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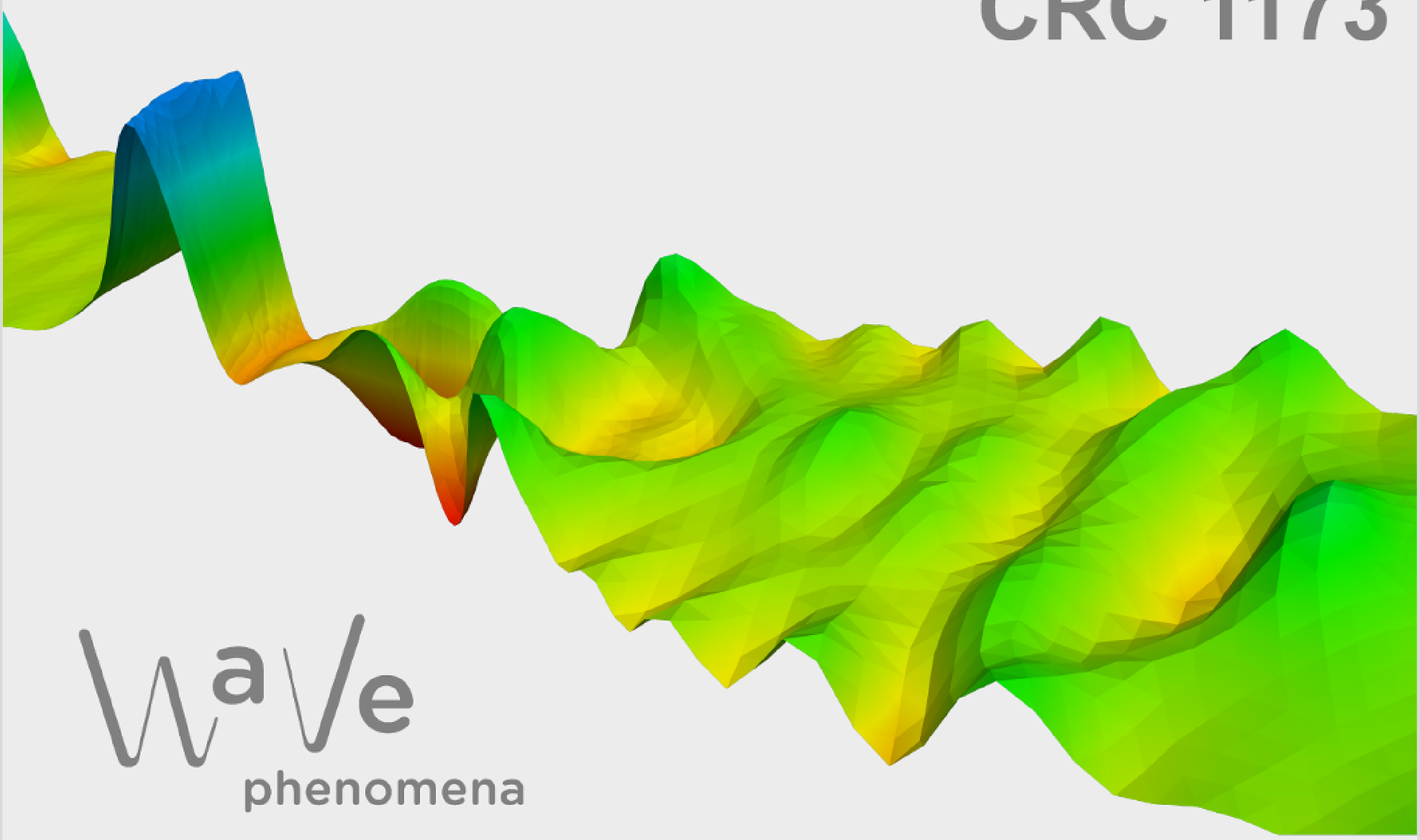

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TIME HARMONIC MAXWELL'S EQUATIONS IN AN OPEN PERIODIC WAVEGUIDE: A LIMITING ABSORPTION PRINCIPLE AND THE RADIATION CONDITION

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Abstract. We study Maxwell's equations in an open waveguide in $\mathbb{R}^3$. The electromagnetic parameters (permittivity, dielectricity, and conductivity) are periodic with respect to x_3 in an infinite cylinder along the x_3-axis and constant outside of the cylinder. The Floquet-Bloch transform is used to reduce the problem to a family of quasi-periodic problems. The quasi-periodic Calderon operator for the exterior of the cylinder reduces the problems to a bounded region. A general functional analytic framework is applied to prove convergence of the solutions when the conductivity tends to zero and allows the formulation of a proper radiation condition. A particular difficuly is introduced because of the singular dependence of the quasi-periodic Calderon operator on the Floquet-Bloch parameter at cut-off values.

MSC: 35Q61

Key words: Maxwell's equation, limiting absorption principle, radiation condition

1. Introduction

We define the infinite cylinders Ω_R and the bounded cylinders W_R by

$$\Omega_R \;:=\; \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < R^2\} \quad \text{and} \quad W_R \;:=\; \{x \in \Omega_R : x_3 \in (0, 2\pi)\}\,,$$

respectively, for $R > 0$. To be consistent we set $W_\infty = \mathbb{R}^2 \times (0, 2\pi) \subset \mathbb{R}^3$. Let the waveguide be given by the cylinder $\Omega_{\hat{R}}$ for some $\hat{R} > 0$. In this paper we use cylindrical coordinates (r, ϕ, x_3) and denote by $\hat{r} = (\cos\phi, \sin\phi, 0)^\top$, $\hat{\phi} = (-\sin\phi, \cos\phi, 0)^\top$, and $\hat{z} = (0, 0, 1)^\top$ the coordinate unit vectors.
Let $\varepsilon, \mu, \sigma \in L^\infty(\mathbb{R}^3)$ be real valued and $2\pi-$periodic with respect to x_3. Furthermore, we assume that $\varepsilon(x) \equiv \varepsilon_0$ and $\mu(x) \equiv \mu_0$ and $\sigma(x) \equiv 0$ for $x_1^2 + x_2^2 > \hat{R}^2$. The constants $\mu_0 > 0$ and $\varepsilon_0 > 0$ are the parameters of the background medium. Furthermore, let $\sigma \geq 0$ in $\mathbb{R}^3$ and let ε and μ be positive and bounded below by some positive constant. The wave number $k > 0$ is defined as $k = \omega\sqrt{\varepsilon_0\mu_0}$. We look for vector fields $E, H \in H_{loc}(\mathrm{curl}, \mathbb{R}^3)$ with

$$\begin{aligned} \mathrm{curl}\, E &= i\omega\mu H + f_h \quad \text{in } \mathbb{R}^3\,, \\ \mathrm{curl}\, H &= -i\omega\varepsilon E + \sigma E \;+\; f_e \quad \text{in } \mathbb{R}^3\,. \end{aligned} \tag{1.1}$$

Here, $f_h, f_e \in L^2(\mathbb{R}^3, \mathbb{C}^3)$ are assumed to have support in $W_{\hat{R}}$. The space $H(\mathrm{curl}, D) = \{u \in L^2(D, \mathbb{C}^3) : \mathrm{curl}\, u \in L^2(D, \mathbb{C}^3)\}$ denotes the usual curl-space and $H_{loc}(\mathrm{curl}, D)$ the local space for any domain $D \subset \mathbb{R}^3$.

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Eliminating H from the system and renaming $u = E$ yields the variational formulation

$$(1.2)\quad \int\limits_{\mathbb{R}^3} \frac{\mu_0}{\mu}\,\operatorname{curl} u \cdot \operatorname{curl}\overline{\psi} - k^2\left(\frac{\varepsilon}{\varepsilon_0} + i\frac{\sigma}{\omega\varepsilon_0}\right) u \cdot \overline{\psi}\,dx = \mu_0 \int\limits_{W_{\hat{R}}} \frac{1}{\mu}\, f_h \cdot \operatorname{curl}\overline{\psi} + i\omega\, f_e \cdot \overline{\psi}\,dx$$

for all $\psi \in H(\operatorname{curl}, \mathbb{R}^3)$ with compact support. We set $q := \frac{\varepsilon}{\varepsilon_0} + i\frac{\sigma}{\omega\varepsilon_0}$ for abbreviation and note that $q \equiv 1$ for $x \notin \Omega_{\hat{R}}$.

In the absorbing case, i.e. when $\sigma > 0$ on $\Omega_{\hat{R}}$, we expect the solution u to decay as $|x_3| \to \infty$, i.e. we look for a solution $u \in H_*(\operatorname{curl}, \mathbb{R}^3)$ where

$$(1.3)\qquad H_*(\operatorname{curl}, \mathbb{R}^3) := \left\{ u : \mathbb{R}^3 \to \mathbb{C}^3 \,\middle|\, u|_{\Omega_R} \in H(\operatorname{curl}, \Omega_R) \text{ for all } R > 0 \right\}.$$

With respect to $r = \sqrt{x_1^2 + x_2^2} \to \infty$ this problem has to be complemented by a suitable radiation condition. A popular one is the angular spectral radiation condition, i.e. the Fourier transform

$$\hat{u}(\tilde{x}, \xi) \;:=\; \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} u(\tilde{x}, x_3)\, e^{-i\xi x_3}\, dx_3\,, \quad \xi \in \mathbb{R}\,,$$

with respect to $x_3 \in \mathbb{R}$ satisfies the two dimensional Sommerfeld radiation condition in the form

$$(1.4)\qquad \partial_r \hat{u}^{(j)}(\tilde{x}, \xi) \;-\; ik(\xi)\, \hat{u}^{(j)}(\tilde{x}, \xi) \;=\; \mathcal{O}(r^{-3/2})\,,\; r \to \infty\,,$$

for almost all $\xi \in \mathbb{R}$ (and every component) uniformly with respect to $\tilde{x}/r$. Here, $\tilde{x} = (x_1, x_2)$ and $k(\xi) = \sqrt{k^2 - \xi^2}$ where the square root function is chosen such that $\operatorname{Re} k(\xi) \geq 0$ and $\operatorname{Im} k(\xi) \geq 0$.
The basic tool in proven existence for the case $\sigma > 0$ is the Floquet-Bloch transform with respect to x_3 which reduces the problem to a family of quasi-periodic problems in the periodicity cell $W_\infty := \mathbb{R}^2 \times (0, 2\pi)$. One solves this problem for (almost) all members of this family and applies the inverse Floquet-Bloch transform.

In the non-absorbing case, i.e. for $\sigma \equiv 0$, we expect the existence of travelling modes, i.e. non-trivial solutions ϕ of (1.2) for $f_h = f_e = 0$ which are quasi-periodic (see below) with respect to x_3 and decay as $r = \sqrt{x_1^2 + x_2^2} \to \infty$. The formulation of a proper radiation condition for this case is more complicated and needs some preparation. Essentially, it requires the splitting of the solution into right- and left going modes when $x_3 \to +\infty$ or $x_3 \to -\infty$, respectively, and a part which decays as $|x_3| \to \infty$ and satisfies the angular spectral radiation condition from above.

The structure of this paper is as follows:

- Section 2: We investigate the quasi-periodic problems which are formulated on the (still unbounded) domain $W_\infty := \mathbb{R}^2 \times (0, 2\pi)$. As usual we reduce them to the bounded domain W_R for some $R \geq \hat{R}$ with the help of the quasi- periodic Calderon operator. Floquet-parameters α which are cut-off values (i.e. $|n + \alpha| = k$ for some $n \in \mathbb{Z}$) or which are critical values (i.e. non-uniqueness holds for the α-quasi-periodic problem) play an essential role in the analysis. Properties of the Calderon operator are shown in Subsection 2.1, and the Fredholm property of the reduced quasi-periodic problems is proven in Subsections 2.2 and 2.3.
- Section 3: For the application of the inverse Floquet-Bloch transform the dependence on the Flochet parameter α is of essential importance. Therefore, we study the behavior of the solution of the quasi-periodic problems for α

being in a neighborhood of cut-off values or critical values. If $\sigma > 0$ it will turn out (under Assumption 3.5) that the solution is continuous at all Floquet parameters α, in particular at the cut-off values. Application of the inverse Floquet-Bloch transform yields existence and uniqueness of a solution in $H_*(\mathrm{curl}, \mathbb{R}^3)$ (Theorem 3.10). To study the dependence on the conductivity σ we take some fixed $\hat{\sigma} > 0$ and set $\sigma = t\hat{\sigma}$ for $t > 0$. Then the solution $u_{\alpha,t}$ of the α-quasi-periodic problem depends also on t. Application of a general functional analytic result (as in [6, 2]) provides an explicit representation of $u_{\alpha,t}$ in the neighborhood of $(\alpha, t) = (\hat{\alpha}, 0)$ where $\hat{\alpha}$ is a critical value.

- Section 4: With this representation we prove convergence of the solution u_t of the problem (1.2), (1.4) with conductivity $\sigma = t\hat{\sigma}$ as t tends to zero. We formulate the corresponding open waveguide radiation condition and prove that the limit u_0 satisfies this radiation condition. Surprisingly, it will turn out that the radiation condition depends on the limiting function $\hat{\sigma}$.
- Section 5: The main convergence result requires an assumption (Assumption 3.7) which we interpret in terms of a transmission-boundary condition for the Maxwell system.
- In the final Section 6 we show that the radiation condition guarantees uniqueness of the non-absorbing problem.

2. The Quasi-periodic Problem

We recall that a function $\psi : \mathbb{R} \to \mathbb{C}$ is $\alpha-$quasi-periodic for some $\alpha \in \mathbb{R}$ if $\psi(x_3 + 2\pi) = e^{i2\pi\alpha}\psi(x_3)$ for all $x_3 \in \mathbb{R}$. Obviously, if ψ is $\alpha-$quasi-periodic then also $\alpha + p-$quasi-periodic for all $p \in \mathbb{Z}$. Therefore, we can restrict ourselves to $\alpha \in [-1/2, 1/2]$.

Quasi-periodic solutions of (1.2) will be determined in the Sobolev space

$$H_{\alpha,*}(\mathrm{curl}, W_\infty) \ := \ \left\{ u : \mathbb{R}^3 \to \mathbb{C}^3 : \begin{array}{l} u|_{W_R} \in H(\mathrm{curl}, W_R) \text{ for all } R > 0\,, \\ u(x_1, x_2, \cdot) \text{ is } \alpha\text{-quasi-periodic} \end{array} \right\}.$$

The space $H_{\alpha,*}(\mathrm{curl}, W_\infty \setminus \overline{W_R})$ is defined analogously. The α-quasi-periodic problem corresponding to (1.2) is to determine $u = u(\cdot, \alpha) \in H_{\alpha,*}(\mathrm{curl}, W_\infty)$ with

$$\int\limits_{W_\infty} \frac{\mu_0}{\mu}\, \mathrm{curl}\, u \cdot \mathrm{curl}\, \overline{\psi} - k^2 q\, u \cdot \overline{\psi}\, dx \ = \ \mu_0 \int\limits_{W_{\hat{R}}} \frac{1}{\mu} f_h \cdot \mathrm{curl}\, \overline{\psi} + i\omega\, f_e \cdot \overline{\psi}\, dx \tag{2.1}$$

for all $\psi \in H_{\alpha,*}(\mathrm{curl}, W_\infty)$ which vanish for $x_1^2 + x_2^2 > R^2$ for some $R > \hat{R}$. Furthermore, we assume that α is not a cut-off value and u satisfies the radiation condition of Rayleigh type, both defined as follows.

Definition 2.1. *(a) $\alpha \in [-1/2, 1/2]$ is called a cut-off value if there exists $n \in \mathbb{Z}$ with $|n + \alpha| = k$.*

(b) Let α not be a cut-off value, i.e. $k_n = k_n(\alpha) := \sqrt{k^2 - (n+\alpha)^2} \neq 0$ for all $n \in \mathbb{Z}$. A solution $u \in H_{\alpha,*}(\mathrm{curl}, W_\infty \setminus \overline{W_R})$ of $\mathrm{curl}^2 u - k^2 u = 0$ in $W_\infty \setminus \overline{W_R}$ (in the sense of (2.1)) for some $R \geq \hat{R}$ satisfies the radiation condition of Rayleigh type if the Fourier coefficients $u_n(x_1, x_2) := \frac{1}{2\pi}\int_0^{2\pi} u(x)\, e^{-i(n+\alpha)x_3} dx_3$ satisfy the two-dimensional Sommerfeld radiation condition (for every component)

$$\frac{\partial u_n(\tilde{x})}{\partial r} \ - \ ik_n u_n(\tilde{x}) \ = \ \mathcal{O}\big(1/|\tilde{x}|^{3/2}\big)\,, \quad r = |\tilde{x}| \to \infty\,, \tag{2.2}$$

uniformly with respect to $\tilde{x}/|\tilde{x}|$. Again, $\tilde{x} = (x_1, x_2)$ and $k_n = \sqrt{k^2 - (n+\alpha)^2}$.

Since this problem is still set up on the unbounded region W_∞ we transform it to a problem on the bounded region $W_{\hat{R}}$ for some $\hat{R} \geq \hat{R}$ with the help of the quasi-periodic Calderon operator which is the analog of the Dirichlet-Neumann operator in the scalar case. Therefore, in the following subsection we consider the boundary value problem in some exterior region $\mathbb{R}^3 \setminus \overline{\Omega_{\hat{R}}}$.

2.1. **The Calderon Operator.** We fix some $\hat{R} \geq \hat{R}$ and set $W^+ = W_\infty \setminus \overline{W_{\hat{R}}}$ and define $\Gamma = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = \hat{R}^2\}$ and $\gamma = \{x \in \Gamma : x_3 \in (0, 2\pi)\}$. Note that W^+, Γ, and γ depend on $\hat{R}$, but we do not indicate this dependence. It is the aim of this subsection to find $w \in H_{\alpha,*}(\mathrm{curl}, W^+)$ with

$$\mathrm{curl}^2\, w - k^2 w = 0 \text{ in } W^+\,, \quad \hat{r} \times w = h \text{ on } \gamma \tag{2.3}$$

where $h : \Gamma \to \mathbb{C}^3$ is some given α-quasi-periodic tangential vector field on Γ. This boundary value problem has to be complemented by the radiation condition of Definition 2.1, part (b). First we define the suitable Sobolev spaces on Γ.

Definition 2.2. *We define the space $H_\alpha^{\pm 1/2}(\gamma)$ of scalar functions and the spaces $H_\alpha^{-1/2}(\mathrm{Div}, \gamma)$ and $H_\alpha^{-1/2}(\mathrm{Curl}, \gamma)$ of vector fields by*

$$H_\alpha^{\pm 1/2}(\gamma) \;:=\; \left\{ \begin{array}{c} p(\phi, x_3) = \displaystyle\sum_{n,m\in\mathbb{Z}} p_{n,m}\, e^{im\phi + (n+\alpha)x_3} : \\ \displaystyle\sum_{n,m\in\mathbb{Z}} |p_{n,m}|^2 [1 + n^2 + m^2]^{\pm 1/2} < \infty \end{array} \right\},$$

$$H_\alpha^{-1/2}(\mathrm{Div}, \gamma) \;:=\; \left\{ \begin{array}{c} h(\phi, x_3) = \displaystyle\sum_{n,m\in\mathbb{Z}} \left[h^\phi_{n,m}\, \hat{\phi} + h^z_{n,m}\, \hat{z}\right] e^{im\phi + (n+\alpha)x_3} : \\ \displaystyle\sum_{n,m\in\mathbb{Z}} \frac{|h^z_{n,m}|^2 + |h^\phi_{n,m}/\hat{R}|^2 + |n h^z_{n,m} + m h^\phi_{n,m}/\hat{R}|^2}{\sqrt{1 + n^2 + m^2}} < \infty \end{array} \right\},$$

$$H_\alpha^{-1/2}(\mathrm{Curl}, \gamma) \;:=\; \left\{ \begin{array}{c} h(\phi, x_3) = \displaystyle\sum_{n,m\in\mathbb{Z}} \left[h^\phi_{n,m}\, \hat{\phi} + h^z_{n,m}\, \hat{z}\right] e^{im\phi + (n+\alpha)x_3} : \\ \displaystyle\sum_{n,m\in\mathbb{Z}} \frac{|h^z_{n,m}|^2 + |h^\phi_{n,m}/\hat{R}|^2 + |n h^\phi_{n,m}/\hat{R} - m h^z_{n,m}|^2}{\sqrt{1 + n^2 + m^2}} < \infty \end{array} \right\}.$$

It can be shown as in [5] that $\left\langle H_\alpha^{-1/2}(\mathrm{Div}, \gamma), H_\alpha^{-1/2}(\mathrm{Curl}, \gamma)\right\rangle$ and $\left\langle H_\alpha^{1/2}(\gamma), H_\alpha^{-1/2}(\gamma)\right\rangle$ are dual pairs with duality forms

$$\langle h, f\rangle = 4\pi^2 \hat{R} \sum_{n,m\in\mathbb{Z}} \left[h^\phi_{n,m} \overline{f^\phi_{n,m}} + h^z_{n,m} \overline{f^z_{n,m}}\right],\; h \in H_\alpha^{-1/2}(\mathrm{Div}, \gamma),\; f \in H_\alpha^{-1/2}(\mathrm{Curl}, \gamma)\,,$$

$$\langle p, q\rangle = 4\pi^2 \hat{R} \sum_{n,m\in\mathbb{Z}} p_{n,m}\, \overline{q_{n,m}}\,,\; p \in H_\alpha^{1/2}(\gamma)\,,\; q \in H_\alpha^{-1/2}(\gamma)\,.$$

Furthermore, the trace operators $u \mapsto \hat{r} \times u|_\gamma$ and $u \mapsto \hat{r} \times (u \times \hat{r})|_\gamma$ are bounded and surjective from $H_\alpha(\mathrm{curl}, W_R)$ into $H_\alpha^{-1/2}(\mathrm{Div}, \gamma)$ and $H_\alpha^{-1/2}(\mathrm{Curl}, \gamma)$, respectively, for every $R > \hat{R}$ which can be shown as, e.g., in [5], Section 5.1.

The Calderon-operator is defined as the mapping which maps $h \in H_\alpha^{-1/2}(\mathrm{Div}, \gamma)$ to the trace $\hat{r} \times \mathrm{curl}\, w$ on γ where $w \in H_{\alpha,*}(\mathrm{curl}, W^+)$ solves the exterior boundary value problem (2.3) and the radiation condition of Definition 2.1. The following two theorems have been proven in [4] (see Theorems 3.1, 3.2, and 3.3).

Theorem 2.3. *Let $h \in H_\alpha^{-1/2}(\mathrm{Div}, \gamma)$ be given by*

$$h(\phi, x_3) = \sum_{n,m\in\mathbb{Z}} \left[h^\phi_{n,m}\,\hat\phi + h^z_{n,m}\,\hat z\right] e^{im\phi+(n+\alpha)x_3}\,, \quad \phi, x_3 \in [0, 2\pi]\,, \tag{2.4}$$

and let α be no cut-off value; i.e. $k_n \neq 0$ for all $n \in \mathbb{Z}$. Then the unique solution $w \in H_{\alpha,*}(\mathrm{curl}, W^+)$ of the exterior boundary value problem (2.3), (2.2) is given in cylindrical coordinates by

$$w(r, \phi, x_3) \;=\; \sum_{n,m\in\mathbb{Z}} \left[w^r_{n,m}(r)\,\hat r + w^\phi_{n,m}(r)\,\hat\phi + w^z_{n,m}(r)\,\hat z\right] e^{im\phi+(n+\alpha)x_3} \tag{2.5}$$

for $r > \hat R$ and $\phi, x_3 \in (0, 2\pi)$ with

$$w^r_{n,m}(r) \;=\; h^\phi_{n,m}\,\frac{i(n+\alpha)}{r}\,G^{(1)}_{n,m}(r) - h^z_{n,m}\,\frac{im\hat R}{r}\,G^{(2)}_{n,m}(r)\,, \tag{2.6}$$

$$w^\phi_{n,m}(r) \;=\; h^\phi_{n,m}\,\frac{m(n+\alpha)}{r}\,G^{(5)}_{n,m}(r) + h^z_{n,m}\,G^{(4)}_{n,m}(r)\,, \tag{2.7}$$

$$w^z_{n,m}(r) \;=\; -h^\phi_{n,m}\,G^{(3)}_{n,m}(r) \tag{2.8}$$

where

$$G^{(1)}_{n,m}(r) \;:=\; \frac{1}{k_n^2}\left[\frac{m^2 H_m(rk_n)}{\hat R k_n H'_{n,m}(\hat R k_n)} - \frac{rk_n H'_m(rk_n)}{H_m(\hat R k_n)}\right], \tag{2.9}$$

$$G^{(2)}_{n,m}(r) \;:=\; \frac{H_m(rk_n)}{\hat R k_n H'_m(\hat R k_n)}\,, \tag{2.10}$$

$$G^{(3)}_{n,m}(r) \;:=\; \frac{H_m(rk_n)}{H_m(\hat R k_n)}\,, \qquad G^{(4)}_{n,m}(r) \;:=\; \frac{H'_m(rk_n)}{H'_m(\hat R k_n)}\,, \tag{2.11}$$

$$G^{(5)}_{n,m}(r) \;:=\; \frac{1}{k_n^2}\left[\frac{H_m(rk_n)}{H_m(Rk_n)} - \frac{rH'_m(rk_n)}{\hat R H'_m(\hat R k_n)}\right], \tag{2.12}$$

Furthermore, for given coefficients $h^z_{n,m}, h^\phi_{n,m} \in \mathbb{C}$ with $\sum_{m,n\in\mathbb{Z}} \frac{a^2_{n,m}}{\sqrt{1+n^2+m^2}} < \infty$ where $a^2_{n,m} = |h^z_{n,m}|^2 + |h^\phi_{n,m}/\hat R|^2 + |nh^z_{n,m} + mh^\phi_{n,m}/\hat R|^2$ and given $\hat\alpha \in [-1/2, 1/2]$ with $|n + \hat\alpha| \neq k$ for all $n \in \mathbb{Z}$ there exists $\delta > 0$ such that the mapping $\alpha \mapsto w$ is holomorphic[1] *as a mapping from $\{\alpha \in \mathbb{C} : |\alpha - \hat\alpha| < \delta\}$ into $H(\mathrm{curl}, W_R)$ for every $R > 0$.*

Theorem 2.4. *Let $h \in H_\alpha^{-1/2}(\mathrm{Div}, \gamma)$ be given by (2.4) and let α be no cut-off value; i.e. $k_n \neq 0$ for all $n \in \mathbb{Z}$. Then Λ_α is expressed as*

$$(\Lambda_\alpha h)(\phi, x_3) \;=\; \sum_{n,m\in\mathbb{Z}} \left[\lambda^z_{n,m}\hat z + \lambda^\phi_{n,m}\hat\phi\right] e^{im\phi+i(n+\alpha)x_3} \tag{2.13}$$

for $h \in H_\alpha^{-1/2}(\mathrm{Div}, \gamma)$ where

$$\lambda^z_{n,m} \;=\; -h^\phi_{n,m}\frac{k^2}{\hat R}\,G^{(1)}_{n,m}(\hat R) \;+\; m\left[\frac{m}{\hat R}\,h^\phi_{n,m} + (n+\alpha)h^z_{n,m}\right] G^{(2)}_{n,m}(\hat R)\,, \tag{2.14}$$

$$\lambda^\phi_{n,m} \;=\; k^2\,h^z_{n,m}\hat R\,G^{(2)}_{n,m}(\hat R) - (n+\alpha)\left[\frac{m}{\hat R}h^\phi_{n,m} + (n+\alpha)h^z_{n,m}\right]\hat R\,G^{(2)}_{n,m}(\hat R). \tag{2.15}$$

Λ_α has the following properties.

(a) *Λ_α, defined by (2.14), (2.15) is well-defined and bounded from $H_\alpha^{-1/2}(\mathrm{Div}, \gamma)$ into itself.*

[1] in the sense of, e.g., [1], Section 8.5,

(b) $\operatorname{Im}\langle \Lambda_\alpha h, \hat r \times h\rangle \geq 0$ *for all* $h \in H_\alpha^{-1/2}(\mathrm{Div},\gamma)$.

(c) $\operatorname{Im}\langle \Lambda_\alpha h, \hat r \times h\rangle = 0$ *implies that the corresponding solution* w *of* (2.5) *is decaying exponentially, i.e. there exist* $\delta, c > 0$ *with*

$$\max_{\phi, x_3 \in (0,2\pi)} |w(r,\phi,x_3)| \;\leq\; c\, e^{-\delta r} \quad \text{for } r \geq \hat R\,. \tag{2.16}$$

Furthermore, $h^\phi_{n,m}$ *and* $h^z_{n,m}$ *vanish for all* $n, m \in \mathbb{Z}$ *with* $|n+\alpha| < k$.

(d) *The Calderon operator* Λ_α *can be decomposed as* $\Lambda_\alpha = \Lambda_\alpha^D + \Lambda_\alpha^C$ *where* Λ_α^D *and* Λ_α^C *are bounded from* $H_\alpha^{-1/2}(\mathrm{Div},\gamma)$ *into itself and* $\operatorname{Div}\Lambda_\alpha^D h = 0$ *and* $\operatorname{Curl}\Lambda_\alpha^C h = 0$ *for all* $h \in H_\alpha^{-1/2}(\mathrm{Div},\gamma)$.

(e) *The operator* $\operatorname{Div}\Lambda_\alpha$ *is bounded from* $H_\alpha^{-1/2}(\mathrm{Div},\gamma)$ *into* $H_\alpha^{-1/2}(\gamma)$ *and*

$$\|\Lambda_\alpha^C h\|_{H_\alpha^{-1/2}(\mathrm{Div},\gamma)} \;\leq\; c\,\|\operatorname{Div}\Lambda_\alpha h\|_{H_\alpha^{-1/2}(\gamma)} \quad \text{for all } h \in H_\alpha^{-1/2}(\mathrm{Div},\gamma)\,. \tag{2.17}$$

(f) *The operator* Λ_α^D *has a decomposition into* $\Lambda_\alpha^D = \hat\Lambda_\alpha^D + \Lambda_\alpha^K$ *where* $(h,\psi) \mapsto \langle \hat\Lambda_\alpha^D h, \psi \times \hat r\rangle$ *is hermitian on* $H_\alpha^{-1/2}(\mathrm{Div},\gamma) \times H_\alpha^{-1/2}(\mathrm{Div},\gamma)$ *and non-negative, i.e.* $\langle \hat\Lambda_\alpha^D h, h \times \hat r\rangle \geq 0$ *for all* h*, and* Λ_α^K *is compact.*

(g) Λ_α *depends holomorphically on* α *in the following sense. Let* $\hat\alpha \in [-1/2, 1/2]$ *not be a cut-off value. Then there exists* $\delta > 0$ *such that the mapping* $\alpha \mapsto \hat\Lambda_\alpha$ *is (strongly) holomorphic*[2] *from* $\{\alpha \in \mathbb{C} : |\alpha - \hat\alpha| < \delta\}$ *into* $\mathcal{B}\big(H_{per}^{-1/2}(\mathrm{Div},\gamma)\big)$ *where* $H_{per}^{-1/2}(\mathrm{Div},\gamma)$ *denotes the space* $H_\alpha^{-1/2}(\mathrm{Div},\gamma)$ *for* $\alpha = 0$ *and* $\hat\Lambda_\alpha v = e^{-i\alpha x_3}\Lambda_\alpha\big(v\, e^{i\alpha x_3}\big)$ *for* $v \in H_0^{-1/2}(\mathrm{Div},\gamma)$.
If $\hat\alpha$ *is a cut-off value, i.e.* $|\hat n + \hat\alpha| = k$ *for some* $\hat n \in \mathbb{N}_0$ *then* $h \mapsto \sum_{(n,m)\notin\mathcal{C}} \big[\lambda^z_{n,m}\hat z + \lambda^\phi_{n,m}\hat\phi\big]\, e^{im\phi + i(n+\alpha)x_3}$ *depends continuously on* α *in a neighborhood of* $\hat\alpha$ *where* $\mathcal{C} := \{(\hat n, m) : |\hat n + \hat\alpha| = k,\ |m| \leq 1\}$.

(h) *Define the operator* D_α *from* $H_\alpha^{1/2}(\gamma)$ *into* $H_\alpha^{-1/2}(\gamma)$ *by* $D_\alpha p := \operatorname{Div}\Lambda_\alpha(\hat r \times \operatorname{Grad} p)$. *Then* $\operatorname{Im}\langle D_\alpha p, p\rangle \geq 0$ *for all* $p \in H_\alpha^{1/2}(\gamma)$*, and* $\operatorname{Im}\langle D_\alpha p, p\rangle = 0$ *implies that* $\operatorname{Re}\langle D_\alpha p, p\rangle \geq 0$.
Furthermore, there is an operator $\tilde D_\alpha$ *from* $H_\alpha^{1/2}(\Gamma)$ *into* $H_\alpha^{-1/2}(\Gamma)$ *which is hermetian and non-negative, i.e.* $\langle \tilde D_\alpha p, p\rangle \geq 0$ *for all* $p \in H_\alpha^{1/2}(\Gamma)$*, and* $D_\alpha - \tilde D_\alpha$ *is compact.*

Remarks 2.5. *(a) The second part of (g) has not been proven in* [4] *but follows directly from the arguments of Theorem 3.2 of* [4].

(b) We note that $\langle \Lambda_\alpha h, \hat r \times h\rangle = 4\pi^2 \sum_{n,m\in\mathbb{Z}} [\lambda^z_{n,m}\overline{h^\phi_{n,m}} - \lambda^\phi_{n,m}\overline{h^z_{n,m}}]$ *with* $\lambda^z_{n,m}, \lambda^\phi_{n,m}$ *from* (2.14), (2.15). *In the proof of Theorem 3.2 (which refers to Theorem 2.4) in* [4] *it has been shown that the imaginary part of every term in the series is non-negative. This implies directly that also the operator* $h \mapsto \sum_{(n,m)\notin\mathcal{C}} \big[\lambda^z_{n,m}\hat z + \lambda^\phi_{n,m}\hat\phi\big]\, e^{im\phi + i(n+\alpha)x_3}$ *is non-negative in the sense of part (b) of the theorem.*

2.2. **The Reduced Quasi-Periodic Problem.** In this subsection we show equivalence of the quasi-periodic problem (2.1), (2.2) to the following problem set up in $W := W_{\hat R} = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < \hat R^2,\ x_3 \in (0, 2\pi)\}$. We recall that $q = \frac{\varepsilon}{\varepsilon_0} + i\frac{\sigma}{\omega\varepsilon_0}$ and allow the general case $\sigma \geq 0$.

In this subsection we fix $\alpha \in [-1/2, 1/2]$ and assume again that α is not a cut-off value i.e. $|n+\alpha| \neq k$ for all $n \in \mathbb{Z}$. We consider the problem to determine

[2] in the sense of, e.g., [1], Section 8.5,

$\tilde u \in H_\alpha(\mathrm{curl}, W)$ such that

$$(2.18)\qquad \begin{aligned} &\int\limits_W \frac{\mu_0}{\mu}\,\mathrm{curl}\,\tilde u \cdot \mathrm{curl}\,\overline{\psi} - k^2 q\,\tilde u \cdot \overline{\psi}\,dx \;+\; \big\langle \Lambda_\alpha(\hat r \times \tilde u), \psi\big\rangle \\ &= \;\mu_0 \int\limits_W \frac{1}{\mu}\,f_h \cdot \mathrm{curl}\,\overline{\psi} + i\omega f_e \cdot \overline{\psi}\,dx \quad \text{for all } \psi \in H_\alpha(\mathrm{curl}, W) \end{aligned}$$

where $\Lambda_\alpha : H_\alpha^{-1/2}(\mathrm{Div}, \gamma) \to H_\alpha^{-1/2}(\mathrm{Div}, \gamma)$ denotes the Calderon operator. Again, $q := \frac{\varepsilon}{\varepsilon_0} + i\frac{\sigma}{\omega\varepsilon_0}$, and we wrote (and do this also in the following) ψ instead of $\hat r \times (\psi \times \hat r)$ in the sesqui-linear form $\langle \cdot, \cdot \rangle$ of (2.18). The space $H_\alpha(\mathrm{curl}, W)$ is the subspace of $H(\mathrm{curl}, W)$ consisting of α-quasi-periodic functions equipped with the inner product $\langle \cdot, \cdot \rangle_{H(\mathrm{curl}, W)}$ of $H(\mathrm{curl}, W)$.
We rewrite (2.18) as

$$b_\alpha(\tilde u, \psi) \;=\; \mu_0 \int\limits_W \frac{1}{\mu}\,f_h \cdot \mathrm{curl}\,\overline{\psi} + i\omega f_e \cdot \overline{\psi}\,dx \quad \text{for all } \psi \in H_\alpha(\mathrm{curl}, W)$$

where the sesqui-linear form $b_\alpha : H_\alpha(\mathrm{curl}, W) \times H_\alpha(\mathrm{curl}, W) \to \mathbb{C}$ is defined by

$$(2.19)\qquad b_\alpha(u, \psi) \;:=\; \int\limits_W \frac{\mu_0}{\mu}\,\mathrm{curl}\,u \cdot \mathrm{curl}\,\overline{\psi} - k^2 q\,u \cdot \overline{\psi}\,dx \;+\; \big\langle \Lambda_\alpha(\hat r \times u), \psi\big\rangle\,.$$

We have the following equivalence.

Theorem 2.6. *Let α not be a cut-off value.*

(a) *Let $u \in H_{\alpha,*}(\mathrm{curl}, W_\infty)$ be a solution of* (2.1), (2.2). *Then $\tilde u := u|_W \in H_\alpha(\mathrm{curl}, W)$ is a solution of* (2.18).

(b) *Let $\tilde u \in H_\alpha(\mathrm{curl}, W)$ be a solution of* (2.18). *Set $u = \tilde u$ in W and $u = \tilde u$ in $W_\infty \setminus W$ where $\tilde u$ is the (unique) solution of* (2.3) *for $h := \hat r \times \tilde u$, i.e. (compare with* (2.5) *for $h^\phi_{n,m} = -\tilde u^z_{n,m}$ and $h^z_{n,m} = \tilde u^\phi_{n,m}$)*

$$(2.20)\qquad u(r, \phi, x_3) \;=\; \sum_{n,m \in \mathbb{Z}} \big[u^r_{n,m}(r)\,\hat r + u^\phi_{n,m}(r)\,\hat\phi + u^z_{n,m}(r)\,\hat z\big]\,e^{im\phi + (n+\alpha)x_3}$$

for $r > \hat R$ and $\phi, x_3 \in (0, 2\pi)$ with

$$\begin{aligned} u^r_{n,m}(r) &= -\tilde u^z_{n,m}\,\frac{i(n+\alpha)}{r}\,G^{(1)}_{n,m}(r) - \tilde u^\phi_{n,m}\,\frac{im\hat R}{r}\,G^{(2)}_{n,m}(r)\,, \\ u^\phi_{n,m}(r) &= -\tilde u^z_{n,m}\,\frac{m(n+\alpha)}{r}\,G^{(5)}_{n,m}(r) + \tilde u^\phi_{n,m}\,G^{(4)}_{n,m}(r)\,, \\ u^z_{n,m}(r) &= \tilde v^z_{n,m}\,G^{(3)}_{n,m}(r)\,. \end{aligned}$$

Then $u \in H_{\alpha,*}(\mathrm{curl}, W_\infty)$ is a solution of (2.1), (2.2).

We omit the proof (which is quite standard). Properties of b_α are collected in the following lemma.

Lemma 2.7. (a) *The sequi-linear form b_α is bounded on $H_\alpha(\mathrm{curl}, W)$, i.e.*

$$\big|b_\alpha(u, \psi)\big| \;\le\; c\,\|u\|_{H(\mathrm{curl},W)}\,\|\psi\|_{H(\mathrm{curl},W)} \quad \textit{for all } u, \psi \in H_\alpha(\mathrm{curl}, W)\,.$$

(b) $\mathrm{Im}\,b_\alpha(u, u) \le 0$ *for all $u \in H_\alpha(\mathrm{curl}, W)$. If $\mathrm{Im}\,q > 0$ on $\Omega_{\hat R}$, i.e. $\sigma > 0$ on $\Omega_{\hat R}$, then $\mathrm{Im}\,b_\alpha(u, u) < 0$ for all $u \neq 0$.*

(c) *Define the closed subspace* $\mathcal{N} := \big\{u \in H_\alpha(\mathrm{curl}, W) : b_\alpha(u,\psi) = 0$ *for all* $\psi \in H_\alpha(\mathrm{curl}, W)\big\}$. *If* $\operatorname{Im} q > 0$ *on* W *then* $\mathcal{N} = \{0\}$. *For* $u \in \mathcal{N}$ *the Fourier coefficients* $u^j_{m,n}$ *of* $u|_\Gamma$ *vanish for all* $n, m \in \mathbb{Z}$ *with* $|n+\alpha| < k$ *and* $j = 1, 2, 3$. *Furthermore,* $\mathcal{N} = \big\{u \in H_\alpha(\mathrm{curl}, W) : b_\alpha(\psi, u) = 0$ *for all* $\psi \in H_\alpha(\mathrm{curl}, W)\big\}$.

Proof. (a) This follows directly from the Cauchy-Schwarz inequality for the volume integral, the boundedness of the trace operators $u \mapsto \hat{r} \times u|_\gamma$ from $H_\alpha(\mathrm{curl}, W)$ into $H_\alpha^{-1/2}(\mathrm{Div}, W)$ and $\psi \mapsto (\hat{r} \times \psi) \times \hat{r}|_\gamma$ from $H_\alpha(\mathrm{curl}, W)$ into $H_\alpha^{-1/2}(\mathrm{Curl}, \gamma)$, and the boundedness of Λ_α from $H_\alpha^{-1/2}(\mathrm{Div}, \gamma)$ into itself.

(b) We conclude from part (b) of Theorem 2.4 that $\operatorname{Im} b_\alpha(u,u) = -k^2 \int_W \operatorname{Im} q\, |u|^2\, dx + \operatorname{Im}\langle \Lambda_\alpha(\hat{r} \times u), u\rangle \leq 0$. If $\operatorname{Im} q > 0$ on $\Omega_{\hat{R}}$ then $\operatorname{Im} b_\alpha(u,u)$ implies that $u = 0$ on $\Omega_{\hat{R}}$. By the previous theorem we conclude that u vanishes everywhere.

(c) The fact that $\mathcal{N} = \{0\}$ for $\sigma > 0$ follows directly from part (b). For arbitrary q and $u \in \mathcal{N}$ we conclude that $0 = \operatorname{Im} b_\alpha(u,u) = -k^2 \int_W (\operatorname{Im} q)|u|^2 dx + \operatorname{Im}\langle \Lambda_\alpha(\hat{r} \times u), u\rangle \leq 0$ and thus, by part (c) of Theorem 2.4, that $(\operatorname{Im} q)u = 0$ in W and $u^\phi_{n,m} = u^z_{n,m} = 0$ for all $|n+\alpha| < k$. For $n \in \mathbb{Z}$ with $|n+\alpha| > k$ we note that $\frac{k_n H'_m(k_n \hat{R})}{H_m(k_n \hat{R})} = \frac{|k_n| K'_m(|k_n|\hat{R})}{K_m(|k_n|\hat{R})}$ is real valued. With the form (2.13), (2.14), (2.15) of Λ_α for $h = \hat{r} \times \psi$ we note that $\langle \Lambda_\alpha(\hat{r} \times \psi), u\rangle$ has the form

$$\begin{aligned}\langle \Lambda_\alpha(\hat{r} \times \psi), u\rangle \;&=\; \sum_{m,n:|n+\alpha|>k} \Big[\psi^z_{n,m} A_{n,m} + \frac{m}{\hat{R}}(n+\alpha) B_{n,m} \psi^\phi\Big]\, \overline{u^z} \\ &\qquad + \psi^\phi_{n,m} C_{n,m} + \frac{m}{\hat{R}}(n+\alpha) B_{n,m} \psi^z\Big]\, \overline{u^\phi}\Big]\end{aligned}$$

for some coefficients $A_{n,m}$, $B_{n,m}$, and $C_{n,m}$ which are real valued. Therefore, $\langle \Lambda_\alpha(\hat{r} \times \psi), u\rangle = \overline{\langle \Lambda_\alpha(\hat{r} \times u), \psi\rangle}$ and thus $\overline{b_\alpha(\psi, u)} = b_\alpha(u,\psi) + 2k^2 \int_W (\operatorname{Im} q)|u|^2 dx = 0$ for all $\psi \in H_\alpha(\mathrm{curl}, W)$. This shows that $\mathcal{N} = \big\{u \in H_\alpha(\mathrm{curl}, W) : b_\alpha(\psi, u) = 0$ for all $\psi \in H_\alpha(\mathrm{curl}, W)\big\}$. □

2.3. **The Helmholtz decomposition.** It is the aim to show the Fredholm property of b_α (see next subsection). As a standard tool we need a suitable Helmholtz decomposition of the solution space. We fix some $\alpha \in [-1/2, 1/2]$ which is not a cut-off value, define $W = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < \hat{R}^2\}$, and b_α as before, and define the spaces $\nabla H^1_\alpha(W)$ and $\mathcal{D}_\alpha$ by

$$\begin{aligned}\nabla H^1_\alpha(W) \;&=\; \big\{\nabla p : p \in H^1_\alpha(W)\big\}\,, \\ \mathcal{D}_\alpha \;&=\; \big\{v \in H_\alpha(\mathrm{curl}, W) : b_\alpha(v, \nabla\phi) = 0 \text{ for all } \phi \in H^1_\alpha(W)\big\}\,.\end{aligned}$$

If $\alpha = 0$ we understand the space $\nabla H^1_\alpha(W)$ as $\big\{\nabla p : p \in H^1_\alpha(W),\ \int_W p\, dx = 0\big\}$.

First we note that $\nabla H^1_\alpha(W)$ and $\mathcal{D}_\alpha$ are closed subspaces of $H_\alpha(\mathrm{curl}, W)$. The space $\mathcal{D}_\alpha$ corresponds to the usual space of functions with vanishing divergence. However, it contains a non-local boundary condition on γ which involves the Calderon operator. We refer to [9], Section 10.3, for a related situation in the exterior of a ball.

Lemma 2.8. *The space* $H_\alpha(\mathrm{curl}, W)$ *has the decomposition into the direct sum*

$$H_\alpha(\mathrm{curl}, W) \;=\; \nabla H^1_\alpha(W) \;\oplus\; \mathcal{D}_\alpha\,.$$

The projections onto $\nabla H^1_\alpha(W)$ *and* $\mathcal{D}_\alpha$ *along this direct sum are bounded.*

Proof. Let $u \in H_\alpha(\mathrm{curl}, W)$. We consider the variational problem to determine $p \in H^1_\alpha(W)$ with

$$b_\alpha(\nabla p, \nabla\phi) \;=\; b_\alpha(u, \nabla\phi) \quad \text{for all } \phi \in H^1_\alpha(W)\,.$$

Once we have solved this problem we have the decomposition $u = \nabla p + (u - \nabla p)$ and $u - \nabla p \in \mathcal{D}_\alpha$. The variational equation takes the form

$$k^2 \int\limits_W q\, \nabla p \cdot \nabla\overline{\phi}\, dx - \big\langle \Lambda_\alpha(\hat r \times \mathrm{Grad}\, p), \mathrm{Grad}\,\phi \big\rangle = -b_\alpha(u, \nabla\phi) \text{ for all } \phi \in H^1_\alpha(W)\,,$$

i.e.
$$k^2 \int\limits_W q\, \nabla p \cdot \nabla\overline{\phi}\, dx + \langle D_\alpha p, \phi \rangle \;=\; -b_\alpha(u, \nabla\phi) \text{ for all } \phi \in H^1_\alpha(W)$$

with the operator D_α from part (h) of Theorem 2.4. From $k^2 \int_W \mathrm{Re}\, q\, |\nabla p|^2\, dx + \mathrm{Re}\langle \tilde D_\alpha p, p\rangle \geq c \|\nabla p\|^2_{L^2(W)}$ and the compactness of $D_\alpha - \tilde D_\alpha$ we conclude that this variational equation is of Fredholm type. To show uniqueness let $p \in H^1_\alpha(W)$ with $k^2 \int_W q\, \nabla p \cdot \nabla\overline{\phi}\, dx + \langle D_\alpha p, \phi\rangle = 0$ for all $\phi \in H^1_\alpha(W)$. Taking $\phi = p$ and the imaginary part yields $k^2 \int_W \mathrm{Im}\, q\, |\nabla p|^2\, dx + \mathrm{Im}\langle D_\alpha p, p\rangle = 0$ and thus $\mathrm{Im}\langle D_\alpha p, p\rangle = 0$ because both terms on the left hand side are non-negative by Theorem 2.4. By the same theorem we conclude that $\mathrm{Re}\langle D_\alpha p, p\rangle \geq 0$. Taking the real part of $k^2 \int_W q\, |\nabla p|^2\, dx + \langle D_\alpha p, p\rangle = 0$ we conclude that $\nabla p = 0$ (because $\mathrm{Re}\, q > 0$) and thus $p = 0$.
Boundedness of the projections follows from the continuous dependence of p from u. The sum is direct because $\nabla p \in \mathcal{D}_\alpha$ for some $p \in H^1_\alpha(W)$ implies $b_\alpha(\nabla p, \nabla\phi) = 0$ for all $\phi \in H^1_\alpha(W)$ and thus $p = 0$ by the previous arguments. □

Lemma 2.9. *The space $\mathcal{D}_\alpha$ is compactly imbedded in $L^2(W, \mathbb{C}^3)$.*

Proof. Let (v_j) be a bounded sequence in $\mathcal{D}_\alpha$. Extend v_j into $H_{\alpha,*}(\mathrm{curl}, W_\infty)$ by (2.5) for $h = \hat r \times v_j|_-$ on γ. Then (v_j) is bounded in $H_\alpha(\mathrm{curl}, W_R)$ for every $R > \hat R$. Let now $\phi \in H^1_\alpha(W_\infty)$ vanish for $x_1^2 + x_2^2 > R^2$ for some $R > \hat R$. Then, because $v_j \in \mathcal{D}_\alpha$,

$$\begin{aligned}
k^2 \int\limits_{W_\infty} q\, v_j \cdot \nabla\bar\phi\, dx \;&=\; k^2 \int\limits_W q\, v_j \cdot \nabla\bar\phi\, dx \;+\; k^2 \int\limits_{W_\infty \setminus W} v_j \cdot \nabla\bar\phi\, dx \\
&=\; \langle \Lambda_\alpha(\hat r \times v_j), \mathrm{Grad}\,\phi\rangle \;-\; \int\limits_{W_\infty \setminus W} [\mathrm{curl}\, v_j \cdot \underbrace{\mathrm{curl}\, \nabla\bar\phi}_{=0} - k^2 v_j \cdot \nabla\bar\phi]\, dx \\
&=\; \int\limits_\gamma (\hat r \times \mathrm{curl}\, v_j|_+) \cdot \mathrm{Grad}\,\bar\phi\, ds \\
&\qquad - \int\limits_{W_\infty \setminus W} [\mathrm{curl}\, v_j \cdot \mathrm{curl}\, \nabla\bar\phi - k^2 v_j \cdot \nabla\bar\phi]\, dx \;=\; 0
\end{aligned}$$

by the definition of Λ_α, Green's formula applied in $W_\infty \setminus \overline{W}$, the α-quasi-periodic boundary conditions, and $\mathrm{curl}^2 v_j - k^2 v_j = 0$ in $W_\infty \setminus \overline{W}$. We choose a (real valued) cut-off function $\psi \in C^\infty(\mathbb{R}^2)$ with $\psi(x_1, x_2) = 1$ for $x_1^2 + x_2^2 \leq \hat R^2$ and $\psi(x_1, x_2) = 0$ for $x_1^2 + x_2^2 \geq (\hat R + 1)^2$ and set $\tilde v_j(x) = v_j(x)\psi(x_1, x_2)$. Let $R > \hat R + 1$. Then $\tilde v_j \in H_\alpha(\mathrm{curl}, W_R)$ is bounded in $H(\mathrm{curl}, W_R)$. Let $\varphi \in H^1_\alpha(W_R)$ be arbitrary. Then, using the previous identity for $\phi = \psi\varphi$,

$$k^2 \int\limits_{W_R} q\, \tilde{v}_j \cdot \nabla\bar{\varphi}\, dx = k^2 \int\limits_{W_R} q\, v_j \cdot \nabla(\psi\bar{\varphi})\, dx - k^2 \int\limits_{W_R} q\, v_j \cdot \nabla\psi\, \bar{\varphi}\, dx = k^2 \int\limits_{W_R} f_j \cdot \bar{\varphi}\, dx$$

with $f_j := -q\, v_j \cdot \nabla\psi$ which is bounded in $L^2(W_R, \mathbb{C}^3)$. Now we apply the improved compactness result of Lemma A.2 (choice (d) of (X, Y)) of [8] which holds also for complex q. This yields the existence of a convergent subsequence of $(\tilde{v}_j)$ in $L^2(W_R, \mathbb{C}^3)$ and thus also of (v_j) in $L^2(W, \mathbb{C}^3)$. □

The following result is needed for proving the Fredholm property of the form b_α.

Lemma 2.10. *There exists $c > 0$ such that $\|\operatorname{Div}\Lambda_\alpha(\hat{r} \times v)\|_{H_\alpha^{-1/2}(\gamma)} \le c\|v\|_{L^2(W)}$ for all $v \in \mathcal{D}_\alpha$.*

Proof. Let $\phi \in H_\alpha^{1/2}(\gamma)$. We extend ϕ to $\phi \in H_\alpha^1(W)$ such that $\|\phi\|_{H^1(W)} \le \hat{c}\|\phi\|_{H_\alpha^{1/2}(\gamma)}$ where $\hat{c}$ is independent of ϕ. Then

$$\begin{aligned}
\big|\langle \operatorname{Div}\Lambda_\alpha(\hat{r} \times v), \phi\rangle\big| &= \big|\langle \Lambda_\alpha(\hat{r} \times v), \operatorname{Grad}\phi\rangle\big| \\
&= \Big|b(v, \nabla\phi) + k^2 \int\limits_W q\, v \cdot \nabla\overline{\phi}\, dx\Big| \;=\; k^2 \Big|\int\limits_W q\, v \cdot \nabla\overline{\phi}\, dx\Big| \\
&\le \; c\,\|v\|_{L^2(W)}\,\|\phi\|_{H^1(W)} \;\le\; c\,\hat{c}\,\|v\|_{L^2(W)}\,\|\phi\|_{H_\alpha^{1/2}(\gamma)}\,.
\end{aligned}$$

Here we used that $b(v, \nabla\phi) = 0$. This yields the estimate. □

By the theorem of Riesz there exists $r_\alpha \in H_\alpha(\mathrm{curl}, W)$ and a bounded operator B_α from $H_\alpha(\mathrm{curl}, W)$ into itself such that

$$\begin{aligned}
\langle B_\alpha u, \psi\rangle_{H(\mathrm{curl},W)} &= b_\alpha(u, \psi) \\
&= \int\limits_W \left[\frac{\mu_0}{\mu}\, \mathrm{curl}\, u \cdot \mathrm{curl}\, \overline{\psi} - k^2 q\, u \cdot \overline{\psi}\right] dx \;+\; \big\langle \Lambda_\alpha(\hat{r} \times u), \psi\big\rangle\,, \qquad (2.21) \\
\langle r_\alpha, \psi\rangle_{H(\mathrm{curl},W)} &= \mu_0 \int\limits_W \left[\frac{1}{\mu}\, f_h \cdot \mathrm{curl}\, \overline{\psi} + i\omega f_e \cdot \overline{\psi}\right] dx
\end{aligned}$$

for all $u, \psi \in H_\alpha(\mathrm{curl}, W)$. We define a second operator $\hat{B}_\alpha$ from $H_\alpha(\mathrm{curl}, W)$ into itself by

$$\langle \hat{B}_\alpha u, \psi\rangle_{H(\mathrm{curl},W)} \;=\; \int\limits_W \frac{\mu_0}{\mu}\, \mathrm{curl}\, u \cdot \mathrm{curl}\, \overline{\psi} + u \cdot \overline{\psi}\, dx \;+\; \big\langle \hat{\Lambda}_\alpha^D(\hat{r} \times u), \psi\big\rangle$$

with the operator $\hat{\Lambda}_\alpha^D$ from Theorem 2.4, part (f). As a consequence of Lemma 2.7 we have

Lemma 2.11. *(a) $\hat{B}_\alpha$ is selfadjoint and coercive, i.e. $\langle \hat{B}_\alpha u, u\rangle_{H(\mathrm{curl},W)} \ge c\|u\|^2_{H(\mathrm{curl},W)}$ for all $u \in H_\alpha(\mathrm{curl}, W)$.*

(b) $B_\alpha - \hat{B}_\alpha$ is compact from $\mathcal{D}_\alpha$ into $H_\alpha(\mathrm{curl}, W)$.

Proof. (a) is obvious by part (f) of Theorem 2.4.

(b) Introducing the compact operator K from $H_\alpha(\mathrm{curl}, W)$ into itself by

$$\langle Ku, \psi\rangle_{H(\mathrm{curl},W)} \;=\; \big\langle \Lambda_\alpha^K(\hat{r} \times u), \psi\big\rangle\,, \quad u, \psi \in H_\alpha(\mathrm{curl}, W)\,,$$

with the compact operator $\Lambda_\alpha^K = \Lambda_\alpha^D - \hat{\Lambda}_\alpha^D$ from Theorem 2.4, part (f), we note that $B_\alpha - \hat{B}_\alpha - K$ is represented as

$$\langle (B_\alpha - \hat{B}_\alpha - K)u, \psi\rangle_{H(\mathrm{curl},W)} \;=\; -\int\limits_W (k^2 q + 1)\, u \cdot \overline{\psi}\, dx \;+\; \big\langle \Lambda_\alpha^C(\hat{r} \times u), \psi\big\rangle$$

for $u, \psi \in H_\alpha(\mathrm{curl}, W)$. For $u \in \mathcal{D}_\alpha$ and $\psi \in H_\alpha(\mathrm{curl}, W)$ we estimate

$$\begin{aligned}
&\big|\langle (B_\alpha - \hat{B}_\alpha - K)u, \psi\rangle_{H(\mathrm{curl},W)}\big| \\
\leq\;& c\left[\|u\|_{L^2(W)}\, \|\psi\|_{L^2(W)} \;+\; \|\Lambda_\alpha^C(\hat{r} \times u)\|_{H_\alpha^{-1/2}(\mathrm{Div},\gamma)}\, \|\psi\|_{H_\alpha^{-1/2}(\mathrm{Curl},\gamma)}\right] \\
\leq\;& c\left[\|u\|_{L^2(W)}\, \|\psi\|_{L^2(W)} \;+\; \|\operatorname{Div}\Lambda_\alpha(\hat{r} \times u)\|_{H_\alpha^{-1/2}(\gamma)}\, \|\psi\|_{H_\alpha^{-1/2}(\mathrm{Curl},\gamma)}\right] \\
\leq\;& c\left[\|u\|_{L^2(W)}\, \|\psi\|_{L^2(W)} \;+\; \|u\|_{L^2(W)}\, \|\psi\|_{H_\alpha^{-1/2}(\mathrm{Curl},\gamma)}\right] \\
\leq\;& c\, \|u\|_{L^2(W)}\, \|\psi\|_{H(\mathrm{curl},W)}
\end{aligned}$$

where we used the estimate (2.17), Lemma 2.10, and the trace theorem. Therefore, $\|(B_\alpha - \hat{B}_\alpha - K)u\|_{H(\mathrm{curl},W)} \leq c\,\|u\|_{L^2(W)}$ which proves compactness of $B_\alpha - \hat{B}_\alpha - K$ by the compact imbedding of $\mathcal{D}_\alpha$ in $L^2(W, \mathbb{C}^3)$. □

Lemma 2.12. *The operator B_α is a Fredholm operator with index zero and Riesz number one. The latter means that $\mathcal{N}(B_\alpha^2) = \mathcal{N}(B_\alpha)$.*

Proof. By the definition of $\mathcal{D}_\alpha$ the restriction of b_α to $\mathcal{D}_\alpha \times \nabla H_\alpha^1(W)$ vanishes. The restrictions of b_α to $\mathcal{D}_\alpha \times \mathcal{D}_\alpha$ and $\nabla H_\alpha^1(W) \times \mathcal{D}_\alpha$ and $\nabla H_\alpha^1(W) \times \nabla H_\alpha^1(W)$ yield bounded operators $B_{1,1} : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ and $B_{1,2} : \nabla H_\alpha^1(W) \to \mathcal{D}_\alpha$ and $B_{2,2} : \nabla H_\alpha^1(W) \to \nabla H_\alpha^1(W)$, respectively, with $\langle B_{1,1}u, v\rangle_{H(\mathrm{curl},W)} = b_\alpha(u, v)$ and $\langle B_{1,2}\nabla\phi, v\rangle_{H(\mathrm{curl},W)} = b_\alpha(\nabla\phi, v)$ and $\langle B_{2,2}\nabla\phi, \nabla\rho\rangle_{H(\mathrm{curl},W)} = b_\alpha(\nabla\phi, \nabla\rho)$ for all $u, v \in \mathcal{D}_\alpha$ and $\phi, \rho \in H_\alpha^1(W)$. Therefore, the operator B_α has a decomposition into

$$B_\alpha \;\cong\; \begin{pmatrix} B_{1,1} & B_{1,2} \\ 0 & B_{2,2} \end{pmatrix} \;:\; \mathcal{D}_\alpha \times \nabla H_\alpha^1(W) \;\longrightarrow\; \mathcal{D}_\alpha \times \nabla H_\alpha^1(W)\,.$$

By the previous lemma the operator $B_{1,1}$ is of the form $B_{1,1} = A + C$ where A is invertible and C is compact. Since the operator $B_{2,2}$ is invertible (see proof of Lemma 2.8) we have a decomposition into

$$\begin{pmatrix} B_{1,1} & B_{1,2} \\ 0 & B_{2,2} \end{pmatrix} \;=\; \begin{pmatrix} A & B_{1,2} \\ 0 & B_{2,2} \end{pmatrix} \;+\; \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$$

where $\begin{pmatrix} A & B_{1,2} \\ 0 & B_{2,2} \end{pmatrix}$ is invertible and $\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$ is compact. In particular, B_α is a Fredholm operator with index zero.

To show that $\mathcal{N}(B_\alpha^2) \subset \mathcal{N}(B_\alpha)$ let $B_\alpha^2 u = 0$. With $v = B_\alpha u$ we have $b_\alpha(v, \psi) = 0$ for all $\psi \in H_\alpha(\mathrm{curl}, W)$ and thus (Lemma 2.7, part (c)) $b_\alpha(\psi, v) = 0$ for all $\psi \in H_\alpha(\mathrm{curl}, W)$. Therefore, $0 = b_\alpha(u, v) = \langle B_\alpha u, v\rangle_{H(\mathrm{curl},W)} = \langle v, v\rangle_{H(\mathrm{curl},W)}$ which implies $B_\alpha u = v = 0$. □

Corollary 2.13. *For every $\alpha \in [-1/2, 1/2]$ the space $H_\alpha(\mathrm{curl}, W)$ has a decomposition in the form*

$$H_\alpha(\mathrm{curl}, W) \;:=\; \mathcal{N}(B_\alpha) \;\oplus\; \mathcal{R}(B_\alpha)\,.$$

If $\sigma = 0$, i.e. $q = \varepsilon/\varepsilon_0$ is real valued then the sum is orthogonal in $H_\alpha(\mathrm{curl}, W)$. Therefore, the projections onto $\mathcal{N}(B_\alpha)$ and $\mathcal{R}(B_\alpha)$ are the orthogonal projections.

Proof. The decomposition follows from the Fredholm property and because the Riesz number is one. To show the orthogonality let $u \in \mathcal{N}(B_\alpha)$. Part (c) of Lemma 2.7 implies $b_\alpha(\psi, u) = 0$ for all $\psi \in H_\alpha(\mathrm{curl}, W)$, i.e. $\langle B_\alpha \psi, u\rangle_{H(\mathrm{curl},W)} = 0$ for all $\psi \in H_\alpha(\mathrm{curl}, W)$, i.e. u is orthogonal to $\mathcal{R}(B_\alpha)$. □

For $\sigma > 0$, i.e. $\operatorname{Im} q > 0$, we have uniqueness and existence of the quasi-periodic problems.

Theorem 2.14. *Let $\sigma \in L^\infty(W)$ with $\sigma \geq \sigma_0$ on $W_{\hat{R}}$ for some constant $\sigma_0 > 0$, and let $\alpha \in [-1/2, 1/2]$ not be a cut-off value. Then there exists a unique solution $\tilde{u} \in H_\alpha(\mathrm{curl}, W)$ of (2.18). Furthermore, its extension as in Theorem 2.6 determines the unique solution $u \in H_{\alpha,*}(\mathrm{curl}, W_\infty)$ of (2.1), (2.2).*

Proof. By the Fredholm property of B_α it suffices to show injectivity of B_α. Let $\tilde{u} \in H_\alpha(\mathrm{curl}, W)$ with $B_\alpha \tilde{u} = 0$, in particular $b_\alpha(\tilde{u}, \tilde{u}) = 0$. Taking the imaginary part and using part (b) of Lemma 2.7 yields $\tilde{u} = 0$. □

We note that if ε, μ, and σ are sufficiently smooth such that the unique continuation principle holds for (1.1) then $\sigma > 0$ on some open subset of $W_{\hat{R}}$ suffices for uniqueness and existence.

3. The Dependence on α and σ and Existence in the Case of Absorption

In this section we study the behavior of the solution $u = u_\alpha$ of (2.1), (2.2) in neighborhoods of cut-off values (see Definition 2.1) and critical values. We recall the sets $W_\infty = \{x \in \mathbb{R}^3 : x_1^2 + x_2^3 > \hat{R}^2\}$ and $W = \{x \in W_\infty : x_3 \in (0, 2\pi)\}$.

Definition 3.1. *A parameter $\alpha \in (-1/2, 1/2]$ which is not a cut-off value (that is, $|n + \alpha| \neq k$ for all $n \in \mathbb{Z}$) is called a critical value if there exists a non-trivial $\alpha-$quasi-periodic (with respect to x_3) solution $\phi \in H_{\alpha,*}(\mathrm{curl}, W_\infty)$ of*

$$\int_{W_\infty} \frac{\mu_0}{\mu} \,\mathrm{curl}\,\phi \cdot \mathrm{curl}\,\overline{\psi} - k^2 \frac{\varepsilon}{\varepsilon_0}\, \phi \cdot \overline{\psi}\, dx \;=\; 0 \tag{3.1}$$

for all $\psi \in H_\alpha(\mathrm{curl}, W_\infty)$ with compact support in $\overline{W_\infty}$ and satisfies also the radiation condition (2.2). The function ϕ is called (propagating) mode. We denote the set of all critical values by $\mathcal{A}$ and the space of all modes corresponding to $\alpha \in \mathcal{A}$ by $\mathcal{M}^\alpha$.

Remark 3.2. *We note that by definition the set $\{\alpha \in [-1/2, 1/2] : |n + \alpha| = k$ for some $n \in \mathbb{Z}\}$ of cut-off values is disjoint to the set $\mathcal{A}$ of critical values.*

This is a standard assumption for treating open waveguide problems, see, e.g., [2, 3, 6]. We observe that Theorem 2.6 implies that the critical values are exactly the parameters α for which the operators B_α from (2.21) for $\sigma = 0$ fail to be isomorphisms from $H_\alpha(\mathrm{curl}, W)$ onto itself. Furthermore, $\mathcal{N}(B_\alpha) = \{\phi|_W : \phi \in \mathcal{M}^\alpha\}$.

Lemma 3.3. *Let α be a critical value. Then the space $\mathcal{M}^\alpha$ of corresponding modes, defined in Definition 3.1 is finite dimensional. The functions $\phi \in \mathcal{M}^\alpha$ are decaying exponentially, i.e. there exists $c, \delta > 0$ with $\max_{x\in\gamma_r} |\phi(x)| \leq c\, e^{-\delta r}$ for $r \geq \hat{R}$ where $\gamma_r = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = r^2,\ 0 < x_3 < 2\pi\}$.*

Proof. First we note that α is not a cut-off value by Definition 3.1. From Theorem 2.6 we observe that $\{\phi|_W : \phi \in \mathcal{M}^\alpha\} = \ker(B_\alpha)$ with B_α from (2.21). The kernel is finite dimensional because of the Fredholm property of B_α. Furthermore, from part (c) of Theorem 2.4 we conclude that ϕ is decaying exponentially fast. □

In the following we will study the dependence of u on α and σ. So far, we allowed $q = \frac{\varepsilon}{\varepsilon_0} + i\frac{\sigma}{\omega\varepsilon_0}$ to be complex valued with arbitrary (periodic) conductivity $\sigma \geq 0$. In this subsection we assume that $\sigma = t\hat{\sigma}$ for some parameter $t \geq 0$ and some fixed periodic $\hat{\sigma} \in L^\infty(\mathbb{R}^3)$ which is strictly positive on $\Omega_{\hat{R}}$ with some positive lower bound and which vanishes in $\mathbb{R}^3 \setminus \Omega_{\hat{R}}$.
FThe operator $B_\alpha : H_\alpha(\mathrm{curl}, W) \to H_\alpha(\mathrm{curl}, W)$ depends now on two parameters α and t, and we write $B_{\alpha,t}$ to indicate this dependence. The corresponding sesqui-linear form is denoted by $b_{\alpha,t} : H_\alpha(\mathrm{curl}, W) \times H_\alpha(\mathrm{curl}, W) \to \mathbb{C}$. We set also $q_t := \frac{\varepsilon}{\varepsilon_0} + i\frac{\hat{\sigma}}{\omega\varepsilon_0}$ to indicate the dependence on t.

It is convenient to transform the $\alpha-$quasi-periodicity from the function space into the sesqui-linear form. Therefore, we define the space of $2\pi-$periodic (with respect to x_3) functions by

$$H_{per}(\mathrm{curl}, W) \;:=\; \big\{ u \in H(\mathrm{curl}, W) : x_3 \mapsto u(x) \text{ is } 2\pi\text{-periodic} \big\}$$

equipped with the canonical inner product $\langle \cdot, \cdot \rangle_{H(\mathrm{curl},W)}$ induced by the inner product in $H(\mathrm{curl}, W)$.[3] Then (2.18) is equivalent to the determination of $v_{\alpha,t} \in H_{per}(\mathrm{curl}, W)$ such that

$$\begin{aligned} &\int\limits_W \left[\frac{\mu_0}{\mu} \, \mathrm{curl}\big(v_{\alpha,t}(x) e^{i\alpha x_3}\big) \cdot \mathrm{curl}\big(\overline{\psi(x) e^{i\alpha x_3}}\big) - k^2 q_t \, v_{\alpha,t} \cdot \overline{\psi} \right] dx \\ &\quad + \; \big\langle \Lambda_\alpha(\hat{r} \times v_{\alpha,t} \, e^{i\alpha x_3}) \, , \psi \, e^{i\alpha x_3} \big\rangle \\ &= \; \mu_0 \int\limits_W \left[\frac{1}{\mu} \, f_h \cdot \mathrm{curl} \, \overline{(\psi(x) e^{i\alpha x_3})} + i\omega f_e \cdot \overline{(\psi(x) e^{i\alpha x_3})} \right] dx \end{aligned} \tag{3.2}$$

for all $\psi \in H_{per}(\mathrm{curl}, W)$. This is the basic equation which we study with respect to the dependence on α and t. Substituting the form (2.13), (2.14), (2.15) of Λ_α for $h = \hat{r} \times v_{\alpha,t} \, e^{i\alpha x_3}$ we write the sesqui-linear form on the left hand side as

$$\begin{aligned} a_{\alpha,t}(v, \psi) \; :=& \; \int\limits_W \frac{\mu_0}{\mu} \, \mathrm{curl}\big(v e^{i\alpha x_3}\big) \cdot \mathrm{curl}\big(\overline{\psi e^{i\alpha x_3}}\big) - k^2 q_t \, v \cdot \overline{\psi} \, dx \\ & + 4\pi^2 k^2 \sum_{n,m \in \mathbb{Z}} v^z_{n,m} \, \overline{\psi^z_{n,m}} \, G^{(1)}_{n,m}(\hat{R}) \; + \; 4\pi^2 k^2 \hat{R}^2 \sum_{n,m \in \mathbb{Z}} v^\phi_{n,m} \overline{\psi^\phi_{n,m}} \, G^{(2)}_{n,m}(\hat{R}) \\ & + 4\pi^2 \hat{R}^2 \sum_{n,m \in \mathbb{Z}} G^{(2)}_{n,m}(\hat{R}) \left[(n+\alpha) \, v^\phi_{n,m} - \frac{m}{\hat{R}} \, v^z_{n,m} \right] \left[\frac{m}{\hat{R}} \, \overline{\psi^z_{n,m}} - (n+\alpha) \, \overline{\psi^\phi_{n,m}} \right]. \end{aligned} \tag{3.3}$$

Here, the Fourier coefficients $v^j_{m,n} \in \mathbb{C}$ are given by

$$\begin{aligned} v^z_{n,m} \; &= \; \frac{1}{4\pi^2} \int_0^{2\pi} \!\! \int_0^{2\pi} v(\hat{R}, \phi, x_3) \cdot \hat{z} \, e^{-im\phi - inx_3} \, d\phi \, dx_3 \, , \quad (n,m) \in \mathbb{Z} \times \mathbb{Z} \, , \\ v^\phi_{n,m} \; &= \; \frac{1}{4\pi^2} \int_0^{2\pi} \!\! \int_0^{2\pi} v(\hat{R}, \phi, x_3) \cdot \hat{\phi} \, e^{-im\phi - inx_3} \, d\phi \, dx_3 \, , \quad (n,m) \in \mathbb{Z} \times \mathbb{Z} \, . \end{aligned} \tag{3.4}$$

Lemma 3.4. *(a) Let* $\hat{\alpha} \in [-1/2, 1/2]$ *not be a cut-off value and* $\hat{t} > 0$. *Then there exists a neighborhood* $I \times T = (\hat{\alpha} - \delta, \hat{\alpha} + \delta) \times (\hat{t} - \delta, \hat{t} + \delta)$ *of* $(\hat{\alpha}, \hat{t})$ *such that there exists*

[3] Apparently, it is $H_{per}(\mathrm{curl}, W) = H_\alpha(\mathrm{curl}, W)$ for $\alpha = 0$ but we want to avoid the notion $H_0(\mathrm{curl}, W)$ because this notation is reserved for the space of functions with $\nu \times v = 0$ on the boundary.

a unique solution $v_{\alpha,t} \in H_{per}(\mathrm{curl}, W)$ of (3.2) for $(\alpha,t) \in I \times T$, and $(\alpha,t) \mapsto v_{\alpha,t}$ is continuously differentiable from $I \times T$ into $H_{per}(\mathrm{curl}, W)$.

(b) Let $\hat\alpha \in [-1/2, 1/2]$ be neither a cut-off value nor a critical value. Then there exists a neighborhood $I \times T = (\hat\alpha - \delta, \hat\alpha + \delta) \times (-\delta, \delta)$ of $(\hat\alpha, 0)$ such that there exists a unique solution $v_{\alpha,0} \in H_{per}(\mathrm{curl}, W)$ of (3.2) for $(\alpha,t) \in I \times T$, and $(\alpha,t) \mapsto v_{\alpha,t}$ is continuously differentiable from $I \times T$ into $H_{per}(\mathrm{curl}, W)$.

Proof. Let $A_{\alpha,t} : H_{per}(\mathrm{curl}, W) \to H_{per}(\mathrm{curl}, W)$ be the operator corresponding to $a_{\alpha,t}$, i.e. $\langle A_{\alpha,t} v, \psi\rangle_{H(\mathrm{curl},W)} = a_{\alpha,t}(v,\psi)$ for all $v, \psi \in H_{per}(\mathrm{curl}, W)$. Since $\hat\alpha$ is not a cut-off value application of part (g) of Theorem 2.4 yields that $\alpha \mapsto \langle \Lambda_\alpha(\hat r \times v\, e^{i\alpha x_3}), \psi\, e^{i\alpha x_3}\rangle$ is holomorphic in $\{\alpha \in \mathbb{C} : |\alpha - \hat\alpha| < \delta\}$ (for some $\delta > 0$) for every $v, \psi \in H_{per}(\mathrm{curl}, W)$. This means that the operator S_α from $H_{per}(\mathrm{curl}, W)$ into itself, defined as $\langle S_\alpha v, \psi\rangle_{H(\mathrm{curl},W)} = \langle \Lambda_\alpha(\hat r \times v\, e^{i\alpha x_3}), \psi\, e^{i\alpha x_3}\rangle$ is weakly holomorphic with respect to α in the sense of, e.g., [1], Section 8.5. By the same reference this mapping is even strongly holomorphic and thus continuously differentiable. Since the integral term depends also smoothly on (α,t) we observe that $(\alpha,t) \mapsto A_{\alpha,t}$ is continuously differentiable. Since $A_{\hat\alpha,\hat t}$ (where $\hat t = 0$ in case(b)) is an isomorphism we conclude that also the solution $v_{\alpha,t}$ depends continuously differentiable on α and t. □

In the following two subsections we study the dependence on α and t in the neighborhoods of cut-off values and critical values.

3.1. **α in a neighborhood of a cut-off value.** Let $\hat\alpha$ be a cut off value, i.e. there exist $\hat n \in \mathbb{Z}$ with $|\hat n + \hat\alpha| = k$. The (non-empty) set $\mathcal{C} := \{(\hat n, m) \in \mathbb{Z}^2 : |\hat n + \hat\alpha| = k,\ |m| \le 1\}$ will play an essential role in the following analysis. If we decompose k in the form $k = \tilde n + \kappa$ with $\kappa \in (-1/2, 1/2]$ and $\tilde n \in \mathbb{Z}_{\ge 0}$, then $\hat\alpha = \pm\kappa$ are the cut-off values. The set $\mathcal{C}$ consists of 3 elements if $k \notin \frac{1}{2}\mathbb{N}$. Indeed, under this assumption there are exactly two cut-off values $\pm\kappa$ in $[-1/2, 1/2]$, one in $(-1/2, 0)$ and the other in $(0, 1/2)$. To each of them there corresponds exactly one $\hat n \in \mathbb{Z}$ with $|\hat n + \hat\alpha| = k$, namely $\hat n = \pm\tilde n$ if $\hat\alpha = \pm\kappa$. If $k \in \mathbb{N}$ or $k \in -1/2 + \mathbb{N}$ then there exists only one cut-off value $\hat\alpha = 0$ or $\hat\alpha = 1/2$, respectively, and $\mathcal{C}$ consists of 6 elements.

Assumption 3.5. *Let $k \notin \frac{1}{2}\mathbb{N}$.*

As noted above the set $\mathcal{C}$ is now given by $\mathcal{C} = \{(\hat n, m) : |m| \le 1\}$ where $\hat n = \pm\tilde n$ if $\hat\alpha = \pm\kappa$. If this assumption is violated then one has to modify the analysis below. We do not treat this case.

In this subsection we study the behavior of the solution $v_{\alpha,t}$ of (3.2) if $\alpha \to \hat\alpha$ for some cut-off value $\hat\alpha \in \{+\kappa, -\kappa\}$, i.e., we have to study the singular behavior of Λ_α as $\alpha \to \hat\alpha$. We decompose the Calderon operator from (2.13)–(2.15) (for $h = \hat r \times u$) into the form

$$
\begin{aligned}
(\Lambda_\alpha(\hat r \times u))(\phi, x_3) \ &= \sum_{(n,m)\notin\mathcal{C}} \left[\lambda^z_{n,m}\hat z + \lambda^\phi_{n,m}\hat\phi\right] e^{im\phi + i(n+\alpha)x_3} \\
&+ \left\{ u^z_{\hat n,0} \frac{k^2}{\hat R}\, G^{(1)}_{\hat n,0}(\hat R, \alpha)\hat z + [k^2 - (\hat n + \alpha)^2] u^\phi_{\hat n,0} \hat R\, G^{(2)}_{\hat n,0}(\hat R, \alpha)\hat\phi \right\} e^{i(\hat n+\alpha)x_3} \\
&+ \sum_{|m|=1} \left[\lambda^z_{\hat n,m}\hat z + \lambda^\phi_{\hat n,m}\hat\phi\right] e^{im\phi + i(\hat n+\alpha)x_3}
\end{aligned}
$$

where

$$\begin{aligned}
\lambda^z_{n,m} &= u^z_{n,m}\frac{k^2}{\hat R}\,G^{(1)}_{n,m}(\hat R,\alpha) \;+\; m\left[-\frac{m}{\hat R}\,u^z_{n,m} + (n+\alpha)u^\phi_{n,m}\right] G^{(2)}_{n,m}(\hat R,\alpha)\,,\\
\lambda^\phi_{n,m} &= [k^2-(n+\alpha)^2]\,u^\phi_{n,m}\hat R\,G^{(2)}_{n,m}(\hat R,\alpha) + m(n+\alpha)\,u^z_{n,m}\,G^{(2)}_{n,m}(\hat R,\alpha)\,.
\end{aligned}$$

We set

$$\begin{aligned}
(\tilde\Lambda_\alpha(\hat r\times u))(\phi,x_3) \;&=\; \sum_{(n,m)\notin\mathcal{C}} \left[\lambda^z_{n,m}\hat z + \lambda^\phi_{n,m}\hat\phi\right] e^{im\phi+i(n+\alpha)x_3}\\
&+ [k^2-(\hat n+\alpha)^2]\sum_{|m|\le 1} u^\phi_{\hat n,m}\hat R\,G^{(2)}_{\hat n,m}(\hat R,\alpha)\hat\phi\, e^{im\phi+i(\hat n+\alpha)x_3}
\end{aligned}\tag{3.5}$$

and write $\Lambda_\alpha(\hat r\times u)$ as

$$\begin{aligned}
&(\Lambda_\alpha(\hat r\times u))(\phi,x_3) \;=\; (\tilde\Lambda_\alpha(\hat r\times u))(\phi,x_3) \;+\; u^z_{\hat n,0}\frac{k^2}{\hat R}G^{(1)}_{\hat n,0}(\hat R,\alpha)\,\hat z\, e^{i(\hat n+\alpha)x_3}\\
&+\sum_{|m|=1}\left\{\left[u^z_{\hat n,m}\left(\frac{k^2}{\hat R}G^{(1)}_{\hat n,m}(\hat R,\alpha) - \frac{1}{\hat R}\,G^{(2)}_{\hat n,m}(\hat R,\alpha)\right) + u^\phi_{\hat n,m} m(\hat n+\alpha)\,G^{(2)}_{\hat n,m}(\hat R,\alpha)\right]\hat z\right.\\
&\left.+\; u^z_{\hat n,m}\,m(\hat n+\alpha)\,G^{(2)}_{\hat n,m}(\hat R,\alpha)\hat\phi\right\} e^{im\phi+i(\hat n+\alpha)x_3}\\
=\;& (\tilde\Lambda_\alpha(\hat r\times u))(\phi,x_3) \;+\; g_0(\alpha)\,u^z_{\hat n,0}\,\hat z\, e^{i(\hat n+\alpha)x_3}\\
&+\sum_{|m|=1}\left\{\left[g_m(\alpha)\,u^z_{\hat n,m} + h_m(\alpha)\,u^\phi_{\hat n,m}\right]\hat z \;+\; h_m(\alpha)\,u^z_{\hat n,m}\hat\phi\right\} e^{im\phi+i(\hat n+\alpha)x_3}
\end{aligned}$$

with

$$\begin{aligned}
g_0(\alpha) &= \frac{k^2}{\hat R}G^{(1)}_{\hat n,0}(\hat R,\alpha)\,,\\
g_m(\alpha) &= \frac{k^2}{\hat R}G^{(1)}_{\hat n,m}(\hat R,\alpha) - \frac{1}{\hat R}\,G^{(2)}_{\hat n,m}(\hat R,\alpha)\,,\quad |m|=1\,,\\
h_m(\alpha) &= m(\hat n+\alpha)\,G^{(2)}_{\hat n,m}(\hat R,\alpha)\,,\quad |m|=1\,.
\end{aligned}$$

The decomposition of Λ_α implies a decomposition of the sesquilinear form $a_{\alpha,t}$ from (3.3). Now we use the representation theorem by Riesz. There exist bounded operators $A_{\alpha,t}$ and $\tilde A_{\alpha,t}$ from $H_{per}(\mathrm{curl},W)$ into itself and elements $w^z_m, w^\phi_m, y_\alpha \in H_{per}(\mathrm{curl},W)$ for $|m|\le 1$ such that

$$\begin{aligned}
\langle A_{\alpha,t}v,\psi\rangle_{H(\mathrm{curl},W)} &= a_{\alpha,t}(v,\psi)\\
&= \int_W \left[\frac{\mu_0}{\mu}\,\mathrm{curl}\big(v(x)e^{i\alpha x_3}\big)\cdot\mathrm{curl}\big(\overline{\psi(x)e^{i\alpha x_3}}\big) - k^2 q_t\, v\cdot\overline{\psi}\right]dx\\
&\quad+\left\langle \Lambda_\alpha(\hat r\times v\,e^{i\alpha x_3})\,,\psi\,e^{i\alpha x_3}\right\rangle,\\
\langle \tilde A_{\alpha,t}v,\psi\rangle_{H(\mathrm{curl},W)} &= \int_W \left[\frac{\mu_0}{\mu}\,\mathrm{curl}\big(v(x)e^{i\alpha x_3}\big)\cdot\mathrm{curl}\big(\overline{\psi(x)e^{i\alpha x_3}}\big) - k^2 q_t\, v\cdot\overline{\psi}\right]dx\\
&\quad+\left\langle \tilde\Lambda_\alpha(\hat r\times v\,e^{i\alpha x_3})\,,\psi\,e^{i\alpha x_3}\right\rangle,\\
\langle \psi, w^z_m\rangle_{H(\mathrm{curl},W)} &= \psi^z_{\hat n,m}\,,\\
\langle \psi, w^\phi_m\rangle_{H(\mathrm{curl},W)} &= \psi^\phi_{\hat n,m}\,,\\
\langle y_\alpha,\psi\rangle_{H(\mathrm{curl},W)} &= \mu_0\int_W\left[\frac{1}{\mu}\,f_h\cdot\mathrm{curl}\big(\overline{\psi e^{i\alpha x_1}}\big) + i\omega f_e\cdot\overline{\psi}\,e^{-i\alpha x_1}\right]dx
\end{aligned}$$

for all $v, \psi \in H_{per}(\mathrm{curl}, W)$ and $|m| \le 1$. Therefore, (3.2) is written as an operator equation in the form

$$\begin{aligned} (3.6) \qquad & \tilde{A}_{\alpha,t} v_{\alpha,t} \;+\; 4\pi^2 g_0(\alpha) \langle v_{\alpha,t}, w_0^z \rangle_{H(\mathrm{curl},W)}\, w_0^z \\ & + 4\pi^2 \sum_{|m|=1} \big\{ \big[g_m(\alpha)\, \langle v_{\alpha,t}, w_m^z \rangle_{H(\mathrm{curl},W)} + h_m(\alpha)\, \langle v_{\alpha,t}, w_m^\phi \rangle_{H(\mathrm{curl},W)} \big] w_m^z \\ & \qquad + h_m(\alpha)\, \langle v_{\alpha,t}, w_m^z \rangle_{H(\mathrm{curl},W)} w_m^\phi \big\} \;=\; y_\alpha\,, \quad \alpha \neq \hat{\alpha}\,, \end{aligned}$$

in $H(\mathrm{curl}, W)$. We note that $A_{\alpha,t}$ and $\tilde{A}_{\alpha,t}$ differ by an operator of rank 5.

Lemma 3.6. *The operator $\tilde{A}_{\alpha,t}$ depends continuously on α in a neighborhood of $\hat{\alpha}$. Furthermore, $\mathrm{Im}\langle \tilde{A}_{\alpha,t} v, v\rangle_{H(\mathrm{curl},W)} \le -k^2 \int_W \mathrm{Im}\, q_t\, |v|^2 dx$.*

Proof. By part (g) of Theorem 2.4 and the fact that $\lim_{\alpha\to\hat{\alpha}} [k^2 - (\hat{n}+\alpha)^2]\, G^{(2)}_{\hat{n},m}(\hat{R}, \alpha) = 0$ for $|m| \le 1$ (Lemma A.1 of [4], formulas (A.4e)) we conclude that the operator $\tilde{\Lambda}_\alpha$ depends continuously on α in a neighborhood of $\hat{\alpha}$ and $|g_j(\alpha)| \to \infty$ as $\alpha \to \hat{\alpha}$ (see Lemma A.1 of [4], formulas (A.4b) and (A.4c)). Furthermore, $\mathrm{Im}\langle \tilde{\Lambda}_\alpha(\hat{r} \times h), h\rangle \le 0$ for all h (because of Remark 2.5). This implies that also $\tilde{A}_{\alpha,t}$ depends continuously on α and $\mathrm{Im}\langle \tilde{A}_{\alpha,t} v, v\rangle_{H(\mathrm{curl},W)} \le -k^2 \int_W \mathrm{Im}\, q_t\, |v|^2 dx$. □

Assumption 3.7. *Let $\tilde{A}_{\hat{\alpha},t}$ be an isomorphism from $H_{per}(\mathrm{curl}, W)$ onto itself and let the 3×3-matrix with entries $\langle \tilde{A}_{\hat{\alpha},t}^{-1} w_{m'}^z, w_m^z \rangle_{H(\mathrm{curl},W)}$ for $m', m \in \{-1, 0, 1\}$ be regular.*

Remark 3.8. *Assumption 3.7 is satisfied for $t > 0$. Indeed, if $\tilde{A}_{\hat{\alpha},t} v = 0$ for some $v \neq 0$ then Lemma 3.6 implies $0 = \mathrm{Im}\langle \tilde{A}_{\hat{\alpha},t} v, v\rangle_{H(\mathrm{curl},W)} \le -k^2 \int_W \mathrm{Im}\, q_t\, |v|^2 dx < 0$, a contradiction. Thus $v = 0$, and the Fredholm property yields that $\tilde{A}_{\hat{\alpha},t}$ is an isomorphism. Analogously, let $\sum_{|m'|\le 1} s_{m'} \langle \tilde{A}_{\hat{\alpha},t}^{-1} w_{m'}^z, w_m^z \rangle_{H(\mathrm{curl},W)} = 0$ for all $|m| \le 1$. With $v := \sum_{|m'|\le 1} s_{m'} \tilde{A}_{\hat{\alpha},t}^{-1} w_{m'}^z$ we have $\langle v, w_m^z \rangle_{H(\mathrm{curl},W)} = 0$ for all $|m| \le 1$ and thus $\big\langle v, \sum_{|m|\le 1} s_m w_m^z \big\rangle_{H(\mathrm{curl},W)} = 0$, i.e. $\langle v, \tilde{A}_{\hat{\alpha},t} v\rangle_{H(\mathrm{curl},W)} = 0$ which implies $v = 0$ by the above arguments. The linear independence of w_m^z implies $s_m = 0$ for all m.*

It is the aim to prove the following convergence result.

Lemma 3.9. *Let $t \ge 0$ and $\hat{\alpha}$ be a cut-off value for which Assumption 3.7 holds.*

(a) *For α sufficiently close to $\hat{\alpha}$ there exists a unique solution $v_{\alpha,t} \in H_{\mathrm{per}}(\mathrm{curl}, W)$ of (3.2), and $v_{\alpha,t}$ converges to the unique solution $v_{\hat{\alpha},t} \in H_{\mathrm{per}}(\mathrm{curl}, W)$ of*

$$(3.7) \qquad \tilde{A}_{\hat{\alpha},t}\, v_{\hat{\alpha},t} \;=\; y_{\hat{\alpha}} \;-\; \sum_{|m|\le 1} s_m^z(\hat{\alpha})\, w_m^z\,,$$

where the three coefficients $s_0^z(\hat{\alpha}), s_{\pm 1}^z(\hat{\alpha}) \in \mathbb{C}$ solve the system (3.14) below. Furthermore, there exists $c = c(t) > 0$ with

$$(3.8) \qquad \big| \langle v_{\alpha,t}, w_m^z \rangle_{H(\mathrm{curl},W)} \big| \;\le\; \frac{c}{|g_m(\alpha)|} \quad \text{for } |m| \le 1 \text{ and all } \alpha\,.$$

(b) *Let $u_{\alpha,t} \in H_{\alpha,*}(\mathrm{curl}, W_\infty)$ be the extension of $v_{\alpha,t}\, e^{i\alpha x_3} \in H_\alpha(\mathrm{curl}, W)$ as constructed in (2.20) of Theorem 2.6. Then $\alpha \mapsto u_{\alpha,t}$ has an extension to $\alpha = \hat{\alpha}$, and this extension is continuous from a neighborhood of $\hat{\alpha}$ into $H(\mathrm{curl}, W_R)$ for all $R > \hat{R}$.*

Proof. (a) Under Assumption 3.7 also $\tilde{A}_{\alpha,t}$ is invertible for α close to $\hat{\alpha}$ since $\alpha \mapsto \tilde{A}_{\alpha,t}$ is continuous. We obtain the solution of (3.6) in the standard way: Set $B = B_{\alpha,t} := \tilde{A}_{\alpha,t}^{-1}$ and write v for $v_{\alpha,t}$ for simplicity. Then v is given by

$$v \;=\; By_\alpha \;-\; s_0^z\, Bw_0^z \;-\; \sum_{|m|=1} \big[s_m^z\, Bw_m^z + s_m^\phi\, Bw_m^\phi \big] \tag{3.9}$$

where $s_m^z = s_m^z(\alpha)$ and $s_m^\phi = s_m^\phi(\alpha)$ are given by

$$s_0^z \;=\; 4\pi^2 g_0(\alpha) \langle v, w_0^z\rangle_{H(\mathrm{curl},W)}\,, \tag{3.10}$$

$$s_m^z \;=\; 4\pi^2 \big[g_m(\alpha) \langle v, w_m^z\rangle_{H(\mathrm{curl},W)} + h_m(\alpha) \langle v, w_m^\phi\rangle_{H(\mathrm{curl},W)} \big]\,, \tag{3.11}$$

$$s_m^\phi \;=\; 4\pi^2 h_m(\alpha) \langle v, w_m^z\rangle_{H(\mathrm{curl},W)}\,. \tag{3.12}$$

Substituting the form of v into these equations yields a system of 5 equations for the 5 unknowns $s_0^z, s_{\pm1}^z, s_{\pm1}^\phi$. With

$$M_{m',m}^{\phi,z} := \langle Bw_{m'}^\phi, w_m^z\rangle_{H(\mathrm{curl},W)}\,, \qquad M_{m',m}^{z,\phi} := \langle Bw_{m'}^z, w_m^\phi\rangle_{H(\mathrm{curl},W)}\,,$$

$$M_{m',m}^{z,z} := \langle Bw_{m'}^z, w_m^z\rangle_{H(\mathrm{curl},W)}\,, \qquad M_{m',m}^{\phi,\phi} := \langle Bw_{m'}^\phi, w_m^\phi\rangle_{H(\mathrm{curl},W)}\,,$$

all depending on α through B, we obtain the $5 \times 5-$system

$$\begin{aligned}
\frac{1}{4\pi^2 g_0}\, s_0^z \;&=\; \langle By_\alpha, w_0^z\rangle_{H(\mathrm{curl},W)} - s_0^z\, M_{0,0}^{z,z} - \sum_{|m'|=1} \big[s_{m'}^z M_{m',0}^{z,z} + s_{m'}^\phi M_{m',0}^{\phi,z} \big]\,,\\
\frac{1}{4\pi^2 g_m}\, s_m^z \;&=\; \langle By_\alpha, w_m^z\rangle_{H(\mathrm{curl},W)} - s_0^z\, M_{0,m}^{z,z} - \sum_{|m'|=1} \big[s_{m'}^z M_{m',m}^{z,z} + s_{m'}^\phi M_{m',m}^{\phi,z} \big]\\
&\quad + \frac{h_m}{g_m} \Big\{ \langle By_\alpha, w_m^\phi\rangle_{H(\mathrm{curl},W)} - s_0^z\, M_{0,m}^{z,\phi} - \sum_{|m'|=1} \big[s_{m'}^z M_{m',m}^{z,\phi} + s_{m'}^\phi M_{m',m}^{\phi,\phi} \big] \Big\},\\
\frac{1}{4\pi^2}\, s_m^\phi \;&=\; h_m \Big\{ \langle By_\alpha, w_m^z\rangle_{H(\mathrm{curl},W)} - s_0^z\, M_{0,m}^{z,z} - \sum_{|m'|=1} \big[s_{m'}^z M_{m',m}^{z,z} + s_{m'}^\phi M_{m',m}^{\phi,z} \big] \Big\}
\end{aligned}$$

for $|m| = 1$. Now we use that $|g_m(\alpha)| \to \infty$ for $|m| \le 1$ and $h_m(\alpha) \to h_m(\hat{\alpha}) \neq 0$ for $|m| = 1$ as $\alpha \to \hat{\alpha}$ and the continuous dependence of B and $M_{m',m}^{z,z}$, etc, on α. Therefore, as $\alpha \to \hat{\alpha}$ the system converges to the $5 \times 5-$system (where now $\alpha = \hat{\alpha}$)

$$\begin{aligned}
\langle By_{\hat{\alpha}}, w_m^z\rangle_{H(\mathrm{curl},W)} \;&=\; s_0^z\, M_{0,m}^{z,z} + \sum_{|m'|=1} \big[s_{m'}^z M_{m',m}^{z,z} + s_{m'}^\phi M_{m',m}^{\phi,z} \big]\,, \quad |m| \le 1\,,\\
\langle By_{\hat{\alpha}}, w_m^z\rangle_{H(\mathrm{curl},W)} \;&=\; \frac{1}{4\pi^2 h_m}\, s_m^\phi + s_0^z\, M_{0,m}^{z,z} + \sum_{|m'|=1} \big[s_{m'}^z M_{m',m}^{z,z} + s_{m'}^\phi M_{m',m}^{\phi,z} \big]
\end{aligned} \tag{3.13}$$

for $|m| = 1$. From this we immediately observe that $s_m^\phi = s_m^\phi(\hat{\alpha}) = 0$ for $|m| = 1$, i.e. we arrive at the $3 \times 3-$system

$$\sum_{|m'|\le 1} s_{m'}^z(\hat{\alpha}) \langle \tilde{A}_{\alpha,t}^{-1} w_{m'}^z, w_m^z\rangle_{H(\mathrm{curl},W)} \;=\; \langle \tilde{A}_{\alpha,t}^{-1} y_{\hat{\alpha}}, w_m^z\rangle_{H(\mathrm{curl},W)}\,, \quad |m| \le 1\,, \tag{3.14}$$

for three unknowns $s_0^z(\hat{\alpha}), s_{\pm1}^z(\hat{\alpha}) \in \mathbb{C}$. Since the $3 \times 3-$matrix with entries $M_{m',m}^{z,z}(\hat{\alpha})$ for $m', m \in \{-1, 0, 1\}$ is invertible by Assumption 3.5 also the extended $5 \times 5-$matrix, characterized by the right hand side of (3.13), is invertible as easily seen. By the continuous dependence we conclude that $s_m^z(\alpha) \to s_m^z(\hat{\alpha})$ for $|m| \le 1$ and

$s^\phi_m(\alpha) \to s^\phi_m(\hat\alpha) = 0$ for $|m| = 1$ as $\alpha \to \hat\alpha$. The representation (3.9) yields convergence

$$v_{\alpha,t} \;\longrightarrow\; v_{\hat\alpha,t} \;=\; v \;=\; \tilde A^{-1}_{\hat\alpha,t}\, y_{\hat\alpha} \;-\; \sum_{|m|\le 1} s^z_m(\hat\alpha)\, \tilde A^{-1}_{\hat\alpha,t} w^z_m \quad \text{as } \alpha \to \hat\alpha\,,$$

where $s^z_m(\hat\alpha)$, $|m| \le 1$, solves the system (3.14). We observe that $v_{\hat\alpha,t}$ solves (3.7) and, furthermore, (3.10), (3.11) yields the estimate (3.8).

(b) Now we consider the extension of $u_{\alpha,t}$ into W_∞ given by (2.5)–(2.8) for $h^\phi_{n,m} = -v^z_{n,m}$ and $h^z_{n,m} = v^\phi_{n,m}$; that is,

$$u(r,\phi,x_3) \;=\; \sum_{n,m\in\mathbb{Z}} [u^r_{n,m}(r)\,\hat r + u^\phi_{n,m}(r)\,\hat\phi + u^z_{n,m}(r)\,\hat z]\, e^{im\phi + i(n+\hat\alpha)x_3}\,, \quad r > \hat R\,, \tag{3.15}$$

with

$$u^r_{n,m}(r) \;=\; -v^z_{n,m}\,\frac{i(n+\alpha)}{r}\, G^{(1)}_{n,m}(r,\alpha) - v^\phi_{n,m}\,\frac{im\hat R}{r}\, G^{(2)}_{n,m}(r,\alpha)\,, \tag{3.16}$$

$$u^\phi_{n,m}(r) \;=\; -v^z_{n,m}\,\frac{m(n+\alpha)}{r}\, G^{(5)}_{n,m}(r,\alpha) + v^\phi_{n,m}\, G^{(4)}_{n,m}(r,\alpha)\,, \tag{3.17}$$

$$u^z_{n,m}(r) \;=\; v^z_{n,m}\, G^{(3)}_{n,m}(r,\alpha)\,. \tag{3.18}$$

We split this series again and write $u_{\alpha,t}$ as

$$\begin{aligned} u_{\alpha,t}(r,\phi,x_3) \;&=\; \tilde u_{\alpha,t}(r,\phi,x_3) \\ &- \sum_{|m|\le 1} v^z_{\hat n,m}(\alpha)\,\frac{\hat n+\alpha}{r}\big\{i\, G^{(1)}_{\hat n,m}(r,\alpha)\,\hat r + m\, G^{(5)}_{\hat n,m}(r,\alpha)\,\hat\phi\big\} e^{im\phi + (\hat n+\alpha)x_3} \end{aligned}$$

for $r > \hat R$ and α in a neighborhood of κ. Here we indicated the dependence on α. The function $\tilde u_{\alpha,t}(r,\phi,x_3)$ depends continuously on α at $\hat\alpha$. Indeed, from Lemma A1 of [4][4] we note that all components in the series for $\tilde u_{\alpha,t}(r,\phi,x_3)$ are continuous and the series is uniformly convergent.

At this point we need the behavior of $g_m(\alpha)$ and $G^{(j)}_{n,m}(r,\alpha)$ for $|m| \le 1$ and $j \in \{1,5\}$ as $\alpha \to \hat\alpha$. Formulas (A.4b), (A.4c), (A4.i), and (A4.j) of [4] yields

$$\begin{aligned} k_{\hat n}(\alpha)^2 \ln k_{\hat n}(\alpha)\, G^{(1)}_{\hat n,0}(r,\alpha) \;&\longrightarrow\; -1\,, \quad \frac{G^{(1)}_{\hat n,\pm1}(r,\alpha)}{\ln k_{\hat n}(\alpha)} \;\longrightarrow\; \hat R^2\Big(\frac{r}{\hat R} + \frac{\hat R}{r}\Big)\,, \\ k_{\hat n}(\alpha)^2 \ln k_{\hat n}(\alpha)\, G^{(5)}_{\hat n,0}(r,\alpha) \;&\longrightarrow\; \ln\frac{r}{\hat R}\,, \quad \frac{G^{(5)}_{\hat n,\pm1}(r,\alpha)}{\ln k_{\hat n}(\alpha)} \;\longrightarrow\; \hat R^2\Big(\frac{r}{\hat R} - \frac{\hat R}{r}\Big)\,, \end{aligned}$$

as $\alpha \to \hat\alpha$ uniformly with respect to $r \in [\hat R, R]$ for every $R > \hat R$. Therefore, $\frac{G^{(j)}_{\hat n,m}(r,\alpha)}{g_m(\alpha)} \to c^{(j)}_m(r)$ for some $c^{(j)}_m(r) \in \mathbb{R}$ uniformly with respect to r.

For $j \in \{1,5\}$ we now write $v^z_{\hat n,m}(\alpha) G^{(j)}_{\hat n,0}(r,\alpha) = [v^z_{\hat n,m}(\alpha) g_m(\alpha)]\frac{G^{(j)}_{\hat n,m}(r,\alpha)}{g_m(\alpha)}$ which converges by (3.8) uniformly with respect to r. □

3.2. **Existence in the case of Absorption.** We can now prove the existence of a solution in $H_*(\mathrm{curl},\mathbb{R}^3)$ of (1.2) in the case of $t > 0$. We recall the definition of the Floquet-Bloch transform. For smooth functions u with compact support, writing $x = (x_1,\tilde x)$ for the argument, the transformation is defined by the formula

$$(\mathcal{F}_{\mathrm{FB}} u)(x_1,\tilde x,\alpha) \;:=\; \sum_{\ell\in\mathbb{Z}} u(x_1 + 2\pi\ell,\tilde x)\, e^{-i\ell 2\pi\alpha}\,. \tag{3.19}$$

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[4] Note that $G^{(j)}_{n,m}(r,\alpha) = G^{(j)}_m(r,n+\alpha)$ with $G^{(j)}_m(r,\xi)$ of [4].

The transformation is an (even unitary) isomorphism from $L^2(\Omega_R)$ onto $L^2\big(W_R\times I\big)$ for every $R>0$ and also an isomorphism from $H(\mathrm{curl},\Omega_R)$ onto the space

$$L^2\big(I,H_\alpha(\mathrm{curl},W_R)\big) \;:=\; \left\{ w\in L^2\big(I,H(\mathrm{curl},W_R)\big) : \begin{array}{l} w(\alpha)\in H_\alpha(\mathrm{curl},W_R) \text{ for} \\ \text{almost all } \alpha\in I \end{array} \right\}$$

where $I=(-1/2,1/2)$. Recall the notions $\Omega_R=\{x\in\mathbb{R}^3 : x_1^2+x_2^2<R^2\}$ and $W_R=\{x\in\Omega_R : x_3\in(0,2\pi)\}$. The inverse $\mathcal{F}_{\mathrm{FB}}^{-1}$ is given by

$$(\mathcal{F}_{\mathrm{FB}}^{-1}u)(x) \;=\; \int_{-1/2}^{1/2} u(x,\alpha)\,d\alpha\,,\ x\in\Omega_R\,, \tag{3.20}$$

for $u\in L^2\big(I,H_\alpha(\mathrm{curl},W_R)\big)$.

Theorem 3.10. *Let $t>0$ and let Assumption 3.5 be satisfied. Then there exists a unique solution $u\in H_*(\mathrm{curl},\mathbb{R}^3)$ of* (1.2) *for $\sigma=t\hat\sigma$ satisfying the angular spectral radial radiation condition* (1.4).

Proof. To show existence we note that by Theorem 2.14 there exists a unique solution $u_{\alpha,t}\in H_{\alpha,*}(\mathrm{curl},W_\infty)$ of (2.1), (2.2). Since Assumption 3.7 is satisfied by Remark 3.8 the solution depends continuously on $\alpha\in[-1/2,1/2]$ by Lemma 3.9. Therefore, the inverse Floquet-Bloch transform

$$u_t(x) \;:=\; \int_{-1/2}^{1/2} u_{\alpha,t}(x)\,d\alpha\,,\quad x\in\mathbb{R}^3\,,$$

is a solution of (1.2) in $H_*(\mathrm{curl},\mathbb{R}^3)$. We have to show that it satisfies the radiation condition (1.4). Let $\xi=n+\alpha\in\mathbb{R}$ be arbitrary such that $n\in\mathbb{Z}$ and $\alpha\in(-1/2,1/2]$ is not a cut-off value. We compute the Fourier transform $\hat u_t(\tilde x,\cdot)$ of $u_t(\tilde x,\cdot)$ as

$$\begin{aligned}\hat u_t(\tilde x,\xi) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} u_t(\tilde x,x_3)e^{-i(n+\alpha)x_3}\,dx_3 \\ &= \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}}\int_0^{2\pi} u_t(\tilde x,x_3+2\pi\ell)e^{-i(n+\alpha)(x_3+2\pi\ell)}\,dx_3 \\ &= \frac{1}{2\pi}\int_0^{2\pi}\big(\mathcal{F}_{\mathrm{FB}}u_t(\tilde x,\cdot)\big)(x_3,\alpha)\,e^{-i(n+\alpha)x_3}\,dx_3 \\ &= u_n(\tilde x,\alpha)\end{aligned}$$

which are the Fourier coefficients of $\big(\mathcal{F}_{\mathrm{FB}}u_t)(\tilde x,\cdot)\big)(x_3,\alpha)=u_{\alpha,t}(\tilde x,x_3)$ with respect to $\{e^{i(n+\alpha)x_3}|n\in\mathbb{Z}\}$. Therefore,

$$\partial_r\hat u_t(\tilde x,\xi)-ik(\xi)\,\hat u_t(\tilde x,\xi) \;=\; \partial_r u_n(\tilde x,\alpha)-ik_n(\alpha)u_n(\tilde x,\alpha)\,, \tag{3.21}$$

i.e. the radiation condition (1.4) for u_t coincides with the radiation condition (2.2) for $u_{\alpha,t}$. □

3.3. **$\boldsymbol{\alpha}$ in a neighborhood of a critical value.** For arbitrary $u,\phi\in H(\mathrm{curl},W_\infty)$ we define

$$Q(u,v) \;:=\; i\int\limits_{W_\infty}\frac{\mu_0}{\mu}\,\big[(\mathrm{curl}\,u\times\overline{v})-(\mathrm{curl}\,\overline{v}\times u)\big]\cdot\hat z\,dx\,. \tag{3.22}$$

Note that $Q(u,u)=2\omega\mu_0\,\mathrm{Re}\int_{W_\infty}\big[u\times\big(\overline{\mathrm{curl}\,u/(i\omega\mu)}\big)\big]\cdot\hat z\,dx=4\omega\mu_0\int_{W_\infty}P\cdot\hat z\,dx$ where $P=\frac12\,\mathrm{Re}\big[u\times\big(\overline{\mathrm{curl}\,u/(i\omega\mu)}\big)\big]$ denotes the Poynting vector. Therefore, $Q(u,u)$ is directly related to the flux in the x_3-direction.
Later we will apply Q to functions in $\mathcal{M}^\alpha$ which are decaying fast. The form Q is

hermitian since μ is real. The sesquilinear form Q is closely related to the deriviative of $A_{\alpha,t}$ with respect to α as the following lemma shows.

Lemma 3.11. *Let $\hat{\alpha}$ be a critical value and $u, \psi \in \mathcal{M}^{\hat{\alpha}}$ two corresponding modes and set $\mathcal{N} := \ker(A_{\hat{\alpha},0})$ for abbreviation. Set $v(x) := u(x)e^{-i\hat{\alpha}x_3}|_W$ and $\varphi(x) := \psi(x)e^{-i\hat{\alpha}x_3}|_W$. Then $v, \varphi \in \mathcal{N}$ and*

$$\langle \partial_\alpha A_{\hat{\alpha},0}\, v, \varphi \rangle_{H(\mathrm{curl},W)} \;=\; Q(u, \psi)\,. \tag{3.23}$$

Furthermore,

$$\langle \partial_t A_{\hat{\alpha},0}\, v, \varphi \rangle_X \;=\; -i\,\omega\mu_0 \int\limits_W \hat{\sigma}\, v \cdot \overline{\varphi}\, dx\,. \tag{3.24}$$

We note that $\hat{\alpha}$ is not a cut-off value by definition of a critical value. By the proof of Lemma 3.4 the operator $A_{\alpha,t}$ depends smoothly on α and t.

Proof. By definition we have

$$\begin{aligned}
\langle A_{\alpha,t} v, \varphi \rangle_{H(\mathrm{curl},W)} &= b_{\alpha,t}\big(v\, e^{i\alpha x_3}, \varphi\, e^{i\alpha x_3}\big) \\
&= \int\limits_W \left[\frac{\mu_0}{\mu}\, \mathrm{curl}(v\, e^{i\alpha x_3}) \cdot \mathrm{curl}\,\overline{(\varphi\, e^{i\alpha x_3})} - k^2 \left(\frac{\varepsilon}{\varepsilon_0} + it\frac{\hat{\sigma}}{\omega\varepsilon_0}\right) v \cdot \overline{\varphi}\right] dx \\
&\quad + \; \big\langle \Lambda_\alpha(\hat{r} \times v\, e^{i\alpha x_3}), \varphi\, e^{i\alpha x_3} \big\rangle\,.
\end{aligned}$$

From this and $k^2 = \omega^2\mu_0\varepsilon_0$, the derivative with respect to t follows immediately. With respect to the derivative $\partial_\alpha A_{\hat{\alpha},0}$ we consider both terms separately. By definition we have

$$\big\langle \Lambda_\alpha(\hat{r} \times v\, e^{i\alpha x_3}), \varphi\, e^{i\alpha x_3} \big\rangle \;=\; \int\limits_\gamma (\hat{r} \times \mathrm{curl}\, w_\alpha) \cdot \overline{(\varphi\, e^{i\alpha x_3})}\, ds$$

where $w_\alpha \in \tilde{H}_{\alpha,*}(\mathrm{curl}, W^+)$ is the solution of (2.3) with $\hat{r} \times w_\alpha = g = \hat{r} \times v\, e^{i\alpha x_3}$ on γ. We note that $w_{\hat{\alpha}} = u$. We set $v_\alpha(x) = w_\alpha(x)e^{-i\alpha x_3}$ in W^+ and thus $\hat{r} \times v_\alpha = \hat{r} \times v$ on γ. We apply Green's formula in W^+ and note that ψ decays fast as $r \to \infty$. Therefore,

$$\big\langle \Lambda_\alpha(\hat{r} \times v\, e^{i\alpha x_3}), \varphi\, e^{i\alpha x_3} \big\rangle \;=\; \int\limits_{W^+} \big[\mathrm{curl}(v_\alpha e^{i\alpha x_3}) \cdot \mathrm{curl}(\overline{\varphi\, e^{i\alpha x_3}}) - k^2 v_\alpha \cdot \overline{\varphi}\big]\, dx$$

and thus

$$\begin{aligned}
\partial_\alpha \big\langle \Lambda_\alpha(\hat{r} \times v\, e^{i\alpha x_3}), \varphi\, e^{i\alpha x_3} \big\rangle &= \int\limits_{W^+} \big[\mathrm{curl}(v_\alpha' e^{i\alpha x_3}) \cdot \mathrm{curl}(\overline{\varphi\, e^{i\alpha x_3}}) - k^2 v_\alpha' \cdot \overline{\varphi}\big]\, dx \\
&+ i \int\limits_{W^+} \big[\mathrm{curl}(v_\alpha x_3 e^{i\alpha x_3}) \cdot \mathrm{curl}(\overline{\varphi\, e^{i\alpha x_3}}) - \mathrm{curl}(v_\alpha e^{i\alpha x_3}) \cdot \mathrm{curl}(\overline{\varphi\, x_3 e^{i\alpha x_3}})\big] dx \\
= \int\limits_{W^+} &\big[\mathrm{curl}(v_\alpha' e^{i\alpha x_3}) \cdot \mathrm{curl}(\overline{\varphi\, e^{i\alpha x_3}}) - k^2 v_\alpha' \cdot \overline{\varphi}\big]\, dx \\
&+ i \int\limits_{W^+} \big[v_\alpha e^{i\alpha x_3} \times \mathrm{curl}(\overline{\varphi\, e^{i\alpha x_3}}) - \overline{\varphi\, e^{i\alpha x_3}} \times \mathrm{curl}(v_\alpha e^{i\alpha x_3})\big] \cdot \hat{z}\, dx
\end{aligned}$$

where $v'_\alpha = \partial_\alpha v_\alpha$. For $\alpha = \hat\alpha$ we obtain

$$\partial_\alpha \big\langle \Lambda_{\hat\alpha}(\hat r \times v\, e^{i\hat\alpha x_3}), \varphi\, e^{i\hat\alpha x_3}\big\rangle \;=\; \int\limits_{W^+} \big[\operatorname{curl}(v'_{\hat\alpha} e^{i\hat\alpha x_3}) \cdot \operatorname{curl}(\overline{\varphi\, e^{i\hat\alpha x_3}}) - k^2 v'_{\hat\alpha} \cdot \overline{\varphi}\big]\, dx$$

$$+\, i \int\limits_{W^+} \big[(\operatorname{curl} u \times \overline{\psi}) - (\operatorname{curl} \overline{\psi} \times u)\big] \cdot \hat z\, dx\,.$$

Now we show that the first integral vanishes. Since $\hat r \times v_\alpha|_\gamma = \hat r \times v|_\gamma$ is constant with respect to α we conclude that $\hat r \times v'_{\hat\alpha} = 0$ on γ. Now we observe that $\varphi\, e^{i\hat\alpha x_3} = \psi$ solves the differential equation (2.3) in W^+. Taking $v'_{\hat\alpha} e^{i\hat\alpha x_3}$ as a test function in its variational formulation yields that the first integral vanishes.
This provides a formula for $\partial_\alpha \big\langle \Lambda_{\hat\alpha}(\hat r \times v\, e^{i\hat\alpha x_3}), \varphi\, e^{i\hat\alpha x_3}\big\rangle$. Now we differentiate the volume integral in the form of $\langle A_{\alpha,t} v, \varphi\rangle_{H(\mathrm{curl},W)}$ with respect to α and obtain in the same way that its derivative is given by $i \int_W \big[(\operatorname{curl} u \times \overline{\psi}) - (\operatorname{curl} \overline{\psi} \times u)\big] \cdot \hat z\, dx$. Adding the results yields the assertion. $\square$

Assumption 3.12. *With the set $\mathcal{A}$ of critical values and the corresponding space $\mathcal{M}^\alpha$ of modes for $\alpha \in \mathcal{A}$ (see Definition 3.1) we assume that, for every $\alpha \in \mathcal{A}$ and $0 \neq v \in \mathcal{M}^\alpha$, the linear form $Q(\cdot, v)$ does not vanish identically on $\mathcal{M}^\alpha$.*

In particular, under this assumption the derivative $P\partial_\alpha A_{\hat\alpha,0}|_{\mathcal{N}}$ of the projection of $A_{\alpha,0}$ at $\hat\alpha$ onto $\mathcal{N}$ is one-to-one and thus invertible for every $\hat\alpha \in \mathcal{A}$.
Under Assumption 3.12 the family $(A_{\alpha,t})_{\alpha,t}$ has the following properties. Let $\hat\alpha \in \mathcal{A}$ be a fixed critical value.

(a) $A_{\alpha,t}$ are Fredholm operators with index zero and Riesz number one for (α, t) is a neighborhood of $(\hat\alpha, 0)$ (see Lemma 2.12). With $\mathcal{N} = \mathcal{N}(A_{\hat\alpha,0})$ and $\mathcal{R} = \mathcal{R}(A_{\hat\alpha,0})$ we have $H_{per}(\mathrm{curl}, W) = \mathcal{N} \oplus \mathcal{R}$.
(b) $(\alpha, t) \mapsto A_{\alpha,t}$ is continuously differentiable in some neighborhood of $(\hat\alpha, 0)$ (see Lemma 3.11).
(c) Let $P : X \to \mathcal{N}$ be the projection corresponding to $X = \mathcal{N} \oplus \mathcal{R}$. Then $M_\alpha := P\partial_\alpha A_{\hat\alpha,0}|_{\mathcal{N}}$ is selfadjoint and an isomorphism from $\mathcal{N}$ onto itself, and $M_t := iP\partial_t A_{\hat\alpha,0}|_{\mathcal{N}}$ is selfadjoint and positive definite on $\mathcal{N}$ (see Lemma 3.11).

The following functional analytic theorem has been shown in [7].

Theorem 3.13. *Let X be a Hilbert space and $A_{\alpha,t}$ be a two-parameter family of operators satisfying the conditions (a)–(c) from above for some $\hat\alpha \in [-1/2, 1/2]$. Let $\alpha \mapsto y_\alpha \in X$ be a family of right hand sides that depends Lipschitz continuously on $\alpha \in [-1/2, 1/2]$. Then there exists $\varepsilon \in (0, t_0)$ such that $A_{\alpha,t}$ is invertible for $(\alpha, t) \in (\hat\alpha - \varepsilon, \hat\alpha + \varepsilon) \times [0, \varepsilon)$, $(\alpha, t) \neq (\hat\alpha, 0)$, and the solution $v_{\alpha,t} = A_{\alpha,t}^{-1} y_\alpha$ has the form*

$$v_{\alpha,t} \;=\; \tilde v_{\alpha,t} \;+\; \sum_{\ell=1}^{m} \frac{\langle P y_{\hat\alpha}, \phi_\ell\rangle_X}{\lambda_\ell(\alpha - \hat\alpha) - it}\, \phi_\ell \tag{3.25}$$

for $(\alpha, t) \in (\hat\alpha - \varepsilon, \hat\alpha + \varepsilon) \times [0, \varepsilon)$, $(\alpha, t) \neq (\hat\alpha, 0)$. In this representation, $\|\tilde v_{\alpha,t}\|_X$ is uniformly bounded with respect to (α, t). The family $\big\{\phi_\ell : \ell = 1, \dots, m\big\}$, $m = \dim \mathcal{N}$, is an orthonormal eigensystem with eigenvalues $\big\{\lambda_\ell : \ell = 1, \dots, m\big\}$ of the following generalized eigenvalue problem in the finite dimensional space $\mathcal{N}$:

$$M_\alpha \phi_\ell \;=\; \lambda_\ell\, M_t \phi_\ell \quad \text{in } \mathcal{N} \quad \text{with normalization} \quad \langle M_t \phi_\ell, \phi_{\ell'}\rangle_X \;=\; \delta_{\ell,\ell'} \tag{3.26}$$

for $\ell, \ell' = 1, \dots, m$.

Lemma 3.14. *Let Assumptions 3.5, 3.7, and 3.12 be satisfied. Then there exists at most a finite number of critical values $\alpha_j \in [-1/2, 1/2]$ for $0 < j \leq J$. These are characterized by the fact that the kernel $\mathcal{N}(A_{\alpha_j,0}) = \{\varphi \in H_{per}(\mathrm{curl}, W) : A_{\alpha_j,0}\varphi = 0\}$ is not trivial. The kernels are finite dimensional.*

Proof. Let $\hat{\alpha} \in [-1/2, 1/2]$ be arbitrary and fixed. Three cases can occur:
Case 1: $\hat{\alpha}$ is neither a cut-off value nor a critical value. Then $A_{\alpha,0}$ depends continuously on α and $A_{\hat{\alpha},0}$ is invertible. Therefore, also $A_{\alpha,0}$ is invertible in a neighborhood U of $\hat{\alpha}$, i.e. no critical value exists in this neighborhood U.
Case 2: $\hat{\alpha}$ is a critical value. By the previous theorem there exists a neighborhood U of $\hat{\alpha}$ such that $\hat{\alpha}$ is the only critical value in U.
Case 3: $\hat{\alpha}$ is a cut-off value. Then we know from Lemma 3.9 that $A_{\alpha,0}$ is invertible in a neighborhood U of $\hat{\alpha}$, i.e, there is no critical value in U.
By the compactness of $[-1/2, 1/2]$ finitely many open sets U of the cases 1, 2 or 3 suffice to cover $[-1/2, 1/2]$. $\square$

We denote, for every $0 < j \leq J$, the space of α_j-quasiperiodic propagating modes

$$\mathcal{M}_j := \mathcal{M}^{\alpha_j} = \left\{ \phi \in H_{\alpha_j,*}(\mathrm{curl}, W_\infty) : \begin{array}{c} \phi \text{ solves (3.1) for } \alpha = \alpha_j \text{ and} \\ \text{the radiation condition (2.1)} \end{array} \right\}. \tag{3.27}$$

There holds $\mathcal{N}(A_{\alpha_j}) = \left\{ \phi\, e^{-i\alpha_j x_3}|_W : \phi \in \mathcal{M}_j \right\}$, we denote the dimension by $m_j := \dim \mathcal{M}_j = \dim \mathcal{N}(A_{\alpha_j})$.

In every space $\mathcal{M}_j$ we choose a basis $\{\phi_{1,j}, \ldots, \phi_{m_j,j}\} \subset \mathcal{M}_j$ by considering the self-adjoint eigenvalue problem (3.26) which has the form to determine $\lambda \in \mathbb{R}$ and nontrivial $\phi \in \mathcal{M}_j$ such that

$$Q(\phi, \psi) \;=\; \lambda\, \omega\mu_0 \int_W \hat{\sigma}\, \phi \cdot \bar{\psi}\, dx \text{ for all } \psi \in \mathcal{M}_j\,.$$

We denote the eigenvalues by $\lambda_{\ell,j}$, $\ell = 1, \ldots, m_j$, and the eigenfunctions by $\phi_{\ell,j}$, $\ell = 1, \ldots, m_j$, normalized such that

$$\omega\mu_0 \int_W \hat{\sigma}\, \phi_{\ell,j} \cdot \overline{\phi_{\ell',j}}\, dx \;=\; \delta_{\ell,\ell'}\,, \quad \ell, \ell' = 1, \ldots, m_j\,.$$

4. The Main Convergence Result

In this section we will prove convergence of the solution u_t corresponding to $\sigma = t\hat{\sigma}$ as t tends to zero. The limit u will satisfy the following radiation condition.

Definition 4.1. *Let Assumptions 3.5, 3.7, and 3.12 be satisfied. A solution $u \in H_{loc}(\mathrm{curl}, \mathbb{R}^3)$ of (1.2) satisfies the open waveguide radiation condition if, for given $R > 2\pi$, the field u has a decomposition into $u = u_{rad} + u_{prop}$ where $u_{rad} \in H_*(\mathrm{curl}, \mathbb{R}^3)$ satisfies the angular spectral radiation condition (1.4), and u_{prop} is of the form*

$$u_{prop}(x) \;=\; \sum_{j=1}^{J} \sum_{\ell \in \mathcal{L}_j^\pm} a_{\ell,j}\, \phi_{\ell,j}(x) \quad \text{for } \pm x_3 > R \text{ and some } a_{\ell,j} \in \mathbb{C}\,. \tag{4.1}$$

Here, $\{\phi_{\ell,j} \in \mathcal{M}_j : \ell = 1, \ldots, m_j\}$ are the eigenfunctions of the self-adjoint eigenvalue problem

$$Q(\phi_{\ell,j}, \psi) \;=\; \lambda_{\ell,j}\, \omega\mu_0 \int\limits_{W_{h_0}} \hat{\sigma}\, \phi_{\ell,j} \cdot \overline{\psi}\, dx \quad \text{for all } \psi \in \mathcal{M}_j, \tag{4.2}$$

normalized as $\omega\mu_0 \int_{W_{h_0}} \hat\sigma\, \phi_{\ell,j}\cdot\overline{\phi_{\ell',j}}\, dx = \delta_{\ell,\ell'}$ *and*

$$\mathcal{L}_j^\pm \ := \ \big\{\ell\in\{1,\dots,m_j\} : Q(\phi_{\ell,j},\phi_{\ell,j}) \gtrless 0\big\} \quad \text{for } j=1,\dots,J\,. \tag{4.3}$$

The mode spaces $\mathcal{M}_j$ and the sesqui-linear form Q are defined in (3.27) *and* (3.22)*, respectively.*

Before we show convergence we apply Theorem 3.13 to the equation (3.2), i.e. $A_{\alpha,t}v_{\alpha,t} = y_\alpha$ in $X = H_{per}(\mathrm{curl}, W)$. Recalling that $\phi_{\ell,j}(x)e^{-i\alpha_j x_3}$ span the nullspaces $\mathcal{N}(A_{\alpha_j,0})$ we have the existence of $\delta>0$ such that for every $j=1,\dots,J$ and $0<|\alpha-\alpha_j|<\delta$ the solution $u_{\alpha,t}(x) = v_{\alpha,t}(x)e^{i\alpha x_3}$ is represented in the form

$$u_{\alpha,t}(x) \ = \ \tilde u_{\alpha,t}(x) \ + \ \sum_{\ell=1}^{m_j} \frac{\langle P_j y_{\alpha_j}, \phi_{\ell,j}\, e^{-i\alpha_j x_3}\rangle_{H(\mathrm{curl},W)}}{\lambda_{\ell,j}(\alpha-\alpha_j) - it}\, \phi_{\ell,j}(x)\, e^{i(\alpha-\alpha_j)x_3} \tag{4.4}$$

for $x\in W$ where $P_j : X\to \mathcal{N}(A_{\alpha_j,0})$ is the projection operator. By Corollary 2.13 it is the orthogonal projection operator. Therefore, we can replace $\langle P_j y_{\alpha_j}, \phi_{\ell,j}\, e^{-i\alpha_j x_3}\rangle_{H(\mathrm{curl},W)}$ by $\langle y_{\alpha_j}, \phi_{\ell,j}\, e^{-i\alpha_j x_3}\rangle_{H(\mathrm{curl},W)}$. We extend $\tilde u_{\alpha,t}$ by $u_{\alpha,t}$ for $\alpha\notin\bigcup_{j=1}^J(\alpha_j-\delta,\alpha_j+\delta)$. Then $\|\tilde u_{\alpha,t}\|_{H(\mathrm{curl},W)}$ is uniformly bounded with respect to $(\alpha,t)\in\big([-1/2,1/2]\setminus\mathcal{A}\big)\times[0,\delta]$. Furthermore, $\tilde u_{\alpha,t}$ converges to some $\tilde u_{\alpha,0}$ as $t\to 0$ in $H(\mathrm{curl},W)$ for every $\alpha\notin\mathcal{A}$.
Now we apply the inverse Floquet-Bloch transform (3.20). Splitting the integral yields

$$u_t(x) = \int_{-1/2}^{1/2} \tilde u_{\alpha,t}(x)\, d\alpha + \sum_{j=1}^J\sum_{\ell=1}^{m_j} \langle y_{\alpha_j}, \phi_{\ell,j}\, e^{-i\alpha_j x_3}\rangle_X \int_{\alpha_j-\varepsilon}^{\alpha_j+\varepsilon} \frac{e^{i(\alpha-\alpha_j)x_3}}{\lambda_{\ell,j}(\alpha-\alpha_j)-it}\, d\alpha\, \phi_{\ell,j}(x). \tag{4.5}$$

By the theorem of dominated convergence we conclude that the first term converges to $\tilde u := \int_{-1/2}^{1/2}\tilde u_{\alpha,0}\, d\alpha$ in $H(\mathrm{curl},\Omega_{\hat R})$ as $t\to 0$. For the integral in the second term we have (see [8], Appendix B)

$$\lim_{t\to 0+}\int_{\alpha_j-\varepsilon}^{\alpha_j+\varepsilon} \frac{e^{i(\alpha-\alpha_j)x_3}}{\lambda_{\ell,j}(\alpha-\alpha_j)-it}\, d\alpha \ = \ \frac{2\pi i}{|\lambda_{\ell,j}|}\left[\frac12 + \mathrm{sign}(\lambda_{\ell,j})\frac1\pi\int_0^{\varepsilon x_3}\frac{\sin\tau}{\tau}\, d\tau\right] \tag{4.6}$$

uniformly with respect to $|x_3|\le R$ for every $R>0$. Now we set

$$\psi^\pm(x_3) \ := \ \frac12 \ \pm \ \frac1\pi\int_0^{\varepsilon x_3}\frac{\sin t}{t}\, dt\,, \quad x_3\in\mathbb{R}\,. \tag{4.7}$$

Then we can take the limit $t\to 0$ in (4.5) and arrive at

$$u_0(x) \ = \ \tilde u(x) \ + \ \sum_{j=1}^J\left[\psi^+(x_3)\sum_{\ell\in\mathcal{L}_j^+} a_{\ell,j}\phi_{\ell,j}(x) \ + \ \psi^-(x_3)\sum_{\ell\in\mathcal{L}_j^-} a_{\ell,j}\phi_{\ell,j}(x)\right], \tag{4.8}$$

where $a_{\ell,j} = \frac{2\pi i}{|\lambda_{\ell,j}|}\langle y_{\alpha_j}, \phi_{\ell,j}\, e^{-i\alpha_j x_3}\rangle_{H(\mathrm{curl},W)}$. The limit is taken in $H(\mathrm{curl}, W^{h_0,R})$ for every $R>0$ where $W^{h_0,R} = (-R,R)\times(0,\pi)\times(0,h_0)$.
Therefore, we have almost shown the following main result.

Theorem 4.2. *Let Assumptions 3.5, 3.7, and 3.12 be satisfied. Let $\sigma = t\hat\sigma$ for $t>0$ and some fixed periodic $\hat\sigma\in L^\infty(\mathbb{R}^3)$ which vanishes for $x_3>h_0$ and is bounded below by some positice constant for $x_3<h_0$. Let $u_t\in H_*(\mathrm{curl},\mathbb{R}^3)$ be the corresponding solution of* (1.2), (1.4) *which is guaranteed by Theorem 3.10. Then u_t converges to a solution $u_0\in H_{loc}(\mathrm{curl},\mathbb{R}^3)$ of* (1.2) *for $\sigma=0$ locally on every*

bounded domain as $t \to 0$. Furthermore, u_0 satisfies the open waveguide radiation condition of Definition 4.1 with

$$a_{\ell,j} \;=\; \frac{2\pi i\mu_0}{|Q(\phi_{\ell,j},\phi_{\ell,j})|}\int\limits_W \left[\frac{1}{\mu}\, f_h\cdot \operatorname{curl}\overline{\phi_{\ell,j}} + i\omega f_e\cdot\overline{\phi_{\ell,j}}\right]dx\,,\quad \ell=1,\dots,m_j\,,$$

for $j=1,\dots,J$.

Proof. We consider (4.8). First we show the form of $a_{\ell,j}$. By the definition (3.6) of y_{α_j} we have

$$\langle y_{\alpha_j}\,,\,\phi_{\ell,j}\,e^{-i\alpha_j x_3}\rangle_{H(\mathrm{curl},W)} \;=\; \mu_0\int\limits_W \left[\frac{1}{\mu}\, f_h\cdot \operatorname{curl}\overline{\phi_{\ell,j}} + i\omega f_e\cdot\overline{\phi_{\ell,j}}\right]dx\,.$$

Furthermore, $Q(\phi_{\ell,j},\phi_{\ell,j})=\lambda_{\ell,j}$ by the definition of the eigenvalue problem (4.2).

Next we show that $\tilde u=\int_{-1/2}^{1/2}\tilde u_{\alpha,0}\,d\alpha$ satisfies the radiation condition (1.4). By (3.21) we have to show the radiation condition (2.2) for the Fourier coefficients $\tilde u_n(\tilde x,\alpha)$ of $\big(\mathcal{F}_{\mathrm{FB}}\tilde u(\tilde x,\cdot)\big)(x_3,\alpha)=\tilde u_{\alpha,0}(x)$ with respect to $\{e^{i(n+\alpha)x_3}:n\in\mathbb{Z}\}$ for every $n\in\mathbb{Z}$ and α. We fix $n\in\mathbb{Z}$ and $\alpha\in[-1/2,1/2]$ which is neither a cut-off value nor a critical value.

We recall from (4.4) that $\tilde u_{\alpha,0}$ is given by $\tilde u_{\alpha,0}=u_{\alpha,0}$ if $\alpha\notin\bigcup_{j=1}^J(\alpha_j-\delta,\alpha_j+\delta)$ and

$$\tilde u_{\alpha,0}(x)=u_{\alpha,0}(x)-\frac{1}{\alpha-\alpha_j}\sum_{\ell=1}^{m_j}\frac{\langle y_{\alpha_j},\phi_{\ell,j}\,e^{-i\alpha_j x_3}\rangle_{H(\mathrm{curl},W)}}{\lambda_{\ell,j}}\,\phi_{\ell,j}(x)\,e^{i(\alpha-\alpha_j)x_3} \tag{4.9}$$

for $0<|\alpha-\alpha_j|<\delta$.
Let $u_n(\tilde x,\alpha)=\frac{1}{2\pi}\int_0^{2\pi}u_{\alpha,0}(x)\,e^{-i(n+\alpha)x_3}dx_3$ be the Fourier coefficients of $u_{\alpha,0}\in H_\alpha(\mathrm{curl},W)$. By Theorem 2.6 the extension of $u_{\alpha,0}$ to $\mathbb{R}^3$ satisfies the radiation condition (2.2). Since the modes $\phi_{\ell,j}$ decay exponentially as $|\tilde x|\to\infty$ we note that also the Fourier coefficients $\tilde u_n(\tilde x,\alpha)$ of $\tilde u_{\alpha,0}(x)=\big(\mathcal{F}_{\mathrm{FB}}\tilde u(\tilde x,\cdot)\big)(x_3,\alpha)$ satisfy (2.2).

Finally, we note that in (4.8) the propagating part approaches $\sum_{j=1}^J\sum_{\ell\in\mathcal{L}_j^\pm}a_{\ell,j}\,\phi_{\ell,j}$ only asymptotically. However, if we choose $\tilde\psi^\pm\in C^\infty(\mathbb{R})$ such that $\tilde\psi^\pm(x_3)=1$ for $\pm x_3\ge R$ and $\tilde\psi^\pm(x_3)=0$ for $\pm x_3\le -R$ then we rewrite (4.8) in the form $u_0=u_{rad}+u_{prop}$ with

$$\begin{aligned} u_{rad}(x) \;&=\; \tilde u(x)\;+\;\sum_{j=1}^J\Bigg[\big(\psi^+(x_3)-\tilde\psi^+(x_3)\big)\sum_{\ell\in\mathcal{L}_j^+}a_{\ell,j}\phi_{\ell,j}(x)\\ &\qquad\qquad+\;\big(\psi^-(x_3)-\tilde\psi^-(x_3)\big)\sum_{\ell\in\mathcal{L}_j^-}a_{\ell,j}\phi_{\ell,j}(x)\Bigg]\,,\\ u_{prop}(x)\;&=\;\sum_{j=1}^J\Bigg[\tilde\psi^+(x_3)\sum_{\ell\in\mathcal{L}_j^+}a_{\ell,j}\phi_{\ell,j}(x)\;+\;\tilde\psi^-(x_3)\sum_{\ell\in\mathcal{L}_j^-}a_{\ell,j}\phi_{\ell,j}(x)\Bigg]\,. \end{aligned}$$

Since $\psi^\pm(x_3)=1+\mathcal{O}(1/|x_3|)$ as $\pm x_3\to+\infty$ and $\psi^\pm(x_3)=\mathcal{O}(1/|x_3|)$ as $\pm x_3\to-\infty$ and $\frac{d\psi^+(x_3)}{dx_3}=\mathcal{O}(1/|x_3|)$ as $|x_3|\to\infty$ we conclude that $u_{rad}\in H_*(\mathrm{curl},\Omega)$. Furthermore, also the second part of u_{rad} satisfies the radiation condition (1.4) because of the exponential decay of $\phi_{\ell,j}(x)$ as $x_1^2+x_2^2\to\infty$. $\square$

5. Interpretation of Assumption 3.7

Again we look at the cutoff value $\hat\alpha$ with $\hat n \in \mathbb{Z}$ such that $|\hat n + \hat\alpha| = k$. Let again $\mathcal{C} = \{(\hat n, m) : m \in \mathbb{Z},\ |m| \le 1\}$ and define $\varphi_{n,m}$ by

$$\varphi_{n,m}(\phi, x_3) \;:=\; \frac{1}{4\pi^2 \hat R}\, e^{im\phi + i(n+\hat\alpha)x_3}\,, \quad \phi, x_3 \in (0, 2\pi)\,,\ n, m \in \mathbb{Z}\,. \tag{5.1}$$

The investigation of Assumption 3.7 requires the study of the equations $\tilde A_{\hat\alpha,0} v = 0$ (for injectivity of $\tilde A_{\hat\alpha,0}$) and $\tilde A_{\hat\alpha,0} v = w^z_m$ for some $|m| \le 1$ where w^z_m had been defined in Section 3.1. We consider both equations simultaneously and study $\tilde A_{\hat\alpha,0} v = \rho\, w^z_m$ for $\rho \in \{0, 1\}$.

Consider a solution $v \in H_{\mathrm{per}}(\mathrm{curl}, W)$ of $\tilde A_{\hat\alpha,0} v = \rho\, w^z_{\hat m}$ for some fixed $|\hat m| \le 1$. Define $u(x) = v(x) e^{i\hat\alpha x_3}$ in W. Then $u \in H_{\hat\alpha}(\mathrm{curl}, W)$ solves

$$\int_W \left[\frac{\mu_0}{\mu}\,\mathrm{curl}\, u \cdot \mathrm{curl}\, \overline{\psi} - k^2 q_0\, u \cdot \overline{\psi}\right] dx + \big\langle \tilde\Lambda_{\hat\alpha}(\hat r \times u)\,, \psi\big\rangle = \rho\, \langle \varphi_{\hat n,\hat m} \hat z, \psi\rangle \tag{5.2}$$

for all $\psi \in H_{\hat\alpha}(\mathrm{curl}, W)$ because $\psi^z_{\hat n,\hat m} = \langle \psi, \varphi_{\hat n,\hat m} \hat z\rangle$. This is the variational form of

$$\mathrm{curl}\left(\frac{\mu_0}{\mu}\,\mathrm{curl}\, u\right) - k^2 q_0\, u \;=\; 0 \quad \text{in } W\,, \tag{5.3}$$

$$-\frac{\mu_0}{\mu}\,\hat r \times \mathrm{curl}\, u|_- \;+\; \tilde\Lambda_{\hat\alpha}(\hat r \times u|_-) \;=\; \rho\, \varphi_{\hat n,\hat m} \hat z \quad \text{on } \gamma\,. \tag{5.4}$$

Let the tangential components $u|_- \cdot \hat\phi$ and $u|_- \cdot \hat z$ have the Fourier coefficients

$$u^\phi_{n,m}(\hat R) := \big\langle u(\hat R, \cdot)|_- \cdot \hat\phi, \varphi_{n,m}\big\rangle\,, \quad u^z_{n,m}(\hat R) := \big\langle u(\hat R, \cdot)|_- \cdot \hat z, \varphi_{n,m}\big\rangle\,, \quad n, m \in \mathbb{Z}\,.$$

In the exterior domain $W^+ = W_\infty \setminus \overline{W}$ we define u by

$$u(r, \phi, x_3) \;=\; \sum_{(n,m)\notin\mathcal{C}} \big[u^r_{n,m}(r)\, \hat r + u^\phi_{n,m}(r)\, \hat\phi + u^z_{n,m}(r)\, \hat z\big]\, e^{im\phi + i(n+\hat\alpha)x_3}\,, \quad r > \hat R\,, \tag{5.5}$$

where $u^r_{n,m}(r)$, $u^\phi_{n,m}(r)$, and $u^z_{n,m}(r)$ are given by (3.15)–(3.18) with $v^\phi_{n,m} = u^\phi_{n,m}(\hat R)$ and $v^z_{n,m} = u^z_{n,m}(\hat R)$ for $\alpha = \hat\alpha$ and $n \ne \hat n$ and

$$\begin{aligned} u^r_{\hat n,m}(r) \;&=\; -i\,\frac{\hat R(\hat n + \hat\alpha)}{2(1-|m|)} u^z_{\hat n,m}(\hat R) \left(\frac{\hat R}{r}\right)^{|m|-1} \\ &\quad + i\left[s_m u^\phi_{\hat n,m}(\hat R) - \frac{\hat R(\hat n + \hat\alpha)}{2(1-|m|)} u^z_{\hat n,m}(\hat R)\right] \left(\frac{\hat R}{r}\right)^{|m|+1}, \quad |m| \ge 2\,, \end{aligned} \tag{5.6}$$

$$\begin{aligned} u^\phi_{\hat n,m}(r) \;&=\; s_m\,\frac{\hat R(\hat n + \hat\alpha)}{2(1-|m|)} u^z_{\hat n,m}(\hat R) \left(\frac{\hat R}{r}\right)^{|m|-1} \\ &\quad + \left[u^\phi_{\hat n,m}(\hat R) - s_m\,\frac{\hat R(\hat n + \hat\alpha)}{2(1-|m|)} u^z_{\hat n,m}(\hat R)\right] \left(\frac{\hat R}{r}\right)^{|m|+1}, \quad |m| \ge 2\,, \end{aligned} \tag{5.7}$$

$$u^z_{\hat n,m}(r) \;=\; u^z_{\hat n,m}(\hat R) \left(\frac{\hat R}{r}\right)^{|m|}, \quad |m| \ge 2\,, \tag{5.8}$$

where $s_m = \mathrm{sign}\, m$. We note that $u^r_{n,m}(r)$, $u^\phi_{n,m}(r)$, and $u^z_{n,m}(r)$, given by (3.15)–(3.18), exist for $\alpha = \hat\alpha$ and $n \ne \hat n$ because $\lim_{\alpha\to\hat\alpha} G^{(j)}_{n,m}(r)$ exist for $j \in \{1, \dots, 5\}$

and $(n,m)\notin\mathcal{C}$. This has been shown in Lemma A.1 of [4].
Furthermore, in Theorem 4.1 of [4] it has been shown that this $u|_{W^+}\in H_{\hat\alpha,*}(\mathrm{curl},W^+)$ solves $\mathrm{curl}^2 u-k^2u=0$ in W^+ and the Sommerfeld radiation condition (2.2) for $n\neq\hat n$ and $u_{\hat n,m}(x)=\mathcal{O}(1/r)$ for $n=\hat n$ and $|m|\geq 2$, and $\hat r\times\mathrm{curl}\,u|_+=\tilde\Lambda_{\hat\alpha}(\hat r\times u)$. Therefore, (5.4) turns into

$$\hat r\times\mathrm{curl}\,u|_+ \;=\; \frac{\mu_0}{\mu}\,\hat r\times\mathrm{curl}\,u|_- \;+\; \rho\,\varphi_{\hat n,\hat m}\hat z \quad \text{on } \gamma\,. \tag{5.9}$$

The continuity with respect to the tangential components of u holds only for the coefficients For $(\hat n,m)\in\mathcal{C}$ we have the boundary conditions $u^\phi_{\hat n,m}(\hat R)|_+=u^z_{\hat n,m}(\hat R)|_+=0$. We formulate also (5.9) in terms of the expansions coefficients of the curl. With $v:=\mathrm{curl}\,u$ (5.9) turns into

We have shown the following result.

Theorem 5.1. *Let $\hat\alpha$ be a cut-off value and $v\in H_{\mathrm{per}}(\mathrm{curl},W)$ be a solution of $\tilde A_{\hat\alpha,0}v=\rho\,w^z_m$ for $\rho\in\{0,1\}$. Define $u\in H_{\hat\alpha}(\mathrm{curl},W)$ by $u(x):=v(x)e^{i\hat\alpha x_3}$ and in W^+ by (5.5) with expansion coefficients $u^r_{n,m}(r)$, $u^\phi_{n,m}(r)$, and $u^z_{n,m}(r)$. Analogously, let $v^\phi_{n,m}(r):=\langle\mathrm{curl}\,u(r,\cdot)\cdot\hat\phi,\varphi_{n,m}\rangle$ and $v^z_{n,m}(r):=\langle\mathrm{curl}\,u(r,\cdot)\cdot\hat z,\varphi_{n,m}\rangle$ be the expansion coefficients of the tangential components of $\mathrm{curl}\,u$ for $n,m\in\mathbb{Z}$.*
Then $u|_{W^+}\in H_{\hat\alpha,*}(\mathrm{curl},W^+)$, and u solves

$$\mathrm{curl}\Big(\frac{\mu_0}{\mu}\,\mathrm{curl}\,u\Big)-k^2q_0\,u \;=\; 0 \quad \text{in } W^\infty\setminus\gamma\,, \tag{5.10}$$

with transmission conditions

$$u^\phi_{n,m}(\hat R)|_+=u^\phi_{n,m}(\hat R)|_-\,,\quad u^z_{n,m}(\hat R)|_+=u^z_{n,m}(\hat R)|_- \quad \text{for } (n,m)\notin\mathcal{C}, \tag{5.11}$$

and

$$v^\phi_{n,m}(\hat R)|_+=v^\phi_{n,m}(\hat R)|_-\,,\quad v^z_{n,m}(\hat R)|_+=v^z_{n,m}(\hat R)|_- \quad \text{for } (n,m)\notin\mathcal{C}, \tag{5.12}$$

and boundary conditions

$$u^\phi_{\hat n,m}(\hat R)|_+=u^z_{\hat n,m}(\hat R)|_+=v^\phi_{\hat n,m}(\hat R)|_+=v^z_{\hat n,m}(\hat R)|_+=0 \quad \text{for } (\hat n,m)\in\mathcal{C}\,, \tag{5.13}$$

and

$$v^z_{\hat n,m}(\hat R)|_-=0\,,\quad \frac{\mu_0}{\mu}\,v^\phi_{\hat n,m}(\hat R)|_-=\rho\,\delta_{m,\hat m} \quad \text{for } (\hat n,m)\in\mathcal{C} \tag{5.14}$$

for the expansion coefficients.

Corollary 5.2. *Let the system (5.10) – (5.14) with $\rho=0$ admit only the trivial solution. Then $\tilde A_{\hat\alpha,0}$ is invertible.*
Let, in this case, $u(\hat m)$ be the unique solution of (5.10) – (5.14) for $\rho=1$ and every $|\hat m|\leq 1$. If the $3\times 3-$matrix with entries $\langle u(\hat m)\cdot\hat z,\varphi_{\hat n,m}\rangle$, $m,\hat m\in\{-1,0,+1\}$ is regular then Assumption 3.7 is satisfied for $t=0$.

Proof. The first assertion is clear. For the second we observe that $\tilde A^{-1}_{\hat\alpha,0}w^z_{\hat m}=u(\hat m)$ and thus $\langle\tilde A^{-1}_{\hat\alpha,0}w^z_{\hat m},w^z_m\rangle_{H(\mathrm{curl},W)}=u(\hat m)^z_{\hat n,m}=\langle u(\hat m)\cdot\hat z,\varphi_{\hat n,m}\rangle$. ☐

6. Uniqueness

Finally we show that the open waveguide radiation condition is strong enough to ensure uniqueness.

Theorem 6.1. *Let Assumptions 3.5, 3.7, and 3.12 hold, and let $u\in H_*(\mathrm{curl},\mathbb{R}^3)$ be a solution of 1.2 for $f_h=f_e=0$ satisfying the open waveguide radiation condition of Definition 4.1. Then u vanishes identically.*

Proof. Choose $\psi^\pm \in C^\infty(\mathbb{R})$ such that $\psi^\pm(x_3) = 1$ for $\pm x_3 \geq R$ and $\psi^\pm(x_3) = 0$ for $\pm x_3 \leq -R$. Set $\psi_{\ell,j} = \psi^\pm$ for $\ell \in \mathcal{L}^\pm$. We write u in the form $u = \tilde{u}_{rad} + \tilde{u}_{prop}$ with

$$\tilde{u}_{prop}(x) \;:=\; \sum_{j=1}^{J}\sum_{\ell=1}^{m_j} a_{\ell,j}\,\psi_{\ell,j}(x_3)\,\phi_{\ell,j}(x) \quad \text{for } x \in \mathbb{R}^3$$

and $\tilde{u}_{rad} := u - \tilde{u}_{prop}$. Then $\tilde{u}_{prop}$ and $\tilde{u}_{rad}$ coincide with u_{prop} and u_{rad}, respectively, for $|x_3| > R$. The field $\tilde{u}_{rad}$ satisfies

$$\operatorname{curl}\left[\frac{\mu_0}{\mu}\operatorname{curl}\tilde{u}_{rad}\right] - k^2\frac{\varepsilon}{\varepsilon_0}\,\tilde{u}_{rad} \;=\; -f \tag{6.1}$$

where

$$\begin{aligned} f \;&:=\; \operatorname{curl}\left[\frac{\mu_0}{\mu}\operatorname{curl}\tilde{u}_{prop}\right] - k^2\frac{\varepsilon_0}{\varepsilon}\,\tilde{u}_{prop} \;=\; \sum_{j=1}^{J}\sum_{\ell=1}^{m_j} a_{\ell,j}\,\varphi_{\ell,j} \quad \text{and} \\ \varphi_{\ell,j} \;&=\; \operatorname{curl}\left[\frac{\mu_0}{\mu}\operatorname{curl}(\psi_{\ell,j}\phi_{\ell,j})\right] - k^2\frac{\varepsilon}{\varepsilon_0}\,\psi_{\ell,j}\phi_{\ell,j} \\ &=\; \operatorname{curl}\left[\frac{\mu_0}{\mu}\nabla\psi_{\ell,j}\times\phi_{\ell,j}\right] \;+\; \frac{\mu_0}{\mu}\,\nabla\psi_{\ell,j}\times\operatorname{curl}\phi_{\ell,j}\,. \end{aligned}$$

Here we used $\operatorname{curl}\big[a\operatorname{curl}(\psi\phi)\big] = \psi\operatorname{curl}(a\operatorname{curl}\phi) + \operatorname{curl}\big[a(\nabla\psi\times\phi)\big] + a\,\nabla\psi\times\operatorname{curl}\phi$ and the fact that $\phi_{\ell,j}$ satisfy the homogeneous differential equation. We consider the terms $\varphi_{\ell,j}$ separately and write ψ and ϕ for $\psi_{\ell,j}$ and $\phi_{\ell,j}$, respectively. Since ϕ is α_j-quasi-periodic and ψ' has compact support we obtain $\mathcal{F}(\psi'\phi)(x,\alpha) = \mathcal{F}(\psi')(x_3,\alpha-\alpha_j)\phi(x)$ and thus

$$\mathcal{F}\varphi(x,\alpha) \;=\; \operatorname{curl}\left[\frac{\mu_0}{\mu}\mathcal{F}(\psi')(x_3,\alpha-\alpha_j)\,\hat{z}\times\phi\right] \;+\; \frac{\mu_0}{\mu}\,\mathcal{F}(\psi')(x_3,\alpha-\alpha_j)\,\hat{z}\times\operatorname{curl}\phi\,.$$

We consider $\alpha \in (-1/2, 1/2] \setminus \{\alpha_j : j = 1,\ldots,J\}$. Then $0 < |\alpha-\alpha_j| < 1$ for all j. Since $\mathcal{F}(\psi')(\cdot,\beta)$ is $\beta-$quasi-periodic we have a Fourier representation in the form

$$\mathcal{F}(\psi')(x_3,\beta) \;=\; \sum_{m\in\mathbb{Z}} b_m(\beta)\,e^{i(m+\beta)x_3}\,.$$

With

$$\tilde{\psi}(x_3,\beta) \;:=\; \sum_{m\in\mathbb{Z}}\frac{b_m(\beta)}{i(m+\beta)}\,e^{i(m+\beta)x_3}\,, \quad x_3 \in (0,2\pi)\,, \quad 0 < |\beta| < 1\,,$$

we obtain $\mathcal{F}(\psi')(x_3,\beta) = \tilde{\psi}'(x_3,\beta)$ and thus

$$\begin{aligned} \mathcal{F}\varphi(x,\alpha) \;&=\; \operatorname{curl}\left[\frac{\mu_0}{\mu}\tilde{\psi}'(x_3,\alpha-\alpha_j)\,\hat{z}\times\phi(x)\right] + \frac{\mu_0}{\mu}\,\tilde{\psi}'(x_3,\alpha-\alpha_j)\,\hat{z}\times\operatorname{curl}\phi(x) \\ &=\; \operatorname{curl}\left[\frac{\mu_0}{\mu}\operatorname{curl}\big(\tilde{\psi}(x_3,\alpha-\alpha_j)\phi(x)\big)\right] - k^2\frac{\varepsilon}{\varepsilon_0}\,\tilde{\psi}(x_3,\alpha-\alpha_j)\phi(x)\,. \end{aligned}$$

We do this for $\psi = \psi_{\ell,j}$ and obtain $\tilde{\psi}_{\ell,j}$. Now we take the Floquet-Bloch transform of (6.1). Then $\mathcal{F}\tilde{u}_{\rm rad}(\cdot,\alpha)$ satisfies (6.1) with right hand side $-\mathcal{F}f(\cdot,\alpha)$ where

$$\begin{aligned}
&\mathcal{F}f(\cdot,\alpha)\\
&= \sum_{j=1}^{J}\sum_{\ell=1}^{m_j} a_{\ell,j}\left(\operatorname{curl}\left[\frac{\mu_0}{\mu}\operatorname{curl}\bigl(\tilde{\psi}_{\ell,j}(x_3,\alpha-\alpha_j)\phi_{\ell,j}\bigr)\right] - k^2\frac{\varepsilon}{\varepsilon_0}\,\tilde{\psi}_{\ell,j}(x_3,\alpha-\alpha_j)\phi_{\ell,j}\right)\\
&= \operatorname{curl}\left[\frac{\mu_0}{\mu}\operatorname{curl} w_\alpha\right] - k^2\frac{\varepsilon}{\varepsilon_0}\,w_\alpha \text{ with } w_\alpha(x) = \sum_{j=1}^{J}\sum_{\ell=1}^{m_j} a_{\ell,j}\,\tilde{\psi}_{\ell,j}(x_3,\alpha-\alpha_j)\,\phi_{\ell,j}(x)\,.
\end{aligned}$$

Therefore, $\mathcal{F}\tilde{u}_{\rm rad}(\cdot,\alpha) - w_\alpha$ is a $\alpha-$quasi-periodic solution of the homogeneous Maxwell equation and satisfies the Rayleigh expansion. The uniqueness result for the quasi-periodic problem yields $\mathcal{F}\tilde{u}_{\rm rad}(\cdot,\alpha) = w_\alpha$ on W_∞ for almost all $\alpha \in (-1/2,1/2]$ (actually, for all $\alpha \notin \{\alpha_j : j = 1,\ldots,J\}$ by defintion of the critical values). Finally, we use that $\alpha \mapsto \mathcal{F}\tilde{u}_{\rm rad}(\cdot,\alpha)|_{W_R}$ is in $L^2\bigl((-1/2,1/2),H(\operatorname{curl},W_R)\bigr)$ for any $R > 0$. The function w_α, however, is too singular at $\alpha \to \alpha_j$. Indeed, for $\beta \approx 0$ we have $\tilde{\psi}_{\ell,j}(x_3,\beta) = \frac{b_0(\beta)}{i\beta}\,e^{i\beta x_3} + \sum_{m\neq 0}\frac{b_m(\beta)}{i(m+\beta)}\,e^{i(m+\beta)x_3}$ and $b_0(\beta) \to \beta_0(0)$ as $\beta \to 0$ and

$$b_0(0) = \frac{1}{2\pi}\int_0^{2\pi}\mathcal{F}(\psi'_{\ell,j})(x_3,0)\,dx_3 = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_0^{2\pi}\psi'_{\ell,j}(x_3+2\pi m)\,dx_3 = \pm\frac{1}{2\pi}$$

if $\ell \in \mathcal{L}_j^\pm$. We note that the series restricts to a finite sum. Therefore, $\beta \mapsto b_0(\beta)/\beta$ is not in $L^2(-\delta,\delta)$, and we conclude that $\sum_{\ell=1}^{m_j} s_{\ell,j}a_{\ell,j}\,\phi_{\ell,j}$ has to vanish for every j where $s_{\ell,j} = \pm 1$ for $\ell \in \mathcal{L}_j^\pm$. The linear independence of $\phi_{\ell,j}$ yields $a_{\ell,j} = 0$ for all ℓ and j, i.e. $u_{prop} \equiv 0$ and $w_\alpha = 0$ for almost all α. Therefore, $\mathcal{F}u_{rad}(\cdot,\alpha) \in H_{\alpha,*}(\operatorname{curl},W_\infty)$ solves the $\alpha-$quasi-periodic homogeneous differential equation and the Rayleigh expansion and thus vanishes for almost all α. Therefore, u_{rad} vanishes which ends the proof. $\square$

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