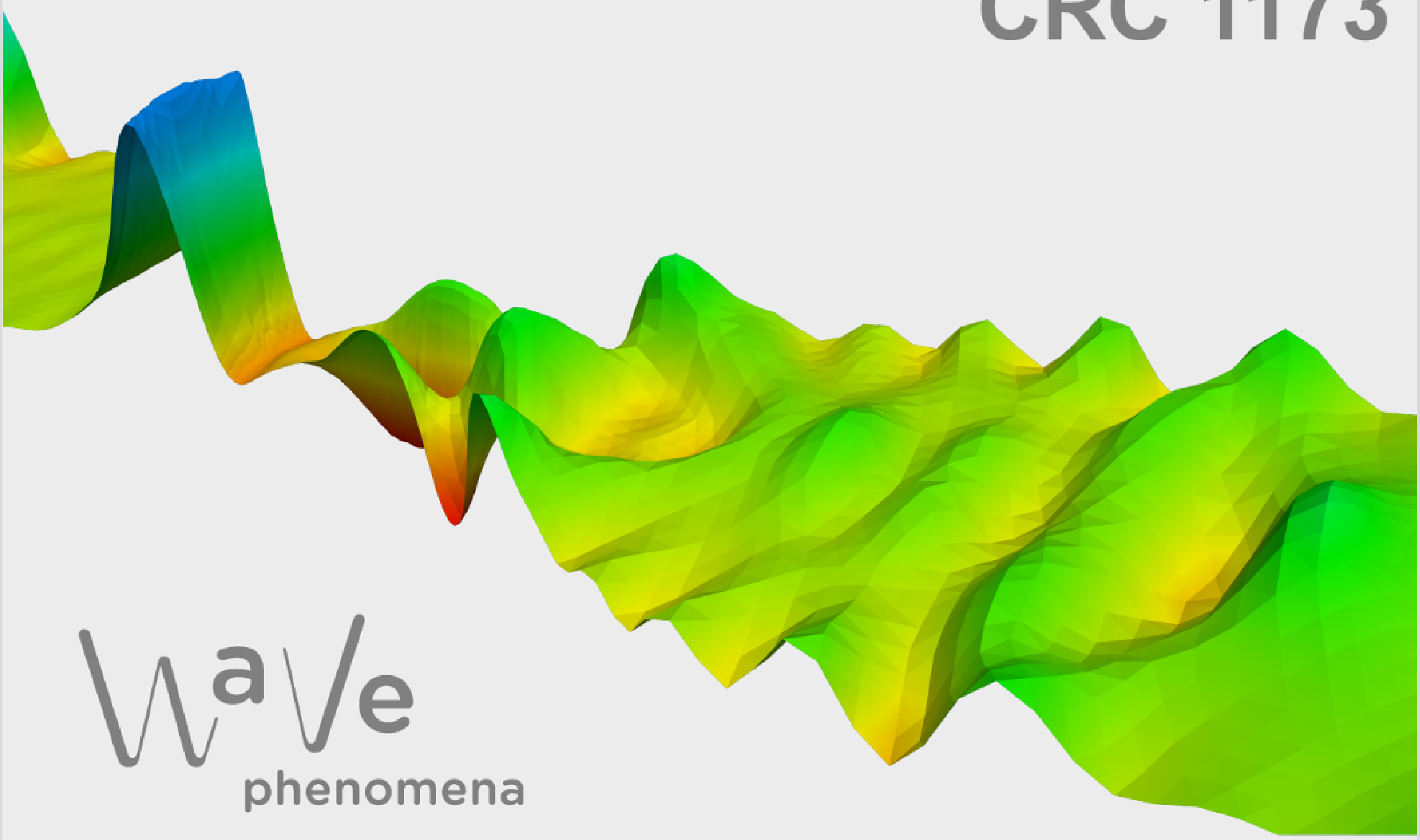
The trouble with the limiting absorption principle for periodic waveguides

Andreas Kirsch

CRC Preprint 2025/19, May 2025

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Participating universities

Funded by

ISSN 2365-662X

THE TROUBLE WITH THE LIMITING ABSORPTION PRINCIPLE FOR PERIODIC WAVEGUIDES

ANDREAS KIRSCH

ABSTRACT. In this paper we consider the open waveguide source problem $\Delta u + k^2\tilde{n}u = -f$ in $\mathbb{R}^2_+ := \{x \in \mathbb{R}^2 : x_2 > 0\}$ with homogeneous Neumann conditions for $x_2 = 0$. Here, $k > 0$ is the wavenumber and $\tilde{n}$ a local perturbation of a refractive index n which is periodic with respect to x_1 and equals to one for $x_2 > h_0$. We derive radiation conditions by replacing k^2 or $\tilde{n}$ or the boundary condition by $k^2 + i\epsilon$ or $\tilde{n} + i\epsilon p$ (for some function p, periodic with respect to x_1) or a boundary condition of impedance type, respectively, and let ϵ tend to zero. We show that in general we derive different radiation conditions. They coincide only in the case that the spaces of propagating modes are one-dimensional.

MSC: 35J05

Key words: Helmholtz equation, limiting absorption principle, radiation condition

1. Introduction

In this paper we consider the following open waveguide source problem. Let $k > 0$ be the wavenumber which is fixed throughout the paper. Set $\mathbb{R}^2_+ := \{x \in \mathbb{R}^2 : x_2 > 0\}$, and let $n \in L^\infty(\mathbb{R}^2_+)$ be the real valued index of refraction which is assumed to be 2π−periodic with respect to x_1. Let $\tilde{n} \in L^\infty(\mathbb{R}^2_+)$ be a local perturbation of n; that is, $\tilde{n}-n$ is supported in $Q_{h_0} := (0, 2\pi) \times (0, h_0)$ for some $h_0 > 0$. We assume that $n(x) \geq n_0$ and $\tilde{n}(x) \geq n_0$ on $\mathbb{R}^2_+$ for some constant $n_0 > 0$ and $n(x) = 1$ for $x_2 > h_0$. Finally, let $f \in L^2(\mathbb{R}^2_+)$ have compact support in Q_{h_0}. It is well known that in general the boundary value problem

$$\Delta u + k^2\tilde{n}u \;=\; -f \quad \text{in } \mathbb{R}^2_+\,, \qquad \frac{\partial u}{\partial x_2} = 0 \quad \text{on } \Gamma_0 := \mathbb{R} \times \{0\}\,, \tag{1}$$

fails to be solvable if in addition the "angular spectrum radiation condition" (see, e.g. [**?**] or (7) below) is assumed. This is due to the fact that the unperturbed homogeneous equation $\Delta u + k^2nu = 0$ can have quasi-periodic (with respect to x_1) solutions $u \in H^2_{loc}(\mathbb{R}^2_+)$ with $\partial u/\partial x_2 = 0$ on Γ_0 which are called guided waves, see Definition 2.1 below. A similar problem occurs for the scattering of plane incident waves $u^{inc}(x) = e^{ik\hat{\theta}\cdot x}$ where $\hat{\theta} = \binom{\sin\theta}{-\cos\theta}$ for $|\theta| < \frac{\pi}{2}$ denotes the direction of the incident wave directed downwards. Then the total field $u = u^{inc} + u^s$ satisfies (1) for $f = 0$. As mentioned already in [5] the upwards propagating radiation condition does not rule out guided waves; that is, is not strict enough. For locally perturbed periodic structures usually (see, e.g. [**?**]) it is assumed that u is the sum $u = u_0 + u_{pert}$ of a α−quasi-periodic solution u_0 of the unperturbed equation $\Delta u_0 + k^2nu_0 = 0$ where $\alpha = k\sin\theta$ and a solution of the source problem (1) for u_{pert} with right hand side $f = k^2(\tilde{n} - n)u_0$ satisfying the angular spectrum radiation

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

condition. However, in general the angular spectrum radiation condition is too strict for the source problem to allow a solution.

A standard tool to derive a correct radiation condition is to justify the Limiting Absorption Principle; that is, consider the problem where k^2 is replaced by $k^2 + i\epsilon$ for $\epsilon > 0$ and show convergence of the corresponding solution as ϵ tends to zero. From the spectral point of view $k^2 + i\epsilon$ belongs to the resolvent set of the selfadjoint operator $-\frac{1}{\tilde{n}}\Delta$ with respect to Neumann boundary conditions and one has to prove convergence of the solution when $k^2 + i\epsilon$ approaches the point k^2 in the spectrum of this operator. We refer to [**?**] for the problem with Dirichlet boundary conditions on Γ_0 and [1, 6, 2] for closed infinite or semi-infinite waveguides. The spectral point of view does not explain the notion of absorption principle. However, if one considers the problem as the TM-mode $E = (0, 0, u(x_1, x_2))^\top$ of an electromagnetic problem in the half space $\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : x_2 > 0\}$ with constant permeability $\mu(x) = \mu_0$ and electric permittivity $\varepsilon(x_1, x_2)$ then $k^2 = \omega^2\varepsilon_0\mu_0$ and $\tilde{n}(x) = \frac{\varepsilon(x)}{\varepsilon_0}$. The modification $k^2 + i\epsilon$ corresponds to adding the conductivity $\sigma(x) = \epsilon\frac{\varepsilon(x)}{\omega\varepsilon_0\mu_0}$ to the system globally in $\mathbb{R}^3_+$. Indeed, the refractive index is then $\frac{1}{\varepsilon_0}\big(\varepsilon(x) + i\frac{\sigma(x)}{\omega}\big) = \big(1 + i\frac{\epsilon}{k^2}\big)\tilde{n}(x)$. An alternative way is to put some conductivity $\epsilon\sigma(x_1, x_2)$ to the inhomogeneous layer $\mathbb{R} \times (0, h_0) \times \mathbb{R}$ only which replaces $\tilde{n}(x)$ by $\tilde{n}(x) + i\epsilon p(x)$ with $p(x) = \frac{\sigma(x)}{\omega\varepsilon_0}$ for $0 < x_2 < h_0$ and $p(x) = 0$ for $x_2 > h_0$. A third possibility is to put absorption to the boundary condition at Γ_0; that is, replace the boundary condition $\frac{\partial u}{\partial x_2} = 0$ on Γ_0 which corresponds to $H \times \nu = 0$ on $\partial\mathbb{R}^3_+$ by $H \times \nu + \epsilon E = 0$ on $\partial\mathbb{R}^3_+$ which is $\frac{\partial u}{\partial x_2} + i\epsilon\omega\mu_0 u = 0$ on Γ_0. Here, $\nu = (0, 1, 0)^\top$ denotes the normal at $\partial\mathbb{R}^3_+$.

It is the aim of the paper to show that these concepts lead to different radiation conditions. Therefore, there is no "one" radiation condition for waveguides in general. We note that the following analysis carries over without difficulty to closed waveguides; that is where $\mathbb{R}^2_+$ is replaced by $\mathbb{R} \times (0, h_0)$ and where an additional homogeneous boundary condition on $\Gamma_{h_0} = \mathbb{R} \times \{h_0\}$ is assumed[1].

The paper is organized as follows. In Sections 2–4 we will consider the unperturbed case; that is $\tilde{n} = n$. In Section 2 we will introduce propagating (or guided) modes. In Section 3 we will recall the Floquet-Bloch transform as a useful tool for treating periodic problems and will prove a representation of the transformed solutions, and in Section 4 we will prove the Limiting Absorption Principles for the three cases introduced above. The perturbed case will be studied in Section 5.

We use the following notations: $W_h = \mathbb{R} \times (0, h)$ for $h > 0$, $\Gamma_h = \mathbb{R} \times \{h\}$ for $h \geq 0$, $Q_h := (0, 2\pi) \times (0, h)$ for $h > 0$, $Q_\infty = (0, 2\pi) \times (0, \infty)$, and $S_h = (0, 2\pi) \times \{h\}$ for $h \geq 0$. Finally, we set Let $W_{R,h} := (-R, R) \times (0, h)$ for $R > 0$ and $h > 0$.

2. Propagating Modes and Formulation of the Problems

We first recall some notations.

Definition 2.1. *(a) $\alpha \in \mathbb{R}$ is called a* cut-off value *if there exists $\ell \in \mathbb{Z}$ with $|\ell + \alpha| = k$.*

[1] This case is even a bit simpler because it avoids the use of the Dirichlet-Neumann operator.

(b) $\alpha \in \mathbb{R}$ is called a propagative wave number *(or quasi-momentum or Floquet spectral value) if there exists a non-trivial $\phi \in H^1_{loc}(\mathbb{R}^2_+)$ such that*

$$\Delta\phi + k^2 n\,\phi \;=\; 0 \quad in\ \mathbb{R}^2_+\,, \qquad \frac{\partial\phi}{\partial x_2} = 0 \quad on\ \Gamma_0\,, \tag{2}$$

ϕ is $\alpha-$quasi-periodic; that is, $\phi(x_1 + 2\pi, x_2) = e^{i\alpha 2\pi}\phi(x_1, x_2)$ for almost all x, and ϕ satisfies the Rayleigh expansion

$$\phi(x) \;=\; \sum_{\ell\in\mathbb{Z}} \phi_\ell\, e^{i(\ell+\alpha)x_1 + i\sqrt{k^2-(\ell+\alpha)^2}|x_2|} \quad for\ x_2 > h_0 \tag{3}$$

for some $\phi_\ell \in \mathbb{C}$ where the convergence is uniform for $x_2 \geq h_0 + \delta$ for all $\delta > 0$. The functions ϕ are called propagating (or guided) modes.

If we decompose k into $k = \hat\ell + \kappa$ with $\hat\ell \in \mathbb{N}\cup\{0\}$ and $\kappa \in (-1/2, 1/2]$ we observe that the cut-off values are given by $\pm\kappa + \ell$ for any $\ell \in \mathbb{Z}$.

Since with α also $\alpha + \ell$ for every $\ell \in \mathbb{Z}$ is a propagative wave number we can restrict ourselves to propagative wave numbers in $(-1/2, 1/2]$.

Under the following assumption it can easily be seen that every propagating mode ϕ corresponding to some propagative wave number α is evanescent; that is, $\phi_\ell = 0$ for all $|\ell + \alpha| \leq k$; that is, there exist $c, \delta > 0$ with $|\phi(x)| \leq c\, e^{-\delta x_2}$ for all $x_2 > h_0$.

Assumption 2.2. *Let $|\ell + \alpha| \neq k$ for all propagative wave numbers α and all $\ell \in \mathbb{Z}$; that is, the cut-off values are no propagative wave numbers.*

Under Assumption 2.2 it can also be shown (see, e.g. [**?**]) that at most a finite number of propagative wave numbers exist in $[-1/2, 1/2]$. Furthermore, if α is a propagative wave number with mode ϕ then $-\alpha$ is a propagative wave number with mode $\overline{\phi}$. Therefore, we can numerate the propagative wave numbers in $[-1/2, 1/2]$ such they are given by $\{\hat\alpha_j : j \in J\}$ where $J \subset \mathbb{Z}$ is finite and symmetric with respect to 0 and $\hat\alpha_{-j} = -\hat\alpha_j$ for $j \in J$. Furthermore, it is known that every eigenspace

$$\hat X_j \;:=\; \big\{\phi \in H^1_{\hat\alpha_j,loc}(\mathbb{R}^2_+) : \phi \text{ satisfies (2) and (3)}\big\} \tag{4}$$

is finite dimensional with some dimension $m_j > 0$. Here, and in the following, $H^1_{\alpha,loc}(\mathbb{R}^2_+) := \{u \in H^1_{loc}(\mathbb{R}^2) : u \text{ is } \alpha-\text{quasi-periodic}\}$.

We now formulate the three modified problems discussed in Section 1.

Problem I: It is the aim to prove convergence of the solution $u_\epsilon \in H^1(\mathbb{R}^2_+)$ of

$$\Delta u_\epsilon + (k^2 + i\epsilon)\,\tilde n\, u_\epsilon \;=\; -f \quad \text{in } \mathbb{R}^2_+\,, \qquad \frac{\partial u_\epsilon}{\partial x_2} = 0 \quad \text{on } \Gamma_0\,, \tag{5}$$

as $\epsilon \to 0$. This problem has been considered (for the Dirichlet boundary condition on Γ_0) in [**?**] or for tubes in $\mathbb{R}^3$ in [3].

Let $H^1_*(\mathbb{R}^2_+)$ be defined as

$$H^1_*(\mathbb{R}^2_+) \;:=\; \big\{u \in H^1_{loc}(\mathbb{R}^2_+) : u \in H^1(W_h) \text{ for all } h > 0\big\}\,.$$

Problem II: Let $p \in L^\infty(W_{h_0})$ be some non-negative periodic (with respect to x_1) function with $p(x) \geq p_0$ on some open subset of Q_{h_0} where $p_0 > 0$. We extend p by zero into $p \in L^\infty(\mathbb{R}^2_+)$. It is the aim to prove convergence of the solution $u_\epsilon \in H^1_*(\mathbb{R}^2_+)$ of

$$\Delta u_\epsilon + k^2(\tilde{n} + i\epsilon p)\, u_\epsilon \;=\; -f \quad \text{in } \mathbb{R}^2_+\,, \qquad \frac{\partial u_\epsilon}{\partial x_2} = 0 \quad \text{on } \Gamma_0\,, \tag{6}$$

satisfying the angular spectrum radiation condition

$$u_\epsilon(x) \;=\; \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} (\mathcal{F}u_\epsilon)(\omega, h_0)\, e^{i\sqrt{k^2-\omega^2}(x_2-h_0)+i\omega x_1}\, d\omega\,, \quad x_2 > h_0\,, \tag{7}$$

where the Fourier transform is defined as

$$(\mathcal{F}\phi)(\omega) \;:=\; \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \phi(s)\, e^{-is\omega}\, ds\,, \quad \omega \in \mathbb{R}\,,$$

considered as an unitary operator from $L^2(\mathbb{R})$ onto itself.

Problem III: Let $q \in L^\infty(\mathbb{R})$ be some non-negative periodic function with $q(x_1) \geq q_0$ on some open interval in $\mathbb{R}$ where $q_0 > 0$. It is the aim to prove convergence of the solution $u_\epsilon \in H^1_*(\mathbb{R}^2_+)$ of

$$\Delta u_\epsilon + k^2 \tilde{n}\, u_\epsilon \;=\; -f \quad \text{in } \mathbb{R}^2_+\,, \qquad \frac{\partial u_\epsilon}{\partial x_2} + i\epsilon q\, u_\epsilon = 0 \quad \text{on } \Gamma_0\,, \tag{8}$$

satisfying again the angulur spectrum radiation condition (7).

Theorem 2.3. *For all $\epsilon > 0$ there exists a unique solution $u_\epsilon \in H^1(\mathbb{R}^2_+)$ of (5). Furthermore, there exists at most one solution $u_\epsilon \in H^1_*(\mathbb{R}^2_+)$ of (6), (7) and (8), (7).*

Proof: For (5) we write $k^2 + i\epsilon$ in polar coordinates as $k^2 + i\epsilon = \rho\, e^{is}$ with $\rho := |k^2 + i\epsilon|$ and some $s \in (0, \pi/2)$. Then we show that the sesqui-linear form

$$a(u, \psi) \;:=\; \int\limits_{\mathbb{R}^2_+} [\nabla u \cdot \nabla\overline{\psi} - (k^2 + i\epsilon)\, \tilde{n}\, u\, \overline{\psi}]\, dx\,, \quad u, \psi \in H^1_+(\mathbb{R}^2_+)\,,$$

is coercive. Indeed, we have that

$$\begin{aligned} \operatorname{Re}\big[e^{(\pi/2-s/2)i} a(u,u)\big] \;&=\; \sin(s/2) \int\limits_{\mathbb{R}^2_+} [|\nabla u|^2 + \rho\, \tilde{n}\, |u|^2]\, dx \\ &\geq\; \sin(s/2)\, \min\{1, \rho\, n_0\}\, \|u\|^2_{H^1(\mathbb{R}^2_+)}\,. \end{aligned}$$

The Theorem of Lax-Milgram yields existence and uniqueness of a solution $u_\epsilon \in H^1(\mathbb{R}^2_+)$ of Problem I.

Let now $u_\epsilon \in H^1_*(\mathbb{R}^2_+)$ be a solution of (6), (7) or (8), (7) for $f = 0$. Application of Green's theorem in the strip W_h yields

$$\int\limits_{W_h} [|\nabla u_\epsilon|^2 - k^2\, (\tilde{n} + i\epsilon_2 p)\, |u_\epsilon|^2]\, dx \;=\; \int\limits_{\Gamma_h} \overline{u_\epsilon}\, \frac{\partial u_\epsilon}{\partial x_2}\, ds \;+\; i\epsilon_3 \int\limits_{\Gamma_0} q\, |u_\epsilon|^2\, ds\,.$$

where $\Gamma_h := \mathbb{R} \times \{h\}$ for $h \geq 0$ and $\epsilon_2 = \epsilon$ and $\epsilon_3 = 0$ for (6) and $\epsilon_3 = \epsilon$ and $\epsilon_2 = 0$ for (8). It is easily seen using Plancherel's formula that the angular spectrum radiation condition (7) implies that $\mathrm{Im}\int_{\Gamma_h}\overline{u_\epsilon}\,\frac{\partial u_\epsilon}{\partial x_2}\,ds\geq 0$. Therefore, taking the imaginary part yields

$$-k^2\epsilon_2\int\limits_{W_h}p\,|u_\epsilon|^2\,dx\;\geq\;\epsilon_3\int\limits_{\Gamma_0}q\,|u_\epsilon|^2\,ds\,.$$

For (6), (7) we conclude that u_ϵ vanishes on some open set in W_h and thus in all of $\mathbb{R}^2_+$ by the unique continuation principle. For (8), (7) we conclude that u_ϵ vanishes on some open interval $(a,b)\times\{0\}\subset\Gamma_0$, and thus also $\partial u_\epsilon/\partial x_2$. We extend u_ϵ and $\tilde{n}$ by zero into $U:=(a,b)\times(-1,0)$. Then $u_\epsilon\in H^2(\mathbb{R}^2_+\cup U)$ and $\Delta u_\epsilon+k^2\tilde{n}u_\epsilon=0$ in $\mathbb{R}^2_+\cup U$. The unique continuation principle implies again that u_ϵ vanishes in $\mathbb{R}^2_+$. $\square$

Existence of solutions for problems (6), (7) and (8), (7) for the unperturbed case $\tilde{n}=n$ will be shown by applying the Floquet-Bloch transform, see Lemma 3.1 below. The perturbed case will be treated in Section 5.

3. Representation of the Floquet-Bloch-Transformed Fields

To treat all three problems simultaneously we introduce three parameters ϵ_j, $j=1,2,3$, for Problems I, II, and III, respectively, for $\tilde{n}=n$; that is, we consider

$$\Delta u_\epsilon+(k^2+i\epsilon_1)\,(n+i\epsilon_2 p)\,u_\epsilon\;=\;-f\quad\text{in }\mathbb{R}^2_+\,,\qquad\frac{\partial u_\epsilon}{\partial x_2}+\epsilon_3\,q\,u_\epsilon=0\quad\text{on }\Gamma_0\,,\tag{9}$$

and the corresponding condition at infinity. The Floquet-Bloch transform is the standard tool to transform the problems to families of quasi-periodic problem in the cell Q_∞. We use the periodic Bloch transform transform, defined as

$$(Fu)(x_1,x_2;\alpha)\;:=\;\sum_{\ell\in\mathbb{Z}}u(x_1+2\pi\ell,x_2)\,e^{-i\alpha(x_1+2\pi\ell)}\,,\quad x=(x_1,x_2)\in\mathbb{R}^2_+\,,$$

for sufficiently smooth functions u. Then it is known (see, e.g. [4]) that F has an extension to an isomorphism from $H^1(W_h)$ onto $L^2\big((-1/2,1/2),H^1_{per}(Q_h)\big):=\big\{u\in L^2\big(Q_h\times(-1/2,1/2)\big):u(\cdot;\alpha)\in H^1_{per}(Q_h)\big\}$. Here, $H^1_{per}(Q_h):=\{v\in H^1(Q_h):v\text{ is }2\pi\text{−periodic wrt }x_1\}$.

Let $u_\epsilon\in H^1_*(\mathbb{R}^2_+)$ be a solution of (9). Then it is known that the Floquet-Bloch transform $u_{\epsilon,\alpha}:=(Fu_\epsilon)(\cdot;\alpha)\in H^1_{loc}(Q_\infty)$ is a $\alpha-$quasi-periodic solutiuon of (9) in the cell Q_∞. The variational can be reduced to the bounded rectangle Q_{h_0} by the use of the Dirichlet-Neumann operator $\Lambda_{\alpha,k^2}:H^{1/2}_{per}(S_{h_0})\to H^{-1/2}_{per}(S_{h_0})$, given by

$$(\Lambda_{\alpha,k^2}\phi)(x_1,h_0)\;:=\;\frac{i}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}}\sqrt{k^2-(\ell+\alpha)^2}\,\phi_\ell(h_0)\,e^{i\ell x_1}\,,\quad x_1\in(0,2\pi)\,,\tag{10}$$

for $\phi\in H^{1/2}_{per}(S_{h_0})$. Here, $\phi_\ell(h_0)$ are the Fourier coefficients of $\phi(\cdot,h_0)$. Then the problem (9) is equivalent to the problem to determine $u_{\epsilon,\alpha}\in H^1_{per}(Q_{h_0})$ with

$$\begin{aligned}(11)\quad&\int\limits_{Q_{h_0}}\left(\nabla u_{\epsilon,\alpha}\cdot\nabla\overline{\psi}-2i\alpha\frac{\partial u_{\epsilon,\alpha}}{\partial x_1}\,\overline{\psi}-[(k^2+i\epsilon_1)\,(n+i\epsilon_2 p)-\alpha^2]\,u_{\epsilon,\alpha}\,\overline{\psi}\right)dx\\&-\int\limits_{S_{h_0}}(\Lambda_{\alpha,k^2+i\epsilon_1}u_{\epsilon,\alpha})\,\overline{\psi}\,ds\;-\;i\epsilon_3\int\limits_{S_0}q\,u_{\epsilon,\alpha}\,\overline{\psi}\,ds\;=\;\int\limits_{Q_{h_0}}f_\alpha\,\overline{\psi}\,dx\quad\text{for all }\psi\in H^1_{per}(Q_{h_0})\,,\end{aligned}$$

where $f_\alpha = (Ff)(\cdot;\alpha)$.

We note that in Problem I the parameter $\epsilon = \epsilon_1$ appears in the volume integral and in the Dirichlet-to-Neumann map while in Problems II and III the parameter $\epsilon = \epsilon_2$ and $\epsilon = \epsilon_3$ appears only in the volume integral over Q_{h_0} or line integral over S_0, respectively. In all three cases the variational equations can be written as $(I - K_{\epsilon_j,\alpha})u_{\epsilon,\alpha} = r_\alpha$ with compact operators $K_{\epsilon_j,\alpha}$ from $H^1_{per}(Q_{h_0})$ into itself[2] when we use the inner product

$$(u,v)_* \;:=\; \int\limits_{Q_{h_0}} [\nabla u \cdot \nabla \overline{v} + u\,\overline{v}]\,dx \;+\; \sum_{\ell\in\mathbb{Z}} \sqrt{1+\ell^2}\, u_\ell(h_0)\,\overline{v_\ell(h_0)}$$

for $u, v \in H^1_{per}(Q_{h_0})$. Also, the operators $K_{\epsilon_j,\alpha}$ and the right hand sides r_α depend continuously on α.

Lemma 3.1. *For $\epsilon_j > 0$ the operators $L_{\epsilon,\alpha} := I - K_{\epsilon,\alpha}$ are invertible for all α and the inverses $L^{-1}_{\epsilon,\alpha}$ depend continuously on α. Furthermore, there exist unique solutions $u_\epsilon \in H^1_*(\mathbb{R}^2_+)$ of problems (6), (7) and (8), (7).*

Proof: Uniqueness of solutions of (11) for all $\alpha \in [-1/2, 1/2]$ is shown exactly as in the proof of Theorem 2.3. Therefore, Fredholm's theory implies that $L_{\epsilon,\alpha}$ are invertible for all α and the inverses $L^{-1}_{\epsilon,\alpha}$ depend continuously on α.

With respect to existence of solutions of (6), (7) and (8), (7) we note that the fact that $u_{\epsilon,\alpha}$ depends continuously on α implies that $\alpha \mapsto u_{\epsilon,\alpha}$ is in $L^2\big((-1/2,1/2), H^1_{per}(Q_h)\big)$; that is, the inverse Floquet-Bloch transform

$$u_\epsilon(x) \;=\; \int\limits_{-1/2}^{1/2} u_{\epsilon,\alpha}(x)e^{i\alpha x_1}\,d\alpha \tag{12}$$

is in $H^1_*(\mathbb{R}^2)$ and represents the solution of (6), (7) or (8), (7), respectively. □

We want to apply the following abstract result taken from [**?**].

Theorem 3.2. *Let H be a (complex) Hilbert space, $I = (-\epsilon_0, \epsilon_0) \subset \mathbb{R}$ and $J = (-\alpha_0, \alpha_0) \subset \mathbb{R}$ open intervals containing 0. Let $K(\epsilon,\alpha) : H \to H$ and $f(\epsilon,\alpha) \in H$, $(\epsilon,\alpha) \in I \times J$, be families of compact operators and elements, respectively, such that $(\epsilon,\alpha) \mapsto K(\epsilon,\alpha)$ is twice continuously differentiable on $I \times J$ and $(z,\alpha) \mapsto f(z,\alpha)$ is Lipschitz continuous on $I \times J$. Set $L(\epsilon,\alpha) = I - K(\epsilon,\alpha)$ and assume the following:*

(a) *The null space $\mathcal{N} := \mathcal{N}\big(L(0,0)\big)$ is not trivial and the Riesz number of $L(0,0)$ is one; that is, the algebraic and geometric multiplicities of the eigenvalue 1 of $K(0,0)$ coincide; that is, $\mathcal{N}\big(L(0,0)^2\big) = \mathcal{N}\big(L(0,0)\big)$. Let $P : H \to \mathcal{N} \subset H$ be the projection operator onto $\mathcal{N}$ corresponding to the direct decomposition $H = \mathcal{N} \oplus \mathcal{R}\big(L(0,0)\big)$,*

(b) *$L(\epsilon,\alpha)$ is one-to-one; that is, also onto, for all $(\epsilon,\alpha) \in I \times J$, $\epsilon > 0$,*

(c) *$A := \frac{1}{i} P \frac{\partial}{\partial\epsilon} K(0,0)|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ is selfadjoint and positive definite and $B := P\frac{\partial}{\partial\alpha} K(0,0)|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ is selfadjoint and one-to-one.*

[2] The notation $K_{\epsilon_j,\alpha}$ means that for given $j \in \{1,2,3\}$ the parameter ϵ_j is positive while the two other parameters are zero.

Let $u(\epsilon,\alpha) \in H$ be the unique solution of $L(\epsilon,\alpha)u(\epsilon,\alpha) = f(\epsilon,\alpha)$ for all $(\epsilon,\alpha) \in I \times J$, $\epsilon > 0$. Then there exists $\epsilon_1 \in (0,\epsilon_0)$ and $\delta \in (0,\alpha_0)$ such that u has the form

$$u(\epsilon,\alpha) \;=\; u^{(1)}(\epsilon,\alpha) \;-\; \sum_{\ell=1}^{m} \frac{f_{\ell,j}}{i\epsilon - \lambda_{\ell,j}\alpha}\,\phi_\ell \quad \text{for } (\epsilon,\alpha) \in (0,\epsilon_1) \times (-\delta,\delta)\,.$$

Here, $\|u^{(1)}(\epsilon,\alpha)\|_H$ is uniformly bounded with respect to (ϵ,α), and $\{\lambda_{\ell,j},\phi_\ell : \ell = 1,\dots,m\}$ is an orthonormal eigensystem of the following generalized eigenvalue problem in the finite dimensional space $\mathcal{N}$ (where $m = \dim \mathcal{N}$):

$$(13) \qquad -B\phi_\ell \;=\; \lambda_{\ell,j}\,A\phi_\ell \quad \text{in } \mathcal{N} \quad \text{with normalization} \quad \big(A\phi_\ell,\phi_{\ell'}\big)_H \;=\; \delta_{\ell,\ell'}$$

for $\ell,\ell' = 1,\dots,m$. Finally, $f_{\ell,j} = \big(Pf(0,0),\phi_\ell\big)_H$ are the expansion coefficients of $A^{-1}Pf(0,0)$ with respect to the inner product $(A\cdot,\cdot)_H$.

We fix $\hat\alpha_j$ for some $j \in J$ and define $H = H^1_{per}(Q_{h_0})$ and $K(\epsilon,\alpha) = K_{\epsilon,\hat\alpha_j+\alpha}$ for $\epsilon = \epsilon_1$ or $\epsilon = \epsilon_2$ or $\epsilon = \epsilon_3$ and $f(\alpha) = r_{\hat\alpha_j+\alpha}$ for $|\alpha| < \alpha_0$ where α_0 is so small such that the interval $[\hat\alpha_j - \alpha_0, \hat\alpha_j + \alpha_0]$ contains no cut-off value. This is possible by Assumption 2.2. Then the smoothness condition concerning $K(\epsilon,\alpha)$ and $f(\alpha)$ are satisfied (see [**?**] or [**?**]). The nullspace $\mathcal{N} = \mathcal{N}(L(0,0))$ is given by $\mathcal{N} = \{\phi|_{Q_{h_0}} : \phi \in X_j\}$ where

$$X_j \;:=\; \big\{\phi \in H^1_{per}(Q_\infty) : \phi(x) = e^{-i\hat\alpha_j x_1}\hat\phi(x) \text{ for some } \hat\phi \in \hat X_j\big\}\,, \quad j \in J\,,$$

is the space $\hat X_j$ of guided modes transformed to periodic functions. Furthermore, X_j is also the nullspace of the adjoint $L(0,0)^*$ which implies that the Riesz number of $L(0,0)$ is one and that the decomposition $H^1_{per}(Q_{h_0}) = \mathcal{N} \oplus \mathcal{R}$ is orthogonal.

The derivative of $K(\epsilon,\alpha)$ with respect to α at $(\epsilon,\alpha) = (0,0)$ is the same in Problems I, II, and III, and its projection to $\mathcal{N}$ is given by

$$(14) \qquad \left(P\frac{\partial L(0,0)}{\partial\alpha}u,\psi\right)_* \;=\; \left(\frac{\partial L(0,0)}{\partial\alpha}u,\psi\right)_* \;=\; -2i\int_{Q_\infty}\left(\frac{\partial u}{\partial x_1} + i\hat\alpha_j u\right)\overline{\psi}\,dx$$

for $u,\psi \in X_j$ where we identified $\mathcal{N}$ with X_j. We refer again to [**?**].

The derivative with respect to ϵ is different for Problem I, II or III. For Problem I we have

$$(15a) \qquad \left(P\frac{\partial L(0,0)}{\partial\epsilon}u,\psi\right)_* \;=\; \left(\frac{\partial L(0,0)}{\partial\epsilon}u,\psi\right)_* \;=\; -i\int_{Q_\infty} n\,u\,\overline{\psi}\,dx$$

for $u,\psi \in X_j$ (see [**?**]) while for Problems II or III we have obviously

$$(15b) \qquad \left(P\frac{\partial L(0,0)}{\partial\epsilon}u,\psi\right)_* \;=\; \left(\frac{\partial L(0,0)}{\partial\epsilon}u,\psi\right)_* \;=\; -ik^2\int_{Q_{h_0}} p\,u\,\overline{\psi}\,dx\,,$$

and

$$(15c) \qquad \left(P\frac{\partial L(0,0)}{\partial\epsilon}u,\psi\right)_* \;=\; \left(\frac{\partial L(0,0)}{\partial\epsilon}u,\psi\right)_* \;=\; -i\int_{S_0} q\,u\,\overline{\psi}\,ds\,,$$

respectively, for $u,\psi \in X_j$. This results in different eigenvalue problems for constructing bases in X_j.

For Problem I the eigenvalue problem has the form (after transformation into $\hat{X}_j$)

$$-2i \int\limits_{Q_\infty} \frac{\partial \hat{\phi}_{\ell,j}}{\partial x_1} \overline{\psi}\, dx \;=\; \lambda_{\ell,j} \int\limits_{Q_\infty} n\, \hat{\phi}_{\ell,j}\, \overline{\psi}\, dx \quad \text{for all } \psi \in \hat{X}_j\,, \tag{16a}$$

for $\ell = 1, \ldots, m_j$ with normalization

$$\int\limits_{Q_\infty} n\, \hat{\phi}_{\ell,j}\, \overline{\hat{\phi}_{\ell',j}}\, dx \;=\; \delta_{\ell,\ell'}\,, \tag{16b}$$

while for Problems II and III the eigenvalue problems have the form

$$-2i \int\limits_{Q_\infty} \frac{\partial \hat{\phi}_{\ell,j}}{\partial x_1} \overline{\psi}\, dx \;=\; \lambda_{\ell,j}\, k^2 \int\limits_{Q_{h_0}} p\, \hat{\phi}_{\ell,j}\, \overline{\psi}\, dx \quad \text{for all } \psi \in \hat{X}_j\,, \tag{16c}$$

for $\ell = 1, \ldots, m_j$ with normalization

$$k^2 \int\limits_{Q_{h_0}} p\, \hat{\phi}_{\ell,j}\, \overline{\hat{\phi}_{\ell'}, j}\, dx \;=\; \delta_{\ell,\ell'}\,. \tag{16d}$$

and

$$-2i \int\limits_{Q_\infty} \frac{\partial \hat{\phi}_{\ell,j}}{\partial x_1} \overline{\psi}\, dx \;=\; \lambda_{\ell,j} \int\limits_{S_0} q\, \hat{\phi}_{\ell,j}\, \overline{\psi}\, ds \quad \text{for all } \psi \in \hat{X}_j\,, \tag{16e}$$

for $\ell = 1, \ldots, m_j$ with normalization

$$\int\limits_{S_0} q\, \hat{\phi}_{\ell,j}\, \overline{\hat{\phi}_{\ell',j}}\, ds \;=\; \delta_{\ell,\ell'}\,, \tag{16f}$$

respectively. Injectivity of the derivative B requires the following assumption.

Assumption 3.3. *Let $\lambda_{\ell,j} \neq 0$ for all $\ell = 1, \ldots, m_j$ and $j \in J$; that is, for every $j \in J$ and every $\hat{\phi} \in \hat{X}_j$ there exists $\psi \in \hat{X}_j$ such that $\int_{Q_\infty} \frac{\partial \hat{\phi}}{\partial x_1} \overline{\psi}\, dx \neq 0$.*

We note from the second form that this assumption is independent of of the chosen inner product. Application of Theorem 3.2 yields the fundamental representation

$$u_{\epsilon,\alpha}(x) \;=\; u^{(1)}_{\epsilon,\alpha}(x) \;-\; \sum_{\ell=1}^{m_j} \frac{f_{\ell,j}}{i\epsilon - \lambda_{\ell,j}(\alpha - \hat{\alpha}_j)}\, \hat{\phi}_{\ell,j}(x)\, e^{-i\hat{\alpha}_j x_1} \tag{17}$$

for $(\epsilon, \alpha) \in (0, \epsilon_0) \times (\hat{\alpha}_j - \alpha_0, \hat{\alpha}_j + \alpha_0)$, and $\|u^{(1)}_{\epsilon,\alpha}\|_{H^1_{per}(Q_{h_0})}$ is uniformly bounded with respect to ϵ and α. Here, $f_{\ell,j} = \big(Pf(0,0), \phi_{\ell,j}\big)_* = \int_{Q_{h_0}} f\, \overline{\hat{\phi}_{\ell,j}}\, dx$.

Remark 3.4. *In the case that $\hat{X}_j$ is one-dimensional; that is, $m_j = 1$, the eigenvalue problems reduce to the same problem. The eigenvalue λ and eigenfunction $\hat{\phi}$ are in each case scaled such that $-2i \int_{Q_\infty} \frac{\partial \hat{\phi}}{\partial x_1} \overline{\hat{\phi}}\, dx = \lambda$.*

4. The Limiting Absorption Principle

We continue with the unperturbed periodic case; that is, $\tilde{n} = n$. For the representation of u_ϵ one has to integrate $u_{\epsilon,\alpha}$ with respect to α, see (12). Since $u^{(1)}_{\epsilon,\alpha}$ converges to $u^{(1)}_{0,\alpha}$ in $H^1(Q_{h_0})$ for every $\alpha \neq \hat{\alpha}_j$ and is uniformly bounded Lebesgue's theorem on dominated convergence shows that $\int_{\hat{\alpha}_j-\delta}^{\hat{\alpha}_j+\delta} u^{(1)}_{\epsilon,\alpha}\, e^{i\alpha x_1} d\alpha$ converges to $\int_{\hat{\alpha}_j-\delta}^{\hat{\alpha}_j+\delta} u^{(1)}_{0,\alpha}\, e^{i\alpha x_1} d\alpha$ in $H^1(W_{R,h_0})$ for every $R > 0$. Again, $W_{R,h_0} := (-R,R) \times (0,h_0)$ for $R > 0$. Furthermore, elementary calculations show that

$$\int\limits_{\hat{\alpha}_j-\delta}^{\hat{\alpha}_j+\delta} \frac{1}{i\epsilon - \lambda_{\ell,j}(\alpha - \hat{\alpha}_j)}\, e^{i(\alpha-\hat{\alpha}_j)x_1}\, d\alpha \;=\; \int\limits_{-\delta}^{\delta} \frac{1}{i\epsilon - \lambda_{\ell,j}\alpha}\, e^{i\alpha x_1}\, d\alpha$$

$$=\; -\frac{2i}{|\lambda_{\ell,j}|} \int\limits_0^{|\lambda_{\ell,j}|\delta/\epsilon} \frac{\cos(\epsilon\beta x_1/|\lambda_{\ell,j}|)}{1+\beta^2}\, d\beta \;-\; 2i\lambda_{\ell,j} \int\limits_0^{\delta x_1} \frac{\beta \sin\beta}{x_1^2\epsilon^2 + \lambda_{\ell,j}^2\beta^2}\, d\beta$$

which converges to

$$-\frac{2i}{|\lambda_{\ell,j}|} \int\limits_0^{\infty} \frac{1}{1+\beta^2}\, d\beta \;-\; \frac{2i}{\lambda_{\ell,j}} \int\limits_0^{\delta x_1} \frac{\sin\beta}{\beta}\, d\beta \;=\; -\frac{i\pi}{|\lambda_{\ell,j}|}\left[1 + (\operatorname{sign}\lambda_{\ell,j})\frac{2}{\pi}\int\limits_0^{\delta x_1} \frac{\sin\beta}{\beta}\, d\beta\right]$$

as ϵ tends to zero uniformly with respect to x_1 from compact sets. Therefore,

$$\int\limits_{\hat{\alpha}_j-\delta}^{\hat{\alpha}_j+\delta} u_{\epsilon,\alpha}\, e^{i\alpha x_1} d\alpha \;\longrightarrow$$

$$\int\limits_{\hat{\alpha}_j-\delta}^{\hat{\alpha}_j+\delta} u^{(1)}_{0,\alpha}\, e^{i\alpha x_1}\, d\alpha \;+\; \pi i \sum_{\ell=1}^{m_j} \frac{f_{\ell,j}}{|\lambda_{\ell,j}|}\left[1 + (\operatorname{sign}\lambda_{\ell,j})\frac{2}{\pi}\int\limits_0^{\delta x_1} \frac{\sin\beta}{\beta}\, d\beta\right] \hat{\phi}_{\ell,j}\,, \quad \epsilon \to 0\,, \tag{18}$$

in $H^1(W_{R,h_0})$ for every $R > 0$. Since the function $\alpha \mapsto u^{(1)}_{0,\alpha}$ for $|\alpha - \hat{\alpha}_j| < \delta$ and zero for $|\alpha - \hat{\alpha}_j| > \delta$ is in $L^2\big((-1/2,1/2), H^1_\alpha(Q_{h_0})\big)$ the inverse Floquet-Bloch transform yields that the first integral on the right hand side of (18) is in $H^1(W_{h_0})$.

We choose any $\sigma_0 > 2\pi + 1$ and functions $\psi^+, \psi^- \in C^\infty(\mathbb{R})$ with $\psi^+(x_1) = 1$ for $x_1 \geq \sigma_0$ and $\psi^+(x_1) = 0$ for $x_1 \leq \sigma_0 - 1$ and, analogously, $\psi^-(x_1) = 1$ for $x_1 \leq -\sigma_0$ and $\psi^-(x_1) = 0$

for $x_1 \geq -\sigma_0 + 1$. Then we decompose the second term on the right hand side of (18) as

$$\pi i \sum_{\ell=1}^{m_j} \frac{f_{\ell,j}}{|\lambda_{\ell,j}|} \left[1 + (\operatorname{sign} \lambda_{\ell,j}) \frac{2}{\pi} \int_0^{\delta x_1} \frac{\sin\beta}{\beta}\, d\beta\right] \hat{\phi}_{\ell,j}$$

$$= \pi i \left[1 + \frac{2}{\pi} \int_0^{\delta x_1} \frac{\sin\beta}{\beta}\, d\beta - 2\psi^+(x_1)\right] \sum_{\ell:\lambda_{\ell,j}>0} \frac{f_{\ell,j}}{|\lambda_{\ell,j}|}\, \hat{\phi}_{\ell,j}$$

$$+ \pi i \left[1 - \frac{2}{\pi} \int_0^{\delta x_1} \frac{\sin\beta}{\beta}\, d\beta - 2\psi^-(x_1)\right] \sum_{\ell:\lambda_{\ell,j}<0} \frac{f_{\ell,j}}{|\lambda_{\ell,j}|}\, \hat{\phi}_{\ell,j}$$

$$+ 2\pi i\, \psi^+(x_1) \sum_{\ell:\lambda_{\ell,j}>0} \frac{f_{\ell,j}}{|\lambda_{\ell,j}|}\, \hat{\phi}_{\ell,j} \;+\; 2\pi i\, \psi^-(x_1) \sum_{\ell:\lambda_{\ell,j}<0} \frac{f_{\ell,j}}{|\lambda_{\ell,j}|}\, \hat{\phi}_{\ell,j}$$

which gives a splitting of the propagating modes in $\hat{X}_j$ into those traveling to the right and the left, respectively. The functions $x_1 \mapsto 1 + \frac{2}{\pi}\int_0^{\delta x_1} \frac{\sin\beta}{\beta}\, d\beta - 2\psi^+(x_1)$ and $x_1 \mapsto 1 - \frac{2}{\pi}\int_0^{\delta x_1} \frac{\sin\beta}{\beta}\, d\beta - 2\psi^-(x_1)$ are in $H^1(\mathbb{R})$. (Note that $\int_0^t \frac{\sin\beta}{\beta}\, d\beta = \pm\frac{\pi}{2} + \mathcal{O}(1/|t|)$ as $t \to \pm\infty$.) Therefore, the right hand side of (18) has a splitting in the form

$$u_{rad,j}(x) \;+\; 2\pi i\, \psi^+(x_1) \sum_{\ell:\lambda_{\ell,j}>0} \frac{f_{\ell,j}}{|\lambda_{\ell,j}|}\, \hat{\phi}_{\ell,j}(x) \;+\; 2\pi i\, \psi^-(x_1) \sum_{\ell:\lambda_{\ell,j}<0} \frac{f_{\ell,j}}{|\lambda_{\ell,j}|}\, \hat{\phi}_{\ell,j}(x)$$

with $u_{rad,j} \in H^1(W_{h_0})$.

Finally, we do this for all propagating wave numbers $\hat{\alpha}_j$. The integral $\int_I u_{\epsilon,\alpha}\, e^{i\alpha x_1} d\alpha$ over the region $I := (-1/2, 1/2) \setminus \cup_j(\hat{\alpha}_j - \delta, \hat{\alpha}_j + \delta)$ converges to $\int_I u_{0,\alpha}\, e^{i\alpha x_1} d\alpha$ in $H^1(Q_{h_0})$ which is also in $H^1(W_{h_0})$ because the integrand is continuous with respect to α. Before we formulate the main convergence result we formulate the open waveguide radiation condition.

Definition 4.1. *Let $\psi_+, \psi_- \in C^\infty(\mathbb{R})$ be any functions with $\psi_\pm(x_1) = 1$ for $\pm x_1 \geq \sigma_0$ (for some $\sigma_0 > 2\pi + 1$) and $\psi_\pm(x_1) = 0$ for $\pm x_1 \leq \sigma_0 - 1$.*
A solution $u \in H^1_{loc}(\mathbb{R}^2_+)$ of (1) satisfies the open waveguide radiation condition with respect to given inner products $(\cdot,\cdot)_{\hat{X}_j}$ in $\hat{X}_j$ if u has a decomposition into $u = u_{rad} + u_{prop}$ into a radiating part $u_{rad} \in H^1_*(\mathbb{R}^2_+)$ and a propagating part u_{prop} which satisfies the following conditions.

(a) *The propagating part u_{prop} has the form*

$$u_{prop}(x) \;=\; \sum_{j\in J} \left[\psi_+(x_1) \sum_{\ell:\lambda_{\ell,j}>0} a_{\ell,j}\, \hat{\phi}_{\ell,j}(x) \;+\; \psi_-(x_1) \sum_{\ell:\lambda_{\ell,j}<0} a_{\ell,j}\, \hat{\phi}_{\ell,j}(x)\right] \tag{19}$$

for $x \in \mathbb{R}^2_+$ and some $a_{\ell,j} \in \mathbb{C}$. Here, for every $j \in J$ the scalars $\lambda_{\ell,j} \in \mathbb{R}$ and $\hat{\phi}_{\ell,j} \in \hat{X}_j$ for $\ell = 1, \ldots, m_j$ are given by the eigenvalues and corresponding eigenfunctions, respectively, of the self adjoint eigenvalue problem

$$-2i \int_{Q_\infty} \frac{\partial \hat{\phi}_{\ell,j}}{\partial x_1}\, \overline{\psi}\, dx \;=\; \lambda_{\ell,j} \left(\hat{\phi}_{\ell,j}\,,\, \psi\right)_{\hat{X}_j} \quad \textit{for all } \psi \in \hat{X}_j\,, \tag{20a}$$

with normalization

$$\left(\hat{\phi}_{\ell,j}\,,\,\hat{\phi}_{\ell',j}\right)_{\hat{X}_j} \;=\; \delta_{\ell,\ell'}\,. \tag{20b}$$

(b) *u_{rad} satisfies the generalized angular spectrum radiation condition*

$$\int\limits_{-\infty}^{\infty}\left|\frac{\partial(\mathcal{F}u_{rad})(\omega,x_2)}{\partial x_2}-i\sqrt{k^2-\omega^2}\,(\mathcal{F}u_{rad})(\omega,x_2)\right|^2 d\omega \;\longrightarrow\; 0 \tag{21}$$

as x_2 tends to infinity where the Fourier transform is defined as

$$(\mathcal{F}\phi)(\omega) \;:=\; \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\phi(s)\,e^{-is\omega}\,ds\,,\quad \omega\in\mathbb{R}\,,$$

considered as an unitary operator from $L^2(\mathbb{R})$ onto itself.

We note that the radiation condition depends on the chosen inner product in $\hat{X}_j$ provided the dimension m_j of $\hat{X}_j$ is larger than one. By obvious modifications of the proofs in [**?**] uniqueness and existence can be shown under this open waveguide radiation condition for any inner product in $\hat{X}_j$.

Noting that the width h_0 of the layer is arbitrary and can be replaced by any larger value h we have shown the following convergence result.

Theorem 4.2. *Let Assumptions 2.2 and 3.3 hold. Let $u_\epsilon\in H^1(\mathbb{R}^2_+)$ or $u_\epsilon\in H^1_*(\mathbb{R}^2_+)$ be the unique solution of (5) or (6), (7) or (8), (7), respectively, for $\epsilon>0$ and $\tilde{n}=n$. Then u_ϵ converges for every $R>0$ and $h>0$ in $H^1(W_{R,h})$ to the unique solution $u_0\in H^1_{loc}(\mathbb{R}^2_+)$ of (1) for $\tilde{n}=n$ which satisfies the open waveguide radiation condition of Definition 4.1. For Problems I, II, and III the inner products are given by $(\phi,\psi)_{\hat{X}_j}=\int_{Q_\infty} n\,\phi\,\overline{\psi}\,dx$ and $(\phi,\psi)_{\hat{X}_j}=k^2\int_{Q_{h_0}} p\,\phi\,\overline{\psi}\,dx$ and $(\phi,\psi)_{\hat{X}_j}=\int_{S_0} q\,\phi\,\overline{\psi}\,ds$, respectively.*

Therefore, in the case that the dimension of one of the mode-spaces $\hat{X}_j$ is larger than one the radiation conditions for equation (1) are different for the three types of Limiting Absorption Principles.

5. The Case of a Local Perturbation

In this section we consider Problems I, II, and III for the local perturbation $\tilde{n}$ instead of n. We will prove the following extension of Theorem 4.2.

Theorem 5.1. *Let Assumptions 2.2 and 3.3 hold.*

(a) *For every $\epsilon>0$ there exist unique solutions $u_\epsilon\in H^1_*(\mathbb{R}^2_+)$ of (6), (7) and (8), (7).*
(b) *Assume that there exists no bound state of (1); that is, there is no non-trivial solution $u\in H^1(\mathbb{R}^2_+)$ of (1). Then the assertion of Theorem 4.2 holds identically with the perturbed index $\tilde{n}$ instead of n.*

Proof: We restrict ourselves to Problem II but note that the same arguments hold also for Problems I and III.

First we introduce the solution operator $\mathcal{L}_\epsilon$ from $L^2(Q_{h_0})$ to $H^1(Q_{h_0})$, defined as $\mathcal{L}_\epsilon f = u_\epsilon|_{Q_{h_0}}$, where $u_\epsilon\in H^1_*(\mathbb{R}^2_+)$ denotes the unique solution of (6), (7) for n instead of $\tilde{n}$. It exists by Lemma 3.1. Then equation (6) can be written in the form $\Delta u_\epsilon+k^2\,(n+i\epsilon p)\,u_\epsilon=$

$-f - k^2\,(\tilde n - n)\,u_\epsilon$ in $\mathbb{R}^2_+$ and $\frac{\partial u_\epsilon}{\partial x_2} = 0$ on Γ_0. Considering the right hand side as a source in $L^2(Q_{h_0})$ we rewrite this equation as a fixpoint equation in the form

$$u_\epsilon \;-\; k^2\,\mathcal{L}_\epsilon\big((\tilde n - n)\,u_\epsilon\big) \;=\; \mathcal{L}_\epsilon f\,. \tag{22}$$

This fixpoint equation is equivalent to (6), (7). Indeed, if $u_\epsilon \in H^1(Q_{h_0})$ is a solution of (22) then we solve (6), (7) for n instead of $\tilde n$ with source $f + k^2(\tilde n - n)\,u_\epsilon \in L^2(Q_{h_0})$. The uniqueness result of Theorem 2.3 yields that the solution with this right hand side solves (6), (7) for $\tilde n$. Therefore, it suffices to study (22) with respect to existence and convergence as $\epsilon \to 0$.

(a) The uniqueness result of Theorem 2.3 implies uniqueness for (22) in $H^1(Q_{h_0})$. Since $H^1(Q_{h_0})$ is compactly embedded in $L^2(Q_{h_0})$ and $\mathcal{L}_\epsilon$ is bounded from $L^2(Q_{h_0})$ into $H^1(Q_{h_0})$ we note that the operator $u \mapsto \mathcal{L}_\epsilon\big((\tilde n - n)u\big)$, is compact from $H^1(Q_{h_0})$ into itself. Therefore, the Fredholm theory implies existence of a solution of (22); that is, of (6), (7) for every $\epsilon > 0$.

(b) We note that Theorem 4.2 implies pointwise convergence of the solution operators $\mathcal{L}_\epsilon$ to the solution operator $\mathcal{L}_0 : L^2(Q_{h_0}) \to H^1(Q_{h_0})$, defined as $\mathcal{L}_0 f = u|_{Q_{h_0}}$ where $u \in H^1_{loc}(\mathbb{R}^2_+)$ is the unique solution of (1) for $\tilde n = n$ satisfying the open waveguide radiation condition corresponding to Problem II. In other words, $\mathcal{L}_\epsilon f \to \mathcal{L}_0 f$ in $H^1(Q_{h_0})$ for every $f \in L^2(Q_{h_0})$ as ϵ tends to zero. Since $\{u \in H^1(Q_{h_0}) : \|u\|_{H^1(Q_{h_0})} = 1\}$ is relatively compact in $L^2(Q_{h_0})$ this implies that $\mathcal{L}_\epsilon$ converges to $\mathcal{L}_0$ in the operator norm of $H^1(Q_{h_0})$, and standard perturbation arguments show convergence of u_ϵ to the solution $u_0 \in H^1(Q_{h_0})$ of $u_0 = \mathcal{L}_0 f + k^2\,\mathcal{L}_0\big((\tilde n - n)\,u_0\big)$ in $H^1(Q_{h_0})$ as $\epsilon \to 0$. Here we use that the equation $u_0 = k^2\,\mathcal{L}_0\big((\tilde n - n)\,u_0\big)$ admits only the trivial solution $u_0 = 0$ by the uniqueness result of [**?**], Theorem 3.3. $\square$

References

[1] S. Fliss and P. Joly, *Solutions of the time-harmonic wave equation in periodic waveguides: Asymptotic behaviour and radiation condition*, Arch. Ration. Mech. Anal., 219 (2016), pp. 349–386.

[2] V. Hoang, *The limiting absorption principle for a periodic semin-infinite waveguide*, SIAM J. Appl. Math., 71 (2011), pp. 791–810.

[3] A. Kirsch, *Scattering by a periodic tube in $\mathbb{R}^3$, Part i: A limiting absorption principle.*, Inverse Problems, 35 (2019).

[4] A. Lechleiter, *The Floquet-Bloch transform and scattering from locally perturbed periodic surfaces*, J. Math. Anal. Appl., 446 (2017), pp. 605–627.

[5] B. Zhang and S. N. Chandler-Wilde, *Acoustic scattering by an inhomogeneous layer on a rigid plate*, SIAM Journal on Applied Mathematics, 58 (1998), pp. 1931–1950.

[6] R. Zhang, *Numerical method for scattering problems in periodic waveguides*, Numerische Mathematik, (2021).

Andreas Kirsch: Department of Mathematics, Karlsruhe Institute of Technology (KIT), 76131 Karlsruhe, Germany