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Moritz Hauck, Alexei Lozinski

CRC Preprint 2025/24, June 2025

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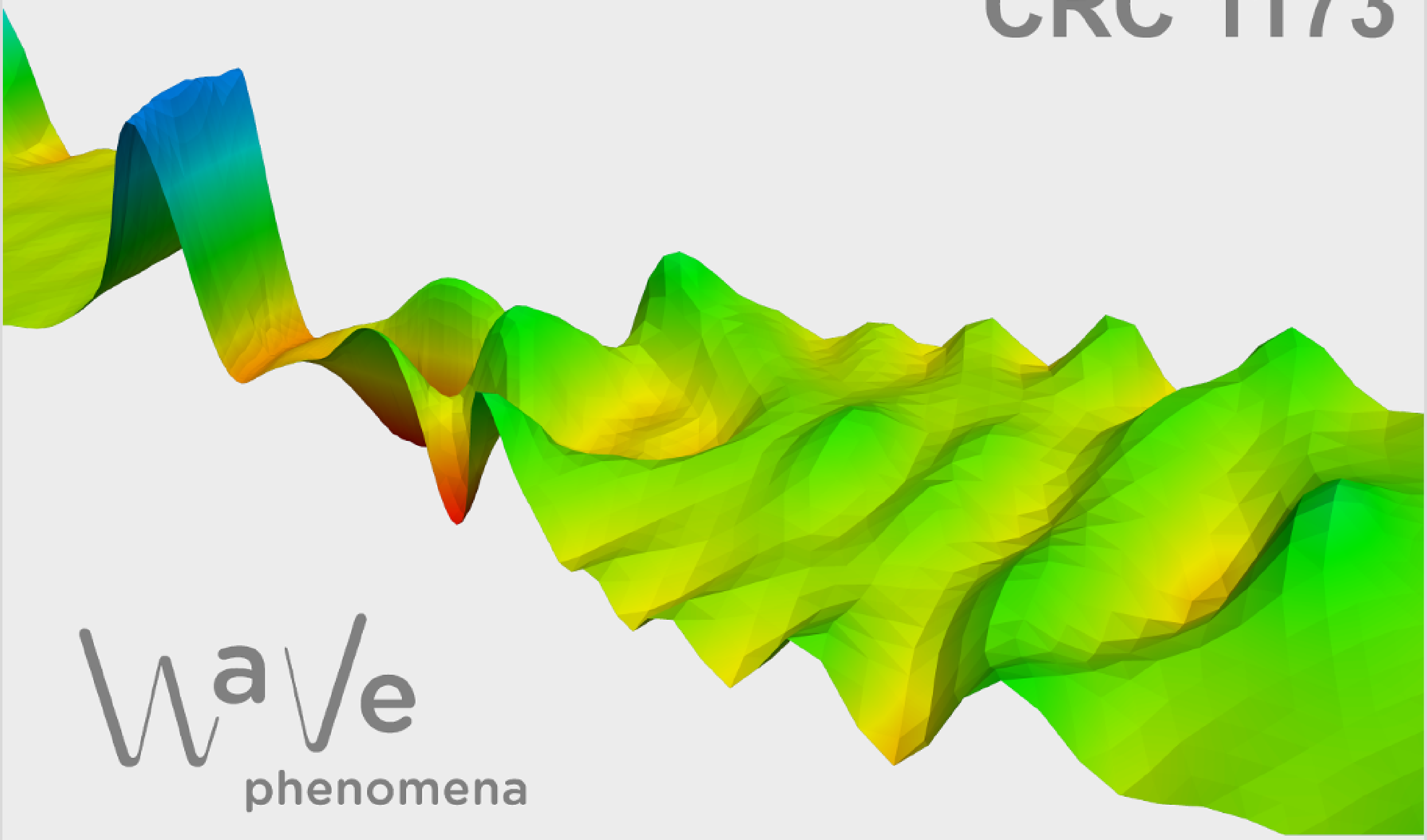
CRC 1173



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ISSN 2365-662X

A LOCALIZED ORTHOGONAL DECOMPOSITION METHOD FOR HETEROGENEOUS STOKES PROBLEMS

MORITZ HAUCK*, ALEXEI LOZINSKI†

ABSTRACT. In this paper, we propose a multiscale method for heterogeneous Stokes problems. The method is based on the Localized Orthogonal Decomposition (LOD) methodology and has approximation properties independent of the regularity of the coefficients. We apply the LOD to an appropriate reformulation of the Stokes problem, which allows us to construct exponentially decaying basis functions for the velocity approximation while using a piecewise constant pressure approximation. The exponential decay motivates a localization of the basis computation, which is essential for the practical realization of the method. We perform a rigorous a priori error analysis and prove optimal convergence rates for the velocity approximation and a post-processed pressure approximation, provided that the supports of the basis functions are logarithmically increased with the desired accuracy. Numerical experiments support the theoretical results of this paper.

1. INTRODUCTION

This paper considers a heterogeneous Stokes problem posed in a bounded Lipschitz polytope $\Omega \subset \mathbb{R}^n$, $n \in \{2, 3\}$, which we assume to be of unit size. Given an external force f, it seeks a velocity u and a pressure p such that

$$\begin{cases} -\nabla \cdot (\nu \nabla u) + \sigma u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

holds, where the coefficients ν and σ, hereafter referred to as viscosity and damping coefficients, respectively, encode the heterogeneity of the problem. They may possibly be rough and involve oscillations on multiple non-separated length scales. Such problems may arise, for example, as part of more complex coupled problems where the viscosity is an unknown. A specific example is magma modeling, where the temperature-dependent viscosity can exhibit large spatial variations, cf. [GP10]. Problems like (1.1) can also be used as approximations to (slow) flow problems around numerous obstacles of possibly small diameter, cf. [ABF99]. In this case, one sets ν as the physical viscosity and σ to zero in the fluid domain, while assigning large values to these coefficients inside the obstacles.

The numerical treatment of such heterogeneous problems with classical finite element methods (FEMs) suffer from suboptimal approximation rates and preasymptotic effects on meshes that do not resolve the coefficients. Since globally resolving all microscopic details of the coefficients may not be computationally feasible, we aim to construct a numerical method with reasonable errors already on coarse meshes. For diffusion-type problems, there is a whole zoo of multiscale methods that use coarse problem-adapted ansatz spaces. Two classes of methods may be distinguished: First, methods that exploit structural properties of the coefficients,

2020 *Mathematics Subject Classification.* 65N12, 65N15, 65N30, 76D07.
Key words and phrases. Stokes problem, flow around obstacles, multiscale method, a priori error analysis, exponential decay.

such as periodicity and scale separation, to construct the problem-adapted basis functions. Their computational cost differs from that of classical FEMs on the same mesh only by the cost of solving a fixed number of local problems. This class includes the Heterogeneous Multiscale Method [EE03], the Two-Scale FEM [MS02], and the Multiscale FEM [HW97]. In contrast, methods in the second class achieve optimal approximation orders under minimal structural assumptions on the coefficients. This is achieved at the expense of a moderate computational overhead compared to classical FEMs. This overhead manifests itself in an enlarged support of the basis functions or an increased number of basis functions per mesh entity. Prominent methods for diffusion-type multiscale problems include the Generalized Multiscale FEM [EGH13, CEL18], the Multiscale Spectral Generalized FEM (MS-GFEM) [BL11, MSD22], Adaptive Local Bases [GGS12], the (Super-) Localized Orthogonal Decomposition (LOD) method [MP14, HP13, HP22b], or Gamblets and operator-adapted wavelets [Owh17, OS19]; see also the review article [AHP21]. We mention that the LOD and the MS-GFEM have been applied to Darcy-type problems, cf. [MHH16, AMS24], which is also of mixed form but different in nature than the Stokes problem; see also [LM09].

Some of the methods of the first class have been successfully generalized to heterogeneous Stokes problems. For example, a homogenization-based method has been proposed in [BEL$^+$13, BEH13] for slowly varying perforated media. As for the Multiscale FEM, its Crouzeix–Raviart version, first proposed in [LBLL14], has been applied to the Stokes problem in perforated domains in [MNLD15, JL24, FAO22, Bal24]. The weak notion of continuity of the Crouzeix–Raviart method and an appropriate choice of the problem-adapted approximation space make it flexible enough to cope with the divergence-free constraint of the Stokes problem. On the contrary, we are not aware of any multiscale method for heterogeneous Stokes problems that works under the minimal structural assumptions on the coefficients, with the exception of [BOD19], which generalizes the concept of operator-adapted wavelets to accommodate differential constraints such as divergence-freeness.

The goal of the present paper is to fill this gap in the literature by adapting the LOD methodology to heterogeneous Stokes problems. The basic idea of the LOD is to decompose the solution space into a fine-scale space and its orthogonal complement with respect to the energy inner product induced by the considered problem. By choosing the fine-scale space to consist of functions that average out on coarse scales, one obtains a finite-dimensional mesh-based complement space that is adapted to the problem at hand and has uniform approximation properties under minimal structural assumptions on the coefficients. It possesses exponentially decaying basis functions whose computation can thus be localized to subdomains, resulting in a practically feasible method. For Stokes problems, the divergence-free constraint poses a major challenge to LOD-type methods, since directly incorporating the constraint can lead to ill-posed problems or slowly decaying basis functions. To overcome this problem, we reformulate the Stokes problem using the space of $(H^1_0(\Omega))^n$-functions, whose divergence is piecewise constant with respect to some coarse mesh, as the solution space for the velocity. The divergence-free velocity is then recovered using a piecewise constant Lagrange multiplier defined on the same mesh. Inspired by the Crouzeix–Raviart Multiscale FEM mentioned above, we then choose the fine-scale space for the velocity as the functions whose averages vanish on all faces of the coarse mesh. We prove that the finite-dimensional orthogonal complement possesses exponentially decaying basis functions, which paves the way to the construction of an LOD-type multiscale method. The use of face averages to define the fine-scale space is novel in the LOD context and allows us to construct the velocity approximation space independent of the pressure. The approximation of the resulting LOD method is exactly divergence-free and thus pressure robust in the sense of [JLM$^+$17]. We also perform an *a priori* error analysis of the proposed method and prove optimal orders of convergence as the mesh size is decreased, provided that the support of the basis functions is allowed to increase logarithmically with the desired accuracy. More specifically, for L^2-right-hand sides we prove first- and second-order convergence for the L^2- and H^1-errors of the velocity approximation, respectively, and first-order convergence for a post-processed pressure approximation. If the right-hand side is H^1-regular, we can squeeze out an additional order of convergence for the velocity approximation. We emphasize that only minimal structural assumptions on the coefficients are necessary for this error analysis.

The paper is organized as follows: In Section 2 we introduce the model problem and a prototypical multiscale method is presented in Section 3. We prove the exponential decay of the prototypical basis functions and localize the basis computation in Section 4. A practical multiscale method is then presented in Section 5. Numerical experiments supporting our theoretical results are given in Section 6.

2. Model problem

This section introduces the weak formulation of the heterogeneous Stokes problem, along with classical results guaranteeing its well-posedness. The weak formulation is based on the Sobolev space $V \coloneqq (H^1_0(\Omega))^n$ endowed with homogeneous Dirichlet boundary conditions on $\partial\Omega$ and the space $M \coloneqq \{q \in L^2(\Omega) \,:\, \int_\Omega q \,\mathrm{d}x = 0\}$ satisfying an integral-mean-zero constraint. In the following, we will always assume that there exist constants $\nu_{\min}, \nu_{\max}$ and $\sigma_{\min}, \sigma_{\max}$ such that

$$0 < \nu_{\min} \le \nu \le \nu_{\max} < \infty, \qquad 0 \le \sigma_{\min} \le \sigma \le \sigma_{\max} < \infty \tag{2.1}$$

holds almost everywhere in Ω. Denoting by $(\cdot,\cdot)_\Omega$ the $L^2(\Omega)$-inner product, the problem's bilinear forms $a\colon V \times V \to \mathbb{R}$ and $b\colon V \times M \to \mathbb{R}$ are defined as

$$a(u,v) \coloneqq (\nu\nabla u, \nabla v)_\Omega + (\sigma u, v)_\Omega, \qquad b(u,q) \coloneqq -(q, \nabla\cdot u)_\Omega.$$

Given a source term $f \in L^2(\Omega)$, the weak formulation of the considered heterogeneous Stokes problem seeks a pair $(u,p) \in V \times M$ such that

$$a(u,v) \;+\; b(v,p) \;=\; (f,v)_\Omega, \tag{2.2a}$$

$$b(u,q) \;=\; 0 \tag{2.2b}$$

holds for all $(v,q) \in V \times M$.

Using the uniform coefficient bounds (2.1) one can show that the bilinear form a is continuous and coercive, i.e., there exist constants $c_a, C_a > 0$ such that

$$|a(v,v)| \ge c_a \|\nabla v\|^2_\Omega, \qquad |a(u,v)| \le C_a \|\nabla u\|_\Omega \|\nabla v\|_\Omega \tag{2.3}$$

holds for all $u, v \in V$, where $\|\cdot\|^2_\Omega \coloneqq (\cdot,\cdot)_\Omega$ denotes the $L^2(\Omega)$-norm. Note that by the Poincaré–Friedrichs inequality, the seminorm $\|\nabla\cdot\|_\Omega$ is equivalent to the full $(H^1(\Omega))^n$-norm. The constants in (2.3) can be specified as $c_a = \nu_{\min}$ and $C_a = \nu_{\max} + C^2_{\mathrm{PF}}\sigma_{\max}$, where $C_{\mathrm{PF}} > 0$ denotes the Poincaré–Friedrichs constant.

To establish the well-posedness of problem (2.2), we need a compatibility condition between the spaces V and M, expressed as the inf–sup condition

$$\inf_{q\in M} \sup_{v\in V} \frac{|b(v,q)|}{\|\nabla v\|_\Omega \|q\|_\Omega} \ge c_b, \tag{2.4}$$

where $c_b > 0$ is typically called the inf–sup constant. This condition is classical and it is typically proved using the so-called Ladyzhenskaya lemma, cf. [Lad63]. It states that for any $q \in M$ there exists $v \in V$ such that

$$\nabla\cdot v = q, \qquad \|\nabla v\|_\Omega \le C_{\mathrm{L}} \|q\|_\Omega, \tag{2.5}$$

which directly implies the inf–sup stability with inf–sup constant $c_b = C_{\mathrm{L}}^{-1}$. After establishing conditions (2.3) and (2.4), the well-posedness of weak formulation (2.2) can be concluded using classical inf–sup theory; see, e.g., [BBF13].

3. Prototypical multiscale method

This section introduces a prototypical multiscale method that achieves optimal order approximations without preasymptotic effects under minimal structural assumptions on the coefficients. To this end, we introduce a geometrically conforming, quasi-uniform, and shape-regular hierarchy of simplicial[1] meshes $\{\mathcal{T}_H\}_H$. Each mesh is a finite subdivision of the closure of Ω, into closed elements K, which are n-dimensional simplices. The parameter $H > 0$ denotes the mesh size and is defined as the maximum diameter of the elements in $\mathcal{T}_H$, i.e., $H \coloneqq \max_{K\in\mathcal{T}_H} \operatorname{diam}(K)$. We further denote the space of $\mathcal{T}_H$-piecewise constant functions by $\mathbb{P}^0(\mathcal{T}_H)$ and write $\Pi_H \colon L^2(\Omega) \to \mathbb{P}^0(\mathcal{T}_H)$ for the corresponding L^2-orthogonal projection. The set of all faces of the mesh $\mathcal{T}_H$ is denoted by $\mathcal{F}_H$ and the subset of interior faces by $\mathcal{F}_H^i$.

For the construction of the prototypical multiscale method we will use an equivalent reformulation of problem (2.2). This reformulation is based on the spaces

$$Z \coloneqq \big\{ v \in V \,:\, \nabla\cdot v \in \mathbb{P}^0(\mathcal{T}_H)\big\}, \qquad M_H \coloneqq M \cap \mathbb{P}^0(\mathcal{T}_H), \tag{3.1}$$

where the space Z partially integrates the divergence-free constraint into the velocity space. Thus, the smaller space M_H is sufficient to enforce that the velocity is divergence-free. The reformulation seeks $(u, p_H) \in Z \times M_H$ such that

$$a(u,v) \quad + \quad b(v,p_H) \quad = \quad (f,v)_\Omega, \tag{3.2a}$$

$$b(u,q_H) \qquad\qquad = \quad 0. \tag{3.2b}$$

holds for all $(v, q_H) \in Z \times M_H$. To prove the well-posedness of this reformulated problem we verify the corresponding inf–sup condition. This inf–sup condition can be shown to hold with the constant c_b from (2.4), using again the Ladyzhenskaya lemma, cf. (2.5). It is easy to verify that the first component of the solution to the reformulated problem coincides with the velocity u from (2.2) and that for the second component we have $p_H = \Pi_H p$, where p is the pressure from (2.2).

Following the construction of the LOD for diffusion-type problems, cf. [MP14, MP20], we consider a decomposition of the space Z into the direct sum of two subspaces. The first one, typically referred to as fine-scale space, contains functions that average out on coarse scales and is defined as

$$W \coloneqq \big\{ v \in Z \,:\, \textstyle\int_F v \,\mathrm{d}\sigma = 0,\ F \in \mathcal{F}_H^i \big\}. \tag{3.3}$$

The second subspace is finite-dimensional and will serve as the approximation space of the prototypical LOD method. It is defined as the orthogonal complement of W with respect to the energy inner product a, i.e.,

$$\tilde{Z}_H \coloneqq \big\{ u \in Z \,:\, a(u,v) = 0,\ v \in W \big\}. \tag{3.4}$$

Note that, since $\tilde{Z}_H$ is constructed as the orthogonal complement of W with respect to the problem-dependent inner product a, it contains problem-specific information that allows reliable approximations even at coarse scales. The use of tildes in the notation of functions and spaces is intended to emphasize that they are adapted to the problem at hand. The following lemma constructs a basis of the space $\tilde{Z}_H$.

[1] One can also use general polygonal/polyhedral meshes. We assume that the meshes are simplicial only to simplify the presentation. The method itself can be applied to more general meshes in a straightforward manner. An extension to meshes with curved elements is also possible.

Lemma 3.1 (Prototypical basis). *The space $\tilde{Z}_H$ has dimension $N := n \cdot \#\mathcal{F}_H^i$ with $\#\mathcal{F}_H^i$ denoting the number of interior faces, and a basis of it is given by*

$$\big\{\tilde{\varphi}_{F,j} \,:\, F \in \mathcal{F}_H^i,\ j = 1,\dots,n\big\} \tag{3.5}$$

with $\tilde{\varphi}_{F,j}$ defined for all $F \in \mathcal{F}_H^i$ and $j = 1,\dots,n$ as the unique solutions to: seek $(\tilde{\varphi}_{F,j}, \xi_{F,j}, \lambda) \in V \times Q \times \mathbb{R}^N$ with $Q := \{q \in M \,:\, \int_K q \,\mathrm{d}x = 0,\ K \in \mathcal{T}_H\}$ such that

$$\begin{aligned}
&a(\tilde{\varphi}_{F,j}, v) \quad + \quad b(v, \xi_{F,j}) \quad + \quad c(v, \lambda) \quad = \quad 0, && \text{(3.6a)}\\
&b(\tilde{\varphi}_{F,j}, \chi) \qquad\qquad\qquad\qquad\qquad\qquad\;\; = \quad 0, && \text{(3.6b)}\\
&c(\tilde{\varphi}_{F,j}, \mu) \qquad\qquad\qquad\qquad\qquad\qquad\;\; = \quad \mu_{F,j} && \text{(3.6c)}
\end{aligned}$$

holds for all $(v, \chi, \mu) \in V \times Q \times \mathbb{R}^N$. Here, we label the entries of a vector $\mu \in \mathbb{R}^N$ using face-index pairs as $\mu = \big\{\mu_{E,k} \,:\, E \in \mathcal{F}_H^i,\ k = 1,\dots,n\big\}$, and the bilinear form $c\colon V \times \mathbb{R}^N \to \mathbb{R}$ is defined as

$$c(v, \mu) := \sum_{E \in \mathcal{F}_H^i} \sum_{k=1}^{n} \mu_{E,k} \int_E v \cdot e_k \,\mathrm{d}\sigma,$$

where $\{e_k \,:\, k = 1,\dots,n\}$ denotes the canonical basis of $\mathbb{R}^n$.

The prototypical basis functions are chosen such that they minimize the energy (associated with the inner product a) under the constraints that their divergence is $\mathcal{T}_H$-piecewise constant and that a Kronecker delta property with respect to the quantities of interest $v \mapsto \int_F v \cdot e_j \,\mathrm{d}\sigma$ is satisfied. The Kronecker delta property ensures that the set (3.5) is linearly independent and therefore is a basis of $\tilde{Z}_H$. Problem (3.6) for the prototypical basis functions differs from the corresponding one in the classical elliptic setting, cf. [AHP21, Sec. 3], mainly in the additional constraint to enforce that the divergence of the basis function is $\mathcal{T}_H$-constant.

Before proving this lemma, we introduce some technical tools that will be used not only in its proof, but also on several other occasions in the remainder of this paper. First, we note that the Ladyzhenskaya lemma stated in (2.5) for the whole domain Ω also holds locally on elements $K \in \mathcal{T}_H$. More precisely, there exists a constant $C'_{\mathrm{L}} > 0$ such that, for every element $K \in \mathcal{T}_H$ and all functions $q \in L^2(K)$ with $\int_K q \,\mathrm{d}x = 0$, there exists $v \in (H_0^1(K))^n$ such that

$$\nabla \cdot v = q, \qquad \|\nabla v\|_K \leq C'_{\mathrm{L}} \|q\|_K \tag{3.7}$$

holds, where the constant depends only on the shape regularity of the mesh. The latter statement can be inferred, for example, from [BCDG16], where the shape-dependence of the Ladyzhenskaya constant is investigated. The locally supported functions v from (3.7), hereafter referred to as element bubble functions, will be frequently used to estimate the bilinear form b.

Furthermore, for estimating the bilinear form c, we introduce face bubble functions denoted by $\{b_{F,j} \,:\, F \in \mathcal{F}_H^i,\ j = 1,\dots,n\}$. Each bubble function is locally supported with $b_{F,j} \in (H_0^1(\omega_F))^n$, where ω_F is the union of the two elements that share face F. The face bubbles are chosen such that $\int_E b_{F,j} \cdot e_k \,\mathrm{d}\sigma = \delta_{EF}\delta_{jk}$ holds for all $E, F \in \mathcal{F}_H^i$ and $j, k = 1,\dots,n$, and the two stability estimates

$$\|b_{F,j}\|_{\omega_F} \leq C_{\mathrm{b}} H^{-n/2+1}, \qquad \|\nabla b_{F,j}\|_{\omega_F} \leq C_{\mathrm{b}} H^{-n/2} \tag{3.8}$$

are satisfied for a constant $C_{\mathrm{b}} > 0$ independent of H. Additionally, we demand that the divergence of the face bubbles is piecewise constant, i.e., $b_{F,j} \in Z$. Note that bubbles with these properties can be constructed using classical face bubbles, which we denote by ψ_F below, cf. [VZ19]. They satisfy $\int_E \psi_F \,\mathrm{d}\sigma = \delta_{EF}$ for any $E, F \in \mathcal{F}_H^i$. We then define $b_{F,j} := \psi_F e_j + v_{F,j}$, where $v_{F,j}$ is supported on ω_F and defined locally on $K \subset \omega_F$ as the $(H_0^1(K))^n$-function satisfying $\nabla \cdot v_{F,j} =$

$-\nabla \cdot (\psi_F e_j) + n_{F,K} \cdot e_j$, where $n_{F,K}$ denotes the unit normal on F outward of K. The function $v_{F,j}$ exist thanks to (3.7) and satisfies $\|\nabla v_{F,j}\|_K \leq C'_{\mathrm{L}} \|\nabla \psi_F\|_K$.

Now let us proceed with the proof of the lemma above.

Proof of Lemma 3.1. We first prove the well-posedness of problem (3.6). To this end, we verify the inf–sup stability of the bilinear form $d\colon V \times (Q \times \mathbb{R}^N)$ defined as $d(v,(\chi,\mu)) \coloneqq b(v,\chi) + c(v,\mu)$, i.e., there exists a constant $c_d > 0$ such that

$$\inf_{(\chi,\mu)\in Q\times\mathbb{R}^N} \sup_{v\in V} \frac{|d(v,(\chi,\mu))|}{\|\nabla v\|_\Omega \|(\chi,\mu)\|} \geq c_d \tag{3.9}$$

holds, where $\|(\chi,\mu)\|^2 \coloneqq \|\chi\|_\Omega^2 + |\mu|^2$ with $|\cdot|$ denoting the Euclidean norm.

Given any $(\chi,\mu) \in Q \times \mathbb{R}^N$, we choose a function $v \in V$ defined as

$$v \coloneqq \sum_{K\in\mathcal{T}_H} v_K + \sum_{F\in\mathcal{F}_H^i} \sum_{j=1}^n \mu_{F,j} b_{F,j},$$

where, for any $K \in \mathcal{T}_H$, $v_K \in (H_0^1(K))^n$ satisfies $\nabla \cdot v_K = \chi$ locally on K as well as $\|\nabla v_K\|_K \leq C'_{\mathrm{L}}\|\chi\|_K$, cf. (3.7). By construction, it holds $|d(v,(\chi,\mu))| = \|(\chi,\mu)\|^2$ and we have for any $K \in \mathcal{T}_H$ that

$$\begin{aligned}
\|\nabla v\|_K^2 &\leq 2(C'_{\mathrm{L}})^2 \|\chi\|_K^2 + 2\Bigg(\sum_{F\in\mathcal{F}_H^i\cap\partial K} \sum_{j=1}^n \mu_{F,j}^2\Bigg)\Bigg(\sum_{F\in\mathcal{F}_H^i\cap\partial K} \sum_{j=1}^n \|\nabla b_{F,j}\|_K^2\Bigg) \\
&\leq 2(C'_{\mathrm{L}})^2 \|\chi\|_K^2 + 2n(n+1)C_{\mathrm{b}}^2 H^{-n} \Bigg(\sum_{F\in\mathcal{F}_H^i\cap\partial K} \sum_{j=1}^n \mu_{F,j}^2\Bigg),
\end{aligned}$$

where we used estimates (3.7) and (3.8). Summing over all elements gives

$$\|\nabla v\|_\Omega^2 \leq \max\{2(C'_{\mathrm{L}})^2, 4n(n+1)C_{\mathrm{b}}^2 H^{-n}\} \|(\chi,\mu)\|^2,$$

which proves inf–sup condition (3.9) with the constant c_d, defined as the reciprocal of the constant in the estimate above.

Next, we prove that (3.5) constitutes a basis of the space $\tilde{Z}_H$. To this end, we first note that conditions (3.6a) and (3.6b) imply that $\tilde{\varphi}_{F,j} \in \tilde{Z}_H$ for all $F \in \mathcal{F}_H^i$ and $j \in \{1,\dots,n\}$. To prove the basis property, we consider an arbitrary $u \in \tilde{Z}_H$ and set $w = u - \sum_{F\in\mathcal{F}_H^i} \sum_{j=1}^n \int_F u \cdot e_j \,\mathrm{d}\sigma \tilde{\varphi}_{F,j}$. Since $w \in W$ and, at the same time, w is a-orthogonal to W, it must hold that $w = 0$. Thus, any $u \in \tilde{Z}_H$ can be written as a unique linear combination of the linearly independent functions $\tilde{\varphi}_{F,j}$, which shows that they form a basis of the space $\tilde{Z}_H$. This concludes the proof. □

Having introduced the prototypical basis functions, we can define a projection operator $\mathcal{R}\colon Z \to \tilde{Z}_H$ which preserves face integrals for any $v \in Z$ by

$$\mathcal{R}v \coloneqq \sum_{F\in\mathcal{F}_H^i} \sum_{j=1}^n \int_F v \cdot e_j \,\mathrm{d}\sigma\, \tilde{\varphi}_{F,j}. \tag{3.10}$$

This operator coincides with the a-orthogonal projection onto $\tilde{Z}_H$, since for any $v \in Z$ and $w \in \tilde{Z}_H$ it holds that $a(v - \mathcal{R}v, w)$, noting that $v - \mathcal{R}v \in W$. This orthogonality property also implies the continuity of $\mathcal{R}$, i.e., for any $v \in Z$ it holds

$$\|\nabla \mathcal{R}v\|_\Omega \leq \sqrt{C_a/c_a}\|\nabla v\|_\Omega, \tag{3.11}$$

where we used (2.3). Furthermore,

$$(\nabla \cdot \mathcal{R}v)|_K = (\nabla \cdot v)|_K \tag{3.12}$$

holds for any $v \in Z$ and $K \in \mathcal{T}_H$. By the definitions of Z, we have that $\nabla \cdot \mathcal{R}v$ and $\nabla \cdot v$ are $\mathcal{T}_H$-piecewise constant functions. Using the fact that $\mathcal{R}$ preserves face integrals, we therefore obtain with the divergence theorem that $\int_K \nabla\cdot\mathcal{R}v\,\mathrm{d}x = \int_K \nabla\cdot v\,\mathrm{d}x$. Identity (3.12) will play an important role in the analysis of the prototypical method, and can be used, for example, to prove that its approximation is exactly divergence-free. Having such an identity is the main reason for defining the fine-scale space W using face integrals as in (3.3).

The desired prototypical method then seeks $(\tilde{u}_H,\tilde{p}_H)\in\tilde{Z}_H\times M_H$ such that

$$a(\tilde{u}_H,\tilde{v}_H) + b(\tilde{v}_H,\tilde{p}_H) = (f,\tilde{v}_H)_\Omega, \tag{3.13a}$$

$$b(\tilde{u}_H,\tilde{q}_H) = 0. \tag{3.13b}$$

holds for all $(\tilde{v}_H,\tilde{q}_H)\in\tilde{Z}_H\times M_H$. For the analysis of this method in the theorem below we need an approximation result for Π_H. Using the Poincaré inequality on convex domains, cf. [PW60], it follows that for all $K\in\mathcal{T}_H$ and $v\in H^1(K)$ it holds

$$\|v-\Pi_H v\|_K \le \pi^{-1}H\|\nabla v\|_K, \tag{3.14}$$

where $\pi>0$ denotes the ratio of the circumference of a circle to its diameter.

Theorem 3.2 (Prototypical method)**.** *The prototypical multiscale method* (3.13) *is well-posed, and its solution is given by* $(\tilde{u}_H,\tilde{p}_H)=(\mathcal{R}u,\Pi_H p)$*, where* (u,p) *solves problem* (2.2)*. Furthermore, there exist constants* $C_\mathrm{e},C'_\mathrm{e}>0$ *independent of* H *such that for any right-hand side* $f\in H^s(\Omega)$ *with* $s\in\{0,1\}$*, we have the error estimates*

$$\|u-\tilde{u}_H\|_\Omega \le C_\mathrm{e}H^{2+s}|f|_{s,\Omega}, \tag{3.15}$$

$$\|\nabla(u-\tilde{u}_H)\|_\Omega \le C'_\mathrm{e}H^{1+s}|f|_{s,\Omega}, \tag{3.16}$$

where $|\cdot|_{s,\Omega}$ *denotes the* $H^s(\Omega)$*-seminorm.*

Proof. First, we prove the inf–sup condition

$$\inf_{\tilde{q}_H\in M_H}\sup_{\tilde{v}_H\in\tilde{Z}_H}\frac{|b(\tilde{v}_H,\tilde{q}_H)|}{\|\nabla\tilde{v}_H\|_\Omega\|\tilde{q}_H\|_\Omega}\ge\tilde{c}_b \tag{3.17}$$

for some constant $\tilde{c}_b>0$, which implies the well-posedness of (3.13). Given any $\tilde{q}_H\in M_H$, we denote by $v\in V$ the function with $\nabla\cdot v=\tilde{q}_H$ and $\|\nabla v\|_\Omega\le C_\mathrm{L}\|\tilde{q}_H\|_\Omega$, cf. (2.5), and define $\tilde{v}_H:=\mathcal{R}v$. One can show that $|b(\tilde{v}_H,\tilde{q}_H)|=|b(v,\tilde{q}_H)|=\|\tilde{q}_H\|^2_\Omega$ using (3.12), and that $\|\nabla\tilde{v}_H\|_\Omega\le\sqrt{C_a/c_a}\|\nabla v\|\le C_\mathrm{L}\sqrt{C_a/c_a}\|\tilde{q}_H\|_\Omega$ by the continuity of $\mathcal{R}$, cf. (3.11), and the particular choice of v. Combining these results, inf–sup condition (3.17) follows directly with the constant $\tilde{c}_b=(C_\mathrm{L}\sqrt{C_a/c_a})^{-1}$. Using that $\mathcal{R}\colon Z\to\tilde{Z}_H$ is a-orthogonal, identity (3.12), as well as that (u,p_H) solves (3.2), it is a straightforward observation that $(\mathcal{R}u,p_H)$ solves (3.13). The uniqueness of the solution to (3.13) then implies that $\tilde{u}_H=\mathcal{R}u$ and $\tilde{p}_H=p_H$.

Second, we derive error estimates for the prototypical multiscale method. To this end, we note that the error $e:=u-\tilde{u}_H$ is an element of the space W. Furthermore, we have for all $K\in\mathcal{T}_H$ that $(\nabla\cdot\tilde{u}_H)|_K=(\nabla\cdot u)|_K=0$, where we apply (3.12) to u and note that $\tilde{u}_H=\mathcal{R}u$. This implies that $\nabla\cdot e=\nabla\cdot\tilde{u}_H=0$. Using the first estimate in (2.3), that $a(\tilde{u}_H,e)=0$, (3.2a) with e as test function, that $b(e,p_H)=0$, and the local Poincaré-type inequality from Lemma A.1, we obtain that

$$c_a\|\nabla e\|^2_\Omega\le a(e,e)=a(u,e)=(f,e)_\Omega\le C_\mathrm{P}H\|f\|_\Omega\|\nabla e\|_\Omega, \tag{3.18}$$

which implies the desired H^1-error estimate for $s=0$.

In the case $s=1$, we add and subtract $\Pi_H f$ in (3.18), the L^2 projection of f onto piecewise constants, which results in

$$c_a\|\nabla e\|^2_\Omega\le(f-\Pi_H f,e)_\Omega+(\Pi_H f,e)_\Omega, \tag{3.19}$$

where the first term on the right-hand side can be bounded using approximation result (3.14) and the Poincaré-type inequality from Lemma A.1 as

$$(f-\Pi_H f,e)_\Omega\le\|f-\Pi_H f\|_\Omega\|e\|_\Omega\le\pi^{-1}C_\mathrm{P}H^2\|\nabla f\|_\Omega\|\nabla e\|_\Omega. \tag{3.20}$$

Abbreviating $f_K \coloneqq (\Pi_H f)|_K$, the second term on the right-hand side of (3.19) can be locally rewritten using the divergence theorem and that $\nabla \cdot e = 0$ as

$$(\Pi_H f, e)_K = \int_K \nabla(f_K \cdot x) \cdot e \,\mathrm{d}x = \int_{\partial K} (f_K \cdot x)(e \cdot n)\,\mathrm{d}\sigma. \tag{3.21}$$

Summing over all elements using that $\int_F e \cdot n_F \,\mathrm{d}\sigma = 0$ for all $F \in \mathcal{F}_H^i$ then gives

$$(\Pi_H f, e)_\Omega = \sum_{F \in \mathcal{F}_H^i} \int_F \big([\Pi_H f]_F \cdot (x - x_F)\big)(e \cdot n_F)\,\mathrm{d}\sigma, \tag{3.22}$$

where we denote by x_F the barycenter of F and by $[\cdot]_F$ the jump of a function across face F chosen consistently with the fixed normal n_F associated to F.

To estimate the term on the right-hand side of the previous equation, we combine the approximation result of (3.14) with Lemma A.2, which gives for any $F \in \mathcal{F}_H^i$ and $K \in \mathcal{T}_H$ with $\partial K \supset F$ that $\|v - v_K\|_F^2 \leq CH\|\nabla v\|_K^2$ holds for any $v \in H^1(K)$ with $v_K \coloneqq (\Pi_H v)|_K$ and the constant $C \coloneqq C_{\mathrm{T}}(1+\pi^{-1})\pi^{-1}$. Applying this estimate to e and f, noting that $\|e\|_F^2 \leq \|e - e_K\|_F^2$ and $[\Pi_H f]_F = -[f - \Pi_H f]_F$ yields that

$$\|e\|_F^2 \leq \tfrac{1}{2} CH \|\nabla e\|_{\omega_F}^2, \qquad \|[\Pi_H f]_F^2\|_F^2 \leq 2CH\|\nabla f\|_{\omega_F}^2.$$

Combining (3.22) with the latter bounds and using the Cauchy–Schwarz inequality, we obtain for the second term on the right-hand side of (3.19) that

$$(\Pi_H f, e)_\Omega \leq 2(n+1)CH^2 \|\nabla f\|_\Omega \|\nabla e\|_\Omega. \tag{3.23}$$

Inserting estimates (3.20) and (3.23) into (3.19) yields the desired H^1-error estimate in the case that $s = 1$. Therefore, taking the maximum of the constants yields estimate (3.16) with $C_{\mathrm{e}}' \coloneqq \max\{c_a^{-1} C_{\mathrm{P}}, c_a^{-1}(\pi^{-1} C_{\mathrm{P}} + 2(n+1)C)\}$.

The L^2-error estimate (3.15) with constant $C_{\mathrm{e}} \coloneqq C_{\mathrm{P}} C_{\mathrm{e}}'$ can be proved by applying Lemma A.1 once again. This completes the proof. □

We emphasize that squeezing out an additional order of convergence for H^1-regular right-hand sides only works in the context of Stokes problems and not for diffusion-type problems. This observation can also be verified numerically.

4. Exponential decay and localization

We emphasize that the prototypical LOD basis functions defined in (3.5) are globally supported. Therefore, their computation would require the solution to global problems, which we consider infeasible. In this section, we show that the prototypical LOD basis functions decay exponentially, which motivates their approximation by locally computable counterparts. A practical multiscale method based on such local approximations is presented in Section 5.

To quantify the decay of the basis functions, we introduce the notion of patches with respect to the coarse mesh $\mathcal{T}_H$. Given an oversampling parameter $\ell \in \mathbb{N}$, where $\mathbb{N}$ denotes the (positive) natural numbers, we define the patch of order ℓ around a union of elements $S \subset \Omega$ recursively by

$$\mathsf{N}^\ell(S) \coloneqq \mathsf{N}^1(\mathsf{N}^{\ell-1}(S)), \quad \ell \geq 2, \qquad \mathsf{N}^1(S) \coloneqq \bigcup\big\{K \in \mathcal{T}_H : \overline{S} \cap \overline{K} \neq \emptyset\big\}, \tag{4.1}$$

and set $\mathsf{N}(S) \coloneqq \mathsf{N}^1(S)$. The notion of patches can also be extended to faces by defining, for any $F \in \mathcal{F}_H$, the patch of order ℓ around F by $\mathsf{N}^\ell(F) = \mathsf{N}^\ell(\omega_F)$, where ω_F denotes the union of the two elements sharing the face F.

The following theorem proves the exponential decay of the prototypical LOD basis functions. The exponential decay proof is based on cut-off techniques, where the H^1-seminorm of the basis function outside a patch of coarse elements is estimated against the norm on a ring of elements at the patch boundary. This estimate can then be reformulated and iterated, leading to the desired exponential decay result;

see also the proofs in [MP20, AHP21]. On an intuitive level, the energy minimization property of the basis functions, while constrained by a Kronecker delta property of the quantities of interest, cf. (3.6), triggers oscillations on the length scale H and leads to an exponential decay with respect to the coarse mesh.

Theorem 4.1 (Exponential decay)**.** *There exists a constant $c > 0$ independent of H, ℓ, F, and j such that for all $F \in \mathcal{F}_H^i$, $j \in \{1, \dots, n\}$, and $\ell \in \mathbb{N}$, it holds that*

$$\|\nabla \tilde{\varphi}_{F,j}\|_{\Omega \setminus \mathsf{N}^\ell(F)} \leq \exp(-c\ell) \|\nabla \tilde{\varphi}_{F,j}\|_\Omega. \tag{4.2}$$

Proof. In the following, we will use the abbreviation $\tilde{\varphi} \coloneqq \tilde{\varphi}_{F,j}$ and consider a cut-off function $\eta \in W^{1,\infty}(\Omega)$ with the properties

$$\begin{cases} \eta \equiv 0 & \text{in } \mathsf{N}^{\ell-1}(F), \\ \eta \equiv 1 & \text{in } \Omega \setminus \mathsf{N}^\ell(F), \\ 0 \leq \eta \leq 1 & \text{in } R \coloneqq \mathsf{N}^\ell(F) \setminus \mathsf{N}^{\ell-1}(F) \end{cases} \tag{4.3}$$

and the bound

$$\sup_{x \in \Omega} |\nabla \eta| \leq C_\eta H^{-1}, \tag{4.4}$$

where $C_\eta > 0$ is a constant independent of H.

Using $\eta\tilde{\varphi}$ as a test function in (3.6a) gives that

$$a(\tilde{\varphi}, \eta\tilde{\varphi}) = -b(\eta\tilde{\varphi}, \xi) - c(\eta\tilde{\varphi}, \lambda),$$

which, using subscripts to denote the restrictions of the bilinear forms a, b, and c to subdomains, can be rewritten as

$$a_{\Omega \setminus \mathsf{N}^\ell(F)}(\tilde{\varphi}, \tilde{\varphi}) = -a_R(\tilde{\varphi}, \eta\tilde{\varphi}) - b_R(\eta\tilde{\varphi}, \xi) - c_R(\eta\tilde{\varphi}, \lambda). \tag{4.5}$$

Here, we dropped the subscript of ξ and used that $\operatorname{supp}(\eta\tilde{\varphi}) \subset (\Omega \setminus \mathsf{N}^\ell(F)) \cup R$, that the divergence of $\eta\tilde{\varphi}$ is piecewise constant on $\Omega \setminus R$, and that $\int_E \eta\tilde{\varphi} \, \mathrm{d}\sigma = 0$ for all faces E not inside R. Using coefficient bound (2.1), we can estimate (4.5) as

$$\begin{aligned} c_a \|\nabla \tilde{\varphi}\|^2_{\Omega \setminus \mathsf{N}^\ell(F)} &\leq -a_R(\tilde{\varphi}, \eta\tilde{\varphi}) - b_R(\eta\tilde{\varphi}, \xi) - c_R(\eta\tilde{\varphi}, \lambda) \\ &=: \Xi_1 + \Xi_2 + \Xi_3. \end{aligned} \tag{4.6}$$

To estimate Ξ_1, we again use coefficient bound (2.1), the bound on η from (4.4), and the Poincaré-type inequality from Lemma A.1 to get that

$$\Xi_1 \leq C'_a \|\nabla \tilde{\varphi}\|_R \|\nabla(\eta\tilde{\varphi})\|_R \leq C'_a(1 + C_\eta C_{\mathrm{P}}) \|\nabla \tilde{\varphi}\|^2_R \tag{4.7}$$

with the constant $C'_a \coloneqq (\nu_{\max} + C_{\mathrm{P}}^2 \sigma_{\max})$. Above we used that, for any element $K \subset R$, there exists a face $E \subset \partial K$ such that $\int_E \eta\tilde{\varphi} \, \mathrm{d}\sigma = \int_E \tilde{\varphi} \, \mathrm{d}\sigma = 0$.

Using similar arguments, we obtain for Ξ_2 that

$$\Xi_2 \leq \|\nabla \cdot (\eta\tilde{\varphi})\|_R \|\xi\|_R \leq (1 + C_\eta C_{\mathrm{P}}) \|\nabla \tilde{\varphi}\|_R \|\xi\|_R. \tag{4.8}$$

To continue the latter estimate, we need a local bound for $\|\xi\|_K$. To this end, we test (3.6a) for any element K with $v_\xi \in (H_0^1(K))^n$ chosen such that $\nabla \cdot v_\xi = \xi$ holds locally in K and $\|\nabla v_\xi\|_K \leq C'_{\mathrm{L}} \|\xi\|_K$ is satisfied, cf. (3.7). This results in

$$\|\xi\|^2_K = a(\tilde{\varphi}, v_\xi) \leq C_a C'_{\mathrm{L}} \|\nabla \tilde{\varphi}\|_K \|\xi\|_K. \tag{4.9}$$

Summing the latter bound over all elements with $K \subset R$, we obtain an estimate for $\|\xi\|_R$ which can be inserted in (4.8) to conclude

$$\Xi_2 \leq (1 + C_\eta C_{\mathrm{P}}) C_a C'_{\mathrm{L}} \|\nabla \tilde{\varphi}\|^2_R.$$

To estimate Ξ_3, we apply the Cauchy–Schwarz inequality for all faces $E \in \mathcal{F}_H^i(R)$, where $\mathcal{F}_H^i(R)$ denotes the set of faces inside R, which yields that

$$\Xi_3 = -\sum_{E\in\mathcal{F}_H^i(R)}\sum_{k=1}^n |\lambda_{E,k}| \int_E \eta\tilde{\varphi}\cdot e_k n\sigma \leq \sum_{E\in\mathcal{F}_H^i(R)}\sum_{k=1}^n |\lambda_{E,k}|\, |E|_{n-1}^{1/2} \|\tilde{\varphi}\cdot e_k\|_E,$$

where $|\cdot|_{n-1}$ denotes the $(n-1)$-dimensional volume. We derive a bound for $|\lambda_{E,k}|$ for all faces $E \in \mathcal{F}_H^i(R)$ by testing (3.6a) with the bubble function $b_{E,k}$, which gives

$$|\lambda_{E,k}| \leq C_a \|\nabla\tilde{\varphi}\|_{\omega_E} \|\nabla b_{E,k}\|_{\omega_E} \leq C_a C_{\mathrm{b}} H^{-n/2} \|\nabla\tilde{\varphi}\|_{\omega_E}. \tag{4.10}$$

Here, we used the second estimates of (2.3) and (3.8) and that $b(b_{E,k},\xi) = 0$ since the divergence of $b_{E,k}$ is piecewise constant and ξ has zero element averages.

Noting that $|E|_{n-1} \leq H^{n-1}$, the latter estimate can be used to show that

$$\begin{aligned}\Xi_3 &\leq C_a C_{\mathrm{b}} H^{-1/2} \sum_{E\in\mathcal{F}_H^i(R)}\sum_{k=1}^n \|\nabla\tilde{\varphi}\|_{\omega_E}\|\tilde{\varphi}\cdot e_k\|_E \\ &\leq C_a C_{\mathrm{b}}\Big(n\sum_{E\in\mathcal{F}_H^i(R)} \|\nabla\tilde{\varphi}\|_{\omega_E}^2\Big)^{1/2}\Big(\frac{1}{H}\sum_{E\in\mathcal{F}_H^i(R)} \|\tilde{\varphi}\|_E^2\Big)^{1/2},\end{aligned}$$

where we applied the discrete Cauchy–Schwarz inequality. Using the trace inequality from Lemma A.2, we obtain that

$$\Xi_3 \leq 2C_a C_{\mathrm{b}}\sqrt{nC_{\mathrm{T}}(1+C_{\mathrm{P}})C_{\mathrm{P}}}\|\nabla\tilde{\varphi}\|_R^2.$$

Inserting the above estimates for Ξ_1, Ξ_2, and Ξ_3 into (4.6) and noting that R can be rewritten as $R = (\Omega \setminus \mathsf{N}^{\ell-1}(F)) \setminus (\Omega \setminus \mathsf{N}^{\ell}(F))$, we obtain that

$$\|\nabla\tilde{\varphi}\|_{\Omega\setminus\mathsf{N}^\ell(F)}^2 \leq C\|\nabla\tilde{\varphi}\|_R^2 = C\|\nabla\tilde{\varphi}\|_{\Omega\setminus\mathsf{N}^{\ell-1}(F)}^2 - C\|\nabla\tilde{\varphi}\|_{\Omega\setminus\mathsf{N}^\ell(F)}^2$$

with the constant

$$C := c_a^{-1}\big(C_a'(1+C_\eta C_{\mathrm{P}}) + (1+C_\eta C_{\mathrm{P}})C_a C_{\mathrm{L}}' + 2C_a C_{\mathrm{b}}\sqrt{nC_{\mathrm{T}}(1+C_{\mathrm{P}})C_{\mathrm{P}}}\big) > 0.$$

As a direct consequence, we obtain that

$$\|\nabla\tilde{\varphi}\|_{\Omega\setminus\mathsf{N}^\ell(F)} \leq \sqrt{\frac{C}{1+C}}\,\|\nabla\tilde{\varphi}\|_{\Omega\setminus\mathsf{N}^{\ell-1}(F)},$$

which, after iterating, yields that

$$\|\nabla\tilde{\varphi}\|_{\Omega\setminus\mathsf{N}^\ell(F)} \leq \Big(\frac{C}{1+C}\Big)^{\ell/2}\|\nabla\tilde{\varphi}\|_\Omega = \exp(-c\ell)\|\nabla\tilde{\varphi}\|_\Omega$$

for the constant $c := \frac{1}{2}\log\big(\frac{1+C}{C}\big) > 0$. This concludes the proof. □

Motivated by the exponential decay of the prototypical LOD basis functions defined in (3.5), we will introduce localizations of them. These localized versions can be practically computed by solving local problems on subdomains and enable the sparse representation of the discrete operator. To formulate these local problems, we introduce localized versions of the spaces V and Q as

$$\begin{aligned} V_F^\ell &:= \{v \in V \,:\, \mathrm{supp}(v) \subset \mathsf{N}^\ell(F)\}, \\ Q_F^\ell &:= \{\chi \in Q \,:\, \mathrm{supp}(\chi) \subset \mathsf{N}^\ell(F)\}. \end{aligned} \tag{4.11}$$

Furthermore, we denote by $R_F^\ell \subset \mathbb{R}^N$ the subspace consisting of vectors μ with $\mu_{F,j} = 0$ for all faces not contained in the interior of the patch $\mathsf{N}^\ell(F)$, where we recall that $\mu_{F,j}$ is a notation for the entries of the vector μ.

Given an oversampling parameter ℓ, the localized basis functions, denoted by

$$\{\tilde{\varphi}_{F,j} \,:\, F \in \mathcal{F}_H^i,\ j = 1,\dots,n\},$$

are determined by the problems: Seek $(\tilde\varphi^\ell_{F,j},\xi^\ell_{F,j},\lambda)\in V^\ell_F\times Q^\ell_F\times R^\ell_F$ such that

$$\begin{aligned}
&(4.12a) && a(\tilde\varphi^\ell_{F,j},v) + b(v,\xi^\ell_{F,j}) + c(v,\lambda) = 0,\\
&(4.12b) && b(\tilde\varphi^\ell_{F,j},\chi) = 0,\\
&(4.12c) && c(\tilde\varphi^\ell_{F,j},\mu) = \mu_{F,j}
\end{aligned}$$

holds for all $(v,\chi,\mu)\in V^\ell_F\times Q^\ell_F\times R^\ell_F$. This saddle point problem is well-posed, which can be shown using similar arguments as in the proof of Lemma 3.1.

In the following, we will frequently use the bound

$$\|\nabla\tilde\varphi^\ell_{F,j}\|_\Omega \le c_a^{-1}C_aC_{\mathrm b}H^{-n/2}, \tag{4.13}$$

which can be proved by testing equation (4.12a) first with $\tilde\varphi^\ell_{F,j}$ and then with $b_{F,j}$, and combining the resulting estimates.

With the localized basis functions at hand, we are now able to define a localized counterpart of the operator $\mathcal R$ from (3.10). This operator, denoted by $\mathcal R^\ell$, is a projection onto the span of the localized basis functions with the property that it preserves face integrals. More precisely, it is for any $v\in Z$ defined as

$$\mathcal R^\ell v := \sum_{F\in\mathcal F_H^i}\sum_{j=1}^n \int_F v\cdot e_j\,\mathrm d\sigma\,\tilde\varphi^\ell_{F,j}. \tag{4.14}$$

We emphasize that, unlike $\mathcal R$, this projection is not a-orthogonal.

The following theorem proves that $\mathcal R^\ell$ approximates $\mathcal R$ exponentially well. Note that using this result for the bubble function $b_{F,j}$ gives an exponential approximation result for the corresponding localized and prototypical basis functions.

Theorem 4.2 (Localization error)**.** *There exist a constant $C_{\mathrm R}>0$ independent of H and ℓ such that for all $v\in Z$ and $\ell\in\mathbb N$, it holds that*

$$\|\nabla(\mathcal R-\mathcal R^\ell)v\|_\Omega \le C_{\mathrm R}\ell^{n/2}\exp(-c\ell)\big(\|\nabla v\|_\Omega + H^{-1}\|v\|_\Omega\big). \tag{4.15}$$

Proof. We consider an arbitrary but fixed $v\in Z$ and abbreviate $e:=(\mathcal R-\mathcal R^\ell)v$ and $v_{F,j}:=\int_F v\cdot e_j\,\mathrm d\sigma$, which allows us to write

$$e = \sum_{F\in\mathcal F_H^i}\sum_{j=1}^n v_{F,j}\,(\tilde\varphi_{F,j}-\tilde\varphi^\ell_{F,j}), \tag{4.16}$$

using the definitions of $\mathcal R$ and $\mathcal R^\ell$. Applying the first bound from (2.3) then gives

$$c_a\|\nabla e\|^2_\Omega \le \sum_{F\in\mathcal F_H^i}\sum_{j=1}^n v_{F,j}a(\tilde\varphi_{F,j}-\tilde\varphi^\ell_{F,j},e). \tag{4.17}$$

To estimate the terms on the right-hand side of the latter inequality, we fix a face $F\in\mathcal F^i_H$ and index $j\in\{1,\dots,n\}$ and recall the definition of the cut-off function in (4.3), which we now denote by η_F. Note that, by the definition of the space $\tilde Z_H$ and since $e\in W$, it holds that $a(\tilde\varphi_{F,j},e)=0$. Using this and observing that $(1-\eta_F)e\in V^\ell_F$ is an admissible test function in equation (4.12a), we obtain that

$$\begin{aligned}
a(\tilde\varphi_{F,j}-\tilde\varphi^\ell_{F,j},e) &= -a(\tilde\varphi^\ell_{F,j},(1-\eta_F)e+\eta_Fe)\\
&= b((1-\eta_F)e,\xi^\ell_{F,j}) + c((1-\eta_F)e,\lambda) - a(\tilde\varphi^\ell_{F,j},\eta_Fe)\\
&=: \Xi_1+\Xi_2+\Xi_3.
\end{aligned}\tag{4.18}$$

In the following, we estimate the terms Ξ_1, Ξ_2, and Ξ_3 separately. To estimate Ξ_1, we note that the divergence of $(1-\eta_F)e$ is piecewise constant on $\Omega\backslash R_F$ and that $\xi^\ell_{F,j}$ has zero element averages. This together with (4.4) and Lemma A.1 gives

$$\Xi_1 \le \|\nabla\cdot(1-\eta_F)e\|_{R_F}\|\xi^\ell_{F,j}\|_{R_F} \le (1+C_{\eta_F}C_{\mathrm P})\|\nabla e\|_{R_F}\|\xi^\ell_{F,j}\|_{R_F}. \tag{4.19}$$

To derive a L^2-bound for $\xi^\ell_{F,j}$, we proceed similarly as in (4.9), which yields that

$$\|\xi^\ell_{F,j}\|_{R_F} \leq C_a C'_{\mathrm{L}} \|\nabla \tilde{\varphi}^\ell_{F,j}\|_{R_F},$$

and inserting this into (4.19) gives

$$\Xi_1 \leq C_a C'_{\mathrm{L}}(1 + C_{\eta_F} C_{\mathrm{P}}) \|\nabla e\|_{R_F} \|\nabla \tilde{\varphi}^\ell_{F,j}\|_{R_F}.$$

To estimate Ξ_2, we apply the discrete Chauchy–Schwarz inequality, the trace inequality from Lemma A.2, the bound $|E|_{n-1} \leq H^{n-1}$, and the local Poincaré-type inequality from Lemma A.1, which yields that

$$\begin{aligned}
\Xi_2 &\leq \Big(\sum_{E \subset R_F} \sum_{k=1}^{n} \lambda^2_{E,k} \Big)^{1/2} \Big(\sum_{E \subset R_F} |E|_{n-1} \|e\|^2_E \Big)^{1/2} \\
&\leq \Big(\sum_{E \subset R_F} \sum_{k=1}^{n} \lambda^2_{E,k} \Big)^{1/2} \Big(C_{\mathrm{T}} \sum_{E \subset R_F} |E|_{n-1} \|e\|_K (\|\nabla e\|_K + H^{-1} \|e\|_K) \Big)^{1/2} \\
&\leq \Big(\sum_{E \subset R_F} \sum_{k=1}^{n} \lambda^2_{E,k} \Big)^{1/2} \sqrt{n C_{\mathrm{T}} C_{\mathrm{P}} (1 + C_{\mathrm{P}})} H^{n/2} \|\nabla e\|_{R_F}.
\end{aligned}$$

To derive a bound for $|\lambda_{E,k}|$, we proceed as in (4.10), but locally, which results in

$$|\lambda_{E,k}| \leq |a(\tilde{\varphi}^\ell_{F,j}, b_{E,k})| \leq C_a C_{\mathrm{b}} H^{-n/2} \|\nabla \tilde{\varphi}^\ell_{F,j}\|_{\omega_E}.$$

Using this estimate, the above estimate for Ξ_2 can be continued as

$$\Xi_2 \leq C_a C_{\mathrm{b}} n \sqrt{C_{\mathrm{T}} C_{\mathrm{P}} (1 + C_{\mathrm{P}})} \|\nabla e\|_{R_F} \|\nabla \tilde{\varphi}^\ell_{F,j}\|_{R_F}.$$

To estimate Ξ_3, we again use similar arguments as in (4.7), which yields that

$$\Xi_3 \leq C'_a \|\nabla(\eta_F e)\|_{R_F} \|\nabla \tilde{\varphi}^\ell_{F,j}\|_{R_F} \leq C'_a (1 + C_{\eta_F} C_{\mathrm{P}}) \|\nabla e\|_{R_F} \|\nabla \tilde{\varphi}^\ell_{F,j}\|_{R_F}$$

with $C'_a \coloneqq (\nu_{\max} + C^2_{\mathrm{P}} \sigma_{\max})$.

Inserting the above estimates for Ξ_1, Ξ_2, and Ξ_3 into (4.18) gives the bound

$$a(\tilde{\varphi}_{F,j} - \tilde{\varphi}^\ell_{F,j}, e) \leq C \|\nabla e\|_{R_F} \|\nabla \tilde{\varphi}^\ell_{F,j}\|_{R_F}$$

with

$$C \coloneqq \big((C_a C'_{\mathrm{L}} + C'_a)(1 + C_{\eta_F} C_{\mathrm{P}}) + C_a C_{\mathrm{b}} n \sqrt{C_{\mathrm{T}} C_{\mathrm{P}} (1 + C_{\mathrm{P}})} \big).$$

We can now apply the exponential decay result of Theorem 4.1 to $\tilde{\varphi}^\ell_{F,j}$ instead of $\tilde{\varphi}_{F,j}$, where we replace Ω in the statement of the theorem by $\mathsf{N}^\ell(F)$. Using this and bound (4.13) for the localized basis function then gives

$$a(\tilde{\varphi}_{F,j} - \tilde{\varphi}^\ell_{F,j}, e) \leq C \exp(-c\ell) \|\nabla e\|_{R_F} \|\nabla \tilde{\varphi}^\ell_{F,j}\|_{\mathsf{N}^\ell(F)} \leq C' H^{-n/2} \exp(-c\ell) \|\nabla e\|_{R_F},$$

where we abbreviated $C' \coloneqq c_a^{-1} C_a C_{\mathrm{b}} C$.

It remains to sum the above estimate over all faces $F \in \mathcal{F}^i_H$ and indices $j \in \{1, \dots, n\}$ as in (4.17). Using the discrete Cauchy–Schwarz inequality, the trace inequality from Lemma A.2, and Young's inequality, we get that

$$\begin{aligned}
c_a \|\nabla e\|^2_\Omega &\leq \sqrt{n} C' \exp(-c\ell) H^{-n/2} \Big(\sum_{F \in \mathcal{F}^i_H} |F|_{n-1} \|v\|^2_F \Big)^{1/2} \Big(\sum_{F \in \mathcal{F}^i_H} \|\nabla e\|^2_{R_F} \Big)^{1/2} \\
&\leq \sqrt{n C_{\mathrm{T}}} C' C_{\mathrm{ol}} \ell^{n/2} \exp(-c\ell) \|\nabla e\|_\Omega (\|\nabla v\|_\Omega + H^{-1} \|v\|_\Omega),
\end{aligned}$$

where $C_{\mathrm{ol}} > 0$ is a constant only depending on the regularity of the mesh $\mathcal{T}_H$. This proves the desired estimate with the constant $C_{\mathrm{R}} \coloneqq a_a^{-1} \sqrt{n C_{\mathrm{T}}} C' C_{\mathrm{ol}}$. □

5. Localized multiscale method

In this section, we introduce the proposed multiscale method for heterogeneous Stokes problems. Its approximation space, denoted by $\tilde Z_H^\ell$, is defined as the span of the localized basis functions (4.12), i.e.,

$$\tilde Z_H^\ell \coloneqq \operatorname{span}\{\tilde\varphi_{F,j}^\ell \,:\, F\in\mathcal{F}_H^i,\ j=1,\dots,n\}. \tag{5.1}$$

The proposed multiscale method then seeks $(\tilde u_H^\ell, \tilde p_H^\ell)\in \tilde Z_H^\ell \times M_H$ such that

$$a(\tilde u_H^\ell,\tilde v_H^\ell) \quad+\quad b(\tilde v_H^\ell,\tilde p_H^\ell) \quad=\quad (f,\tilde v_H^\ell)_\Omega, \tag{5.2a}$$

$$b(\tilde u_H^\ell,\tilde q_H^\ell) \quad=\quad 0 \tag{5.2b}$$

holds for all $(\tilde v_H^\ell,\tilde q_H^\ell)\in(\tilde Z_H, M_H)$.

The remainder of this section is devoted to the error analysis of this method. We emphasize that the pressure approximation $\tilde p_H^\ell$ is piecewise constant, and therefore, e.g., first-order convergence can only be expected if $p\in H^1(\Omega)$. However, such regularity requirements are generally not satisfied for heterogeneous Stokes problems (or, if they are, the H^1-norm of p can be very large). As a remedy, we introduce a post-processing step that uses the local pressure contributions $\xi_{F,j}^\ell$ computed together with the LOD basis functions in (4.12), as

$$\tilde p_H^{\ell,\mathrm{pp}} \coloneqq \tilde p_H^\ell + \sum_{F\in\mathcal{F}_H^i}\sum_{j=1}^n c_{F,j}\,\xi_{F,j}^\ell, \tag{5.3}$$

where $c_{F,j}$ is the coefficient of $\tilde\varphi_{F,j}^\ell$ in the representation of $\tilde u_H^\ell$. The following theorem proves the well-posedness of the method (5.2) and its uniform convergence properties for the velocity and post-processed pressure approximations under minimal regularity assumptions, provided that the ℓ is chosen sufficiently large. In addition, it is proved that the piecewise constant (not post-processed) pressure approximation $\tilde p_H^\ell$ converges exponentially to $\Pi_H p$ as ℓ is increased.

Theorem 5.1 (Localized method)**.** *The localized multiscale method* (5.2) *is well-posed. Furthermore, there exist constants $C_u, C_u', C_p, C_p' > 0$ independent of H and ℓ such that for any right-hand side $f\in H^s(\Omega)$ with $s\in\{0,1\}$, it holds that*

$$\|\nabla(u-\tilde u_H^\ell)\|_\Omega \le C_u\big(H^{1+s}|f|_{s,\Omega} + H^{-1}\ell^{n/2}\exp(-c\ell)\|f\|_\Omega\big), \tag{5.4}$$

$$\|u-\tilde u_H^\ell\|_\Omega \le C_u'\big(H + H^{-1}\ell^{n/2}\exp(-c\ell)\big)\|\nabla(u-\tilde u_H^\ell)\|_\Omega \tag{5.5}$$

for the velocity approximation, where we recall that $|\cdot|_{s,\Omega}$ denotes the $H^s(\Omega)$-seminorm. For the pressure approximation, we have the error estimates

$$\|\Pi_H p - \tilde p_H^\ell\|_\Omega \le C_p H^{-1}\ell^{n/2}\exp(-c\ell)\|f\|_\Omega, \tag{5.6}$$

$$\|p-\tilde p_H^{\ell,\mathrm{pp}}\|_\Omega \le C_p'\big(H + H^{-1}\ell^{n/2}\exp(-c\ell)\big)\|f\|_\Omega. \tag{5.7}$$

Proof. We begin this proof by showing the inf–sup condition

$$\inf_{\tilde q_H^\ell\in M_H}\ \sup_{\tilde v_H^\ell\in\tilde Z_H^\ell} \frac{|b(\tilde v_H^\ell,\tilde q_H^\ell)|}{\|\nabla\tilde v_H^\ell\|_\Omega\|\tilde q_H^\ell\|_\Omega} \ge \tilde c_b^\ell \tag{5.8}$$

for some constant $\tilde c_b^\ell>0$ to be specified later, which implies the well-posedness of problem (5.2). To this end, we first show the continuity of the operator $\mathcal{R}^\ell$. Using the continuity of $\mathcal{R}$, cf. (3.11), the approximation result from Theorem 4.2, and the Poincaré–Friedrichs inequality on Ω, it follows that

$$\|\nabla\mathcal{R}^\ell v\|_\Omega \le \big(\sqrt{C_a/c_a} + C_{\mathrm{R}}\ell^{n/2}\exp(-c\ell)(1+C_{\mathrm{PF}}H^{-1})\big)\|\nabla v\|_\Omega.$$

Given any $\tilde q_H^\ell\in M_H$, we denote by $v\in V$ the function satisfying $\nabla\cdot v = \tilde q_H^\ell$ and $\|\nabla v\|_\Omega \le C_{\mathrm{L}}\|\tilde q_H^\ell\|_\Omega$, cf. (2.5). Choosing $\tilde v_H^\ell \coloneqq \mathcal{R}^\ell v$, we obtain noting that (3.12) also holds for the operator $\mathcal{R}^\ell$ that $b(\tilde v_H^\ell, \tilde q_H^\ell) = b(v, \tilde q_H^\ell) = \|\tilde q_H^\ell\|_\Omega^2$. The desired inf–sup condition (5.8) then follows with the constant

$$\tilde c_b^\ell = C_{\mathrm{L}}^{-1}\big(\sqrt{C_a/c_a} + C_{\mathrm{R}}\ell^{n/2}\exp(-c\ell)(1 + C_{\mathrm{PF}}H^{-1})\big)^{-1}. \tag{5.9}$$

To prove the H^1-error estimate for the velocity approximation, we denote by $\tilde Z_H^{\ell,0} := \{v \in \tilde Z_H^\ell \,:\, \nabla\cdot v = 0\}$ the subspace of divergence-free functions of the localized approximation space. Problem (5.2) can then be equivalently reformulated as the unique solution to: seek $\tilde u_H^\ell \in \tilde Z_H^{\ell,0}$ such that

$$a(\tilde u_H^\ell, \tilde v_H^\ell) = (f, \tilde v_H^\ell)_\Omega$$

holds for $\tilde v_H^\ell \in \tilde Z_H^{\ell,0}$.

Similarly, also the prototypical multiscale method (3.13) can be equivalently reformulated in the corresponding subspace of divergence-free functions defined as $\tilde Z_H^0 := \{v \in \tilde Z_H \,:\, \nabla\cdot v = 0\}$, i.e., we seek $\tilde u_H \in \tilde Z_H^0$ such that

$$a(\tilde u_H, \tilde v_H) = (f, \tilde v_H)_\Omega$$

holds for all $\tilde v_H \in \tilde Z_H^0$.

Interpreting $\tilde u_H^\ell \in \tilde Z_H^{\ell,0}$ as a non-conforming, non-consistent approximation of $\tilde u_H \in \tilde Z_H^0$, we can apply Strang's second lemma, cf. [EG04, Lem. 2.25], which gives

$$\begin{aligned}\|\nabla(\tilde u_H - \tilde u_H^\ell)\|_\Omega \le \big(1 + c_a^{-1}C_a\big) &\inf_{\tilde v_H^\ell \in \tilde Z_H^{\ell,0}} \|\nabla(\tilde u_H - \tilde v_H^\ell)\|_\Omega \\ &+ c_a^{-1} \sup_{\tilde v_H^\ell \in \tilde Z_H^{\ell,0}} \frac{|(f, \tilde v_H^\ell)_\Omega - a(\tilde u_H, \tilde v_H^\ell)|}{\|\nabla \tilde v_H^\ell\|_\Omega}.\end{aligned} \tag{5.10}$$

To estimate the infimum on the right-hand side of (5.10), we choose $\tilde v_H^\ell = \mathcal{R}^\ell u$, and note that $\mathcal{R}^\ell u \in \tilde Z_H^{\ell,0}$. The latter holds since property (3.12) also holds for operator $\mathcal{R}^\ell$. Furthermore, since we can identify $\tilde u_H = \mathcal{R}u$, cf. Theorem 3.2, we have that $\tilde u_H - \tilde v_H^\ell = (\mathcal{R} - \mathcal{R}^\ell)u$, which allows to apply Theorem 4.2. This gives

$$\inf_{\tilde v_H^\ell \in \tilde Z_H^{\ell,0}} \|\nabla(\tilde u_H - \tilde v_H^\ell)\|_\Omega \le C_{\mathrm{R}}\ell^{n/2}\exp(-c\ell)(\|\nabla u\|_\Omega + H^{-1}\|u\|_\Omega).$$

The supremum on the right-hand hand side of (5.10) can be estimated noting that for any $\tilde v_H^\ell \in \tilde Z_H^{\ell,0}$ and $\tilde v_H \in \tilde Z_H^0$ it holds that

$$|(f, \tilde v_H^\ell)_\Omega - a(\tilde u_H, \tilde v_H^\ell)| = |(f, \tilde v_H^\ell - \tilde v_H)_\Omega - a(\tilde u_H, \tilde v_H^\ell - \tilde v_H)|.$$

Given $\tilde v_H^\ell \in \tilde Z_H^{\ell,0}$ we choose $\tilde v_H = \mathcal{R}\tilde v_H^\ell$ and use that $\tilde v_H - \tilde v_H^\ell = (\mathcal{R} - \mathcal{R}^\ell)\tilde v_H^\ell \in W$, which implies that $a(\tilde u_H, \tilde v_H^\ell - \tilde v_H) = 0$. Applying Theorem 4.2 and the Poincaré–Friedrichs inequality then yields the estimate

$$\sup_{\tilde v_H^\ell \in \tilde Z_H^{\ell,0}} \frac{|(f, \tilde v_H^\ell)_\Omega - a(\tilde u_H, \tilde v_H^\ell)|}{\|\nabla \tilde v_H^\ell\|_\Omega} \le C_{\mathrm{PF}}C_{\mathrm{R}}\ell^{n/2}\exp(-c\ell)(1 + H^{-1}C_{\mathrm{PF}})\|f\|_\Omega.$$

We are now ready to continue (5.10). Applying the Poincaré–Friedrichs inequality again and using the stability estimate $\|\nabla u\|_\Omega \le c_a^{-1}C_{\mathrm{PF}}\|f\|_\Omega$, we obtain

$$\|\nabla(\tilde u_H - \tilde u_H^\ell)\|_\Omega \le C\ell^{n/2}\exp(-c\ell)\|f\|_\Omega, \tag{5.11}$$

with the constant $C := c_a^{-1}(2 + c_a^{-1}C_a)C_{\mathrm{R}}C_{\mathrm{PF}}(1 + H^{-1}C_{\mathrm{PF}}) > 0$. Estimate (5.4) can now be concluded with $C_u := \max\{C_{\mathrm{e}}', C\} > 0$ using the triangle inequality and the H^1-convergence of the prototypical method, cf. Theorem 3.2. Note that to simplify the constant, we have used that Ω is of unit size, i.e., $H \le 1$.

Next, we show that the piecewise constant pressure approximation $\tilde p_H^\ell$ converges exponentially to $\Pi_H p$ in the L^2-norm as ℓ is increased. For this, we recall that $\tilde p_H = \Pi_H p$ solves (3.13), and consider $v \in V$ satisfying $\nabla\cdot v = \tilde p_H - \tilde p_H^\ell$ as well as the estimate $\|\nabla v\|_\Omega \leq C_{\mathrm{L}}\|\tilde p_H - \tilde p_H^\ell\|_\Omega$, cf. (2.5). Since $\tilde p_H - \tilde p_H^\ell$ is piecewise constant, it holds that $\nabla\cdot\mathcal{R}v = \nabla\cdot\mathcal{R}^\ell v = \tilde p_H - \tilde p_H^\ell$. Using $\mathcal{R}v$ and $\mathcal{R}^\ell v$ as the test functions in (5.2a) and (3.13a), and subtracting the resulting equations gives

$$\begin{aligned}\|\tilde p_H - \tilde p_H^\ell\|_\Omega^2 &= (\tilde p_H, \nabla\cdot\mathcal{R}v)_\Omega - (\tilde p_H^\ell, \nabla\cdot\mathcal{R}^\ell v)_\Omega\\ &= (f, \mathcal{R}v - \mathcal{R}^\ell v) - a\left(\tilde u_H^\ell, (\mathcal{R}-\mathcal{R}^\ell)v\right) - a(\tilde u_H - \tilde u_H^\ell, \mathcal{R}v).\end{aligned}$$

To estimate the right-hand side of the latter equation, we apply the Poincare–Friedrichs inequality and (2.3), (4.15), and (5.11) to obtain that

$$\begin{aligned}&\|\tilde p_H - \tilde p_H^\ell\|_\Omega^2\\ &\quad\leq C_{\mathrm{PF}}(1+c_a^{-1}C_a)\|f\|_\Omega\|\nabla(\mathcal{R}-\mathcal{R}^\ell)v\|_\Omega + C_a\|\nabla(\tilde u_H - \tilde u_H^\ell)\|_\Omega\|\nabla\mathcal{R}v\|_\Omega\\ &\quad\leq C_p H^{-1}\ell^{n/2}\exp(-c\ell)\|f\|_\Omega\|\tilde p_H - \tilde p_H^\ell\|_\Omega.\end{aligned}$$

Note that in the last step we have used the stability bound (2.5) to estimate v, which proves the desired estimate (5.6) with the constant

$$C_p := C_{\mathrm{L}}\big(C_{\mathrm{PF}}(1+c_a^{-1}C_a)C_{\mathrm{R}}(1+C_{\mathrm{PF}}) + C_a C\sqrt{C_a/c_a}\big).$$

To prove the L^2-error estimate for the velocity approximation, we use an Aubin–Nietsche-type duality argument. Denoting by $(\bar u, \bar p) \in V\times M$ the solution to (2.2) for the right-hand side $g := u - \tilde u_H^\ell$, we obtain for any $v \in Z_H^{\ell,0}$ that

$$\|u - \tilde u_H^\ell\|_\Omega^2 = (g, u - \tilde u_H^\ell)_\Omega = a(u - \tilde u_H^\ell, \bar u - v) \leq C_a\|\nabla(u - \tilde u_H^\ell)\|_\Omega\|\nabla(\bar u - v)\|_\Omega.$$

Choosing v as the approximation of the proposed method to $\bar u$, and applying the already established H^1-error estimate (5.4) for $s = 0$ to $\bar u$, we get that

$$\|u - \tilde u_H^\ell\|_\Omega^2 \leq C_a C_u\big(H + H^{-1}\ell^{n/2}\exp(-c\ell)\big)\|\nabla(u - \tilde u_H^\ell)\|_\Omega\|g\|_\Omega.$$

Estimate (5.5) can then be concluded with the constant $C_u' := C_a C_u$.

To prove the L^2-error estimate for the post-processed pressure approximation, we consider an arbitrary but fixed element $K \in \mathcal{T}_H$ and a function $v \in (H_0^1(K))^n$ to be specified later. For any $F \in \mathcal{F}_H^i$ and $j \in \{1,\dots,n\}$, we then test equation (4.12a) determining the respective localized basis functions with v. Multiplying the resulting equation by $c_{F,j}$, the coefficient of $\tilde\varphi_{F,j}^\ell$ in the basis representation of $\tilde u_H^\ell$, cf. (5.3), and summing up yields that

$$a_K(\tilde u_H^\ell, v) + b_K(v, \tilde p_H^{\ell,\mathrm{pp}} - \tilde p_H^\ell) = 0,$$

where the subscript denotes the restriction of bilinear forms a and b to K.

Moreover, testing (2.2) with the same v and using that by the divergence theorem there holds $b_K(v, \Pi_H p) = (\Pi_H p)|_K \int_K \nabla\cdot v\,\mathrm{d}x = 0$, we obtain that

$$a_K(u, v) + b_K(v, p - \Pi_H p) = (f, v)_K.$$

Subtracting the latter two equations results in

$$a_K(u - \tilde u_H^\ell, v) + b_K(v, p - \Pi_H p - (\tilde p_H^{\ell,\mathrm{pp}} - \tilde p_H^\ell)) = (f, v)_K. \tag{5.12}$$

Abbreviating $q := p - \Pi_H p - (\tilde p_H^{\ell,\mathrm{pp}} - \tilde p_H^\ell)$, the Ladyzhenskaya lemma, cf. (3.7), asserts the existence of a function $v \in (H_0^1(K))^n$ such that $\nabla\cdot v = q$ holds locally in K and which satisfies $\|\nabla v\|_K \leq C_L'\|q\|_K$. Choosing this v in equation (5.12) and using the uniform coefficient bounds (2.1) as well as the local Poincaré–Friedrichs inequality $\|v\|_K \leq H\|\nabla v\|_K$ for all $v \in (H_0^1(\Omega))^n$, we obtain that

$$\begin{aligned}\|q\|_K^2 &= (f, v)_K - a_K(u - \tilde u_H^\ell, v)\\ &\leq C_{\mathrm{L}}'\big(H\|f\|_K + \sigma_{\max}H\|u - \tilde u_H^\ell\|_K + \nu_{\max}\|\nabla(u - \tilde u_H^\ell)\|_K\big)\|q\|_K.\end{aligned}$$

Summing this inequality over all $K \in \mathcal{T}_H$, we arrive at

$$\|q\|_\Omega \leq \sqrt{3}C_{\mathrm{L}}'\big(H\|f\|_\Omega + \sigma_{\max}H\|u - \tilde u_H^\ell\|_\Omega + \nu_{\max}\|\nabla(u - \tilde u_H^\ell)\|_\Omega\big). \tag{5.13}$$

The triangle inequality inequality finally yields that

$$\|p - \tilde{p}_H^{\ell,\mathrm{pp}}\|_\Omega \leq \|\Pi_H p - \tilde{p}_H^\ell\|_\Omega + \|q\|_\Omega,$$

where the first term can be bounded using (5.6) and the second term using (5.13) and estimates (5.4) and (5.5) for $s = 0$. This proves estimate (5.7) with the constant $C_p' \coloneqq C_p + \sqrt{3}C_\mathrm{L}'(1 + C_u C_u' \sigma_{\max}(1 + \nu_{\max} + \ell^{n/2}\exp(-c\ell)))$, which is independent of H and can be bounded independently of ℓ. This proves the assertion. □

If the oversampling parameter is suitably coupled to the coarse mesh size, one obtains uniform spatial convergence, as can be seen in the following corollary.

Corollary 5.2 (Uniform convergence). *Let $f \in H^s(\Omega)$, $s \in \{0, 1\}$. Then, if the oversampling parameter ℓ is increased logarithmically with the coarse mesh size H, i.e., $\ell \in \mathcal{O}\big(\log(1/H)\big)$, we have for a constant $C > 0$ independent of H and ℓ that*

$$\|\nabla(u - \tilde{u}_H^\ell)\|_\Omega \leq CH^{1+s}, \qquad \|u - \tilde{u}_H^\ell\|_\Omega \leq CH^{2+s}, \qquad \|p - \tilde{p}_H^{\ell,\mathrm{pp}}\|_\Omega \leq CH.$$

6. Implementation and numerical experiments

In this section, we discuss the implementation of the proposed multiscale method and present numerical experiments that support the theoretical results of this paper.

Implementation. For a practical implementation of the method, the local but still infinite-dimensional patch problems (4.12) need to be discretized using the fine meshes $\mathcal{T}_{h,F}$ of the patches $\mathsf{N}^\ell(F)$. These meshes need to resolve all microscopic features of the coefficients to obtain a reliable approximation. The overall structure of the algorithm is summarized in Table 1.

Remark 6.1 (Computational complexity). Typically, the LOD method is used in a multi-query context, where the basis functions are constructed once (the offline stage, i.e., steps (1) – (2) in the algorithm of Table 1), while the actual computation of an approximate solution (the online stage, i.e., step (3) in Table 1) is performed many times for different right-hand sides f. This is a common feature with other multiscale numerical methods; we refer, e.g., to the MsFEM for the Stokes problem in [MNLD15, Bal24], where possible applications of offline-online strategies are described. Furthermore, it is possible to combine the LOD method (similarly for the MsFEM) with the Reduced Basis approach to treat parameter-dependent coefficients; see [AH15]. This is one example where the high cost of the offline stage is well justified because the resulting basis functions can be reused many times.

However, using the LOD as a one-time solver may also be justified in some cases. For example, if the direct solution of the linear system of equations of the fine-scale FEM is prohibitively expensive in terms of memory usage. The LOD, on the other hand, requires only fine-scale solutions on relatively small subdomains. Furthermore, the construction of the LOD basis functions can be easily parallelized. Finally, the LOD as solver can be advantageous over a direct solver for the global fine-scale FEM (even when used on a single instance of f) if a linear solver of superlinear complexity is considered in both cases. Indeed, suppose one uses a

Table 1. Main steps of the fully discrete LOD algorithm.

(1) Generate coarse mesh $\mathcal{T}_H$ and ℓ-th order patches $\mathsf{N}^\ell(F)$ around its faces.
(2) For each patch (in parallel), create a fine local mesh $\mathcal{T}_{h,F}$, assemble local matrices, and solve problems (4.12) for LOD basis functions.
(3) Assemble and solve coarse LOD Stokes system (5.2) and perform post-processing of pressure as in (5.3).

direct solver that solves a linear system of size $N \times N$, resulting from a FEM-discretization of a Stokes-type system, in $\mathcal{O}(N^q)$ CPU time, where $q > 1$. In the offline phase of the LOD one computes $\mathcal{O}(H^{-d})$ basis functions (Ω is of unit size). Computing each function takes $\mathcal{O}((\frac{\ell H}{h})^{dq})$ CPU time, so the entire offline phase costs $(\ell H)^{d(q-1)} \times \mathcal{O}(h^{-dq})$. The cost of the online phase is usually negligible. On the other hand, the cost of solving the linear system of equations of the fine-scale FEM is $\mathcal{O}(h^{-dq})$. Under the above assumptions, the LOD is thus cheaper by a factor of $(\ell H)^{d(q-1)} \ll 1$, where ℓ typically scales as $\log(1/H)$ to get convergence of optimal order. The same complexity considerations apply to the above-mentioned MsFEM-type methods for Stokes problems, noting that their offline cost is usually much lower than that of LOD, since they do not require oversampling, and thus the complexity estimates hold with $\ell = 1$. On the other hand, the accuracy of MsFEM can be rather low in practice, cf. the numerical results in [JL24, Bal24], and the theoretical analysis can only be done under the periodicity assumptions on the coefficients. This can be contrasted with the LOD method proposed here, which provides guaranteed accurate approximations under only minimal structural assumptions on the coefficients, as seen in the numerical results below.

For our numerical experiments, we use a fine-scale discretization based on the Crouzeix–Raviart FEM (CR-FEM), cf. [CR73]. The CR-FEM is particularly well suited since its piecewise constant pressure approximation space allows for definition (3.1), the resulting reformulated Stokes problem (3.2), and the problems defining the localized basis functions (4.12) to be easily adapted to the fully discrete setting. The fully discrete velocity approximation obtained by (5.2) is divergence-free in the weak discrete sense, i.e., its divergence vanishes on all elements of the fine mesh. This does not mean, however, that it is divergence-free on Ω, since the CR velocity space is non-conforming. For simplicity, we assume that the global fine mesh, denoted by $\mathcal{T}_h$, is obtained by (multiple) uniform red refinement of the coarse mesh $\mathcal{T}_H$. The patch problems (4.12) are then discretized using discrete versions of the infinite-dimensional spaces V_F^ℓ and Q_F^ℓ, cf. (4.11), defined on the local meshes $\mathcal{T}_{h,F}$ obtained by restricting $\mathcal{T}_h$ to the respective patches (the global mesh $\mathcal{T}_h$ is never used directly). To practically enforce that the Lagrange multipliers $\xi_{F,j}^\ell$ in (4.12) have zero element averages, another Lagrange multiplier, which is piecewise constant with respect to $\mathcal{T}_H$, must be added.

The error estimates for the prototypical multiscale method from Theorem 3.2 can be transferred to the case of a CR fine-scale discretization with some minor modifications. For example, one needs to generalize the Poincaré-like inequality from Lemma A.1 to the case of fine-scale CR functions which are not $H_0^1(\Omega)$-conforming. This can be easily done using the concept of conforming companions to CR-functions, cf. [Gal14, Chap. 5.2]. With this tool at hand, one can redo the proof of Theorem 3.2, redefining e as the error between the fine-scale CR-FEM solution and the fully discrete prototypical LOD solution. The only part of the proof that deserves special attention is the integration by parts in (3.21), used to show the second-order H^1-convergence of the method provided $f \in H^1(\Omega)$. Here, due to the non-conformity, we need to apply integration by parts on each element of the fine mesh $\mathcal{T}_h$, and (3.21) becomes

$$(\Pi_H f, e)_K = \int_{\partial K} (f_K \cdot x)(e \cdot n)\,\mathrm{d}\sigma + \sum_{F_h \in \mathcal{F}_h^i(K)} \int_{F_h} (f_K \cdot x)([e]_{F_h} \cdot n_{F_h})\,\mathrm{d}\sigma$$

where $\mathcal{F}_h^i(K)$ regroups the faces of the fine mesh $\mathcal{T}_h$ inside the coarse element K, and $[e]_{F_h}$ in the integral over the face F_h denotes the jump of e across F_h consistent with the normal n_{F_h}, arbitrarily chosen on F_h. Summing this over all elements, using $\int_{F_h} [e]_{F_h} \cdot n_{F_h}\,\mathrm{d}\sigma = 0$ for all $F_h \in \mathcal{F}_h^i$ (i.e., all the internal faces of $\mathcal{T}_h$), and

$\int_F \{e\}_F \cdot n_F \,\mathrm{d}\sigma = 0$ on all $F \in \mathcal{F}_H^i$ with $\{\cdot\}$ denoting from now on the average of a discontinuous function on a face, we obtain the following counterpart of (3.22):

$$(\Pi_H f, e)_\Omega = \sum_{F\in\mathcal{F}_H^i} \int_F ([\Pi_H f]_F \cdot (x - x_F))(\{e\}_F \cdot n_F)\,\mathrm{d}\sigma + \sum_{F_h\in\mathcal{F}_h^i} \int_{F_h} (\{\Pi_H f\}_{F_h} \cdot (x - x_{F_h}))([e]_{F_h} \cdot n_{F_h})\,\mathrm{d}\sigma\,.$$

We recall that x_F stands here for the barycenter of F and the similar notation is used for the barycenters of the fine mesh faces. The first term in the equation above can be estimated as in the original proof, following (3.22). The second term can be bounded by $Ch\|\Pi_H f\|_\Omega \|\nabla e\|_\Omega$, which needs to be added to the right-hand side of (3.23). The exponential decay and approximation result from Theorems 4.1 and 4.2 can be easily adapted to the fully discrete setting by inserting an appropriate interpolation to the CR space on $\mathcal{T}_h$ wherever necessary, cf. [MP20, Chap. 4.4]. Adapting the proof of Theorem 5.1 using the aforementioned results in the fully discrete setting, then gives, for example, the H^1-error estimate:

$$\|\nabla(u_h - \tilde{u}^\ell_{H,h})\|_\Omega \le C\big(H^2\|\nabla f\|_\Omega + (H^{-1}\ell^{n/2}\exp(-c\ell) + h)\|f\|_\Omega\big), \tag{6.1}$$

where $\tilde{u}^\ell_{H,h}$ and u_h denote the fully discrete LOD solution and fine-scale CR-FEM solution, respectively. and $C > 0$ is a constant independent of H, ℓ, and h. An error estimate against the continuous solution can be inferred using the triangle inequality, estimate (6.1), and classical *a priori* convergence results for the CR-FEM. Note that in error estimates against the continuous solution, the term $Ch\|f\|_\Omega$ is dominated by the error of the fine-scale CR-FEM, proportional to $h(|u|_{2,\Omega}+|p|_{1,\Omega})$. Nevertheless, some of our numerical experiments, where we compute the error against the fine-scale CR-FEM solution (not detailed here), confirm the presence of this term in the error estimates. Note also that the respective first-order H^1-estimate for the case $f \in L^2(\Omega)$ can be generalized to the fully discrete setting directly, without the above considerations. In particular, it does not include the h-term.4

Numerical experiments. We consider the domain $\Omega = (0,1)^2$ and introduce a hierarchy of meshes generated by uniform red refinement of the initial mesh shown in Figure 6.1 (left). For simplicity, we denote the meshes in the hierarchy by $\mathcal{T}_{2^0}, \mathcal{T}_{2^{-1}}, \dots$, where the subscript refers to the side length of the squares formed by joining opposing triangles. The coefficient ν is chosen to be piecewise constant with respect to the mesh $\mathcal{T}_\epsilon$ with element values obtained as realizations of independent random variables uniformly distributed in the interval $[0.1, 1]$. Note that ϵ is assumed to be a negative power of two. For elements whose midpoints have a distance less than 4ϵ from a parabola, the corresponding element values are set to 10; see Figure 6.1 (right). The coefficient σ is set to zero for simplicity.

Note that all numerical experiments presented below can be reproduced using the code available at `https://github.com/moimmahauck/Stokes_LOD_CR`.

Exponential decay of basis functions. For the first numerical experiment we use the coefficient ν as described above for the value $\epsilon = 2^{-6}$. The mesh for the fine-scale discretization is chosen to be $\mathcal{T}_{2^{-8}}$, which sufficiently resolves the coefficient. This relatively large fine mesh size is necessary to compute the prototypical LOD basis functions needed to evaluate the localization errors.

In Figure 6.2 (left), we illustrate the modulus of an exemplary basis function using a logarithmic color scale. A trained eye observes an exponential decay of the modulus with respect to the underlying coarse mesh, which we indicated in light gray. This supports the exponential decay result of Theorem 4.1. Next, we numerically investigate the localization error when replacing a prototypical basis

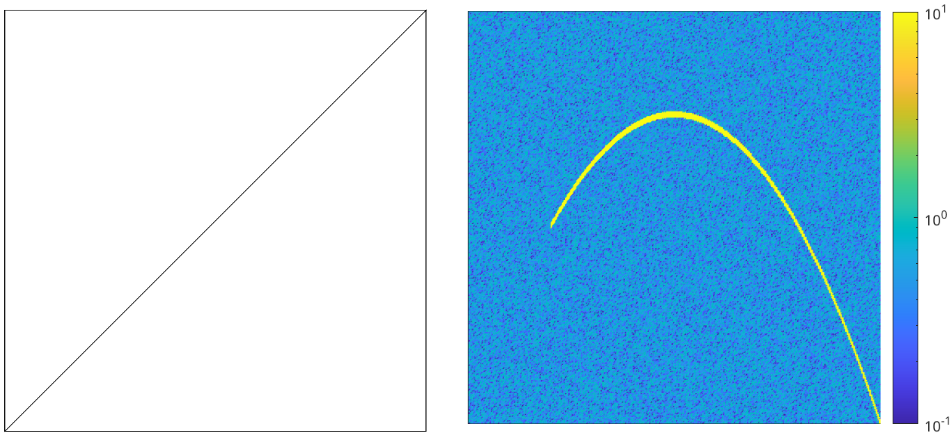

FIGURE 6.1. Initial mesh for the mesh generation (left). Multiscale coefficient used in all numerical experiments (right).

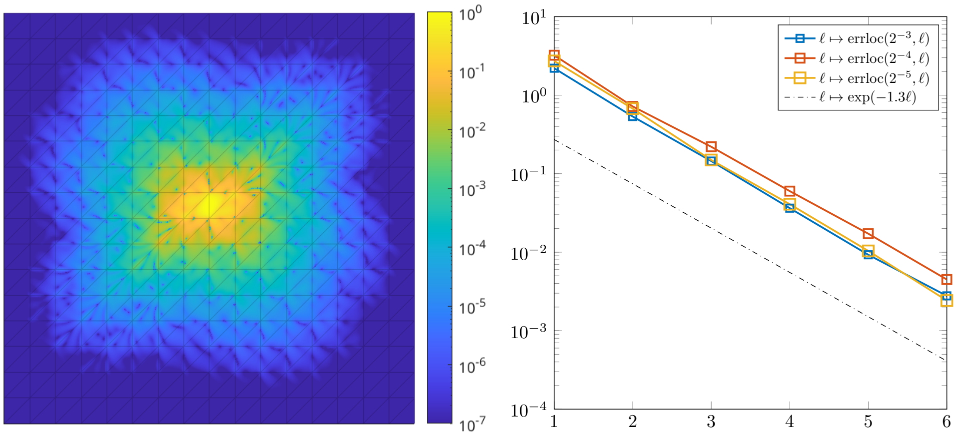

FIGURE 6.2. Decay of the modulus of a prototypical LOD basis function, plotted using a logarithmic color scale (left). H^1-errors of the localized approximation of the prototypical LOD basis functions for several coarse mesh sizes H, plotted as a function of the oversampling parameter ℓ (right).

function by its localized counterpart. For a given coarse mesh size H and localization parameter ℓ, we define the H^1-localization error as

$$\mathrm{errloc}(H, \ell) \coloneqq \max_{F \in \mathcal{F}_H^i} \max_{j=1,\dots,n} \|\nabla(\tilde{\varphi}_{F,j,h} - \tilde{\varphi}^\ell_{F,j,h})\|_\Omega.$$

In Figure 6.2 (right), one clearly observes an exponential decay of the H^1-norm localization error as the localization parameter ℓ is increased. This supports the exponential approximation result from Theorem 4.2.

Optimal order convergence. For the second numerical experiment, we use the coefficient ν from above for the value $\epsilon = 2^{-8}$ and choose the mesh $\mathcal{T}_{2^{-10}}$ for the fine-scale discretization. Furthermore, as right-hand side we use the function

$$f(x, y) \coloneqq (-y, x)^T.$$

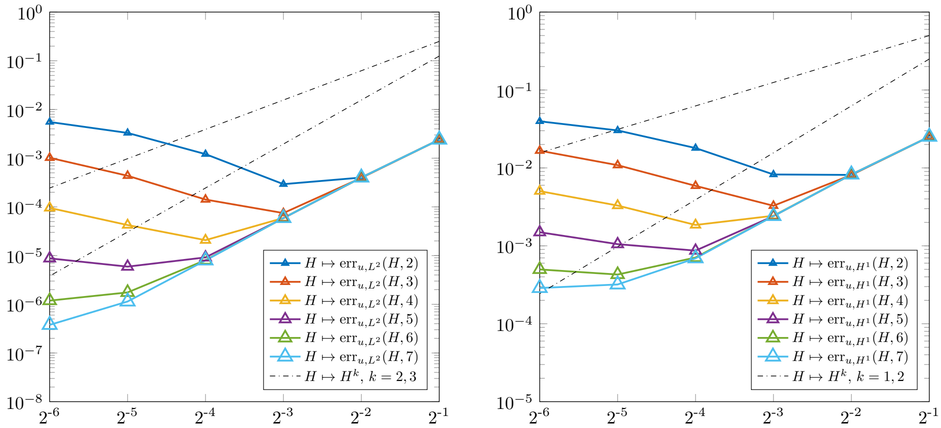

FIGURE 6.3. L^2-errors (left) and H^1-errors (right) for the velocity approximation for several oversampling parameters ℓ, plotted as functions of the coarse mesh size H.

In the following, we investigate the errors

$$\mathrm{err}_{u,H^1}(H,\ell) \coloneqq \|\nabla(u_h - \tilde{u}^\ell_{H,h})\|_\Omega, \qquad \mathrm{err}_{u,L^2}(H,\ell) \coloneqq \|u_h - \tilde{u}^\ell_{H,h}\|_\Omega,$$
$$\mathrm{err}_{p,L^2}(H,\ell) \coloneqq \|p_h - \tilde{p}^{\ell,\mathrm{pp}}_{H,h}\|_\Omega, \qquad \mathrm{err}_{\Pi_H p,L^2}(H,\ell) \coloneqq \|\Pi_H p_h - \tilde{p}^\ell_{H,h}\|_\Omega,$$

where we recall that above (u_h, p_h) denotes the reference solution computed on the fine mesh. For the L^2- and H^1-errors of the velocity approximation, we observe in Figure 6.3 the (almost) third and second order convergence, respectively, provided that the localization parameter is chosen sufficiently large. Recalling that $f \in H^1(\Omega)$, this is in line with the prediction from Theorem 5.1. For a fixed oversampling parameter, we observe that after a certain error level is reached, the error increases again as the mesh size is decreased. This is a well-known effect that occurs for some LOD methods such as [MP14, Mai21]. It can be overcome with a more sophisticated localization strategy; see, e.g., [HP13, HP22a, DHM23]. Transposing the idea of these works into the context of the present article, one would consider a quasi-interpolation operator to a finite-dimensional subspace of Z with local basis functions, whose kernel coincides with the space W, and perform a localization of the correctors to this operator, cf. [HP13]. The quasi-interpolation operator typically involves certain bubble functions, cf. [HP22a, DHM23], and their construction is not straightforward in the present case of Stokes equations due to the divergence constraint. In future work, we hope to propose an alternative improved localization strategy that avoids the explicit use of bubble functions.

Now we turn to the pressure approximation. For the L^2-error of the post-processed pressure approximation we observe in Figure 6.4 (left) the first-order convergence, again provided that the localization parameter is chosen sufficiently large. This observation is in line with Theorem 5.1. For the piecewise constant pressure approximation $\tilde{p}^\ell_{H,h}$, we observe the exponential L^2-convergence towards $\Pi_H p_h$ in Figure 6.4 (right), which is also consistent with Theorem 5.1. Note that the outliers are due to combinations of H and ℓ where all patches are global, which implies that the pressure approximation coincides with $\Pi_H p_h$, cf. Theorem 3.2.

ACKNOWLEDGMENT

M. Hauck acknowledges support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173. The work on

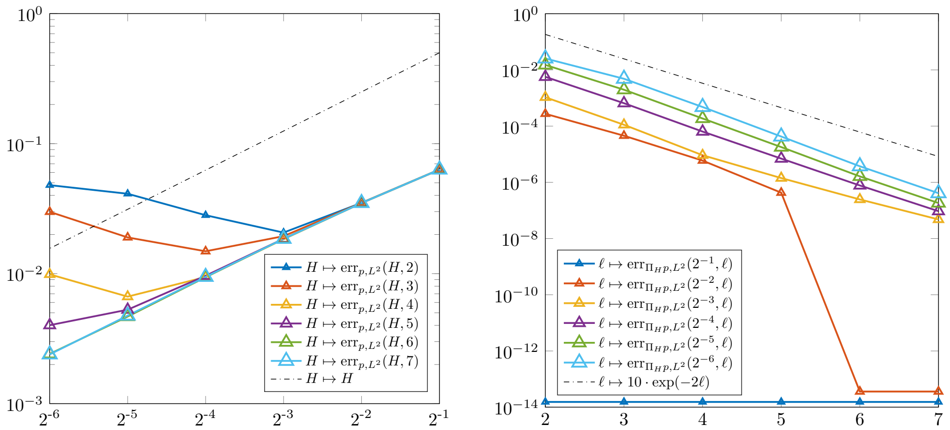

FIGURE 6.4. L^2-errors of the post-processed pressure approximation for several oversampling parameters ℓ, plotted as functions of the coarse mesh size H (left). L^2-errors of the pressure approximation computed with respect to $\Pi_H p$ for several coarse mesh sizes H, plotted as a function of the oversampling parameter ℓ (right).

the revision was done while both authors stayed at the Institute for Mathematical and Statistical Innovation (IMSI) at Chicago for a workshop on Reduced-Order Modeling for Complex Engineering Problems. The authors gratefully acknowledge the hospitality of IMSI and their stimulating working enviroment. The authors would also like to thank Axel Målqvist for helpful discussions on the construction and analysis of the proposed method and the anonymous reviewers for their constructive and insightful remarks that have allowed us to clarify several points in the manuscript.

Appendix A. Collection of frequently used bounds

Lemma A.1 (Local Poincaré-type inequality)**.** *There exists $C_{\mathrm{P}} > 0$ independent of H such that, for all $K \in \mathcal{T}_H$ and all $v \in H^1(K)$ satisfying $\int_F v \,\mathrm{d}\sigma = 0$ for at least one face with $F \subset \partial K$, it holds that*

$$\|v\|_K \leq C_{\mathrm{P}} H \|\nabla v\|_K. \tag{A.1}$$

Proof. This result can be derived from [EG04, Lem. B.66] using a transformation to the reference element and the corresponding estimates in [EG04, Lem. 1.101]. □

Lemma A.2 (Trace inequality)**.** *There exists a constant $C_{\mathrm{T}} > 0$ independent of H such that, for all $K \in \mathcal{T}_H$ and $v \in H^1(K)$, it holds for any face with $F \subset \partial K$ that*

$$\|v\|_F^2 \leq C_{\mathrm{T}} \big(\|\nabla v\|_K + H^{-1} \|v\|_K \big) \|v\|_K.$$

Proof. The proof of this result can be done as in [DPE12, Lem. 1.49], invoking the quasi-uniformity we assumed for the sequence of meshes considered. □

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* Institute for Applied and Numerical Mathematics, Karlsruhe Institute of Technology, Englerstr. 2, 76131 Karlsruhe, Germany

Email address: moritz.hauck@kit.edu

† Université Marie et Louis Pasteur, CNRS, LmB (UMR 6623), F-25000 Besançon, France

Email address: alexei.lozinski@univ-fcomte.fr