$$
\begin{aligned}
a_n^{(2)}(\mathbf{w}_n,\mathbf{w}'_n) = &-\langle c_s^2\rho(\operatorname{id}-\pi^l_n P_{L^2_0})(\mathbf{q}\cdot\mathbf{w}_\tau),\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}'_n\rangle-\langle c_s^2\rho\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}_n,(\operatorname{id}-\pi^l_n P_{L^2_0})(\mathbf{q}\cdot\mathbf{w}'_\tau)\rangle\\
&-\langle c_s^2\rho(M_n\mathbf{w}_n+\tilde{O}_n\mathbf{u}_n),M_n\mathbf{w}'_n+\tilde{O}_n\mathbf{u}'_n\rangle
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\langle B_n^{(2)}\mathbf{u}_n,\mathbf{u}'_n\rangle_{\mathbb{X}_n} :=\\
&C_2(\langle\mathbf{v}_\tau,\mathbf{v}'_\tau\rangle+\langle S_n\mathbf{u}_n,S_n\mathbf{u}'_n\rangle+\langle c_s^2\rho\tilde{O}_n\mathbf{u}_n,\tilde{O}_n\mathbf{u}'_n\rangle\\
&+\langle c_s^2\rho M_n\mathbf{w}_n,M_n\mathbf{w}'_n\rangle+\langle\operatorname{mean}(\mathbf{q}\cdot\mathbf{w}_\tau),\operatorname{mean}(\mathbf{q}\cdot\mathbf{w}'_\tau)\rangle)\\
&\\
&+\langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_\tau,\operatorname{div}\mathbf{v}'_\tau\rangle+\langle c_s^2\rho\operatorname{div}\mathbf{v}_\tau,\mathbf{q}\cdot\mathbf{v}'_\tau\rangle+\langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_\tau,\mathbf{q}\cdot\mathbf{v}'_\tau\rangle\\
&-\langle\rho(\omega+i\Omega\times)\mathbf{v}_\tau,(\omega+i\Omega\times)\mathbf{v}'_\tau\rangle-\langle\rho(\underline{\underline{m}}+i\omega\gamma)\mathbf{v}_\tau,\mathbf{v}'_\tau\rangle\\
&-\langle\rho(\omega+i\Omega\times)\mathbf{v}_\tau,i\mathbf{D}^n_{\mathbf{b}}\mathbf{v}'_n\rangle-\langle\rho i\mathbf{D}^n_{\mathbf{b}}\mathbf{v}_n,(\omega+i\Omega\times)\mathbf{v}'_\tau\rangle\\
&\\
&-\langle\rho(\omega+i\Omega\times)\mathbf{v}_\tau,(\omega+i\mathbf{D}^n_{\mathbf{b}}+i\Omega\times)\mathbf{w}'_n\rangle-\langle\rho(\underline{\underline{m}}+i\omega\gamma)\mathbf{v}_\tau,\mathbf{w}'_\tau\rangle+\langle c_s^2\rho\mathbf{q}\cdot\mathbf{v}_\tau,\mathbf{q}\cdot\mathbf{w}'_\tau\rangle\\
&-\langle c_s^2\rho(\operatorname{div}+\pi^l_n P_{L^2_0}\mathbf{q}\cdot)\mathbf{v}_\tau,M_n\mathbf{w}'_n+\tilde{O}_n\mathbf{u}'_n\rangle+\langle c_s^2\rho(\operatorname{id}-\pi^l_n P_{L^2_0})(\mathbf{q}\cdot\mathbf{v}_\tau),\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}'_n\rangle\\
&+\langle c_s^2\rho\operatorname{div}\mathbf{v}_\tau,(\operatorname{id}-\pi^l_n P_{L^2_0})\mathbf{q}\cdot\mathbf{w}'_n\rangle-\langle c_s^2\rho\pi^l_n P_{L^2_0}(\mathbf{q}\cdot\mathbf{v}_\tau),\pi^l_n P_{L^2_0}(\mathbf{q}\cdot\mathbf{w}'_\tau)\rangle\\
&\\
&+\langle\rho(\omega+i\mathbf{D}^n_{\mathbf{b}}+i\Omega\times)\mathbf{w}_n,(\omega+i\Omega\times)\mathbf{v}'_\tau\rangle+\langle\rho(\underline{\underline{m}}+i\omega\gamma)\mathbf{w}_\tau,\mathbf{v}'_\tau\rangle-\langle c_s^2\rho\mathbf{q}\cdot\mathbf{w}_\tau,\mathbf{q}\cdot\mathbf{v}'_\tau\rangle\\
&+\langle c_s^2\rho(M_n\mathbf{w}_n+\tilde{O}_n\mathbf{u}_n),(\operatorname{div}+\pi^l_n P_{L^2_0}\mathbf{q}\cdot)\mathbf{v}'_\tau\rangle-\langle c_s^2\rho\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}_n,(\operatorname{id}-\pi^l_n P_{L^2_0})(\mathbf{q}\cdot\mathbf{v}'_\tau)\rangle\\
&-\langle c_s^2\rho(\operatorname{id}-\pi^l_n P_{L^2_0})(\mathbf{q}\cdot\mathbf{w}_\tau),\operatorname{div}\mathbf{v}'_\tau\rangle+\langle c_s^2\rho\pi^l_n P_{L^2_0}(\mathbf{q}\cdot\mathbf{w}_\tau),\pi^l_n P_{L^2_0}(\mathbf{q}\cdot\mathbf{v}'_\tau)\rangle\\
&\\
&-\langle c_s^2\rho(\operatorname{id}-\pi^l_n P_{L^2_0})(\mathbf{q}\cdot\mathbf{w}_\tau),\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}'_n\rangle-\langle c_s^2\rho\operatorname{div}^n_{\boldsymbol{\nu}}\mathbf{w}_n,(\operatorname{id}-\pi^l_n P_{L^2_0})(\mathbf{q}\cdot\mathbf{w}'_\tau)\rangle\\
&-\langle c_s^2\rho(M_n\mathbf{w}_n+\tilde{O}_n\mathbf{u}_n),M_n\mathbf{w}'_n+\tilde{O}_n\mathbf{u}'_n\rangle.
\end{aligned}
$$

The terms added with $K_n^{(1)}$ and $K_n^{(2)}$ are subtracted through the operator K_n, which is given by

$$
\begin{aligned}
&\langle K_n\mathbf{u}_n,\mathbf{u}'_n\rangle_{\mathbb{X}_n} :=\\
&\quad-(1+C_2)\langle\mathbf{v}_\tau,\mathbf{v}'_\tau\rangle-(C_1+C_2)\langle S_n\mathbf{u}_n,S_n\mathbf{u}'_n\rangle \qquad (78a)\\
&\quad-(1+C_2)\langle c_s^2\rho M_n\mathbf{w}_n,M_n\mathbf{w}'_n\rangle-C_2\langle\operatorname{mean}(\mathbf{q}\cdot\mathbf{w}_\tau),\operatorname{mean}(\mathbf{q}\cdot\mathbf{w}'_\tau)\rangle \qquad (78b)
\end{aligned}
$$

$$- (1 + C_2)\langle c_s^2 \rho \tilde{O}_n \mathbf{u}_n, \tilde{O}_n \mathbf{u}_n' \rangle. \tag{78c}$$

In *Step 4*, we defined the compact operator K in (K-a)-(K-b) and set $B \coloneqq AT - K$, i.e.

$$\langle B\mathbf{u}, \mathbf{u}' \rangle_{\mathbb{X}} \coloneqq$$

$$\langle c_s^2 \rho \operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{v}' \rangle - \langle \rho i \partial_{\mathbf{b}} \mathbf{v}, i \partial_{\mathbf{b}} \mathbf{v}' \rangle + \langle c_s^2 \rho P_{L_0^2}(\mathbf{q} \cdot \mathbf{w}), P_{L_0^2}(\mathbf{q} \cdot \mathbf{w}') \rangle \tag{79a}$$

$$- \langle \rho i \partial_{\mathbf{b}} \mathbf{v}, (\omega + i\partial_{\mathbf{b}} + i\Omega \times) \mathbf{w}' \rangle + \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega \times) \mathbf{w}, i\partial_{\mathbf{b}} \mathbf{v}' \rangle \tag{79b}$$

$$+ \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega \times) \mathbf{w}, (\omega + i\partial_{\mathbf{b}} + i\Omega \times) \mathbf{w}' \rangle + \langle \rho(i\omega\gamma + \underline{\underline{m}}) \mathbf{w}, \mathbf{w}' \rangle \tag{79c}$$

$$+ \langle \mathbf{v}, \mathbf{v}' \rangle + C_1 \langle \mathbf{v}, \mathbf{v}' \rangle + \langle c_s^2 \rho M \mathbf{w}, M \mathbf{w}' \rangle \tag{79d}$$

$$+ C_2(\langle \mathbf{v}, \mathbf{v}' \rangle + \langle \mathbf{v}, \mathbf{v}' \rangle + \langle c_s^2 \rho M \mathbf{w}, M \mathbf{w}' \rangle + \langle \operatorname{mean}(\mathbf{q} \cdot \mathbf{w}), \operatorname{mean}(\mathbf{q} \cdot \mathbf{w}') \rangle) \tag{79e}$$

$$+ \langle c_s^2 \rho \mathbf{q} \cdot \mathbf{v}, \operatorname{div} \mathbf{v}' \rangle + \langle c_s^2 \rho \operatorname{div} \mathbf{v}, \mathbf{q} \cdot \mathbf{v}' \rangle - \langle \rho(\omega + i\Omega \times) \mathbf{v}, (\omega + i\Omega \times) \mathbf{v}' \rangle \tag{79f}$$

$$- \langle \rho(\omega + i\Omega \times) \mathbf{v}, i\partial_{\mathbf{b}} \mathbf{v}' \rangle - \langle \rho i \partial_{\mathbf{b}} \mathbf{v}, (\omega + i\Omega \times) \mathbf{v}' \rangle - i\omega \langle \gamma \rho \mathbf{v}, \mathbf{v}' \rangle - \langle \rho \underline{\underline{m}} \mathbf{v}, \mathbf{v}' \rangle \tag{79g}$$

$$- \langle \rho \underline{\underline{m}} \mathbf{v}, \mathbf{w}' \rangle - i\omega \langle \gamma \rho \mathbf{v}, \mathbf{w}' \rangle - \langle c_s^2 \rho P_{L_0^2}(\mathbf{q} \cdot \mathbf{v}), P_{L_0^2}(\mathbf{q} \cdot \mathbf{w}') \rangle \tag{79h}$$

$$- \langle \rho(\omega + i\Omega \times) \mathbf{v}, (\omega + i\partial_{\mathbf{b}} + i\Omega \times) \mathbf{w}' \rangle - \langle c_s^2 \rho (\operatorname{div} + P_{L_0^2} \mathbf{q} \cdot) \mathbf{v}, M \mathbf{w}' \rangle \tag{79i}$$

$$+ \langle c_s^2 \rho \operatorname{mean}(\mathbf{q} \cdot \mathbf{v}), \operatorname{div} \mathbf{w}' \rangle \tag{79j}$$

$$+ \langle \rho \underline{\underline{m}} \mathbf{w}, \mathbf{v}' \rangle + i\omega \langle \gamma \rho \mathbf{w}, \mathbf{v}' \rangle + \langle c_s^2 \rho P_{L_0^2}(\mathbf{q} \cdot \mathbf{w}), P_{L_0^2}(\mathbf{q} \cdot \mathbf{v}') \rangle \tag{79k}$$

$$+ \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega \times) \mathbf{w}, (\omega + i\Omega \times) \mathbf{v}' \rangle + \langle c_s^2 \rho M \mathbf{w}, (\operatorname{div} + P_{L_0^2} \mathbf{q} \cdot) \mathbf{v}' \rangle \tag{79l}$$

$$- \langle c_s^2 \rho \operatorname{div} \mathbf{w}, \operatorname{mean}(\mathbf{q} \cdot \mathbf{v}') \rangle \tag{79m}$$

$$- \langle c_s^2 \rho \operatorname{mean}(\mathbf{q} \cdot \mathbf{w}), \operatorname{div} \mathbf{w}' \rangle - \langle c_s^2 \rho \operatorname{div} \mathbf{w}, \operatorname{mean}(\mathbf{q} \cdot \mathbf{w}') \rangle - \langle c_s^2 \rho M \mathbf{w}, M \mathbf{w}' \rangle. \tag{79n}$$