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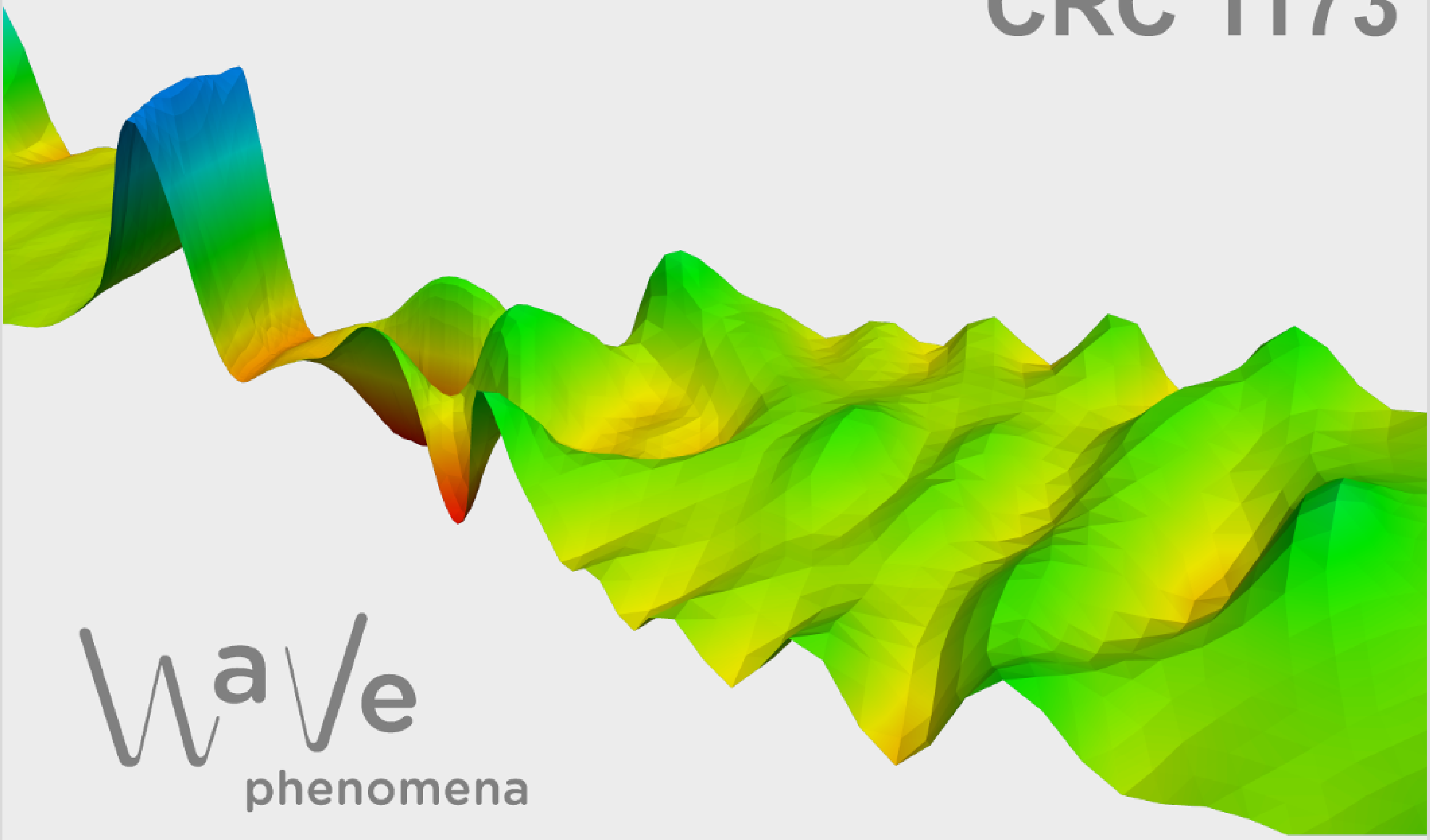

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Global solutions to nonconservative NLS with non-decaying initial data

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Abstract

We investigate the long-time behavior of solutions to nonconservative nonlinear Schrödinger equations (NLS) of the form

$$i\partial_t u = -\partial_x^2 u - u^p, \quad p \in \mathbb{N},\, p \geq 2.$$

We focus on initial data that are neither localized nor periodic. Our approach is based on the work [13] by Jaquette, Lessard and Takayasu, where they used a perturbative analysis around the explicit spatially homogeneous solution of the associated ODE. We establish global existence results and asymptotic decay of solutions in a general Banach algebra setting. Applications include small data global well-posedness results for almost periodic initial data, which seem to be the first in a nonconservative NLS framework. Applications to (almost) periodic initial data with localized perturbations are also presented.
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1 Introduction

The nonlinear Schrödinger equation (NLS) is a fundamental model in mathematical physics that describes the evolution of complex wave functions under the influence of dispersion and nonlinearity. Traditionally studied in conservative settings, the NLS has seen wide applications ranging from optics and quantum mechanics to fluid dynamics (see [1],[10],[20],[24],[23]). In this work, we consider a nonconservative variant of the NLS, given by

$$\begin{cases} i\partial_t u = -\partial_x^2 u - u^p, & t \geq 0,\, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $p \geq 2$ is an integer. The question of global existence for such nonlinear Schrödinger equations has been studied extensively in the past decades. Classical approaches are based on energy estimates and dispersive properties of the Schrödinger semigroup. Pioneering contributions were given by Strauss [22], Klainerman and Ponce [15], and Shatah [21], where global existence was proven for quadratic nonlinearities in dimensions $d \geq 4$, and in dimension $d = 3$ for nonlinearities of order strictly greater than two. In [8] and [9], Germain, Masmoudi and Shatah considered Nonlinear Schrödinger Equations with quadratic nonlinearities of the form $\alpha u^2 + \beta \bar{u}^2$ in dimensions $d = 2, 3$. They proved global well-posedness for small and localized initial data using the concept of space-time resonances. For further results in this setting we refer to [12], [11], and [14].
The main goal of this paper is to investigate the well-posedness and long-time behavior of solutions to the nonconservative NLS (1.1) in function spaces beyond the usual setting of L^2-based Sobolev spaces. In particular, we are interested in solutions of (1.1) with initial data that are neither decaying nor periodic. To give some motivation for this topic, we refer the reader to [17], [2], [3]. Our framework mainly exploits the boundedness properties of the Schrödinger semigroup in function spaces that are Banach algebras for pointwise multiplication. We use a perturbative strategy around the explicit spatially homogeneous solution of the associated x-independent ODE

$$z' = iz^p, \quad t \geq 0. \tag{1.2}$$

In other words, we consider solutions of (1.1) that remain close to the solution z of (1.2). For the corresponding initial value we then obtain a sort of small data global existence result. We derive smallness conditions under which the full PDE remains globally well-posed and its solutions decay asymptotically to zero. Our calculations are strongly based on the work [13] of Jaquette, Lessard and Takayasu. They studied the global dynamics of the NLS on the d-dimensional torus with more general nonconservative nonlinearities. Their approach allows us to extend the global existence theory to a variety of initial data classes that are neither periodic nor localized. We first give a more abstract version of our result. Then we can apply our abstract result in particular to almost periodic functions as well as their localized perturbations. To the best of our knowledge, this is the first global existence result for almost periodic initial data in a nonconservative setting. Even in the conservative case, there are only a few results so far. In [6], Boutet and Egorova considered the defocusing (conservative) NLS

$$\begin{cases} i\partial_t u = -\partial_x^2 u + |u|^{2k} u, & t \geq 0,\, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

They showed global well-posedness of (1.3) in the cubic case $k = 1$ with limit periodic initial data using the complete integrability of the cubic NLS. Twenty years later, Oh showed global well-posedness of (1.3) for arbitrary $k \in \mathbb{N}$ with

limit periodic initial data [18]. Also in the context of other complete integrable PDEs like the KdV equation there are just few results on global existence with almost periodic initial data, e.g. [7],[5].
The paper is organized as follows: In Section 2 we present our general result on global well-posedness of (1.1). In Section 3 we give a short introduction to almost periodic functions and apply the general result in this context. Finally, in Section 4, we give some more applications in the setting of periodic initial data with localized perturbation.

Notation: $A \lesssim_{\epsilon_1,...,\epsilon_n} B$ denotes the inequality $A \leq C(\epsilon_1, ..., \epsilon_n)B$ for a constant C depending on $\epsilon_1, ..., \epsilon_n$.

2 General theory

In the following, let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach space of complex-valued functions $f : \mathbb{R} \to \mathbb{C}$. We assume that $\mathcal{A}$ is a Banach algebra for pointwise multiplication and that $\mathcal{A}$ contains the constant functions. Furthermore, we assume that the Schrödinger semigroup $S(t) = e^{it\partial_x^2}$, $t \geq 0$, is strongly continuous and bounded on $\mathcal{A}$. We consider the Nonlinear Schrödinger Equation (NLS)

$$\begin{cases} i\partial_t u = -\partial_x^2 u - u^p, & t \geq 0,\, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

for an integer $p \geq 2$. By our assumptions there exists an $M \geq 1$ such that

$$\|S(t)f\|_{\mathcal{A}} \leq M\|f\|_{\mathcal{A}}$$

for all $f \in \mathcal{A}$ and $t \geq 0$. Furthermore we have a constant $C_{\mathcal{A}} > 0$ with

$$\|fg\|_{\mathcal{A}} \leq C_{\mathcal{A}}\|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}}$$

for all $f, g \in \mathcal{A}$.
With these estimates at hand and a standard fixed point argument, one can easily prove local well-posedness of (1.1) with initial data in $\mathcal{A}$.

Theorem 2.1. *Let $p \geq 2$ and $u_0 \in \mathcal{A}$. Then there exists a positive time $T > 0$ such that* (1.1) *has a unique mild solution $u \in C([0,T], \mathcal{A})$ with initial data u_0:*

$$u(t) = S(t)u_0 + i\int_0^t S(t-s)u(s)^p \mathrm{d}s, \quad t \in [0,T]. \tag{2.1}$$

The guaranteed time of existence T depends only on the norm $\|u_0\|_{\mathcal{A}}$. More precisely,

$$T \gtrsim_{M,C_{\mathcal{A}},p} \|u_0\|_{\mathcal{A}}^{1-p}.$$

By standard arguments one gets the following blow-up alternative.
Let $T^* = T^*(u_0) > 0$ be the maximal time of existence for a mild solution $u \in C([0,T^*),\mathcal{A})$ of (1.1) with initial data $u_0 \in \mathcal{A}$. Then either $T^* < \infty$ and

$$\limsup_{t\nearrow T^*} \|u(t)\|_{\mathcal{A}} = \infty,$$

or $T^* = \infty$.
To show global well-posedness of (1.1) we first consider the spatially homogeneous problem.

Lemma 2.2. *Let $z_0 \in \mathbb{C}$ and $p \in \mathbb{N}$ with $p \geq 2$. We consider the ODE*

$$\begin{cases} z' = iz^p, & t \geq 0, \\ z(0) = z_0. \end{cases} \tag{2.2}$$

If $\sin((p-1)\arg(z_0)) > -1$, then (2.2) has a unique global solution $z : [0,\infty) \to \mathbb{C}$, given by

$$z(t) = \frac{z_0}{(1-i(p-1)z_0^{p-1}t)^{1/(p-1)}}, \quad t \geq 0. \tag{2.3}$$

In particular, there is a constant $C_1 = C_1(\arg(z_0),p) > 0$ with

$$|z(t)| \leq C_1|z_0| \tag{2.4}$$

for all $t \geq 0$.

Proof. We write z_0 in polar coordinates $z_0 = r_0e^{i\theta_0}$. Let $\sin((p-1)\theta_0) > -1$. Then one can easily check that the function z given in (2.3) solves (2.2) on $[0,\infty)$. We have

$$\begin{aligned} |z(t)|^{2(p-1)} &= \frac{|z_0|^{2(p-1)}}{|1-i(p-1)z_0^{p-1}t|^2} \\ &= \frac{r_0^{2(p-1)}}{1+2(p-1)\sin((p-1)\theta_0)r_0^{p-1}t+(p-1)^2r_0^{2(p-1)}t^2} \end{aligned} \tag{2.5}$$

for $t \geq 0$. To estimate the denominator in (2.5) we consider two cases.
1) If $\sin((p-1)\theta_0) \geq 0$, then we have

$$1+2(p-1)\sin((p-1)\theta_0)r_0^{p-1}t+(p-1)^2r_0^{2(p-1)}t^2 \geq 1$$

for all $t \geq 0$.
2) If $\sin((p-1)\theta_0) \leq 0$, we can compute the global minimum of the denominator and get

$$1+2(p-1)\sin((p-1)\theta_0)r_0^{p-1}t+(p-1)^2r_0^{2(p-1)}t^2 \geq 1-\sin^2((p-1)\theta_0) > 0$$

for all $t \geq 0$ due to the assumptions on z_0. So in both cases, the denominator in (2.5) is bounded from below by a positive constant depending only on p and θ_0. This yields the claim. $\square$

Remark 2.3. *i) The solution $z(t)$ behaves like $t^{-\frac{1}{p-1}}$ as $t \to \infty$.*

ii) In the case $\sin((p-1)\arg(z_0)) \geq 0$, the above calculation shows $C_1 = 1$.

iii) For $\sin((p-1)\arg(z_0)) = -1$ the function z has a singularity at $t = \Big((p-1)r_0^{p-1}\Big)^{-1}$. Hence, there cannot exist a global solution to (2.2) in this case.

iv) The function z also solves (1.2) for negative times. To prevent blow-up in this case, the condition on z_0 has to be changed to $\sin((p-1)\arg(z_0)) < 1$.

v) We can also consider (2.2) with changed sign in front of the nonlinearity. Define the function $\tilde{z} : [0, \infty) \to \mathbb{C}$, $\tilde{z}(t) = z(-t)$. If $\sin((p-1)\arg(z_0)) < 1$, then $\tilde{z}$ solves the ODE

$$\begin{cases} \tilde{z}' = -i\tilde{z}^p, & t \geq 0, \\ \tilde{z}(0) = z_0. \end{cases}$$

Now we are interested in solutions of (1.1) that are close to the function z provided by Lemma 2.2.

Proposition 2.4. *Let $z : [0, \infty) \to \mathbb{C}$ be given by (2.3) with $z_0 \in \mathbb{C} \setminus \{0\}$ and $u \in C([0,T], \mathcal{A})$ be a mild solution of (1.1) with initial data $u_0 \in \mathcal{A}$. We define $\tilde{u} \in C([0,T], \mathcal{A})$ by*

$$u(t) = z(t) + z(t)^p \tilde{u}(t).$$

Then $\tilde{u}$ satisfies the modified Duhamel formula

$$\tilde{u}(t) = S(t)\tilde{u}(0) + i \int_0^t S(t-s) \sum_{m=2}^{p} \binom{p}{m} z(s)^{m(p-1)} \tilde{u}(s)^m \mathrm{d}s \tag{2.6}$$

for all $t \in [0,T]$.

Proof. Due to $z_0 \neq 0$ and the algebra property of $\mathcal{A}$ we have that $\tilde{u}$ and the right-hand side of (2.6) are well defined. To prove the claim, we set

$$v(t) = S(-t)u(t), \qquad \tilde{v}(t) = S(-t)\tilde{u}(t)$$

for all $t \in [0,T]$. One can easily check the identity

$$v(t) = z(t) + z(t)^p \tilde{v}(t) \tag{2.7}$$

for all $t \in [0,T]$. After inserting (2.7) into the Duhamel formula (2.1) we get

$$v(t) = u_0 + i \int_0^t S(-s) u(s)^p \mathrm{d}s.$$

So we see that $v \in C^1([0,T],\mathcal{A})$ with

$$\begin{aligned}\partial_t v(t) &= iS(-t)u^p(t) = iS(-t)\left(z(t)+z(t)^p\tilde{u}(t)\right)^p \\ &= iz(t)^p + ipz(t)^{2p-1}S(-t)\tilde{u}(t) + iz(t)^p S(-t)\sum_{m=2}^{p}\binom{p}{m} z(t)^{m(p-1)}\tilde{u}(s)^m\end{aligned} \tag{2.8}$$

for all $t \in [0,T]$. Here, we used the definition of $\tilde{u}$. On the other hand, equation (2.7) yields us

$$\begin{aligned}\partial_t v(t) &= z'(t) + pz(t)^{p-1}z'(t)\tilde{v}(t) + z(t)^p\partial_t\tilde{v}(t) \\ &= iz(t)^p + ipz(t)^{2p-1}\tilde{v}(t) + z(t)^p\partial_t\tilde{v}(t)\end{aligned} \tag{2.9}$$

for all $t \in [0,T]$ where we used that z solves (2.2). If we now compare equations (2.8) and (2.9), we obtain that $\tilde{v}$ solves the equation

$$\partial_t\tilde{v}(t) = iS(-t)\sum_{m=2}^{p}\binom{p}{m} z(t)^{m(p-1)}\tilde{u}(s)^m$$

for all $t \in [0,T]$. By writing this in a mild formulation, we get

$$\tilde{v}(t) = \tilde{v}(0) + i\int_0^t S(-s)\sum_{m=2}^{p}\binom{p}{m} z(s)^{m(p-1)}\tilde{u}(s)^m \mathrm{d}s.$$

Applying $S(t)$ on both sides yields the claim. □

To formulate our next theorem, we need some definitions.

Definition 2.5. *Let $r, \rho_0, \rho_1 > 0$ and $p \in \mathbb{N}$ with $p \geq 2$. We define the sets*

$$\begin{aligned}B(\rho_0) &:= \left\{z \in \mathbb{C}\setminus\{0\} : \sin((p-1)\arg(z)) > -1,\ |z| \leq \rho_0\right\}, \\ B(\rho_0,\rho_1) &:= \left\{z_0 + \phi \in \mathcal{A} : z_0 \in B(\rho_0), \|\phi\|_{\mathcal{A}} \leq \rho_1 |z_0|^p\right\}.\end{aligned}$$

Furthermore, we define the positive constants

$$P(r,\rho_0) := \sum_{m=2}^{p}\binom{p}{m}\left(r\rho_0^{p-1}\right)^{m-1}$$

and

$$C_2(\theta,p) := \int_0^\infty \frac{1}{1+2\sin((p-1)\theta)s+s^2}\mathrm{d}s \tag{2.10}$$

for all $\theta \in \mathbb{R}$ with $\sin((p-1)\theta) > -1$.

As a concrete example, we have $C_2 = \frac{\pi}{2}$ if $\sin((p-1)\theta) = 0$ and $C_2 \leq \frac{\pi}{2}$ if $\sin((p-1)\theta) \geq 0$. Note that the integral in (2.10) diverges for $\sin((p-1)\theta)$ close to -1.

Theorem 2.6. *Let $\rho_0, \rho_1 > 0$ and $p \in \mathbb{N}$ with $p \geq 2$. Furthermore, let $u_0 = z_0 + \phi \in B(\rho_0, \rho_1)$ and $u \in C([0, T^*), \mathcal{A})$ be a maximal mild solution of* (1.1) *with initial data u_0. Assume that there exists some $r > 0$ such that*

$$M\rho_1 \exp\left\{ MC_{\mathcal{A}}^{p-1} C_1^{(p-2)(p-1)} C_2 \frac{P(r,\rho_0)}{p-1} \right\} < r, \tag{2.11}$$

where $C_1 = C_1(\arg(z_0), p)$ and $C_2 = C_2(\arg(z_0), p)$ are given by Lemma 2.2 and (2.10)*. Then we have $T^* = \infty$ and $\lim_{t\to\infty} \|u(t)\|_{\mathcal{A}} = 0$. More specifically, let $z : [0,\infty) \to \mathbb{C}$ be given by* (2.3)*. Then the estimate*

$$\|u(t) - z(t)\|_{\mathcal{A}} \leq r|z(t)|^p$$

holds for all $t \geq 0$.

Proof. For the proof, we write $z_0 = r_0 e^{i\theta_0}$ with $\sin((p-1)\theta_0) > -1$ and $r_0 \leq \rho_0$. We consider the solution $\tilde{u}$ of the modified Duhamel formula (2.6). In this context, we have $\phi = z_0^p \tilde{u}(0)$. Using the boundedness of the semigroup and the algebra property of $\mathcal{A}$ we get

$$\|\tilde{u}(t)\|_{\mathcal{A}} \leq M\|\tilde{u}(0)\|_{\mathcal{A}} + MC_{\mathcal{A}}^{p-1} \int_0^t \sum_{m=2}^{p} \binom{p}{m} |z(s)|^{m(p-1)} \|\tilde{u}(s)\|_{\mathcal{A}}^m \mathrm{d}s.$$

Now we choose an $r > 0$ according to the assumption and define

$$T := \sup\{t \geq 0 : \|\tilde{u}(t)\|_{\mathcal{A}} \leq r\}.$$

By the choice of r and ϕ we have

$$\|\tilde{u}(0)\|_{\mathcal{A}} = |z_0|^{-p} \|\phi\|_{\mathcal{A}} \leq \rho_1 < r,$$

where the last inequality follows from (2.11). This guaranties $T > 0$. Combining this with estimate (2.4) and $r_0 \leq \rho_0$ we obtain

$$\begin{aligned}
\|\tilde{u}(t)\|_{\mathcal{A}} &\leq M\rho_1 + MC_{\mathcal{A}}^{p-1} \int_0^t \sum_{m=2}^{p} \binom{p}{m} |z(s)|^{(m-2)(p-1)} |z(s)|^{2(p-1)} \|\tilde{u}(s)\|_{\mathcal{A}}^m \mathrm{d}s \\
&\leq M\rho_1 + MC_{\mathcal{A}}^{p-1} \int_0^t \sum_{m=2}^{p} \binom{p}{m} (C_1\rho_0)^{(m-2)(p-1)} |z(s)|^{2(p-1)} r^{m-1} \|\tilde{u}(s)\|_{\mathcal{A}} \mathrm{d}s \\
&\leq M\rho_1 + MC_{\mathcal{A}}^{p-1} C_1^{(p-2)(p-1)} \rho_0^{-(p-1)} P(r,\rho_0) \int_0^t |z(s)|^{2(p-1)} \|\tilde{u}(s)\|_{\mathcal{A}} \mathrm{d}s
\end{aligned}$$

for all $t \in [0, T]$, where we used the definition of $P(r, \rho_0)$. At this point we want to apply Grönwall's inequality. With the calculations from the proof of

Lemma 2.2 and $r_0 \le \rho_0$ we compute

$$\begin{aligned}
&\rho_0^{-(p-1)} P(r,\rho_0) \int_0^t |z(s)|^{2(p-1)} \mathrm{d}s \\
\le& \rho_0^{-(p-1)} P(r,\rho_0) \int_0^\infty \frac{r_0^{2(p-1)}}{1 + 2(p-1)\sin((p-1)\theta_0) r_0^{p-1} s + (p-1)^2 r_0^{2(p-1)} s^2} \mathrm{d}s \\
=& \rho_0^{-(p-1)} \frac{P(r,\rho_0)}{p-1} \int_0^\infty \frac{r_0^{p-1}}{1 + 2\sin((p-1)\theta_0)s + s^2} \mathrm{d}s \\
\le& \frac{P(r,\rho_0)}{p-1} C_2.
\end{aligned}$$

Thus we obtain

$$\|\tilde{u}(t)\|_{\mathcal{A}} \le M\rho_1 \exp\left\{ M C_{\mathcal{A}}^{p-1} C_1^{(p-2)(p-1)} C_2 \frac{P(r,\rho_0)}{p-1} \right\}$$

for all $t \in [0,T]$. So, by assumption (2.11) we have $\|\tilde{u}(t)\|_{\mathcal{A}} < r$ for all $t \in [0,T]$. The definition of T and the blow-up alternative thus imply $T = T^* = \infty$. In particuar, we have

$$\|u(t) - z(t)\|_{\mathcal{A}} = |z(t)|^p \|\tilde{u}(t)\|_{\mathcal{A}} \le r|z(t)|^p$$

for all $t \ge 0$ which implies $\lim_{t\to\infty} \|u(t)\|_{\mathcal{A}} = 0$ due to $\lim_{t\to\infty} z(t) = 0$. □

Now we can state the main theorem.

Theorem 2.7. *Let $p \in \mathbb{N}$ with $p \ge 2$ and $z_0 \in \mathbb{C}\setminus\{0\}$ with $\sin((p-1)\arg(z_0)) > -1$. We define*

$$C(\arg(z_0)) := M^{-1} \exp\left\{ -M C_{\mathcal{A}}^{p-1} C_1^{(p-2)(p-1)} C_2 \frac{2^p - p - 1}{p-1} \right\} \tag{2.12}$$

with $C_1 = C_1(\arg(z_0), p)$ and $C_2 = C_2(\arg(z_0), p)$ as in Theorem 2.6. If $u_0 \in \mathcal{A}$ satisfies

$$\|u_0 - z_0\|_{\mathcal{A}} < C(\arg(z_0))|z_0|,$$

then the initial value problem (1.1) *has a unique global solution $u \in C([0,\infty), \mathcal{A})$ with $\lim_{t\to\infty} \|u(t)\|_{\mathcal{A}} = 0$. More specifically, let $z : [0,\infty) \to \mathbb{C}$ be given by* (2.3). *Then the estimate*

$$\|u(t) - z(t)\|_{\mathcal{A}} \le |z_0|^{-p+1} |z(t)|^p$$

holds for all $t \ge 0$.

Proof. We set $\phi = u_0 - z_0$. Due to $\|\phi\|_{\mathcal{A}} < C(\arg(z_0))|z_0|$, we find a $\rho_1 > 0$ with

$$\|\phi\|_{\mathcal{A}} \le \rho_1 |z_0|^p < C(\arg(z_0))|z_0|. \tag{2.13}$$

Furthermore, we set $\rho_0 = |z_0|$ and $r = \rho_0^{-p+1}$. By that choice, we have $u_0 \in B(\rho_0, \rho_1)$ and $P(r, \rho_0) = P(1,1) = 2^p - p - 1$ which implies

$$M\rho_1 \exp\left\{ MC_{\mathcal{A}}^{p-1} C_1^{(p-2)(p-1)} C_2 \frac{P(r,\rho_0)}{p-1} \right\} = \rho_1 C(\arg(z_0))^{-1} < |z_0|^{-p+1} = r$$

where we used (2.13). The claim now follows from Theorem 2.6. $\square$

We mention that if $\sin((p-1)\arg(z_0)) \geq 0$, then we have

$$C(\arg(z_0)) \geq M^{-1} \exp\left\{ -MC_{\mathcal{A}}^{p-1} \frac{\pi}{2} \frac{2^p - p - 1}{p-1} \right\}$$

in (2.12).

Remark 2.8. *i) We can also consider negative times. As mentioned in Remark 2.3 the assumption* $\sin((p-1)\arg(z_0)) > -1$ *then has to be changed to* $\sin((p-1)\arg(z_0)) < 1$*. Moreover, the Schrödinger semigroup* $S(t)$ *has to be strongly continuous and bounded for* $t \leq 0$*. Then, by the same arguments we get version similar to Theorem 2.7 for negative times (with modified constants* C_1 *and* C_2*). In particular, if* $\sin((p-1)\arg(z_0)) \in (-1,1)$ *we can concatenate both solutions and gets a unique global solution* $u \in C(\mathbb{R}, \mathcal{A})$ *of* (1.1) *with* $\lim_{t\to\pm\infty} \|u(t)\|_{\mathcal{A}} = 0$*. We omit the details.*

ii) Using the function $\tilde{z}$ *from Remark 2.3 we can do the same analysis again and get global solutions to* (1.1) *with changed sign in front of the nonlinearity*

$$\begin{cases} i\partial_t u = -\partial_x^2 u + u^p, & t \geq 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

We omit the details.

Remark 2.9. *To prove Theorem 2.7, we only used the boundedness of the Schrödinger semigroup and the algebra property of* $\mathcal{A}$*. So, the above theory can immediately be generalized to an arbitrary bounded* C_0*-semigroup* $(T(t))_{t\geq 0}$ *with generator* iA *and an arbitrary complex Banach algebra* $\mathcal{B}$ *with a neutral element. Thus, we obtain global solutions for the general initial value problem*

$$\begin{cases} i\partial_t u = -Au - u^p, & t \geq 0, \\ u(0) = u_0, \end{cases}$$

for an integer $p \geq 2$*, where the power-type nonlinearity* u^p *is understood in the sense of the multiplication operation in* $\mathcal{B}$*. The modifications we mentioned in Remark 2.8 also work in this general case, of course.*

3 Application to almost periodic functions

In this section we want to apply our previous results in the context of almost periodic functions. We will show global well-posedness of (1.1) using the boundedness of the Schrödinger semigroup $S(t) = e^{it\partial_x^2}$, $t \geq 0$, and the respective algebraic structure. Furthermore, we will consider almost periodic initial data with localized perturbation.

3.1 Almost periodic functions

A function $f : \mathbb{R} \to \mathbb{C}$ of the form

$$f(x) = \sum_{k=1}^{n} c_k e^{i\xi_k x}, \quad x \in \mathbb{R},$$

where c_k are complex numbers and ξ_k real numbers for all $1 \leq k \leq n$, is called a *trigonometric polynomial.*

Definition 3.1. *We call a function $f : \mathbb{R} \to \mathbb{C}$ almost periodic, if it can be uniformly approximated by trigonometric polynomials. More precisely, given any $\epsilon > 0$, there exists a trigonometric polynomial P_ϵ such that*

$$\sup_{x\in\mathbb{R}} |f(x) - P_\epsilon(x)| < \epsilon.$$

We write $AP(\mathbb{R})$ to denote the space of almost periodic functions.

One can easily check that almost periodic functions are continuous and bounded on $\mathbb{R}$ (Theorem 1.2 in [4]). We now introduce the Fourier series of an almost periodic function.

Definition 3.2. *Let $f \in AP(\mathbb{R})$. We define the Fourier coefficients of f by*

$$\hat{f}(\xi) := \lim_{L\to\infty} \frac{1}{L} \int_0^L f(x)e^{-i\xi x} \mathrm{d}x$$

for $\xi \in \mathbb{R}$.

We note that $\hat{f}(\xi)$ is well defined for all $\xi \in \mathbb{R}$ and that there exist at most countable many $\xi \in \mathbb{R}$ with $\hat{f}(\xi) \neq 0$ (Theorem 1.12, 1.15 in [4]). We define the frequency set of f by $\sigma(f) := \{\xi \in \mathbb{R} : \hat{f}(\xi) \neq 0\}$ and write

$$f(x) \sim \sum_{\xi\in\sigma(f)} \hat{f}(\xi)e^{i\xi x},$$

where the right hand side is called the *Fourier series associated to f*. In general, the Fourier series does not converge uniformly to f. Nevertheless, there are relations between f and its associated Fourier series, that hold for arbitrary $f \in AP(\mathbb{R})$.

Proposition 3.3 (Theorem 1.18-1.20 in [4]). *Let $f \in AP(\mathbb{R})$. Then the following statements are true.*

a) We have Parseval's equality

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L |f(x)|^2 \mathrm{d}x = \sum_{\xi\in\sigma(f)} |\hat{f}(\xi)|^2.$$

b) If the Fourier series of f is uniformly convergent, then it converges to f.

c) Let $g \in AP(\mathbb{R})$ with $f \neq g$. Then f and g have distinct Fourier series.

Now, we want to classify almost periodic functions depending on the frequency set. Consider a countable set of real numbers $\Omega = \{\omega_1, \omega_2, ...\}$. We say that Ω is *linearly independent (over $\mathbb{Q}$)* if any relation of the form:

$$\sum_{j=1}^{n} r_j\omega_j = 0, \quad r_j \in \mathbb{Q}, n \in \mathbb{N}, n \leq |\Omega|,$$

implies that $r_j = 0$ for all $1 \leq j \leq n$. Observe that we get an equivalent definition if we only require $r_j \in \mathbb{Z}$ here. Furthermore, we define $\overline{\Omega}$ by

$$\overline{\Omega} := \left\{ \xi \in \mathbb{R} : \xi = \sum_{j=1}^{n} r_j\omega_j \text{ with } r_j \in \mathbb{Z}, n \in \mathbb{N}, n \leq |\Omega| \right\},$$

which is still a countable set.

Definition 3.4. *Let $\Lambda \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}$ be linearly independent. We say that Ω constitutes a basis for the set Λ if $\Lambda \subset \overline{\Omega}$.*

One can easily check that every countable set of real numbers contains a basis. In patricular, for every $f \in AP(\mathbb{R})$ there is a basis Ω of the frequency set $\sigma(f)$. If this Ω can be chosen such that it contains just one element, then the function f is periodic. If Ω can be chosen as a finite set, then we call f a *quasi-periodic* function.
In the following, we will consider a special type of almost periodic functions.

Definition 3.5. *Let $\Omega \subset \mathbb{R}$ be linearly independent. We define the space*

$$\mathcal{A}_\Omega := \left\{ f \in AP(\mathbb{R}) : \sigma(f) \subset \overline{\Omega} \text{ and } \|f\|_{\mathcal{A}_\Omega} < \infty \right\},$$

where the $\mathcal{A}_\Omega$-norm is given by

$$\|f\|_{\mathcal{A}_\Omega} := \sum_{\xi\in\overline{\Omega}} |\hat{f}(\xi)|.$$

We present some basic properties of $\mathcal{A}_\Omega$.

Lemma 3.6 (Lemma 2.1 in [19])**.** *Let $f \in \mathcal{A}_\Omega$. Then we have*

$$f(x) = \sum_{\xi \in \overline{\Omega}} \hat{f}(\xi) e^{i\xi x},$$

where the sum converges uniformly in $x \in \mathbb{R}$. Moreover $\mathcal{A}_\Omega$ is a Banach algebra with algebra constant $C_{\mathcal{A}_\Omega} = 1$.

For $f \in \mathcal{A}_\Omega$, the linear propagator $S(t) = e^{it\partial_x^2}$ is given by

$$S(t)f := \sum_{\xi \in \overline{\Omega}} \hat{f}(\xi) e^{-i\xi^2 t} e^{i\xi \cdot}, \quad t \in \mathbb{R}.$$

A simple calculation shows that $(S(t))_{t\in\mathbb{R}}$ is a strongly continuous group on $\mathcal{A}_\Omega$ with

$$\|S(t)f\|_{\mathcal{A}_\Omega} = \sum_{\xi \in \overline{\Omega}} \left| \hat{f}(\xi) e^{-i\xi^2 t} \right| = \sum_{\xi \in \overline{\Omega}} |\hat{f}(\xi)| = \|f\|_{\mathcal{A}_\Omega} \tag{3.1}$$

for all $t \in \mathbb{R}$. So, by Lemma 3.6 and (3.1) all assumptions for the general result in Section 2 are satisfied ($C_{\mathcal{A}} = M = 1$) and we can apply Theorem 2.7 for positive and negative times.

Theorem 3.7. *Let $\Omega \subset \mathbb{R}$ be linearly independent, $p \in \mathbb{N}$ with $p \geq 2$ and $z_0 \in \mathbb{C} \setminus \{0\}$ with $\sin((p-1)\arg(z_0)) > -1$. We define*

$$C(\arg(z_0)) := \exp\left\{ -C_1^{(p-2)(p-1)} C_2 \frac{2^p - p - 1}{p - 1} \right\}$$

with $C_1 = C_1(\arg(z_0), p)$ and $C_2 = C_2(\arg(z_0), p)$ as in Theorem 2.6. If $u_0 \in \mathcal{A}_\Omega$ satisfies

$$\|u_0 - z_0\|_{\mathcal{A}_\Omega} < C(\arg(z_0))|z_0|,$$

then the initial value problem (1.1) *has a unique global solution $u \in C([0,\infty), \mathcal{A}_\Omega)$ with $\lim_{t\to\infty} \|u(t)\|_{\mathcal{A}_\Omega} = 0$. More specifically, let $z : [0,\infty) \to \mathbb{C}$ be given by* (2.3)*. Then the estimate*

$$\|u(t) - z(t)\|_{\mathcal{A}_\Omega} \leq |z_0|^{-p+1} |z(t)|^p$$

holds for all $t \geq 0$.

Remark 3.8. *Via Remark 2.8 we obtain a similar result for negative times if* $\sin((p-1)\arg(z_0)) < 1$.

3.2 Almost periodic functions with localized perturbation

Now we want to consider almost periodic initial data with an additional localized perturbation. To do that we will define a space of localized functions similar to

$\mathcal{A}_\Omega$ from the previous subsection.
Let $\mathcal{S}(\mathbb{R})$ denote the space of Schwartz functions. The Fourier transform of $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) \mathrm{d}x, \quad \xi \in \mathbb{R}. \tag{3.2}$$

Let $\mathcal{S}'(\mathbb{R})$ denote the space of tempered distributions. The Fourier transform of $T \in \mathcal{S}'(\mathbb{R})$ is defined by

$$\mathcal{F}(T)(f) := \hat{T}(f) := T(\hat{f}), \quad f \in \mathcal{S}(\mathbb{R}).$$

Definition 3.9. *We define the space*

$$\mathcal{A}(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : \hat{f} \in L^1(\mathbb{R}) \right\},$$

with the norm $\|f\|_{\mathcal{A}(\mathbb{R})} := \|\hat{f}\|_{L^1(\mathbb{R})}$.

By the Lemma of Riemann-Lebesgue, we see that every $f \in \mathcal{A}(\mathbb{R})$ can be represented by a continuous function that vanishes at infinity. We denote this function again by f.
To apply our theory we need the following algebra property.

Proposition 3.10. *Let $f, g \in \mathcal{A}(\mathbb{R})$. Then we have $fg \in \mathcal{A}(\mathbb{R})$ with*

$$\|fg\|_{\mathcal{A}(\mathbb{R})} \leq \|f\|_{\mathcal{A}(\mathbb{R})} \|g\|_{\mathcal{A}(\mathbb{R})}. \tag{3.3}$$

Proof. By the above remark, fg is well-defined as the product of two continuous functions vanishing at infinity. In particular, fg is again a tempered distribution. By standard Fourier analysis we get $\mathcal{F}(fg) = \hat{f} * \hat{g}$ which lies in $L^1(\mathbb{R})$. So we can calculate

$$\|fg\|_{\mathcal{A}(\mathbb{R})} = \|\hat{f} * \hat{g}\|_{L^1(\mathbb{R})} \leq \|\hat{f}\|_{L^1(\mathbb{R})} \|\hat{g}\|_{L^1(\mathbb{R})} = \|f\|_{\mathcal{A}(\mathbb{R})} \|g\|_{\mathcal{A}(\mathbb{R})}$$

using Young's convolution inequality. □

Let $\Omega \subset \mathbb{R}$ be linearly independent. Since $\mathcal{A}(\mathbb{R})$ and $\mathcal{A}_\Omega$ have trivial intersection, we can define the sum space

$$\mathcal{A}_\Omega(\mathbb{R}) := \mathcal{A}(\mathbb{R}) + \mathcal{A}_\Omega$$

with norm $\|f + g\|_{A_\Omega(\mathbb{R})} = \|f\|_{\mathcal{A}(\mathbb{R})} + \|g\|_{\mathcal{A}_\Omega}$.

Proposition 3.11. *Let $f \in \mathcal{A}(\mathbb{R})$ and $g \in \mathcal{A}_\Omega$. Then the product fg is also in $\mathcal{A}(\mathbb{R})$ with*

$$\|fg\|_{\mathcal{A}(\mathbb{R})} \leq \|f\|_{\mathcal{A}(\mathbb{R})} \|g\|_{\mathcal{A}_\Omega}. \tag{3.4}$$

In particular $\mathcal{A}_\Omega(\mathbb{R})$ is a Banach algebra with algebra constant $C_{\mathcal{A}_\Omega(\mathbb{R})} = 1$.

Proof. Since f and g are both continuous and bounded on $\mathbb{R}$, the product fg defines again a tempered distribution. By Lemma 3.6 we know that the Fourier series of g converges uniformly to g. In particular we have

$$\mathcal{F}(fg) = \sum_{\xi\in\overline{\Omega}} \hat{g}(\xi)\mathcal{F}\left(fe^{i\xi\cdot}\right) = \sum_{\xi\in\overline{\Omega}} \hat{g}(\xi)\hat{f}\left(\cdot - \frac{\xi}{2\pi}\right)$$

in the sense of distributions. So we can compute

$$\begin{aligned}\|fg\|_{\mathcal{A}(\mathbb{R})} = \|\mathcal{F}(fg)\|_{L^1(\mathbb{R})} &\leq \sum_{\xi\in\overline{\Omega}} |\hat{g}(\xi)| \left\|\hat{f}\left(\cdot - \frac{\xi}{2\pi}\right)\right\|_{L^1(\mathbb{R})} = \sum_{\xi\in\overline{\Omega}} |\hat{g}(\xi)| \|\hat{f}\|_{L^1(\mathbb{R})} \\ &= \|f\|_{\mathcal{A}(\mathbb{R})}\|g\|_{\mathcal{A}_\Omega}\end{aligned}$$

which proves (3.4). The second claim now follows immediately by Lemma 3.6, (3.3) and (3.4).

□

For $f + g \in \mathcal{A}_\Omega(\mathbb{R})$, the linear propagator $S(t) = e^{it\partial_x^2}$ is given by

$$S(t)(f+g) = \mathcal{F}^{-1}\left(e^{-i\xi^2 t}\hat{f}\right) + \sum_{\xi\in\overline{\Omega}} \hat{g}(\xi)e^{-i\xi^2 t}e^{i\xi\cdot}, \quad t \in \mathbb{R}.$$

As in (3.1) we have

$$\|S(t)(f+g)\|_{\mathcal{A}_\Omega(\mathbb{R})} = \|f+g\|_{\mathcal{A}_\Omega(\mathbb{R})}$$

for all $t \in \mathbb{R}$ and together with Proposition 3.11, all assumptions in Section 2 are satisfied ($C_{\mathcal{A}} = M = 1$). So, we can apply Theorem 2.7 also in this case for positive and negative times.

Theorem 3.12. *Let $\Omega \subset \mathbb{R}$ be linearly independent, $p \in \mathbb{N}$ with $p \geq 2$ and $z_0 \in \mathbb{C} \setminus \{0\}$ with $\sin((p-1)\arg(z_0)) > -1$. We define*

$$C(\arg(z_0)) := \exp\left\{-C_1^{(p-2)(p-1)}C_2\frac{2^p - p - 1}{p-1}\right\}$$

with $C_1 = C_1(\arg(z_0), p)$ and $C_2 = C_2(\arg(z_0), p)$ as in Theorem 2.6. If $u_0 \in \mathcal{A}_\Omega(\mathbb{R})$ satisfies

$$\|u_0 - z_0\|_{\mathcal{A}_\Omega(\mathbb{R})} < C(\arg(z_0))|z_0|,$$

then the initial value problem (1.1) has a unique global solution $u \in C([0,\infty), \mathcal{A}_\Omega(\mathbb{R}))$ with $\lim_{t\to\infty}\|u(t)\|_{\mathcal{A}_\Omega(\mathbb{R})} = 0$. More specifically, let $z : [0,\infty) \to \mathbb{C}$ be given by (2.3). Then the estimate

$$\|u(t) - z(t)\|_{\mathcal{A}_\Omega(\mathbb{R})} \leq |z_0|^{-p+1}|z(t)|^p$$

holds for all $t \geq 0$.

Remark 3.13. *Via Remark 2.8 we obtain a similar result for negative times if* $\sin((p-1)\arg(z_0)) < 1$.

Remark 3.14. *We can also consider almost periodic functions* $f : \mathbb{R}^d \to \mathbb{C}$ *in higher dimensions. The spaces* $\mathcal{A}_\Omega, \mathcal{A}(\mathbb{R})$ *and* $\mathcal{A}_\Omega(\mathbb{R})$ *can be generalized in an obvious way, and this does not change our arguments at all. So, equivalent versions of Theorems 3.7 and 3.12 also hold in higher dimensions. We omit the details.*

4 Further applications

We add some more examples for periodic initial data with localized perturbation. At this point, we want to mention the works [17],[2] and [3] one more time, where the authors considered the classical NLS in the so-called "tooth spaces" $H^r(\mathbb{R}) + H^s(\mathbb{T})$ defined below.

Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus and $\langle\xi\rangle = \left(1+\xi^2\right)^{\frac{1}{2}}$ the Japanese bracket. For $s \geq 0$, the Sobolev spaces $H^s(\mathbb{T})$ are defined by

$$H^s(\mathbb{T}) := \left\{ f \in L^2(\mathbb{T}) : \|f\|_{H^s(\mathbb{T})} := \left(\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty \right\},$$

where the Fourier transform of a periodic function $f : \mathbb{T} \to \mathbb{C}$ is given by

$$\hat{f}(k) := \int_0^1 e^{-2\pi i k x} f(x) \mathrm{d}x, \quad k \in \mathbb{Z}.$$

Analogously, for $r \geq 0$, the Sobolev spaces $H^r(\mathbb{R})$ are defined by

$$H^r(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \|f\|_{H^r(\mathbb{R})} := \left(\int_{\mathbb{R}} \langle\xi\rangle^{2r} |\hat{f}(\xi)|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} < \infty \right\},$$

where the Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$ is given by (3.2).

Similarly as in the previous Section, we shall apply our theory to the sum spaces

$$\mathcal{A}(\mathbb{R}) + H^s(\mathbb{T}), \quad \text{for } s > \frac{1}{2},$$
$$H^r(\mathbb{R}) + H^s(\mathbb{T}), \quad \text{for } s \geq r > \frac{1}{2}.$$

It is well known that $H^s(\mathbb{T})$ and $H^r(\mathbb{R})$ are Banach algebras for $s, r > \frac{1}{2}$. We will prove that the above sum spaces have this property as well.

Proposition 4.1. *Let* $g \in H^s(\mathbb{T})$ *for some* $s > \frac{1}{2}$.

(i) Let $f \in \mathcal{A}(\mathbb{R})$. *Then the product* fg *is again in* $\mathcal{A}(\mathbb{R})$ *with*

$$\|fg\|_{\mathcal{A}(\mathbb{R})} \lesssim_s \|f\|_{\mathcal{A}(\mathbb{R})} \|g\|_{H^s(\mathbb{T})}. \tag{4.1}$$

In particular $\mathcal{A}(\mathbb{R}) + H^s(\mathbb{T})$ *is a Banach algebra.*

(ii) Let $f \in H^r(\mathbb{R})$ for some $\frac{1}{2} < r \leq s$. Then the product fg is again in $H^r(\mathbb{R})$ with

$$\|fg\|_{H^r(\mathbb{R})} \lesssim_r \|f\|_{H^r(\mathbb{R})}\|g\|_{H^s(\mathbb{T})}. \tag{4.2}$$

In particular, $H^r(\mathbb{R}) + H^s(\mathbb{T})$ is a Banach algebra.

Proof. First let $f \in \mathcal{A}(\mathbb{R})$. By the same calculations as in the proof of Proposition 3.11 we get

$$\|fg\|_{\mathcal{A}(\mathbb{R})} \leq \|f\|_{\mathcal{A}(\mathbb{R})} \sum_{k\in\mathbb{Z}} |\hat{g}(k)|.$$

Due to $s > \frac{1}{2}$ we can apply the Cauchy-Schwarz inequality to get

$$\sum_{k\in\mathbb{Z}} |\hat{g}(k)| = \sum_{k\in\mathbb{Z}} \langle k\rangle^{-2s}\langle k\rangle^{2s}|\hat{g}(k)| \lesssim_s \|g\|_{H^s(\mathbb{T})}.$$

This proves (4.1). Now let $f \in H^r(\mathbb{R})$. By Lemma 4.5 in [16] we have

$$\|fg\|_{H^r(\mathbb{R})} \lesssim_s \|f\|_{H^r(\mathbb{R})}\|g\|_{H^r(\mathbb{T})}.$$

So, (4.2) follows by the trivial bound $\|g\|_{H^r(\mathbb{T})} \leq \|g\|_{H^s(\mathbb{T})}$ due to $r \leq s$. □

In both cases, the linear propagator $S(t) = e^{it\partial_x^2}$ is given by

$$S(t)(f+g) = \mathcal{F}^{-1}\left(e^{-i\xi^2 t}\hat{f}\right) + \sum_{k\in\mathbb{Z}} \hat{g}(k)e^{-ik^2t}e^{ik\cdot}, \quad t \in \mathbb{R}.$$

As in (3.1), we get that $S(t)$ is an isometry for all $t \in \mathbb{R}$. Hence, all assumptions for the general result in Section 2 are satisfied again ($M = 1$) and we can apply Theorem 2.7 to both cases.

Theorem 4.2. *Let $\frac{1}{2} < r \leq s$ and $\mathcal{A} \in \{\mathcal{A}(\mathbb{R}) + H^s(\mathbb{T}), H^r(\mathbb{R}) + H^s(\mathbb{T})\}$. Then the assertion of Theorem 2.7 holds with $M = 1$.*

Remark 4.3. *i) In these cases, the algebra constant $C_{\mathcal{A}}$ depends on the parameters r and s and diverges for r, s close to $\frac{1}{2}$.*

ii) We can also consider periodic functions with localized perturbation in higher dimensions. The spaces $H^s(\mathbb{T}^d), H^r(\mathbb{R}^d)$ and $\mathcal{A}(\mathbb{R}^d)$ are defined analogously and by the same arguments we get a corresponding version of Theorem 4.2 in higher dimensions. Only the condition on the parameters r and s changes to $\frac{d}{2} < r \leq s$ to guarantee the algebra property. We omit the details.

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