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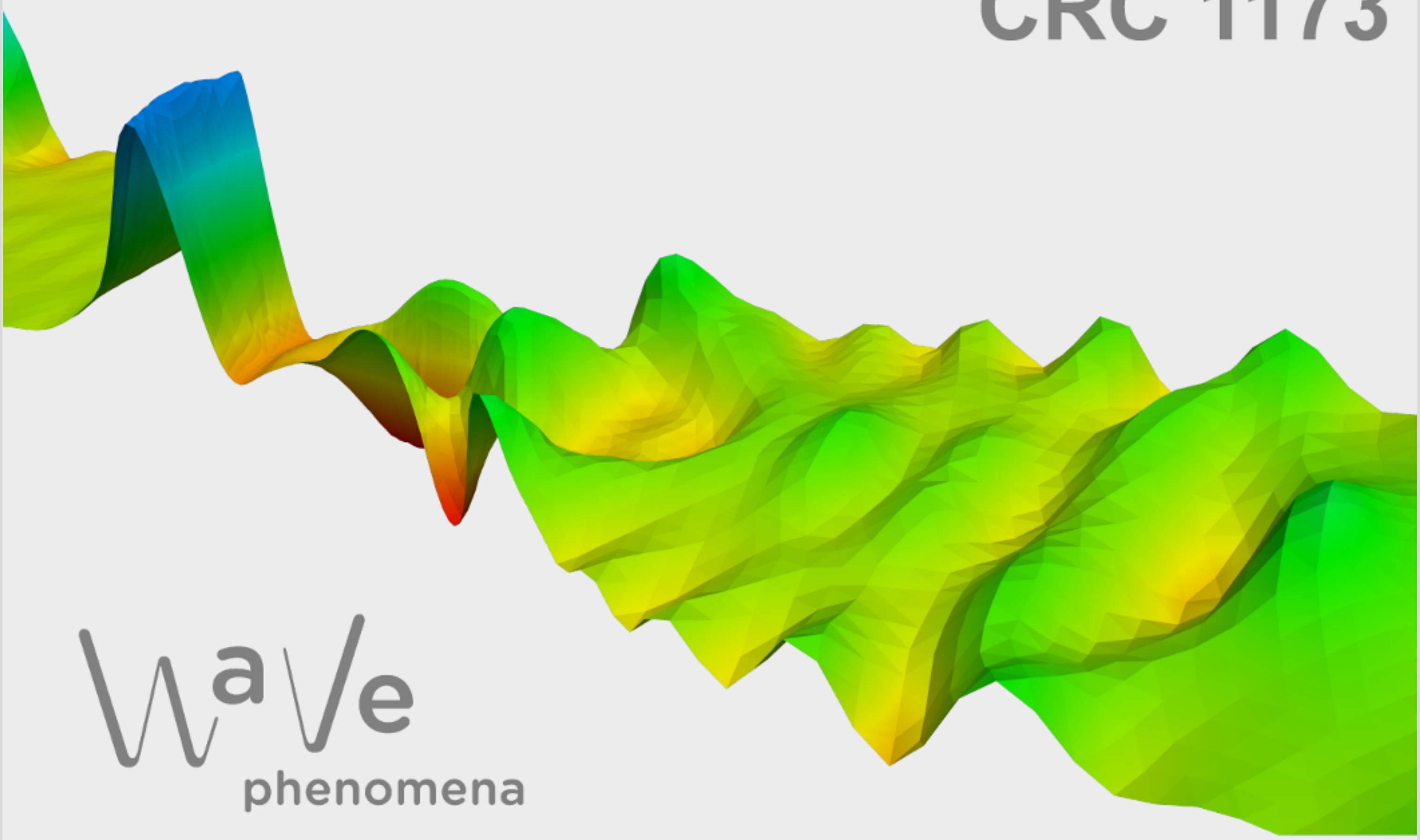

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A LIMITING ABSORPTION PRINCIPLE FOR THE SCATTERING BY A PERIODIC LAYER IN THE CASE OF A CUT-OFF VALUE

ANDREAS KIRSCH

Abstract. In this paper we consider the propagation of waves in an open waveguide in the half space $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$ under Dirichlet- or Neumann boundary condition for $x_2 = 0$. The index of refraction $n = n(x)$ is periodic along the axis of the waveguide (which we choose to be the x_1-axis) and equal to one for $x_2 > h_0$ for some $h_0 > 0$. We show first existence and uniqueness of a solution for the absorbing case, i.e. where the index of refraction is given by $n(x) + i\varepsilon q(x)$ with $\varepsilon > 0$ and some function q which is periodic with respect to x_1, vanishes for $x_2 > h_0$, and satisfies the angular spectral representation radiation condition. Then we prove convergence of the solution as ε tends to zero. We show that the limit solves the source problem for $n(x)$ and satisfies a radiation condition which depends, first, on the choice of the absorption function q and, second, whether or not a cut-off value is critical with a non-evanescent mode.

MSC: 35J05

Key words: Helmholtz equation, open waveguide, limiting absorption principle, radiation condition

1. Introduction

In this paper we study the boundary value problem

$$\Delta u + k^2 n\, u \;=\; -f \quad \text{in } \mathbb{R}^2_+\,, \qquad u = 0 \quad \text{or} \quad \partial_{x_2} u = 0 \quad \text{for } x_2 = 0\,, \tag{1}$$

where $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$. We assume that the (real valued) index of refraction $n \in L^\infty(\mathbb{R}^2_+)$ is $2\pi-$periodic with respect to x_1 and equals to 1 for $x_2 > h_0$ for some $h_0 > 0$ and $n(x) \geq n_0$ in $\mathbb{R}^2_+$ for some $n_0 > 0$. Furthermore, $k > 0$ denotes the (real) wave number which is fixed throughout the paper and $f \in L^2(\mathbb{R}^2_+)$ with compact support in W^{h_0} where $W^h := \mathbb{R} \times (0, h) \subset \mathbb{R}^2_+$ denotes the layer of height $h > 0$. The boundary value problem (1) has to be complemented by a suitable radiating condition. Its derivation is the main subject of this paper.

In this paper we continue earlier contributions as in [13, 11] and investigate the scattering of sources by a periodic inhomogeneous layer. It is the aim to derive "natural" radiation conditions arizing from limiting absorption principles. For the scattering of electromagnetic waves by bounded objects the Sommerfeld radiation condition (for scalar problems) or the Silver-Müller radiation condition (for Maxwell's equations) are certainly the natural conditions. The situation is more complicated for scattering problems by periodic structures due to the presense of guided waves. The investigation of radiation problems for periodic structures has a long history, and it is impossible to list all of the relevant literature. Instead, we refer to [1, 17] for a comprehensive introduction of electromagnetic

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scattering theory for diffraction gratings. Radiation conditions play an important role in the modeling of these problems because they assure uniqueness (from the mathematical point of view) and that energy is transported away from the structure (from the physical point of view). Prominent examples of radiation conditions for periodic structures are the Rayleigh expansion (for the scattering by plane waves), the upwards propagating radiation condition suggested by Chandler-Wilde in several papers (for rough surfaces which include locally perturbed periodic structures, see, e.g. [4]) or the angular spectrum representation condition. This notion is popular in the physics literature, see e.g. [8], but has been used also (without using this name) in, e.g., [3, 2, 9]. All of the mentioned radiation conditions, however, are appropreate only if no guided waves exist as, e.g., in the case of the scattering by a conductor described as the graph of a periodic function. There is much less literature for cases where guided waves exist. The above concepts have to be modified to include guided waves. For layered media, in [18, 5, 6] a splitting of the field into a sum of guided waves and a part which satisfied a kind of Sommerfeld radiation condition is suggested. A completely different approach, based on a modal radiation condition, is suggested in [7], using earlier concepts from, e.g., [16].

As in, e.g. [13, 10] and [14] (for closed waveguides) we consider first the case when the refractive index n is absorbing, i.e. $n = n(x)$ is replaced by $n_\varepsilon(x) := n(x) + i\varepsilon q(x)$ with $\varepsilon > 0$ and some fixed $q \in L^\infty(\mathbb{R}^2)$ which is $2\pi-$periodic with respect to x_1, $q(x) \geq q_0$ on W^{h_0} for some $q_0 > 0$ and $q(x) = 0$ for $x_2 > h_0$. For this absorbing case we expect the solution u_ε to be in $H^1(W^h)$ for all $h > 0$. Therefore, the Fourier transform with respect to x_1 is well defined, and the angular spectral representation (see (4) below) is the natural radiation condition because no guided waves exist for absorbing layers.

As a main result in this paper we prove convergence of u_ε as $\varepsilon \to 0$. It will turn out that the limiting field u_0 satisfies a radiation condition which will in general depend on the absorption function q – although this function does not appear in the boundary value problem (1). As a consequence we emphasize that for these kind of problems there does not exist a unique radiation condition but rather a class of radiation conditions which depends on the limiting procedure. We illustrate this in Example 4.4 below. This rather strange situation has been observed already in [14, 12].

We recall that the solution of (1) (for n replaced by $n_\varepsilon := n + i\varepsilon q$) is understood in the variational sense. Set $H^1_{loc}(\mathbb{R}^2_+) = \{u|_{\mathbb{R}^2_+} : u \in H^1_{loc}(\mathbb{R}^2)\}$ for the case of a Neumann boundary condition and $H^1_{loc}(\mathbb{R}^2_+) = \{u|_{\mathbb{R}^2_+} : u \in H^1_{loc}(\mathbb{R}^2)\,,\ u = 0 \text{ for } x_2 = 0\}$ for the case of a Dirichlet boundary condition.

Definition 1.1. *A function $u \in H^1_{loc}(\mathbb{R}^2_+)$ is called a variational solution of (1) (for $n = n_\varepsilon$) if*

$$\int\limits_{\mathbb{R}^2_+} \left[\nabla u \cdot \nabla \psi - k^2 n_\varepsilon\, u\, \psi\right] dx \;=\; \int\limits_{W^{h_0}} f\, \psi\, dx \tag{2}$$

for all $\psi \in H^1_{loc}(\mathbb{R}^2_+)$ with compact support in $\{x \in \mathbb{R}^2 : x_2 \geq 0\}$.

By choosing $\psi \in H^1(\mathbb{R}^2_+)$ in (2) with compact support in $\mathbb{R}^2_+ \setminus \overline{W^{h_0}}$ we note that u is a classical solution of the Helmholtz equation $\Delta u + k^2 u = 0$ for $x_2 > h_0$ and thus analytic.

As mentioned above, if $\varepsilon > 0$ we expect the solution to decay as $x_1 \to \pm\infty$. More precisely, we search for a solution in

$$H^1_*(\mathbb{R}^2_+) \;:=\; \{u \in H^1_{loc}(\mathbb{R}^2_+) : u|_{W^h} \in H^1(W^h) \text{ for all } h > 0\}\,. \tag{3}$$

A common radiation condition (see, e.g. [3, 2, 9]) is the angular spectrum representation, i.e. the Fourier transform $(\mathcal{F}u)(\xi, x_2)$ of $u(\cdot, x_2)$ with respect to x_1 (which exists because $u(\cdot, x_2) \in H^{1/2}(\mathbb{R})$) has the representation

$$(\mathcal{F}u)(\xi, x_2) \;=\; (\mathcal{F}u)(\xi, h_0)\, e^{i\sqrt{k^2-\xi^2}(x_2-h_0)}\,, \quad x_2 > h_0\,, \tag{4}$$

for almost all $\xi \in \mathbb{R}$. The Floquet-Bloch transform (see Subsection 2.2) reduces the equation $\Delta u + k^2 n_\varepsilon u = -f$ to a family of quasi-periodic problems. Therefore, we study quasi-periodic problems in the following section.

2. The Quasi-periodic Problems and the Case of Absorption

2.1. **The Quasi-periodic Problems.** A function $v \in L^\infty(\mathbb{R}^2_+)$ is called quasi-periodic with respect to x_1 with parameter $\alpha \in \mathbb{R}$ if $v(x_1 + 2\pi, x_2) = e^{2\pi\alpha i} v(x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2_+$.
We set $Q^\infty := (0, 2\pi) \times (0, \infty)$ and $Q^h := (0, 2\pi) \times (0, h)$ for $h > 0$ and define

$$H^1_{\alpha,loc}(Q^\infty) \;:=\; \big\{u|_{Q^\infty} : u \in H^1_{loc}(\mathbb{R}^2_+),\; u(\cdot, x_2) \text{ is } \alpha\text{−quasi-periodic}\big\}\,.$$

Recall that $H^1_{\alpha,loc}(Q^\infty)$ contains the boundary conditions in the Dirchlet case. The quasi-periodic analog to (1), (4) is the problem to determine, for any given $\alpha \in \mathbb{R}$ and $g_\alpha \in L^2(Q^{h_0})$, a α−quasi-periodic solution $u_\alpha \in H^1_{\alpha,loc}(Q^\infty)$ of

$$\Delta u_\alpha + k^2 n_\varepsilon u_\alpha = -g_\alpha \quad \text{in } Q^\infty \qquad u_\alpha = 0 \quad \text{or} \quad \partial_{x_2} u_\alpha = 0 \quad \text{for } x_2 = 0\,, \tag{5}$$

which satisfies the Rayleigh expansion

$$u_\alpha(x) \;=\; \sum_{\ell\in\mathbb{Z}} u_\ell\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h_0)+i(\ell+\alpha)x_1}\,, \quad x_2 > h_0\,, \tag{6}$$

for some $u_\ell \in \mathbb{C}$ where the convergence is uniform for $x_2 \ge h$ for all $h > h_0$. We note that this problem is well defined for all $\varepsilon \ge 0$, i.e. in particular for the case $n_\varepsilon = n$ of no absorption. The coefficients u_ℓ are actually the Fourier coefficients $u_\ell(\alpha, h_0) := \frac{1}{2\pi}\int_0^{2\pi} u_\alpha(x_1, h_0) e^{-i(\ell+\alpha)x_1} dx_1$ of $u_\alpha(\cdot, h_0)$.
It is well known that this problem can be reduced to a problem on the bounded domain Q^h (for any $h \ge h_0$) with the Dirichlet-to-Neumann operator. We set $H^1_\alpha(Q^h) := \{u \in H^1(Q^h) : u(\cdot, x_2) \text{ is } \alpha\text{−quasi-periodic}\}$ and include the boundary conditions $u = 0$ for $x_2 = 0$ in the case of Dirichlet boundary conditions.

Lemma 2.1. *Let $\alpha \in \mathbb{R}$ and $g_\alpha \in L^2(Q^{h_0})$ be fixed. Let $h \ge h_0$ be arbitrary.*
(a) Let $u_\alpha \in H^1_{\alpha,loc}(Q^\infty)$ be a solution of (5), (6)*. Then $u_\alpha|_{Q^h} \in H^1_\alpha(Q^h)$ solves*

$$\int_{Q^h} \big[\nabla u_\alpha \cdot \nabla\bar\psi - k^2 n_\varepsilon\, u_\alpha\, \bar\psi\big]\, dx - 2\pi i \sum_{\ell\in\mathbb{Z}} u_\ell(\alpha, h)\, \overline{\psi_\ell(\alpha, h)}\, \sqrt{k^2 - (\ell+\alpha)^2} = \int_{Q^{h_0}} g_\alpha\, \bar\psi\, dx \tag{7}$$

for all $\psi \in H^1_\alpha(Q^h)$ where $u_\ell(\alpha, h) = \frac{1}{2\pi}\int_0^{2\pi} u_\alpha(x_1, h) e^{-i(\ell+\alpha)x_1} dx_1$ are the Fourier coefficients of $u_\alpha(\cdot, h)$.

(b) Let $u_\alpha \in H^1_\alpha(Q^h)$ solve (7). Extend u_α into Q^∞ by defining

$$u_\alpha(x) \;=\; \sum_{\ell\in\mathbb{Z}} u_\ell(\alpha,h)\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-h)+i(\ell+\alpha)x_1}\,, \quad x_2>h\,. \tag{8}$$

Then $u_\alpha \in H^1_{\alpha,loc}(Q^\infty)$ solves (5), (6).

We omit the proof because it is standard. As mentioned above, the connection between the quasi-periodic problems (5), (6) and the original problem (1), (4) is given by the Floquet-Bloch transform which we recall next.

2.2. **The Floquet-Bloch Transform.** Recall that the Floquet-Bloch transform $F : L^2(\mathbb{R}) \to L^2\big((0,2\pi)\times(-1/2,1/2)\big)$ is defined by (for $f\in\mathcal{S}(\mathbb{R})$)

$$(Ff)(t,\alpha) \;=\; \sum_{m\in\mathbb{Z}} f(t+2\pi m)\, e^{-i\alpha 2\pi m}\,, \quad t,\alpha\in\mathbb{R}\,.$$

This formula directly shows that for smooth functions f and fixed α the transformed function $t\mapsto (Ff)(t,\alpha)$ is $\alpha-$quasi-periodic while for fixed t the function $\alpha\mapsto(Ff)(t,\alpha)$ is periodic with period 1. It is hence sufficient to consider $L^2\big((0,2\pi)\times(-1/2,1/2)\big)$ as image space of F. The inverse transform is given by

$$(F^{-1}g)(t+2\pi\ell) \;=\; \int_{-1/2}^{1/2} g(t,\alpha)\, e^{2\pi\ell\alpha i} d\alpha\,, \quad t\in(0,2\pi)\,,\; \ell\in\mathbb{Z}\,.$$

There is a simple relationship to the Fourier transform $\mathcal{F}: L^2(\mathbb{R})\to L^2(\mathbb{R})$ given by (for $f\in\mathcal{S}(\mathbb{R})$)

$$(\mathcal{F}f)(\xi) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\xi x} dx\,, \quad \xi\in\mathbb{R}\,. \tag{9}$$

Indeed, for $\alpha\in[-1/2,1/2]$ and $\ell\in\mathbb{Z}$ we have

$$\begin{aligned}(\mathcal{F}f)(\alpha+\ell) \;&=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ix(\alpha+\ell)}dx \;=\; \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_0^{2\pi} f(x+2\pi m)\, e^{-i(x+2\pi m)(\alpha+\ell)}dx\\ &=\; \frac{1}{2\pi}\int_0^{2\pi}\sum_{m\in\mathbb{Z}} f(x+2\pi m)\, e^{-i\alpha 2\pi m} e^{-ix(\ell+\alpha)}dx \qquad (10)\\ &=\; \frac{1}{2\pi}\int_0^{2\pi}(Ff)(x,\alpha)\, e^{-ix(\ell+\alpha)}dx \;=\; \hat{f}_\ell(\alpha)\end{aligned}$$

where $\hat{f}_\ell(\alpha)$ are the Fourier coefficients of the $\alpha-$quasi-periodic function $(Ff)(\cdot,\alpha)$. In particular,

$$(Ff)(x,\alpha) \;=\; \sum_{\ell\in\mathbb{Z}} \hat{f}_\ell(\alpha)\, e^{ix(\ell+\alpha)} \;=\; \sum_{\ell\in\mathbb{Z}} (\mathcal{F}f)(\alpha+\ell)\, e^{ix(\ell+\alpha)}\,. \tag{11}$$

In view of our scattering problem, we apply the Floquet-Bloch transform to the variable x_1 and consider x_2 as a parameter. Recalling that $W^h := \mathbb{R}\times(0,h)$ and $Q^h := (0,2\pi)\times$

$(0,h)$ and $I=(-1/2,1/2)$ one can then show that F is an isometry from $L^2(W^h)$ onto $L^2(Q^h\times I)$,

$$\|Ff\|^2_{L^2(Q^h\times I)} \;=\; \int\limits_{-1/2}^{1/2}\int\limits_{Q^h} |(Ff)(x,\alpha)|^2\,dx\,d\alpha \;=\; \int\limits_{W^h} |f(x)|^2\,dx \;=\; \|f\|^2_{L^2(W^h)}\,.$$

It has been shown (see, e.g. [15, Section 6]) that F is also an isomorphism from $H^1(W^h)$ onto

$$L^2\big(I,H^1_\alpha(Q^h)\big) \;:=\; \left\{u\in L^2\big(I,H^1(Q^h)\big):\ \begin{array}{l} x_1\mapsto u(x,\alpha) \text{ is} \\ \alpha-\text{quasi-periodic}\end{array}\right\}.$$

2.3. **Equivalence and Existence in the Case of Absorption.** The following connection between (5), (6) and (1), (4) is well known (see, e.g., [14] for closed waveguides)

Lemma 2.2. *Let $\varepsilon\ge 0$ be fixed and $f\in L^2(W^{h_0})$ with compact support and $g_\alpha := (Ff)(\cdot,\alpha)\in L^2(Q^{h_0})$ its Floquet-Bloch transform.*
(a) Let $u\in H^1_*(\mathbb{R}^2_+)$ be a solution of (1), (4) and $u_\alpha(x):=(Fu)(x,\alpha)$ be its Floquet-Bloch transform. Then $u_\alpha\in H^1_\alpha(Q^\infty)$ solves (5), (6) with right hand side g_α for almost all $\alpha\in I$. Furthermore, $\alpha\mapsto u_\alpha$ belongs to $L^2\big(I,H^1(Q^h)\big)$ for every $h>0$.

(b) Let $u_\alpha\in H^1_\alpha(Q^\infty)$ solve (5), (6) with right hand side g_α for almost all $\alpha\in I$, and let the mapping $\alpha\mapsto u_\alpha$ belong to $L^2\big(I,H^1(Q^h)\big)$ for every $h>0$. Then the inverse transform $u=\int_I u_\alpha d\alpha$ belongs to $H^1_*(\mathbb{R}^2_+)$ and is a solution of (1), (4).

We did not indicate the dependence on ε because ε is kept fixed. We note that the relationship between (4) and the Rayleigh expansion (6) is given by (10). Also we observe that $\alpha\mapsto g_\alpha$ is infinitely often differentiable (even analytic) because the Floquet-Bloch transform reduces to a finite sum.

This lemma holds for $\varepsilon>0$ but also for $\varepsilon=0$. In the latter case, however, existence of a solution $u\in H^1_*(\mathbb{R}^2_+)$ or, equivalently, L^2-boundedness of $\alpha\mapsto u_\alpha$ is not assured in general, but only for special right hand sides (which are orthogonal to the modes, see, e.g., [13] or, for closed waveguides, [14]). It is convenient to introduce the space of periodic functions as

$$H^1_{\text{per}}(Q^h) \;:=\; \big\{u\in H^1(Q^h): u(\cdot,x_2) \text{ is } 2\pi-\text{periodic}\big\}\,. \tag{12}$$

In the case of a Dirichlet boundary condition the condition $u=0$ for $x_2=0$ is added to the definition of $H^1_{\text{per}}(Q^h)$. In the absorbing case $\varepsilon>0$ we have existence.

Theorem 2.3. *Let $\varepsilon>0$, i.e. $\operatorname{Im} n_\varepsilon>0$ on W^{h_0}. Then there exists a unique solution $u_\varepsilon\in H^1_*(\mathbb{R}^2_+)$ of (1), (4).*

Proof. For every $\alpha\in I$ we consider the quasi-periodic problem (7) for $h=h_0$ where $g_\alpha=(Ff)(\cdot,\alpha)$ is the Floquet-Bloch transform of f. The mapping $\alpha\mapsto g_\alpha$ is smooth because f has compact support. By Lemma 2.1 this problem is equivalent to the quasi-periodic problem (5), (6) in Q^∞. We replace $u_\alpha\in H^1_\alpha(Q^{h_0})$ and $\psi\in H^1_\alpha(Q^{h_0})$ by $v_\alpha(x)=e^{-i\alpha x_1}u_\alpha(x)$ and $\phi(x)=e^{-i\alpha x_1}\psi(x)$, respectively. Then $v_\alpha,\phi\in H^1_{\text{per}}(Q^{h_0})$, and

(7) transforms into

$$
\begin{aligned}
a_\alpha(v_\alpha,\phi) \;:=\; & \int\limits_{Q^{h_0}} \big[\nabla(e^{i\alpha x_1}v_\alpha)\cdot\nabla\overline{(e^{i\alpha x_1}\phi)} - k^2 n_\varepsilon\, v_\alpha\,\bar\phi\big]\,dx \\
& -\; 2\pi i\sum_{\ell\in\mathbb{Z}} v_\ell(\alpha,h_0)\,\overline{\phi_\ell(\alpha,h_0)}\,\sqrt{k^2-(\ell+\alpha)^2} \;=\; \int\limits_Q g_\alpha\,\overline{(e^{i\alpha x_1}\phi)}\,dx
\end{aligned}
\tag{13}
$$

for all $\phi\in H^1_{\text{per}}(Q^{h_0})$. Here, $v_\ell(\alpha,h_0)=\frac{1}{2\pi}\int_0^{2\pi}v_\alpha(x_1,h_0)e^{-i\ell x_1}dx_1$ are the Fourier coefficients of $v_\alpha(\cdot,h_0)$ with respect to $\{e^{i\ell x_1}:\ell\in\mathbb{Z}\}$ (which coincide with the Fourier coefficients of $u_\alpha(\cdot,h_0)$ with respect to $\{e^{i(\ell+\alpha)x_1}:\ell\in\mathbb{Z}\}$). Let $q_0>0$ with $q(x)\ge q_0$ for all $x\in W^{h_0}$. Then

$$
\begin{aligned}
\operatorname{Re} a_\alpha(u,u) \;&=\; \int\limits_{Q^{h_0}}\big[|\nabla(e^{i\alpha x_1}u)|^2-k^2 n\,|u|^2\big]\,dx + 2\pi\sum_{|\ell+\alpha|>k}|u_\ell(\alpha,h_0)|^2\sqrt{(\ell+\alpha)^2-k^2} \\
&\ge\; \|\nabla(e^{i\alpha x_1}u)\|^2_{L^2}-k^2\|n\|_\infty\|u\|^2_{L^2} \;=\; \|i\alpha u+\nabla u\|^2_{L^2}-k^2\|n\|_\infty\|u\|^2_{L^2} \\
&\ge\; \frac12\|\nabla u\|^2_{L^2}-\left(k^2\|n\|_\infty+\frac14\right)\|u\|^2_{L^2}\,, \\
-\operatorname{Im} a_\alpha(u,u) \;&=\; k^2\varepsilon\int\limits_{Q^{h_0}} q\,|u|^2\,dx + 2\pi\sum_{|\ell+\alpha|<k}|u_\ell(\alpha,h_0)|^2\sqrt{k^2-(\ell+\alpha)^2} \\
&\ge\; k^2\varepsilon\,q_0\,\|u\|^2_{L^2}\,,
\end{aligned}
$$

where we used the elementary estimate $\|i\alpha u+\nabla u\|^2\ge\|\nabla u\|^2+\frac14\|u\|^2-\|\nabla u\|\|u\| = \frac12\|\nabla u\|^2+\frac12(\|\nabla u\|-\|u\|)^2-\frac14\|u\|^2\ge\frac12\|\nabla u\|^2-\frac14\|u\|^2$. Let now $t>0$ such that $t(k^2\|n\|_\infty+1/4)<k^2\varepsilon\,q_0$. Then

$$
\begin{aligned}
\operatorname{Re}\big[(t+i)\,a_\alpha(u,u)\big] \;&=\; t\operatorname{Re} a_\alpha(u,u)-\operatorname{Im} a_\alpha(u,u) \\
&\ge\; \frac t2\,\|\nabla u\|^2_{L^2}-t\left(k^2\|n\|_\infty+\frac14\right)\|u\|^2_{L^2}+k^2\varepsilon\,q_0\,\|u\|^2_{L^2} \\
&=\; \frac t2\,\|\nabla u\|^2_{L^2}\;+\;\left(k^2\varepsilon\,q_0-tk^2\|n\|_\infty-\frac t4\right)\|u\|^2_{L^2}\;\;\ge\;c\,\|u\|^2_{H^1}\,.
\end{aligned}
$$

Therefore, $(t+i)a_\alpha$ is coercive, uniformly with respect to $\alpha\in[-1/2,1/2]$, and depends continuously on α. Therefore, the theorem of Lax-Milgram yields the existence and uniqueness of a solution $v_\alpha\in H^1_{\text{per}}(Q^{h_0})$ of $a_\alpha(v_\alpha,\phi)=\int_{Q^{h_0}}g_\alpha\,\overline{(e^{i\alpha x_1}\phi)}\,dx$ for all $\phi\in H^1_{\text{per}}(Q^{h_0})$, and thus of (7), which depends continuously on α. In particular, $\alpha\mapsto u_\alpha$ is in $L^2\big(I,H^1(Q^{h_0})\big)$. Application of part (b) of Lemma 2.2 ends the proof. □

Remark 2.4. *The assumption that q is uniformly bounded below on W^{h_0} by some positive constant q_0 can be relaxed. If q is only bounded below on some open subset $U\subset Q^{h_0}$ then we still have uniqueness. Indeed, if $a_\alpha(v_\alpha,\phi)=0$ for all $\phi\in H^1_{\text{per}}(Q^{h_0})$ then $a_\alpha(v_\alpha,v_\alpha)=0$, i.e. v_α and thus u_α vanishes on U. The unique continuation principle implies that u_α vanishes everywhere. Since a_α has the Fredholm property (see Subsection 3.2) we have again also existence for all α and continuous dependence, and can take the inverse Floquet-Bloch transform.*

3. Modes

3.1. **Critical Values, Cut-off Values, and Mode Spaces.** The arguments in the proof of Theorem 2.3 do not work if $\varepsilon = 0$. Indeed, in this case of no absorption there exist (in general) parameters α for which the homogeneous equation (5) (i.e. for $g_\alpha = 0$) admits non-trivial quasi-periodic solutions ϕ satisfying also the Rayleigh expansion (6).

Definition 3.1. *(a) $\alpha \in [-1/2, 1/2]$ is called **cut-off value** if there exists $\ell \in \mathbb{Z}$ with $|\alpha + \ell| = k$.*
*(b) $\alpha \in [-1/2, 1/2]$ is called **critical value** if there exists a non-trivial solution $\phi \in H^1_{\alpha,loc}(Q^\infty)$ of $\Delta\phi + k^2 n\phi = 0$ in Q^∞ satisfying the Rayleigh expansion (6). The set of critical values $\alpha \in [-1/2, 1/2]$ is denoted by $\mathcal{A}$. For $\alpha \in \mathcal{A}$ we define the space $\mathcal{M}(\alpha)$ of modes by*

$$\mathcal{M}(\alpha) \; := \; \left\{\phi \in H^1_{\alpha,loc}(Q^\infty) : \phi \textit{ satisfies } \Delta\phi + k^2 n\phi = 0 \textit{ in } Q^\infty \textit{ and (6)}\right\} . \tag{14}$$

If we decompose k into the form $k = \tilde{\ell} + \kappa$ with $\tilde{\ell} \in \mathbb{Z}_{\geq 0}$ and $\kappa \in (-1/2, 1/2]$ then $\pm\kappa$ are the cut-off values. It is obvious that α is a critical value with mode ϕ if, and only if, $-\alpha$ is a critical value with mode $\bar{\phi}$.

Lemma 3.2. *Let $\alpha \in \mathcal{A}$ and $\phi \in \mathcal{M}(\alpha)$.*
- (a) *Then the coefficients ϕ_ℓ in the Rayleigh expansion* (6) *vanish for all $\ell \in \mathbb{Z}$ with $|\ell + \alpha| < k$.*
- (b) *Let $\phi_\ell = 0$ for all $\ell \in \mathbb{Z}$ with $|\ell + \alpha| \leq k$. Then ϕ is evanescent, i.e. for every $h > h_0$ there exist $c, \sigma > 0$ with $|\phi(x)| \leq c\, e^{-\sigma x_2}$ for $x_2 \geq h$.*

Proof. (a) From Lemma 2.1 (for $\varepsilon = 0$ and $g_\alpha = 0$ and some $h \geq h_0$) we conclude that

$$\int\limits_{Q^h} [\nabla\phi \cdot \nabla\bar{\psi} - k^2 n\, \phi\, \bar{\psi}]\, dx \;-\; 2\pi i \sum_{\ell\in\mathbb{Z}} \phi_\ell(\alpha, h)\, \overline{\psi_\ell(\alpha, h)}\, \sqrt{k^2 - (\ell + \alpha)^2} \;=\; 0 \tag{15}$$

for all $\psi \in H^1_\alpha(Q^h)$. Taking $\psi = \phi$ and the imaginary part yields the assertion.
(b) From part (a) we have for $h = h_0$

$$\phi(x) \;=\; \sum_{|\ell+\alpha|>k} \phi_\ell\, e^{i(\ell+\alpha)x_1}\, e^{-\sqrt{(\ell+\alpha)^2-k^2}(x_2-h_0)} \quad \text{for } x_2 > h_0 \,.$$

Since $\sqrt{(\ell + \alpha)^2 - k^2} \;\geq\; 2\sigma$ for some $\sigma > 0$ and all ℓ with $|\ell + \alpha| > k$ we obtain $|\phi(x)| \leq e^{-\sigma(x_2-h_0)} \sum_{|\ell+\alpha|>k} |\phi_\ell|\, e^{-\frac{1}{2}\sqrt{(\ell+\alpha)^2-k^2}(h-h_0)}$ which yields the assertion with $c = e^{\sigma h_0} \sum_{|\ell+\alpha|>k} |\phi_\ell|\, e^{-\frac{1}{2}\sqrt{(\ell+\alpha)^2-k^2}(h-h_0)}$. ◻

From this result we conclude that every $\phi \in \mathcal{M}(\alpha)$ is evanescent if $\alpha \in \mathcal{A}$ is not a cut-off value. However, if $\alpha \in \mathcal{A}$ is a cut-off value, i.e. $|\ell_0 + \alpha| = k$ for some $\ell_0 \in \mathbb{Z}$, and $\phi \in \mathcal{M}(\alpha)$ then ϕ is evanescent if, and only if, $\phi_{\ell_0} = 0$. We illustrate this with two examples.

Example 3.3. *In the case of the Neumann boundary condition $\partial_{x_2} u = 0$ for $x_2 = 0$ the simplest example is just $\phi(x) = e^{ikx_1}$, $x \in \mathbb{R}^2_+$. We observe that ϕ is quasi-periodic with parameter κ where again $k = \tilde{\ell} + \kappa$. Furthermore, κ is obviously a cut-off value and also critical because ϕ satisfies the Helmholtz equation, the boundary condition $\partial_{x_2}\phi = 0$ for $x_2 = 0$, and the Rayleigh expansion. In this case $\mathcal{M}(\kappa) = \mathrm{span}\{\phi\}$, and ϕ is not evanescent.*

The following example, again for the Neumann boundary condition, constructs a critical value α which is also a cut-off value with a two-dimensional mode space $\mathcal{M}(\alpha)$. The mode space is spanned by an evanescent mode and a non-evanescent mode.

Example 3.4. *This example deals again with the Neumann boundary condition. Let first $k>0$ be arbitrary, $h_0 = 1$ and $n = 1+\pi^2/k^2$ for $0<x_2<1$ and $n=1$ for $x_2>1$. Then*

$$\phi_1(x_1,x_2) \;=\; e^{ikx_1}\cdot\begin{cases} 1\,, & x_2>1\,,\\ -\cos(\pi x_2)\,, & 0<x_2<1\,,\end{cases}$$

solves $\Delta\phi + k^2 n\phi = 0$ in $\mathbb{R}^2_+$ and $\partial_{x_2}\phi(x_1,0)=0$. The function is $\kappa-$quasiperiodic if k is decomposed into $k=\tilde\ell+\kappa$ with $\tilde\ell\in\mathbb{Z}_{\geq 0}$ and $\kappa\in(-1/2,1/2]$. In particular, κ is a critical value, i.e. $\kappa\in\mathcal{A}$, and also a cut-off value. The mode $\phi_1\in\mathcal{M}(\kappa)$ is non-evanescent. Note that $k>0$ is arbitrary and n depends explicitely on k through $n=1+\pi^2/k^2$.
We show now the existence of $k>0$ such that for the same $n=1+\pi^2/k^2$ there is another mode in $\mathcal{M}(\kappa)$ which is evanescent. Indeed, the function

$$\phi_2(x_1,x_2) \;=\; e^{i(k+1)x_1}\cdot\begin{cases} \cos\sqrt{\pi^2+k^2-(k+1)^2}\,e^{-\sqrt{(k+1)^2-k^2}(x_2-1)}\,, & x_2>1\,,\\ \cos\big(\sqrt{\pi^2+k^2-(k+1)^2}x_2\big)\,, & 0<x_2<1\,,\end{cases}$$

is $\kappa-$quasi-periodic and a mode if k solves

$$\sqrt{(k+1)^2-k^2}\cos t_k \;-\; t_k\sin t_k \;=\; 0\,. \tag{16}$$

where $t_k = \sqrt{\pi^2+k^2-(k+1)^2}$. Using $\sqrt{(k+1)^2-k^2} = \sqrt{\pi^2-t_k^2}$ we have to find $t\in\big(0,\sqrt{\pi^2-1}\big)$ such that

$$f(t) \;:=\; \sqrt{\pi^2-t^2}\cos t - t\sin t \;=\; 0\,.$$

From $f(0)=\pi>0$ and $f(\pi/2)=-\pi/2<0$ we conclude that such a zero t_k exists. Then $k=\frac{1}{2}(\pi^2-t_k^2-1)$ satisfies (16), i.e. also ϕ_2 is a mode corresponding to κ which is evanescent.

We define the subspace $\mathcal{M}_{evan}(\alpha)$ of $\mathcal{M}(\alpha)$ for $\alpha\in\mathcal{A}$ by

$$\mathcal{M}_{evan}(\alpha) \;:=\; \big\{\phi\in\mathcal{M}(\alpha):\phi\text{ is evanescent}\big\} \;=\; \big\{\phi\in\mathcal{M}(\alpha):\phi_\ell=0\text{ for }|\ell+\alpha|\leq k\big\}\,.$$

Lemma 3.5. *Let $k\notin\frac{1}{2}\mathbb{N}$ and $\alpha\in\mathcal{A}$. Then the codimension of $\mathcal{M}_{evan}(\alpha)$ in $\mathcal{M}(\alpha)$ is zero or one.*

Proof. Assume that $\mathcal{M}_{evan}(\alpha)\neq\mathcal{M}(\alpha)$. Let $\tilde\phi\in\mathcal{M}(\alpha)\setminus\mathcal{M}_{evan}(\alpha)$. From the remark following Lemma 3.2 α must be a cut-off value, i.e. there exists $\ell_0\in\mathbb{Z}$ with $|\alpha+\ell_0|=k$. From $k\notin\frac{1}{2}\mathbb{N}$ it is easily seen that ℓ_0 is unique. Therefore, ϕ has the form

$$\tilde\phi(x) \;=\; \tilde\phi_{\ell_0}(\alpha,h_0)\,e^{i(\ell_0+\alpha)x_1} \;+\; \sum_{|\ell+\alpha|>k}\tilde\phi_\ell(\alpha,h_0)\,e^{i(\ell+\alpha)x_1}\,e^{-\sqrt{(\ell+\kappa)^2-k^2}\,(x_2-h_0)}\,,\quad x_2\geq h_0\,,$$

and $\tilde\phi_{\ell_0}(\alpha,h_0)\neq 0$. Without loss of generality we can assume that $\tilde\phi_{\ell_0}(\alpha,h_0)=1$. Then every $\phi\in\mathcal{M}(\alpha)$ has the decomposition $\phi=\phi_{\ell_0}(\alpha,h_0)\tilde\phi + [\phi-\phi_{\ell_0}(\alpha,h_0)\tilde\phi]$ and $[\phi-\phi_{\ell_0}(\alpha,h_0)\tilde\phi]\in\mathcal{M}_{evan}(\alpha)$. Therefore, $\mathcal{M}(\alpha)\subset\operatorname{span}\{\tilde\phi\}+\mathcal{M}_{evan}(\alpha)$. ◻

We note that $\tilde\phi \in \mathcal{M}(\alpha)$ constructed in the proof has the form

$$\tilde\phi(x) \;=\; e^{i(\ell_0+\alpha)x_1} \;+\; \sum_{|\ell+\alpha|>k} \tilde\phi_\ell(\alpha,h)\, e^{i(\ell+\alpha)x_1}\, e^{-\sqrt{(\ell+\kappa)^2-k^2}\,(x_2-h)}\,, \quad x_2 \geq h\,,$$

for any $h \geq h_0$. In particular, $\tilde\phi_{\ell_0}(\alpha,h) = 1$ for all $h \geq h_0$.

We will see that the existence of critical cut-off values with $\mathcal{M}_{evan}(\alpha) \neq \mathcal{M}(\alpha)$ lead to a slower decay of the radiating part of the solution of (1).

3.2. **The Mode Spaces are Finite Dimensional.** We fix $h \geq h_0$. From Lemma 2.1 (for $\varepsilon = 0$ and $g_\alpha = 0$) we obtain for the transformed modes $\tilde\phi \in H^1_{\rm per}(Q^h)$, defined by $\tilde\phi(x) = e^{-i\alpha x_1}\phi(x)$, the equivalent characterization

$$\int\limits_{Q^h} [\nabla(e^{i\alpha x_1}\tilde\phi)\cdot\nabla\overline{(e^{i\alpha x_1}\psi)} - k^2 n\,\tilde\phi\,\bar\psi]\,dx \;-\; 2\pi i \sum_{\ell\in\mathbb{Z}} \tilde\phi_\ell\,\overline{\psi_\ell}\,\sqrt{k^2-(\ell+\alpha)^2} \;=\; 0\,,$$

i.e.

$$\int\limits_{Q^h} [\nabla\tilde\phi\cdot\nabla\bar\psi - 2i\alpha\,\partial_{x_1}\tilde\phi\,\bar\psi + (\alpha^2-k^2 n)\,\tilde\phi\,\bar\psi]\,dx \;-\; 2\pi i \sum_{\ell\in\mathbb{Z}} \tilde\phi_\ell\,\overline{\psi_\ell}\,\sqrt{k^2-(\ell+\alpha)^2} \;=\; 0 \tag{17}$$

for all $\psi \in H^1_{\rm per}(Q)$ where we dropped the argument (α,h) in the Fourier coefficients. We equip $H^1_{\rm per}(Q^h)$ with the inner product

$$\langle u,v\rangle_* \;:=\; \int\limits_{Q^h} \nabla u\cdot\nabla\bar v\,dx \;+\; 2\pi\sum_{\ell\in\mathbb{Z}} u_\ell\,\overline{v_\ell}\,(1+|\ell|)\,, \quad u,v\in H^1_{\rm per}(Q)\,. \tag{18}$$

Then this inner product generates a norm which is equivalent to the usual norm in $H^1(Q^h)$. This well known result can be easily shown by, e.g., an indirect proof. By the Theorem of Riesz there exists a bounded operator $K(\alpha)$ from $H^1_{\rm per}(Q^h)$ into itself with

$$\begin{aligned}\langle K(\alpha)u,v\rangle_* \;=\; & \int\limits_{Q^h} [2i\alpha\,\partial_{x_1}u\,\bar v - (\alpha^2-k^2 n)\,u\,\bar v]\,dx \\ & +\; 2\pi\sum_{\ell\in\mathbb{Z}} u_\ell\,\overline{v_\ell}\,\big[i\sqrt{k^2-(\ell+\alpha)^2} + |\ell| + 1\big]\end{aligned} \tag{19}$$

for $u,v \in H^1_{\rm per}(Q^h)$. With this operator $K(\alpha)$ equation (17) is written as $\langle\tilde\phi - K(\alpha)\tilde\phi,\psi\rangle_* = 0$ for all $\psi \in H^1_{\rm per}(Q^h)$, i.e.

$$\tilde\phi - K(\alpha)\tilde\phi \;=\; 0 \quad \text{for } \tilde\phi \in H^1_{\rm per}(Q^h)\,. \tag{20}$$

Therefore, α is critical if, and only if, $I - K(\alpha)$ is not invertible. Further properties of $K(\alpha)$ are collected in the following lemma.

Lemma 3.6. (a) *$K(\alpha)$ is compact for every α and depends continuously on α.*

(b) *Let α be a critical value (could also be a cut-off value), i.e. $\alpha \in \mathcal{A}$. Then 1 is a semi-simple eigenvalue of $K(\alpha)$, i.e. $\mathcal{N}\big([I-K(\alpha)]^2\big) = \mathcal{N}\big(I-K(\alpha)\big)$. The decomposition of $H^1_{per,0-}(Q)$ in the form*

$$H^1_{\rm per}(Q^h) \;=\; \mathcal{N}\big(I-K(\alpha)\big) \;\oplus\; \mathcal{R}\big(I-K(\alpha)\big)$$

is orthogonal.

(c) *The operator $K(\alpha)$ is infinitely often differentiable at all α which are not cut-off values.*

Proof. (a) Compactness follows from the compact embedding of $H^1_{\text{per}}(Q^h)$ into $L^2(Q^h)$ and the boundedness of $-\sqrt{(\ell+\alpha)^2-k^2}+|\ell|+1$ for large values of $|\ell|$. Continuity follows from the continuity of every term and the convergence of
$\sum_\ell |u_\ell \overline{v_\ell}| \le \big[\sum_\ell |u_\ell|^2\big]^{1/2}\big[\sum_\ell |v_\ell|^2\big]^{1/2} \le c\|u|_\gamma\|_{H^{1/2}(\gamma)}\|v_\gamma\|_{H^{1/2}(\gamma)}$ where $\gamma=(0,2\pi)\times\{h\}$.
(b) First we show $\mathcal{N}\big(I-K(\alpha)\big)=\mathcal{N}\big(I-K(\alpha)^*\big)$. Indeed, Let $\phi\in\mathcal{N}\big(I-K(\alpha)\big)$ and $\psi\in H^1_{\text{per},0-}(Q^h)$ arbitrary. Then

$$\begin{aligned}\big\langle (I-K(\alpha))\psi,\phi\big\rangle_* &= \int\limits_{Q^h}\big[\nabla\psi\cdot\nabla\bar\phi-2i\alpha\,\frac{\partial\psi}{\partial x_1}\,\bar\phi-(k^2n-\alpha^2)\,\psi\,\bar\phi\big]\,dx\\ &\quad+\sum_{|\ell+\alpha|\ge k}\sqrt{(\ell+\alpha)^2-k^2}\,\psi_\ell\,\overline{\phi_\ell}\\ &= \overline{\big\langle (I-K(\alpha)\phi,\psi\big\rangle_*} \;=\; 0\,.\end{aligned}$$

Therefore, $(I-K(\alpha)^*)\phi=0$. This shows $\mathcal{N}\big(I-K(\alpha)\big)=\mathcal{N}\big(I-K(\alpha)^*\big)$.
Let now $u\in\mathcal{N}\big((I-K(\alpha))^2\big)$ and set $v=\big(I-K(\alpha)\big)u$. Then $v\in\mathcal{N}\big(I-K(\alpha)\big)=\mathcal{N}\big(I-K(\alpha)^*\big)$ by the previous argument. Therefore, $\|v\|_*^2=\big\langle (I-K(\alpha))u,v\big\rangle_*=\big\langle u,(I-K(\alpha)^*)v\big\rangle_*=0$. Therefore, $v=0$; that is, $u\in\mathcal{N}\big(I-K(\alpha)\big)$. The orthogonality follows by the same arguments. Indeed, if $u\in\mathcal{N}\big(I-K(\alpha)\big)=\mathcal{N}\big(I-K(\alpha)^*\big)$ and $v=\big(I-K(\alpha)\big)\psi\in\mathcal{R}\big(I-K(\alpha)\big)$ then $\langle v,u\rangle_*=\big\langle (I-K(\alpha))\psi,u\big\rangle_*=\big\langle\psi,(I-K(\alpha)^*)u\big\rangle_*=0$.
(c) The integral is certainly smooth at all α. Furthermore, $\frac{d}{d\alpha}\sqrt{k^2-|\ell+\alpha|^2}=-(\ell+\alpha)/\sqrt{k^2-|\ell+\alpha|^2}$ which is also bounded for large values of ℓ because $|\ell+\alpha|\neq k$ for all ℓ. The same holds for all derivatives. Then we can argue as in Part (a). □

By a theorem of Riesz the null space $\mathcal{N}(I-K(\alpha))$ is finite dimensional. This implies the following corollary.

Corollary 3.7. *Every space $\mathcal{M}(\alpha)$ of modes for $\alpha\in\mathcal{A}$ is finite dimensional.*

3.3. **Construction of a Basis.** As the next step we construct a convenient basis of $\mathcal{M}(\alpha)$. As in [14] the following sesqui-linear form will play an important role.

$$E(u,v) \;:=\; i\int\limits_{Q^\infty}\big[u\,\partial_{x_1}\bar v-\bar v\,\partial_{x_1}u\big]\,dx\quad\text{for } u,v\in H^1(Q^\infty)\,. \tag{21}$$

We collect some properties.

Lemma 3.8. *Let $u\in\mathcal{M}(\alpha)$ and $v\in\mathcal{M}(\beta)$ for $\alpha,\beta\in\mathcal{A}$ and at least one of them is evanescent.*

(a) *Then $E(u,v)$ exists and*

$$E(u,v) \;=\; 2\pi\, i\int\limits_{x_1=b}\big[u\,\partial_{x_1}\bar v-\bar v\,\partial_{x_1}u\big]\,ds\quad \textit{for any } b\in\mathbb{R} \tag{22}$$

where we write $\int_{x_1=b}$ for the line integral $\int_C$ with $C=\{b\}\times(0,\infty)$.

(b) $E(u,v) = 0$ *if* $\alpha, \beta \in \mathcal{A} \setminus \{-1/2\}$ *and* $\alpha \neq \beta$.

Proof. Existence of the integrals follows because the product $u\bar{v}$ is evanescent.
(a) First we note that the line integrals $\int_{x_1=b}\big[u\,\partial_{x_1}\bar{v} - \bar{v}\,\partial_{x_1}u\big]ds$ are independent of b which follows from Green's second theorem, applied in the region $(b_1,b_2)\times(0,R)$ and letting R tend to infinity.
We apply Green's first theorem in Q^∞ to the functions $x_1u(x)$ and $\bar{v}(x)$. With $(\Delta + k^2n)(x_1u) = 2\partial_{x_1}u$ we obtain

$$\int\limits_{Q^\infty}\big[\nabla(x_1u)\cdot\nabla\bar{v} - k^2nx_1u\,\bar{v} + 2\bar{v}\,\partial_{x_1}u\big]dx \;=\; \int\limits_{x_1=2\pi}\partial_{x_1}(x_1u)\,\bar{v}\,ds\,,$$

i.e.

$$\int\limits_{Q^\infty}x_1\big[\nabla u\cdot\nabla\bar{v} - k^2nu\,\bar{v}\big] + \big[u\,\partial_{x_1}\bar{v} + 2\bar{v}\,\partial_{x_1}u\big]dx \;=\; \int\limits_{x_1=2\pi}\big[2\pi\partial_{x_1}u\,\bar{v} + u\,\bar{v}\big]ds\,.$$

Now we interchang the roles of u and $\bar{v}$ and take the difference. This yields

$$\int\limits_{Q^\infty}\big[\bar{v}\,\partial_{x_1}u - u\,\partial_{x_1}\bar{v}\big]\,dx \;=\; 2\pi\int\limits_{x_1=2\pi}\big[\bar{v}\,\partial_{x_1}u - u\,\partial_{x_1}\bar{v}\big]\,ds$$

which proves part (a).
(b) By (22) and the quasi-periodicity of u and v we obtain

$$\int\limits_{x_1=0}\big[\bar{v}\,\partial_{x_1}u - u\,\partial_{x_1}\bar{v}\big]\,ds = \int\limits_{x_1=2\pi}\big[\bar{v}\,\partial_{x_1}u - u\,\partial_{x_1}\bar{v}\big]\,ds = e^{i(\alpha-\beta)2\pi}\int\limits_{x_1=0}\big[\bar{v}\,\partial_{x_1}u - u\,\partial_{x_1}\bar{v}\big]\,ds$$

which shows that the integrals vanish because $e^{i(\alpha-\beta)2\pi} \neq 1$. □

With this form E we contruct a basis of $\mathcal{M}(\alpha)$ for any $\alpha \in \mathcal{A}$. We will see below that the following basis will follow naturally from the Limiting Absorption Principle as in, e.g., [13] or [14].

We make the following assumptions.

Assumption 3.9. *Let* $k \notin \frac{1}{2}\mathbb{N}$ *and decompose* k *into* $k = \tilde{\ell} + \kappa$ *for some* $\tilde{\ell} \in \mathbb{Z}_{\geq 0}$ *and* $\kappa \in (-1/2, 1/2) \setminus \{0\}$*. Then* $\pm\kappa$ *are the cut-off values. We assume that* κ *(and thus also* $-\kappa$*) are critical values, i.e.* $\pm\kappa \in \mathcal{A}$*.*

The assumption that the cut-off values are critical (as in Examples 3.3 and 3.4) decribes the new situation in this paper. If the set of cut-off values is disjoint from the set of critical values we refer to [11] for a complete discussion. If $k \in \frac{1}{2}\mathbb{N}$ then $\kappa = 0$ or $\kappa = 1/2$, and the space $\mathcal{M}_{evan}(\kappa)$ can have co-dimension 2. This situation requires a different discussion.

Assumption 3.10. *For every* $\alpha \in \mathcal{A}$ *and every* $v \in \mathcal{M}_{evan}(\alpha)$ *with* $v \neq 0$ *let the linear form* $E(\cdot, v)$ *be not trivial on* $\mathcal{M}_{evan}(\alpha)$*.*

Let $\alpha \in \mathcal{A}$ and let $\mathcal{M}_{evan}(\alpha) \neq \{0\}$. With the hermetian sesqui-linear form E from (21) we consider the self-adjoint eigenvalue problem to determine $\lambda_\ell \in \mathbb{R}$ and non-trivial $\phi^\ell \in \mathcal{M}_{evan}(\alpha)$, $\ell = 1, \ldots, m := \dim\mathcal{M}_{evan}(\alpha)$, with

$$E(\phi^\ell, \psi) \;=\; \lambda_\ell\, k^2\int\limits_Q q\,\phi^\ell\,\bar{\psi}\,dx \quad \text{for all } \psi \in \mathcal{M}_{evan}(\alpha)\,. \tag{23}$$

The eigenfunctions ϕ^ℓ are orthogonal with respect to the inner product $\langle u,v\rangle_q := k^2\langle qu,v\rangle_{L^2(Q)}$. We normalize the eigenfunctions such that $\langle \phi^\ell,\phi^j\rangle_q = \delta_{\ell,j}$. Note that the eigenfunctions depend on the function q which appears in the definition of $n_\varepsilon = n + i\varepsilon q$.

In the case $\mathcal{M}(\alpha) \neq \mathcal{M}_{evan}(\alpha)$ we extend this basis to a basis of $\mathcal{M}(\alpha)$.

Lemma 3.11. *Let Assumptions 3.9 and 3.10 hold.*

(a) *Then $E(\phi^\ell,\phi^\ell) = \lambda_\ell \neq 0$ for all $\ell = 1,\ldots,m$.*

(b) *Let $\alpha \in \mathcal{A}$ and $\mathcal{M}_{evan}(\alpha) \neq \{0\}$ and ϕ^ℓ, $\ell = 1,\ldots,m$, be the eigenfunctions of the eigenvalue problem (23). Then $\{\phi^\ell : \ell = 1,\ldots,m\}$ is an orthonormal (with respect to $\langle u,v\rangle_q$) basis of $\mathcal{M}_{evan}(\alpha)$.*

(c) *Let $\alpha \in \mathcal{A}$ be a cut-off value, i.e. $\alpha = \pm\kappa$, such that $\mathcal{M}(\pm\kappa) \neq \mathcal{M}_{evan}(\pm\kappa)$. Then there exist a unique $\hat\phi^\pm \in \mathcal{M}(\pm\kappa)$ with $E(\hat\phi^\pm,\psi) = 0$ for all $\psi \in \mathcal{M}_{evan}(\pm\kappa)$ and $\hat\phi^\pm$ has the form*

$$\hat\phi^\pm(x) \;=\; e^{\pm ikx_1} \;+\; \hat\phi^\pm_{evan}(x)\,, \quad x_2 > h_0\,, \tag{24}$$

and $\hat\phi^\pm_{evan}$ is evanescent. We set $\hat\phi^\pm_{evan} := \hat\phi^\pm$ for $x_2 < h_0$. Therefore, if $\{\phi^{\ell,\pm} : \ell = 1,\ldots,m\}$ is a basis of $\mathcal{M}_{evan}(\pm\kappa)$ determined by part (b) then $\{\phi^{\ell,\pm} : \ell = 1,\ldots,m\} \cup \{\hat\phi^\pm\}$ is a basis of $\mathcal{M}(\pm\kappa)$.

We note that $\hat\phi^\pm_{evan}$ is a $\pm k-$quasi-periodic solution of the Helmholtz equation in $Q^\infty \setminus \{x : x_2 = h_0\}$ satisfying the transmission conditions $[\hat\phi^\pm_{evan}] = e^{\pm ikx_1}$ and $[\partial_{x_2}\hat\phi^\pm_{evan}] = 0$ for $x_2 = h_0$.

Proof. (a) We assume that $\lambda_\ell = 0$ for some ℓ. Then $E(\phi^\ell,\psi) = \lambda_\ell k^2 \int_Q q\,\phi^\ell\bar\psi\,dx = 0$ for all $\psi \in \mathcal{M}_{evan}(\alpha)$, i.e. $E(\phi^\ell,\cdot)$ is trivial on $\mathcal{M}_{evan}(\alpha)$, a contradiction to Assumption 3.10.

(b) This is obvious.

(c) Let $\alpha = \kappa$ and $\tilde\phi \in \mathcal{M}(\kappa)$ be the function constructed in the proof of Lemma 3.5 normalized such that $\tilde\phi_{\tilde\ell}(\kappa,h_0) = 1$. [1] We set

$$\hat\phi^+ \;=\; \tilde\phi \;-\; \sum_{\nu=1}^m \frac{E(\tilde\phi,\phi^{\nu,+})}{\lambda_{\nu,+}}\,\phi^{\nu,+}\,,$$

where again $\{\phi^{\nu,+} : \nu = 1,\ldots,m\}$ is the basis of $\mathcal{M}_{evan}(\kappa)$ constructed by the eigenvalue problem (23). Then $\hat\phi^+$ has the desired properties. Indeed, for any $j = 1,\ldots,m$ we compute

$$E(\hat\phi^+,\phi^{j,+}) \;=\; E(\tilde\phi,\phi^{j,+}) \;-\; \sum_{\nu=1}^m \frac{E(\tilde\phi,\phi^{\nu,+})}{\lambda_{\nu,+}}\,E(\phi^{\nu,+},\phi^{j,+}) \;=\; 0\,.$$

Furthermore, $\hat\phi^+_{\tilde\ell}(\kappa,h_0) = \tilde\phi_{\tilde\ell}(\kappa,h_0) = 1$ because $\phi^{\nu,+}_{\tilde\ell}(\kappa,h_0) = 0$ for all $\nu = 1,\ldots,m$. Therefore, $\hat\phi^+$ has the form (24) □

[1] Recall that $k = \tilde\ell + \kappa$. Note that $\tilde\phi_{\tilde\ell}(\kappa,h_0) = 1$ implies $\tilde\phi_{\tilde\ell}(\kappa,h) = 1$ for all $h \geq h_0$ by the remark following Lemma 3.5.

4. Finiteness of the Set of Critical Values And The Limiting Absorption Principle

4.1. **Statement of the Main Results.** A first goal is to show that the set $\mathcal{A}$ of critical values is finite.

Theorem 4.1. *Let Assumptions 3.9 and 3.10 hold. Then there exist only finitely many critical values in the interval $[-1/2, 1/2]$. They are symmetric with respect to 0, i.e. α is critical if, and only if, $-\alpha$ is critial. We number them by $\hat{\alpha}_j$, $j \in J$, where $J \subset \mathbb{Z} \setminus \{0\}$ is a finite set which is symmetric with respect to 0. By Assumption 3.9 the cut-off values $\pm\kappa$ are included in the set of critical values. Therefore, $|J| \geq 2$, and we denote these particular critical values by $\hat{\alpha}_1 = \kappa$ and $\hat{\alpha}_{-1} = -\kappa$.*
The corresponding mode spaces $\mathcal{M}(\hat{\alpha}_j)$ are finite dimensional for every $j \in J$. We set $m_j := \dim \mathcal{M}_{evan}(\hat{\alpha}_j)$.

We will prove this theorem in Subsection 4.5. For every $j \in J$ we determine the basis $\{\phi^{\ell,j} : \ell = 1, \ldots, m_j\}$ of $\mathcal{M}_{evan}(\hat{\alpha}_j)$ by the eigenvalue problem (23) and extend this basis to a basis of $\mathcal{M}(\hat{\alpha}_j)$ if $j = \pm 1$ and $\mathcal{M}_{evan}(\pm\kappa) \neq \mathcal{M}(\pm\kappa)$ by contructing $\hat{\phi}^{\pm} \in \mathcal{M}(\pm\kappa)$ as in Lemma 3.11.

We consider now the equation (7) for $n_\varepsilon = n + i\varepsilon q$ and $\varepsilon > 0$. The second goal of this paper is the proof of the following convergence property.

Theorem 4.2. *Let Assumptions 3.9 and 3.10 hold. For any $R_0 > 2\pi + 1$ let $\xi^{\pm} \in C^\infty(\mathbb{R})$ be a pair of functions with $\xi^{\pm}(x_1) = 1$ for $\pm x_1 \geq R_0$ and $\xi^{\pm}(x_1) = 0$ for $\pm x_1 \leq R_0 - 1$. The solutions $u_\varepsilon \in H^1_*(\mathbb{R}^2_+)$ of (1), (4) (which exist and are unique by Theorem 2.3) converge to some function u_0 which has the form $u_0 = u_0^{prop} + u_0^{rad}$ where*

$$(25)\quad u_0^{prop}(x) \;=\; \sum_{\sigma\in\{+,-\}} \sum_{j\in J} \xi^\sigma(x_1) \sum_{\ell:\sigma\lambda_{\ell,j}>0} \frac{2\pi i}{|\lambda_{\ell,j}|} \langle f, \phi^{\ell,j}\rangle_{L^2(W^{h_0})}\, \phi^{\ell,j}(x)\,, \quad x \in \mathbb{R}^2_+\,,$$

$$(26)\quad u_0^{rad}(x) \;=\; \tilde{u}^{rad}(x) \;+ \sum_{\sigma\in\{+,-\}} \xi^\sigma(x_1)\, \frac{e^{i\pi/4}}{\sqrt{2\pi k|x_1|}} \langle f, \hat{\phi}^\sigma\rangle_{L^2(W^{h_0})} \hat{\phi}^\sigma(x)\,, \quad x \in \mathbb{R}^2_+\,,$$

respectively, for some $\tilde{u}^{rad} \in H^1_*(\mathbb{R}^2_+)$. If $\mathcal{M}_{evan}(\pm\kappa) = \mathcal{M}(\pm\kappa)$ then $u_0^{rad} = \tilde{u}^{rad}$. Here, $\phi^{\ell,j}$ are the evanescent modes corresponding to the critical values $\hat{\alpha}_j \in (-1/2, 1/2]$, $j \in J$, and $\hat{\phi}^{\pm}$ are the non-evanescent modes of $\hat{\alpha}_{\pm 1} = \pm\kappa$ if $\mathcal{M}_{evan}(\pm\kappa) \neq \mathcal{M}(\pm\kappa)$. The convergence is understood in $H^1\big((-R, R) \times (0, h)\big)$ for all $R > 0$ and $h \geq h_0$.
Finally, the part $\tilde{u}^{rad}(x)$ decays as $\mathcal{O}(1/|x_1|)$ as $x_1 \to \pm\infty$ for every $x_2 > 0$.

Before we prove these theorems we want to illustrate the previous result with two examples.

Example 4.3. *We consider the simplest case of a homogeneous half space with Neumann boundary conditions as in Example 3.3. Let $f \in L^2(W^{h_0})$ have compact support. Then the unique solution of the problem $\Delta u + k^2 u = -f$ in $\mathbb{R}^2_+$, $\partial_{x_2} u = 0$ for $x_2 = 0$, satisfying the Sommerfeld radiation condition is given by*

$$u(x) \;=\; \frac{i}{4} \int\limits_{W^{h_0}} \left[H_0^{(1)}(k|x-y|) + H_0^{(1)}(k|x-y^*|)\right] f(y)\, dy\,, \quad x \in \mathbb{R}^2_+\,,$$

where $y^* = (y_1, -y_2)^\top$. *From the asymptotics*

$$\frac{i}{4} H_0^{(1)}(k|x-y|) \;=\; \frac{e^{i\pi/4}}{2\sqrt{2\pi k}} \frac{e^{ik|x|}}{\sqrt{|x|}} e^{-ik\hat{x}\cdot y} \;+\; \mathcal{O}(|x|^{-3/2}) \text{ as } |x| \to \infty$$

(where $\hat{x} = x/|x|$*) we obtain*

$$u(x) \;=\; \frac{e^{i\pi/4}}{2\sqrt{2\pi k}} \frac{e^{ik|x|}}{\sqrt{|x|}} \int\limits_{W^{h_0}} \left[e^{-ik\hat{x}\cdot y} + e^{-ik\hat{x}\cdot y^*}\right] f(y)\, dy \;+\; \mathcal{O}(|x|^{-3/2}) \text{ as } |x| \to \infty\,.$$

Therefore, for fixed x_2 *and* $x_1 \to +\infty$ *we obtain (since* $\hat{x} \approx (1,0)$*)*

$$u(x) \;=\; \frac{e^{i\pi/4}}{\sqrt{2\pi k}} \frac{e^{ikx_1}}{\sqrt{x_1}} \int\limits_{W^{h_0}} e^{-iky_1} f(y)\, dy \;+\; \mathcal{O}(|x_1|^{-3/2}\,.$$

This coincides with (26) *because in this case no evanescent modes exist and* $\hat{\phi}^{\pm}(x) = e^{\pm ikx_1}$.

With respect to radiation conditions the following example illustrates a fundamental difference to source problems for inhomogeneous media with refractive indices $n = n(x)$ where $n(x) - 1$ has bounded support. While for these problems the limits as $\varepsilon \to 0$ of the solutions corresponding to $n + i\varepsilon q$ (for some $q = q(x) > 0$ on the support of $n - 1$) are independent of q in the case considered here the limit does depend on the choice of q.

Example 4.4. *Let* $k > 0$ *and* $m \in \mathbb{Z}$ *with* $m + 1/2 > k$. *Set* $\omega := m + 1/2$ *and define* $\phi_\pm$ *by*

$$\phi_\pm(x) \;=\; e^{\pm i\omega x_1} \cdot \begin{cases} \cos\sqrt{k^2 n - \omega^2}\, e^{-\sqrt{\omega^2 - k^2}(x_2 - 1)}\,, & x_2 > 1\,, \\ \cos\!\left(\sqrt{k^2 n - \omega^2} x_2\right), & 0 < x_2 < 1\,, \end{cases}$$

where the constant $n > 0$ *is chosen such* $\partial_{x_2}\phi_\pm$ *is continuous at* $x_2 = 1$, *i.e.* n *satisfies*

$$\sqrt{\omega^2 - k^2}\cos\sqrt{k^2 n - \omega^2} - \sqrt{k^2 n - \omega^2}\sin\sqrt{k^2 n - \omega^2} \;=\; 0\,. \tag{27}$$

Such a value of n *exists. Indeed, set* $\psi(t) := \sqrt{\omega^2 - k^2}\cos t - t \sin t$. *Then* $\psi(0) > 0$ *and* $\psi(\pi/2) < 0$, *i.e. there exists* $\hat{t} \in (0, \pi/2)$ *with* $\psi(\hat{t}) = 0$. *Then* $n = \frac{\hat{t}^2 + \omega^2}{k^2}$ *solves* (27). *We note further that* $\phi_\pm$ *are evanescent and* $\alpha-$*quasi-periodic with* $\alpha = 1/2$. *Therefore,* $\mathcal{M}(1/2) = \mathcal{M}_{evan}(1/2) = \operatorname{span}\{\phi_+, \phi_-\}$. *Furthermore,* $E(\phi_+, \phi_+) > 0$ *and* $E(\phi_-, \phi_-) < 0$ *and* $E(\phi_+, \phi_-) = 0$ *because* $\int_0^{2\pi} e^{i2\omega x_1} dx_1 = 0$.
If $q > 0$ *is a constant then also* $\langle \phi_+, \phi_- \rangle_q = k^2 q \int_{Q^\infty} \phi_+ \overline{\phi_-} dx = 0$. *Therefore, in this case* $\phi^1 = \phi_+/\|\phi_+\|_q$ *and* $\phi^2 = \phi_-/\|\phi_-\|_q$ *are the eigenfunctions of* (23) *with* $\lambda_\ell = E(\phi^\ell, \phi^\ell)$. *The solution* u_0 *of the source problem* (1) *arizing from the limit* $\varepsilon \to 0$ *for* $n + i\varepsilon q$ *has the asymptotic form* $u_0(x) \approx c\,\phi^1(x)$ *as* $x_1 \to \infty$ *for some* $c \in \mathbb{C}$.
If, however, $q = \tilde{q}(x_1) > 0$ *is not constant but* 2π *periodic (and constant with respect to* x_2*) such that* $\int_0^{2\pi} \tilde{q}(x_1) e^{i2\omega x_1} dx_1 \neq 0$ *then* ϕ_+, ϕ_- *are not orthogonal anymore with respect to* $\langle \cdot, \cdot \rangle_{\tilde{q}}$, *and the eigenfunctions* $\{\tilde{\phi}^1, \tilde{\phi}^2\}$ *and eigenvalues* $\{\tilde{\lambda}_1, \tilde{\lambda}_2\}$ *of* (23) *are different from* $\{\phi^1, \phi^2\}$ *and* $\{\lambda_1, \lambda_2\}$, *respectively. A simple argument shows that* $\tilde{\lambda}_1$ *and* $\tilde{\lambda}_2$ *are of different sign. Let, e.g.,* $\tilde{\lambda}_1 > 0$. *Then* $\tilde{\phi}^1$ *has the form* $\tilde{\phi}^1 = a\phi^1 + b\phi^2$ *with* $a, b \neq 0$. *Therefore, the solution* $\tilde{u}_0$ *of the source problem* (1) *arizing from the limit* $\varepsilon \to 0$ *for* $n + i\varepsilon\tilde{q}$ *has the asymptotic form* $\tilde{u}_0(x) \approx \tilde{c}\,\tilde{\phi}^1(x) = \tilde{c}[a\phi^1 + b\phi^2]$ *as* $x_1 \to \infty$ *for some* $\tilde{c} \in \mathbb{C}$.

We conclude that u_0 and $\tilde{u}_0$ are both solutions of (1)*, and they are linearly independent. Both arize from limiting absorption principles, and there is no preferable solution. In this way there is no unique radiation condition but only a class of radiation conditions which depend on the inner products $\langle\cdot,\cdot\rangle_q$ in* (23)*.*

We return to the proof of Theorems 4.1 and 4.2, fix any $h \geq h_0$ and write equation (7) in the form

$$v(\alpha,\varepsilon) - K(\alpha,\varepsilon)v(\alpha,\varepsilon) \;=\; r(\alpha) \tag{28}$$

for the transformed field $v(\alpha,\varepsilon) = e^{-i\alpha x_1}u_{\alpha,\varepsilon} \in H^1_{\text{per}}(Q^h)$ where the operator $K(\alpha,\varepsilon)$ from $H^1_{per}(Q^h)$ into itself is now defined in the same way as $K(\alpha)$ from (19), but depends now on the two parameters α and ε. The right hand side $r(\alpha)$ is given by

$$\langle r(\alpha),\psi\rangle_* \;=\; \int\limits_{Q^{h_0}} e^{-i\alpha x_1} g_\alpha(x)\,\overline{\psi(x)}\,dx\,, \quad \psi \in H^1_{\text{per}}(Q^h)\,, \tag{29}$$

where g_α is the Floquet-Bloch transform of the right hand side f of (1), i.e. $g_\alpha = (Ff)(\cdot,\alpha)$. We note that $K(\alpha,\varepsilon)$ and $r(\alpha)$ depend on h.

For the behavior of the solution as $\varepsilon \to 0$ the smoothness of $(\alpha,\varepsilon) \mapsto K(\alpha,\varepsilon)$ and $\alpha \mapsto r(\alpha)$ is important. Since f has compact support we conclude that g_α and thus also $r(\alpha)$ depends smoothly on α.

4.2. **Smoothness of $(\alpha,\varepsilon) \mapsto K(\alpha,\varepsilon)$.** The operator $K(\alpha,\varepsilon)$ is certainly infinitely often differentiable with respect to ε (it is even linear with respect to ε). From Lemma 3.6 we know that $K(\alpha,\varepsilon)$ is also infinitely often differentiable with respect to α at $\hat{\alpha}$ if $\hat{\alpha}$ is not a cut-off value, i.e. $|\ell+\hat{\alpha}| \neq k$ for all ℓ. Let now $\hat{\alpha}$ be a cut-off value and $\ell_0 \in \mathbb{Z}$ with $|\ell_0+\hat{\alpha}| = k$. Recall that $\ell_0 = \pm\tilde{\ell}$ if $\hat{\alpha} = \pm\kappa$. For α in a neighborhood of $\hat{\alpha}$ we decompose $K(\alpha,\varepsilon)$ into $K(\alpha,\varepsilon) = \tilde{K}(\alpha,\varepsilon) + \rho(\alpha)B$ with

$$\begin{aligned}\langle\tilde{K}(\alpha,\varepsilon)u,v\rangle_* \;&:=\; \int\limits_{Q^h}\big[2i\alpha\,\partial_{x_1}u\,\bar{v} - (\alpha^2 - k^2 n_\varepsilon)\,u\,\bar{v}\big]\,dx \\ &\quad + 2\pi\sum_{\ell\neq\ell_0} u_\ell\,\overline{v_\ell}\,\Big[i\sqrt{k^2-(\ell+\alpha)^2} + |\ell| + 1\Big] \;+\; 2\pi\,u_{\ell_0}\,\overline{v_{\ell_0}}\,(1+|\ell_0|)\,,\end{aligned} \tag{30}$$

$$\rho(\alpha) \;:=\; 2\pi i\,\sqrt{k^2-(\ell_0+\alpha)^2}\,, \tag{31}$$

$$\langle Bu,v\rangle_* \;:=\; u_{\ell_0}\,\overline{v_{\ell_0}} \tag{32}$$

for $u,v \in H^1_{\text{per}}(Q^h)$ where we dropped the argument (α,h) in the coefficients. We note that $Bu = \langle u,b\rangle_* b$ where $b \in H^1_{\text{per}}(Q^h)$ is defined by $\langle u,b\rangle_* = u_{\ell_0} = \frac{1}{2\pi}\int_0^{2\pi} u(x_1,h)e^{-i\ell_0 x_1}dx_1$ for $u \in H^1_{\text{per}}(Q^h)$. Then $\tilde{K}(\alpha,\varepsilon)$ is infinitely often differentiable in a neighborhood of $(\hat{\alpha},0)$.

4.3. **An Abstract Representation Theorem.** In this section we consider (28) in an abstract setting. Without loss of generality we assume that the critical value is $\hat{\alpha} = 0$.

Setting and Assumptions: In the following theorem let H be a (complex) Hilbert space, $K(\alpha,\varepsilon): H \to H$ and $r(\alpha) \in H$ for $(\alpha,\varepsilon) \in (-\delta_0,\delta_0)\times[0,\delta_0) \subset \mathbb{R}\times\mathbb{R}$ be families of compact operators and elements, respectively. Set $L(\alpha,\varepsilon) = I - K(\alpha,\varepsilon)$ and assume that 1 is a semi-simple eigenvalue of $K(0,0)$; that is, $\mathcal{N}\big(L(0,0)^2\big) = \mathcal{N}\big(L(0,0)\big)$ where

$\mathcal{N}(L) = \{x \in H : Lx = 0\}$ denotes the nullspace of an operator L. Furthermore, let $P : H \to \mathcal{N} \subset H$ be the projection onto the finite dimensional space $\mathcal{N} := \mathcal{N}\big(L(0,0)\big)$ with respect to the direct decomposition $H = \mathcal{N} \oplus \mathcal{R}$ where $\mathcal{R} := \mathcal{R}\big(L(0,0)\big)$. We assume that this decomposition is orthogonal with orthogonal projection operator $P : H \to \mathcal{N}$. Let $Q := I - P$ the projection onto $\mathcal{R}$.
Furthermore, we assume that $K(\alpha,\varepsilon) : H \to H$ has the form $K(\alpha,\varepsilon) = \tilde{K}(\alpha,\varepsilon) + \rho(\alpha)\, B$ where $\tilde{K}(\alpha,\varepsilon)$ depends smoothly, i.e. infinitely often differentiably, on $(\alpha,\varepsilon) \in (-\delta_0,\delta_0) \times [0,\delta_0)$ and $\rho : (-\delta_0,\delta_0) \to \mathbb{C}$ is continuous with $\rho(0) = 0$ such that $\rho(\alpha) \neq 0$ for $\alpha \neq 0$ and $\alpha \mapsto \rho(\alpha)^2$ is smooth with $\frac{d}{d\alpha}[\rho(\alpha)^2]|_{\alpha=0} \neq 0$. (Essentially, ρ is the square root function.) Furthermore, $B : H \to H$ is a one-dimensional operator, given by $Bv = \langle v, b\rangle\, b$ for $v \in H$ and some $b \in H$.
We denote the projections of the partial derivatives of $\tilde{L}(\alpha,\varepsilon) := I - \tilde{K}(\alpha,\varepsilon)$ at $(0,0)$ by $M_\alpha := P\partial_\alpha \tilde{L}(0,0)$ and $M_\varepsilon = iP\partial_\varepsilon \tilde{L}(0,0)$. We assume that $M_\alpha|_{\mathcal{N}}$ and $M_\varepsilon|_{\mathcal{N}}$ are self-adjoint, and M_ε is positive (i.e. $\langle M_\varepsilon u, u\rangle > 0$ for $u \in \mathcal{N}$, $u \neq 0$).

Let $\mathcal{N}_0$ be the orthogonal complement of Pb in $\mathcal{N}$, i.e. $\mathcal{N}_0 := \{\phi \in \mathcal{N} : \langle \phi, Pb\rangle = 0\} = \{\phi \in \mathcal{N} : \langle \phi, b\rangle = 0\}$. (If $Pb = 0$ then $\mathcal{N}_0 = \mathcal{N}$.) Let $P_0 : \mathcal{N} \to \mathcal{N}_0$ be the orthogonal projection, given by $P_0 v = v - \langle v, Pb\rangle \frac{Pb}{\|Pb\|^2} = -\langle v, b\rangle \frac{Pb}{\|Pb\|^2}$, $v \in \mathcal{N}$. (The operator P_0 is the identity if $Pb = 0$.) Then $P_0 P$ is the orthogonal projection from H onto $\mathcal{N}_0$.

Since also the operators $P_0 M_\alpha|_{\mathcal{N}_0}$ and $P_0 M_\varepsilon|_{\mathcal{N}_0}$ are self-adjoint and $P_0 M_\varepsilon|_{\mathcal{N}_0}$ is positive we can consider the following self-adjoint eigenvalue problem in the finite dimensional space $\mathcal{N}_0$: Determine $\lambda_\ell \in \mathbb{R}$ and non-trivial $\phi_\ell \in \mathcal{N}_0$, $\ell = 1,\ldots,m$, (where $m = \dim \mathcal{N}_0$) with

$$P_0 M_\alpha \phi_\ell \;=\; \lambda_\ell P_0 M_\varepsilon \phi_\ell\,. \tag{33}$$

The eigenfunctions corresponding to different eigenvalues are orthogonal with respect to $\langle u, v\rangle_{M_\varepsilon} := \langle u, M_\varepsilon v\rangle$. We normalize the eigenfunctions by $\langle \phi_j, \phi_\ell\rangle_{M_\varepsilon} = \delta_{j,\ell}$.

If $\mathcal{N} \neq \mathcal{N}_0$ we extend this basis to a basis in $\mathcal{N}$ by defining $\hat{\phi} \in \mathcal{N}$ by

$$\hat{\phi} \;:=\; \frac{1}{\|Pb\|^2}\left[Pb \;-\; \sum_{\ell=1}^{m} \frac{\langle M_\alpha Pb, \phi_\ell\rangle}{\lambda_\ell}\,\phi_\ell\right]. \tag{34}$$

We note that $\hat{\phi} \in \mathcal{N}$ is uniquely determined by $\langle \hat{\phi}, Pb\rangle = 1$ and $\langle P_0 M_\alpha \hat{\phi}, \phi_\ell\rangle = 0$ for all $\ell = 1,\ldots,m$.

We assume in addition that $P_0 M_\alpha|_{\mathcal{N}_0} : \mathcal{N}_0 \to \mathcal{N}_0$ is one-to-one, i.e. $\lambda_\ell = \langle M_\alpha \phi_\ell, \phi_\ell\rangle \neq 0$ for all $\ell = 1,\ldots,m$.

Under these assumptions the following auxiliary result holds.

Lemma 4.5. *Let $A_j(\alpha,\varepsilon)$ be smooth and uniformly bounded for $j = 1,2$. Set $U^\delta := \{(\alpha,\varepsilon) \in (-\delta,\delta) \times [0,\delta) : (\alpha,\varepsilon) \neq (0,0)\}$ and define $D(\alpha,\varepsilon) : \mathcal{N}_0 \to \mathcal{N}_0$ by*

$$D(\alpha,\varepsilon) \;=\; \big[P_0 P\tilde{L}(\alpha,\varepsilon) - P_0 P\tilde{L}(\alpha,\varepsilon)\big(A_1(\alpha,\varepsilon) + \rho(\alpha)A_2(\alpha,\varepsilon)\big)Q\tilde{L}(\alpha,\varepsilon)\big]_{\mathcal{N}_0}\,.$$

(a) *Then $D(\alpha,\varepsilon)$ is invertible for $(\alpha,\varepsilon) \in U^\delta$ for sufficiently small $\delta > 0$ and $\|D(\alpha,\varepsilon)^{-1}\| \le c/\sqrt{\alpha^2 + \varepsilon^2}$.*

(b) *For $r \in \mathcal{N}_0$ the solution $w = w(\alpha,\varepsilon) \in \mathcal{N}_0$ of $D(\alpha,\varepsilon)w(\alpha,\varepsilon) = r$ has a decomposition into $w(\alpha,\varepsilon) = w_0(\alpha,\varepsilon) + w_1(\alpha,\varepsilon) + \rho(\alpha)\, w_2(\alpha,\varepsilon)$ where w_0 is given by*

$$w_0(\alpha,\varepsilon) \;=\; \sum_{\ell=1}^{m} \frac{\langle r, \phi_\ell\rangle}{\lambda_\ell \alpha - i\varepsilon}\, \phi_\ell\,,$$

and $w_j(\alpha,\varepsilon)$ are smooth in U^δ and $w_j(\alpha,\varepsilon) = \mathcal{O}(\|r\|)$ for $j = 1, 2$ uniformly with respect to (α,ε).

(c) *If $r = r(\alpha,\varepsilon) \in \mathcal{N}_0$ is of the form $r(\alpha,\varepsilon) = r_1(\alpha,\varepsilon) + \rho(\alpha)\, r_2(\alpha,\varepsilon)$ with smooth $r_j(\alpha,\varepsilon) = \mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$ then the solution $w(\alpha,\varepsilon)$ of $D(\alpha,\varepsilon)w(\alpha,\varepsilon) = r(\alpha,\varepsilon)$ has a decomposition into $w(\alpha,\varepsilon) = w_1(\alpha,\varepsilon) + \rho(\alpha)\, w_2(\alpha,\varepsilon)$ where $w_j(\alpha,\varepsilon)$ are smooth in U^δ and uniformly bounded.*

Proof. (a), (b) Using the linearization of $P\tilde{L}(\alpha,\varepsilon)$ and $P\tilde{L}(\alpha,\varepsilon) = P[\tilde{L}(\alpha,\varepsilon) - \tilde{L}(0,0)] = \mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$ and $\tilde{L}(\alpha,\varepsilon)|_{\mathcal{N}_0} = [\tilde{L}(\alpha,\varepsilon) - \tilde{L}(0,0)]|_{\mathcal{N}_0} = \mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$ we have

$$D(\alpha,\varepsilon) \;=\; [\alpha P_0 M_\alpha - i\varepsilon P_0 M_\varepsilon] \;-\; B_1(\alpha,\varepsilon) \;-\; \rho(\alpha) B_2(\alpha,\varepsilon)$$

where $B_j(\alpha,\varepsilon)$ are smooth with $B_j(\alpha,\varepsilon) = \mathcal{O}(\alpha^2+\varepsilon^2)$. Set $S(\alpha,\varepsilon) = [\alpha P_0 M_\alpha - i\varepsilon P_0 M_\varepsilon]^{-1}$ which is given by

$$S(\alpha,\varepsilon) r \;=\; \sum_{\ell=1}^{m} \frac{\langle r, \phi_\ell\rangle}{\lambda_\ell \alpha - i\varepsilon}\, \phi_\ell\,.$$

From this we observe that $\|S(\alpha,\varepsilon)\| \approx 1/\sqrt{\alpha^2+\varepsilon^2}$. The equation $D(\alpha,\varepsilon)w(\alpha,\varepsilon) = r$ is therefore equivalent to

$$\big[I - S(\alpha,\varepsilon)B_1(\alpha,\varepsilon) - \rho(\alpha)\, S(\alpha,\varepsilon)B_2(\alpha,\varepsilon)\big] w(\alpha,\varepsilon) \;=\; S(\alpha,\varepsilon) r = w_0(\alpha,\varepsilon)\,.$$

Note that $S(\alpha,\varepsilon)$ and $B_j(\alpha,\varepsilon)$ are smooth in U^δ and $S(\alpha,\varepsilon)B_j(\alpha,\varepsilon) = \mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$. The Neumann series argument yields $w(\alpha,\varepsilon) = S(\alpha,\varepsilon)r + w_1(\alpha,\varepsilon) + \rho(\alpha) w_2(\alpha,\varepsilon)$ where $w_j(\alpha,\varepsilon)$ are smooth with $w_j(\alpha,\varepsilon) = \mathcal{O}(\|r\|)$.
(c) In this case we have $w_0(\alpha,\varepsilon) = \sum_{\ell=1}^{m} \frac{\langle r_1(\alpha,\varepsilon), \phi_j\rangle}{\lambda_\ell\alpha - i\varepsilon}\, \phi_\ell + \rho(\alpha) \sum_{\ell=1}^{m} \frac{\langle r_2(\alpha,\varepsilon), \phi_\ell\rangle}{\lambda_\ell\alpha - i\varepsilon}\, \phi_\ell$, and the sums are smooth in U^δ and uniformly bounded. □

Now we can formulate and prove the main functional analytic theorem.

Theorem 4.6. *Let the assumptions at the beginning of this subsection hold. Set again $U^\delta := \{(\alpha,\varepsilon) \in (-\delta,\delta) \times [0,\delta) : (\alpha,\varepsilon) \neq (0,0)\}$ for $\delta > 0$.*

(a) *Then the operators $L(\alpha,\varepsilon)$ are invertible for $(\alpha,\varepsilon) \in U^\delta$ for sufficiently small $\delta > 0$.*

(b) *Let the right hand side $r(\alpha) \in H$ have the form $r(\alpha) = \tilde{r}(\alpha) + \eta(\alpha)\, \rho(\alpha)\, b$ where $\eta(\alpha) \in \mathbb{C}$ and $\tilde{r}(\alpha) \in H$ are smooth. Then the unique solution $u(\alpha,\varepsilon) = L(\alpha,\varepsilon)^{-1} r(\alpha)$ of $u(\alpha,\varepsilon) - K(\alpha,\varepsilon)u(\alpha,\varepsilon) = r(\alpha)$ for $(\alpha,\varepsilon) \in U^\delta$ has a decomposition in the form*

$$u(\alpha,\varepsilon) \;=\; \sum_{\ell=1}^{m} \frac{\langle r(0), \phi_\ell\rangle}{\lambda_\ell \alpha - i\varepsilon}\, \phi_\ell \;+\; u_1(\alpha,\varepsilon) \;+\; s(\alpha,\varepsilon)\, u_2(\alpha,\varepsilon) \tag{35}$$

where $s(\alpha,\varepsilon)$ and $u_j(\alpha,\varepsilon)$ are continuous in U^δ and $u_j(\alpha,\varepsilon)$ are uniformly bounded for $(\alpha,\varepsilon) \in U^\delta$, and $s(\alpha,\varepsilon) \in \mathbb{C}$ satisfies

(36) $$|s(\alpha,\varepsilon)| \;\leq\; \frac{c}{|\rho(\alpha)|+\varepsilon} \quad \textit{for all } (\alpha,\varepsilon) \in U^\delta\,.$$

Finally, $u_1(\alpha,0)$ has the form $u_1(\alpha,0) = u_{11}(\alpha) + \rho(\alpha)u_{12}(\alpha)$ with smooth $u_{1j}(\alpha)$ and

(37) $$\lim_{\varepsilon\to 0}[s(\alpha,\varepsilon)\,u_2(\alpha,\varepsilon)] \;=\; -\frac{\langle r(0),\hat\phi\rangle}{\rho(\alpha)}\,\hat\phi \;+\; u_{21}(\alpha) \;+\; \rho(\alpha)\,u_{22}(\alpha) \quad \textit{for } \alpha \neq 0$$

where $u_{2j}(\alpha)$ are smooth in $(-\delta,\delta)$ and $\hat\phi$ is given by (34).

If $\mathcal{N} = \mathcal{N}_0$ then (35) *holds with $s(\alpha,\varepsilon) = 0$.*

Proof. We use the splitting $H = \mathcal{N} \oplus \mathcal{R}$ where again $\mathcal{N} := \mathcal{N}\big(L(0,0)\big)$ and $\mathcal{R} := \mathcal{R}\big(L(0,0)\big)$ with corresponding projections P and $Q := I - P$, respectively. Then the equation $L(\alpha,\varepsilon)u(\alpha,\varepsilon) = r(\alpha)$ is equivalent to the set of equations

$$\begin{aligned}
[P\tilde L(\alpha,\varepsilon) - \rho(\alpha)PB]u^{\mathcal{N}}(\alpha,\varepsilon) \;&+\; [P\tilde L(\alpha,\varepsilon) - \rho(\alpha)PB]u^{\mathcal{R}}(\alpha,\varepsilon) &&= P\tilde r(\alpha) + \eta(\alpha)\rho(\alpha)Pb\,,\\
QL(\alpha,\varepsilon)u^{\mathcal{N}}(\alpha,\varepsilon) \;&+\; QL(\alpha,\varepsilon)u^{\mathcal{R}}(\alpha,\varepsilon) &&= Qr(\alpha)
\end{aligned}$$

for $u(\alpha,\varepsilon) = u^{\mathcal{N}}(\alpha,\varepsilon) + u^{\mathcal{R}}(\alpha,\varepsilon)$ with $\big(u^{\mathcal{N}}(\alpha,\varepsilon), u^{\mathcal{R}}(\alpha,\varepsilon)\big) \in \mathcal{N}\times\mathcal{R}$.
We consider first the case $Pb \neq 0$, i.e. $\mathcal{N} \neq \mathcal{N}_0$, and remark on the (simpler) case $Pb = 0$ below.

We observe that $P_0PBv = \langle v,b\rangle P_0Pb = 0$ and $\langle PBv,\hat\phi\rangle = \langle v,b\rangle\langle Pb,\hat\phi\rangle = \langle v,b\rangle$ because $\langle Pb,\hat\phi\rangle = 1$. Applying P_0 to the first quation and multiplying the first equation by $\hat\phi$ results in the equivalent system (note that $\langle Pz,\hat\phi\rangle = \langle z.\hat\phi\rangle$ for any $z \in H$)

(38) $$P_0P\tilde L(\alpha,\varepsilon)u^{\mathcal{N}}(\alpha,\varepsilon) \;+\; P_0P\tilde L(\alpha,\varepsilon)u^{\mathcal{R}}(\alpha,\varepsilon) = P_0P\tilde r(\alpha)\,,$$

(39) $$\langle \tilde L(u^{\mathcal{N}}(\alpha,\varepsilon) + u^{\mathcal{R}}(\alpha,\varepsilon)),\hat\phi\rangle - \rho(\alpha)\langle u^{\mathcal{N}}(\alpha,\varepsilon) + u^{\mathcal{R}}(\alpha,\varepsilon), b\rangle = \langle \tilde r(\alpha),\hat\phi\rangle + \eta\rho(\alpha)\,,$$

(40) $$QL(\alpha,\varepsilon)u^{\mathcal{N}}(\alpha,\varepsilon) \;+\; QL(\alpha,\varepsilon)u^{\mathcal{R}}(\alpha,\varepsilon) = Qr(\alpha)\,,$$

We make an ansatz for $u^{\mathcal{N}}(\alpha,\varepsilon)$ in the form $u^{\mathcal{N}}(\alpha,\varepsilon) = w(\alpha,\varepsilon) + u_0^{\mathcal{N}}(\alpha,\varepsilon) + s(\alpha,\varepsilon)\hat\phi$ with $w(\alpha,\varepsilon) := \sum_{j=1}^m \frac{\langle r(0),\phi_j\rangle}{\lambda_j\alpha - i\varepsilon}\,\phi_j$ and $u_0^{\mathcal{N}}(\alpha,\varepsilon) \in \mathcal{N}_0$ and $s(\alpha,\varepsilon) \in \mathbb{C}$. Then (38), (40) is written as

(41) $$P_0P\tilde L(u_0^{\mathcal{N}} + w) \;+\; P_0P\tilde L u^{\mathcal{R}} \;+\; s\,P_0P\tilde L\hat\phi \;=\; P_0P\tilde r\,,$$

(42) $$QL(u_0^{\mathcal{N}} + w) \;+\; QLu^{\mathcal{R}} \;+\; s\,QL\hat\phi \;=\; Qr\,.$$

It is easily seen that $QL(0,0)|_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$ is an isomorphism from $\mathcal{R}$ onto itself. Therefore, also $QL(\alpha,\varepsilon)|_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$ are isomorphisms from $\mathcal{R}$ onto itself for $(\alpha,\varepsilon) \in \overline{U^\delta} = [-\delta,\delta]\times[0,\delta]$ for sufficiently small $\delta > 0$. We set $A(\alpha,\varepsilon) := \big[QL(\alpha,\varepsilon)|_{\mathcal{R}}\big]^{-1} : \mathcal{R} \to \mathcal{R}$ for abbreviation and recall that $QL(\alpha,\varepsilon) = Q\tilde L(\alpha,\varepsilon) + \rho(\alpha)QB$. The Neuman series representation of the inverse (and combining terms with even and odd powers of ρ) yields the form $A(\alpha,\varepsilon) = A_1(\alpha,\varepsilon) + \rho(\alpha)A_2(\alpha,\varepsilon)$ where A_j are smooth and bounded uniformly for $(\alpha,\varepsilon) \in \overline{U^\delta}$. Therefore, we can express $u^{\mathcal{R}}$ as

$$u^{\mathcal{R}} \;=\; (A_1 + \rho A_2)Qr - s\,(A_1 + \rho A_2)QL\hat\phi - (A_1 + \rho A_2)QLu_0^{\mathcal{N}} - (A_1 + \rho A_2)Q\tilde L w\,.$$

Here we have used that $Bw = \langle w, b\rangle b = \langle w, Pb\rangle b = 0$ because $w \in \mathcal{N}_0$.
The first term is (smooth and) bounded, and we write $Qr = Q\tilde{r} + \eta\rho\, Qb$. For the last term we note that $\tilde{L}(\alpha,\varepsilon)w(\alpha,\varepsilon) = [\tilde{L}(\alpha,\varepsilon) - \tilde{L}(0,0)]w(\alpha,\varepsilon)$. Since $\|w(\alpha,\varepsilon)\| = \mathcal{O}(1/\sqrt{\alpha^2+\varepsilon^2})$ we conclude that also the last term is smooth in U^δ and bounded. Therefore, we can write $u^{\mathcal{R}}$ in the form

$$u^{\mathcal{R}} \;=\; \tilde{u}_1^{\mathcal{R}} + \rho\, \tilde{u}_2^{\mathcal{R}} - s\,(A_1 + \rho A_2)QL\hat{\phi} - (A_1+\rho A_2)QLu_0^{\mathcal{N}} \tag{43}$$

where $\tilde{u}_j^{\mathcal{R}}$ are smooth in U^δ and uniformly bounded. We substitute this into (41) and arrive at

$$\begin{aligned} &[P_0P\tilde{L} - P_0P\tilde{L}(A_1+\rho A_2)QL]u_0^{\mathcal{N}} \\ =\;& P_0P\tilde{r} - P_0P\tilde{L}w - P_0P\tilde{L}(\tilde{u}_1^{\mathcal{R}} + \rho\,\tilde{u}_2^{\mathcal{R}}) + s\,P_0P\tilde{L}(A_1+\rho A_2)QL\hat{\phi} - s\,P_0P\tilde{L}\hat{\phi} \\ =\;& P_0P\tilde{r} - P_0P\tilde{L}w - P_0P\tilde{L}(\tilde{u}_1^{\mathcal{R}} + \rho\,\tilde{u}_2^{\mathcal{R}}) - s\,[P_0P\tilde{L} - P_0P\tilde{L}(A_1+\rho A_2)QL]\hat{\phi}\,. \end{aligned} \tag{44}$$

We define $u_1^{\mathcal{N}}, u_2^{\mathcal{N}} \in \mathcal{N}_0$ as the solutions of

$$\big(P_0P\tilde{L} - P_0P\tilde{L}(A_1+\rho A_2)QL\big)u_1^{\mathcal{N}} \;=\; P_0P\tilde{r} - P_0P\tilde{L}w - P_0P\tilde{L}(\tilde{u}_1^{\mathcal{R}} + \rho\,\tilde{u}_2^{\mathcal{R}})\,, \tag{45}$$

$$\big(P_0P\tilde{L} - P_0P\tilde{L}(A_1+\rho A_2)QL\big)u_2^{\mathcal{N}} \;=\; -\big(P_0P\tilde{L} - P_0P\tilde{L}(A_1+\rho A_2)QL\big)\hat{\phi}\,. \tag{46}$$

Then $u_0^{\mathcal{N}} = u_1^{\mathcal{N}} + su_2^{\mathcal{N}}$. On the left hand sides of (45) and (46) we can replace QL by $Q\tilde{L}$ because $QBu_j^{\mathcal{N}} = \langle u_j^{\mathcal{N}}, b\rangle Qb = \langle u_j^{\mathcal{N}}, Pb\rangle Qb = 0$. The first two terms on the right hand side of (45) are written as

$$P_0P\tilde{r} - P_0P\tilde{L}w = [P_0P\tilde{r}(\alpha) - P_0P\tilde{r}(0)] + [\alpha P_0M_\alpha - i\varepsilon P_0M_\varepsilon - P_0P\tilde{L}(\alpha,\varepsilon)]w(\alpha,\varepsilon)$$

where we used the definition of w as the solution of $[\alpha P_0M_\alpha - i\varepsilon P_0M_\varepsilon]w = P_0Pr(0)$. This expression is smooth in U^δ and of order $\mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$. Since $P_0P\tilde{L}(\alpha,\varepsilon) = P_0P[\tilde{L}(\alpha,\varepsilon) - \tilde{L}(0,0)]$ also the remaining terms are of order $\mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$, and we have a representation of the right hand sides of (45) and (46) as $r_1(\alpha,\varepsilon)+\rho(\alpha)r_2(\alpha,\varepsilon)$ and $r_3(\alpha,\varepsilon)+\rho(\alpha)r_4(\alpha,\varepsilon)$, respectively, where $r_j(\alpha,\varepsilon)$ are smooth in U^δ and of order $\mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$.
Application of Lemma 4.5 yields a representation of $u_j^{\mathcal{N}}(\alpha,\varepsilon)$ in the form

$$u_j^{\mathcal{N}}(\alpha,\varepsilon) \;=\; u_{j1}^{\mathcal{N}}(\alpha,\varepsilon) + \rho(\alpha)\, u_{j2}^{\mathcal{N}}(\alpha,\varepsilon) \tag{47}$$

for $j = 1, 2$ where $u_{ij}^{\mathcal{N}}(\alpha,\varepsilon) \in \mathcal{N}_0$ are smooth in U^δ and uniformly bounded. Now we substitute $u_0^{\mathcal{N}} = u_1^{\mathcal{N}} + s\,u_2^{\mathcal{N}}$ into (43) and arrive at $u^{\mathcal{R}} = u_1^{\mathcal{R}} + s\,u_2^{\mathcal{R}}$ where

$$u_j^{\mathcal{R}}(\alpha,\varepsilon) \;=\; u_{j1}^{\mathcal{R}}(\alpha,\varepsilon) + \rho(\alpha)\, u_{j2}^{\mathcal{R}}(\alpha,\varepsilon) \tag{48}$$

and $u_{j\ell}^{\mathcal{R}}(\alpha,\varepsilon) \in \mathcal{R}$ are smooth in U^δ and uniformly bounded. Using $L\hat{\phi} = \tilde{L}\hat{\phi} + \rho\langle\hat{\phi}, b\rangle b = \tilde{L}\hat{\phi} + \rho b$ and $Lu_{2j}^{\mathcal{N}} = \tilde{L}u_{2j}^{\mathcal{N}} + \rho\langle u_{2j}^{\mathcal{N}}, b\rangle b = \tilde{L}u_{2j}^{\mathcal{N}}$ and sorting terms we note that $u_{21}^{\mathcal{R}}(\alpha,\varepsilon)$ has the form

$$\begin{aligned} u_{21}^{\mathcal{R}}(\alpha,\varepsilon) \;&=\; -\left[A_1Q\tilde{L}\hat{\phi} + A_1Q\tilde{L}u_{21}^{\mathcal{N}} + \rho^2A_2Qb + \rho^2A_2Q\tilde{L}u_{22}^{\mathcal{N}}\right] \\ &= \rho(\alpha)^2u_3^{\mathcal{R}}(\alpha,\varepsilon) + u_4^{\mathcal{R}}(\alpha,\varepsilon) \end{aligned} \tag{49}$$

where $u_3^{\mathcal{R}}(\alpha,\varepsilon)$ is bounded and $u_4^{\mathcal{R}}(\alpha,\varepsilon) = \mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$.

The scalar quantity $s = s(\alpha,\varepsilon)$ has to be determined from (39). We substitute $u^{\mathcal{N}} = w + u_1^{\mathcal{N}} + s(u_2^{\mathcal{N}} + \hat{\phi})$ and $u^{\mathcal{R}} = u_1^{\mathcal{R}} + su_2^{\mathcal{R}}$ into (39) and collect the terms with the factor

s. This yields $s(\alpha,\varepsilon) = s_{num}(\alpha,\varepsilon)/s_{den}(\alpha,\varepsilon)$ where

$$s_{den} = \langle \tilde{L}(u_2^{\mathcal{N}} + \hat{\phi}), \hat{\phi}\rangle + \langle \tilde{L} u_2^{\mathcal{R}}, \hat{\phi}\rangle - \rho - \rho\langle u_2^{\mathcal{R}}, b\rangle\,, \tag{50}$$

$$s_{num} = \langle \tilde{r}, \hat{\phi}\rangle + \eta\rho - \langle \tilde{L}(u_1^{\mathcal{N}} + w), \hat{\phi}\rangle - \langle \tilde{L} u_1^{\mathcal{R}}, \hat{\phi}\rangle + \rho\langle u_1^{\mathcal{R}}, b\rangle\,. \tag{51}$$

Here we used $\langle u_j^{\mathcal{N}}, b\rangle = \langle w, b\rangle = 0$ and $\langle \hat{\phi}, b\rangle = 1$. Multiplication of (46) by $u_2^{\mathcal{N}} \in \mathcal{N}_0$ yields (with $\langle P_0 P z, u_2^{\mathcal{N}}\rangle = \langle z, u_2^{\mathcal{N}}\rangle$ for all $z \in H$)

$$\begin{aligned}\left\langle \tilde{L}(u_2^{\mathcal{N}} + \hat{\phi}), u_2^{\mathcal{N}}\right\rangle &= \left\langle \tilde{L}(A_1 + \rho A_2) Q L(u_2^{\mathcal{N}} + \hat{\phi}), u_2^{\mathcal{N}}\right\rangle \\ &= \left\langle \tilde{L}(A_1 + \rho A_2) Q \tilde{L}(u_2^{\mathcal{N}} + \hat{\phi}), u_2^{\mathcal{N}}\right\rangle \;+\; \rho\left\langle \tilde{L}(A_1 + \rho A_2) Q b, u_2^{\mathcal{N}}\right\rangle.\end{aligned}$$

Addition of this equation to (50) yields

$$\begin{aligned} s_{den} \;=\; &\langle \tilde{L}(u_2^{\mathcal{N}} + \hat{\phi}), u_2^{\mathcal{N}} + \hat{\phi}\rangle \;+\; \langle \tilde{L} u_2^{\mathcal{R}}, \hat{\phi}\rangle \;-\; \rho \;-\; \rho\langle u_2^{\mathcal{R}}, b\rangle \\ &-\; \left\langle \tilde{L}(A_1 + \rho A_2) Q \tilde{L}(u_2^{\mathcal{N}} + \hat{\phi}), u_2^{\mathcal{N}}\right\rangle \;-\; \rho\left\langle \tilde{L}(A_1 + \rho A_2) Q b, u_2^{\mathcal{N}}\right\rangle. \end{aligned} \tag{52}$$

We determine the behavior at $(\alpha,\varepsilon) = (0,0)$ for the terms appearing in this expression of s_{den}.
First we note that $P\tilde{L}(\alpha,\varepsilon) = \mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$ and $\tilde{L}(\alpha,\varepsilon)|_{\mathcal{N}} = \mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$. This yields already $\left\langle P\tilde{L}(A_1 + \rho A_2) Q \tilde{L}(u_2^{\mathcal{N}} + \hat{\phi}), u_2^{\mathcal{N}}\right\rangle = \mathcal{O}(\alpha^2+\varepsilon^2)$. The first term in (52) behaves as

$$\begin{aligned}\langle P\tilde{L}(u_2^{\mathcal{N}} + \hat{\phi}), u_2^{\mathcal{N}} + \hat{\phi}\rangle &= \alpha\langle M_\alpha(u_2^{\mathcal{N}} + \hat{\phi}), u_2^{\mathcal{N}} + \hat{\phi}\rangle \\ &\quad - i\varepsilon\langle M_\varepsilon(u_2^{\mathcal{N}} + \hat{\phi}), u_2^{\mathcal{N}} + \hat{\phi}\rangle \;+\; \mathcal{O}(\alpha^2+\varepsilon^2)\,.\end{aligned}$$

Then we recall $u_2^{\mathcal{R}} = u_{21}^{\mathcal{R}} + \rho u_{22}^{\mathcal{R}} = \rho(\rho u_3^{\mathcal{R}} + u_{22}^{\mathcal{R}}) + u_4^{\mathcal{R}}$ with bounded $\rho u_3^{\mathcal{R}} + u_{22}^{\mathcal{R}}$ and $u_4^{\mathcal{R}} = \mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$. Therefore, $\langle \tilde{L} u_2^{\mathcal{R}}, \hat{\phi}\rangle = \langle P\tilde{L} u_2^{\mathcal{R}}, \hat{\phi}\rangle = \mathcal{O}(\alpha^2+\varepsilon^2) + \rho(\alpha)\mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$ and $\rho\langle u_2^{\mathcal{R}}, b\rangle = \mathcal{O}(\rho(\alpha)^2) + \rho(\alpha)\mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$. Finally, the last term in (52) behaves as $\left\langle P\tilde{L}(A_1 + \rho A_2)\hat{\phi}, u_2^{\mathcal{N}}\right\rangle = \mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$. Summarizing, s_{den} has the form

$$\begin{aligned} s_{den}(\alpha,\varepsilon) \;=\; &\alpha\langle M_\alpha(u_2^{\mathcal{N}} + \hat{\phi}), u_2^{\mathcal{N}} + \hat{\phi}\rangle \;-\; i\varepsilon\langle M_\varepsilon(u_2^{\mathcal{N}} + \hat{\phi}), u_2^{\mathcal{N}} + \hat{\phi}\rangle \\ &+\; a_1(\alpha,\varepsilon) \;-\; \rho(\alpha)\big[1 + a_2(\alpha,\varepsilon) + \rho(\alpha) a_3(\alpha,\varepsilon)\big] \end{aligned} \tag{53}$$

with $a_1(\alpha,\varepsilon) = \mathcal{O}(\alpha^2+\varepsilon^2)$ and $a_2(\alpha,\varepsilon) = \mathcal{O}(\sqrt{\alpha^2+\varepsilon^2})$ and $a_3(\alpha,\varepsilon)$ is continuous in Q^δ and bounded. We show the existence of $\delta > 0$ and $c > 0$ such that

$$|s_{den}(\alpha,\varepsilon)| \;\geq\; c\left(|\rho(\alpha)| + \varepsilon\right) \quad \text{for all } (\alpha,\varepsilon) \in U^\delta\,. \tag{54}$$

If this claim does not hold there exist sequences $\alpha_j \to 0$ and $\varepsilon_j \to 0$ and $c_j \to 0$ such that $s_{den}(\alpha_j,\varepsilon_j) = c_j\left(|\rho(\alpha_j)| + \varepsilon_j\right)$, i.e.

$$\begin{aligned} &\alpha_j\langle M_\alpha(u_2^{\mathcal{N}}(j) + \hat{\phi}), u_2^{\mathcal{N}}(j) + \hat{\phi}\rangle - \varepsilon_j\big[c_j + i\langle M_\varepsilon(u_2^{\mathcal{N}}(j) + \hat{\phi}), u_2^{\mathcal{N}}(j) + \hat{\phi}\rangle\big] \\ &+\; a_1(\alpha_j,\varepsilon_j) \;-\; \rho(\alpha_j)\left[1 + a_2(\alpha_j,\varepsilon_j) + \rho(\alpha_j) a_3(\alpha_j,\varepsilon_j) + c_j\frac{|\rho(\alpha_j)|}{\rho(\alpha_j)}\right] = 0 \end{aligned} \tag{55}$$

where $u_2^{\mathcal{N}}(j) := u_2^{\mathcal{N}}(\alpha_j,\varepsilon_j)$. We set $\hat{\alpha}_j = \alpha_j/\sqrt{\alpha_j^2+\varepsilon_j^2}$ and $\hat{\varepsilon}_j = \varepsilon_j/\sqrt{\alpha_j^2+\varepsilon_j^2}$. Then $\hat{\alpha}_j^2 + \hat{\varepsilon}_j^2 = 1$ and there exist convergent subsequences $\hat{\alpha}_j \to \hat{\alpha}$ and $\hat{\varepsilon}_j \to \hat{\varepsilon}$ and $u_2^{\mathcal{N}}(j) \to \hat{u}_2^{\mathcal{N}}$ in H for some $\hat{u}_2^{\mathcal{N}} \in \mathcal{N}_0$. (Note that $\mathcal{N}_0$ is finite dimensional.) Then $\langle M_\alpha(u_2^{\mathcal{N}}(j)+\hat{\phi}), u_2^{\mathcal{N}}(j)+\hat{\phi}\rangle$ and $\langle M_\varepsilon(u_2^{\mathcal{N}}(j)+\hat{\phi}), u_2^{\mathcal{N}}(j)+\hat{\phi}\rangle$ converge to $\langle M_\alpha(\hat{u}_2^{\mathcal{N}}+\hat{\phi}), \hat{u}_2^{\mathcal{N}}+\hat{\phi}\rangle$ and $\langle M_\varepsilon(\hat{u}_2^{\mathcal{N}}+\hat{\phi}), \hat{u}_2^{\mathcal{N}}+\hat{\phi}\rangle$, respectively, which are real.

Division of (55) by $\sqrt{\alpha_j^2+\varepsilon_j^2}$ and letting $j\to\infty$ shows convergence $\rho(\alpha_j)/\sqrt{\alpha_j^2+\varepsilon_j^2}$ to some $\hat\rho$ with $\operatorname{Im}\hat\rho \ge 0$ and

$$\hat\alpha\langle M_\alpha(\hat u_2^{\mathcal N}+\hat\phi),\hat u_2^{\mathcal N}+\hat\phi\rangle \;-\; i\hat\varepsilon\langle M_\varepsilon(\hat u_2^{\mathcal N}+\hat\phi),\hat u_2^{\mathcal N}+\hat\phi\rangle \;=\; \hat\rho\,.$$

Since M_ε is positive and $\hat u_2^{\mathcal N}+\hat\phi\neq 0$ and $\operatorname{Im}\hat\rho\ge 0$ we conclude that $\hat\varepsilon=0$, i.e. $\frac{\varepsilon_j}{\sqrt{\alpha_j^2+\varepsilon_j^2}}\to 0$. This implies $\frac{\varepsilon_j}{\alpha_j}\to 0$ and thus also $\varepsilon_j/\rho(\alpha_j)\to 0$. Division of (55) by $\rho(\alpha_j)$ and using $\alpha_j/\rho(\alpha_j)\to 0$ and $\varepsilon_j/\rho(\alpha_j)\to 0$ and $a_1(\alpha_j,\varepsilon_j)/\rho(\alpha_j)\to 0$ yields $[1+\cdots]\to 0$, a contradiction. This proves (54).

Therefore, $s=s(\alpha,\varepsilon)$ is well defined. Since the numerator s_{num} is uniformly bounded we have an estimate of $s(\alpha,\varepsilon)$ of the form (36).

We recall the representation

$$\begin{aligned} u(\alpha,\varepsilon) &= u^{\mathcal N}(\alpha,\varepsilon)+u^{\mathcal R}(\alpha,\varepsilon)\\ &= w(\alpha,\varepsilon)+u_1^{\mathcal N}(\alpha,\varepsilon)+u_1^{\mathcal R}(\alpha,\varepsilon)+s(\alpha,\varepsilon)\big[u_2^{\mathcal N}(\alpha,\varepsilon)+\hat\phi+u_2^{\mathcal R}(\alpha,\varepsilon)\big]\\ &= w(\alpha,\varepsilon)+u_{11}^{\mathcal N}(\alpha,\varepsilon)+u_{11}^{\mathcal R}(\alpha,\varepsilon)+\rho(\alpha)[u_{12}^{\mathcal N}(\alpha,\varepsilon)+u_{12}^{\mathcal R}(\alpha,\varepsilon)]\\ &\quad + s(\alpha,\varepsilon)u_{21}^{\mathcal R}(\alpha,\varepsilon)+s(\alpha,\varepsilon)\rho(\alpha)\big[u_{22}^{\mathcal N}(\alpha,\varepsilon)+u_{22}^{\mathcal R}(\alpha,\varepsilon)\big]\\ &\quad + s(\alpha,\varepsilon)\left[\hat\phi+u_{21}^{\mathcal N}(\alpha,\varepsilon)\right]. \end{aligned} \tag{56}$$

The term with $s(\alpha,\varepsilon)\rho(\alpha)$ is certainly bounded. Furthermore, we recall that $u_{21}^{\mathcal R}(\alpha,\varepsilon)=\rho(\alpha)^2u_3^{\mathcal R}(\alpha,\varepsilon)+u_4^{\mathcal R}(\alpha,\varepsilon)$ with bounded $u_3^{\mathcal R}(\alpha,\varepsilon)$ and $u_4^{\mathcal R}(\alpha,\varepsilon)=\mathcal O(\sqrt{\alpha^2+\varepsilon^2})$. Therefore, also $s(\alpha,\varepsilon)u_{21}^{\mathcal R}(\alpha,\varepsilon)$ is bounded. Therefore, $u(\alpha,\varepsilon)$ has a representation in the form (35) with $u_2(\alpha,\varepsilon)=\hat\phi+u_{21}^{\mathcal N}(\alpha,\varepsilon)$.

Finally, we consider the limit $\varepsilon\to 0$. We note that $w(\alpha,0)=\frac{1}{\alpha}\sum_{\ell=1}^m\frac{\langle r(0),\phi_\ell\rangle}{\lambda_\ell}\,\phi_\ell$. From the discussion of the pair (45), (46) we note that the functions $r_j(\alpha,0)$ appearing on their right hand sides are now smooth in $(-\delta,\delta)$ with $r_j(0,0)=0$. This implies that all of the functions $u_{ij}^{\mathcal N}(\alpha,0)$ and $u_{ij}^{\mathcal R}(\alpha,0)$ and $u_3^{\mathcal R}(\alpha,0)$ and $u_4^{\mathcal R}(\alpha,0)$ from (47), (48), and (49) are smooth in $(-\delta,\delta)$. We go back to the definitions (50) and (51) of s_{den} and s_{num}, respectively. With $u_2^{\mathcal R}=\rho(\rho u_3^{\mathcal R}+u_{22}^{\mathcal R})+u_4^{\mathcal R}$ we conclude that $s_{den}(\alpha,0)$ has the form $s_{den}(\alpha,0)=-\rho(\alpha)+z_2(\alpha)+\rho(\alpha)z_3(\alpha)$ with smooth functions z_2, z_3 which satify $z_j(0)=0$. Analogously, since $\langle\tilde L(\alpha,0)u_1^{\mathcal R}(\alpha,0),\hat\phi\rangle=\langle P\tilde L(\alpha,0)u_{11}^{\mathcal R}(\alpha,0),\hat\phi\rangle+\rho(\alpha)\langle P\tilde L(\alpha,0)u_{12}^{\mathcal R}(\alpha,0),\hat\phi\rangle$ we conclude that $s_{num}(\alpha,0)=\langle\tilde r(\alpha),\hat\phi\rangle+\rho(\alpha)z_0(\alpha)+z_1(\alpha)$ with smooth functions z_0, z_1 and $z_1(0)=0$. Therefore,

$$s(\alpha,0)\;=\;\frac{\langle\tilde r(\alpha),\hat\phi\rangle+\rho(\alpha)z_0(\alpha)+z_1(\alpha)}{-\rho(\alpha)+z_2(\alpha)+\rho(\alpha)z_3(\alpha)}\;=\;-\frac{\langle\tilde r(0),\hat\phi\rangle}{\rho(\alpha)}\;+\;s_1(\alpha)+\rho(\alpha)\,s_2(\alpha)$$

with smooth $s_j(\alpha)$. Therefore,

$$s(\alpha,0)\,u_2(\alpha,0)\;=\;\left[-\frac{\langle\tilde r(0),\hat\phi\rangle}{\rho(\alpha)}\;+\;s_1(\alpha)+\rho(\alpha)\,s_2(\alpha)\right]\big[\hat\phi+u_{21}^{\mathcal N}(\alpha,0)\big]\,.$$

Next we recall that $u_2^{\mathcal{N}} = u_{21}^{\mathcal{N}} + \rho\, u_{22}^{\mathcal{N}}$ solves (46). For $\varepsilon = 0$ we devide (46) by α and let α tend to zero. This gives the equation $P_0 M_\alpha u_2^{\mathcal{N}}(0,0) = -P_0 M_\alpha \hat{\phi}$ and thus

$$u_{21}^{\mathcal{N}}(0,0) \;=\; u_2^{\mathcal{N}}(0,0) \;=\; -\sum_{j=1}^{m} \frac{\langle P_0 M_\alpha \hat{\phi}, \phi_j\rangle}{\lambda_j}\, \phi_j \;=\; -\sum_{j=1}^{m} \frac{\langle M_\alpha \hat{\phi}, \phi_j\rangle}{\lambda_j}\, \phi_j \;=\; 0\,.$$

Collecting the terms with $\rho(\alpha)$ yields the representation (37). The form of $u_1(\alpha,0)$ is obtained from (56) when we use $s(\alpha,0)\rho(\alpha) = -\langle \tilde{r}(0), \hat{\phi}\rangle \;+\; s_1(\alpha)\,\rho(\alpha) + \rho(\alpha)^2\, s_2(\alpha)$. The proof is finished for the case $Pb \neq 0$.

Let now $Pb = 0$. Then all the arguments of this previous proof are valid if one replaces Pb and $s(\alpha,\varepsilon)$ by 0. $\square$

4.4. **Checking the Assumptions of Theorem 4.6.** Let $\hat{\alpha}$ be a critical value, i.e. $\hat{\alpha} \in \mathcal{A}$, and $h \geq h_0$. We check the assumptions of Theorem 4.6 for both cases, i.e. $\hat{\alpha}$ is not a cut-off value or $\hat{\alpha}$ is a cut-off value, i.e. $\hat{\alpha} = \pm\kappa$ if we decompose k again as $k = \tilde{\ell} + \kappa$ with $\tilde{\ell} \in \mathbb{Z}_{\geq 0}$ and $\kappa \in (-1/2, 1/2]$. By Assumption 3.9 we assume $\kappa \in (-1/2, 1/2)$, $\kappa \neq 0$, and $\pm\kappa \in \mathcal{A}$. Of course, the critical value $\hat{\alpha}$ has to be moved to $\hat{\alpha} = 0$ when we apply Theorem 4.6.
Case 1: $\hat{\alpha} \in \mathcal{A}$ is not a cut-off value. Then $\mathcal{M}(\hat{\alpha}) = \mathcal{M}_{evan}(\hat{\alpha})$, and $L(\alpha,\varepsilon)$ is smooth with respect to both variables. In this case Theorem 4.6 has to be applied with $b = 0$, thus also $\mathcal{N}_0 = \mathcal{N} = \mathcal{N}\big(L(\hat{\alpha},0)\big)$. To clarify the connection between $\mathcal{M}(\hat{\alpha}) = \mathcal{M}_{evan}(\hat{\alpha})$ and the null space $\mathcal{N}$ of $L(\hat{\alpha},0)$ we define the operator $J_{\hat{\alpha}} : \mathcal{M}(\hat{\alpha}) \to H^1_{\mathrm{per}}(Q^h)$ by

$$(J_{\hat{\alpha}}\phi)(x) \;:=\; e^{-i\hat{\alpha}x_1}\phi(x)\,, \quad x \in Q^h\,,\; \phi \in \mathcal{M}(\hat{\alpha})\,.$$

Then $J_{\hat{\alpha}}$ maps $\mathcal{M}(\hat{\alpha})$ onto $\mathcal{N} = \mathcal{N}(L(\hat{\alpha},0))$ as seen above from the construction of $L(\alpha,\varepsilon)$. To connect the basis of $\mathcal{M}(\hat{\alpha})$, contructed in Lemma 3.11, with the basis of $\mathcal{N} := \mathcal{N}\big(L(\hat{\alpha},0)\big)$, determined by the eigenfunctions ϕ_ℓ of (33), we need the following result. It contains the case of $\hat{\alpha}$ being a cut-off value (needed for Case 2).

Lemma 4.7. (a) *Let $\hat{\alpha} \in \mathcal{A}$ not be a cut-off value. For $w, \phi \in \mathcal{M}_{evan}(\hat{\alpha})$ we have*

$$\left\langle \frac{\partial}{\partial\alpha} L(\hat{\alpha},0) J_{\hat{\alpha}} w, J_{\hat{\alpha}}\phi \right\rangle_* \;=\; -2i \int_{Q^\infty} \bar{\phi}\, \partial_{x_1} w\, dx \;=\; E(w,\phi)\,. \tag{57}$$

Note that in this case $\mathcal{M}(\hat{\alpha})$ and $\mathcal{M}_{evan}(\hat{\alpha})$ coincide.

(b) *If $\hat{\alpha} = \pm\kappa$ is a cut-off value then (57) holds for $\tilde{L}(\pm\kappa,0)$ replacing $L(\hat{\alpha},0)$. Furthermore, if $\hat{\phi}^\pm \in \mathcal{M}(\pm\kappa)$ is the unique element from part (c) of Lemma 3.11 which is orthogonal to $\mathcal{M}_{evan}(\pm\kappa)$ (with respect to E) then $\big\langle \frac{\partial}{\partial\alpha} \tilde{L}(\pm\kappa,0) J_{\pm\kappa}\hat{\phi}^\pm, J_{\pm\kappa}\phi \big\rangle_* = 0$ for all $\phi \in \mathcal{M}_{evan}(\pm\kappa)$.*

(c) *For $v, \psi \in H^1_{\mathrm{per}}(Q^h)$. we have*

$$\frac{\partial}{\partial\varepsilon} \big\langle \tilde{L}(\hat{\alpha},0) v, \psi \big\rangle_* \;=\; \frac{\partial}{\partial\varepsilon} \big\langle L(\hat{\alpha},0) v, \psi \big\rangle_* \;=\; -k^2 i \int_{Q^{h_0}} q\, v\, \bar{\psi}\, dx \tag{58}$$

Proof. (a) Since ϕ and w are evanescent we have $\phi_\ell = w_\ell = 0$ for $|\ell+\hat\alpha| \le k$. With $v := J_{\hat\alpha} w$ and $\psi := J_{\hat\alpha}\phi$ we have

$$\begin{aligned}\langle L(\alpha,\varepsilon)v,\psi\rangle_* \;&=\; \int\limits_{Q^h} \Big[\nabla v\cdot\nabla\overline{\psi} - 2i\alpha\,\frac{\partial v}{\partial x_1}\,\overline{\psi} + (\alpha^2-k^2 n_\varepsilon)\,v\,\overline{\psi}\Big]\,dx \\ &\quad + 2\pi \sum_{|\ell+\hat\alpha|>k} w_\ell\,\overline{\phi_\ell}\,\sqrt{(\ell+\alpha)^2-k^2}\end{aligned}$$

for α close to $\hat\alpha$ (such that also $|\ell+\alpha|>k$). Therefore,

$$\begin{aligned}\frac{\partial}{\partial\alpha}\big\langle L(\hat\alpha,0)v,\psi\big\rangle_* \;&=\; -2i\int\limits_{Q^h}\big[\partial_{x_1}v + i\hat\alpha v\big]\overline{\psi}\,dx + 2\pi\sum_{|\ell+\hat\alpha|>k} w_\ell\,\overline{\phi_\ell}\,\frac{\ell+\hat\alpha}{\sqrt{(\ell+\hat\alpha)^2-k^2}} \\ &=\; -2i\int\limits_{Q^h}\bar\phi\,\partial_{x_1}w\,dx \;+\; 2\pi\sum_{|\ell+\hat\alpha|>k} w_\ell\,\overline{\phi_\ell}\,\frac{\ell+\hat\alpha}{\sqrt{(\ell+\hat\alpha)^2-k^2}}\,.\end{aligned}$$

Since w has the representation

$$w(x) \;=\; \sum_{|\ell+\hat\alpha|>k} w_\ell\, e^{i(\ell+\hat\alpha)x_1} e^{-\sqrt{(\ell+\hat\alpha)^2-k^2}(x_2-h)}$$

and analogously for ϕ we compute

$$\begin{aligned}-2i\int\limits_{Q^\infty\setminus Q^h}\bar\phi\,\partial_{x_1}w\,dx \;&=\; 4\pi\sum_{|\ell+\hat\alpha|>k} w_\ell\,\overline{\phi_\ell}\,(\ell+\hat\alpha)\int_h^\infty e^{-2\sqrt{(\ell+\hat\alpha)^2-k^2}(x_2-h)}dx_2 \\ &=\; 2\pi\sum_{|\ell+\hat\alpha|>k} w_\ell\,\overline{\phi_\ell}\,\frac{\ell+\hat\alpha}{\sqrt{(\ell+\hat\alpha)^2-k^2}}\end{aligned}$$

which shows the first identity in (57). The second is shown by partial integration with respect to x_1 (note that $x_1 \mapsto w(x)\bar\phi(x)$ is $2\pi-$periodic).

(b) The proof follows exactly the same lines because $\frac{\partial}{\partial\alpha}\big\langle (I-\tilde K(\hat\alpha))J_{\hat\alpha}w, J_{\hat\alpha}\phi\big\rangle_*$ has the same form as $\frac{\partial}{\partial\alpha}\big\langle (I-K(\hat\alpha))J_{\hat\alpha}w, J_{\hat\alpha}\phi\big\rangle_*$ whenever $w_{\ell_0}\,\overline{\phi_{\ell_0}}$ vanishes.

(c) This part is obvious. □

We continue with Case 1 and note that $(v,\psi) \mapsto i\frac{\partial}{\partial\varepsilon}\big\langle L(\hat\alpha,0))v,\psi\big\rangle_*$ is hermetian and positive on $\mathcal{N} := \mathcal{N}\big(L(\hat\alpha,0)\big)$. Indeed, $i\,\frac{\partial}{\partial\varepsilon}\big\langle L(\hat\alpha,0)v,v\big\rangle_* = 0$ implies that v vanishes on the support of q. Since $v\in\mathcal{N}\big(L(\hat\alpha,0)\big)$ is of the form $v(x) = e^{-i\hat\alpha x_1}\phi(x)$ with some ϕ satisfying the equation $\Delta\phi + k^2 n\phi = 0$ the unique continuation principle implies $v=0$ everywhere.

Therefore, this lemma implies that the eigenvalue problem (23) is equivalent to

$$\frac{\partial}{\partial\alpha}\big\langle L(\hat\alpha,0)\phi_\ell,\psi\big\rangle_* \;=\; \lambda_\ell\, i\,\frac{\partial}{\partial\varepsilon}\big\langle L(\hat\alpha,0)\phi_\ell,\psi\big\rangle_* \quad \text{for all } \psi\in\mathcal{N} \tag{59}$$

for $\phi_\ell = J_{\hat\alpha}\phi^\ell = e^{-i\hat\alpha x_1}\phi^\ell|_{Q^h}\in\mathcal{N}$ which coincides with (33) in the case $b=0$.

Assumption 3.10 and Lemma 3.8 implies that $M_\alpha = P\frac{\partial}{\partial\alpha}L(\hat\alpha,0)|_{\mathcal{N}}$ is onto to one.

Therefore, in this Case 1 all of the assumptions of Theorem 4.6 are satisfied.

Case 2: $\hat\alpha \in \mathcal{A}$ is a cut-off value, i.e. $\hat\alpha = \pm\kappa$. By (30) the operator $L(\alpha,\varepsilon)$ has a decomposition into $L(\alpha,\varepsilon) = \tilde L(\alpha,\varepsilon) + \rho^\pm(\alpha)\, b^\pm$ in a neighborhood of $\pm\kappa$ where $b^\pm \in H^1_{\text{per}}(Q^h)$ had been defined by $\langle \psi, b^\pm\rangle_* = \psi_{\pm\tilde\ell}(\pm\kappa, h)$, see the definition of B in (32), and $\rho(\alpha) = 2\pi i\sqrt{k^2 - (\pm\tilde\ell + \alpha)^2}$, see (31). We note that ρ^2 is smooth with $\frac{d}{d\alpha}[\rho(\pm\kappa)^2] = \pm 8\pi^2 k \neq 0$. The subspaces $\mathcal{M}_{evan}(\pm\kappa)$ are either equal to $\mathcal{M}(\pm\kappa)$ or are subspaces of co-dimension one. Therefore, the subspace $\mathcal{N}_0$ of $\mathcal{N} := \mathcal{N}(L(\pm\kappa, 0))$, defined as in Theorem 4.6 by $\mathcal{N}_0 := \{u \in \mathcal{N} : \langle u, Pb^\pm\rangle = 0\}$, is exactly the image $J_{\pm\kappa}\mathcal{M}_{evan}(\pm\kappa) = \{J_{\pm\kappa}u : u \in \mathcal{M}_{evan}(\pm\kappa)\}$ of $\mathcal{M}_{evan}(\pm\kappa)$, i.e. $\mathcal{N}_0 = J_{\pm\kappa}\mathcal{M}_{evan}(\pm\kappa)$.
The eigenvalue problem (23) for the construction of a basis of $\mathcal{M}_{evan}(\pm\kappa)$ is again equivalent to the eigenvalue problem (59) in $\mathcal{N}_0$, i.e. for $\phi_\ell = J_{\pm\kappa}\phi^\ell$ and $\psi \in \mathcal{N}_0$.
Furthermore, part (b) of Lemma 4.7 implies also that the restrictions $J_{\pm\kappa}\hat\phi^\pm \in \mathcal{N}$ of $\hat\phi^\pm \in \mathcal{M}(\pm\kappa)$ from Lemma 3.11 satisfy $\langle M_\alpha J_{\pm\kappa}\hat\phi^\pm, J_{\pm\kappa}\phi\rangle = 0$ for all $\phi \in \mathcal{M}_{evan}(\pm\kappa)$. Since also $\langle J_{\pm\kappa}\hat\phi^\pm, b^\pm\rangle_* = \hat\phi^\pm_{\pm\tilde\ell}(\pm\kappa, h) = 1$ we conclude that $J_{\pm\kappa}\hat\phi^\pm \in \mathcal{N}$ is exactly the function $\hat\phi \in \mathcal{N}$ from the abstract theorem.

Therefore, also in this Case 2 the assumptions of Theorem 4.6 are satisfied.

4.5. **Proof of Theorem 4.1.** Let $\hat\alpha \in [-1/2, 1/2]$ be fixed. Two cases can occur.

Case 1: $\hat\alpha \notin \mathcal{A}$, i.e. $\hat\alpha$ is not a critical value. Then $L(\hat\alpha, 0)$ is an isomorphism from $H^1_{\text{per}}(Q^h)$ onto itself. Since $L(\alpha,\varepsilon)$ depends continuously on (α,ε) the operators $L(\alpha,\varepsilon)$ are isomorphisms for all (α,ε) in a neighborhood of $(\hat\alpha, 0)$. Furthermore, since $r(\alpha)$ depends continuously on α, also $(\alpha,\varepsilon) \mapsto L(\alpha,\varepsilon)^{-1}r(\alpha)$ is continuous.

Case 2: $\hat\alpha \in \mathcal{A}$, i.e. $\hat\alpha$ is a critical value. Application of part (a) of Theorem 4.6 yields that $L(\alpha,\varepsilon) = I - K(\alpha,\varepsilon)$ is invertible for $(\alpha,\varepsilon) \neq (\hat\alpha, 0)$ in a neighborhood of $(\hat\alpha, 0)$.

Therefore, for every $\hat\alpha \in [-1/2.1/2]$ there exists a neighborhood U of $\hat\alpha$ which contains at most one critical value. Since $[-1/2, 1/2]$ is compact finitely many set U suffice to cover $[-1/2, 1/2]$, and the proof of Theorem 4.1 is complete.

As a next step towards the proof of Theorem 4.2 we apply part (b) of Theorem 4.6 to (28).

4.6. **Local Representation of the Solution in Q^h.** Since $r(\alpha)$ is smooth the assumptions on $r(\alpha)$ are trivially satisfied.
We define the punctured neighborhood $U_j^\delta = \big\{(\alpha,\varepsilon) \in (\hat\alpha_j - \delta, \hat\alpha_j + \delta) \times [0,\delta) : (\alpha,\varepsilon) \neq (\hat\alpha_j, 0)\big\}$ of $(\hat\alpha_j, 0)$.
We fix $j \in J$ and consider first the case $j = \pm 1$, i.e. $\hat\alpha_j = \pm\kappa$. We recall that

$$\rho^\pm(\alpha) = 2\pi i\sqrt{k^2 - (\pm\tilde\ell + \alpha)^2} = 2\pi i\sqrt{\kappa \mp \alpha}\sqrt{2\tilde\ell + \kappa \pm \alpha}$$

which has a square root singularity at $\pm\kappa$. Application of part (b) of Theorem 4.6 provides a representation of the solution $v(\alpha,\varepsilon)$ of (28) as

$$\begin{aligned} v(\alpha,\varepsilon) &= \sum_{\ell=1}^{m_{\pm 1}} \frac{\langle r(\pm\kappa), e^{\mp i\kappa x_1}\phi^{\ell,\pm 1}\rangle_*}{\lambda_{\ell,\pm 1}(\alpha \mp \kappa) - i\varepsilon}\, e^{\mp i\kappa x_1}\phi^{\ell,\pm 1} \;+\; v_1^{\pm 1}(\alpha,\varepsilon) \;+\; s^\pm(\alpha,\varepsilon)\, v_2^\pm(\alpha,\varepsilon) \\ &= \sum_{\ell=1}^{m_{\pm 1}} \frac{\langle g_{\pm\kappa}, \phi^{\ell,\pm 1}\rangle_{L^2(Q^{h_0})}}{\lambda_{\ell,\pm 1}(\alpha \mp \kappa) - i\varepsilon}\, e^{\mp i\kappa x_1}\phi^{\ell,\pm 1} \;+\; v_1^{\pm 1}(\alpha,\varepsilon) \;+\; s^\pm(\alpha,\varepsilon)\, v_2^\pm(\alpha,\varepsilon) \end{aligned}$$

for $(\alpha,\varepsilon)\in U^\delta_{\pm 1}$ where we used the definition (29) of $r(\alpha)$. We wrote $\phi^{\ell,\pm 1}$ for $\phi^{\ell,\pm 1}|_{Q^h}$ for simplicity. The parts $v_1^{\pm 1}(\alpha,\varepsilon)$ and $v_2^{\pm}(\alpha,\varepsilon)$ are uniformly bounded for $(\alpha,\varepsilon)\in U^\delta_{\pm 1}$, and $s^\pm(\alpha,\varepsilon)\in\mathbb{C}$ satisfies

$$|s^\pm(\alpha,\varepsilon)| \;\leq\; \frac{c}{|\rho^\pm(\alpha)|+\varepsilon} \quad \text{for all } (\alpha,\varepsilon)\in U^\delta_{\pm 1}\,. \tag{60}$$

Finally, $v_1^{\pm 1}(\alpha,0)$ has the form $v_1^{\pm 1}(\alpha,0)=v_{11}^{\pm 1}(\alpha)+\rho^\pm(\alpha)v_{12}^{\pm 1}(\alpha)$ and

$$\begin{aligned}\lim_{\varepsilon\to 0}[s^\pm(\alpha,\varepsilon)\,v_2^\pm(\alpha,\varepsilon)] &= -\frac{\langle r(\pm\kappa), e^{\mp i\kappa x_1}\hat\phi^\pm\rangle_*}{\rho^\pm(\alpha)}\,e^{\mp i\kappa x_1}\hat\phi^\pm \;+\; v_{21}^{\pm 1}(\alpha)+\rho^\pm(\alpha)v_{22}^{\pm 1}(\alpha)\\ &= -\frac{\langle g_{\pm\kappa}(\pm\kappa),\hat\phi^\pm\rangle_{L^2(Q^{h_0})}}{\rho^\pm(\alpha)}\,e^{\mp i\kappa x_1}\hat\phi^\pm \;+\; v_{21}^{\pm 1}(\alpha)+\rho^\pm(\alpha)v_{22}^{\pm 1}(\alpha)\end{aligned}$$

for $0<|\alpha|<\delta$ where $v_{ij}^{\pm 1}(\alpha)\in H^1_{\mathrm{per}}(Q^h)$ depend smoothly on α. We note that $\langle g_{\hat\alpha_j},\phi^{\ell,j}\rangle_{L^2(Q^{h_0})}=\langle f,\phi^{\ell,j}\rangle_{L^2(W^{h_0})}$ by the definition of the Floquet-Bloch transform g_α of f. Transformation to the $\alpha-$quasi-periodic solution $u_{\alpha,\varepsilon}\in H^1_\alpha(Q^h)$ of (7) (where we indicated the dependence on ε) yields the representation

$$u_{\alpha,\varepsilon} \;=\; \sum_{\ell=1}^{m_{\pm 1}}\frac{\langle f,\phi^{\ell,\pm 1}\rangle_{L^2(W^{h_0})}}{\lambda_{\ell,\pm 1}(\alpha\mp\kappa)-i\varepsilon}\,e^{i(\alpha\mp\kappa)x_1}\phi^{\ell,\pm 1} \;+\; u_1^{\pm 1}(\alpha,\varepsilon) \;+\; s^\pm(\alpha,\varepsilon)\,u_2^{\pm 1}(\alpha,\varepsilon) \tag{61}$$

for $(\alpha,\varepsilon)\in U^\delta_{\pm 1}$ where $u_j^{\pm 1}(\alpha,\varepsilon)=e^{i\alpha x_1}v_j^{\pm 1}(\alpha,\varepsilon)$ are uniformly bounded and

$$\lim_{\varepsilon\to 0}[s^\pm(\alpha,\varepsilon)\,u_2^\pm(\alpha,\varepsilon)] \;=\; -\frac{\langle f,\hat\phi^\pm\rangle_{L^2(W^{h_0})}}{\rho^\pm(\alpha)}\,e^{i(\alpha\mp\kappa)x_1}\hat\phi^\pm \;+\; u_{21}^{\pm 1}(\alpha)+\rho^\pm(\alpha)u_{22}^{\pm 1}(\alpha) \tag{62}$$

for all $0<|\alpha|<\delta$. The convergence in understood in $H^1(Q^h)$. Since the functions are quasi-periodic the convergence holds even in $H^1(Q^{R,h})$ for all $R>0$ where $Q^{R,h}:=(-R,R)\times(0,h)$.

For $j\in J$ with $|j|\geq 2$ the same representation as in (61) holds without the last term, i.e.

$$u_{\alpha,\varepsilon} \;=\; \sum_{\ell=1}^{m_j}\frac{\langle f,\phi^{\ell,j}\rangle_{L^2(W^{h_0})}}{\lambda_{\ell,j}(\alpha-\hat\alpha_j)-i\varepsilon}\,e^{i(\alpha-\hat\alpha_j)x_1}\phi^{\ell,j} \;+\; u_1^j(\alpha,\varepsilon)$$

in U^δ_j where $u_1^j(\alpha,\varepsilon)\in H^1_\alpha(Q^h)$ are uniformly bounded. Furthermore, $u_i^j(\alpha,0)$ have the forms $u_i^j(\alpha,0)=u_{i1}^j(\alpha)+\rho(\alpha)u_{i2}^j(\alpha)$ with smooth $u_{ik}^j(\alpha)$.
We now set $U^\delta_0:=(-1/2-\delta,1/2+\delta)\setminus\{\hat\alpha_j : j\in J\}$ and choose functions $\eta_j\in C^\infty(\mathbb{R})$, $j\in J\cup\{0\}$, with $\operatorname{supp}\eta_j\subset U^\delta_j$ and $\sum_j\eta_j(\alpha)=1$ for all $\alpha\in[-1/2,1/2]$ (partition of unity). Therefore,

$$\begin{aligned}u_{\alpha,\varepsilon} &= \eta_0(\alpha)\,u_{\alpha,\varepsilon} \;+\; \sum_{j\in J}\eta_j(\alpha)\,u_{\alpha,\varepsilon}\\ &= \eta_0(\alpha)\,u_{\alpha,\varepsilon} \;+\; \sum_{j\in J}\eta_j(\alpha)\sum_{\ell=1}^{m_j}\frac{\langle f,\phi^{\ell,j}\rangle_{L^2(W^{h_0})}}{\lambda_{\ell,j}(\alpha-\hat\alpha_j)-i\varepsilon}\,e^{i(\alpha-\hat\alpha_j)x_1}\phi^{\ell,j}\\ &\quad + \sum_{j\in J}\eta_j(\alpha)\big[u_1^j(\alpha,\varepsilon)+\delta_{|j|,1}\,s^{\operatorname{sign}j}(\alpha,\varepsilon)\,u_2^j(\alpha,\varepsilon)\big]\,,\quad (\alpha,\varepsilon)\in[-1/2,1/2]\times(0,\delta)\,.\end{aligned}$$

4.7. **Proof of Theorem 4.2.** Recall that u_ε is the inverse Floquet-Bloch transform of $\alpha \mapsto u_{\alpha,\varepsilon}$, i.e.

$$
\begin{aligned}
u_\varepsilon \;&=\; \int\limits_{-1/2}^{1/2} u_{\alpha,\varepsilon}\, d\alpha \;=\; \sum_{j\in J}\sum_{\ell=1}^{m_j} \langle f, \phi^{\ell,j}\rangle_{L^2(W^{h_0})} \int\limits_{-1/2}^{1/2} \frac{\eta_j(\alpha)}{\lambda_{\ell,j}(\alpha-\hat\alpha_j)-i\varepsilon}\, e^{i(\alpha-\hat\alpha_j)x_1} d\alpha\, \phi^{\ell,j} \\
&\quad + \sum_{j\in J} \int\limits_{-1/2}^{1/2} \eta_j(\alpha)\left[u_1^j(\alpha,\varepsilon) + \delta_{|j|,1}\, s^{\operatorname{sign} j}(\alpha,\varepsilon)\, u_2^j(\alpha,\varepsilon)\right] d\alpha \;+\; \int\limits_{-1/2}^{1/2} \eta_0(\alpha)\, u_{\alpha,\varepsilon}\, d\alpha
\end{aligned}
\tag{63}
$$

in W^h where $u_1^j(\alpha,\varepsilon)$ and $u_2^j(\alpha,\varepsilon)$ are uniformly bounded in $H^1(Q^h)$. We now consider the limit as ε tend to zero. The first integral had been is investigated already in several papers, we repeat the arguments for the convenience of the reader. First we write

$$
\int\limits_{-1/2}^{1/2} \frac{\eta_j(\alpha)}{\lambda_{\ell,j}(\alpha-\hat\alpha_j)-i\varepsilon}\, e^{i(\alpha-\hat\alpha_j)x_1}\, d\alpha \;=\; \int\limits_{-\delta}^{\delta} \frac{\eta_j(\alpha+\hat\alpha_j)-1}{\lambda_{\ell,j}\alpha - i\varepsilon}\, e^{i\alpha x_1}\, d\alpha + \int\limits_{-\delta}^{\delta} \frac{e^{i\alpha x_1}}{\lambda_{\ell,j}\alpha - i\varepsilon}\, d\alpha
$$

and note that the first integral converges to $\frac{1}{\lambda_{\ell,j}} \int_{-\delta}^{\delta} \frac{\eta_j(\alpha+\hat\alpha_j)-1}{\alpha}\, e^{i\alpha x_1}\, d\alpha$ (uniformly with respect to x_1) which represents a H^1-function because $\frac{\eta_j(\alpha+\hat\alpha_j)-1}{\alpha} = \frac{\eta_j(\alpha+\hat\alpha_j)-\eta_j(\hat\alpha_j)}{\alpha}$ is smooth). Partial integration shows that this part decays as $1/|x_1|$ as $x_1 \to \pm\infty$. In the appendix we show that

$$
\lim_{\varepsilon\to 0} \int\limits_{\delta}^{\delta} \frac{1}{\lambda_{\ell,j}\alpha - i\varepsilon}\, e^{i\alpha x_1}\, d\alpha \;=\; \frac{2\pi i}{|\lambda_{\ell,j}|}\, \psi^\sigma(x_1)
\tag{64}
$$

uniformly with respect to $|x_1| \le R$ for every $R > 0$ where $\sigma = \operatorname{sign}\lambda_{\ell,j}$ and

$$
\psi^\pm(x_1) \;=\; \frac{1}{2} \;\pm\; \frac{1}{\pi}\int_0^{\delta x_1} \frac{\sin t}{t}\, dt\,, \quad x_1 \in \mathbb{R}\,.
\tag{65}
$$

For the remaining integrals in (63) we use Lebesgue's theorem on dominated convergence. Indeed, $\|\eta_j(\alpha) u_1^j(\alpha,\varepsilon)\|_{H^1(Q^h)} \le c$ and $\|\eta_0(\alpha) u_{\alpha,\varepsilon}\|_{H^1(Q^h)} \le c$ (because η_0 vanishes in neighborhoods of $\hat\alpha_j$) and $\|\eta_{\pm 1}(\alpha) s^\pm(\alpha,\varepsilon) u_2^{\pm 1}(\alpha,\varepsilon)\|_{H^1(Q^{R,h})} \le c(R)/|\rho^\pm(\alpha)|$ for every $R > 0$ where again $Q^{R,h} = (-R,R)\times(0,h)$. This term is intergrable with respect to α. Furthermore, we have pointwise convergence for almost all α. Therefore, we conclude that $\int_{-1/2}^{1/2} \eta_j(\alpha)\, u_1^j(\alpha,\varepsilon)\, d\alpha$ and $\int_{-1/2}^{1/2} \eta_0(\alpha)\, u_{\alpha,\varepsilon}\, d\alpha$ converge to $\int_{-1/2}^{1/2} \eta_j(\alpha)\, u_1^j(\alpha,0)\, d\alpha$ and

$\int_{-1/2}^{1/2} \eta_0(\alpha)\, u_{\alpha,0}\, d\alpha$, respectively, in $H^1(W^h)$, and $\int_{-1/2}^{1/2} \eta_{\pm 1}(\alpha)\, s^\pm(\alpha,\varepsilon)\, u_2^{\pm 1}(\alpha,\varepsilon)\, d\alpha$ converges to $\int_{-1/2}^{1/2} \eta_{\pm 1}(\alpha)\, s^\pm(\alpha,0)\, u_2^{\pm 1}(\alpha,0)\, d\alpha$ in $H^1(Q^{R,h})$ for all $R > 0$. We write again

$$\begin{aligned}
\int_{-1/2}^{1/2} \eta_{\pm 1}(\alpha)\, s^\pm(\alpha,0)\, u_2^{\pm 1}(\alpha,0)\, d\alpha \;&=\; -\langle f, \hat\phi^\pm\rangle_{L^2(W^{h_0})} \int_{-1/2}^{1/2} \frac{\eta_{\pm 1}(\alpha)}{\rho^\pm(\alpha)}\, e^{i(\alpha\mp\kappa)x_1}\, d\alpha\; \hat\phi^\pm \\
&+ \int_{-1/2}^{1/2} \eta_{\pm 1}(\alpha)\, \left[u_{21}^{\pm 1}(\alpha) + \rho^\pm(\alpha) u_{22}^{\pm 1}(\alpha)\right] d\alpha \\
&= \langle f, \hat\phi^\pm\rangle_{L^2(W^{h_0})}\, \varphi^\pm(x_1)\, \hat\phi^\pm \;-\; \langle f, \hat\phi^\pm\rangle_{L^2(W^{h_0})} \int_{\pm\kappa-\delta}^{\pm\kappa+\delta} \frac{\eta_{\pm 1}(\alpha) - 1}{\rho^\pm(\alpha)}\, e^{i(\alpha\mp\kappa)x_1}\, d\alpha\; \hat\phi^\pm \\
&+ \int_{-1/2}^{1/2} \eta_{\pm 1}(\alpha)\, \left[u_{21}^{\pm 1}(\alpha) + \rho^\pm(\alpha) u_{22}^{\pm 1}(\alpha)\right] d\alpha
\end{aligned} \tag{66}$$

with

$$\varphi^\pm(x_1) \;=\; -\int_{\pm\kappa-\delta}^{\pm\kappa+\delta} \frac{1}{\rho^\pm(\alpha)}\, e^{i(\alpha\mp\kappa)x_1}\, d\alpha\,, \quad x_1 \in \mathbb{R}\,. \tag{67}$$

The second and third term on the right hand side of (66) represent functions in $H^1(W^h)$ because the integrands are bounded. They decay as $\mathcal{O}(1/|x_1|)$ which is seen as follows. First we note that $\frac{\eta_{\pm 1}(\alpha)-1}{\rho^\pm(\alpha)} = \frac{\eta_{\pm 1}(\alpha)-1}{\alpha}\frac{\alpha}{\rho^\pm(\alpha)} = A^\pm(\alpha)\rho^\pm(\alpha)$ for some smooth function $A^\pm(\alpha)$. The integral $\int_{\pm\kappa-\delta}^{\pm\kappa+\delta} A^\pm(\alpha)\rho^\pm(\alpha) d\alpha$ decays as $\mathcal{O}(1/|x_1|)$ as seen from the last part of the proof of Lemma A.2. In the same way it is seen that the last integral of (66) decays as $\mathcal{O}(1/|x_1|)$.
Summarizing we have convergence of u_ε to some u_0 which is of the form

$$\begin{aligned}
u_0(x) \;&=\; \tilde u(x) \;+\; \sum_{j\in J} \sum_{\lambda_{\ell,j}>0} \frac{2\pi i}{|\lambda_{\ell,j}|}\, \langle f, \phi^{\ell,j}\rangle_{L^2(W^{h_0})} \psi^+(x_1)\, \phi^{\ell,j}(x) \\
&\quad + \sum_{j\in J} \sum_{\lambda_{\ell,j}<0} \frac{2\pi i}{|\lambda_{\ell,j}|}\, \langle f, \phi^{\ell,j}\rangle_{L^2(W^{h_0})} \psi^-(x_1)\, \phi^{\ell,j}(x) \\
&\quad + \langle f, \hat\phi^+\rangle_{L^2(W^{h_0})} \varphi^+(x_1)\, \hat\phi^+(x) \;+\; \langle f, \hat\phi^-\rangle_{L^2(W^{h_0})} \varphi^-(x_1)\, \hat\phi^-(x)
\end{aligned} \tag{68}$$

where $\tilde u \in H^1(W^h)$ decays as $1/|x_1|$ as $x_1 \to \pm\infty$. The convergence is in $H^1(Q^{R,h})$ for every $R > 0$ where again $Q^{R,h} = (-R,R) \times (0,h)$.

Since also $h \geq h_0$ is arbitrary we have convergence in $H^1(Q^{R,h})$ for all $R > 0$ and $h \geq h_0$ of $u_\varepsilon \in H^1_*(\mathbb{R}^2_+)$ to some function u_0 which has the form (68).

It remains to transform the representation (68) into the form (25), (26). We decompose $\psi^\pm(x_1)$ and $\varphi^\pm(x_1)$ as

$$\begin{aligned}
\psi^\pm(x_1) &= \xi^\pm(x_1) \;+\; [\psi^\pm(x_1) - \xi^\pm(x_1)]\,, \\
\varphi^\pm(x_1) &= \frac{e^{i\pi/4}}{\sqrt{2\pi k|x_1|}}\xi^\pm(x_1) \;+\; \left[\varphi^\pm(x_1) - \frac{e^{i\pi/4}}{\sqrt{2\pi k|x_1|}}\xi^\pm(x_1)\right]
\end{aligned}$$

and use the asymptic behavior $\psi^\pm(x_1)-\xi^\pm(x_1) = \mathcal{O}(1/|x_1|)$ and $\varphi^\pm(x_1)-\frac{e^{i\pi/4}}{\sqrt{2\pi k|x_1|}}\xi^\pm(x_1) = \mathcal{O}(1/|x_1|)$ as $|x_1| \to \infty$ (Lemma A.1 and Lemma A.2). Therefore, we can split u_0 from (68) into the form $u_0 = u_0^{prop} + u_0^{rad}$ with u_0^{prop}, u_0^{rad} from (25), (26), respectively, with

$$\begin{aligned}
\tilde{u}^{rad}(x) &= \sum_{\sigma\in\{+,-\}} [\psi^\sigma(x_1) - \xi^\sigma(x_1)] \sum_{j\in J} \sum_{\sigma\lambda_{\ell,j}>0} \frac{2\pi i}{|\lambda_{\ell,j}|} \langle f, \phi^{\ell,j}\rangle_{L^2(W^{h_0})}\, \phi^{\ell,j}(x) \\
&\quad + \tilde{u}(x) \;+ \sum_{\sigma\in\{+,-\}} \left[\varphi^\sigma(x_1) - \frac{e^{i\pi/4}}{\sqrt{2\pi k}}\xi^\sigma(x_1)\right] \langle f, \hat{\phi}^\sigma\rangle_{L^2(W^{h_0})} \frac{1}{\sqrt{|x_1|}}\, \hat{\phi}^\sigma(x)\,.
\end{aligned}$$

Then $\tilde{u}^{rad} \in H^1_*(\mathbb{R}^2_+)$, and $\tilde{u}^{rad}$ decays as $1/|x_1|$. The proof of Theorem 4.2 is complete.

5. The Open Waveguide Radiation Condition and Uniqueness

5.1. **The Open Waveguide Radiation Condition.** Theorem 4.2 describes the behavior of the solution u_0 in the x_1-direction. It remains to construct a radiation condition which describes also the behavior as $x_2 \to \infty$. We show first the following representation of u_0^{rad} from Theorem 4.2 in the half plane $x_2 > h_0$.

Lemma 5.1. *The radiating part u_0^{rad} from (26) can be expressed as*

$$\begin{aligned}
u_0^{rad}(x) &= \frac{i}{4} \int\limits_{\mathbb{R}^2_{h_0}} R_0(y)\big[H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\big]\, dy \\
&\quad + \frac{i}{2} \int\limits_{\Gamma_{h_0}} u_0^{rad}(y)\, \partial_{y_2} H_0^{(1)}(k|x-y|)\, ds(y) \quad \text{in } \mathbb{R}^2_{h_0}
\end{aligned} \tag{69}$$

where now $y^* = (y_1, 2h_0 - y_2)^\top$, $\mathbb{R}^2_{h_0} = \mathbb{R}\times(h_0,\infty)$, and $\Gamma_{h_0} = \mathbb{R}\times\{h_0\}$, and $R_0 \in L^2(\mathbb{R}^2_{h_0})$ is given by

$$R_0(x) \;=\; i \sum_{j\in J}\sum_{\ell=1}^{m_j} \frac{\langle f, \phi^{\ell,j}\rangle_{L^2(W^{h_0})}}{\lambda_{\ell,j}} \int\limits_{-\delta}^{\delta} \big[2\,\partial_{x_1}\phi^{\ell,j}(x) + i\alpha\, \phi^{\ell,j}(x)\big]\, e^{i\alpha x_1} d\alpha\,. \tag{70}$$

Proof. We go back to the representation (63) of u_ε and define

$$\begin{aligned} u_\varepsilon^{prop}(x) &= \sum_{j\in J}\sum_{\ell=1}^{m_j}\langle f,\phi^{\ell,j}\rangle_{L^2(W^{h_0})}\int_{-\delta}^{\delta}\frac{1}{\lambda_{\ell,j}\alpha-i\varepsilon}\,e^{i\alpha x_1}\,d\alpha\;\phi^{\ell,j}(x)\,,\\ u_\varepsilon^{rad} &= \sum_{j\in J}\sum_{\ell=1}^{m_j}\langle f,\phi^{\ell,j}\rangle_{L^2(W^{h_0})}\int_{-\delta}^{\delta}\frac{\eta_j(\alpha+\hat\alpha_j)-1}{\lambda_{\ell,j}\alpha-i\varepsilon}\,e^{i\alpha x_1}d\alpha\;\phi^{\ell,j}\\ &\quad+\sum_{j\in J}\int_{-1/2}^{1/2}\eta_j(\alpha)\left[u_1^j(\alpha,\varepsilon)+\delta_{|j|,1}\,s^{\operatorname{sign} j}(\alpha,\varepsilon)\,u_2^j(\alpha,\varepsilon)\right]d\alpha+\int_{-1/2}^{1/2}\eta_0(\alpha)\,u_{\alpha,\varepsilon}\,d\alpha\,. \end{aligned}$$

We recall from the proof of Theorem 4.2 that u_ε^{prop} and u_ε^{rad} converge to u_0^{prop} and u_0^{rad}, respectively, where u_0^{prop} and u_0^{rad} are given by (25) and (26), respectively.
We prove first a representation of u_ε^{rad} for $x_2>h_0$ analogously to (69). We observe that u_ε^{rad} satisfies the boundary value problem

$$(\Delta+k^2)u_\varepsilon^{rad}=-R_\varepsilon \text{ for } x_2>h_0\,,\quad u_\varepsilon^{rad}=S_\varepsilon \text{ for } x_2=h_0\,, \tag{71}$$

where

$$\begin{aligned} R_\varepsilon(x) &= \sum_{j\in J}\sum_{\ell=1}^{m_j}\langle f,\phi^{\ell,j}\rangle_{L^2(W^{h_0})}\int_{-\delta}^{\delta}\frac{1}{\lambda_{\ell,j}\alpha-i\varepsilon}\,(\Delta+k^2)\big[e^{i\alpha x_1}\,\phi^{\ell,j}(x)\big]\,d\alpha\\ &= \sum_{j\in J}\sum_{\ell=1}^{m_j}\langle f,\phi^{\ell,j}\rangle_{L^2(W^{h_0})}\int_{-\delta}^{\delta}\frac{i\alpha}{\lambda_{\ell,j}\alpha-i\varepsilon}\,\big[2\,\partial_{x_1}\phi^{\ell,j}(x)+i\alpha\,\phi^{\ell,j}(x)\big]\,e^{i\alpha x_1}d\alpha\,, \end{aligned}$$

and $S_\varepsilon=u_\varepsilon^{rad}|_{\Gamma_{h_0}}$. We note that R_ε is a linear combination of terms of the form

$$\int_{-\delta}^{\delta}\frac{2i\alpha}{\lambda_{\ell,j}\alpha-i\varepsilon}\,e^{i\alpha x_1}\,d\alpha\,\partial_{x_2}\phi^{\ell,j}(x)\quad\text{and}\quad\int_{-\delta}^{\delta}\frac{\alpha^2}{\lambda_{\ell,j}\alpha-i\varepsilon}\,e^{i\alpha x_1}\,d\alpha\,\phi^{\ell,j}(x)\,.$$

As inverse Floquet-Bloch transforms of smooth functions the integrals are $L^2(\mathbb{R})-$functions with respect to x_1. Since $\phi^{\ell,j}$ decay exponentially with respect to x_2 we conclude $R_\varepsilon\in L^2(\mathbb{R}^2_{h_0})$. Furthermore, $S_\varepsilon\in H^{1/2}(\Gamma_{h_0})$. From the radiation condition (4) for u_ε we have

$$\begin{aligned} &\partial_{x_2}(\mathcal{F}u_\varepsilon^{rad})(\xi,x_2)-i\sqrt{k^2-\xi^2}\,(\mathcal{F}u_\varepsilon^{rad})(\xi,x_2)\\ =\;&-\big[\partial_{x_2}(\mathcal{F}u_\varepsilon^{prop})(\xi,x_2)-i\sqrt{k^2-\xi^2}\,(\mathcal{F}u_\varepsilon^{prop})(\xi,x_2)\big] \end{aligned}$$

which converges to zero as $x_2\to\infty$ for every fixed $\xi\in\mathbb{R}$. Therefore, u_ε^{rad} satisfies the radiation condition

$$\partial_{x_2}(\mathcal{F}u_\varepsilon^{rad})(\xi,x_2)-i\sqrt{k^2-\xi^2}\,(\mathcal{F}u_\varepsilon^{rad})(\xi,x_2)\;\longrightarrow\;0\,,\quad x_2\to\infty\,, \tag{72}$$

for almost all $\xi \in \mathbb{R}$. Application of Lemma A.3 yields the representation

$$
\begin{aligned}
u_\varepsilon^{rad}(x) \;&=\; \frac{i}{4}\int\limits_{\mathbb{R}^2_{h_0}} R_\varepsilon(y)\big[H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\big]\,dy \\
&+\; \frac{i}{2}\int\limits_{\Gamma_{h_0}} u_\varepsilon^{rad}(y)\,\partial_{y_2}H_0^{(1)}(k|x-y|)\,ds(y) \quad \text{in } \mathbb{R}^2_{h_0}\,.
\end{aligned} \tag{73}
$$

We will now show that we can take the limit in (73) as $\varepsilon \to 0$. It is easy to see that R_ε converges to R_0, given by (70) uniformly in the sense that

$$
|R_\varepsilon(x) - R_0(x)| \;\le\; c\,\varepsilon|\ln\varepsilon|\,e^{-\sigma x_2} \quad \text{for } x \in \mathbb{R}^2_{h_0}\,.
$$

Indeed

$$
\int\limits_{-\delta}^{\delta}\left|\frac{i\alpha}{\lambda_{\ell,j}\alpha - i\varepsilon} - \frac{i}{\lambda_{\ell,j}}\right|\big[2\,|\partial_{x_1}\phi^{\ell,j}(x)| + |\alpha|\,|\phi^{\ell,j}(x)|\big]\,d\alpha \;\le\; c\,\frac{\varepsilon}{|\lambda_{\ell,j}|}\int\limits_{-\delta}^{\delta}\frac{1}{\sqrt{\lambda_{\ell,j}^2\alpha^2+\varepsilon^2}}\,d\alpha\,e^{-\sigma x_2}
$$

and $\int\limits_{-\delta}^{\delta}\frac{d\alpha}{\sqrt{\lambda_{\ell,j}^2\alpha^2+\varepsilon^2}} \le c\,|\ln\varepsilon|$.

Next, we recall from Theorem 4.2 that u_ε^{rad} converges to u_0^{rad} locally, i.e. in $H^1(Q^{R,h})$ for every $R>0$ and $h>h_0$. Furthermore, since $\frac{\eta_j(\alpha+\hat\alpha_j)-1}{\lambda_{\ell,j}\alpha-i\varepsilon}$ and $u_\ell^j(\cdot,\alpha,\varepsilon)$ (for $\ell = 1,2$) and $\eta_0(\alpha)u_{\alpha,\varepsilon)}$ are uniformly (with respect to α and ε) bounded in $H^1(Q^h)$ and are quasi-periodic we have global boundedness of S_ε in the sense that, for any $m \in \mathbb{Z}$,

$$
\begin{aligned}
\|u_\varepsilon^{rad}\|_{H^1(Q_m^h)} \;&\le\; \sum_{j\in J}\sum_{\ell=1}^{m_j}|\langle f,\phi^{\ell,j}\rangle_{L^2(W^{h_0})}|\int\limits_{-\delta}^{\delta}\frac{|\eta_j(\alpha+\hat\alpha_j)-1|}{|\lambda_{\ell,j}\alpha - i\varepsilon|}\,d\alpha\;\|\phi^{\ell,j}\|_{H^1(Q_m^h)} \\
&+\sum_{j\in J}\int\limits_{-1/2}^{1/2}|\eta_j(\alpha)|\,\|u_1^j(\alpha,\varepsilon)\|_{H^1(Q_m^h)} + \int\limits_{-1/2}^{1/2}|\eta_0(\alpha)|\,\|u_{\alpha,\varepsilon}\|_{H^1(Q_m^h)}\,d\alpha \\
&+\sum_{|j|=1}\int\limits_{\hat\alpha_j-\delta}^{\hat\alpha_j+\delta}\frac{c}{|\rho^{\operatorname{sign} j}(\alpha)|}\,\|u_2^j(\cdot,\alpha,\varepsilon)\|_{H^1(Q_m^h)}\,d\alpha \\
&\le\; c' \quad \text{for all } \varepsilon>0 \text{ and } m\in\mathbb{Z}\,,
\end{aligned}
$$

where $Q_m^h = (2\pi m, 2\pi m + 2\pi)\times(0,h)$ and c' is independent of ε and m. Therefore, for fixed $x \in \mathbb{R}^2_{h_0}$,

$$
\begin{aligned}
|u_\varepsilon^{rad}(x) - u_0^{rad}(x)| \;&\le\; c\,\varepsilon|\ln\varepsilon|\int\limits_{\mathbb{R}^2_{h_0}} e^{-\sigma y_2}\big|H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\big|\,dy \\
&+\frac{1}{2}\sum_{m\in\mathbb{Z}}\int\limits_{2\pi m}^{2\pi(m+1)}|u_\varepsilon^{rad}(y_1,h_0) - u_0^{rad}(y_1,h_0)|\,\big|\partial_{y_2}H_0^{(1)}(k|x-y|)|_{y_2=h_0}\big|\,dy_1\,.
\end{aligned}
$$

First we consider the integral over $\mathbb{R}^2_{h_0}$ which we split into the bounded ball $\{y : |y-x| < 1\}$ and the unbounded complement $\{y : |y-x| > 1\}$. The integral over $\{y : |y-x| < 1\}$ converges because of the weak singularity at $y = x$. For the region $\{y : |y-x| > 1\}$ we use the estimate

$$\big|H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\big| + \big|\partial_{y_2} H_0^{(1)}(k|x-y|)\big| \;\leq\; c\frac{x_2+y_2}{|x-y|^{3/2}} \tag{74}$$

for $x, y \in \mathbb{R}^2_{h_0}$, $|x-y| \geq 1$ (see [4]). Therefore,

$$\int\limits_{|y-x|>1} e^{-\sigma y_2}\big|H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\big|\, dy \;\leq\; c \int\limits_{|y-x|>1} e^{-\sigma y_2}\frac{x_2+y_2}{|x-y|^{3/2}}\, dy$$

which shows convergence of this part.
For the line integral we split the series into $|m| \leq M$ and $|m| > M$. We estimate

$$\begin{aligned}
&\sum_{|m|\geq M} \int\limits_{2\pi m}^{2\pi(m+1)} |u_\varepsilon^{rad}(y_1,h_0) - u_0^{rad}(y_1,h_0)|\,\big|\partial_{y_2} H_0^{(1)}(k|x-y|)|_{y_2=h_0}\big|\, dy_1 \\
\leq\;& c\,(x_2+h_0) \sum_{|m|\geq M} \int\limits_{2\pi m}^{2\pi(m+1)} \frac{|u_\varepsilon^{rad}(y_1,h_0) - u_0^{rad}(y_1,h_0)|}{[(x_1-y_1)^2 + (x_2-h_0)^2]^{3/4}}\, dy_1 \\
\leq\;& c\,(x_2+h_0) \sup_{m\in\mathbb{Z}} \|u_\varepsilon^{rad}(\cdot,h_0) - u_0^{rad}(\cdot,h_0)\|_{L^2(I_m)} \sum_{|m|\geq M} \Bigg[\int\limits_{2\pi m}^{2\pi(m+1)} \frac{dy_1}{|x_1-y_1|^3}\Bigg]^{1/2} \\
\leq\;& \tilde{c} \sum_{|m|\geq M} \frac{1}{|m|^{3/2}}
\end{aligned}$$

where $I_m = (2\pi m, 2\pi m + 2\pi)$. Here we used the Cauchy-Schwarz inequality and that $\int_{2\pi m}^{2\pi(m+1)} \frac{dy_1}{|x_1-y_1|^3}$ behaves as $\mathcal{O}(|m|^{-3})$. Therefore, for given small $\eta > 0$ we can choose M such that $\tilde{c}\sum_{|m|\geq M} \frac{1}{|m|^{3/2}} \leq \eta$. For this M we use the convergence of u_ε^{rad} to u_0^{rad} in $L^2\big(-2\pi(M-1), 2\pi(M-1)\big)$ and the continuity of $\partial_{y_2} H_0^{(1)}(k|x-y|)|$ for $x_2 > h_0$ and $y_2 = h_0$. This yields convergence of $\sum_{|m|<M} \int_{I_m} |u_\varepsilon^{rad}(y_1,h_0) - u_0^{rad}(y_1,h_0)|\,\big|\partial_{y_2} H_0^{(1)}(k|x-y|)|_{y_2=h_0}\big|\, dy_1 = \int_{-2\pi(M-1)}^{2\pi(M-1)} |u_\varepsilon^{rad}(y_1,h_0) - u_0^{rad}(y_1,h_0)|\,\big|\partial_{y_2} H_0^{(1)}(k|x-y|)|_{y_2=h_0}\big|\, dy_1$ to zero as $\varepsilon \to 0$. □

This result motivates the following formulation of the radiation condition. We recall that $\phi^{\ell,j}$ are the evanescent modes corresponding to the critical values $\hat{\alpha}_j \in (-1/2, 1/2]$, $j \in J$, and $\hat{\phi}^{\pm}$ are the non-evanescent modes of $\hat{\alpha}_{\pm 1} = \pm\kappa$ (if $\mathcal{M}_{evan}(\pm\kappa) \neq \mathcal{M}(\pm\kappa)$).

Definition 5.2. *Let Assumptions 3.9 and 3.10 hold. Fix $R_0 > 2\pi + 1$, and let $\xi^{\pm} \in C^\infty(\mathbb{R})$ with $\xi^{\pm}(x_1) = 1$ for $\pm x_1 \geq R_0$ and $\xi^{\pm}(x_1) = 0$ for $\pm x_1 \leq R_0 - 1$. A solution $u \in H^1_{loc}(\mathbb{R}^2_+)$ of (1); that is, of*

$$\Delta u + k^2 n\, u \;=\; -f \quad \text{in } \mathbb{R}^2_+\,, \qquad u = 0 \text{ or } \partial_{x_2} u = 0 \text{ for } x_2 = 0\,, \tag{75}$$

satisfies the ***open waveguide radiation condition*** *if u has a decomposition in the form $u = u^{rad} + u^{prop}$ where*

(a) u^{prop} *and* u^{rad} *have the forms*

$$u^{prop}(x) \;=\; \sum_{\sigma\in\{+,-\}}\sum_{j\in J}\xi^\sigma(x_1)\sum_{\ell:\sigma\lambda_{\ell,j}>0} a_{\ell,j}\,\phi^{\ell,j}(x)\,, \tag{76}$$

$$u^{rad}(x) \;=\; \tilde u^{rad}(x) \;+\; \sum_{\sigma\in\{+,-\}}\xi^\sigma(x_1)\,\frac{b^\sigma}{\sqrt{|x_1|}}\,\hat\phi^\sigma(x)\,,\quad x\in\mathbb{R}^2_+\,, \tag{77}$$

respectively, for some $a_{\ell,j}, b^\pm\in\mathbb{C}$ *and* $\tilde u^{rad}\in H^1_*(\mathbb{R}^2_+)$. *If* $\mathcal{M}_{evan}(\pm\kappa)=\mathcal{M}(\pm\kappa)$ *then* $b^\pm=0$.

(b) *For* $x_2>h_0$ *the radiating part* u^{rad} *can also be expressed as*

$$u^{rad}(x) \;=\; v^{rad}(x) \;+\; \frac{i}{2}\int\limits_{\Gamma_{h_0}} u^{rad}(y)\,\partial_{y_2}H_0^{(1)}(k|x-y|)\,ds(y) \quad in\ \mathbb{R}^2_{h_0}\,, \tag{78}$$

where $v^{rad}\in H^1_*(\mathbb{R}^2_{h_0})$ *satisfies the generalized angular spectrum radiation condition*

$$\int\limits_{-\infty}^{\infty}\bigl|\partial_{x_2}(\mathcal{F}v^{rad})(\xi,x_2)-i\sqrt{k^2-\xi^2}\,(\mathcal{F}v^{rad})(\xi,x_2)\bigr|^2 d\xi \;\longrightarrow\; 0\,,\quad x_2\to\infty\,. \tag{79}$$

Here, the space $H^1_*(\mathbb{R}^2_{h_0})$ *is defined analogously to* $H^1_*(\mathbb{R}^2_+)$ *in* (3).

Corollary 5.3. *Let Assumptions 3.9 and 3.10 hold. The solution* u_0 *of Theorem 4.2 satisfies the open wavegiode radiation condition of the previous definition with* $a_{\ell,j}=\frac{2\pi i}{|\lambda_{\ell,j}|}\,\langle f,\phi^{\ell,j}\rangle_{L^2(W^{h_0})}$ *and* $b^\pm=\frac{e^{i\pi/4}}{\sqrt{2\pi k}}\,\langle f,\hat\phi^\pm\rangle_{L^2(W^{h_0})}$.

Proof. It remains to show part (b) of the radiation condition. Application of Lemma 5.1 yields the form (78) with

$$v^{rad}(x) \;=\; \frac{i}{4}\int\limits_{\mathbb{R}^2_{h_0}} R_0(y)\bigl[H_0^{(1)}(k|x-y|)-H_0^{(1)}(k|x-y^*|)\bigr]\,dy\,.$$

Since R_0, given by (70), satisfies the assumptions of Lemma A.3 we conclude that v^{rad} satisfies the radiation condition (79) which ends the proof. ☐

5.2. **Uniqueness.** We show finally that the radiation condition implies uniqueness.

Theorem 5.4. *Let Assumptions 3.9 and 3.10 hold and let* u *be a solution of* (1) *corresponding to* $f=0$ *satisfying the open waveguide radiation condition of Definition 5.2. Then* u *has to vanish identically.*

Proof. Let u be a solution corresponding to $f=0$ satisfying the open waveguide radiation condition of Definition 5.2. Again, we observe that $\tilde u^{rad}$ has the form

$$\tilde u^{rad}(x) \;=\; u(x) \;-\; \sum_{\sigma\in\{+,-\}}\xi^\sigma(x_1)\left[\frac{b^\sigma}{\sqrt{|x_1|}}\,\hat\phi^\sigma(x) \;+\; \sum_{j\in J}\sum_{\ell:\sigma\lambda_{\ell,j}>0} a_{\ell,j}\,\phi^{\ell,j}(x)\right]. \tag{80}$$

In the form of $(\Delta+k^2n)\tilde u^{rad}$ the expressions $(\Delta+k^2n)(\varphi^\pm\hat\phi^\pm)$ and $(\Delta+k^2n)(\xi^\pm\phi^{\ell,j})$ appear where $\varphi^\pm(x_1)=\xi^\pm(x_1)/\sqrt{|x_1|}$. Let the pair (ψ,ϕ) be $(\varphi^\pm,\hat\phi^\pm)$ or $(\xi^\pm,\phi^{\ell,j})$. Then $(\Delta+k^2n)(\psi(x_1)\phi(x))=2\psi'(x_1)\partial_{x_1}\phi(x)+\psi''(x_1)\phi(x)$. Since $\tilde u^{rad}$ and the function $2\psi'\partial_{x_1}\phi+\psi''\phi$ are in $L^2(W^h)$ for all $h>0$ we can take the Floquet-Bloch transform.

Therefore $(\Delta+k^2n)(F\tilde{u}^{rad})(x,\alpha)$ is a linear combinations of terms of the form $F(2\psi'\partial_{x_1}\phi+\psi''\phi)(x,\alpha)$. Note that ϕ is $\hat{\alpha}_j$ quasi-periodic and therefore, $F(2\psi'\partial_{x_1}\phi+\psi''\phi)(x,\alpha) = 2(F\psi')(x_1,\alpha-\hat{\alpha}_j)\partial_{x_1}\phi(x)+(F\psi'')(x_1,\alpha-\hat{\alpha}_j)\phi(x)$. We assume that $\alpha\in(-1/2,1/2]\setminus\mathcal{A}$ and set $\beta=\alpha-\hat{\alpha}_j$ for the moment. Then $0<|\beta|<1$ and $(F\psi')(x_1,\beta)$ has a Fourier expansion in the form

$$(F\psi')(x_1,\beta) \;=\; \sum_{m\in\mathbb{Z}}\psi_m(\beta)\,e^{i(m+\beta)x_1} \;=\; \sum_{m\in\mathbb{Z}}\mathcal{F}(\psi')(m+\beta)\,e^{i(m+\beta)x_1}$$

with the Fourier transform $\mathcal{F}(\psi')$ of $\psi'\in L^2(\mathbb{R})$ (see (11)). Now we observe that

$$(F\psi')(x_1,\beta) \;=\; \frac{d}{dx_1}\tilde{\psi}(x_1,\beta) \quad\text{with}\quad \tilde{\psi}(x_1,\beta) \;=\; \sum_{m\in\mathbb{Z}}\frac{\mathcal{F}(\psi')(m+\beta)}{i(m+\beta)}\,e^{i(m+\beta)x_1}$$

for $0<|\beta|<1$. Therefore, we can write

$$\begin{aligned} & 2F\psi'(x_1,\alpha-\hat{\alpha}_j)\partial_{x_1}\phi(x)+F\psi''(x_1,\alpha-\hat{\alpha}_j)\phi(x) \\ =\;& 2\tilde{\psi}'(x_1,\alpha-\hat{\alpha}_j)\partial_{x_1}\phi(x)+\tilde{\psi}''(x_1,\alpha-\hat{\alpha}_j)\phi(x) \\ =\;& (\Delta+k^2n)[\tilde{\psi}(x_1,\alpha-\hat{\alpha}_j)\phi(x)] \end{aligned}$$

for $\alpha\in(-1/2,1/2]\setminus\mathcal{A}$. Now we substitute $(\psi,\phi)=(\varphi^\pm,\hat{\phi}^\pm)$ and $(\psi,\phi)=(\xi^\pm,\phi^{\ell,j})$ and denote the corresponding functions $\tilde{\psi}$ by $\tilde{\varphi}^\pm$ and $\tilde{\xi}^\pm$, respectively. Then we have from the Floquet-Bloch transform of (80) that $v(\cdot,\alpha)$, defined by

$$\begin{aligned} v(x,\alpha) \;:=\;& (F\tilde{u}^{rad})(x,\alpha) \;-\; \sum_{\sigma\in\{+.-\}} b^\sigma\,\tilde{\varphi}^\sigma(x_1,\alpha-\sigma\kappa)\,\hat{\phi}^\sigma(x) \\ & -\;\sum_{\sigma\in\{+.-\}}\sum_{j\in J}\sum_{\ell:\sigma\lambda_{\ell,j}>0} a_{\ell,j}\,\tilde{\xi}^\sigma(x_1,\alpha-\hat{\alpha}_j)\phi^{\ell,j}(x)\,, \end{aligned}$$

satisfies $(\Delta+k^2n)v(\cdot,\alpha)=0$. Furthermore, $v(\cdot,\alpha)$ is $\alpha-$quasi-periodic and satisfy the Rayleigh expansion. The uniqueness result for $\alpha\notin\mathcal{A}$ implies $v(\cdot,\alpha)=0$, i.e.

$$\begin{aligned} (F\tilde{u}^{rad})(x,\alpha) \;=\;& \sum_{\sigma\in\{+,-\}} b^\sigma\,\tilde{\varphi}^\sigma(x_1,\alpha-\sigma\kappa)\,\hat{\phi}^\sigma(x) \\ & +\;\sum_{\sigma\in\{+,-\}}\sum_{j\in J}\sum_{\ell:\sigma\lambda_{\ell,j}>0} a_{\ell,j}\tilde{\xi}^\sigma(x_1,\alpha-\hat{\alpha}_j)\phi^{\ell,j}(x) \end{aligned}$$

for all x and all $\alpha\in(-1/2,1/2]\setminus\mathcal{A}$. Now we fix $\iota\in J$ and consider the behavior as $\alpha\approx\hat{\alpha}_\iota$. Let first $|\iota|>1$ and $\sigma=+$. The left hand side is in $L^2\big((-1/2,1/2),L^2(Q^h)\big)$ by part (b) of the radiation condition and the mapping property of the Floquet-Bloch transform. The term $\tilde{\xi}^+(x_1,\alpha-\hat{\alpha}_\iota)$ has been investigated in Lemma A.4 in the appendix (set $\beta:=\alpha-\hat{\alpha}_\iota$), and we observe that this term behaves as $1/(\alpha-\hat{\alpha}_\iota)$ and is thus not in L^2 (locally at $\alpha\approx\hat{\alpha}_\iota$). Since all the other terms are locally (in a neighborhood of $\hat{\alpha}_\iota$) in L^2 we conclude that $\sum_{\lambda_{\ell,\iota}>0}a_{\ell,\iota}\phi_{\ell,\iota}$ has to vanish identically, i.e. $a_{\ell,\iota}=0$ for all $\ell=1,\ldots,m_\iota$ with $\lambda_{\ell,\iota}>0$. The same arguments, applied to $\tilde{\xi}^-$ yields that $a_{\ell,\iota}=0$ for all $\ell=1,\ldots,m_\iota$ with $\lambda_{\ell,\iota}<0$.

This holds for all ι with $|\iota|>1$. For $\iota=\pm1$ we have $\hat{\alpha}_\iota=\pm\kappa$ and, as before, $\alpha\mapsto\tilde{\xi}^\pm(x_1,\alpha\mp\kappa)$ behaves as $1/(\alpha\mp\kappa)$ and thus $a_{\ell,\pm1}=0$. For $j=\pm1$ there can also be the term $\tilde{\varphi}^\pm(x_1,\alpha\mp\kappa)$. Again, by Lemma A.4 we conclude that this term behaves as

$1/\sqrt{|\alpha-\hat{\alpha}_\iota|}$), and is again not in L^2. This shows that also $b^\pm = 0$, i.e. also $F\tilde{u}^{rad}(\cdot,\alpha)$ vanishes for almost all α which implies that $u = \tilde{u}^{rad} = 0$. □

Appendix A. Investigation of some integrals

Lemma A.1. *Let $\delta > 0$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Set $\sigma = \operatorname{sign}\lambda$. Then*

$$\int_\delta^\delta \frac{1}{\lambda\alpha - i\varepsilon} e^{i\alpha x_1}\, d\alpha \;\longrightarrow\; \frac{2\pi i}{|\lambda|}\,\psi^\sigma(x_1)$$

as $\varepsilon \to 0$ uniformly with respect to $|x_1| \le R$ for every $R > 0$. Here,

$$\psi^\pm(x_1) \;=\; \frac{1}{2} \;\pm\; \frac{1}{\pi}\int_0^{\delta x_1} \frac{\sin t}{t}\, dt\,, \quad x_1 \in \mathbb{R}\,. \tag{81}$$

The functions $\psi^\pm$ behave asymptotically as $\psi^\pm(x_1) = 1 + \mathcal{O}(1/|x_1|)$ as $\pm x_1 \to \infty$ and $\psi^\pm(x_1) = \mathcal{O}(1/|x_1|)$ as $\pm x_1 \to -\infty$.

Proof. We calculate

$$\begin{aligned}\int_{-\delta}^{\delta} \frac{e^{i\alpha x_1}}{\lambda\alpha - i\varepsilon}\, d\alpha &= \int_{-\delta}^{\delta} \frac{[\cos(\alpha x_1) + i\sin(\alpha x_1)][\lambda\alpha + i\varepsilon]}{\lambda^2\alpha^2 + \varepsilon^2}\, d\alpha \\ &= 2i\varepsilon\int_0^\delta \frac{\cos(\alpha x_1)}{\lambda^2\alpha^2+\varepsilon^2}\, d\alpha \;+\; 2i\lambda\int_0^\delta \frac{\alpha\sin(\alpha x_1)}{\lambda^2\alpha^2+\varepsilon^2}\, d\alpha\,,\end{aligned}$$

where we used that the integral over odd integrands vanishes. Let us start with an analysis of the first term, using the substitution $\alpha = t\,\varepsilon/|\lambda|$,

$$2i\varepsilon\int_0^\delta \frac{\cos(\alpha x_1)}{\lambda^2\alpha^2+\varepsilon^2}\, d\alpha \;=\; \frac{2i\varepsilon^2}{|\lambda|}\int_0^{\delta|\lambda|/\varepsilon} \frac{\cos(t\varepsilon x_1/|\lambda|)}{t^2\varepsilon^2+\varepsilon^2}\, dt \;=\; \frac{2i}{|\lambda|}\int_0^{\delta|\lambda|/\varepsilon} \frac{\cos(t\varepsilon x_1/|\lambda|)}{1+t^2}\, dt\,.$$

Therefore, using the transformation $t = \alpha x_1$ in the second integral,

$$\int_{-\delta}^{\delta} \frac{e^{i\alpha x_1}}{\lambda\alpha - i\varepsilon}\, d\alpha \;=\; \frac{2i}{|\lambda|}\int_0^{\delta|\lambda|/\varepsilon} \frac{\cos(t\varepsilon x_1/|\lambda|)}{1+t^2}\, dt \;+\; 2i\lambda\int_0^{\delta x_1} \frac{t\sin t}{\lambda^2 t^2 + \varepsilon^2 x_1^2}\, dt\,.$$

In the limit $\varepsilon \to 0$, we therefore find

$$\int_{-\delta}^{\delta} \frac{e^{i\alpha x_1}}{\lambda\alpha - i\varepsilon}\, d\alpha \;\longrightarrow\; \frac{2i}{|\lambda|}\int_0^\infty \frac{1}{1+t^2}\, dt \;+\; \frac{2i}{\lambda}\int_0^{\delta x_1} \frac{\sin t}{t}\, dt \;=\; \frac{\pi i}{|\lambda|}\left[1 + \operatorname{sign}\lambda\,\frac{2}{\pi}\int_0^{\delta x_1}\frac{\sin t}{t}\, dt\right].$$

The convergence is uniform in x_1 on compact subsets of $\mathbb{R}$. □

Lemma A.2. *Let $\delta > 0$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Then*

$$\varphi^+(x_1) \;=\; -\int\limits_{\kappa-\delta}^{\kappa+\delta} \frac{1}{\rho^+(\alpha)}\, e^{i(\alpha-\kappa)x_1}\, d\alpha \;=\; \begin{cases} \dfrac{e^{i\pi/4}}{\sqrt{2\pi k|x_1|}} \;+\; \mathcal{O}(1/|x_1|)\,, & x_1 \to +\infty\,, \\ \qquad \mathcal{O}(1/|x_1|)\,, & x_1 \to -\infty\,, \end{cases}$$

and

$$\varphi^-(x_1) \;=\; -\int\limits_{-\kappa-\delta}^{-\kappa+\delta} \frac{1}{\rho^-(\alpha)}\, e^{i(\alpha+\kappa)x_1}\, d\alpha \;=\; \begin{cases} \dfrac{e^{i\pi/4}}{\sqrt{2\pi k|x_1|}} \;+\; \mathcal{O}(1/|x_1|)\,, & x_1 \to -\infty\,, \\ \qquad \mathcal{O}(1/|x_1|)\,, & x_1 \to +\infty\,. \end{cases}$$

Proof. We consider the integral $-\int_{\kappa-\delta}^{\kappa+\delta} \frac{1}{\rho^+(\alpha)}\, e^{i(\alpha-\kappa)x_1}\, d\alpha$ and recall that $k = \tilde{\ell} + \kappa$ and $\rho^+(\alpha) = 2\pi i \sqrt{\kappa-\alpha}\sqrt{2\tilde{\ell}+\kappa+\alpha}$. Therefore,

$$\frac{1}{\rho^+(\alpha)} \;=\; \frac{1}{2\pi i\sqrt{2k}\,\sqrt{\kappa-\alpha}} \;+\; \sqrt{\kappa-\alpha}\, f(\alpha)$$

with

$$f(\alpha) \;=\; \frac{1}{2\pi i[\sqrt{2k} + \sqrt{2\tilde{\ell}+\kappa+\alpha}]\,\sqrt{2k}\,\sqrt{2\tilde{\ell}+\kappa+\alpha}}\,.$$

We note that f is smooth for $|\alpha-\kappa| < \delta$. We compute, analogously to the proof of the previous lemma,

$$\begin{aligned} \frac{1}{i}\int\limits_{\kappa-\delta}^{\kappa+\delta} \frac{1}{\sqrt{\kappa-\alpha}}\, e^{i(\alpha-\kappa)x_1}\, d\alpha \;&=\; -i\int\limits_{-\delta}^{\delta} \frac{1}{\sqrt{\beta}}\, e^{-i\beta x_1}\, d\beta \\ &=\; -i\int\limits_{0}^{\delta}\left[\frac{\cos(\beta x_1) - i\sin(\beta x_1)}{\sqrt{\beta}} + \frac{\cos(\beta x_1) + i\sin(\beta x_1)}{i\sqrt{\beta}}\right] d\beta \\ &=\; -(1+i)\int\limits_{0}^{\delta} \frac{\cos(\beta x_1) + \sin(\beta x_1)}{\sqrt{\beta}}\, d\beta \\ &=\; -\frac{\sqrt{2}\, e^{i\pi/4}}{\sqrt{|x_1|}} \int\limits_{0}^{\delta|x_1|} \frac{\cos t + (\operatorname{sign} x_1)\sin t}{\sqrt{t}}\, dt \end{aligned}$$

where we used the transformation $t = \beta|x_1|$ in the last step. Since $\int\limits_0^{\delta|x_1|} \frac{\cos t}{\sqrt{t}}\, dt = \sqrt{\frac{\pi}{2}} + \mathcal{O}(1/\sqrt{|x_1|})$ and $\int\limits_0^{\delta|x_1|} \frac{\sin t}{\sqrt{t}}\, dt = \sqrt{\frac{\pi}{2}} + \mathcal{O}(1/\sqrt{|x_1|})$ as $|x_1| \to \infty$ we obtain

$$-\frac{1}{2\pi i\sqrt{2k}}\int\limits_{\kappa-\delta}^{\kappa+\delta} \frac{1}{\sqrt{\kappa-\alpha}}\, e^{i(\alpha-\kappa)x_1}\, d\alpha \;=\; \begin{cases} \dfrac{e^{i\pi/4}}{\sqrt{2\pi k|x_1|}} \;+\; \mathcal{O}(1/|x_1|)\,, & x_1 \to +\infty\,, \\ \qquad \mathcal{O}(1/|x_1|)\,, & x_1 \to -\infty\,, \end{cases}$$

For the integral $\int_{\kappa-\delta}^{\kappa+\delta} \sqrt{\kappa-\alpha}\, f(\alpha)\, e^{i(\alpha-\kappa)x_1}\, d\alpha$ we use partial integration and obtain

$$\int\limits_{\kappa-\delta}^{\kappa+\delta} \sqrt{\kappa-\alpha}\, f(\alpha)\, e^{i(\alpha-\kappa)x_1}\, d\alpha \;=\; \frac{1}{ix_1} \int\limits_{\kappa-\delta}^{\kappa+\delta} \sqrt{\kappa-\alpha}\, f(\alpha)\, \frac{d}{d\alpha} e^{i(\alpha-\kappa)x_1}\, d\alpha$$

$$=\; \frac{1}{ix_1} \left[\sqrt{\kappa-\alpha}\, f(\alpha)\, e^{i(\alpha-\kappa)x_1} \Big|_{\kappa-\delta}^{\kappa+\delta} - \int\limits_{\kappa-\delta}^{\kappa+\delta} \frac{d}{d\alpha}\big(\sqrt{\kappa-\alpha}\, f(\alpha)\big)\, e^{i(\alpha-\kappa)x_1}\, d\alpha \right]$$

which behaves as $\mathcal{O}(1/|x_1|)$ since $1/\sqrt{\kappa-\alpha}$ is integrable.
The analysis for $\varphi^-(x_1)$ follows the same arguments. □

Lemma A.3. *Let $v \in H^1_*(\mathbb{R}^2_{h_0})$ for some $h_0 > 0$ such that u satisfies the radiation condition (72) and such that $f := -(\Delta + k^2)v \in L^2(\mathbb{R}^2_{h_0})$. Furthermore, assume that the f satisfies an estimate of the form $|f(x)| \le |f_1(x_1)|\, e^{-\sigma x_2}$ for $x \in \mathbb{R}^2_{h_0}$ and some $\sigma > 0$ and $f_1 \in L^2(\mathbb{R})$. Then v can be represented as*

$$(82)\qquad \begin{aligned} v(x) \;&=\; \frac{i}{4} \int\limits_{\mathbb{R}^2_{h_0}} f(y)\big[H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\big]\, dy \\ &+\; \frac{i}{2} \int\limits_{\Gamma_{h_0}} v(y)\, \partial_{y_2} H_0^{(1)}(k|x-y|)\, ds(y) \quad \textit{for } x \in \mathbb{R}^2_{h_0} \end{aligned}$$

where $\Gamma_{h_0} = \mathbb{R} \times \{h_0\}$. The first part of v satisfies the stronger radiation condition (79) while the line integral satisfies even (4).

Proof. Let $\tilde v$ be the right hand side of (82). The volume potential which we denote by $\tilde v_1$, satisfies $(\Delta + k^2)\tilde v_1 = -f$ in $\mathbb{R}^2_{h_0}$ and $\tilde v_1 = 0$ on Γ_{h_0}. This is well known, but we repeat the argument. We fix x_0 and consider x in a neighborhood of x_0. We split the region of integration into $\{y : |y - x_0| < R\} \cup \{y : |y - x_0| > R\}$. The formula

$$(\Delta + k^2) \int\limits_{\substack{\mathbb{R}^2_{h_0} \\ |y-x_0|<R}} f(y)\, \frac{i}{4} \left[H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\right] dy \;=\; -f(x)$$

is standard. For the remaining integral we use (74) and estimate $|f(y)|\big[H_0^{(1)}(k|x-y|) - H_0^{(1)}(k|x-y^*|)\big] \le c|f_1(y_1)|e^{-\sigma y_2} \frac{x_2+y_2}{|x-y|^{3/2}}$ for $|y - x_0| > R$ and the same kind of estimate for the derivatives. Lebesgues theorem on dominated convergence implies that we can interchange integration and the Helmholtz operator which shows $(\Delta + k^2)\tilde v_1 = -f$ in $\mathbb{R}^2_{h_0}$. The line integral which we denote by $\tilde v_2$, satisfies $(\Delta + k^2)\tilde v_1 = 0$ in $\mathbb{R}^2_{h_0}$ and $\tilde v_1 = v$ on Γ_{h_0}. This follows from the jump conditions of the double layer potential on Γ_{h_0} with $H^{1/2}-$densities.

We show that $\tilde v$ satisfies the radiation condition (72). This is seen by observing that both integrals are convolution-type integrals with respect to y_1. Setting $\varphi_1(x_1, x_2, y_2) = \frac{i}{4}\big[H_0^{(1)}(k\sqrt{x_1^2 + (x_2-y_2)^2}) - H_0^{(1)}(k\sqrt{x_1^2 + (x_2+y_2)^2})\big]$ and $\varphi_2(x_1, x_2) =$

$\frac{i}{2}\partial_{y_2} H_0^{(1)}(k\sqrt{x_1^2+(x_2-h_0)^2})$ we express the right hand side $\tilde{v}$ of (82) as

$$\tilde{v}(x) \;=\; \int_{h_0}^{\infty} [f(\cdot,y_2) * \varphi_1(\cdot,x_2,y_2)](x_1)\,dy_2 \;+\; [v(\cdot,h_0) * \varphi_2(\cdot,x_2)](x_1)\,.$$

We apply the convolution theorem and use

$$\begin{aligned}
(\mathcal{F}\varphi_1)(\xi,x_2,y_2) \;&=\; \frac{i}{2\sqrt{k^2-\xi^2}}\left[e^{i\sqrt{k^2-\xi^2}|x_2-y_2|} - e^{i\sqrt{k^2-\xi^2}(x_2+y_2)}\right],\\
(\mathcal{F}\varphi_2)(\xi,x_2) \;&=\; e^{i\sqrt{k^2-\xi^2}(x_2-h_0)}\,.
\end{aligned}$$

Therefore, the Fourier transform of $\tilde{v}$ is given by

$$\begin{aligned}
(\mathcal{F}\tilde{v})(\xi,x_2) \;&=\; \frac{i}{2\sqrt{k^2-\xi^2}}\int_{h_0}^{\infty}(\mathcal{F}f)(\xi,y_2)\left[e^{i\sqrt{k^2-\xi^2}|x_2-y_2|} - e^{i\sqrt{k^2-\xi^2}(x_2+y_2)}\right]dy_2\\
&\quad+\;(\mathcal{F}v)(\xi,h_0)\,e^{i\sqrt{k^2-\xi^2}(x_2-h_0)}\,.
\end{aligned} \tag{83}$$

Splitting the interval (h_0,∞) into $(h_0,x_2)\cup(x_2,\infty)$ we obtain easily

$$\begin{aligned}
\partial_{x_2}(\mathcal{F}\tilde{v})(\xi,x_2) - i\sqrt{k^2-\xi^2}\,(\mathcal{F}\tilde{v})(\xi,x_2) \;&=\; \int_{x_2}^{\infty}(\mathcal{F}f)(\xi,y_2)\,e^{i\sqrt{k^2-\xi^2}(y_2-x_2)}\,dy_2\\
&=\; \int_0^{\infty}(\mathcal{F}f)(\xi,y_2+x_2)\,e^{i\sqrt{k^2-\xi^2}y_2}\,dy_2
\end{aligned} \tag{84}$$

and thus

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left|\partial_{x_2}(\mathcal{F}\tilde{v})(\xi,x_2) - i\sqrt{k^2-\xi^2}\,(\mathcal{F}\tilde{v})(\xi,x_2)\right|^2 d\xi\\
\le\;& \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\left|(\mathcal{F}f)(\xi,y_2+x_2)\right|\left|(\mathcal{F}f)(\xi,y_2'+x_2)\right|d\xi\,dy_2\,dy_2'\\
\le\;& \int_0^{\infty}\int_0^{\infty}\sqrt{\int_{-\infty}^{\infty}\left|(\mathcal{F}f)(\xi,y_2+x_2)\right|^2 d\xi}\sqrt{\int_{-\infty}^{\infty}\left|(\mathcal{F}f)(\xi,y_2'+x_2)\right|^2 d\xi}\;dy_2\,dy_2'\\
=\;& \left[\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\|f(\cdot,x_2+y_2)\|_{L^2(\mathbb{R})}\,dy_2\right]^2 \;\le\; \frac{1}{2\pi}\,\|f_1\|_{L^2(\mathbb{R})}^2\left[\int_0^{\infty} e^{-\sigma(x_2+y_2)}dy_2\right]^2,
\end{aligned}$$

and this converges to zero as $x_2\to\infty$. Here we used the Plancherel formula. Therefore, $\tilde{v}$ satisfies (79). Since this is stronger than (72) we obtain $\tilde{v}=v$ by the uniqueness of the Dirichlet boundary value problem in $\mathbb{R}^2_{h_0}$. □

Lemma A.4. *Set* $\varphi^\pm(x_1) = \frac{\xi^\pm(x_1)}{\sqrt{|x_1|}}$ *for* $x_1 \in \mathbb{R}$. *Then*

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \frac{\mathcal{F}\big(\frac{d}{dx_1}\xi^\pm\big)(m+\beta)}{i(m+\beta)}\, e^{i(m+\beta)x_1} &= \pm\frac{1}{2\pi i\beta}\, e^{\mp i\beta x_1} + g_1^\pm(x_1,\beta)\,,\\ \sum_{m\in\mathbb{Z}} \frac{\mathcal{F}\big(\frac{d}{dx_1}\varphi^\pm\big)(m+\beta)}{i(m+\beta)}\, e^{i(m+\beta)x_1} &= \frac{1}{\sqrt{2\pi|\beta|}}\,[1 \mp (\operatorname{sign}\beta) i]\, e^{i\beta x_1} + g_2^\pm(x_1,\beta)\end{aligned}$$

for $x_1 \in [0,2\pi]$ *and* $0 < |\beta| < 1$ *where* $g_j^\pm \in C^1\big[[0,2\pi]\times[-1,1]\big]$ *for* $j = 1,2$.

Proof. The series on the left hand side of the first line can be written as

$$\begin{aligned}&\frac{\mathcal{F}\big(\frac{d}{dx_1}\xi^\pm\big)(\beta)}{i\beta}\, e^{i\beta x_1} \;+\; \sum_{m\neq 0} \frac{\mathcal{F}\big(\frac{d}{dx_1}\xi^\pm\big)(m+\beta)}{i(m+\beta)}\, e^{i(m+\beta)x_1}\\ =\;& \frac{1}{i\beta\, 2\pi} \int_{-R_0}^{R_0} \frac{d}{dt}\xi^\pm(t)\, e^{-it\beta} dt\, e^{i\beta x_1} \;+\; \sum_{m\neq 0} \frac{\mathcal{F}\big(\frac{d}{dx_1}\xi^\pm\big)(m+\beta)}{i(m+\beta)}\, e^{i(m+\beta)x_1}\end{aligned}$$

because $\frac{d}{dx_1}\xi^\pm$ vanishes outside of $(-R_0,R_0)$. Partial integration of the first term yields $\frac{1}{i\beta}\int_{-R_0}^{R_0} \frac{d}{dt}\xi^\pm(t)\, e^{-it\beta}dt = \frac{1}{i\beta}\,\xi^\pm(t)e^{-i\beta t}\big|_{-R_0}^{R_0} + \int_{-R_0}^{R_0} \xi^\pm(t)\, e^{-it\beta}dt$ which yields the desired result because because $\xi^\pm(\pm R_0) = 1$ and $\xi^\pm(\mp R_0) = 0$, and $\int_{-R_0}^{R_0}\xi^\pm(t)\, e^{-it\beta}dt$ and the contribution of the series $\sum_{m\neq 0}$ are smooth with respect to x_1 and β.

Analogously, the series on the left hand side of the second assertion can be written as

$$\begin{aligned}&\frac{\mathcal{F}\big(\frac{d}{dx_1}\varphi^\pm\big)(\beta)}{i\beta}\, e^{i\beta x_1} \;+\; \sum_{m\neq 0} \frac{\mathcal{F}\big(\frac{d}{dx_1}\varphi^\pm\big)(m+\beta)}{i(m+\beta)}\, e^{i(m+\beta)x_1}\\ =\;& \frac{1}{i\beta\, 2\pi} \int_{-\infty}^{\infty} \frac{d}{dt}\varphi^\pm(t)\, e^{-it\beta} dt\, e^{i\beta x_1} \;+\; \sum_{m\neq 0} \frac{\mathcal{F}\big(\frac{d}{dx_1}\varphi^\pm\big)(m+\beta)}{i(m+\beta)}\, e^{i(m+\beta)x_1}\,.\end{aligned}$$

We consider, for some $R > R_0 + 1$,

$$\begin{aligned}\frac{1}{i\beta}\int_{-\infty}^{R} \frac{d}{dt}\varphi^+(t)\, e^{-it\beta}dt &= \frac{1}{i\beta}\,\varphi^+(t)e^{-it\beta}\big|_0^R + \int_0^R \varphi^+(t)\, e^{-it\beta}dt\\ &= \frac{1}{i\beta}\,\frac{1}{\sqrt{R}}\, e^{-iR\beta} + \int_0^{R_0} \frac{\xi^+(t)-1}{\sqrt{t}}\, e^{-it\beta}dt + \int_0^R \frac{1}{\sqrt{t}}\, e^{-it\beta}dt\,.\end{aligned}$$

The first term tends to zero as $R \to \infty$. Therefore,

$$\frac{1}{i\beta 2\pi}\int_{-\infty}^{\infty} \frac{d}{dt}\varphi^+(t)\, e^{-it\beta}dt = \frac{1}{2\pi}\int_0^{R_0} \frac{\xi^+(t)-1}{\sqrt{t}}\, e^{-it\beta}dt + \frac{1}{2\pi}\int_0^{\infty} \frac{1}{\sqrt{t}}\, e^{-it\beta}dt\,.$$

Finally, with the substitution $s = t|\beta|$ we have

$$\begin{aligned}\int_0^\infty \frac{1}{\sqrt{t}}\, e^{-it\beta} dt &= \frac{1}{\sqrt{|\beta|}} \int_0^\infty \frac{1}{\sqrt{s}}\, e^{-is(\operatorname{sign}\beta)} ds \;=\; \frac{1}{\sqrt{|\beta|}} \int_0^\infty \frac{\cos s - i(\operatorname{sign}\beta)\sin s}{\sqrt{s}}\, ds \\ &= \frac{\sqrt{2\pi}}{\sqrt{|\beta|}}\,[1 - (\operatorname{sign}\beta) i]\end{aligned}$$

because $\int_0^\infty \frac{\cos s}{\sqrt{s}}\, ds = \int_0^\infty \frac{\sin s}{\sqrt{s}}\, ds = \sqrt{\pi/2}$. The analysis for φ^- is done analogously. □

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