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Spectral gap properties of perturbed periodic media

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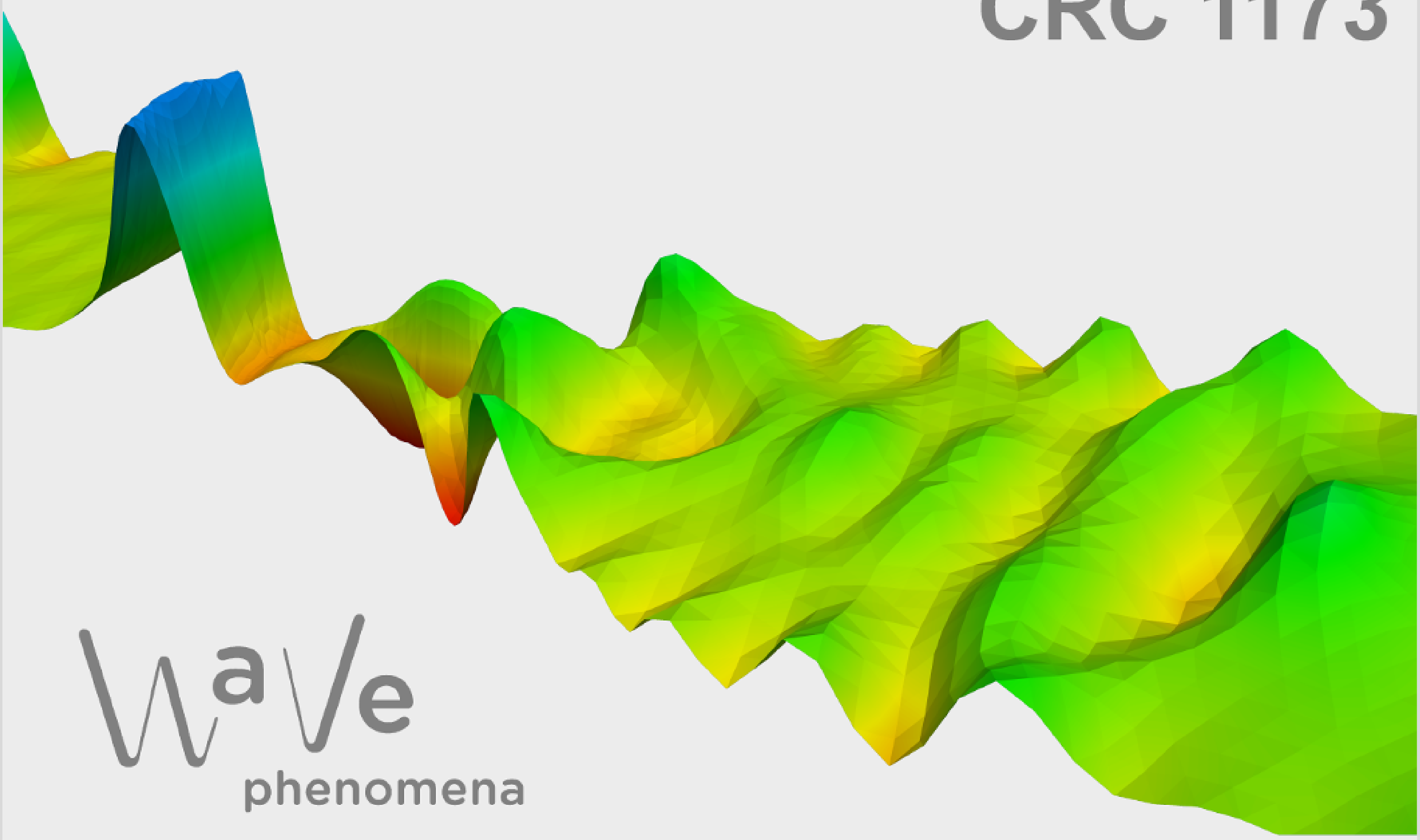

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SPECTRAL GAP PROPERTIES OF PERTURBED PERIODIC MEDIA

A. KIRSCH AND B. SCHWEIZER

Abstract. We analyze periodic operators on $\mathbb{R}^n$ and small perturbations of these operators. The perturbation is periodic in $n-1$ directions and has bounded support in the remaining direction. We show that, when the perturbation has a sign, every spectral gap for the unperturbed operator is reduced by the perturbation. We develop a general theory that can be applied to elliptic operators, to systems such as that of linear elasticity, and to Maxwell's equations.

Keywords: band gap; Bloch waves; perturbation of periodic operators

MSC: 47A25, 78A40, 35L05

1. Introduction

In a periodic medium, a self-adjoint differential operator has a spectrum that typically consists of bands and band gaps. This qualitative structure can be understood with a Floquet-Bloch transform: One can show that the spectrum in the periodic medium is given as the union of the spectra of differential operators in a periodicity cell. The spectra in the periodicity cell are discrete, the union is taken over the Floquet parameter; this leads to a spectrum that consists of intervals.

We ask: What happens to band-gaps when the periodic medium in n dimensions is perturbed in a non-periodic fashion? We restrict ourselfs to perturbations that are periodic in the first $n-1$ directions and that are confined to $x_n \in (0, 2\pi)$. The differential operators in this text are quite general. The two basic examples are (a) elliptic systems, the spectrum then describes the behavior of waves in the medium, and (b) Maxwell's equations, the spectrum then describes the propagating electro-magnetic modes in the medium. Our main result states, loosely speaking: A non-periodic perturbation with a sign reduces the spectral gap, no matter how small the perturbation is chosen.

To be more specific in the description of our main result, let us present the motivating example for our theory. The example is a scalar elliptic equation and our results are (under additional assumptions) already known in this problem, see [4, 5, 6, 7]. Our aim is to develop methods that are more general, the operator M below can also be the Maxwell operator or the elliptic operator of linear elasticity. Nonetheless, let us describe the simplest case here: For coefficients $a, b : \mathbb{R}^n \to \mathbb{R}$, both 2π-periodic in each direction in $\mathbb{R}^n$ and both positive, we study the family of operators

$$M^\lambda u \;:=\; -\nabla \cdot (a\,\nabla u) - \lambda\, b\, u\,.$$

Solutions are always functions $u : \mathbb{R}^n \to \mathbb{C}^m$, in this guiding example with $m = 1$, the parameter $\lambda \in \mathbb{R}$ is the spectral parameter.

We now treat the case that the coefficient b is perturbed, more precisely, it is replaced by the coefficient $b + \delta q^2$, where $q : \mathbb{R}^n \to \mathbb{R}$ is a bounded function with support in $\mathbb{R}^{n-1} \times (0, 2\pi)$, periodic in $n-1$ directions, and $\delta > 0$ is a small parameter

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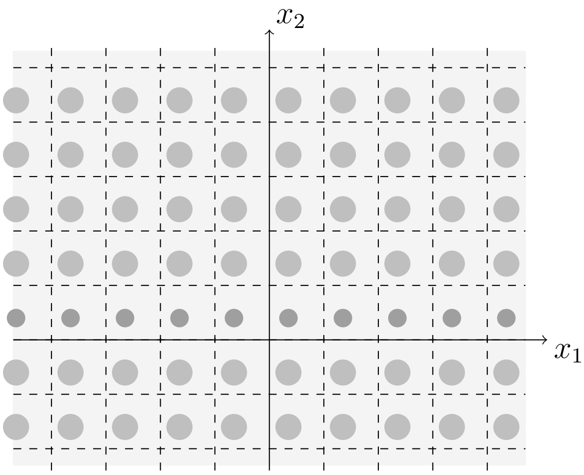

FIGURE 1. Sketch of a perturbed periodic medium in dimension $n = 2$. The medium is characterized by the coefficients of a differential operator. The underlying coefficients, a and b in the guiding example, are periodic in every direction. The perturbation, in the example given by the coefficient q, is confined to a strip $\mathbb{R}^{n-1} \times W = \mathbb{R} \times (0, 2\pi)$. Our result is that the perturbation reduces spectral gaps of the underlying medium.

that measures the strength of the perturbation. We emphasize that q is localized in direction x_n and hence, in particular, not periodic in this direction. The family of perturbed operators is therefore

$$M^{\lambda,\delta} u \ := \ -\nabla \cdot (a \nabla u) - \lambda (b + \delta q^2) u \,.$$

The choice to write q^2 for the perturbation indicates that the sign of the perturbation is relevant.

In applications, one is interested in spectral gaps of operator families as above; essentially, a number $\lambda \in \mathbb{R}$ is in a spectral gap when the operator $M^{\lambda,\delta}$ is invertible. For the application, this means that there are no propagating wave solutions for the system, for the frequency corresponding to λ. Since the perturbation modifies the properties of the periodic background medium along a hyperplane, one may expect that there are propagating wave solutions that are concentrated along this hyperplane. With this interpretation in mind, one may expect that spectral gaps are getting smaller when δ is increased. Our main result is that, indeed, the upper end of the spectral gap is pushed down by the small perturbation.

Mathematically, we define spectral gaps as maximal intervals $(a_-, a_+) \subset \mathbb{R}$ such that, for every $\lambda \in (a_-, a_+)$, the operator $M^{\lambda,\delta}$ is invertible. Let (a_-, a_+) be a spectral gap for the unperturbed problem ($\delta = 0$). Our main result is that, for every $\delta > 0$, there exists $\lambda \in (a_-, a_+)$ near a_+ such that $M^{\lambda,\delta}$ is not invertible. In particular, the interval (a_-, a_+) is *not* a spectral gap for the operator family $M^{\lambda,\delta}$ for $\delta > 0$.

The precise statement is formulated as Theorem 3.2. It treats a general situation with an unperturbed operator M (the above example uses $M : u \mapsto -\nabla \cdot (a \nabla u)$, in the case of the Maxwell system, essentially $M : u \mapsto \operatorname{curl}\operatorname{curl} u$) a mass-matrix operator E (in the above example $E : u \mapsto b\, u$), and a perturbation δQ^2 (in the above example the operator $u \mapsto \delta q^2 u$). We investigate the spectral gaps of the family $M - \lambda(E + \delta Q^2)$ in dependence of δ.

1.1. **The heart of the argument.** We provide a rough sketch of the main argument for the theorem, this idea was already used in [4, 5, 6, 7]. The entire argument is based on Floquet-Bloch transforms; we assume that the underlying operators are periodic with the periodicity cell W^n for $W = (0, 2\pi)$. On the one hand, we can transform in all variables $x_1, ..., x_n$; we write $k \in I^n$ with $I = [-1/2, 1/2]$ for the dual variable. This transformation is only useful for the operators M and E. On the other hand, we can transform the system in the first $n-1$ variables $x_1, ..., x_{n-1}$; the dual variable is then denoted as $m \in I^{n-1}$. This transformation can be performed for the periodic operator $M - \lambda E$ and also for the perturbed operator $M - \lambda(E + \delta Q^2)$. For a parameter $m^\circ \in I^{n-1}$ we write, e.g., $(M - \lambda E)_{m^\circ}$ to indicate the differential operator in the Floquet-Bloch representation for the parameter m° (Section 2 is, to a large extend, concerned with making the corresponding construction precise).

Step 1, made precise in Lemma 3.4: We show that a certain scalar product becomes large in absolute values in the limit $\lambda \nearrow a_+$. For an appropriately chosen function $r \in L^2(W^n)$, its trivial extension $R^* r \in L^2(W^{n-1} \times \mathbb{R})$, and for a critical dual parameter $m^\circ \in I^{n-1}$, there holds

$$\limsup_{\lambda \to a_+} \left| \langle (M - \lambda E)^{-1}_{m^\circ} R^* r, R^* r \rangle_{L^2(W^{n-1} \times \mathbb{R})} \right| = \infty \,, \tag{1.1}$$

compare relation (3.4). The relation coincides with our intuition: The fact that a_+ is a boundary point of the spectral gap implies that the inverse of $M - \lambda E$ must be large in some way.

Step 2, made precise in Lemma 3.5: We introduce, compare (3.14), an operator A as follows:

$$A := \lambda \delta \, Q R (M - \lambda E)^{-1}_{m^\circ} R^* Q : L^2(W^n) \to L^2(W^n) \,. \tag{1.2}$$

This operator is self-adjoint. Since it contains the inverse of $M - \lambda E$, the operator has a large norm for λ close to a_+ by (1.1). This implies that, for fixed $\delta > 0$, there is $\lambda < a_+$ such that A has the norm 1. To sketch loosely the rest of the argument, we might say: 1 is an eigenvalue of A, hence there is an eigenfunction $w \in L^2(W^n)$ with $Aw = w$. Let us consider $v := \lambda \sqrt{\delta}\, (M - \lambda E)^{-1}_{m^\circ} R^* Q w$. Then there holds, on the one hand, $(M - \lambda E)_{m^\circ} v = \lambda \sqrt{\delta}\, R^* Q w$ by definition of v. On the other hand, $\sqrt{\delta} Q R v = A w = w$. Together, these two facts imply $\lambda \delta R^* Q Q R v = \lambda \sqrt{\delta} R^* Q w = (M - \lambda E)_{m^\circ} v$. This shows

$$(M - \lambda E - \lambda \delta R^* Q Q R)_{m^\circ}(v) = 0 \,. \tag{1.3}$$

In particular, $(M - \lambda E - \lambda \delta R^* Q Q R)_{m^\circ}$ has no bounded inverse, which is actually the conclusion of Lemma 3.5 for some $\lambda < a_+$, see (3.12). This implies the theorem since there exists $\lambda < a_+$ which is not in a spectral gap of the family $M - \lambda(E + \delta Q^2)$.

Let us include a more technical remark concerning our methods. It is tempting to work with operators that have a compact resolvent and to exploit spectral theorems for compact self-adjoint operators. This is what we tried first, but we did not succeed to fit the Maxwell system in this framework; the operator curl curl has a compact resolvent only in the space of divergence-free functions, but this space does not behave well under the Floquet-Bloch transform. We solved the problem by working only with Fredholm operators.

1.2. **References.** Our results are inspired by the work of Brown et al., [4, 5, 6, 7], where the above statement was derived for the elliptic operator M of the introduction. Let us describe the newest of these publications, [6], which generalizes older results. Starting from Maxwell's equations and considering TE-modes, the underlying problem regards a scalar field $H : \mathbb{R}^2 \to \mathbb{C}$ in two dimensions and the relevant coefficient is the strictly positive permittivity $\varepsilon : \mathbb{R}^2 \to \mathbb{R}$. The operator acting on the unknown H is the elliptic operator $-\nabla \cdot (\varepsilon^{-1}\nabla)$. The authors consider perturbations of the coefficient ε, the perturbation must be non-negative, non-trivial and periodic in the first direction.

In comparison with our result we may say: We treat more general equations and arbitrary dimensions, our approach does not require Floquet-Bloch theory in H^{-1} and we do not have to make assumptions on the band functions λ_s (see Assumption 3 in [6] where a quadratic lower bound is assumed for the band functions). Nevertheless, as described above, we use the same underlying ideas, compare the description in Section 2.2 of [4].

We emphasize that our goal is not to show the creation of isolated eigenvalues by the perturbation, which is the topic, e.g., in [9, 15] and many other publications. Another topic of interest, also different from the questions asked here, regards the formation of gaps as in [3]. For the periodic Maxwell operator in three dimensions, the existence of band gaps is the topic of [10].

The book [11] gives an overview about many aspects of photonic crystals, explains the appearance of band gaps in periodic crystals and the corresponding wave phenomena, it covers defects in periodic media and provides many examples of how waves can travel along defects. At this point, let us emphasize that the contribution at hand is not about the creation of band gaps by a defect, but it is about how a band gap changes when a small defect is introduced. A more mathematical overview on photonic band gap optical materials is [14].

In [2], perturbations of a one-dimensional self-adjoint periodic Sturm–Liouville problem are studied. Under integrability assumptions on the perturbation, it is shown that the essential spectrum and the absolutely continuous spectrum remain unchanged, and that at most finitely many eigenvalues appear in the spectral gaps. In [8], a localized perturbation of a periodic Schrödinger operator is treated.

The aim of [1] is in a different direction: The authors use the method of Floquet-Bloch transforms in order to homogenize periodic equations and in order to study the limit of spectral values in the homogenization limit.

2. The functional analytic setting

We are interested in partial differential equations, the unknowns are functions $u : \mathbb{R}^n \to \mathbb{C}^m$. We write $x \in \mathbb{R}^n$ for the independent variable so that $u : x \mapsto u(x)$. Since we will work below with the Floquet-Bloch transform, we will seek for solutions always in the underlying space $u \in L^2(\mathbb{R}^n, \mathbb{C}^m)$. We will suppress the image space $\mathbb{C}^m$ when it is clear from the context.

Notation. For normed linear spaces X and Y we write $\mathcal{L}(X, Y)$ for the space of linear and bounded maps $T : X \to Y$, and set $\mathcal{L}(X) := \mathcal{L}(X, X)$. The kernel of an operator T is denoted as $\mathcal{N}(T)$. For a Banach-space Y, for convenience of notation, we consider its anti-dual space Y': For elements $u \in Y'$, $v \in Y$ and $\lambda \in \mathbb{C}$ holds $\langle u, \lambda v\rangle_{Y',Y} = \bar{\lambda}\langle u, v\rangle_{Y',Y}$. This definition allows to write the dual pairing as $\langle \cdot, \cdot\rangle_{Y',Y} : Y' \times Y \to \mathbb{C}$ and we have for both, the scalar product $\langle \cdot, \cdot\rangle_X$ in a Hilbert space X and the dual pairing the following property regarding complex conjugation: For $\lambda, \mu \in \mathbb{C}$ and admissible elements u, v, there holds $\langle \lambda u, \mu v\rangle = \lambda\overline{\mu}\langle u, v\rangle$. Accordingly, we define the symbol $\langle \cdot, \cdot\rangle_{Y,Y'}$ as $\langle u, v\rangle_{Y,Y'} := \overline{\langle v, u\rangle_{Y',Y}}$ for $u \in Y$ and $v \in Y'$. A bounded linear operator $T : Y \to Y'$ is self-adjoint when, for all admissible u, v, there holds $\langle Tu, v\rangle_{Y',Y} = \langle u, Tv\rangle_{Y,Y'}$ for all $u, v \in Y$. For Hilbert spaces X the dual pairing is the inner product.

We will consider the situation that X is a Hilbert space and that $Y \subset X$ is a linear subspace. In this situation, we denote the embedding by $\iota : Y \to X$, but we oftentimes suppress the embedding and identify $u \in Y$ with $u \in X$. An element $u \in X$ is accordingly identified with the element $\iota^* u \in Y'$, which acts as $\langle \iota^* u, v\rangle_{Y',Y} = \langle u, \iota v\rangle_X$. Also here, we oftentimes suppress the embedding and write u instead of $\iota^* u$. Accordingly, an element $u \in Y$ is identified with $\iota^*\iota u \in Y'$.

Two real intervals are of particular importance throughout this text: We use

$$W := (0, 2\pi) \qquad \text{and} \qquad I := [-1/2, 1/2]\,. \tag{2.1}$$

2.1. **Function spaces and differential operators.** In order to define operators on functions $u \in X = L^2(\mathbb{R}^n) = L^2(\mathbb{R}^n, \mathbb{C}^m)$, we assume that we are given a reflexive Banach space $Y \subset X$ which is dense in X. In our examples, Y is a subspace of functions with additional regularity.

The operator M. With the anti-dual space Y' of Y, we have the Gelfand triple $Y \subset X \equiv X' \subset Y'$. The differential operator M of interest is given by $M \in \mathcal{L}(Y, Y')$.

The operator E and the family M^λ. We consider a second operator $E : X \to X$. In many applications, E is the identity. We will later demand that E is bounded, self-adjoint and coercive. With a parameter $\lambda \in \mathbb{C}$ we define a family of operators M^λ as

$$M^\lambda \;:=\; M - \lambda E\,. \tag{2.2}$$

In this setting, we identify E with the map $\iota^* \circ E \circ \iota : Y \to Y'$. Suppressing embeddings, the map can also be described by the formula $\langle Eu, v\rangle_{Y',Y} = \langle Eu, v\rangle_X$ for all $u, v \in Y$. With this interpretation, (2.2) defines $M^\lambda \in \mathcal{L}(Y, Y')$ for every $\lambda \in \mathbb{R}$.

The restriction R. We are interested in perturbations of the family M^λ. We want to study perturbations that are supported in the unbounded strip $\mathbb{R}^{n-1} \times W$ with $W = (0, 2\pi)$. For a precise definition we use the restriction operator $R : L^2(\mathbb{R}^n) \ni u \mapsto u|_{\mathbb{R}^{n-1}\times W} \in L^2(\mathbb{R}^{n-1} \times W)$, which is a bounded linear operator. We also use the adjoint operator $R^* : L^2(\mathbb{R}^{n-1} \times W) \to L^2(\mathbb{R}^n)$; this operator extends a function on $\mathbb{R}^{n-1} \times W$ trivially to $\mathbb{R}^n$ (extension of the function by zero). By our convention regarding embeddings, we do not distinguish between R and the operator $R \circ \iota : Y \to L^2(\mathbb{R}^{n-1} \times W)$.

The operators S^δ. Indexed by a further real parameter $\delta \geq 0$, we assume that we are given a family of bounded linear operators $S^\delta \in \mathcal{L}(L^2(\mathbb{R}^{n-1} \times W, \mathbb{C}^m))$. We assume that δ is a measure for the size of S^δ in the sense that, for some $C > 0$, there holds $\|S^\delta\| \leq C\delta$ for every $\delta \geq 0$. We restrict ourselfs to the effect of a multiplication with a non-negative function and assume $S^\delta = \delta Q Q$, where the self-adjoint operator Q is given by the multiplication with a bounded non-negative function q, hence $Q : u \mapsto q\, u$.

We can now define the object of our investigations. We consider the family of bounded linear operators $Y \to Y'$, parametrized by the parameters $\lambda \in \mathbb{C}$ and $\delta \geq 0$,

$$M^{\lambda,\delta} \;:=\; M - \lambda\,(E + R^* S^\delta R)\,. \tag{2.3}$$

2.2. **Examples.** Our main applications are elliptic systems and Maxwell's equations.

Perturbed elliptic problem. The most elementary example for our theory is the scalar elliptic equation that was already sketched in the introduction. We want to cleairfy how this example is cast in the abstract language of our main result.

Definition 2.1 (Perturbed elliptic scalar problem)**.** *Let $n \in \mathbb{N}$ be arbitrary and let $m = 1$. Coefficients are $a, b : \mathbb{R}^n \to \mathbb{R}$, both bounded and 2π-periodic in each direction, both a and b with a positive lower bound. Let a perturbation be given by $0 \neq q \in L^\infty(\mathbb{R}^n)$ which is bounded, non-negative, supported in $\mathbb{R}^{n-1} \times W$ and 2π-periodic in the first $n-1$ directions. The spaces are $X := L^2(\mathbb{R}^n, \mathbb{C}^m)$ and $Y := H^1(\mathbb{R}^n, \mathbb{C}^m)$, the operators are*

$$Mu := -\nabla \cdot (a\nabla u)\,, \quad \textit{i.e.} \quad \langle Mu, \varphi\rangle_{Y',Y} = \int_{\mathbb{R}^n} a\,\nabla u \cdot \nabla\bar\varphi\,, \tag{2.4}$$

$$Eu := bu\,, \quad \textit{i.e.} \quad \langle Eu, \varphi\rangle_{Y',Y} = \int_{\mathbb{R}^n} b\,u\,\bar\varphi\,, \tag{2.5}$$

$$S^\delta v := \delta\, q^2 v\,, \quad \textit{i.e.} \quad \langle R^* S^\delta R u, \varphi\rangle_{Y',Y} = \int_{\mathbb{R}^n} \delta\, q^2\, u\,\bar\varphi\,. \tag{2.6}$$

Systems are actually described by the same formulas (2.4)–(2.6), but they are interpreted in a more general way: In the case of systems one considers arbitrary $m \in \mathbb{N}$, the symbol ∇u stands for the Jacobi matrix of u, the coefficient a is such that $a(x)$ is a fourth order tensor for every $x \in \mathbb{R}^n$, the coefficients $b(x)$ and $q(x)$ are second order tensors for every x, see Section 4.1.

Perturbed Maxwell system. Our second application is the perturbed Maxwell operator. The two unknowns are a magnetic and an electric field, $\vec H, \vec E : \mathbb{R}^3 \to \mathbb{C}^3$. The system can be written as

$$\operatorname{curl} \vec H + i\omega\varepsilon\vec E = -i\omega\delta\varepsilon_1 \vec E\,, \quad \operatorname{curl}\vec E - i\omega\mu\vec H = 0\,.$$

We eliminate $\vec H$ by inserting the second relation into the first, set $u = \vec E$ and use $\lambda = \omega^2$. We obtain an equivalent description with an operator family $M^{\lambda,\delta}$,

$$M^{\lambda,\delta} u \;:=\; \operatorname{curl}\big(\mu^{-1}\operatorname{curl} u\big) \;-\; \lambda(\varepsilon + \delta\varepsilon_1)u \;=\; 0\,. \tag{2.7}$$

The Maxwell system can be described with our abstract framework as follows.

Definition 2.2 (Perturbed Maxwell system)**.** *Let the coefficient functions $\mu, \varepsilon \in L^\infty(\mathbb{R}^3, \mathbb{R})$ have positive lower bounds and let them be 2π-periodic in every direction. Let a perturbation be given by a non-negative function $0 \neq \varepsilon_1 \in L^\infty(\mathbb{R}^3, \mathbb{R})$ that is 2π-periodic in the first two directions and satisfies $\varepsilon_1(x) = 0$ for $x = (x_1, x_2, x_3)$ with $x_3 \notin W$.*

The Maxwell system is encoded with the choices $X := L^2(\mathbb{R}^3, \mathbb{C}^3)$ and $Y := H(\operatorname{curl}, \mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3, \mathbb{C}^3) \mid \operatorname{curl} u \in L^2(\mathbb{R}^3, \mathbb{C}^3)\}$ and the operators

$$\begin{aligned}
Mu \;&:=\; \operatorname{curl}(\mu^{-1}\operatorname{curl} u)\,, \quad \textit{i.e.} \quad \langle Mu, \psi\rangle_{Y',Y} = \int_{\mathbb{R}^3} \mu^{-1}\operatorname{curl} u \cdot \operatorname{curl}\bar\psi\,,\\
Eu \;&:=\; \varepsilon\, u\,, \quad \textit{i.e.} \quad \langle Eu, \psi\rangle_{Y',Y} = \int_{\mathbb{R}^3} \varepsilon\, u\,\bar\psi\,,\\
S^\delta v \;&:=\; \delta\,\varepsilon_1\, v\,, \quad \textit{i.e.} \quad \langle R^* S^\delta R u, \psi\rangle_{Y',Y} = \int_{\mathbb{R}^3} \delta\,\varepsilon_1\, u\,\bar\psi\,.
\end{aligned}$$

We apply our theory to this system of equations in Section 4.2.

2.3. **The Floquet-Bloch transform.** Domains for spatial variables are introduced as products of the one-dimensional sets $\mathbb{R}$ and $W = (0, 2\pi)$. We use $W^n = (0, 2\pi)^n$ as a periodicity cell in dimension n and $L^2(W^n)$ for functions on this unit cell. For the dual variable of the Floquet-Bloch transform we use the unit interval $I = [-1/2, 1/2]$ and products thereof, in particular, $I^n = [-1/2, 1/2]^n$.

Transformation in all variables. The n-dimensional Floquet-Bloch transform is a map

$$\mathcal{F}_{\{1,\dots,n\}} : L^2(\mathbb{R}^n) \to L^2(W^n \times I^n) = L^2(I^n, L^2(W^n)) . \tag{2.8}$$

This transformation is defined as the extension of the following map, which associates to a smooth function u with compact support the function $\hat{u}$:

$$\hat{u}(x, k) \ := \sum_{\ell \in \mathbb{Z}^n} u(x + 2\pi\ell)\, e^{-i(x+2\pi\ell)\cdot k} . \tag{2.9}$$

The independent variables for $\hat{u}$ are $x \in W^n$ and $k \in I^n$. We note that, for smooth u and for arbitrary k, the map $\hat{u}(\cdot, k) = (\mathcal{F}_{\{1,\dots,n\}} u)(\cdot, k)$ is 2π-periodic in every direction. We work here with this periodic form of the Floquet-Bloch transform instead of the classicical one, given by $e^{ik\cdot x}\mathcal{F}_{\{1,\dots,n\}}$, which provides quasi-periodic images.

The map $\mathcal{F}_{\{1,\dots,n\}}$ of (2.8) is an isomorphism from $L^2(\mathbb{R}^n)$ onto $L^2(W^n \times I^n)$. For almost every $x \in W^n$, the inverse is given by

$$(\mathcal{F}^{-1}_{\{1,\dots,n\}} v)(x) \ = \ \int_{I^n} v(x, k)\, e^{ik\cdot x}\, dk \quad \text{for all } v \in L^2(W^n \times I^n) . \tag{2.10}$$

The transform $\mathcal{F} := \mathcal{F}_{\{1,\dots,n\}}$ is unitary in L^2-spaces,

$$\int_{I^n} \big\langle (\mathcal{F}v)(\cdot, k), (\mathcal{F}\phi)(\cdot, k) \big\rangle_{L^2(W^n)}\, dk \ = \ \langle v, \phi\rangle_{L^2(\mathbb{R}^n)} \quad \text{for all } v, \phi \in L^2(\mathbb{R}^n) . \tag{2.11}$$

In contrast to the more standard transform to quasi-periodic functions, the periodic Floquet-Bloch transform does not commute with gradients. Instead, one has the formula $\nabla(e^{ik\cdot x}\mathcal{F}u) = e^{ik\cdot x}\mathcal{F}(\nabla u)$, which leads to

$$\int_{I^n} \big\langle \nabla(e^{ik\cdot x}\mathcal{F}v)(\cdot, k), \nabla(e^{ik\cdot x}\mathcal{F}\phi)(\cdot, k) \big\rangle_{L^2(W^n)}\, dk \ = \ \langle \nabla v, \nabla\phi\rangle_{L^2(\mathbb{R}^n)} \tag{2.12}$$

for all $v, \phi \in H^1(\mathbb{R}^n)$.

Transformation in less than $\boldsymbol{n}$ *variables.* It is also possible to perform the Floquet-Bloch transform only in some of the n directions. Let us derive formulas for the transformation in x_n, writing the variables as $x = (x_1, x_2, \dots, x_n) =: (\tilde{x}, x_n)$. The transform in the last variable is a map $\mathcal{F}_{\{n\}} : L^2(\mathbb{R}^n) \to L^2\big((\mathbb{R}^{n-1} \times W) \times I\big)$. On smooth functions with compact support, it is defined, for $x = (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times W$ and $k_n \in I$, by

$$(\mathcal{F}_{\{n\}} u)(x, k_n) := \sum_{\ell \in \mathbb{Z}} u(\tilde{x}, x_n + 2\pi\ell)\, e^{-i(x_n+2\pi\ell)k_n} .$$

It defines a unitary isomorphism.

The partial Floquet-Bloch transform in the first $n-1$ variables is defined accordingly as a map $\mathcal{F}_{\{1,\dots,n-1\}} : L^2(\mathbb{R}^n) \to L^2\big((W^{n-1} \times \mathbb{R}) \times I^{n-1}\big)$. For smooth arguments u, it is defined, for $x = (\tilde{x}, x_n) \in W^{n-1} \times \mathbb{R}$ and $m \in I^{n-1}$, by

$$(\mathcal{F}_{\{1,\dots,n-1\}} u)(x, m) := \sum_{\ell \in \mathbb{Z}^{n-1}} u(\tilde{x} + 2\pi\ell, x_n)\, e^{-i(\tilde{x}+2\pi\ell)\cdot m} .$$

The transformation $\mathcal{F}_{\{1,\dots,n-1\}}$ is a unitary isomorphism, for all $v,\phi \in L^2(\mathbb{R}^n)$ holds

$$\langle v,\phi\rangle_{L^2(\mathbb{R}^n)} \;=\; \int_{I^{n-1}} \big\langle (\mathcal{F}_{\{1,\dots,n-1\}} v)(\cdot,m), (\mathcal{F}_{\{1,\dots,n-1\}} \phi)(\cdot,m)\big\rangle_{L^2(W^{n-1}\times\mathbb{R})}\, dm\,.$$

2.4. **Floquet-Bloch representation of operators.** The Floquet-Bloch transformation relates a function u to a transformed function $\hat{u}$. The transformation can therefore also be used to express operators M that act on u with operators that act on $\hat{u}$.

Transformation in all variables. Let us start with a loose description of the Floquet-Bloch transformation of an equation. When we write the equation as $Mu = f$, we seek for a function $u : x \mapsto u(x)$ in the space Y so that $Mu = f$ holds as an equality in Y'. When we apply a Floquet-Bloch transformation, we seek, for almost every $k \in I^n = [-1/2, 1/2]^n$, a periodic solution $\hat{u}(\cdot,k) : W^n \ni x \mapsto \hat{u}(x,k)$ of an equation $M_k \hat{u}(\cdot,k) = \hat{f}(\cdot,k)$.

We note that we may identify a function $\hat{u}(\cdot,k)$ on W^n with its periodic extension,

$$\hat{u}(x + 2\pi\,\ell, k) := \hat{u}(x,k) \qquad \forall x \in W^n\,,\ \ell \in \mathbb{Z}^n\,. \tag{2.13}$$

We must demand that the periodic extension of $\hat{u}$ is locally in Y; this property is expressed with $\hat{u}(\cdot,k) \in Y_{\rm per}$ for a reflexive Banach space $Y_{\rm per} \subset L^2(W^n)$. We can then define

$$L^2(I^n, Y_{\rm per}) := \left\{ \hat{u} \in L^2(I^n, L^2(W^n)) \middle| \hat{u}(\cdot,k) \in Y_{\rm per} \text{ for all } k\,,\ \|\hat{u}\|_{L^2(I^n,Y_{\rm per})} < \infty \right\}\,,$$

where the norm is defined by

$$\|\hat{u}\|^2_{L^2(I^n,Y_{\rm per})} \;:=\; \int_{I^n} \|\hat{u}(\cdot,k)\|^2_{Y_{\rm per}}\, dk\,. \tag{2.14}$$

The space $L^2(I^n, Y_{\rm per})$ is a linear subspace of $L^2(I^n, L^2(W^n))$ and (2.14) defines a norm on this subspace, $L^2(I^n, Y_{\rm per})$ is a Banach space.

Definition 2.3 (Admissible subspace $Y_{\rm per}$)**.** *We need a compatibility of $Y_{\rm per}$ with Y. We say that $Y_{\rm per} \subset L^2(W^n)$ is admissible when the restriction of the Floquet-Bloch transform defines an isomorphism*

$$\mathcal{F}_{\{1,\dots,n\}} : Y \to L^2(I^n, Y_{\rm per})\,. \tag{2.15}$$

The next definition introduces a representation of an operator $M : Y \to Y'$ with the Floquet-Bloch transformation. Loosely speaking, we demand that, for the transformed operator, the variables x and k are decoupled. This is possible when M is a differential operator with periodic coefficients.

Definition 2.4 (Floquet-Bloch representation)**.** *We say that an operator $M : Y \to Y'$ possesses a continuous Floquet-Bloch representation in n directions when, for an admissible space $Y_{\rm per}$, there exists a family of bounded operators $M_k : Y_{\rm per} \to Y'_{\rm per}$, indexed by $k \in I^n$, depending continuously on k, such that M is represented by the family $(M_k)_k$ in the following Floquet-Bloch sense: For arbitrary $u, v \in Y$ and their transformations $\hat{u} = \mathcal{F}_{\{1,\dots,n\}}\, u$ and $\hat{v} = \mathcal{F}_{\{1,\dots,n\}}\, v$ holds*

$$\langle Mu, v\rangle_{Y',Y} \;=\; \int_{I^n} \big\langle M_k \hat{u}(\cdot,k), \hat{v}(\cdot,k)\big\rangle_{Y'_{\rm per},Y_{\rm per}}\, dk\,. \tag{2.16}$$

We note that this definition carries over to the operator $E : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ when we consider this map as as operator from Y into Y'. Continuity of $I^n \ni k \mapsto E_k$ from I^n into $\mathcal{L}(Y_{\rm per}, Y'_{\rm per})$ is satisfied if $k \mapsto E_k : L^2(W^n) \to L^2(W^n)$ is continuous.

Remark 2.5. *We demand that the family M_k is continuous on the compact set $k \in I^n$. This ensures that the family M_k is uniformly bounded in $\mathcal{L}(Y_{\rm per}, Y'_{\rm per})$. In particular, the right hand side of* (2.16) *is well defined since the maps $k \mapsto \hat{u}(\cdot, k)$ and $k \mapsto \hat{v}(\cdot, k)$ are of class $L^2(I^n, Y_{\rm per})$.*

Transformation in $\boldsymbol{n-1}$ *variables.* We write again $x = (x_1, x_2, ..., x_n) = (\tilde{x}, x_n)$ for the independent coordinates. The dual variable will always be decomposed as $k = (m, \kappa)$ with $m \in I^{n-1}$ and $\kappa \in \mathbb{R}$. In the next definition, we always think of $\tilde{Y}_{\rm per}$ as being a subspace of functions in $L^2(W^{n-1} \times \mathbb{R})$ that are periodic in the first $n-1$ directions and that have some additional smoothness such that the periodic extension is locally in Y.

Definition 2.6 (Admissible subspace $\tilde{Y}_{\rm per}$)**.** *Let $Y_{\rm per} \subset L^2(W^n)$ be admissible for Y. We say that a subspace $\tilde{Y}_{\rm per} \subset L^2(W^{n-1} \times \mathbb{R})$ is admissible when the following holds:*

(a) *The $(n-1)$-variables Floquet-Bloch transform on Y defines an isomorphism*

$$\mathcal{F}_{\{1,...,n-1\}} : Y \to L^2(I^{n-1}, \tilde{Y}_{\rm per}) . \tag{2.17}$$

(b) *The 1-variable Floquet-Bloch transform on $\tilde{Y}_{\rm per}$ defines an isomorphism*

$$\mathcal{F}_{\{n\}} : \tilde{Y}_{\rm per} \to L^2(I, Y_{\rm per}) , \tag{2.18}$$

We note that a consequence of (2.18) is that also

$$\mathcal{F}_{\{n\}} : L^2(I^{n-1}, \tilde{Y}_{\rm per}) \to L^2(I^n, Y_{\rm per}) \tag{2.19}$$

is an isomorphism. Indeed, the map is independent of the parameter $m \in I^{n-1}$.

The next definition can be interpreted as: We demand that the operator S is periodic in the first $n-1$ directions.

Definition 2.7 (Floquet-Bloch representation in $n-1$ directions)**.** *We say that an operator $S : Y \to Y'$ possesses a continuous Floquet-Bloch representation in the first $n-1$ directions when, for spaces $Y_{\rm per}$ and $\tilde{Y}_{\rm per}$ as in Definition 2.6, there exists a family of bounded operators $S_m : \tilde{Y}_{\rm per} \to \tilde{Y}'_{\rm per}$, depending continuously on m, such that S is represented by the family $(S_m)_m$ in the Floquet-Bloch sense: For all $u, v \in Y$ there holds, with $\hat{u} = \mathcal{F}_{\{1,...,n-1\}}\, u$ and $\hat{v} = \mathcal{F}_{\{1,...,n-1\}}\, v$:*

$$\langle Su, v\rangle_{Y',Y} \;=\; \int_{I^{n-1}} \big\langle S_m \hat{u}(\cdot, m), \hat{v}(\cdot, m)\big\rangle_{\tilde{Y}'_{\rm per},\tilde{Y}_{\rm per}}\, dm\,. \tag{2.20}$$

2.5. **Implications of Floquet-Bloch representations.**

Lemma 2.8 (Relation between two Floquet-Bloch representations)**.** *Let the spaces $Y_{\rm per}$ and $\tilde{Y}_{\rm per}$ be admissible in the sense of Definitions 2.3 and 2.6 and let M possess a Floquet-Bloch representation $(M_k)_k$ in n directions in the sense of* (2.16)*. We define $M_m : \tilde{Y}_{\rm per} \to \tilde{Y}'_{\rm per}$ through*

$$\langle M_m \tilde{u}, \tilde{v}\rangle_{\tilde{Y}'_{\rm per},\tilde{Y}_{\rm per}} \;:=\; \int_I \big\langle M_{(m,\kappa)}(\mathcal{F}_{\{n\}}\tilde{u})(\cdot,\kappa)\,,\, (\mathcal{F}_{\{n\}}\tilde{v})(\cdot,\kappa)\big\rangle_{Y'_{\rm per},Y_{\rm per}}\, d\kappa \tag{2.21}$$

for $\tilde{u}, \tilde{v} \in \tilde{Y}_{\rm per}$ and $m \in I^{n-1}$. Then the family M_m is a Floquet-Bloch representation of M in the first $n-1$ directions as described in Definition 2.7.

Proof. We have to check property (2.20) for M, i.e., we have to verify that, for $u, v \in Y$, the expression $\langle Mu, v\rangle_{Y',Y}$ coincides, for $\hat{u} = \mathcal{F}_{\{1,\dots,n-1\}}\, u$ and $\hat{v} = \mathcal{F}_{\{1,\dots,n-1\}}\, v$, with

$$\begin{aligned}
&\int_{I^{n-1}} \big\langle M_m \hat{u}(\cdot, m), \hat{v}(\cdot, m)\big\rangle_{\tilde{Y}'_{\text{per}}, \tilde{Y}_{\text{per}}}\, dm \\
&\quad\overset{(2.21)}{=} \int_{I^{n-1}} \int_I \big\langle M_{(m,\kappa)}(\mathcal{F}_{\{n\}}\hat{u}(\cdot, m))(\cdot, \kappa)\,,\, (\mathcal{F}_{\{n\}}\hat{v}(\cdot, m))(\cdot, \kappa)\big\rangle_{Y'_{\text{per}}, Y_{\text{per}}}\, d\kappa\, dm \\
&\quad= \int_{I^n} \big\langle M_k(\mathcal{F}_{\{1,\dots,n\}}u)(\cdot, k)\,,\, (\mathcal{F}_{\{1,\dots,n\}}v)(\cdot, k)\big\rangle_{Y'_{\text{per}}, Y_{\text{per}}}\, dk\,.
\end{aligned}$$

The desired equality holds by (2.16). □

With the following assumption we clearify the situation of interest. Item 3 demands the periodicity of M and E in all directions and the periodicity of the operators S^δ in $n-1$ directions. As we will show in Section 4, the examples of Definition 2.1 (both in the scalar and in the vectorial case) and the Maxwell system of Definition 2.2 fit in this framework.

Assumption 2.9. *We assume that the following holds.*

(1) **Space Y and operators M, E, and S^δ.** *For a reflexive Banach space $Y \subset L^2(\mathbb{R}^n)$ we are given an operator $M \in \mathcal{L}(Y, Y')$. Additionally, we are given $E \in \mathcal{L}(L^2(\mathbb{R}^n))$ and a family of operators $S^\delta \in \mathcal{L}(L^2(\mathbb{R}^{n-1} \times W))$.*
(2) **Spaces Y_{per} and $\tilde{Y}_{\text{per}}$.** *Two reflexive Banach spaces Y_{per} and $\tilde{Y}_{\text{per}}$ are admissible for Y in the sense of Definitions 2.3 and 2.6.*
(3) **Operators M_k, E_k, and S^δ_m.** *The operators M and E possess continuous Floquet-Bloch representations $(M_k)_k$ and $(E_k)_k$ in n directions in the sense of Definition 2.4. The operators S^δ possess Floquet-Bloch representations $(S^\delta_m)_m$ in $n-1$ directions in the sense of Definition 2.7.*
(4) **Properties of E_k.** *For every $k \in I^n$, the operator $E_k : L^2(W^n) \to L^2(W^n)$ is a self-adjoint coercive isomorphism. The mapping $k \mapsto E_k$ is continuously differentiable from I^n into $\mathcal{L}(L^2(W^n))$.*
(5) **Properties of M_k.** *For every $k \in I^n$ the following holds: The operator M_k is self-adjoint, i.e. $\langle M_k u, v\rangle_{Y'_{\text{per}}, Y_{\text{per}}} = \langle u, M_k v\rangle_{Y_{\text{per}}, Y'_{\text{per}}}$ for all $u, v \in Y_{\text{per}}$. The operator M_k+E_k is uniformly coercive, i.e. $\langle (M_k+E_k)u, u\rangle_{Y'_{\text{per}}, Y_{\text{per}}} \geq c\|u\|^2_{Y_{\text{per}}}$ for some $c > 0$ and all $u \in Y_{\text{per}}$. The operators $M_k - \lambda E_k : Y_{\text{per}} \to Y'_{\text{per}}$ are Fredholm operators with index 0 for all $0 \neq \lambda \in \mathbb{C}$. The set of all $\lambda \in \mathbb{C}$ such that $M_k - \lambda E_k$ has a non-trivial kernel has no accumulation point except (possibly) $\lambda = 0$. The mapping $k \mapsto M_k$ is continuously differentiable from I^n into $\mathcal{L}(Y_{\text{per}}, Y'_{\text{per}})$.*
(6) **Properties of S^δ_m.** *For every $m \in I^{n-1}$ the operator $S^\delta_m : L^2(W^n) \to L^2(W^n)$ is linear, bounded, self-adjoint. We assume $S^\delta_m = \delta Q Q$, where $Q \neq 0$ is a multiplication operator that multiplies the argument with a fixed non-negative function $q \in L^\infty(W^n)$. Accordingly, for some $C > 0$, there holds $\|S^\delta_m\| \leq C\,\delta$.*

Our proof shows that the Fredholm property and the accumulation point property of Item 5 is needed actually only in a neighborhood of the critical point, later on denoted as a_+.

Remark 2.10 (Inherited properties)**.** *Formulas* (2.16) *and* (2.21) *allow to transfer the properties of E_k and $M_k + E_k$ from Items 4 and 5 to E and E_m, and $M + E$ and $M_m + E_m$, respectively. In particular, also E and E_m are self-adjoint and coercive on*

$L^2(W^n)$ and $L^2(W^{n-1}\times\mathbb{R})$, respectively, and $M+E$ and M_m+E_m are self-adjoint and coercive as mappings from Y to Y' and $\tilde Y_{\rm per}$ to $\tilde Y'_{\rm per}$, respectively.

To indicate the required calculations, we show how to obtain coercivity of M_m+E_m. From (2.21) *for M_m and E_m we obtain, for $\tilde u\in\tilde Y_{\rm per}$*

$$\begin{aligned}\langle (M_m+E_m)\tilde u,\tilde u\rangle_{\tilde Y'_{\rm per},\tilde Y_{\rm per}} &= \int_I \big\langle \big((M_k+E_k)\circ\mathcal{F}_{\{n\}}\big)\tilde u,\mathcal{F}_{\{n\}}\tilde u\big\rangle_{Y'_{\rm per},Y_{\rm per}}\,d\kappa\\ &\geq c\int_I \|\mathcal{F}_{\{n\}}\tilde u\|^2_{Y_{\rm per}}\,d\kappa \;\geq\; c\,\|\tilde u\|^2_{\tilde Y_{\rm per}}\,,\end{aligned}$$

where we used the isomorphism (2.18) *in the last inequality. We recall that constants c and C may change from one line to the next.*

Notation regarding restrictions. We recall that $\tilde Y_{\rm per}$ is a subspace of $L^2(W^{n-1}\times\mathbb{R})$. Strictly speaking, in Item 6, we should not write S^δ_m, but we should write $R_1S^\delta_mR_1^*$: $L^2(W^n)\to L^2(W^n)$ with the restriction operator $R_1 : L^2(W^{n-1}\times\mathbb{R})\ni u\mapsto u|_{W^n}\in L^2(W^n)$ and its dual, the trivial extension R_1^*. In the following, we do not distinguish between R and R_1 and write R for both restriction operators. When the action of the operator is clear from the context, we suppress the restriction operator entirely.

Invertibility properties. When an operator M^λ can be represented in the Floquet-Bloch sense with a family $(M^\lambda_k)_{k\in I^n}$, then properties of M^λ imply properties of $(M^\lambda_k)_{k\in I^n}$ and vice versa. We now investigate how invertibility properties carry over from one representation to the other.

Lemma 2.11 (Invertibility of M^λ when all M^λ_k are invertible)**.** *Let Assumption 2.9 hold and let $\lambda\in\mathbb{R}$ be fixed. We assume that, for every $k\in I^n$, the map $M^\lambda_k = M_k-\lambda E_k : Y_{\rm per}\to Y'_{\rm per}$ is invertible. In this situation, also $M^\lambda : Y\to Y'$ is invertible.*

Proof. Let $f\in Y'$ be arbitrary. The map $\mathcal{F}_{\{1,\dots,n\}} : Y\to L^2(I^n,Y_{\rm per})$ is an isomorphism by (2.15). We introduce $\tilde f := f\circ\mathcal{F}^{-1}_{\{1,\dots,n\}} : L^2(I^n,Y_{\rm per})\to\mathbb{C}$. As an anti-linear functional on $L^2(I^n,Y_{\rm per})$, the map $\tilde f$ has a representation $\hat f\in L^2(I^n,Y'_{\rm per})$. We use this representation and define, for almost every $k\in I^n$,

$$\hat u(k) \;:=\; (M^\lambda_k)^{-1}(\hat f(k)) \;\in\; Y_{\rm per}\,.$$

The continuity of $k\mapsto M^\lambda_k$ required in Definition 2.4 implies uniform boundedness of the operator family $(M^\lambda_k)^{-1} : Y'_{\rm per}\to Y_{\rm per}$ and thus $\hat u\in L^2(I^n,Y_{\rm per})$. The inverse transform yields $u := \mathcal{F}^{-1}_{\{1,\dots,n\}}\hat u\in Y$. The fact that M and E are represented by the families M_k and E_k allows to calculate with (2.16), for a test-function $\varphi\in Y$,

$$\begin{aligned}\big\langle M^\lambda u,\varphi\big\rangle_{Y',Y} &= \int_{I^n}\big\langle M^\lambda_k\hat u(\cdot,k),\hat\varphi(\cdot,k)\big\rangle_{Y'_{\rm per},Y_{\rm per}}\,dk\\ &= \int_{I^n}\Big\langle \hat f(\cdot,k),\hat\varphi(\cdot,k)\Big\rangle_{Y'_{\rm per},Y_{\rm per}}\,dk \;=\; \big\langle \hat f,\hat\varphi\big\rangle_{L^2(I^n,Y_{\rm per})',L^2(I^n,Y_{\rm per})}\\ &= \tilde f(\hat\varphi) \;=\; (f\circ\mathcal{F}^{-1}_{\{1,\dots,n\}})(\hat\varphi) \;=\; f(\varphi)\,.\end{aligned}$$

This shows $M^\lambda u = f$ and we have found the inverse element $u\in Y$.

We have shown surjectivity of M^λ. Since the operator is selfadjoint, this implies also the injectivity of M^λ. □

Lemma 2.12 (Invertibility of all M^λ_k when M^λ is invertible)**.** *Let Assumption 2.9 hold and let $0\neq\lambda\in\mathbb{C}$ be fixed. When $M^\lambda : Y\to Y'$ is invertible, then, for every*

$k \in I^n$, the map $M_k^\lambda = M_k - \lambda E_k : Y_{\rm per} \to Y'_{\rm per}$ is invertible. Furthermore, the mapping $k \mapsto (M_k^\lambda)^{-1} : Y'_{\rm per} \to Y_{\rm per}$ is continuous.

Proof. For a contradiction argument we assume that, for some $k = k^\circ \in I^n$, the map $M_k^\lambda = M_k - \lambda E_k : Y_{\rm per} \to Y'_{\rm per}$ is not invertible. Our aim is to conclude that $M^\lambda : Y \to Y'$ is not invertible, which yields the desired contradiction.

The operator $M_{k^\circ}^\lambda$ is a Fredholm operator with index 0 by Assumption 2.9, Item 5. Since we assumed that the operator is not invertible, it has a kernel; we find $u^\circ \in Y_{\rm per}$ with $\|u^\circ\|_{Y_{\rm per}} = 1$ and $M_{k^\circ}^\lambda u^\circ = 0$. Starting from the element $u^\circ \in Y_{\rm per}$, it is our aim to construct a bounded family $\hat u_\varepsilon \in L^2(I^n, Y_{\rm per})$ and the corresponding family $u_\varepsilon := \mathcal{F}^{-1}_{\{1,\dots,n\}} \hat u_\varepsilon \in Y$ with $u_\varepsilon \not\to 0$ in Y and $M^\lambda u_\varepsilon \to 0$ in Y' as $\varepsilon \to 0$. This shows the non-invertibility of M^λ and concludes the proof.

We now construct $\hat u_\varepsilon$; for notational simplicity we assume here that k° is an inner point of I^n, the same construction with minor modifications can be used also for boundary points. For arbitrary (small) $\varepsilon > 0$ we construct a localization function as a normalized characteristic function on a small cube: We set $p_\varepsilon(k) := \varepsilon^{-n/2}$ if $|k_j - k_j^\circ| < \varepsilon/2$ for all $j = 1, \dots, n$, and $p_\varepsilon(k) := 0$ for other arguments $k = (k_1, ..., k_n)$. With the characteristic function p_ε we define

$$\hat u_\varepsilon(\cdot, k) \ :=\ u^\circ(\cdot)\, p_\varepsilon(k)\,, \quad \hat u_\varepsilon \in L^2(I^n, Y_{\rm per})\,.$$

Because of $\|u^\circ\|_{Y_{\rm per}} = 1$ we have constructed a function with

$$\|\hat u_\varepsilon\|^2_{L^2(I^n, Y_{\rm per})} \ =\ \int_{I^n} \|u^\circ\|^2_{Y_{\rm per}} |p_\varepsilon(k)|^2\, dk \ =\ \int_{I^n} |p_\varepsilon(k)|^2\, dk \ =\ 1\,.$$

The norm of the family $u_\varepsilon = \mathcal{F}^{-1}_{\{1,\dots,n\}} \hat u_\varepsilon \in Y$ is bounded from below by a positive constant, since $\mathcal{F}^{-1}_{\{1,\dots,n\}}$ is an isomorphism.

For the family u_ε we now calculate $M^\lambda u_\varepsilon$ with the help of the Floquet-Bloch representation of M^λ, see (2.16). For arbitrary $v \in Y$ and the transformed function $\hat v = \mathcal{F}_{\{1,\dots,n\}} v$ holds

$$\begin{aligned}
\left|\langle M^\lambda u_\varepsilon, v\rangle_{Y',Y}\right| &= \left| \int_{I^n} \left\langle M_k^\lambda \hat u_\varepsilon(\cdot, k), \hat v(\cdot, k)\right\rangle_{Y'_{\rm per}, Y_{\rm per}} dk \right| \\
&= \left| \int_{I^n} \left\langle p_\varepsilon(k)\, M_k^\lambda\, u^\circ, \hat v(\cdot, k)\right\rangle_{Y'_{\rm per}, Y_{\rm per}} dk \right| \\
&\le \left| \int_{I^n} p_\varepsilon(k) \left\langle (M_k^\lambda - M_{k^\circ}^\lambda)\, u^\circ, \hat v(\cdot, k)\right\rangle_{Y'_{\rm per}, Y_{\rm per}} dk \right|\,,
\end{aligned}$$

where we used $M_{k^\circ}^\lambda u^\circ = 0$.

We now exploit that M_k^λ is continuous in k by Definition 2.4. The difference $M_k^\lambda - M_{k^\circ}^\lambda$ is small in $\mathcal{L}(Y_{\rm per}, Y'_{\rm per})$ on the support of p_ε for small $\varepsilon > 0$, which provides, as $\varepsilon \to 0$:

$$\left|\langle M^\lambda u_\varepsilon, v\rangle_{Y',Y}\right| \ \le\ o(1)\, \|v\|_Y\,,$$

where the quantity $o(1)$ is independent of v. This shows $M^\lambda u_\varepsilon \to 0$ in Y' as $\varepsilon \to 0$, which is the desired result.

We have obtained that $M_k^\lambda = M_k - \lambda E_k$ is an isomorphism from $Y_{\rm per}$ onto $Y'_{\rm per}$ for every $k \in I^n$. Since these operators depend continuously on k, their inverses are continuous. This implies the second claim of the lemma. □

3. Spectral gaps

Definition 3.1 (Spectral gap). *We consider a family of operators $M^{\lambda,\delta}$, indexed by $\lambda \in \mathbb{C}$, for a fixed value $\delta \geq 0$. We say that a nontrivial interval $(a_-, a_+) \subset \mathbb{R}$ is* contained in a spectral gap *for the operator family $M^{\lambda,\delta} : Y \to Y'$ when*

$$M^{\lambda,\delta} : Y \to Y' \quad \text{is invertible for all } \lambda \in (a_-, a_+)\,. \tag{3.1}$$

We say that $(a_-, a_+) \subset \mathbb{R}$ is a spectral gap, *when (a_-, a_+) is contained in a spectral gap and when it is maximal with this property, i.e.: For every $\eta > 0$, neither $(a_- - \eta, a_+)$ nor $(a_-, a_+ + \eta)$ is contained in a spectral gap.*

Our main result is the following theorem. We show that the upper end of the spectral gap is pushed down or eigenvalues are created inside the gap. Let us emphasize that it is not excluded that the spectral gap is shifted towards 0; in this sense, the spectral gap of the perturbed system need not be smaller in length.

The assumption that is formulated in the theorem is satisfied for every system with a unique continuation property.

Theorem 3.2 (A spectral gap is reduced by a perturbation). *Let Assumption 2.9 hold and let (a_-, a_+) be a spectral gap of the family $M^\lambda = M - \lambda E$. We use $a_0 := (a_- + a_+)/2$, the mid-point of the spectral gap. We assume that, for every $k \in I^n$ with $\mathcal{N}(M_k^{a_+}) \neq \{0\}$, there exists an eigenfunction $\phi \in \mathcal{N}(M_k^{a_+})$ such that $\langle Q\phi, \phi\rangle = \int_{W^n} q\,|\phi|^2 \neq 0$.*

Then there exists $\delta_0 > 0$ such that, for every $\delta \in (0, \delta_0)$, the interval (a_0, a_+) is not contained in a spectral gap of the family $M^{\lambda,\delta} = M - \lambda(E + R^* S^\delta R)$.

The theorem is a consequence of the subsequent three lemmas. The first lemma provides a value $k^\circ \in I^n$ for which invertibility fails.

Lemma 3.3 (Critical k). *Let Assumption 2.9 hold and let (a_-, a_+) be a spectral gap of the family M^λ with $a_+ > 0$. Then there exists $k^\circ \in I^n$ such that*

$$M_{k^\circ}^{a_+} = M_{k^\circ} - a_+ E_{k^\circ} : Y_{\text{per}} \to Y'_{\text{per}} \tag{3.2}$$

is not invertible.

In view of Lemmas 2.11 and 2.12, the conclusion of the lemma is essentially equivalent to: M^{a_+} is not invertible.

Proof. Step 1: A closed set Σ of spectral values. We introduce $\sigma(M_k) := \{\lambda \in \mathbb{R} \mid M_k - \lambda E_k : Y_{\text{per}} \to Y'_{\text{per}}$ is not invertible$\}$ and consider the bounded set

$$\Sigma := \bigcup_{k \in I^n} \sigma(M_k) \cap [a_+, a_+ + 1] \subset \mathbb{R}\,.$$

We claim that Σ is closed. To show this, let $\lambda_j \to \lambda \in \mathbb{R}$ be a convergent sequence with $\lambda_j \in \Sigma$. By definition of Σ there holds $\lambda_j \in \sigma(M_{k_j}) \cap [a_+, a_+ + 1]$ for some $k_j \in I^n$. By compactness of I^n, we find a subsequence $k_j \to k$ for some $k \in I^n$. The operator $M_k - \lambda E_k$ cannot be invertible since otherwise also $M_{k_j} - \lambda_j E_{k_j}$ had to be invertible for sufficiently large j. We exploit here that $I^n \ni k \mapsto M_k \in \mathcal{L}(Y_{\text{per}}, Y'_{\text{per}})$ and $k \mapsto E_k$ are continuous. We therefore find $\lambda \in \sigma(M_k)$ and hence $\lambda \in \Sigma$. Since λ_j was arbitrary, this shows that Σ is closed.

Step 2: Conclusion. For a contradiction argument, we assume the following property: For every $k \in I^n$, the operator $M_k^{a_+} = M_k - a_+ E_k : Y_{\text{per}} \to Y'_{\text{per}}$ is invertible. This property implies $a_+ \notin \Sigma$. Since Σ is closed, we find $\eta > 0$ such that $\lambda \notin \Sigma$ for every $\lambda \in [a_+, a_+ + \eta]$. With this parameter $\eta > 0$, for every $\lambda \in [a_+, a_+ + \eta]$ and every $k \in I^n$, the operator M_k^λ is invertible. We can apply Lemma 2.11 to conclude that M^λ is invertible. We therefore find that also $(a_-, a_+ + \eta)$ is contained in a spectral gap. This is in contradiction to the maximality of (a_-, a_+). □

The second lemma is treating the unperturbed operator after a Floquet-Bloch transform in $n-1$ coordinates. We are therefore treating the underlying spatial domain $W^{n-1} \times \mathbb{R}$.

Lemma 3.4 (Large inverse for the unperturbed operator)**.** *Let the situation be as in Theorem 3.2. Let $k^\circ = (m^\circ, \kappa^\circ)$ with $m^\circ \in I^{n-1}$ and $\kappa^\circ \in I$ be such that $M_{k^\circ}^{a_+}$ has a non-trivial kernel. Let $0 \neq \phi \in \mathcal{N}(M_{k^\circ}^{a_+})$ be an element of the kernel and let $r \in L^2(W^n)$ be a function with*

$$\langle r e^{-i\kappa^\circ x_n}, \phi \rangle_{L^2(W^n)} \;\neq\; 0\,. \tag{3.3}$$

Then there holds, for every sequence $(\lambda_i)_i$ with $\lambda_i < a_+$ and $\lambda_i \to a_+$:

$$\limsup_{\lambda \to a_+} \left| \langle (M - \lambda E)_{m^\circ}^{-1} R^* r, R^* r \rangle_{L^2(W^{n-1}\times\mathbb{R})} \right| \;=\; \infty\,. \tag{3.4}$$

Proof. Step 1: Reformulation of the task. The scalar product in (3.4) has the entry $v := (M - \lambda E)_{m^\circ}^{-1} R^* r \in \tilde{Y}_{\mathrm{per}}$. In this proof, we must analyze v, the solution of the problem $(M - \lambda E)_{m^\circ} v = R^* r$, with respect to its dependence on λ. We transform this equation with the one-dimensional Floquet-Bloch transform in the last variable. We write $v_\kappa(\cdot) = (\mathcal{F}_{\{n\}} v)(\cdot, (m^\circ, \kappa)) \in L^2(W^n)$ and $r_\kappa \in L^2(W^n)$ for the transformed functions in the point κ. We recall that m° is fixed in this proof, κ varies and we use $k = (m^\circ, \kappa)$.

Since the function $R^* r$ vanishes for $x_n \notin W$, the Floquet-Bloch transform of $R^* r$ in the variable x_n is the function $r_\kappa(x) = r(x) e^{-i\kappa x_n}$. The transformed solution $v_\kappa \in Y_{\mathrm{per}}$ solves

$$(M_k - \lambda E_k) v_\kappa \;=\; r_\kappa\,, \tag{3.5}$$

compare Lemma 2.8. The essential task of this proof is to find a lower bound for the expression

$$\langle v_\kappa, r_\kappa \rangle_{L^2(W^n)} = \langle (M - \lambda E)_k^{-1} r_\kappa, r_\kappa \rangle_{L^2(W^n)}\,.$$

We claim that, for every $\hat{\kappa} \in I$, there exist constants $c > 0$ and $\eta > 0$ such that, for all $|\kappa - \hat{\kappa}| < \eta$ and all $\lambda \in (a^+ - \eta, a^+)$:

$$\langle v_\kappa, r_\kappa \rangle_{L^2(W^n)} \;\geq\; -c \|r_\kappa\|_{L^2(W^n)}^2\,. \tag{3.6}$$

Let us note that (3.6) follows directly when $\hat{\kappa}$ has the property that $(M - a^+ E)_{(m^\circ, \hat{\kappa})}$ is invertible. Indeed, in this case, there exists $\eta = \eta(\hat{\kappa})$ and $c = c(\hat{\kappa}) > 0$ such that $\|(M_k - \lambda E_k)^{-1}\| \;\leq\; c$ for all $|\kappa - \hat{\kappa}| < \eta$ and $\lambda \in (a^+ - \eta, a^+)$. This provides (3.6).

Step 2: Description with eigenfunctions. We now consider $\hat{\kappa}$ such that $(M - a^+ E)_{(m^\circ, \hat{\kappa})}$ is not invertible. We apply Theorem A.4 with $X := L^2(W^n)$ and $Y := Y_{\mathrm{per}} \subset X$ and the operator families $M(\kappa) := M_{(m^\circ, \kappa)}$ and $E(\kappa) := E_{(m^\circ, \kappa)}$. All assumptions of Theorem A.4 are satisfied, its application yields the existence of $\eta = \eta(\hat{\kappa}) > 0$ and a finite number N of mappings $\kappa \mapsto \mu_{j,\kappa}$ for $j = 1, \ldots, N$ and $|\kappa - \hat{\kappa}| < \eta$ such that $\mu_{j,\kappa}$ are eigenvalues of M_k with mass matrix E_k for $j = 1, \ldots, N$, satisfying $\mu_{j,\hat{\kappa}} = a_+$, the mappings $\kappa \mapsto \mu_{j,\kappa}$ are Lipschitz continuous for all $j = 1, \ldots, N$. We denote by $\phi_{j,\kappa} \in Y_{\mathrm{per}}$ corresponding eigenfunctions,

$$M_k \phi_{j,\kappa} = \mu_{j,\kappa}\, E_k \phi_{j,\kappa}\,,$$

normalized as $\langle \phi_{j,\kappa}, E_k \phi_{\ell,\kappa} \rangle_{L^2(W^n)} = \delta_{j,\ell}$. Theorem A.4 provides also the continuity of the eigenprojections P_κ (orthogonal with respect to $\langle u, v \rangle_{E_k} = \langle u, E_k v \rangle_{L^2(W^n)}$) from $L^2(W^n)$ onto the N-dimensional space

$$\mathcal{M}_k \ := \ \left\{ \phi \in Y_{\rm per} \mid (M_k - \mu_{j,\kappa} E_k)\phi = 0 \text{ for some } j \in \{1, \dots, N\} \right\} \ \subset \ L^2(W^n)\,.$$

The projection is given, for arbitrary $\psi \in L^2(W^n)$, by

$$P_\kappa \psi \ = \ \sum_{j=1}^N \langle \psi, \phi_{j,\kappa} \rangle_{E_k} \, \phi_{j,\kappa}\,. \tag{3.7}$$

We recall that we consider $\lambda < a_+$ and that (a_-, a_+) is a spectral gap. The eigenvalues therefore satisfy $\lambda < a_+ \le \mu_{j,\kappa}$.

Step 3: Verification of (3.6) *and an improvement thereof when* $(M - a^+ E)_{(m^\circ, \hat\kappa)}$ *is not invertible.* We write the solution of (3.5) in the form

$$v_\kappa \ = \ \sum_{j=1}^N \frac{\langle r_\kappa, \phi_{j,\kappa} \rangle_{L^2}}{\mu_{j,\kappa} - \lambda} \, \phi_{j,\kappa} \ + \ v_\kappa^\perp\,. \tag{3.8}$$

The equation for $v_\kappa^\perp$ is then

$$\begin{aligned} (M_k - \lambda E_k) v_\kappa^\perp \ &= \ r_\kappa \ - \ \sum_{j=1}^N \frac{\langle r_\kappa, \phi_{j,\kappa} \rangle_{L^2}}{\mu_{j,\kappa} - \lambda} (M_k - \lambda E_k) \phi_{j,\kappa} \\ &= \ r_\kappa \ - \ \sum_{j=1}^N \langle r_\kappa, \phi_{j,\kappa} \rangle_{L^2(W^n)} \, E_k \phi_{j,\kappa}\,. \end{aligned} \tag{3.9}$$

Claim 3a: With the projection P_k *of* (3.7), *we claim that there holds* $P_k v_\kappa^\perp = 0$. Indeed, multiplying (3.9) with $\phi_{\ell,\kappa}$, using that $M_k - \lambda E_k$ is self-adjoint and that the $\phi_{j,\kappa}$ are E_k-orthogonal, we find

$$(\mu_{\ell,\kappa} - \lambda) \langle v_\kappa^\perp, E_k \phi_{\ell,\kappa} \rangle_{L^2(W^n)} \ = \ \langle r_\kappa, \phi_{\ell,\kappa} \rangle_{L^2(W^n)} - \sum_{j=1}^N \langle r_\kappa, \phi_{j,\kappa} \rangle_{L^2} \, \delta_{j,\ell} \ = \ 0\,.$$

This provides $\langle v_\kappa^\perp, E_k \phi_{\ell,\kappa} \rangle_{L^2(W^n)} = 0$ for all ℓ and thus $P_k v_\kappa^\perp = 0$. Claim 3a is shown.

Using Claim 3a, we can replace equation (3.9) by the equivalent equation

$$(M_k - \lambda E_k + E_k P_\kappa) v_\kappa^\perp \ = \ r_\kappa \ - \ \sum_{j=1}^N \langle r_\kappa, \phi_{j,\kappa} \rangle_{L^2} \, E_k \phi_{j,\kappa}\,. \tag{3.10}$$

Claim 3b: The operator $M_k - \lambda E_k + E_k P_\kappa$ *is an isomorphism.* The claim regards κ close to $\hat\kappa$ and λ close to a^+. Since only small perturbations are treated, it is actually sufficient to show that $M_k - \lambda E_k + E_k P_\kappa$ is an isomorphism for $\kappa = \hat\kappa$ and $\lambda = a^+$.

Let us consider injectivity. For $k = (m^\circ, \hat\kappa)$ we study the relation $(M_k - a^+ E_k + E_k P_{\hat\kappa}) u = 0$. Multiplication with $P_{\hat\kappa} u$ yields, because of $\mu_{j,\hat\kappa} = a_+$ for all j,

$$0 \ = \ \langle (M_k - a^+ E_k + E_k P_{\hat\kappa}) u, P_{\hat\kappa} u \rangle_{L^2} \ = \ \langle E_k P_{\hat\kappa} u, P_{\hat\kappa} u \rangle_{L^2}\,,$$

and thus $P_{\hat\kappa} u = 0$. The original equation for u simplifies to $(M_k - a^+ E_k) u = 0$ and we conclude the u is in the eigenspace, $u \in \mathcal{M}_k$. In this situation, the projection acts trivially and we obtain $0 = P_{\hat\kappa} u = u$. This provides the injectivity.

The operator $M_k - \lambda E_k + E_k P_\kappa$ is a finite-dimensional perturbation and hence a compact perturbation of a Fredholm operator with index 0. We therefore conclude that the entire operator is again Fredholm with index 0 and thus, by injectivity, an isomorphism for all $|\kappa - \hat\kappa| \le \eta$ and $\lambda \in (a^+ - \eta, a^+)$. Claim 3b is shown.

Claim 3b implies $\|(M_k - \lambda E_k + E_k P_\kappa)^{-1}\| \le c$ and hence, since $v_\kappa^\perp$ satisfies (3.10),

$$\|v_\kappa^\perp\|_{Y_{\rm per}} \;\le\; c\|r_\kappa\|_{L^2} \quad \text{for all } |\kappa - \hat\kappa| \le \eta \text{ and } \lambda \in (a^+ - \eta, a^+)\,.$$

We insert our findings into (3.8) to obtain

$$\begin{aligned} \langle v_\kappa, r_\kappa\rangle_{L^2(W^n)} \;&=\; \sum_{j=1}^N \frac{|\langle r_\kappa, \phi_{j,\kappa}\rangle_{L^2}|^2}{\mu_{j,\kappa} - \lambda} \;+\; \langle v_\kappa^\perp, r_\kappa\rangle_{L^2(W^n)} \\ &\ge\; \sum_{j=1}^N \frac{|\langle r_\kappa, \phi_{j,\kappa}\rangle_{L^2}|^2}{\mu_{j,\kappa} - \lambda} \;-\; c\|r_\kappa\|^2_{L^2(W^n)}\,. \end{aligned} \tag{3.11}$$

This holds for $|\kappa - \hat\kappa| < \eta$ and $\lambda \in (a^+ - \eta, a^+)$. We note that (3.11) also provides (3.6) for $\hat\kappa$ since $\mu_{j,\kappa} \ge a^+ > \lambda$ holds for all j and κ.

Step 4: Conclusion. From the compactness of I we conclude that we can choose η and c independent of $\hat\kappa$, which means that (3.6) holds for some constant c for all $\kappa \in I$ and all $\lambda \in (a^+ - \eta, a^+)$, and the stronger estimate (3.11) holds for all $\kappa \in (\hat\kappa - \eta, \hat\kappa + \eta)$ with $\hat\kappa$ such that $(M - a^+E)_{(m^\circ,\hat\kappa)}$ is not invertible.

When we integrate the lower bounds with respect to $\kappa \in I$, we use the stronger estimate (3.11) in the interval $(\kappa^\circ - \eta, \kappa^\circ + \eta)$ and the weaker estimate (3.6) in the remaining part of I. The one-dimensional Floquet-Bloch transform is an unitary isomorphism and we can calculate

$$\begin{aligned} &\langle (M - \lambda E)^{-1}_{m^\circ} R^* r, R^* r\rangle_{L^2(W^{n-1}\times\mathbb{R})} \\ &\quad= \langle v, R^* r\rangle_{L^2(W^{n-1}\times\mathbb{R})} \;=\; \int_I \langle v_\kappa, r_\kappa\rangle_{L^2(W^n)}\, d\kappa \\ &\quad\ge -c\,\|r_\kappa\|^2_{L^2(W^n)} \;+\; \int_{\kappa^\circ-\eta}^{\kappa^\circ+\eta} \sum_{\ell=1}^N \frac{|\langle r_\kappa, \phi_{\ell,\kappa}\rangle_{L^2(W^n)}|^2}{\mu_{\ell,\kappa} - \lambda}\, d\kappa\,. \end{aligned}$$

In this lower bound, we can now use the fact that the maps $\kappa \mapsto \mu_{j,\kappa}$ are locally Lipschitz continuous. Because of $\mu_{j,\kappa^\circ} = a_+$, we have

$$\mu_{j,\kappa} - \lambda \;\le\; \mu_{j,\kappa^\circ} - \lambda \;+\; c_L|\kappa - \kappa^\circ| \;=\; a_+ - \lambda \;+\; c_L|\kappa - \kappa^\circ|$$

for $|\kappa - \kappa^\circ| < \eta$ where $c_L > 0$ denotes the Lipschitz constant. This leads to the lower bound

$$\begin{aligned} &\langle (M - \lambda E)^{-1}_{m^\circ} R^* r, R^* r\rangle_{L^2(W^{n-1}\times\mathbb{R})} \\ &\quad\ge \int_{\kappa^\circ-\eta}^{\kappa^\circ+\eta} \frac{1}{a_+ - \lambda + c_L|\kappa - \kappa^\circ|} \sum_{j=1}^N \left|\left\langle r_\kappa, \phi_{j,\kappa}\right\rangle_{L^2(W^n)}\right|^2 d\kappa \;-\; c\,\|r\|^2_{L^2(W^n)}\,. \end{aligned}$$

At this point we use the requirement (3.3), which provides the information that $\langle r_{\kappa^\circ}, \phi_{j,\kappa^\circ}\rangle_{L^2(W^n)} = \langle r e^{-i\kappa^\circ x_n}, \phi_{j,\kappa^\circ}\rangle_{L^2(W^n)} \ne 0$ for some $j \in \{1, \dots, N\}$. Indeed, $\langle r e^{-i\kappa^\circ x_n}, \phi_{j,\kappa^\circ}\rangle_{L^2(W^n)} = 0$ for all $j \in \{1, \dots, N\}$ would imply $\langle r e^{-i\kappa^\circ x_n}\phi\rangle_{L^2(W^n)} = 0$ for all $\phi \in \mathcal{N}(M^{a_+}_{k^\circ})$, a contradiction to the assumption. This yields the estimate

$$\begin{aligned} &\langle (M - \lambda E)^{-1}_{m^\circ} R^* r, R^* r\rangle_{L^2(W^{n-1}\times\mathbb{R})} \\ &\quad\ge c' \int_{\kappa^\circ-\eta'}^{\kappa^\circ+\eta'} \frac{1}{a_+ - \lambda + c_L|\kappa - \kappa^\circ|}\, d\kappa \;-\; c\,\|r\|^2_{L^2(W^n)}\,. \end{aligned}$$

The assertion follows since the integral tends to infinity as $\lambda \to a_+$. □

Our third lemma uses the above constructions in order to derive a result on the perturbed operator. Again, we treat the operator after a Floquet-Bloch transform in $n-1$ coordinates.

Lemma 3.5 (The inverse of the perturbed operator has a large norm)**.** *Let the situation be as in Theorem 3.2 with (a_-, a_+) a spectral gap of the family $M^\lambda = M - \lambda E$ and $a_0 = \frac{1}{2}(a_+ + a_-)$ the mid-point. Let $k^\circ = (m^\circ, \kappa^\circ)$ be such that $M_{k^\circ}^{a_+}$ has a non-trivial kernel $\mathcal{N}(M_{k^\circ}^{a_+}) \subset Y_{\mathrm{per}}$. We use the assumption of Theorem 3.2 that there exists $\phi \in \mathcal{N}(M_{k^\circ}^{a_+})$ with $\langle Q\phi, \phi\rangle \neq 0$. Then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, there exists $\lambda \in (a_0, a_+)$ such that the operator*

$$(M - \lambda E - \lambda\delta R^* QQR)_{m^\circ} \tag{3.12}$$

has no bounded inverse.

Proof. Step 1: Simplification and the function f. With the two real parameters $\lambda \in [a_0, a_+)$ and $\delta > 0$, we define the real-valued function

$$f(\lambda, \delta) \ := \ \lambda\delta \sup_{\|w\|=1} \langle (M - \lambda E)_{m^\circ}^{-1} R^* Q w, R^* Q w\rangle_{L^2(W^{n-1}\times\mathbb{R})}\,, \tag{3.13}$$

where w ranges in $L^2(W^n)$ and the norm is $\|w\| = \|w\|_{L^2(W^n)}$.

Since the operator is invertible for $\lambda = a_0$, by boundedness of Q, we can choose $\delta_0 > 0$ such that $f(a_0, \delta) \le \frac{1}{2}$ holds for every $\delta \in (0, \delta_0)$.

From now on, we consider a fixed parameter $\delta \in (0, \delta_0)$.

By assumption we can choose $\phi \in \mathcal{N}(M_{k^\circ}^{a_+})$ with $\|\phi\|_{L^2(W^n)} = 1$ and $\langle Q\phi, \phi\rangle \neq 0$. We set $w := \phi\, e^{i\kappa^\circ x_n}$ and $r := Qw = Q\phi\, e^{i\kappa^\circ x_n}$, the latter since Q is a multiplication operator. Then $\langle r e^{-i\kappa^\circ x_n}, \phi\rangle_{L^2(W^n)} = \langle Q\phi, \phi\rangle \neq 0$, hence (3.3) is satisfied. Lemma 3.4 yields $\sup_{a_0<\lambda<a_+} f(\lambda, \delta) = \infty$.

By the continuity of $f(\cdot, \delta)$ there exists $\lambda \in (a_0, a_+)$ with $f(\lambda, \delta) = 1$. We claim that, for this value of λ, the operator $(M - \lambda E - \lambda\delta R^* QQR)_{m^\circ}$ has no bounded inverse. When this claim is shown, the proof is complete.

Step 2: The operator A and conclusion. We have to study the operator

$$A := \lambda\delta\, QR(M - \lambda E)_{m^\circ}^{-1} R^* Q : L^2(W^n) \to L^2(W^n) \tag{3.14}$$

in more detail. By the choice of λ, the operator A has the property

$$1 \ = \ f(\lambda, \delta) \ = \ \sup_{\|w\|=1} \langle Aw, w\rangle_{L^2(W^n)}\,.$$

Since A is self adjoint, this relation implies $\|A\| = 1$. Accordingly, there exists a sequence $w_j \in L^2(W^n)$ with $\|w_j\|_{L^2(W^n)} = 1$ and

$$\langle Aw_j, w_j\rangle_{L^2(W^n)} \to 1 \quad \text{as } j \to \infty\,.$$

We calculate

$$\begin{aligned}
\|Aw_j - w_j\|_{L^2(W^n)}^2 &= \|Aw_j\|_{L^2(W^n)}^2 \ - \ 2\,\mathrm{Re}\langle Aw_j, w_j\rangle_{L^2(W^n)} \ + \ \|w_j\|_{L^2(W^n)}^2 \\
&\le \|A\| \ - \ 2\,\mathrm{Re}\langle Aw_j, w_j\rangle_{L^2(W^n)} \ + \ 1 \\
&= 2 \ - \ 2\,\mathrm{Re}\langle Aw_j, w_j\rangle_{L^2(W^n)}\,.
\end{aligned}$$

The right hand side converges to zero. Since the left-hand side is nonnegative, this implies $Aw_j - w_j \to 0$. The function $v_j := \lambda\sqrt{\delta}\,(M - \lambda E)_{m^\circ}^{-1} R^* Q w_j$ then satisfies $\sqrt{\delta} QRv_j - w_j = Aw_j - w_j \to 0$ and hence also $\sqrt{\delta} R^* QQRv_j - R^* Q w_j \to 0$.

On the other hand, there holds $(M - \lambda E)_{m^\circ} v_j = \lambda\sqrt{\delta}\, R^* Q w_j$ and thus

$$(M - \lambda E - \lambda\delta R^* QQR)_{m^\circ}\, v_j \to 0\,. \tag{3.15}$$

However, v_j does not tend to zero; indeed, otherwise we would find $w_j = \sqrt{\delta} Q^\delta R v_j - (\sqrt{\delta} Q R v_j - w_j) \to 0$, in contradiction to $\|w_j\|_{L^2(W^n)} = 1$. Relation (3.15) therefore implies that $(M - \lambda E - \lambda\delta R^* QQR)_{m^\circ}$ is not boundedly invertible. □

Proof of Theorem 3.2. Lemma 3.3 provides $k^\circ = (m^\circ, \kappa^\circ)$ as needed in Lemma 3.5. In particular, we find a non-trivial element $\phi \in \mathcal{N}(M_{k^\circ}^{a+})$.

Lemma 3.5 provides $\lambda \in (a_0, a_+)$ such that $(M - \lambda E - \lambda\delta R^* QQR)_{m^\circ}$ is not invertible. Lemma 2.12 shows that λ is not in the spectral gap of the perturbed operator. This concludes the proof of Theorem 3.2. □

4. Applications

We next show that our result has many applications: scalar elliptic equations, elliptic systems with arbitrary dimensions in image and pre-image, elasticity systems and Maxwell's equations. In all applications we conclude: When periodic coefficient are perturbed in such a way that the coefficients are modified along a hyperplane, then spectral gaps become smaller at the upper end.

4.1. **Application to an elliptic system.** Let the two dimensions $n, m \in \mathbb{N}$ be arbitrary. The coefficient fields are $a : \mathbb{R}^n \to \mathbb{R}^{m\times m\times n\times n}$ and $b : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, they are measurable, bounded and 2π-periodic in every direction. For every $x \in \mathbb{R}^n$, the coefficient a is a fourth order tensor in the sense that an arbitrary matrix $\xi \in \mathbb{R}^{n\times m}$ is mapped to a matrix $a(x)\,\xi \in \mathbb{R}^{n\times m}$, given as

$$(a(x)\,\xi)_i^k = \sum_{j,l} a_{i,j}^{k,l}(x)\,\xi_j^l\,. \tag{4.1}$$

We use the convention that indices that are related to directions in $\mathbb{R}^n$ are i and j, they are lower indices and they run from 1 to n. Indices that are related to vector coordinates in $\mathbb{R}^m$ are k and l, they are upper indices and they run from 1 to m.

We always assume that b is positive definite: For some $\gamma > 0$, for every vector $\zeta \in \mathbb{R}^m$ and every $x \in \mathbb{R}^n$ holds: $\zeta \cdot b(x)\zeta = \sum_{k,l} b^{k,l}(x)\zeta_k\zeta_l \geq \gamma\|\zeta\|^2$. Furthermore, we assume that, for every x, the matrix $b(x)$ is symmetric.

Regarding positivity of a, we consider two different requirements. We say that a is strongly elliptic when the following holds for some $\gamma > 0$: For every $x \in \mathbb{R}^n$ and every matrix $\xi \in \mathbb{R}^{n\times m}$

$$\xi \cdot a(x)\,\xi = \sum_{i,j,k,l} \xi_i^k a_{i,j}^{k,l}(x)\,\xi_j^l \geq \gamma\|\xi\|^2\,. \tag{4.2}$$

In order to treat additionally the system of elasticity, we consider also an alternative concept in the case $n = m$. We say that a is weakly elliptic when (i) for every x and every skew-symmetrix matrix ξ there holds $a(x)\,\xi = 0$, and (ii) for some $\gamma > 0$ holds

$$\xi \cdot a(x)\,\xi \geq \gamma\|\xi\|^2 \tag{4.3}$$

for every $x \in \mathbb{R}^n$ and every *symmetric* matrix $\xi \in \mathbb{R}^{n\times m}$.

With this notation, we can generalize the perturbed elliptic problem of Definition 2.1: The dimensions $n, m \in \mathbb{N}$ are now arbitrary, the coefficients are $a : \mathbb{R}^n \to \mathbb{R}^{m\times m\times n\times n}$ and $b : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, both measurable, bounded and 2π-periodic in every direction, b positive and symmetric. The tensor a satisfies either the strong ellipticity (4.2) or the weak ellipticity (4.3) (in the latter case, the dimensions must coincide, $n = m$). The perturbation is given by $0 \neq q \in L^\infty(\mathbb{R}^n, \mathbb{R}^{m\times m})$ supported in $\mathbb{R}^{n-1} \times W$ with $q(x)$ a symmetric matrix for every x, the map q is 2π-periodic in the first $n-1$ directions.

We denote the derivative of a function $u : \mathbb{R}^n \to \mathbb{R}^m$ with ∇u: For every x, the matrix is $\nabla u(x) = (\partial_j u_l(x))_j^l$. Accordingly, $(a\nabla u)(x) = \left(\sum_{j,l} a_{i,j}^{k,l}(x)\, \partial_j u_l(x)\right)_i^k$. In the first line of the subsequent formulas, the dot in the integrand indicates the scalar product in the space $\mathbb{R}^{n\times m}$ of matrices. With this interpretation of products, we use the operators M, E and S^δ of the definitions (2.4)–(2.6).

Proposition 4.1 (The perturbed elliptic problem fits in the abstract framework)**.** *We consider the perturbed elliptic problem of Definition 2.1 with the above generalization as a system. We assume that the assumptions on a, b and q are satisfied: boundedness and periodicity, positivity of b, strong or weak ellipticity of a. This perturbed elliptic system satisfies all properties of Assumption 2.9.*

Proof. The underlying spaces are $X = L^2(\mathbb{R}^n, \mathbb{C}^m)$ and $Y = H^1(\mathbb{R}^n, \mathbb{C}^m)$. There holds $Y \subset L^2(\mathbb{R}^n)$, $M \in \mathcal{L}(Y, Y')$, $E \in \mathcal{L}(L^2(\mathbb{R}^n))$, hence also $E \in \mathcal{L}(Y, Y')$, $S^\delta \in \mathcal{L}(L^2(\mathbb{R}^{n-1} \times W))$. This verifies Item 1 of Assumption 2.9.

We introduce Y_{per} as the normed space

$$Y_{\text{per}} := \left\{ u|_{W^n} \middle|\, u : \mathbb{R}^n \to \mathbb{C}^n \text{ is } 2\pi\text{-periodic in every direction, } u \in H^1_{\text{loc}}(\mathbb{R}^n) \right\} ,$$
$$\|u\|_{Y_{\text{per}}} := \|u\|_{H^1(W^n)} .$$

The subordinate space is given by

$$\tilde{Y}_{\text{per}} := \left\{ u|_{W^{n-1}\times\mathbb{R}} \middle|\, u \in H^1_{\text{loc}}(\mathbb{R}^n, \mathbb{C}^n) \text{ is } 2\pi\text{-periodic in } x_1,\ldots, x_{n-1}, \right.$$
$$\left. u|_{W^{n-1}\times\mathbb{R}} \in H^1(W^{n-1} \times \mathbb{R}) \right\} ,$$
$$\|u\|_{\tilde{Y}_{\text{per}}} := \|u\|_{H^1(W^{n-1}\times\mathbb{R})} .$$

We must check that the two reflexive Banach spaces Y_{per} and $\tilde{Y}_{\text{per}}$ are admissible for Y in the sense of Definitions 2.3 and 2.6. Regarding Definition 2.3, we must verify (2.15), namely that

$$\mathcal{F}_{\{1,\ldots,n\}} : Y \to L^2(I^n, Y_{\text{per}}) \tag{4.4}$$

is an isomorphism. We consider a function $u \in Y$ with derivative $g = \nabla u \in L^2(\mathbb{R}^n)$. The function u is mapped to $\hat{u} = \hat{u}(x, k) = (\mathcal{F}_{\{1,\ldots,n\}} u)(x, k)$, similarly g to $\hat{g} = \mathcal{F}_{\{1,\ldots,n\}} g$ (component-wise, for every entry of g). The derivative of the transformed function is given by the formula $\nabla(e^{ik\cdot x}\hat{u}) = e^{ik\cdot x}\hat{g}$. This verifies that the map of (4.4) is well-defined. Vice versa, an arbitrary function $\hat{u} \in L^2(I^n, Y_{\text{per}})$ is also of class $L^2(I^n, L^2(W^n))$ and posesses therefore a pre-image u. The gradient of the pre-image can be calculated from $\nabla(e^{ik\cdot x}\hat{u}) = e^{ik\cdot x}\hat{g}$; in particular, the pre-image has an $L^2(W^n)$-gradient and is therefore of class Y. This argument provides also estimates and we conclude that the map of (4.4) is an isomorphism.

In the same way one verifies (2.17) and (2.18):

$$\mathcal{F}_{\{1,\ldots,n-1\}} : Y \to L^2(I^{n-1}, \tilde{Y}_{\text{per}})$$
$$\mathcal{F}_{\{n\}} : \tilde{Y}_{\text{per}} \to L^2(I, Y_{\text{per}}) .$$

are isomorphisms. This verifies Item 2 of Assumption 2.9.

We next have to show that the operators M and E possess continuous Floquet-Bloch representations $(M_k)_k$ and $(E_k)_k$ in n directions in the sense of Definition 2.4 and that S^δ possesses a Floquet-Bloch representations $(S^\delta_m)_m$ in $n-1$ directions in the sense of Definition 2.7. We define the operators $M_k, E_k : Y_{\rm per} \to Y'_{\rm per}$ and $S^\delta_m : L^2(W^n) \to L^2(W^n)$ as

$$\begin{aligned} \langle M_k u, \varphi\rangle &:= \int_{W^n} a(x)\, \nabla(u(x)e^{ik\cdot x}) \cdot \nabla(\bar\varphi(x) e^{-ik\cdot x})\, dx\,, \\ \langle E_k u, \varphi\rangle &:= \int_{W^n} b\, u\, \bar\varphi\,, \qquad \langle S^\delta_m u, \varphi\rangle := \int_{W^n} \delta\, q^2\, u\, \bar\varphi\,. \end{aligned} \tag{4.5}$$

We verify the most interesting relation, namely the representation property (2.16) for M; the formulas for E and S^δ follow in the same way and are simpler since no derivative is involved. We calculate, using, in this order: definition of M_k, rule for gradients of transformed functions, the coefficient a is periodic in all directions and hence multiplication commutes with the Floquet-Bloch transform, unitarity of the Floquet-Bloch transform, definition of M.

$$\begin{aligned} &\int_{I^n} \big\langle M_k \hat u(\cdot,k), \hat v(\cdot,k)\big\rangle_{Y'_{\rm per},Y_{\rm per}}\, dk \\ &= \int_{I^n}\int_{W^n} a(x)\, \nabla(\hat u(x,k) e^{ik\cdot x}) \cdot \nabla(\overline{\hat\varphi(x,k)} e^{-ik\cdot x})\, dx\, dk \\ &= \int_{I^n}\int_{W^n} a(x)\, e^{ik\cdot x} \mathcal{F}_{\{1,\dots,n\}}(\nabla u)(x,k) \cdot e^{-ik\cdot x} \mathcal{F}_{\{1,\dots,n\}}(\nabla\bar\varphi)(x,k)\, dx\, dk \\ &= \int_{I^n}\int_{W^n} \mathcal{F}_{\{1,\dots,n\}}(a\nabla u)(x,k) \cdot \mathcal{F}_{\{1,\dots,n\}}(\nabla\bar\varphi)(x,k)\, dx\, dk \\ &= \int_{\mathbb{R}^n} a(x)\, \nabla u(x) \cdot \nabla\bar\varphi(x)\, dx \\ &= \langle M u, v\rangle_{Y',Y}\,. \end{aligned}$$

This shows Item 3 of Assumption 2.9.

For every $k \in I^n$, the operator $E_k : L^2(W^n) \to L^2(W^n)$ is given as a multiplication with the positive definite matrix $b = b(x)$. This shows that E_k is a self-adjoint coercive isomorphism. The mapping $k \mapsto E_k$ is independent of k, hence, in particular, differentiable. We obtain Item 4.

For every k, by its definition, M_k is self-adjoint. We have to show that $M_k + E_k$ is uniformly coercive. In the case of strong ellipticity, this follows by a direct comparison of the expression $\langle M_k u, u\rangle$ with $\|u\|^2_{Y_{\rm per}}$. In the case of weak ellipticity, the expression $\langle M_k u, u\rangle$ controls only the symmetric part of ∇u. By Korn's inequality, this controls indeed all derivatives and we conclude coercivity.

For every k, the operator M_k is an elliptic operator with well-defined resolvent $(M_k + E_k)^{-1} : Y'_{\rm per} \to Y_{\rm per}$. This implies that, for every $\lambda \in \mathbb{C}$, we can write $(M_k + E_k)^{-1}(M_k - \lambda E_k) = \mathrm{id} - (M_k + E_k)^{-1}(1+\lambda)E_k$. The right hand side is a compact perturbation of the identity in $Y_{\rm per}$ since the embedding $Y_{\rm per} \to L^2(W^n)$ is compact. This implies that the left hand side is a Fredholm operator with index 0 in $\mathcal{L}(L^2(W^n))$. Since $(M_k + E_k)^{-1} : Y'_{\rm per} \to Y_{\rm per}$ is an isometry, we can conclude that the operator $M_k - \lambda E_k : Y_{\rm per} \to Y'_{\rm per}$ is also a Fredholm operator with index 0. The fact that $\lambda = 0$ is the only possible accumulation point of the spectral values is a consequence of the compactness of the resolvent.

Its formula shows that the map $k \mapsto M_k$ is continuously differentiable from I^n into $\mathcal{L}(Y_{\rm per}, Y'_{\rm per})$. We have thus verified Item 5 of Assumption 2.9.

For every $m \in I^{n-1}$, the operator $S^\delta_m : L^2(W^n) \to L^2(W^n)$ is given by the multiplication with the function δq^2. In particular, it is bounded and self-adjoint. This shows Item 6. □

In order to apply Theorem 3.2, we finally have to check that there exists an eigenfunction $\phi \in \mathcal{N}(M_k^{a_+})$ such that $\langle Q\phi, \phi\rangle = \int_{W^n} q\,|\phi|^2 \neq 0$. This is a consequence of the fact that eigenfunctions are not vanishing on any open set (unique continuation property). We conclude the following result on spectral gaps.

Theorem 4.2 (Application to the perturbed elliptic problem)**.** *Let* $(a_-, a_+) \subset (0,\infty)$ *with mid-point* $a_0 = (a_- + a_+)/2$ *be a spectral gap of the operator*

$$u \;\mapsto\; -\nabla\cdot(a\,\nabla u) - \lambda\, b\, u\,,$$

where $a, b \in L^\infty(\mathbb{R}^n)$ *and* $q \in L^\infty(\mathbb{R}^{n-1}\times W)$ *satisfy the above assumptions. Then there there exists* $\delta_0 > 0$ *such that, for every* $\delta \in (0, \delta_0]$*, the interval* (a_0, a_+) *is not contained in a spectral gap of the operator family*

$$u \;\mapsto\; -\nabla\cdot(a\,\nabla u) - \lambda\,(b + \delta q^2)\, u\,.$$

Application to a problem in linear elasticity. The time-harmonic Lamé system is covered in the above result. Let us sketch the setting and provide the necessary arguments in dimension $n = m$. We use the density $\rho_0 \in L^\infty(\mathbb{R}^n, \mathbb{R})$ and the Lamé-parameters $\mu_0, \lambda_0 \in L^\infty(\mathbb{R}^n, \mathbb{R})$, the three functions are assumed to be periodic in every direction and non-negative, ρ_0 and μ_0 with a positive lower bound (we write μ_0 and λ_0 for the Lamé-parameters in order to avoid confusion with the spectral parameter λ). Every matrix $\xi \in \mathbb{R}^{n\times n}$ can be symmetrized, we use $\xi^{\mathrm{sym}} := (\xi + \xi^T)/2$. We define the tensor a as follows: For every $x \in \mathbb{R}^n$ and every matrix $\xi \in \mathbb{R}^{n\times n}$, we set

$$a(x)\xi = 2\mu_0(x)\xi^{\mathrm{sym}} + \lambda_0(x)\,\mathrm{trace}(\xi)\,\mathrm{id}\,. \tag{4.6}$$

In particular, for every skew-symmetric matrix $\xi \in \mathbb{R}^{n\times n}$ (that is: $\xi^T = -\xi$), there holds $a(x)\xi = 0$. We note that the weak ellipticity is satisfied since, for every symmetric matrix $\xi \in \mathbb{R}^{n\times n}$, using the dot here to indicate the scalar product in the space of matrices:

$$\xi\cdot a(x)\xi = 2\mu_0\xi\cdot\xi + \lambda_0|\mathrm{trace}(\xi)|^2 \geq \gamma\|\xi\|^2\,. \tag{4.7}$$

Using the symmetrized gradient $\nabla^{\mathrm{sym}}u = (\nabla u + (\nabla u)^T)/2$, we can write the operator also as

$$\mathrm{div}(a\nabla u) = \mathrm{div}(a\nabla^{\mathrm{sym}}u) = \mathrm{div}(2\mu_0\nabla^{\mathrm{sym}}u + \lambda_0\,\mathrm{trace}(\nabla u)\,\mathrm{id})\,.$$

With the above choices, the elliptic operator of Theorem 4.2 encodes the Lamé elasticity system with strain $\epsilon = \nabla^{\mathrm{sym}}u$, stress $\sigma = 2\mu_0(x)\epsilon + \lambda_0(x)\,\mathrm{trace}(\epsilon)\,\mathrm{id}$, and balance of momentum described by the operator $Mu = -\,\mathrm{div}(\sigma)$.

Theorem 3.2 yields the spectral gap result for the elasticity system:

Theorem 4.3 (Application to the perturbed system of elasticity)**.** *We consider bounded, non-negative and measurable periodic coefficient functions* $\mu_0, \lambda_0, \rho_0 : \mathbb{R}^n \to \mathbb{R}$*, we assume that* μ_0 *and* ρ_0 *have positive lower bounds. Let* a *be the tensor of the Lamé system described in* (4.6)*. Let* $(a_-, a_+) \subset (0,\infty)$ *with mid-point* a_0 *be a spectral gap of the operator*

$$u \;\mapsto\; \mathrm{div}(a\nabla u) + \lambda\,\rho_0\, u\,.$$

Let $0 \neq \rho_1$ *be a non-negative function, periodic in the first* $n-1$ *directions and supported on* $\mathbb{R}^{n-1}\times W$*. Then there exists* $\delta_0 > 0$ *such that, for all* $\delta \in (0, \delta_0]$*, the interval* (a_0, a_+) *is not contained in a spectral gap of the elasticity system with perturbed mass,*

$$u \;\mapsto\; \mathrm{div}(a\nabla u) + \lambda\,(\rho_0 + \delta\rho_1)\, u\,.$$

4.2. **Application to the Maxwell system.** In this section we consider the Maxwell system of Definition 2.2,

$$\langle Mu,\psi\rangle_{Y',Y} = \int_{\mathbb{R}^3} \mu^{-1}\operatorname{curl} u\cdot\operatorname{curl}\bar\psi\,, \qquad \langle Eu,\psi\rangle_{Y',Y} \;=\; \int_{\mathbb{R}^3} \varepsilon\, u\,\bar\psi\,,$$
$$\langle R^* S^\delta R u,\psi\rangle_{Y',Y} \;=\; \int_{\mathbb{R}^3} \delta\,\varepsilon_1\, u\,\bar\psi\,.$$

The underlying space is $X = L^2(\mathbb{R}^3,\mathbb{C}^3)$. The domain of the operators is $Y = H(\operatorname{curl},\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3,\mathbb{C}^3) \mid \operatorname{curl} u \in L^2(\mathbb{R}^3,\mathbb{C}^3)\}$. With the space of locally integrable functions, $H_{\text{loc}}(\operatorname{curl},\mathbb{R}^3) = \{u \in L^2_{\text{loc}}(\mathbb{R}^3,\mathbb{C}^3) \mid \operatorname{curl} u \in L^2_{\text{loc}}(\mathbb{R}^3,\mathbb{C}^3)\}$ we define the space Y_{per} as

$$Y_{\text{per}} := \{u|_{W^n} |\, u:\mathbb{R}^n\to\mathbb{C}^n \text{ is } 2\pi\text{-periodic in every direction, } u\in H_{\text{loc}}(\operatorname{curl},\mathbb{R}^n)\}\ ,$$
$$\|u\|_{Y_{\text{per}}} := \|u\|_{H(\operatorname{curl},W^n)}\ ,$$

and $\tilde Y_{\text{per}}$ as

$$\tilde Y_{\text{per}} := \{u|_{W^{n-1}\times\mathbb{R}} |\, u \in H_{\text{loc}}(\operatorname{curl},\mathbb{R}^n,\mathbb{C}^n) \text{ is } 2\pi\text{-periodic in } x_1,\ldots,\, x_{n-1},$$
$$u|_{W^{n-1}\times\mathbb{R}} \in H(\operatorname{curl}, W^{n-1}\times\mathbb{R})\Big\}\ ,$$
$$\|u\|_{\tilde Y_{\text{per}}} := \|u\|_{H(\operatorname{curl},W^{n-1}\times\mathbb{R})}\,.$$

The operators $M_k, E_k : Y_{\text{per}} \to Y'_{\text{per}}$ and $S^\delta_m : L^2(W^3)\to L^2(W^3)$ are defined just as in (4.5); the factor in the definition of E_k is ε instead of b, the factor in the definition of S_m is ε_1 instead of q^2, and in the definition of M_k gradients must be replaced by rotations,

$$\begin{aligned}\langle M_k u,\varphi\rangle &:= \int_{W^n} \mu^{-1}\ \operatorname{curl}(u(x)e^{ik\cdot x})\cdot\operatorname{curl}(\bar\varphi(x)e^{-ik\cdot x})\,dx\,,\\ \langle E_k u,\varphi\rangle &:= \int_{W^n}\varepsilon\, u\,\bar\varphi\,, \qquad \langle S^\delta_m u,\varphi\rangle := \int_{W^n}\delta\,\varepsilon_1\, u\,\bar\varphi\,.\end{aligned} \tag{4.8}$$

Proposition 4.4 (The Maxwell system fits in the abstract framework)**.** *Under the assumptions on μ,ε and ε_1 in Definition 2.2, the above description of the Maxwell system satisfies all requirements of Assumption 2.9.*

Proof. Most items of Assumption 2.9 are shown as in Proposition 4.1.

When we want to verify that $\mathcal{F}_{\{1,\ldots,n\}} : Y \to L^2(I^n, Y_{\text{per}})$ is an isomorphism, we must calculate derivatives. We exploit the product rule to calculate rotations, $\operatorname{curl}(e^{ik\cdot x}\hat u(x,k)) = e^{ik\cdot x}\operatorname{curl}\hat u(x,k) + \nabla(e^{ik\cdot x})\times\hat u(x,k)$. This identity implies that the function $e^{ik\cdot x}\hat u(x,k)$ has a curl in L^2 when $\operatorname{curl}\hat u(x,k)$ and $\hat u(x,k)$ are both in L^2. Since the Floquet-Bloch transform is an isometry in L^2, we can also conclude that, vice versa, $\hat u(x,k)$ has a curl in L^2 when $\operatorname{curl}(e^{ik\cdot x}\hat u(x,k))$ and $\hat u(x,k)$ are both in L^2. This observation is also used in the analysis of $\mathcal{F}_{\{1,\ldots,n-1\}}$ and $\mathcal{F}_{\{n\}}$.

The only other assumption that requires additional arguments is Item 5 of Assumption 2.9. The fact that M_k is self-adjoint follows from its formula in (4.8). That $M_k + E_k$ is uniformly coercive follows from

$$\langle (M_k+E_k)u,u\rangle_{Y'_{\text{per}},Y_{\text{per}}} \geq \int_{W^n}\mu^{-1}\ \operatorname{curl}(u(x)e^{ik\cdot x})\cdot\operatorname{curl}(\bar u(x)e^{-ik\cdot x})\,dx$$
$$+\int_{W^n}\varepsilon\,|u(x)e^{ik\cdot x}|^2\,dx \geq \min(\mu^{-1},\varepsilon)\|u(x)e^{ik\cdot x}\|^2_{H(\operatorname{curl})} \geq c\|u\|^2_{Y_{\text{per}}}\,.$$

The fact that the mapping $k\mapsto M_k$ is continuously differentiable from I^n into $\mathcal{L}(Y_{\text{per}},Y'_{\text{per}})$ follows also from (4.8).

It remains to show that the operators $M_k - \lambda E_k : Y_{\rm per} \to Y'_{\rm per}$ are Fredholm operators with index 0 for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$. This property is verified for real and positive λ in Lemma B.1 of [13], the argument can easily be extended to $0 \neq \lambda \in \mathbb{C}$. We note that, in [13], the roles of ε and μ are exchanged and the parameter λ is denoted as ω^2. Another technical difference is that, while we let the operator curl act on $u(x)e^{ik\cdot x}$, in [13], u is in the space of k-quasiperiodic functions (and α is used instead of k as the parameter for quasi-periodicity).

The eigenvalues have no accumulation point except for (possibly) $\lambda = 0$. This is a consequence of the proof of Lemma B.1 of [13]. The underlying space can be decomposed with a Helmholtz decomposition. On the space of divergence-free functions, the Maxwell operator is a compact perturbation of the identity. On the space of gradients, the Maxwell operator is multiplication with $-\omega^2$. This shows that 0 is the only possible accumulation point of eigenvalues. □

Theorem 3.2 yields the subsequent spectral gap result for the Maxwell system.

Theorem 4.5 (Application to the perturbed Maxwell system). *Let $\mu, \varepsilon \in L^\infty(\mathbb{R}^n)$ and $\varepsilon_1 \in L^\infty(\mathbb{R}^{n-1} \times W)$ satisfy the assumptions of Definition 2.2. Let $(a_-, a_+) \subset (0,\infty)$ with mid-point a_0 be a spectral gap of the operator family*

$$u \ \mapsto \ \operatorname{curl}(\mu^{-1} \operatorname{curl} u) - \lambda\, \varepsilon\, u\,.$$

Then there exists $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0]$, the interval (a_0, a_+) is not contained in a spectral gap of the operator family

$$u \ \mapsto \ \operatorname{curl}(\mu^{-1} \operatorname{curl} u) - \lambda\, (\varepsilon + \delta\varepsilon_1)\, u\,.$$

Appendix A. Lipschitz dependence of eigenvalues

In the proof our main result, we use a well-known property of parameter dependent eigenvalue problems: The Lipschitz dependence of eigenvalues, formulated in a classical form in Theorem A.1 below. The Lipschitz dependence of eigenvalues is well-known and frequently used, but we did not find a reference with full proof in the required setting, namely that of Theorem A.4 for Fredholm operators. For the convenience of the reader, we provide all proofs. The ideas are taken from [12].

A.1. **Classical eigenvalue problem.** We are interested in a family of operators, indexed with a real parameter κ. We always assume that, with real numbers $\hat\kappa, \eta_0 \in \mathbb{R}$, $\eta_0 > 0$, the parameter ranges in a real interval, $\kappa \in I_0 := (\hat\kappa - \eta_0, \hat\kappa + \eta_0)$. We consider the problem for $\kappa = \hat\kappa$ as the unperturbed problem.

For a linear operator $A : X \to X$, we write $\sigma(A)$ for the spectrum of A. Let us assume that, for every $0 \neq \lambda \in \mathbb{C}$, the operator $A - \lambda\,\mathrm{id}$ is a Fredholm operator with index 0. Under this assumption, the set $\sigma(A) \setminus \{0\} \subset \mathbb{R}$ consists only of eigenvalues. Furthermore, for every eigenvalue, algebraic and geometric multiplicity coincide.

Theorem A.1 (Lipschitz dependence of eigenvalues). *Let X be a Hilbert space over $\mathbb{C}$. For an interval $I_0 = (\hat\kappa - \eta_0, \hat\kappa + \eta_0)$, let $A \in C^1(I_0, \mathcal{L}(X))$ be a family of operators. For every $\kappa \in I_0$ we assume that (i) $A(\kappa) : X \to X$ is self-adjoint and (ii) for every $0 \neq \lambda \in \mathbb{C}$, the operator $A(\kappa) - \lambda\mathrm{id}$ is a Fredholm operator with index 0.*

Let $\hat\lambda \neq 0$ be an isolated eigenvalue of $A(\hat\kappa)$ and let $m := \dim \mathcal{N}(A(\hat\kappa) - \hat\lambda\,\mathrm{id}) < \infty$ be the dimension of the corresponding eigenspace. Then there exists a parameter $0 < \eta \leq \eta_0$ such that, on the possibly reduced interval $I := (\hat\kappa - \eta, \hat\kappa + \eta)$, there exist m Lipschitz continuous functions $\lambda_j : I \to \mathbb{R}$, $j = 1, \ldots, m$, such that, for every $j \leq m$, $\lambda_j(\hat\kappa) = \hat\lambda$ and, for every $\kappa \in I$, the real number $\lambda_j(\kappa)$ is an eigenvalue of

$A(\kappa)$. In addition, the functions λ_j provide a continuation of the eigenspace in the following sense: The linear subspace

$$\mathcal{M}(\kappa) := \bigoplus_{j=1}^{m} \mathcal{N}\big(A(\kappa) - \lambda_j(\kappa)\,\mathrm{id}\big)$$

satisfies $\mathcal{M}(\hat\kappa) = \mathcal{N}(A(\hat\kappa) - \hat\lambda\,\mathrm{id})$, for every $\kappa \in I$ holds $\dim(\mathcal{M}(\kappa)) = m$, and the orthogonal projection $P(\kappa)$ from X onto $\mathcal{M}(\kappa) \subset X$ is differentiable as a map $P : I \ni \kappa \mapsto P(\kappa) \in \mathcal{L}(X)$.

Remark: A well know counterexample by Rellich, presented as Example II, 5.3 in [12], shows that the eigenvectors $\phi_j(\kappa)$ corresponding to the eigenvalues $\lambda_j(\kappa)$ are not necessarily continuous in κ. It is therefore important to study the projection $P(\kappa)$ to the entire continuation of the eigenspace, $\mathcal{M}(\kappa)$. We also note that Example II, 5.9 of [12] shows that the eigenvalues λ_j are not necessarily continuously differentiable in κ. In this sense, the Lipschitz dependence of the eigenvalues is optimal.

The proof of the theorem is based on two lemmas. The first lemma provides a formula for the projection onto eigenspaces. If we think of the above situation with a parameter κ, we might say: The subsequent lemma is for a fixed point $\kappa \in I$.

Regarding notation: We denote eigenvalues in the theorem by λ_j, it is possible (and it might be necessary) that eigenvalues are repeated, $\lambda_j = \lambda_i$ for $j \neq i$. In the subsequent lemma, eigenvalues are denoted by μ_j, every eigenvalue appears only once.

Lemma A.2 (Formula for projections)**.** *Let X be a Hilbert space over $\mathbb{C}$ and let $A \in \mathcal{L}(X)$ be self-adjoint with the property that $A - \lambda\,\mathrm{id}$ is Fredholm with index 0 for every $0 \neq \lambda \in \mathbb{C}$. Let $D \subset \mathbb{C}$ be an open disc with $0 \notin \bar{D}$, we regard the boundary $\Gamma = \partial D$ as a positively oriented simple curve in $\mathbb{C}$. We assume that there are no spectral values on the boundary, $\partial D \cap \sigma(A) = \emptyset$, and that D contains finitely many eigenvalues of A, denoted as $\mu_1, \dots, \mu_n$. In this situation, the orthogonal projection $P \in \mathcal{L}(X)$ onto the subspace $\mathcal{M} := \bigoplus_{j=1}^{n} \mathcal{N}(A - \mu_j\,\mathrm{id})$ coincides with a complex integral: For arbitrary $u \in X$ holds*

$$Pu \;=\; \frac{1}{2\pi i}\int_\Gamma (A - \lambda\,\mathrm{id})^{-1} u\, d\lambda\,. \tag{A.1}$$

Proof. Step 1: Preparations. For every eigenvalue μ_j, let $\psi_{\ell,j}$ with index $\ell = 1, \dots, m_j$ be the eigenvectors of A corresponding to μ_j. We may assume that these eigenvectors are normalized such that $\{\psi_{\ell,j} \mid \ell = 1, \dots, m_j,\ j = 1, \dots, n\}$ is an orthonormal basis of $\mathcal{M}$. By its definition, the space $\mathcal{M}$ is invariant under A in the sense that $A(\mathcal{M}) \subset \mathcal{M}$. Since A is self-adjoint, this implies that also the orthogonal complement $\mathcal{M}^\perp$ is invariant under A.

For $u \in X$, the projection onto $\mathcal{M}$ is

$$Pu = \sum_{j=1}^{n} \sum_{\ell=1}^{m_j} \langle u, \psi_{\ell,j}\rangle_X\ \psi_{\ell,j}\,.$$

The application of $(A - \lambda\,\mathrm{id})^{-1}$ can be written, with $u^\perp = (\mathrm{id} - P)u \in \mathcal{M}^\perp$, as

$$(A - \lambda\,\mathrm{id})^{-1} u = (A - \lambda\,\mathrm{id})^{-1} u^\perp \;+\; \sum_{j=1}^{n} \frac{1}{\mu_j - \lambda} \sum_{\ell=1}^{m_j} \langle u, \psi_{\ell,j}\rangle_X\ \psi_{\ell,j}\,.$$

In order to obtain (A.1), we must establish a relation of these two quantities.

Step 2: Application of the Cauchy integral formula. Let $u, v \in X$ be arbitrary and let $u^\perp$ be defined as above; these vectors are kept fixed until the end of the proof. For every $\lambda \in D \setminus \{\mu_1, \dots, \mu_n\}$, we consider the expression

$$\begin{aligned} f(\lambda) &:= \big\langle (A - \lambda\,\mathrm{id})^{-1} u, v \big\rangle_X \\ &= \big\langle (A - \lambda\,\mathrm{id})^{-1} u^\perp, v \big\rangle_X + \sum_{j=1}^{n} \frac{1}{\mu_j - \lambda} \sum_{\ell=1}^{m_j} \langle u, \psi_{\ell,j} \rangle_X \, \langle \psi_{\ell,j}, v \rangle_X \,. \end{aligned}$$

We begin with an analysis of the first term, $g : \lambda \mapsto \big\langle (A - \lambda\,\mathrm{id})^{-1} u^\perp, v \big\rangle_X$. This function can actually be extended as a continuous function to all of D and also to $\bar{D}$. This follows from the fact that $A - \lambda\,\mathrm{id} : \mathcal{M}^\perp \to \mathcal{M}^\perp$ is an invertible operator for every $\lambda \in \bar{D}$ (it is a Fredholm operator with index 0, restricted to the complement of the kernel, and it is considered as a map to its image). By the analytic dependence on λ, the function g is holomorphic in $\lambda \in D$.

As a consequence, f is a meromorphic function in D with poles in the eigenvalues μ_j, $j = 1, \dots, n$, the corresponding residua are $\sum_{\ell=1}^{m_j} \langle u, \psi_{\ell,j} \rangle_X \, \langle \psi_{\ell,j}, v \rangle_X$. The residual theorem yields

$$\begin{aligned} \frac{1}{2\pi i} \int_\Gamma \big\langle (A - \lambda\,\mathrm{id})^{-1} u, v \big\rangle_X \, d\lambda &= \frac{1}{2\pi i} \int_\Gamma f(\lambda) \, d\lambda \\ = \sum_{j=1}^{n} \sum_{\ell=1}^{m_j} \langle u, \psi_{\ell,j} \rangle_X \, \langle \psi_{\ell,j}, v \rangle_X &= \big\langle P u, v \big\rangle_X \,. \end{aligned}$$

Since this holds for all $v \in X$, the characterization (A.1) is shown. □

The next lemma provides, for a differentiable family of projections, a differentiable family of coordinates for the images.

Lemma A.3 (Coordinates for a family of projections)**.** *Let X be a complex Hilbert space and let $I = (\hat{\kappa} - \eta, \hat{\kappa} + \eta)$ be a real interval. Let $P \in C^1(I, \mathcal{L}(X))$ be a family of operators such that $P(\kappa) : X \to X$ is an orthogonal projection for every $\kappa \in I$.*

Then there exists a differentiable family $I \ni \kappa \mapsto U(\kappa) \in \mathcal{L}(X)$ such that $U(\hat{\kappa}) =$ id and, for every $\kappa \in I$, $U(\kappa)$ is a unitary operator that provides coordinates for the subspace $P(\kappa)X$ in the sense that

$$U(\kappa) P(\hat{\kappa}) U(\kappa)^* = P(\kappa)\,. \tag{A.2}$$

Proof. Step 1: Preparation. We prepare the proof by collecting consequences of the fact that every $P(\kappa)$ is a projection. From $P(\kappa) \circ P(\kappa) = P(\kappa)$ we obtain for the derivative $P'(\kappa) = \partial_\kappa P(\kappa)$, suppressing the argument, $P'P + PP' = P'$ and, as a consequence, $PP'P = 0$. For $Q := P'P - PP'$ we therefore obtain

$$PQ = -PP'\,, \quad QP = P'P\,, \quad QP - PQ = P'\,. \tag{A.3}$$

The orthogonality of P implies, for all elements $u, v \in X$, the identity $\langle u, Pv \rangle = \langle Pu, Pv \rangle = \langle Pu, v \rangle$ and hence $P^* = P$. Differentiating the identity also implies $(P')^* = P'$ and, in turn, $Q^* = -Q$.

Step 2: Unitary operators as solutions of an ODE. We use an ordinary differential equation (ODE) to find $U \in C^1(I, \mathcal{L}(X))$, namely the linear initial value problem

$$U'(\kappa) = Q(\kappa)\, U(\kappa) \; \forall \kappa \in I\,, \qquad U(\hat{\kappa}) = \mathrm{id}\,. \tag{A.4}$$

We claim that the solution is a family of unitary operators. To verify this claim, we study the operator family $V := U^*$. By forming the dual of (A.4) and exploiting

$Q(\kappa)^* = -Q(\kappa)$, we find that V solves the ODE

$$V'(\kappa) \; = \; -V(\kappa)\, Q(\kappa) \; \forall \kappa \in I\,, \qquad V(\hat\kappa) \; = \; \mathrm{id}\,. \tag{A.5}$$

The product rule yields $(VU)' = V'U + VU' = -VQU + VQU = 0$ and thus $V(\kappa)U(\kappa) = \mathrm{id}$ for all κ.

The expression UV satisfies the initial condition $(UV)(\hat\kappa) = \mathrm{id}$ and the differential equation $(UV)' = U'V + UV' = Q(UV) - (UV)Q$. The same linear differential equation is solved by the constant function id. Uniqueness of the ODE implies $U(\kappa)V(\kappa) = \mathrm{id}$ for all κ. We conclude that U is invertible with $U^* = U^{-1}$, and hence the claim.

Step 3: Property regarding projections. It remains to show (A.2). We compute

$$(PU)' \; = \; P'U + PU' \; = \; (P' + PQ)U \; = \; Q(PU)\,.$$

This shows that $P(\kappa)U(\kappa)$ satisfies the same ODE (namely $Y' = QY$) as $U(\kappa)P(\hat\kappa)$, compare (A.4). Both functions have the initial condition $P(\hat\kappa)$. We conclude that the two functions coincide, $P(\kappa)U(\kappa) = U(\kappa)P(\hat\kappa)$ for all κ. This provides (A.2). □

We can now prove the theorem on the continuation of eigenvalues.

Proof of Theorem A.1. We are given an eigenvalue $\hat\lambda \in \mathbb{C} \setminus \{0\}$ of $A(\hat\kappa)$ with multiplicity m. Our aim is to show that the eigenvalue can be continued with m functions $\lambda_j : I \to \mathbb{R}$.

Step 1: Construction of the projections. We choose a small circle $D = B_\varepsilon(\hat\lambda) \subset \mathbb{C}$ with positively oriented boundary curve $\Gamma = \partial D$ such that $0 \notin \bar D$ and $\hat\lambda$ is the only eigenvalue of $A(\hat\kappa)$ in $\bar D$.

By our choice of Γ, the operator $A(\hat\kappa) - \lambda\,\mathrm{id}$ is invertible for all $\lambda \in \Gamma$. This fact, together with the compactness of Γ, allows to find $\eta > 0$ such that $A(\kappa) - \lambda\,\mathrm{id}$ is invertible for all $\lambda \in \Gamma$ and all $\kappa \in I = (\hat\kappa - \eta, \hat\kappa + \eta)$. This implies that $P(\kappa) := \frac{1}{2\pi i}\int_\Gamma (A(\kappa) - \lambda\,\mathrm{id})^{-1}\, d\lambda$ exists for all $\kappa \in I$ and that P is differentiable with respect to κ.

For every $\kappa \in I$, we denote the eigenvalues of $A(\kappa)$ in the disc D by $\mu_\ell(\kappa)$, $\ell = 1, \dots, n$ for some $n = n(\kappa) \le m$. Lemma A.2 implies that this $P(\kappa)$ is the orthogonal projection onto the finite sum of eigenspaces $\mathcal{M}(\kappa) := \bigoplus_{\ell=1}^n \mathcal{N}(A(\kappa) - \mu_\ell(\kappa)\,\mathrm{id})$.

Step 2: Coordinates. We can now apply Lemma A.3 to the family of projections $P(\kappa)$. The equation $U(\kappa)P(\hat\kappa)U(\kappa)^* = P(\kappa)$ shows that $U(\kappa)$ maps $\mathcal{M}(\hat\kappa)$ into $\mathcal{M}(\kappa)$ and $U(\kappa)^* = U(\kappa)^{-1}$ maps $\mathcal{M}(\kappa)$ into $\mathcal{M}(\hat\kappa)$, i.e. $U(\kappa)$ is an isomorphism from $\mathcal{M}(\hat\kappa)$ onto $\mathcal{M}(\kappa)$.

A consequence of the fact that the subspaces are isomorphic is that the dimension is constant. We recall that $\mathcal{M}(\hat\kappa) = \mathcal{N}(A(\hat\kappa) - \hat\lambda \mathrm{id})$ and hence $\dim \mathcal{M}(\hat\kappa) = m$. This implies $\dim \mathcal{M}(\kappa) = m$ for all κ.

The operator $B(\kappa) := U(\kappa)^* A(\kappa) U(\kappa)|_{\mathcal{M}(\hat\kappa)}$ is an operator on the finite dimensional space $Z := \mathcal{M}(\hat\kappa)$, we constructed a family of self-adjoint operators $B(\kappa) \in \mathcal{L}(Z)$ for $\kappa \in I$. If λ is an eigenvalue of $A(\kappa)$ in D with eigenvector $\phi \in \mathcal{M}(\kappa)$, then λ is an eigenvalue of $B(\kappa)$ with eigenfunction $U(\kappa)^*\phi \in Z$. Conversely, if λ is an eigenvalue of $B(\kappa)$ with eigenvector $\psi \in Z$, then λ is an eigenvalue of $A(\kappa)$, with $\lambda \in D$, with eigenvector $U(\kappa)\psi \in \mathcal{M}(\kappa)$. It is therefore sufficient to consider the eigenvalues of $B(\kappa) : Z \to Z$ inside D and their dependence on the parameter κ.

Step 3: Lipschitz continuity. It remains to consider the following situation: Let Z be a finite dimensional complex Hilbert space and let, for every $\kappa \in I$, the operator $B(\kappa) : Z \to Z$ be self-adjoint, the map $B : I \to \mathcal{L}(Z)$ being differentiable.

We consider the ordered eigenvalues $\lambda_j(\kappa)$ with $\lambda_1(\kappa) \le \ldots \le \lambda_m(\kappa)$, repeating multiple eigenvalues according to their multiplicity. Courant's min-max principle characterizes the eigenvalues for $j = 1, \ldots, m$ as

$$\lambda_j(\kappa) \;=\; \max_{\operatorname{codim} M = j-1} \; \min_{0 \ne \psi \in M} \frac{\langle B(\kappa)\psi, \psi\rangle_X}{\|\psi\|_X^2}\,.$$

From this formula, we can conclude the Lipschitz continuity of the eigenvalues. Indeed, from the differentiability of $B(\kappa)$ we have $\|B(\kappa_1) - B(\kappa_2)\| \le C_L\,|\kappa_1 - \kappa_2|$, and therefore $\langle B(\kappa_1)\psi, \psi\rangle_X \le \langle B(\kappa_2)\psi, \psi\rangle_X + C_L|\kappa_1 - \kappa_2|\|\psi\|_X^2$. Taking, for a fixed subspace M, a minimum, we find

$$\min_{0 \ne \psi \in M} \frac{\langle B(\kappa_1)\psi, \psi\rangle_X}{\|\psi\|_X^2} \;\le\; \min_{0 \ne \psi \in M} \frac{\langle B(\kappa_2)\psi, \psi\rangle_X}{\|\psi\|_X^2} \;+\; C_L|\kappa_1 - \kappa_2|\,.$$

Choosing M as the subspace that maximizes the left hand side, we obtain $\lambda_j(\kappa_1) \le \lambda_j(\kappa_2) + c|\kappa_1 - \kappa_2|$. Reversing the roles of κ_1 and κ_2 implies Lipschitz continuity of the eigenvalues. This completes the proof. □

A.2. **A spectral result in a generalized setting.** We want to generalize our results to the following situation: Let $Y \subset X \subset Y'$ be a Gelfand triple, $M \in \mathcal{L}(Y, Y')$ and $E \in \mathcal{L}(X)$ both self-adjoint, E coercive. With the embeddings of the Gelfand triple, E can also be regarded as an operator $E \in \mathcal{L}(Y, Y')$ using the relation $\langle Eu, v\rangle_{Y',Y} = \langle E\iota u, \iota v\rangle_X$ for all $u, v \in Y$. With this interpretation, it makes sense to study the eigenvalue problem for $\psi \in Y$ and $\lambda \in \mathbb{R}$,

$$M\psi = \lambda\, E\psi\,. \tag{A.6}$$

Our result is that the dependence of eigenvalues λ on a parameter κ can be described as in Theorem A.1. To prove this result, we formulate an equivalent problem to which we apply Theorem A.1.

A central point of the proof is the choice of the scalar product on X and the corresponding interpretation of the operator $E \in \mathcal{L}(Y, Y')$. Since E is self-adjoint and coercive, we can use on X the expression $\langle u, v\rangle_E = \langle Eu, v\rangle_X$ as scalar product and equivalent norm. Choosing this scalar product means that we introduce also a new embedding $\iota^* : X \to Y'$: We want that ι^* is the adjoint to $\iota : Y \to X$ with respect to $\langle\cdot,\cdot\rangle_E$, hence, for $u \in X$ and $v \in Y$, it must satisfy the relation with the exclamation mark,

$$\langle \iota^* u, v\rangle_{Y',Y} \overset{(!)}{=} \langle u, \iota v\rangle_E \;=\; \langle Eu, \iota v\rangle_X\,. \tag{A.7}$$

This implies a relation for the operator $E \in \mathcal{L}(Y, Y')$, which is defined by the first equality, for $u, v \in Y$,

$$\langle Eu, v\rangle_{Y',Y} \;=\; \langle E\iota u, \iota v\rangle_X \;=\; \langle \iota u, \iota v\rangle_E \;=\; \langle \iota^*\iota u, v\rangle_{Y',Y}\,. \tag{A.8}$$

This shows that, with ι^* defined with the E-scalar product on X, the operator $E \in \mathcal{L}(Y, Y')$ is given as

$$E = \iota^*\iota\,. \tag{A.9}$$

From now on, we will use only the E-scalar product on X and exploit $E = \iota^*\iota$.

We will demand coercivity of $M + E : Y \to Y'$, which is defined as follows: For some $c > 0$, there holds $\langle (M + E)v, v\rangle_{Y',Y} \ge c\|v\|_Y^2$ for every $v \in Y$.

Theorem A.4 (Lipschitz dependence of eigenvalues in the generalized case)**.** *Let X be a complex Hilbert space, let $Y \subset X$ be an embedded Banach space and let Y' be the dual space of Y. For an interval $I_0 = (\hat\kappa - \eta_0, \hat\kappa + \eta_0)$, let $M \in C^1(I_0, \mathcal{L}(Y, Y'))$*

and $E \in C^1(I_0, \mathcal{L}(X))$ be two families of operators. For every $\kappa \in I_0$ we assume that (i) $M(\kappa)$ and $E(\kappa)$ are both self-adjoint and (ii) $E(\kappa)$ and $M(\kappa) + E(\kappa)$ are both coercive (both uniformly with respect to κ) and (iii) for every $0 \neq \lambda \in \mathbb{C}$, the operator $M(\kappa) - \lambda E$ is a Fredholm operator with index 0.

Let $\hat\lambda \neq 0$ be an isolated eigenvalue of $M(\hat\kappa)$ with mass matrix $E(\hat\kappa)$ and let $m := \dim \mathcal{N}\big(M(\hat\kappa) - \hat\lambda E(\hat\kappa)\big) < \infty$ be the dimension of the corresponding eigenspace. Then there exists a parameter $0 < \eta \leq \eta_0$ such that, on the possibly reduced interval $I := (\hat\kappa - \eta, \hat\kappa + \eta)$, there exist m Lipschitz continuous functions $\lambda_j : I \to \mathbb{R}$, $j = 1, \ldots, m$, such that, for every $j \leq m$, $\lambda_j(\hat\kappa) = \hat\lambda$ and, for every $\kappa \in I$, the real number $\lambda_j(\kappa)$ is an eigenvalue of $M(\kappa)$ with mass-operator $E(\kappa)$. In addition, the functions λ_j provide a continuation of the eigenspace in the following sense: The linear subspace

$$\mathcal{M}(\kappa) := \bigoplus_{j=1}^{m} \mathcal{N}\big(M(\kappa) - \lambda_j(\kappa)E(\kappa)\big)$$

satisfies $\mathcal{M}(\hat\kappa) = \mathcal{N}(A(\hat\kappa) - \hat\lambda\,\mathrm{id})$, for every $\kappa \in I$ holds $\dim(\mathcal{M}(\kappa)) = m$, and the orthogonal projection $P(\kappa)$ from X onto $\mathcal{M}(\kappa) \subset X$ is differentiable as a map $P : I \ni \kappa \mapsto P(\kappa) \in \mathcal{L}(X)$.

Proof. We consider the following family of operators:

$$A(\kappa) \;:=\; \iota\,[M(\kappa) + E(\kappa)]^{-1}\,\iota^* \;:\; X \to X\,.$$

We recall that, as a map $Y \to Y'$, there holds $E = \iota^*\iota$. The family of operators is well-defined since $M(\kappa) + E(\kappa)$ is coercive, there holds $A \in C^1(I_0, \mathcal{L}(X))$, since M, E and ι^* depend differentiably on κ. Furthermore, since M and E are, also A is self-adjoint for every κ.

Step 1: Equivalence of the eigenvalue problems. We consider the self-adjoint eigenvalue problem $A\psi = \rho\psi$, that is,

$$\iota\,[M(\kappa) + \iota^*\iota]^{-1}\iota^*\psi(\kappa) \;=\; \rho(\kappa)\psi(\kappa)\,. \tag{A.10}$$

for $\rho(\kappa) \in \mathbb{R}$ and $\psi(\kappa) \in X$. We want to show that the problems (A.6) and (A.10) are equivalent. In the corresponding calculations, we drop the argument κ for simplicity.

Let $\psi \in X$ be an eigenfunction for (A.10) with eigenvalue $\rho \neq 0$. We define $\mu := \rho^{-1} - 1 \in \mathbb{R}$ and $\phi := [M + \iota^*\iota]^{-1}\iota^*\psi \in Y$. Relation (A.10) and the definition of ϕ imply $\iota\phi = \rho\psi$ and $\iota^*\psi = [M + \iota^*\iota]\phi$. We obtain, for arbitrary $\varphi \in Y$,

$$\begin{aligned}\langle \iota^*\iota\phi, \varphi\rangle_{Y',Y} \;&=\; \langle \iota\phi, \iota\varphi\rangle_E \;=\; \rho\langle \psi, \iota\varphi\rangle_E \;=\; \rho\langle \iota^*\psi, \varphi\rangle_{Y',Y}\\ &=\; \rho\langle [M + \iota^*\iota]\phi, \varphi\rangle_{Y',Y}\,.\end{aligned}$$

This provides $M\phi = \mu\,\iota^*\iota\phi = \mu E\phi$ and we see that μ is an eigenvalue of (A.6) with eigenvector ϕ.

Conversely, let μ and ϕ satisfy $M\phi = \mu\iota^*\iota\phi$. We set $\rho := (\mu+1)^{-1}$ and $\psi := \rho^{-1}\iota\phi$. The choice of ρ implies $\rho(M + \iota^*\iota)\phi = \iota^*\iota\phi$. We conclude $\phi = (M + \iota^*\iota)^{-1}\iota^*\psi$ and thus $\rho\psi = \iota\phi = \iota(M + \iota^*\iota)^{-1}\iota^*\psi$. This shows that ρ is an eigenvalue of (A.10) with eigenvector ψ.

We have obtained that the problems (A.6) and (A.10) are equivalent.

Step 2: Fredholm property of the family A. We can apply Theorem A.1 to the family $A = A(\kappa)$ when we verify, for every κ and every $\rho \neq 0$, that the operator $A(\kappa) - \rho\,\mathrm{id}$ is a Fredholm operator with index 0. From Step 1 we conclude that the kernel of $A(\kappa) - \rho\,\mathrm{id}$ is finite dimensional; the kernel is the complement of the range since $A(\kappa)$ is self-adjoint.

Therefore, we can apply Theorem A.1 which yields the existence of m Lipschitz continuous functions $\kappa \mapsto \rho_j(\kappa)$ such that $\rho_j(\kappa)$ are eigenvalues of $A(\kappa)$ and $\rho_j(\hat{\kappa}) = \hat{\rho} := (1+\hat{\mu})^{-1}$. Then $\mu_j(\kappa) := \rho_j(\kappa)^{-1} - 1$ yields the Lipschitz continuation of the eigenvalues.

Step 3: Projections. It remains to look at the eigenprojection. Let $\{\phi_{\ell,j}(\kappa) \mid \ell = 1, \dots, m_j\}$ be an orthonormal (with respect to $\langle \cdot, \cdot \rangle_{E(\kappa)}$) basis of $\mathcal{N}\big(M(\kappa) - \mu_j(\kappa)E(\kappa)\big)$. We have seen above that $\psi_{\ell,j}(\kappa) := (\mu_j(\kappa)+1)\iota\phi_{\ell,j}(\kappa) \in X$ is a basis of $\bigoplus_{j=1}^m \mathcal{N}\big(A(\kappa) - \rho_j(\kappa)\mathrm{id}\big)$. This shows $\mathcal{M}(\kappa) = \bigoplus_{j=1}^m \mathcal{N}\big(A(\kappa) - \rho_j(\kappa)\mathrm{id}\big)$. By the second part of Theorem A.1 the orthogonal projection from X onto $\bigoplus_{j=1}^m \mathcal{N}\big(A(\kappa) - \rho_j(\kappa)\mathrm{id}\big) = \mathcal{M}(\kappa)$ is differentiable as a mapping from $(\hat{\kappa} - \eta, \hat{\kappa} + \eta)$ into $\mathcal{L}(X)$. ☐

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