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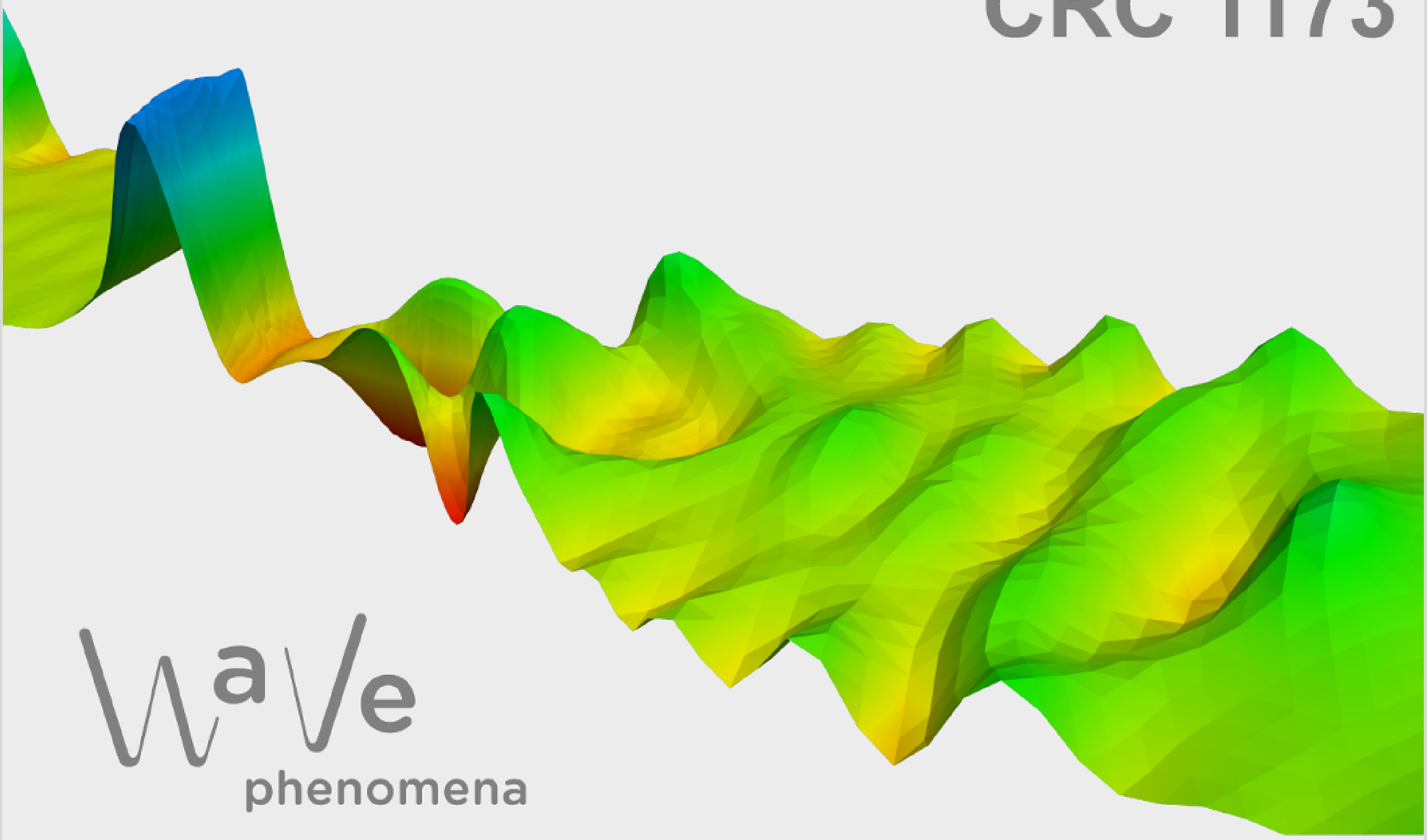

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STABLE BLOWUP FOR SUPERCRITICAL WAVE MAPS INTO PERTURBED SPHERES

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Abstract. We consider wave maps from (1+d)-dimensional Minkowski space, $d \geq 3$, into rotationally symmetric manifolds which arise from small perturbations of the sphere $\mathbb{S}^d$. We prove the existence of co-rotational self-similar finite time blowup solutions with smooth blowup profiles. Furthermore, we show the nonlinear asymptotic stability of these solutions under suitably small co-rotational perturbations on the full space.

1. Introduction

We consider maps $U : \mathbb{R}^{1+d} \to N$, where $\mathbb{R}^{1+d}$ is the $(1+d)$-dimensional Minkowski space and N a Riemannian manifold, specifically, a d-dimensional, rotationally symmetric warped product manifold, see [9], [29], [35] for the general definition. Such a map U is called *wave map*, if it is a critical point of the following Lagrangian

$$\mathcal{L}[U] := \frac{1}{2}\int_{\mathbb{R}^{1+d}} |\nabla_x U(t,x)|_h^2 - |\partial_t U(t,x)|_h^2 \, dt\, dx,$$

where h is a metric on N. The Euler-Lagrange equations associated to this functional, which are in this case called the *wave maps equation*, are given in local coordinates by [1]

$$\Box\, U^a + \Gamma^a_{bc}(U)\, \partial^\mu U^b \, \partial_\mu U^c = 0, \quad a = 1, \dots, d \tag{1.1}$$

and they constitute a system of semilinear wave equations. Here, $\Box = -\partial_t^2 + \Delta_x$ is the linear wave operator on $\mathbb{R}^{1+d}$ and we raise indices with respect to the Minkowski metric $m^{\mu\nu} = m_{\mu\nu} = \operatorname{diag}(-1, 1, \dots, 1)$. Furthermore, Γ^a_{bc} are the Christoffel symbols associated to the metric h on N. Eq. (1.1) is invariant under the scaling transformation

$$U_\lambda(t,x) := U(t/\lambda, x/\lambda), \quad \lambda > 0,$$

and by inspection of the scaling of the associated energy

$$E[U](t) := \frac{1}{2}\int_{\mathbb{R}^d} |\partial_t U(t,x)|_h^2 + |\nabla_x U(t,x)|_h^2 \, dx$$

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[1] as usual Greek indices are running from 0 to d, Latin ones from 1 to d and the Einstein summation convention is in force.

one finds

$$E[U_\lambda](t) = \lambda^{d-2} E[U](t/\lambda).$$

This implies that the wave maps equation is energy critical in dimension two and energy supercritical for $d \geq 3$. In the following, we restrict ourselves to the latter case.
It is well-known that in the energy supercritical case and for large classes of warped product target manifolds Eq. (1.1) admits finite-time blowup via self-similar solutions [35, 7, 15]. However, besides their existence little is known about the role of these solutions in the generic time evolution. Most results are available for $N = \mathbb{S}^d$, where existence [34], [38], [5] and stability of self-similar blowup has been established in the past years in various settings [13, 18, 11, 8, 10, 3, 24, 20, 19, 16], see also Section 1.2. The main goal of this paper is to prove *stable self-similar blowup* in all supercritical dimensions for more general targets which can be viewed as small perturbations of the sphere.

1.1. **The main result.** To state our main results we introduce the following families of warped product manifolds.

Definition 1.1. Let $\alpha \in C^\infty(\mathbb{R})$ be a non-trivial real-valued 2π-periodic function, which is even, nonnegative and satisfies $\alpha(0) = \alpha(\pi) = 0$. Set $\epsilon_0 := (\max_{u \in [0,\pi]} \alpha(u))^{-1} > 0$. Then we define for $\epsilon \in \mathbb{R}$, $|\epsilon| < \epsilon_0$, the warped product manifold (S^d_ϵ, h) by

$$S^d_\epsilon := (0, \pi) \times_{w_\epsilon} \mathbb{S}^{d-1} \tag{1.2}$$

equipped with the warping function $w_\epsilon : \mathbb{R} \to \mathbb{R}$,

$$w_\epsilon(u) := \sin(u)(1 + \epsilon\, \alpha(u)). \tag{1.3}$$

In coordinates $(u, \Omega) \in (0, \pi) \times \mathbb{S}^{d-1}$ the metric on S^d_ϵ is given by

$$h = du^2 + w_\epsilon(u)^2 d\Omega^2,$$

with $d\Omega^2$ denoting the standard round metric on $\mathbb{S}^{d-1} \hookrightarrow \mathbb{R}^d$.

The conditions in Definition 1.1 ensure that w_ϵ is a smooth, odd and 2π-periodic function which is strictly positive in the open interval $(0, \pi)$. In addition, $w_\epsilon'(0) = 1$ as well as $w_\epsilon'(\pi) = -1$. This implies that S^d_ϵ is a smooth, $d-$dimensional Riemannian manifold, see e.g. [1]. For $\epsilon = 0$, we have $w_0(u) = \sin(u)$ and in this case S^d_ϵ can be identified with $\mathbb{S}^d \setminus \{\mathfrak{N}, \mathfrak{S}\}$, where $\mathfrak{N}$ and $\mathfrak{S}$ denote the north and south pole of the sphere. For small $|\epsilon| > 0$ the warping function w_ϵ can be viewed as a small perturbation of the warping function of the $d-$sphere $\mathbb{S}^d$ so that we will call S^d_ϵ a *perturbed sphere* in that case.

It is convenient to consider on S^d_ϵ so-called normal coordinates $U = (U^1, \dots, U^d)$, where

$$U^j := u\, \Omega^j, \quad \text{for} \quad j = 1, \dots, d, \tag{1.4}$$

see, e.g. [35]. In this way, S^d_ϵ can be identified (when including the north pole corresponding to the limiting value $u = 0$) with the ball $B^d_\pi(0) \subset \mathbb{R}^d$. Hence, we consider the initial value problem

$$\begin{aligned} \Box\, U^a + \Gamma^a_{bc}(U)\, \partial^\mu U^b\, \partial_\mu U^c = 0, \quad a \in \{1, \dots, d\} \\ U(0, \cdot) = U_0, \quad \partial_t U(0, \cdot) = U_1 \end{aligned} \tag{1.5}$$

on $I \times \mathbb{R}^d$, where $I \subset \mathbb{R}$ is an interval containing zero and data $U_0, U_1 : \mathbb{R}^d \to \mathbb{R}^d$. We restrict our attention to co-rotational maps $U(t, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ which are by definition of the form

$$U(t,x) = u(t,|x|)\frac{x}{|x|}, \tag{1.6}$$

for a *radial profile* $u : I \times [0,\infty) \to \mathbb{R}$ which has to satisfy the equation

$$\left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r\right) u(t,r) + \frac{d-1}{r^2} w_\epsilon(u(t,r))\, w_\epsilon'(u(t,r)) = 0 \tag{1.7}$$

together with the boundary condition $u(t,0) = 0$ for all $t \in I$. As co-rotational symmetry is conserved by the flow it suffices to study the evolution of u governed by Eq. (1.7).
It is well-known that in the case $\epsilon = 0$, Eq. (1.7) admits self-similar solutions in all space dimensions $d \geq 3$. The so-called *ground state self-similar solution* is known in closed form, see [38], [5], and given by

$$u_0^T(t,r) = f_0\left(\frac{r}{T-t}\right) \quad \text{for} \quad f_0(\rho) = 2\arctan\left(\frac{\rho}{\sqrt{d-2}}\right), \quad T > 0. \tag{1.8}$$

In the past years, the stability of u_0^T has been studied intensively. In this paper, we address the problem of stable self-similar blowup for wave maps into S^d_ϵ for $\epsilon \neq 0$. Our first result proves the existence of globally (in space) smooth self-similar blowup profiles for Eq. (1.7) in any space dimension $d \geq 3$, and for any $\epsilon \in \mathbb{R}$ sufficiently small.

In the following statements, we consider for a fixed function α as in Definition 1.1 the family of corresponding warping functions w_ϵ, $|\epsilon| \leq \epsilon_0$.

Theorem 1.2. *Let $d \geq 3$. There exists an $\epsilon^* \in \mathbb{R}$, $0 < \epsilon^* \leq \epsilon_0$ such that for every $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \epsilon^*$, Eq. (1.7) admits a self-similar solution $u_\epsilon^T \in C^\infty([0,T) \times [0,\infty)) \cap L^\infty([0,T] \times [0,\infty))$ of the form*

$$u_\epsilon^T(t,r) = f_\epsilon\left(\frac{r}{T-t}\right), \quad T > 0,$$

such that its gradient blows up in $r = 0$, i.e.,

$$\lim_{t \to T^-} |\partial_r u_\epsilon^T(t,0)| = +\infty.$$

The profile f_ϵ can be written as

$$f_\epsilon = f_0 + \phi_\epsilon$$

with f_0 defined in Eq. (1.8) and a perturbation $\phi_\epsilon \in L^\infty([0,\infty)) \cap C^\infty([0,\infty))$. The function ϕ_ϵ depends Lipschitz continuously on the parameter ϵ in the sense that

$$\left\|(\cdot)^{-1}(\phi_\epsilon - \phi_\kappa)\right\|_{W^{2,\infty}([0,\infty))} \lesssim |\epsilon - \kappa| \tag{1.9}$$

for all $|\epsilon|, |\kappa| \leq \epsilon^*$ and $\phi_0 = 0$. Furthermore, f_ϵ satisfies $f_\epsilon(0) = 0$ and the limit $\lim_{\rho\to\infty} f_\epsilon(\rho)$ exists. Finally, for every $k \in \mathbb{N}_0$ there are constants $C_{1,k}, C_{2,k} > 0$ depending on ϵ such that

$$|f_\epsilon^{(k)}(\rho)| \leq C_{1,k}\langle\rho\rangle^{-k} \quad \text{and} \quad |(f_\epsilon + (\cdot)f_\epsilon')^{(k)}(\rho)| \leq C_{2,k}\langle\rho\rangle^{-1-k}$$

for all $\rho \in [0,\infty)$.

Theorem 1.2 is obtained by functional analytic methods using a perturbative construction which exploits the invertibility of the linearization around f_0 in suitable function spaces. This approach allows us to prove rigorous results on the stability of these solutions under small co-rotational perturbations. We phrase the result in normal coordinates and consider without loss of generality small perturbations around the blowup solution with blowup time $T = 1$.

Theorem 1.3. *Let $d \geq 3$ and choose $\epsilon^* > 0$ as in Theorem 1.2. For $\epsilon \in \mathbb{R}$, $|\epsilon| \leq \epsilon^*$ define*

$$U_\epsilon^T(t,x) := f_\epsilon\left(\frac{|x|}{T-t}\right)\frac{x}{|x|}.$$

Consider co-rotational initial data of the form

$$U_0 = U_\epsilon^1(0,\cdot) + \nu_0, \quad U_1 = \partial_t U_\epsilon^1(0,\cdot) + \nu_1,$$

where for $i \in \{0,1\}$, $\nu_i : \mathbb{R}^d \to \mathbb{R}^d$, $\nu_i(x) = x v_i(|x|)$ and $v_i : [0,\infty) \to \mathbb{R}$ are such that $v_i(|\cdot|) \in \mathcal{S}(\mathbb{R}^d)$. Let $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy

$$\frac{d}{2} < s \leq \frac{d}{2} + \frac{1}{2d+2}, \quad k > d+2. \tag{1.10}$$

Then there exists an $\overline{\epsilon} \in \mathbb{R}$, $0 < \overline{\epsilon} \leq \epsilon^*$ such that for every $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \overline{\epsilon}$ the following holds: There are $\delta, M_0 > 0$ such that for (ν_0,ν_1) as above satisfying

$$\|(\nu_0,\nu_1)\|_{\dot{H}^s\cap\dot{H}^k(\mathbb{R}^d,\mathbb{R}^d)\times\dot{H}^{s-1}\cap\dot{H}^{k-1}(\mathbb{R}^d,\mathbb{R}^d)} < \frac{\delta}{M_0}, \tag{1.11}$$

there exists a $T = T_\epsilon \in [1-\delta, 1+\delta]$ and a unique co-rotational map $U \in C^\infty([0,T)\times\mathbb{R}^d,\mathbb{R}^d)$ that satisfies Eq. (1.5) for all $(t,x) \in [0,T)\times\mathbb{R}^d$. The gradient of U blows up at the origin as $t \to T^-$ and we have the decomposition

$$U(t,x) = U_\epsilon^T\left(\frac{x}{T-t}\right) + \nu\left(t, \frac{x}{T-t}\right),$$

for a function $\nu : [0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ which satisfies

$$\|\nu(t,\cdot)\|_{\dot{H}^r(\mathbb{R}^d,\mathbb{R}^d)} + \|((T-t)\partial_t + \Lambda)\nu(t,\cdot)\|_{\dot{H}^{r-1}(\mathbb{R}^d,\mathbb{R}^d)} \to 0 \tag{1.12}$$

as $t \to T^-$ for all $r \in [s,k]$ where Λ is defined as the operator $\Lambda = x\cdot\nabla$. In addition,

$$U(t,(T-t)\cdot) \to U_\varepsilon$$

uniformly on compact subsets of $\mathbb{R}^d$ as $t \to T^-$.

Theorem 1.2 and Theorem 1.3 guarantee for the considered class of target manifolds the existence of *stable* blowup via co-rotational self-similar solutions in all space dimensions $d \geq 3$. In low space dimensions, this is the first result of this kind for targets other than the sphere, see Section 1.2. The assumptions on the initial data and the underlying topology arise naturally by reformulating the problem as a classical stability problem using similarity coordinates, see Section 1.3. Some further remarks are in order.

Remark 1.4. *(Spectral problem) In the proof of Theorem 1.3 spectral analysis presents the key difficulty. The ODE problem that arises in the investigation of the stability of self-similar blowup in nonlinear wave equations is notoriously hard and current techniques require explicit knowledge of the profile, see e.g.* [14] *for a recent exposition. In this paper, we exploit* *the perturbative nature of our profile construction which enables us to extract the necessary information on the spectral structure of the linearization around f_ϵ even though the profile is not known in closed form.*

Remark 1.5. *(Function spaces) We have chosen to investigate the stability problem in a strip $[0,T) \times \mathbb{R}^d$ using the framework of homogenous intersection Sobolev spaces based on recent work by Glogić* [24]*. However, an analogous stability result could equally be established in backward light cones (at least at a regularity level strictly above scaling) using known results for $\epsilon = 0$, see below.*

We note that the approach developed in this paper is not limited to the wave maps equation. In fact, our results suggest that not only the existence of self-similar blowup solutions in nonlinear wave equations but also their dynamical stability properties are stable under suitable small (scale invariant) perturbations of the nonlinearity.

1.2. **Blowup in energy supercritical wave maps - Known results.** As the literature on wave maps is vast, we restrict the discussion to works which are relevant in the context of our main results. In particular, we only discuss the energy supercritical case $d \geq 3$.
Concerning the existence of self-similar blowup for wave maps into the sphere, the first result was obtained by Shatah [34] for maps from $\mathbb{R}^{3+1}$ into $\mathbb{S}^3$. Later, Turok and Spergel [38] found in the same setting an explicit example of a self-similar solution (which is expected to be the one from Shatah). Bizoń and Biernat [5] provided the analogous closed form expression in all space dimensions $d \geq 4$, which is given in Eq. (1.8). For $3 \leq d \leq 6$, f_0 can indeed be considered as the "ground state" of a family of self-similar blowup profiles whose existence has been proven by Bizoń [4] and Biernat, Bizoń and Maliborski [2].
Numerical investigations by Bizoń, Chmaj and Tabor [6], see also [5], indicate that blowup via f_0 is generic (at least within the class of co-rotational initial data). The first rigorous proof of the stability of f_0 under small co-rotational perturbations (in $d = 3$ and restricted to a backward light cone) has been established in the series of works by the first author [13], jointly with Aichelburg and the second author [18] and with Costin and Xia [11], respectively. Generalizations to higher dimensions were obtained in [8], [10], where the latter work by Costin, the first author and Glogić provides an important simplification of the ODE analysis underlying the spectral problem. Recent works by the first author and Wallauch [20, 19] establish stability in backward light cones at the optimal regularity level in $d = 3, 4$.
In another line of research the stability is studied in so-called hyperboloidal similarity coordinates, which allow to follow the evolution past the blowup time, at least outside the singularity. This has been initiated by the first two authors together with Biernat in [3] and generalized by the first author and Ostermann in [16]. As already mentioned above, the stability analysis on a strip $[0,T) \times \mathbb{R}^d$ has been implemented recently by Glogić [24] in all space dimensions $d \geq 3$.
Finally, non-self-similar blowup occurs at least in dimensions $d \geq 7$, as has been demonstrated by Ghoul, Ibrahim and Nguyen [22].

The existence of finite-time blowup for wave maps from $\mathbb{R}^{d+1}$ into general warped product manifolds $N^m = (0,a) \times_w \mathbb{S}^{m-1}$, $m \in \mathbb{N}$, $m \geq 2$, $a > 0$, has been addressed by Shatah and Tahvildar-Zadeh [35] and Cazenave, Shatah and Tahvildar-Zadeh [7]: In odd space dimensions $d \geq 3$, by using variational methods and ODE tools, the authors construct $k-$equivariant self-similar profiles ($k = 1$ corresponds to the co-rotational case) locally in a backward light cone which lead to finite time blowup via finite speed of propagation. The admissible values of k and m depend on the properties of the corresponding warping function w. By lifting these solutions to one space dimension higher, finite time blowup is established in even space dimensions $d \geq 4$. The stability properties of the solutions constructed in [35] and [7] are unknown, except for $N = \mathbb{S}^d$. Finally, for $d \geq 8$, the first author and Glogić [15] provided examples of negatively curved targets which admit an explicit self-similar blowup solution and proved its stability in backward light cones in the case $d = 9$.

1.3. **Reformulation of the problem and outline of the proof.** It is common to write co-rotational maps as $U(t, x) = xv(t, |x|)$ since v then satisfies a radial nonlinear wave equation with a smooth nonlinearity, see e.g. [35].
To realize this, we change variables and define $\widetilde{v}(t, r) := r^{-1}u(t, r)$ which transforms Eq. (1.7) into a $(d + 2)$-dimensional radial semilinear wave equation with a now smooth nonlinearity

$$\left(\partial_t^2 - \partial_r^2 - \frac{d+1}{r}\partial_r\right)\widetilde{v}(t, r) - \frac{d-1}{r^3}\left(r\,\widetilde{v}(t, r) - w_\epsilon(r\,\widetilde{v}(t, r))\,w_\epsilon'(r\,\widetilde{v}(t, r))\right) = 0. \tag{1.13}$$

In the investigation of Eq. (1.13) it is convenient to define $n := d + 2$ and $v(t, x) := \widetilde{v}(t, |x|)$ for $x \in \mathbb{R}^n$ and then formulate the equation as a wave equation on $\mathbb{R}^n$,

$$(\partial_t^2 - \Delta_x)v(t, x) = \frac{n-3}{|x|^3}\left(|x|v(t, x) - w_\epsilon(|x|v(t, x))w_\epsilon'(|x|v(t, x))\right), \quad x \in \mathbb{R}^n. \tag{1.14}$$

We then introduce similarity coordinates

$$\tau = \log\left(\frac{T}{T-t}\right), \quad \xi = \frac{x}{T-t}$$

for $T > 0$ and $(t, x) \in [0, T) \times \mathbb{R}^n$ and rewrite Eq. (1.14) as a first order system of the form

$$\partial_\tau \Psi(\tau) = \mathbf{L}\Psi(\tau) + \mathbf{N}_\epsilon(\Psi(\tau)), \tag{1.15}$$

where $\mathbf{L}$ generates the free wave evolution and $\mathbf{N}_\epsilon$ denotes the nonlinearity which depends on ϵ via the warping function w_ϵ, see Section 2 for the details. We study the problem in intersection Sobolev spaces of radial functions

$$\mathcal{H}_r^{s,k} = (\dot{H}_r^s(\mathbb{R}^n) \cap \dot{H}_r^k(\mathbb{R}^n)) \times (\dot{H}_r^{s-1}(\mathbb{R}^n) \cap \dot{H}_r^{k-1}(\mathbb{R}^n))$$

for suitable exponents $\frac{n}{2} - 1 < s < \frac{n}{2} < k$, $k \in \mathbb{N}$.

1.3.1. *Existence of self-similar blowup profiles.* For $\epsilon = 0$, the blowup solution (1.8) transforms into a static solution $\mathbf{\Psi}_0$ of (1.15), i.e.,

$$\mathbf{L}\mathbf{\Psi}_0 + \mathbf{N}_{\mathbf{0}}(\mathbf{\Psi}_0) = 0. \tag{1.16}$$

In order to find for $\epsilon \neq 0$ a solution to

$$\mathbf{L}\mathbf{\Psi}_\epsilon + \mathbf{N}_\epsilon(\mathbf{\Psi}_\epsilon) = 0 \tag{1.17}$$

we insert the ansatz $\mathbf{\Psi}_\epsilon = \mathbf{\Psi}_0 + \mathbf{\Phi}_\epsilon$ into (1.17) and write the resulting equation for the perturbation $\mathbf{\Phi}_\epsilon$ as

$$-\mathbf{L}_0\mathbf{\Phi}_\epsilon = \mathbf{V}_\epsilon(\mathbf{\Psi}_0)\mathbf{\Phi}_\epsilon + \widetilde{\mathbf{N}}_\epsilon(\mathbf{\Phi}_\epsilon) + \mathcal{R}_\epsilon(\mathbf{\Psi}_0) \tag{1.18}$$

where $\mathbf{L}_0 := \mathbf{L} + \mathbf{V}_0(\mathbf{\Psi}_0)$ denotes the linearization around $\mathbf{\Psi}_0$ for $\epsilon = 0$, $\mathbf{V}_\epsilon(\mathbf{\Psi}_0)$ is a potential term depending on ϵ via the warping function, $\mathcal{R}_\epsilon(\mathbf{\Psi}_0)$ denotes the remainder and $\widetilde{\mathbf{N}}_\epsilon$ the nonlinearity, which is quadratically small in its argument. The spectral analysis of [24] implies that $\mathbf{L}_0$, defined as an unbounded operator on a suitable domain $\mathcal{D}(\mathbf{L}_0) \subset \mathcal{H}_r^{s,k} \to \mathcal{H}_r^{s,k}$, is invertible.

In order to control the right-hand side of (1.18) we prove a parameter-dependent version of Schauder type estimates originally stated in [24]. For this we assume

$$\frac{n}{2} - 1 < s \leq \frac{n}{2} - 1 + \frac{1}{2n-2}, \quad k > n, \tag{1.19}$$

see Appendix A. Using this, we show that the first two terms on the right hand side of Eq. (1.18) define maps $\mathcal{H}_r^{s,k} \to \mathcal{H}_r^{s,k+1}$ which are Lipschitz-continuous with respect to the parameter ϵ and with respect to the argument. Moreover, the remainder is in $\mathcal{H}_r^{s,k+1}$ and depends Lipschitz-continuously on ϵ, see Lemma 3.2 - 3.4.

With these prerequisites, we solve Eq. (1.18) for suitably small $\epsilon > 0$ by applying Banach's fixed point theorem. The smoothness of the constructed $\mathbf{\Phi}_\epsilon$ can be shown inductively using the mapping properties of the involved operators together with Sobolev embedding. The claimed decay properties of the resulting smooth solution $\mathbf{\Psi}_\epsilon = (\psi_{\epsilon,1}, \psi_{\epsilon,2})$ to Eq. (1.17) do not follow from this perturbative approach (although some decay can be obtained from Strauss type inequalities). Instead, we use ODE analysis similar to the classcial work of Kavian and Weissler [27] to show that $\psi_{\epsilon,j} = \mathcal{O}(|\xi|^{-j})$ at infinity. This property will be crucial in the analysis. Finally, translating the result back to the original coordinates and variables yields Theorem 1.2.

1.3.2. *Stability of self-similar blowup.* We now consider the time-dependent problem (1.15) and linearize around $\mathbf{\Psi}_\epsilon$ by making the ansatz $\Psi(\tau) = \mathbf{\Psi}_\epsilon + \Phi_\epsilon(\tau)$ to obtain

$$\begin{cases} \partial_\tau \Phi_\epsilon(\tau) = \mathbf{L}\Phi_\epsilon(\tau) + \mathbf{L}'_\epsilon \Phi_\epsilon(\tau) + \widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\tau)), \\ \Phi_\epsilon(0) = \mathbf{U}_{\epsilon,T}, \end{cases} \tag{1.20}$$

where $\mathbf{L}$ is again the free wave operator, $\mathbf{L}'_\epsilon \mathbf{u} = (0, V_\epsilon(\psi_{\epsilon,1})u_1)$ for $\mathbf{u} = (u_1, u_2) \in \mathcal{H}_r^{s,k}$ includes the potential term and $\widehat{\mathbf{N}}_\epsilon$ denotes the nonlinear remainder. It is known that under our assumptions on the exponents (s,k), $\mathbf{L}$ generates a strongly continuous one-parameter semigroup $(\mathbf{S}(\tau))_{\tau \geq 0}$ on $\mathcal{H}_r^{s,k}$, which decays exponentially to zero as $\tau \to \infty$, see [24], [14]. The decay properties of $\psi_{\epsilon,1}$ at infinity allow us to show compactness of the perturbation $\mathbf{L}'_\epsilon$. This implies in particular that the full linear operator $\mathbf{L}_\epsilon := \mathbf{L} + \mathbf{L}'_\epsilon$ generates a semigroup $(\mathbf{S}_\epsilon(\tau))_{\tau \geq 0}$. In order to obtain suitable growth bounds for the linearized evolution, we investigate the spectrum of the generator. From time-translation invariance we expect $\lambda = 1$ to be an eigenvalue. We make this rigorous in Lemma 4.3 and show that

$$\mathbf{g}_\epsilon := \begin{pmatrix} \psi_{\epsilon,1} + \Lambda \psi_{\epsilon,1} \\ 2\psi_{\epsilon,1} + 3\Lambda\psi_{\epsilon,1} + \Lambda^2 \psi_{\epsilon,1} \end{pmatrix}$$

is the unique eigenfunction corresponding to $\lambda = 1$. The decay of $\psi_{\varepsilon,1}$ is crucial in order to show that $\mathbf{g}_\epsilon \in \mathcal{D}(\mathbf{L}_\epsilon)$. For $\epsilon = 0$, the spectral properties of $\mathbf{L}_0$ are well-known. More precisely, we know from [24] that there exists an $\tilde{\omega} > 0$ such that

$$\sigma(\mathbf{L}_0) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < -\tilde{\omega}\} \cup \{1\}.$$

By a perturbation argument, using the fact that $1 \in \sigma(\mathbf{L}_\epsilon)$, we infer that

$$\sigma(\mathbf{L}_\epsilon) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < -\omega_0\} \cup \{1\}$$

for some $0 < \omega_0 < \tilde{\omega}$ and sufficiently small $\epsilon \in \mathbb{R}$. The spectral structure in combination with resolvent bounds proven uniformly in the parameter imply

$$\|\mathbf{S}_\epsilon(\tau)(\mathbf{I}-\mathbf{P}_\epsilon)\mathbf{u}\|_{s,k} \lesssim e^{-\omega_0\tau}\,\|(\mathbf{I}-\mathbf{P}_\epsilon)\mathbf{u}\|_{s,k}\,,$$

and $\mathbf{P}_\epsilon\,\mathbf{S}_\epsilon(\tau) = e^{\tau}\,\mathbf{P}_\epsilon$ for $\mathbf{P}_\epsilon$ denoting the spectral projection onto the eigenspace spanned by $\mathbf{g}_\epsilon$, see [30], Theorem A.1. For the nonlinearity $\widehat{\mathbf{N}}_\epsilon$, we establish Lipschitz bounds by imposing again the above assumptions (1.19) on the Sobolev exponents and apply the estimates of Appendix A. With these results at hand we construct in Section 4.2 global, exponentially decaying strong $\mathcal{H}^{s,k}_r$-solutions of Eq. (1.20). For this, we use the standard approach and first suppress the exponential growth induced by the symmetry eigenvalue $\lambda = 1$ by a correction with values in ran$\mathbf{P}_\epsilon$. In a second step we account for this using the $T-$dependence of the initial condition to determine the suitable blowup time T_ϵ. In Proposition 4.10 we upgrade the constructed strong solutions to classical ones. By defining

$$v^T_\epsilon(t,x) := \frac{1}{T-t}\psi_\epsilon\left(\frac{x}{T-t}\right), \quad \psi_\epsilon(\xi) := |\xi|^{-1} f_\epsilon(|\xi|) \tag{1.21}$$

for $x \in \mathbb{R}^n$ and $t \in [0,T)$ we get the following result.

Theorem 1.6. *Let $n \geq 5$ and choose $\epsilon^* > 0$ as in Theorem 1.2. For $\epsilon \in \mathbb{R}$, $|\epsilon| \leq \epsilon^*$, let v^T_ϵ be defined as in Eq.* (1.21) *and let $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy*

$$\frac{n}{2} - 1 < s \leq \frac{n}{2} - 1 + \frac{1}{2n-2}, \quad k > n. \tag{1.22}$$

Then there exist $\omega > 0$ and $0 < \bar{\epsilon} \leq \epsilon^*$ such that for every $\epsilon \in \mathbb{R}$, $|\epsilon| \leq \bar{\epsilon}$ there are $\delta > 0$ and $M > 1$ such that the following holds: For any pair of radial, real-valued functions $\varphi_0, \varphi_1 \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\|(\varphi_0,\varphi_1)\|_{\dot{H}^s\cap\dot{H}^k(\mathbb{R}^n)\times\dot{H}^{s-1}\cap\dot{H}^{k-1}(\mathbb{R}^n)} < \frac{\delta}{M}, \tag{1.23}$$

there exists $T = T_\epsilon \in [1-\delta, 1+\delta]$ and a unique radial solution $v \in C^\infty([0,T)\times\mathbb{R}^n)$ to Eq. (1.14) *with*

$$v(0,\cdot) = v^1_\epsilon(0,\cdot) + \varphi_0, \quad \partial_t v(0,\cdot) = \partial_t v^1_\epsilon(0,\cdot) + \varphi_1.$$

Moreover, v blows up at $(T,0)$ and can be decomposed as

$$v(t,x) = v^T_\epsilon(t,x) + \frac{1}{T-t}\varphi\left(\log\left(\frac{T}{T-t}\right), \frac{x}{T-t}\right)$$

for all $(t,x) \in [0,T)\times\mathbb{R}^n$, where $\varphi \in C^\infty([0,T)\times\mathbb{R}^n)$ is radially symmetric and satisfies

$$\|\varphi(-\log(T-t)+\log T,\cdot)\|_{\dot{H}^r(\mathbb{R}^n)} \lesssim \delta(T-t)^\omega \tag{1.24}$$

for all $r \in [s,k]$. Furthermore,

$$\|(\partial_0 + \Lambda + 1)\varphi(-\log(T-t)+\log T,\cdot)\|_{\dot{H}^{r-1}(\mathbb{R}^n)} \lesssim \delta(T-t)^\omega. \tag{1.25}$$

Finally, we rephrase the results of Theorem 1.6 in terms of normal coordinates using the equivalence of norms of corotational maps and their radial profiles, see [23], to obtain Theorem 1.3.

1.4. **Notation.** For two real numbers $A, B \in \mathbb{R}$ we write $A \lesssim B$ if there exists an absolute constant $C > 0$ such that $A \leq CB$ holds. For a constant $\omega \in \mathbb{R}$ we denote by $\mathbb{H}_\omega := \{z \in \mathbb{C} : \mathrm{Re}\,(z) > \omega\}$ the open right half-plane of ω.
Furthermore, if $\mathcal{H}$ is a Hilbert space and $(L, \mathcal{D}(L))$ a closed linear operator on $\mathcal{H}$ we denote by $R_L(\lambda) := (\lambda - L)^{-1}$ the resolvent operator for λ lying in the resolvent set $\rho(L)$.
For $n \in \mathbb{N}$, we denote by

$$C_r^\infty(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : f \text{ is radial}\}$$

the set of smooth radially symmetric functions and by $C_{c,r}^\infty(\mathbb{R}^n)$ the subset of those, which have compact support. By

$$C_e^\infty[0,\infty) := \{f \in C^\infty([0,\infty)) : f^{(2j+1)}(0) = 0 \text{ for } j \in \mathbb{N}_0\},$$

we define set of smooth "even" functions and note that there is a one-to-one correspondence between $C_r^\infty(\mathbb{R}^n)$ and $C_e^\infty[0,\infty)$. As usual, $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}_r(\mathbb{R}^n)$ denote the set of Schwartz functions and radial Schwartz functions, respectively.
We define the Fourier transform $\mathcal{F}u$ of $u \in C_c^\infty(\mathbb{R}^n)$ by

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(x) e^{-i\xi\cdot x}\, dx.$$

For $u, v \in C_c^\infty(\mathbb{R}^n)$ and $s \geq 0$ we define as usual the inner product

$$\langle u, v\rangle_{\dot{H}^s(\mathbb{R}^n)} := \langle |\cdot|^s \mathcal{F}u, |\cdot|^s \mathcal{F}v\rangle_{L^2(\mathbb{R}^n)},$$

and the corresponding norm $\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} := \langle u, u\rangle_{\dot{H}^s(\mathbb{R}^n)}$. If $k \in \mathbb{N}_0$ is a nonnegative integer

$$\|u\|_{\dot{H}^k(\mathbb{R}^n)} \simeq \sum_{|\beta|=k} \left\|\partial^\beta u\right\|_{L^2(\mathbb{R}^n)}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. Finally, for $s \geq 0$ the homogeneous radial Sobolev space $\dot{H}_r^s(\mathbb{R}^n)$ denotes the space which is obtained by completion of radial test functions $C_{c,r}^\infty(\mathbb{R}^n)$ with respect to the above defined norm.

2. Similarity variables and functional analytic setup

For $d \geq 3$ and a fixed function α, see Definition 1.1, let S_ϵ^d, for $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \epsilon_0$, denote the corresponding family of perturbed spheres. For the warping function w_ϵ as defined in Eq. (1.3) we consider for $x \in \mathbb{R}^n, n := d + 2$ the equation

$$\partial_t^2 v(t,x) - \Delta_x v(t,x) = \frac{n-3}{|x|^3}\left(|x|v(t,x) - w_\epsilon(|x|v(t,x)) w_\epsilon'(|x|v(t,x))\right). \tag{2.1}$$

2.1. **Reformulation of the problem in similarity coordinates.** Restricting to $t \in [0, T)$ for $T > 0$ and $x \in \mathbb{R}^n$ we rewrite Eq. (2.1) in similarity coordinates

$$\tau := \log\left(\frac{T}{T-t}\right) \qquad \xi := \frac{x}{T-t}.$$

Note that this transformation maps the strip $S_T := [0, T) \times \mathbb{R}^n$ into the upper half-space $H_+ := [0, \infty) \times \mathbb{R}^n$. Additionally, we define a new rescaled dependent variable via

$$\psi(\tau, \xi) := Te^{-\tau}\, v(T - Te^{-\tau}, Te^{-\tau}\xi).$$

Consequently, the evolution of v inside S_T corresponds to the evolution of ψ inside H_+. Furthermore, the differential operators with respect to t and x are given in the new variables by

$$\partial_t = \frac{e^\tau}{T}(\partial_\tau + \Lambda), \quad \text{and} \quad \partial_{x_i} = \frac{e^\tau}{T}\partial_{\xi_i},$$

for $i \in \{1, \dots, n\}$, where the operator Λ is again defined via $\Lambda f(\xi) := \xi \cdot \nabla f(\xi)$. Eq. (2.1) then transforms into

$$\begin{aligned}(\partial_\tau^2 + 3\partial_\tau + 2\partial_\tau\Lambda - \Delta_\xi + \Lambda^2 + 3\Lambda + 2)\psi(\tau,\xi)& \\ = \frac{n-3}{|\xi|^3}\left(|\xi|\psi(\tau,\xi) - w_\epsilon(|\xi|\psi(\tau,\xi))\, w_\epsilon'(|\xi|\psi(\tau,\xi))\right).&\end{aligned} \tag{2.2}$$

By defining

$$\psi_1(\tau,\xi) := \psi(\tau,\xi), \quad \psi_2(\tau,\xi) := (\partial_\tau + \Lambda + 1)\psi(\tau,\xi)$$

and $\Psi(\tau) := (\psi_1(\tau,\cdot), \psi_2(\tau,\cdot))$ we get an evolution equation for Ψ of the form

$$\partial_\tau \Psi(\tau) = \widetilde{\mathbf{L}}\Psi(\tau) + \mathbf{N}_\epsilon(\Psi(\tau)), \tag{2.3}$$

where

$$\widetilde{\mathbf{L}} := \begin{pmatrix} -\Lambda - 1 & 1 \\ \Delta & -\Lambda - 2 \end{pmatrix} \tag{2.4}$$

is the wave operator in similarity coordinates and the nonlinearity is given by

$$\mathbf{N}_\epsilon(\mathbf{u}) = \begin{pmatrix} 0 \\ N_\epsilon(u_1) \end{pmatrix} \quad N_\epsilon(u_1)(\xi) := \frac{n-3}{|\xi|^3}\eta_\epsilon(|\xi| u_1(\xi)), \tag{2.5}$$

for $\mathbf{u} = (u_1, u_2)$ where

$$\eta_\epsilon(y) := y - w_\epsilon(y) w_\epsilon'(y). \tag{2.6}$$

2.2. **Time-independent solutions.** For $\epsilon = 0$ a static, smooth solution to Eq. (2.3) is given by

$$\mathbf{\Psi}_0 := \begin{pmatrix} \psi_{0,1} \\ \psi_{0,2} \end{pmatrix}, \text{ with } \psi_{0,1}(\xi) = |\xi|^{-1} f_0(|\xi|) \text{ and } \psi_{0,2} = \psi_{0,1} + \Lambda\psi_{0,1}, \tag{2.7}$$

where the profile f_0 is defined in Eq. (1.8). More precisely,

$$\mathbf{0} = \widetilde{\mathbf{L}}\mathbf{\Psi}_0 + \mathbf{N}_0(\mathbf{\Psi}_0). \tag{2.8}$$

In order to prove the existence of a static solution for small $\epsilon \neq 0$, i.e., a smooth function $\mathbf{\Psi}_\epsilon$ which satisfies

$$\mathbf{0} = \widetilde{\mathbf{L}}\mathbf{\Psi}_\epsilon + \mathbf{N}_\epsilon(\mathbf{\Psi}_\epsilon), \tag{2.9}$$

we make the following perturbative ansatz

$$\mathbf{\Psi}_\epsilon = \mathbf{\Psi}_0 + \mathbf{\Phi}_\epsilon.$$

By plugging this into Eq. (2.9) we obtain an equation for the perturbation $\mathbf{\Phi}_\epsilon$ which we write in the following form

$$\begin{aligned}\mathbf{0} &= \widetilde{\mathbf{L}}\mathbf{\Phi}_\epsilon + \mathbf{N}_\epsilon(\mathbf{\Psi}_0+\mathbf{\Phi}_\epsilon) - \mathbf{N}_0(\mathbf{\Psi}_0)\\ &= \widetilde{\mathbf{L}}_0\mathbf{\Phi}_\epsilon + \mathbf{V}_\epsilon(\mathbf{\Psi}_0)\mathbf{\Phi}_\epsilon + \mathcal{R}_\epsilon(\mathbf{\Psi}_0) + \widetilde{\mathbf{N}}_\epsilon(\mathbf{\Phi}_\epsilon),\end{aligned} \tag{2.10}$$

where $\widetilde{\mathbf{L}}_0 := \widetilde{\mathbf{L}} + \mathbf{V}_0(\mathbf{\Psi}_0)$,

$$\mathbf{V}_0(\mathbf{\Psi}_0)\mathbf{u} := \begin{pmatrix} 0 \\ V_0(\psi_{0,1})\,u_1\end{pmatrix} \quad \text{for} \quad V_0(\psi_{0,1})(\xi) := \frac{n-3}{|\xi|^2}\eta_0'(|\xi|\psi_{0,1}(\xi)),$$

$$\mathbf{V}_\epsilon(\mathbf{\Psi}_0)\mathbf{u} := \begin{pmatrix} 0 \\ V_\epsilon(\psi_{0,1})\,u_1\end{pmatrix} \quad \text{for} \quad V_\epsilon(\psi_{0,1})(\xi) := \frac{n-3}{|\xi|^2}\left(\eta_\epsilon'(|\xi|\psi_{0,1}(\xi)) - \eta_0'(|\xi|\psi_{0,1}(\xi))\right),$$

$$\mathcal{R}_\epsilon(\mathbf{\Psi}_0)(\xi) := \begin{pmatrix} 0 \\ \frac{n-3}{|\xi|^3}\left(\eta_\epsilon(|\xi|\psi_{0,1}(\xi)) - \eta_0(|\xi|\psi_{0,1}(\xi))\right)\end{pmatrix}$$

and

$$\widetilde{\mathbf{N}}_\epsilon(\mathbf{u}) := \begin{pmatrix} 0 \\ \widetilde{N}_\epsilon(u_1)\end{pmatrix}$$

for

$$\widetilde{N}_\epsilon(u_1)(\xi) := \frac{n-3}{|\xi|^3}\left(\eta_\epsilon(|\xi|(\psi_{0,1}(\xi)+u_1(\xi))) - \eta_\epsilon(|\xi|\psi_{0,1}(\xi)) - \eta_\epsilon'(|\xi|\psi_{0,1}(\xi))|\xi|u_1(\xi)\right).$$

We are going to formulate Eq. (2.10) as a fixed point problem in suitable function spaces which we discuss in the following.

2.3. **Intersection Sobolev spaces.** For two exponents $0 \le s_1 < s_2$ we define an inner product on $C_c^\infty(\mathbb{R}^n)$ by

$$\langle u, v\rangle_{\dot H^{s_1}\cap \dot H^{s_2}(\mathbb{R}^n)} := \langle u, v\rangle_{\dot H^{s_1}(\mathbb{R}^n)} + \langle u, v\rangle_{\dot H^{s_2}(\mathbb{R}^n)},$$

Moreover, for $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$ and $1 \le s_1 < s_2$ we set

$$\langle \mathbf{u}, \mathbf{v}\rangle_{s_1,s_2} := \langle u_1, v_1\rangle_{\dot H^{s_1}\cap \dot H^{s_2}(\mathbb{R}^n)} + \langle u_2, v_2\rangle_{\dot H^{s_1-1}\cap \dot H^{s_2-1}(\mathbb{R}^n)}.$$

We define the Hilbert space of radial functions $\mathcal{H}_r^{s_1,s_2}$ as the completion of $C_{c,r}^\infty(\mathbb{R}^n)\times C_{c,r}^\infty(\mathbb{R}^n)$ under the norm induced by $\langle\cdot,\cdot\rangle_{s_1,s_2}$,

$$\|\mathbf{u}\|_{s_1,s_2} := \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle}_{s_1,s_2}.$$

In the following, we gather some properties, which are used throughout the paper.
First, we note that for $0 \le \widetilde{s_1} \le s_1 < s_2 \le \widetilde{s_2}$ there is the continuous embedding

$$\dot H_r^{\widetilde{s_1}}(\mathbb{R}^n) \cap \dot H_r^{\widetilde{s_2}}(\mathbb{R}^n) \hookrightarrow \dot H_r^{s_1}(\mathbb{R}^n) \cap \dot H_r^{s_2}(\mathbb{R}^n), \tag{2.11}$$

see Lemma 4.4 in [24]. In particular, $\mathcal{H}_r^{\widetilde{s_1},\widetilde{s_2}} \hookrightarrow \mathcal{H}_r^{s_1,s_2}$. Moreover, for $m \in \mathbb{N}_0, 0 \le s_1 < \frac{n}{2}, s_2 > \frac{n}{2} + m$,

$$\dot H_r^{s_1}(\mathbb{R}^n) \cap \dot H_r^{s_2}(\mathbb{R}^n) \hookrightarrow C_r^m(\mathbb{R}^n). \tag{2.12}$$

Hence,

$$\mathcal{H}_r^{s_1,s_2} \hookrightarrow C_r^m(\mathbb{R}^n) \times C_r^{m-1}(\mathbb{R}^n), \tag{2.13}$$

for $n \geq 3$, $m \geq 1, 1 \leq s_1 < \frac{n}{2}, s_2 > \frac{n}{2} + m$, where the rightmost space is equipped with the natural topology inherited from $W^{m,\infty}(\mathbb{R}^n) \times W^{m-1,\infty}(\mathbb{R}^n)$, see again Lemma 4.4 in [24].
The next Lemma shows that if a radial function has sufficient decay at infinity it belongs to (intersection) radial homogeneous Sobolev spaces.

Lemma 2.1. *Let $n \in \mathbb{N}_{\geq 2}, k > 0$ and $f \in C_r^\infty(\mathbb{R}^n)$. If we have for all $\beta \in \mathbb{N}_0^n$*

$$|\partial^\beta f(x)| \lesssim |x|^{-k-|\beta|} \tag{2.14}$$

for every $x \in \mathbb{R}^n$ then f belongs to $\dot{H}_r^s(\mathbb{R}^n)$ for every $s \geq 0$ with $s > \frac{n}{2} - k$.
In particular, if $1 \leq s_1 < s_2$, $s_1 > \frac{n}{2} - 1$ and $\mathbf{f} = (f_1, f_2) \in C_r^\infty(\mathbb{R}^n) \times C_r^\infty(\mathbb{R}^n)$ satisfies for every multi-index $\beta \in \mathbb{N}_0^n$

$$|\partial^\beta f_1(x)| \lesssim |x|^{-1-|\beta|} \quad \text{and} \quad |\partial^\beta f_2(x)| \lesssim |x|^{-2-|\beta|} \tag{2.15}$$

for every $x \in \mathbb{R}^n$ then it belongs to $\mathcal{H}_r^{s_1,s_2}$.

The proof of this statement is given in Appendix B.

Corollary 2.2. *We infer that for $n \geq 5$ the blowup solution $\mathbf{\Psi}_0$ defined in Eq. (2.7) belongs to $\mathcal{H}_r^{s_1,s_2}$ for all $\frac{n}{2} - 1 < s_1 < s_2$.*

Furthermore, we have the following Banach algebra property.

Lemma 2.3. *Let $n \geq 5$ and let $0 \leq s_1 < \frac{n}{2} < s_2$. Then $\dot{H}_r^{s_1}(\mathbb{R}^n) \cap \dot{H}_r^{s_2}(\mathbb{R}^n)$ is closed under multiplication, in particular,*

$$\|fg\|_{\dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^n)} \|g\|_{\dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^n)} \tag{2.16}$$

for all $f, g \in \dot{H}_r^{s_1}(\mathbb{R}^n) \cap \dot{H}_r^{s_2}(\mathbb{R}^n)$.

An elementary proof of this Lemma is given in [25], Appendix A.2.
Finally, the following version of a generalized Strauss-inequality has been proven for functions in $C_{c,r}^\infty(\mathbb{R}^n)$ in [24], Proposition B.1 (the general statement follows from a density argument together with the embedding from (2.11)).

Lemma 2.4. *Let $n \geq 2, \frac{1}{2} < s < \frac{n}{2}$ and $0 \leq s_1 < \frac{n}{2} < s_2$. Then for every $\beta \in \mathbb{N}_0^n$ with $|\beta| + \frac{n}{2} < s_2$ and $|\beta| + s \geq s_1$*

$$\left\| |\cdot|^{\frac{n}{2}-s} \partial^\beta u \right\|_{L^\infty(\mathbb{R}^n)} \lesssim \|u\|_{\dot{H}^{|\beta|+s}(\mathbb{R}^n)} \tag{2.17}$$

for all $u \in \dot{H}_r^{s_1}(\mathbb{R}^n) \cap \dot{H}_r^{s_2}(\mathbb{R}^n)$.

2.4. **Known results for $\epsilon = 0$.** The following results, which we summarize for later reference, have been proven in Propositions 4.1, 6.1 and 6.2 in [24].

Proposition 2.5. *Let $n \geq 5$, $\frac{n}{2} - 1 < s < \frac{n}{2}$ and $k > \frac{n}{2} + 1$, $k \in \mathbb{N}$. Then the operators $\widetilde{\mathbf{L}}, \widetilde{\mathbf{L}}_0 : C_{c,r}^\infty(\mathbb{R}^n) \times C_{c,r}^\infty(\mathbb{R}^n) \subset \mathcal{H}_r^{s,k} \to \mathcal{H}_r^{s,k}$ are closable. For the closed operators $(\mathbf{L}, \mathcal{D}(\mathbf{L}))$ and $(\mathbf{L}_0, \mathcal{D}(\mathbf{L}_0))$ we have $\mathcal{D}(\mathbf{L}_0) = \mathcal{D}(\mathbf{L})$ and the following properties hold:*

i) *(Free wave evolution) The operator $\mathbf{L}$ generates a strongly continuous semigroup $(\mathbf{S}(\tau))_{\tau \geq 0}$ of bounded operators on $\mathcal{H}_r^{s,k}$, which satisfies*

$$\|\mathbf{S}(\tau)\mathbf{u}\|_{s,k} \leq e^{\left(\frac{n}{2}-1-s\right)\tau} \|\mathbf{u}\|_{s,k} \tag{2.18}$$

for all $\mathbf{u} \in \mathcal{H}_r^{s,k}$ and $\tau \geq 0$.

ii) *(Spectral gap property of* $\mathbf{L}_0$*) There exists* $0 < \widetilde{\omega} < s + 1 - \frac{n}{2}$ *such that*

$$\{\lambda \in \sigma(\mathbf{L}_0) : \operatorname{Re}\lambda \geq -\widetilde{\omega}\} = \{1\}.$$

The point $\lambda = 1$ *is a simple eigenvalue with an explicit eigenfunction*

$$\mathbf{g}_0 := \begin{pmatrix} g_0 \\ 2g_0 + \Lambda g_0 \end{pmatrix}, \quad \textit{where} \quad g_0(\xi) = \psi_{0,1}(\xi) + \Lambda\psi_{0,1}(\xi) = \frac{1}{|\xi|^2 + n - 4}.$$

Moreover, the map $\mathbf{P}_0 : \mathcal{H}^{s,k}_r \to \mathcal{H}^{s,k}_r$ *defined by*

$$\mathbf{P}_0 := \frac{1}{2\pi i} \int_{\partial B_{1/2}(1)} \mathbf{R}_{\mathbf{L}_0}(\lambda)\, d\lambda$$

is a bounded projection with $\operatorname{ran}(\mathbf{P}_0) = \ker(\mathbf{I} - \mathbf{L}_0) = \langle \mathbf{g}_0 \rangle$.

iii) *(Linearized evolution) The operator* $\mathbf{L}_0$ *generates a strongly continuous semigroup* $(\mathbf{S}_0(\tau))_{\tau \geq 0}$ *of bounded operators on* $\mathcal{H}^{s,k}_r$*. Furthermore, for any* $0 < \omega_0 < \tilde{\omega}$ *there is a* $C \geq 1$ *such that*

$$\|\mathbf{S}_0(\tau)(\mathbf{I} - \mathbf{P}_0)\mathbf{u}\|_{s,k} \leq Ce^{-\omega_0 \tau} \|(\mathbf{I} - \mathbf{P}_0)\mathbf{u}\|_{s,k} \tag{2.19}$$

for all $\mathbf{u} \in \mathcal{H}^{s,k}_r$ *and all* $\tau \geq 0$.

We remark that the exponent s in the above Proposition is chosen to be strictly greater than $\frac{n}{2} - 1$ so that one has exponential decay of the free part $\mathbf{L}$. The fact that 0 does not lie in the spectrum of $\mathbf{L}_0$ will be key for the subsequent analysis.

Now throughout the main parts of the proofs provided in Section 3 and Section 4, we assume $n \geq 5$ and work in $\mathcal{H}^{s,k}_r(\mathbb{R}^n)$ assuming that $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy

$$\frac{n}{2} - 1 < s \leq \frac{n}{2} - 1 + \frac{1}{2n-2}, \quad k > n. \tag{2.20}$$

This will allow us to prove a parameter dependent Schauder type estimate akin to [24], which we state in Proposition A.1, Appendix A.

3. Existence of blowup solutions into perturbed spheres

In the following, we construct for sufficiently small ϵ a solution $\mathbf{\Phi}_\epsilon$ to Eq. (2.10). By Proposition 2.5, the operator $\mathbf{L}_0$ is bounded invertible and thus (2.10) can be phrased as a fixed point problem.

First, we properly define the operators appearing on the right hand side of the equation. To prove the required Lipschitz bounds, we will make extensive use of Proposition A.1 in Appendix A and assume that the exponents $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy Eq. (2.20). We note that under this assumption, $\mathcal{H}^{s,k}_r \hookrightarrow C^2_r(\mathbb{R}^n) \times C^1_r(\mathbb{R}^n)$ and thus the nonlinearity, the potential and the remainder can be defined pointwise as in Eq. (2.10).

In the following we use the notation

$$\mathcal{B}_\delta := \{\mathbf{u} \in \mathcal{H}^{s,k}_r : \|\mathbf{u}\|_{s,k} \leq \delta\} \subset \mathcal{H}^{s,k}_r.$$

To proceed, we state an auxiliary lemma for the function η_ϵ, see Eq. (2.6), that will be useful in the following.

Lemma 3.1. *Let $a, b, c \in \mathbb{R}$ and $|\epsilon| \leq \epsilon_0$. Then*

$$\eta_\epsilon(a) = a^3 \int_0^1 \int_0^1 \int_0^1 x^2 y\, \eta_\epsilon'''(axyz)\, dz\, dy\, dx, \tag{3.1}$$

$$\eta_\epsilon'(a) = a^2 \int_0^1 \int_0^1 x\, \eta_\epsilon'''(axy)\, dy\, dx, \tag{3.2}$$

and

$$\begin{aligned} &\eta_\epsilon(a+c) - \eta_\epsilon(a+b) - \eta_\epsilon'(a)(c-b) \\ &= (c-b) \int_0^1 \int_0^1 \int_0^1 (b + x(c-b))(a + y(b + x(c-b)))\eta_\epsilon'''(z(a + y(b + x(c-b))))\, dz\, dy\, dx. \end{aligned} \tag{3.3}$$

Moreover, for every $\ell \in \mathbb{N}_0$ there exists a constant C_ℓ such that

$$|\eta_\epsilon^{(\ell+3)}(x) - \eta_\kappa^{(\ell+3)}(y)| \leq C_\ell \left(|\epsilon - \kappa| + |x - y|\right) \tag{3.4}$$

holds for every $x, y \in \mathbb{R}$ and $|\epsilon|, |\kappa| \leq \epsilon_0$.

Proof. The first three equalities follow from the fact that $\eta_\epsilon(0) = \eta_\epsilon'(0) = \eta_\epsilon''(0) = 0$ together with the fundamental theorem of calculus. We only prove the first equality, since the other two follow from similar ideas.

$$\begin{aligned} \eta_\epsilon(a) = \int_0^a \eta_\epsilon'(x)\, dx = \int_0^a \int_0^x \eta_\epsilon''(y)\, dy\, dx = \int_0^a \int_0^x \int_0^y \eta_\epsilon'''(z)\, dz\, dy\, dx \\ = a^3 \int_0^1 \int_0^1 \int_0^1 x^2 y\, \eta_\epsilon'''(axyz)\, dz\, dy\, dx. \end{aligned}$$

For (3.4) one observes that $\eta_\epsilon^{(\ell+3)}$ can be written for every $\ell \in \mathbb{N}_0$ as

$$\eta_\epsilon^{(\ell+3)} = \eta_0^{(\ell+3)} + \epsilon\, F_1^{(\ell+3)} + \epsilon^2 F_2^{(\ell+3)}$$

where η_0''', F_1''' and F_2''' are smooth 2π-periodic functions.

□

Lemma 3.2. *Let $n \geq 5$ and $(s, k) \in \mathbb{R} \times \mathbb{N}$ satisfy (2.20). Then, for any $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \epsilon_0$, $\widetilde{\mathbf{N}}_\epsilon : \mathcal{H}_r^{s,k} \to \mathcal{H}_r^{s,k+1}$ and the estimate*

$$\|\widetilde{\mathbf{N}}_\epsilon(\mathbf{u}) - \widetilde{\mathbf{N}}_\kappa(\mathbf{v})\|_{s,k+1} \lesssim \left(\|\mathbf{u}\|_{s,k} + \|\mathbf{v}\|_{s,k}\right) \|\mathbf{u} - \mathbf{v}\|_{s,k} + \left(\|\mathbf{u}\|_{s,k}^2 + \|\mathbf{v}\|_{s,k}^2\right) |\epsilon - \kappa| \tag{3.5}$$

holds for all $|\epsilon|, |\kappa| \leq \epsilon_0$ and all $\mathbf{u}, \mathbf{v} \in \mathcal{B}_\delta \subset \mathcal{H}_r^{s,k}$ for $0 < \delta \leq 1$.

Proof. We start by showing the Lipschitz bound

$$\|\widetilde{\mathbf{N}}_\epsilon(\mathbf{u}) - \widetilde{\mathbf{N}}_\epsilon(\mathbf{v})\|_{s,k+1} \lesssim \left(\|\mathbf{u}\|_{s,k} + \|\mathbf{v}\|_{s,k}\right) \|\mathbf{u} - \mathbf{v}\|_{s,k} \tag{3.6}$$

for all $|\epsilon| \leq \epsilon_0$ and all $\mathbf{u}, \mathbf{v} \in \mathcal{B}_\delta$, $0 < \delta \leq 1$. With the help of (3.3) we write

$$\begin{aligned}
&\widetilde{N}_\epsilon(u_1)(\xi) - \widetilde{N}_\epsilon(v_1)(\xi) = \\
&\frac{n-3}{|\xi|^3}\Big(\eta_\epsilon\big(|\xi|(\psi_{0,1}(\xi)+u_1(\xi))\big) - \eta_\epsilon\big(|\xi|(\psi_{0,1}(\xi)+v_1(\xi))\big) - \eta_\epsilon'(|\xi|\psi_{0,1}(\xi))|\xi|(u_1(\xi)-v_1(\xi))\Big) \\
&= (n-3)\int_0^1\int_0^1\int_0^1 (u_1(\xi)-v_1(\xi))\big(v_1(\xi)+x(u_1(\xi)-v_1(\xi))\big) \\
&\cdot \big(\psi_{0,1}(\xi)+y\big(v_1(\xi)+x(u_1(\xi)-v_1(\xi))\big)\big)\cdot \eta_\epsilon'''\left(z|\xi|\big(\psi_{0,1}(\xi)+y(v_1+x(u_1-v_1))\big)\right) dz\,dy\,dx.
\end{aligned}$$

Since $F_\epsilon(x) := \eta_\epsilon'''(x)$ satisfies the assumptions of Proposition A.1 the estimate (3.6) follows from the previous equation and (A.3). We now show that

$$\|\widetilde{\mathbf{N}}_\epsilon(\mathbf{u}) - \widetilde{\mathbf{N}}_\kappa(\mathbf{u})\|_{s,k+1} \lesssim \|\mathbf{u}\|_{s,k}^2\,|\epsilon-\kappa| \tag{3.7}$$

for every $\mathbf{u} \in \mathcal{B}_\delta$ with $0 < \delta \leq 1$ and every $|\epsilon|, |\kappa| \leq \epsilon_0$. For this, we use again Eq. (3.3) to obtain

$$\begin{aligned}
\widetilde{N}_\epsilon(u_1)(\xi) - \widetilde{N}_\kappa(u_1)(\xi) = (n-3)\,u_1(\xi)^2 \int_0^1\int_0^1\int_0^1 x\,(\psi_{0,1}(\xi)+yxu_1(\xi)) \\
\cdot\left(\eta_\epsilon'''\left(z|\xi|\big(\psi_{0,1}(\xi)+yxu_1(\xi)\big)\right) - \eta_\kappa'''\left(z|\xi|\big(\psi_{0,1}(\xi)+yxu_1(\xi)\big)\right)\right) dz\,dy\,dx
\end{aligned}$$

and Eq. (3.7) follows from an application of Proposition A.1. By combining Eq. (3.6) and Eq. (3.7) one obtains (3.5). The mapping properties of $\widetilde{\mathbf{N}}_\epsilon$ follow from Eq. (3.5) and the fact that $\widetilde{\mathbf{N}}_\epsilon(\mathbf{0}) = \mathbf{0}$. ☐

For the potential, we have the following result.

Lemma 3.3. *Let $n \geq 5$ and $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy* (2.20). *For any $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \epsilon_0$, $\mathbf{V}_\epsilon(\mathbf{\Psi}_0) : \mathcal{H}_r^{s,k} \to \mathcal{H}_r^{s,k+1}$. Furthermore,*

$$\|\mathbf{V}_\epsilon(\mathbf{\Psi}_0)\mathbf{u} - \mathbf{V}_\kappa(\mathbf{\Psi}_0)\mathbf{u}\|_{s,k+1} \lesssim |\epsilon-\kappa|\,\|\mathbf{u}\|_{s,k}$$

for all $|\epsilon|, |\kappa| \leq \epsilon_0$ and all $\mathbf{u} \in \mathcal{H}_r^{s,k}$.

Proof. By definition of $\mathbf{V}_\epsilon(\mathbf{\Psi}_0)$ we have to prove the estimate

$$\|(V_\epsilon(\psi_{0,1}) - V_\kappa(\psi_{0,1}))\,u_1\|_{\dot{H}^{s-1}\cap\dot{H}^k(\mathbb{R}^n)} \lesssim |\epsilon-\kappa|\,\|u_1\|_{\dot{H}^s\cap\dot{H}^k(\mathbb{R}^n)}.$$

In a similar way to the previous result one shows with (3.2) that

$$\begin{aligned}
&((V_\epsilon(\psi_{0,1}) - V_\kappa(\psi_{0,1}))u_1)(\xi) \simeq \\
&\qquad \psi_{0,1}(\xi)^2 u_1(\xi)\int_0^1\int_0^1 x\,(\eta_\epsilon'''(|\xi|\psi_{0,1}(\xi)xy) - \eta_\kappa'''(|\xi|\psi_{0,1}(\xi)xy))\,dy\,dx
\end{aligned}$$

holds for every $\xi \in \mathbb{R}^n$. The claim then again follows by an application of Proposition A.1. ☐

Finally, we consider the remainder.

Lemma 3.4. *Let $n \geq 5$ and $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy* (2.20). *For any $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \epsilon_0$, $\mathcal{R}_\epsilon(\mathbf{\Psi}_0) \in \mathcal{H}_r^{s,k+1} \subset \mathcal{H}_r^{s,k}$. Furthermore,*

$$\|\mathcal{R}_\epsilon(\mathbf{\Psi}_0) - \mathcal{R}_\kappa(\mathbf{\Psi}_0)\|_{s,k+1} \lesssim |\epsilon-\kappa|$$

for all $|\epsilon|, |\kappa| \leq \epsilon_0$.

Proof. By the definition of $\mathcal{R}_\epsilon(\mathbf{\Psi}_0)$ we have to estimate $\xi \mapsto \frac{n-3}{|\xi|^3}\left(\eta_\epsilon(|\xi|\psi_{0,1}(\xi)) - \eta_\kappa(|\xi|\psi_{0,1}(\xi))\right)$ in the $\dot{H}^{s-1} \cap \dot{H}^k-$norm. To do so we write with the help of (3.1)

$$\begin{aligned}&\frac{n-3}{|\xi|^3}\left(\eta_\epsilon(|\xi|\psi_{0,1}(\xi)) - \eta_\kappa(|\xi|\psi_{0,1}(\xi))\right)\\ &\qquad =(n-3)\psi_{0,1}(\xi)^3 \int_0^1\int_0^1\int_0^1 x^2 y\left(\eta_\epsilon'''(|\xi|\psi_{0,1}(\xi)xyz) - \eta_\kappa'''(|\xi|\psi_{0,1}(\xi)xyz)\right) dz\, dy\, dx\end{aligned}$$

and conclude the proof by again applying the Schauder estimate from Proposition A.1 to the function $F_\epsilon(x) := \eta_\epsilon'''(x)$. ☐

In order to prove gradient blowup as claimed in Theorem 1.2, we impose an a priori smallness condition on the perturbation.

Lemma 3.5. *Let $n \geq 5$ and $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy (2.20). There is a $\delta_0^* > 0$ such that any $\mathbf{u} = (u_1, u_2) \in \mathcal{B}_{\delta_0^*}$ satisfies*

$$|u_1(0)| < \psi_{0,1}(0),$$

where

$$\psi_{0,1}(\xi) = 2|\xi|^{-1}\arctan\left(\tfrac{|\xi|}{\sqrt{n-4}}\right).$$

Proof. First, note that $\psi_{0,1}(0) > 0$. By Sobolev embedding, there is a $C > 0$ such that

$$|u_1(0)| \leq \|u_1\|_{L^\infty(\mathbb{R}^n)} \leq C\|\mathbf{u}\|_{s,k} \leq C\delta_0^*$$

for all $\mathbf{u} \in \mathcal{B}_{\delta_0^*} \subset \mathcal{H}_r^{s,k}$. Now choose δ_0^* sufficiently small. ☐

The next result crucially relies on the characterization of the spectrum of the linearized operator for $\epsilon = 0$, see Proposition 2.5.

Proposition 3.6. *Let $n \geq 5$ and $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy (2.20). There exist $0 < \delta^* \leq \min\{1, \delta_0^*\}$ and $0 < \epsilon^* \leq \epsilon_0$ (depending on δ^*) such that for every $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \epsilon^*$ the map*

$$\mathbf{K}_\epsilon : \mathcal{B}_{\delta^*} \to \mathcal{B}_{\delta^*}, \quad \mathbf{K}_\epsilon(\mathbf{\Phi}) := (-\mathbf{L}_0)^{-1}(\mathbf{V}_\epsilon(\mathbf{\Psi}_0)\mathbf{\Phi} + \mathcal{R}_\epsilon(\mathbf{\Psi}_0) + \widetilde{\mathbf{N}}_\epsilon(\mathbf{\Phi})) \tag{3.8}$$

is a well-defined contraction. In particular, δ^* and ϵ^* can be chosen such that

$$\|\mathbf{K}_\epsilon(\mathbf{\Phi}) - \mathbf{K}_\epsilon(\mathbf{\Psi})\|_{s,k} \leq \frac{1}{2}\|\mathbf{\Phi} - \mathbf{\Psi}\|_{s,k} \tag{3.9}$$

holds for every $|\epsilon| \leq \epsilon^*$ and all $\mathbf{\Phi}, \mathbf{\Psi} \in \mathcal{B}_{\delta^*}$.

Proof. Using the fact that $\mathbf{L}_0 : \mathcal{D}(\mathbf{L}_0) \subset \mathcal{H}^{s,k} \to \mathcal{H}^{s,k}$ is bounded invertible together with the embedding (2.11) there exists a constant $C > 0$, such that

$$\|\mathbf{K}_\epsilon(\mathbf{\Phi})\|_{s,k} \leq C\left(\|\mathbf{V}_\epsilon(\mathbf{\Psi}_0)\Phi\|_{s,k+1} + \|\mathcal{R}_\epsilon(\mathbf{\Psi}_0)\|_{s,k+1} + \|\widetilde{\mathbf{N}}_\epsilon(\mathbf{\Phi})\|_{s,k+1}\right)$$

holds for every $\mathbf{\Phi} \in \mathcal{B}_{\delta^*}$ and every $|\epsilon| \leq \epsilon^*$ assuming that $0 < \delta^* \leq \min\{1, \delta_0^*\}$ and $0 < \epsilon^* \leq \epsilon_0$. We then have by the previous results:

$$\|\mathbf{V}_\epsilon(\mathbf{\Psi}_0)\mathbf{\Phi}\|_{s,k+1} + \|\mathcal{R}_\epsilon(\mathbf{\Psi}_0)\|_{s,k+1} + \|\widetilde{\mathbf{N}}_\epsilon(\mathbf{\Phi})\|_{s,k+1} \leq (C_1\,\epsilon^* + C_2\,\epsilon^*/\delta^* + C_3\,\delta^*)\;\delta^*$$

for constants $C_j > 0$, $j \in \{1,2,3\}$ independent of ϵ^* and δ^*. Now choose $\delta^* < 1/(C_3\,C)$ and $\epsilon^* = \epsilon^*(\delta^*)$ sufficiently small to guarantee $C(C_1\,\epsilon^* + C_2\,\epsilon^*/\delta^* + C_3\,\delta^*) \leq 1$. Then

$$\|\mathbf{K}_\epsilon(\mathbf{\Phi})\|_{s,k} \leq \delta^*$$

for all $\mathbf{\Phi} \in \mathcal{B}_{\delta^*}$ so that $\mathbf{K}_\epsilon$ is well-defined on $\mathcal{B}_{\delta^*}$. For the contraction property we take $\mathbf{\Phi}, \mathbf{\Psi} \in \mathcal{B}_{\delta^*}$ and get

$$\begin{aligned}\|\mathbf{K}_\epsilon(\mathbf{\Phi}) - \mathbf{K}_\epsilon(\mathbf{\Psi})\|_{s,k} &\leq C\left(\|\mathbf{V}_\epsilon(\mathbf{\Psi}_0)\ (\mathbf{\Phi} - \mathbf{\Psi})\|_{s,k+1} + \|\widetilde{\mathbf{N}}_\epsilon(\mathbf{\Phi}) - \widetilde{\mathbf{N}}_\epsilon(\mathbf{\Psi})\|_{s,k+1}\right) \\ &\leq C\, C_1\, \epsilon^* \, \|\mathbf{\Phi} - \mathbf{\Psi}\|_{s,k} + C\, C_4\, \delta^* \, \|\mathbf{\Phi} - \mathbf{\Psi}\|_{s,k},\end{aligned}$$

for $C_4 > 0$. Now choose $\delta^* > 0$ possibly even smaller so that $C\, C_4\, \delta^* < 1/2$ is satisfied. Hence the claim follows by adjusting ϵ^* and requiring in addition that $C\, C_1\, \epsilon^* + C\, C_4\, \delta^* \leq 1/2$. □

This result immediately implies the existence part in the following statement.

Proposition 3.7. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (2.20). Let δ^* and ϵ^* be as in Proposition 3.6. Then for every $|\epsilon| \leq \epsilon^*$ there is a unique $\mathbf{\Phi}_\epsilon \in \mathcal{B}_{\delta^*} \cap \mathcal{D}(\mathbf{L}_0)$ satisfying*

$$\mathbf{L}_0\mathbf{\Phi}_\epsilon + \mathbf{V}_\epsilon(\mathbf{\Psi}_0)\mathbf{\Phi}_\epsilon + \mathcal{R}_\epsilon(\mathbf{\Psi}_0) + \widetilde{\mathbf{N}}_\epsilon(\mathbf{\Phi}_\epsilon) = 0 \tag{3.10}$$

in $\mathcal{H}^{s,k}_r$. Furthermore, $\mathbf{\Phi}_\epsilon$ depends Lipschitz continuously on ϵ in the sense that

$$\|\mathbf{\Phi}_\epsilon - \mathbf{\Phi}_\kappa\|_{s,k} \lesssim |\epsilon - \kappa|$$

holds for all $|\epsilon|, |\kappa| \leq \epsilon^*$. Finally, $\mathbf{\Phi}_\epsilon \in \bigcap_{\ell \geq s} \dot{H}^\ell(\mathbb{R}^n) \times \dot{H}^{\ell-1}(\mathbb{R}^n) \subset C^\infty_r(\mathbb{R}^n) \times C^\infty_r(\mathbb{R}^n)$ and $\mathbf{\Phi}_\epsilon$ solves the equation in a classical sense, i.e., $\mathbf{L}_0\mathbf{\Phi}_\epsilon = \widetilde{\mathbf{L}}_0\mathbf{\Phi}_\epsilon$.

Proof. The existence of a unique $\mathbf{\Phi}_\epsilon \in \mathcal{B}_{\delta^*} \cap \mathcal{D}(\mathbf{L}_0)$ satisfying the Eq. (3.10) is a direct consequence of Proposition 3.6 and the contraction mapping principle. For the Lipschitz bound we take $|\epsilon|, |\kappa| \leq \epsilon^*$ and use (3.9) to obtain

$$\|\mathbf{\Phi}_\epsilon - \mathbf{\Phi}_\kappa\|_{s,k} = \|\mathbf{K}_\epsilon(\mathbf{\Phi}_\epsilon) - \mathbf{K}_\kappa(\mathbf{\Phi}_\kappa)\|_{s,k} \leq \frac{1}{2}\|\mathbf{\Phi}_\epsilon - \mathbf{\Phi}_\kappa\|_{s,k} + \|\mathbf{K}_\epsilon(\mathbf{\Phi}_\kappa) - \mathbf{K}_\kappa(\mathbf{\Phi}_\kappa)\|_{s,k}$$

from which $\|\mathbf{\Phi}_\epsilon - \mathbf{\Phi}_\kappa\|_{s,k} \leq 2\|\mathbf{K}_\epsilon(\mathbf{\Phi}_\kappa) - \mathbf{K}_\kappa(\mathbf{\Phi}_\kappa)\|_{s,k}$ follows. Now, for every $\mathbf{\Phi} \in \mathcal{B}_{\delta^*}$

$$\begin{aligned}\|\mathbf{K}_\epsilon(\mathbf{\Phi}) - \mathbf{K}_\kappa(\mathbf{\Phi})\|_{s,k} &\lesssim \|\mathcal{R}_\epsilon(\mathbf{\Psi}_0) - \mathcal{R}_\kappa(\mathbf{\Psi}_0)\|_{s,k} + \|(\mathbf{V}_\epsilon(\mathbf{\Psi}_0) - \mathbf{V}_\kappa(\mathbf{\Psi}_0))\,\mathbf{\Phi}\|_{s,k} \\ &\quad + \|\widetilde{\mathbf{N}}_\epsilon(\mathbf{\Phi}) - \widetilde{\mathbf{N}}_\kappa(\mathbf{\Phi})\|_{s,k} \lesssim |\epsilon - \kappa|\end{aligned}$$

due to Lemma 3.2 and Lemma 3.3.

For the regularity we use the Sobolev embedding from (2.13) and show that $\mathbf{\Phi}_\epsilon \in \mathcal{H}^{s,\ell}_r$ for every $\ell \geq k$. Since $\mathbf{\Phi}_\epsilon \in \mathcal{H}^{s,k}_r$, Lemma 3.2 and Lemma 3.3 yield $\mathbf{V}_\epsilon(\mathbf{\Psi}_0)\mathbf{\Phi}_\epsilon + \mathcal{R}_\epsilon(\mathbf{\Psi}_0) + \widetilde{\mathbf{N}}_\epsilon(\mathbf{\Phi}_\epsilon) \in \mathcal{H}^{s,k+1}_r$. We now want to use Eq. (3.8) to deduce that $\mathbf{\Phi}_\epsilon$ must belong to $\mathcal{H}^{s,k+1}_r$. For this we will at this particular point not suppress the dependency of $\mathbf{L}_0$ on the parameters s and k and will therefore denote by $\mathbf{L}_{0,s_1,s_2}$ the closure of $\widetilde{\mathbf{L}}_{0,s_1,s_2}$ in $\mathcal{H}^{s_1,s_2}_r$. Eq. (3.8) then writes

$$\mathbf{\Phi}_\epsilon = (-\mathbf{L}_{0,s,k})^{-1}\left(\mathbf{V}_\epsilon(\mathbf{\Psi}_0)\mathbf{\Phi}_\epsilon + \mathcal{R}_\epsilon(\mathbf{\Psi}_0) + \widetilde{\mathbf{N}}_\epsilon(\mathbf{\Phi}_\epsilon)\right)$$

and all what is left to show is that if we restrict the resolvent of $\mathbf{L}_{0,s,k}$ (in 0) to the subspace $\mathcal{H}^{s,k+1}_r$ then this operator equals the resolvent (in 0) of the operator $\mathbf{L}_{0,s,k+1}$. But this follows from the fact that $\mathbf{L}_{0,s,k+1}$ is a restriction of $\mathbf{L}_{0,s,k}$ to the subspace $\mathcal{H}^{s,k+1}_r$ (see for example [12], p.21, Lemma 3.5 or [24], p.29, Lemma C.1.). Via repeating this argument we conclude inductively that $\mathbf{\Phi}_\epsilon$ belongs to $\mathcal{H}^{s,\ell}_r$ for every $\ell \geq k$ and is therefore smooth by (2.13).

Finally, we show that $\mathbf{L}_0$ acts as a classical differential operator on $\mathbf{\Phi}_\epsilon$. Since $\mathbf{\Phi}_\epsilon$ belongs to $\mathcal{D}(\mathbf{L}_0)$ there exists a sequence $(\mathbf{u}_j)_{j\in\mathbb{N}} \subset C^\infty_{c,r}(\mathbb{R}^n) \times C^\infty_{c,r}(\mathbb{R}^n)$ such that

$$\mathbf{u}_j \to \mathbf{\Phi}_\epsilon \quad \text{and} \quad \widetilde{\mathbf{L}}_0\mathbf{u}_j \to \mathbf{L}_0\mathbf{\Phi}_\epsilon \quad \text{in} \quad \mathcal{H}^{s,k}_r \quad \text{for} \quad j\to\infty.$$

Now, by the embedding of $\mathcal{H}^{s,k}_r$ into $C^2(\mathbb{R}^n)\times C^1(\mathbb{R}^n)$ we infer by the above convergence of $\mathbf{u}_j$ towards $\mathbf{\Phi}_\epsilon$ that each term in $\widetilde{\mathbf{L}}_0\mathbf{u}_j$ converges pointwise towards $\widetilde{\mathbf{L}}_0\mathbf{\Phi}_\epsilon$. Again by Sobolev embedding, the limit $\widetilde{\mathbf{L}}_0\mathbf{u}_j \to \mathbf{L}_0\mathbf{\Phi}_\epsilon$ holds pointwise, which implies the claim. This last part of the proof shows in fact that $\widetilde{\mathbf{L}}_0$ acts as a classical differential operator on any element in its domain, irrespective of higher regularity. ☐

Corollary 3.8. *Let $n\ge 5$ and $(s,k)\in\mathbb{R}\times\mathbb{N}$ satisfy* (2.20). *Let δ^* and ϵ^* be as in Proposition 3.7. For $\epsilon\in\mathbb{R}$ with $|\epsilon|\le\epsilon^*$ let $\mathbf{\Phi}_\epsilon\in\mathcal{B}_{\delta^*}$ be the corresponding solution of Eq.* (3.10). *Then*

$$\mathbf{\Psi}_\epsilon := \mathbf{\Psi}_0 + \mathbf{\Phi}_\epsilon \in \bigcap_{\ell\ge s} \dot H^\ell_r(\mathbb{R}^n)\times \dot H^{\ell-1}_r(\mathbb{R}^n) \subset C^\infty_r(\mathbb{R}^n)\times C^\infty_r(\mathbb{R}^n)$$

solves Eq. (2.9) *in a classical sense. Furthermore,*

$$\|\mathbf{\Psi}_\epsilon - \mathbf{\Psi}_\kappa\|_{s,k} \lesssim |\epsilon-\kappa| \tag{3.11}$$

for all $|\epsilon|,|\kappa|\le\epsilon^*$.

Finally, we characterize the decay of the solution $\mathbf{\Psi}_\epsilon = (\psi_{\epsilon,1},\psi_{\epsilon,2})$ at infinity. Corollary 3.8 and the generalized Strauss inequality (2.17) imply that $|\psi_{\epsilon,1}(\xi)| \lesssim \langle\xi\rangle^{\frac{n}{2}-s}$ for $s>\frac{n}{2}-1$ and hence the decay rate is strictly less than 1. In the following, we provide a more refined analysis based on ODE arguments to improve this. This is along the lines of [27].

Proposition 3.9. *Let $n\ge 5$, $\frac{n}{2}-1 < s < \frac{n-1}{2}$ and $|\epsilon|\le\epsilon^*$. Suppose that $\mathbf{\Psi}=(\psi_1,\psi_2)\in \bigcap_{\ell\ge s}\dot H^\ell_r(\mathbb{R}^n)\times\dot H^{\ell-1}_r(\mathbb{R}^n)\subset C^\infty_r(\mathbb{R}^n)\times C^\infty_r(\mathbb{R}^n)$ is a classical solution to Eq.* (2.9). *Then its components satisfy for every $\beta\in\mathbb{N}_0^n$,*

$$|\partial^\beta\psi_1(\xi)| \lesssim_\epsilon \langle\xi\rangle^{-1-|\beta|} \quad \text{and} \quad |\partial^\beta\psi_2(\xi)| \lesssim_\epsilon \langle\xi\rangle^{-2-|\beta|} \tag{3.12}$$

for all $\xi\in\mathbb{R}^n$. Furthermore, the limits $\lim_{|\xi|\to\infty}|\xi|\psi_1(\xi)$ and $\lim_{|\xi|\to\infty}|\xi|^2(\psi_1(\xi)+\Lambda\psi_1(\xi))$ exist.

Proof. Since ψ_1 and ψ_2 are smooth radial functions we know that there exist $\widetilde\psi_1,\widetilde\psi_2\in C^\infty_e[0,\infty)$ such that $\psi_1(\xi)=\widetilde\psi_1(|\xi|)$ and $\psi_2(\xi)=\widetilde\psi_2(|\xi|)$ for all $\xi\in\mathbb{R}^n$. From (2.3) we obtain $\widetilde\psi_2(\rho)=\widetilde\psi_1(\rho)+\rho\,\widetilde\psi_1'(\rho)$ for every $\rho\ge 0$ so that $\widetilde\psi_1$ solves the following ordinary differential equation:

$$(1-\rho^2)u''(\rho) + \left(\frac{n-1}{\rho} - 4\rho\right)u'(\rho) - 2u(\rho) + \frac{n-3}{\rho^3}\eta_\epsilon(\rho\,u(\rho)) = 0. \tag{3.13}$$

We now argue as in [27], p.1388, Theorem 3.1. With (3.1) the equation can be written as

$$(1-\rho^2)u''(\rho) + \left(\frac{n-1}{\rho} - 4\rho\right)u'(\rho) - 2u(\rho) + u^3(\rho)F_\epsilon(\rho\,u(\rho)) = 0 \tag{3.14}$$

for an $F_\epsilon\in C^\infty_e(\mathbb{R})$ which is uniformly bounded in ρ and $|\epsilon|\le\epsilon^*$. After substituting $r=\log(\rho)$ in (3.14) we obtain the following equation for $v(r)=u(e^r)$

$$v''(r) + 3v'(r) + 2v(r) - v^3(r)F_\epsilon(e^r\,v(r)) = e^{-2r}(v''(r) + (n-2)v'(r)). \tag{3.15}$$

We will now show that v behaves asymptotically like a solution to the linear autonomous part from equation (3.15). For proving this it is essential to write (3.15) in the following way

$$v''(r) + b(r)\, v'(r) + c(r)\, v(r) = 0 \tag{3.16}$$

with

$$b(r) = \frac{3-(n-2)e^{-2r}}{1-e^{-2r}} \quad \text{and} \quad c(r) = \frac{2 - v^2(r)\, F_\epsilon(e^r\, v(r))}{1-e^{-2r}}.$$

Now we want to investigate the asymptotic behavior of b and c at infinity. It is immediately evident that one has

$$|b(r) - 3| \lesssim e^{-2r}.$$

The asymptotic behavior of c now follows from the uniform boundedness of F_ϵ and the fact that we can already extract some decay out of v due to the generalized Strauss inequality (2.17). Indeed, by this inequality we know that ψ_1 has a decaying rate of at least $\gamma := \frac{n}{2} - s$. Since s satisfies (2.20) we note that γ is strictly greater than $1/2$. Therefore, we obtain for large r due to the boundedness of F_ϵ

$$|c(r) - 2| \lesssim e^{-2\gamma r}. \tag{3.17}$$

We now define

$$U(r) = \begin{pmatrix} 2e^{-r} - e^{-2r} & e^{-r} - e^{-2r} \\ 2e^{-2r} - 2e^{-r} & 2e^{-2r} - e^{-r} \end{pmatrix} \quad \text{and} \quad V(r) = \begin{pmatrix} v(r) \\ v'(r) \end{pmatrix}$$

and show that $W(r) := U(-r)V(r)$ converges pointwise for $r \to \infty$. Using (3.16) it follows that V solves

$$V'(r) = \begin{pmatrix} 0 & 1 \\ -c(r) & -b(r) \end{pmatrix} V(r).$$

Together with the fact that U solves

$$U'(r) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} U(r), \quad U(0) = I$$

we obtain

$$\frac{d}{dr}\left[U(-r)V(r)\right] = U(-r) \begin{pmatrix} 0 & 0 \\ 2 - c(r) & 3 - b(r) \end{pmatrix} V(r).$$

With that we obtain $W'(r) = A(r)W(r)$ for

$$A(r) = U(-r) \begin{pmatrix} 0 & 0 \\ 2 - c(r) & 3 - b(r) \end{pmatrix} U(r).$$

From the differential equation that $W(r)$ solves we obtain for all $r \geq r_0$

$$W(r) = W(r_0) + \int_{r_0}^{r} A(t)W(t)\, dt \tag{3.18}$$

and can therefore deduce with Grönwall's inequality

$$\|W(r)\| \leq \|W(r_0)\| \exp\left(\int_{r_0}^{r} \|A(t)\|\ dt \right) \tag{3.19}$$

where we use $\|\cdot\|$ for an arbitrary matrix norm. We now obtain

$$\|A(r)\| \leq \|U(-r)\| \max\{|b(r)-3|, |c(r)-2|\} \|U(r)\| \lesssim e^{-(2\gamma-1)r}$$

so that A is integrable at infinity. In particular, we can then deduce the boundedness of W from (3.19). Therefore, we obtain from (3.18) the convergence of $W(r)$ as r goes to infinity. If we now denote this limit by (w_1, w_2) we get from the definition of U and V the following two asymptotic behaviors

$$\begin{aligned}
[2v(r)+v'(r)]\, e^{r} - [v(r)+v'(r)]\, e^{2r} &\to w_1, \\
[2v(r)+2v'(r)]\, e^{2r} - [2v(r)+v'(r)]\, e^{r} &\to w_2.
\end{aligned}$$

Comparing these two lines gives us

$$\begin{aligned}
[2v(r)+v'(r)]\, e^{r} &\to 2w_1 + w_2, \\
[v(r)+v'(r)]\, e^{2r} &\to w_1 + w_2.
\end{aligned}$$

In particular, $[v(r)+v'(r)]\, e^{r} \to 0$ and therefore $v(r)e^{r} \to 2w_1 + w_2$ as well as $v'(r)e^{r} \to -2w_1 - w_2$. Therefore, we have shown the following estimates for large r:

$$|v(r)| \leq C\, e^{-r}, \quad |v'(r)| \leq C\, e^{-r}, \quad |v(r)+v'(r)| \leq C\, e^{-2r}$$

and that the limits $\lim\limits_{r\to\infty} v(r)e^{r}$ and $\lim\limits_{r\to\infty} (v(r)+v'(r))e^{2r}$ exist. Transforming everything back into the original variable gives us

$$|\widetilde{\psi}_1(\rho)| \leq C\, \rho^{-1}, |\widetilde{\psi}_1'(\rho)| \leq C\, \rho^{-2}, \quad |\widetilde{\psi}_2(\rho)| \leq C\, \rho^{-2}$$

for all large enough $\rho > 0$ and the limits

$$\lim_{|\xi|\to\infty} |\xi| \psi_1(\xi), \qquad \lim_{|\xi|\to\infty} |\xi|^2 (\psi_1(\xi) + \Lambda \psi_1(\xi))$$

exist.

The decay of the higher derivatives of ψ_1 can be shown inductively by using that $\widetilde{\psi}_1$ solves equation (3.14) and by the fact that all of the higher derivatives are initially bounded via the generalized Strauss inequality.

The decay of the higher derivatives of the second component ψ_2 can be shown as follows: First, we substitute $\rho = \frac{r}{T-t}$ in (3.13) and get the following equation for $\widetilde{\psi}_1$

$$\begin{aligned}
0 = \left(1 - \frac{r^2}{(T-t)^2}\right) u''\left(\frac{r}{T-t}\right) + \left(\frac{(n-1)(T-t)}{r} - \frac{4r}{T-t}\right) u'\left(\frac{r}{T-t}\right) \\
- 2u\left(\frac{r}{T-t}\right) + \frac{(n-3)(T-t)^3}{r^3} \eta_\epsilon\left(\frac{r}{T-t} u\left(\frac{r}{T-t}\right)\right).
\end{aligned} \tag{3.20}$$

If we now differentiate this equation with respect to t we get

$$
\begin{aligned}
0 =&\frac{r}{(T-t)^2}\left(1-\frac{r^2}{(T-t)^2}\right)\widetilde{\psi}_1'''\left(\frac{r}{T-t}\right)-\frac{2r^2}{(T-t)^3}\widetilde{\psi}_1''\left(\frac{r}{T-t}\right)\\
&-\left(\frac{d+1}{r}+\frac{4r}{(T-t)^2}\right)\widetilde{\psi}_1'\left(\frac{r}{T-t}\right)+\frac{r}{(T-t)^2}\left(\frac{(d+1)(T-t)}{r}-\frac{4r}{T-t}\right)\widetilde{\psi}_1''\left(\frac{r}{T-t}\right)\\
&-\frac{2r}{(T-t)^2}\widetilde{\psi}_1'\left(\frac{r}{T-t}\right)-\frac{3(d-1)(T-t)^2}{r^3}\eta_\epsilon\left(\frac{r}{T-t}\widetilde{\psi}_1\left(\frac{r}{T-t}\right)\right)\\
&+\frac{(d-1)(T-t)^3}{r^3}\left(\frac{r}{(T-t)^2}\widetilde{\psi}_1\left(\frac{r}{T-t}\right)+\frac{r^2}{(T-t)^3}\widetilde{\psi}_1'\left(\frac{r}{T-t}\right)\right)\eta_\epsilon'\left(\frac{r}{T-t}\widetilde{\psi}_1\left(\frac{r}{T-t}\right)\right).
\end{aligned}
$$

Then we multiply with $T-t$ and write the equation again in the ρ variable

$$
\begin{aligned}
0 =&\rho\left(1-\rho^2\right)\widetilde{\psi}_1'''(\rho)-2\rho^2\widetilde{\psi}_1''(\rho)-\left(\frac{d+1}{\rho}+4\rho\right)\widetilde{\psi}_1'(\rho)\\
&+\rho\left(\frac{(d+1)}{\rho}-4\rho\right)\widetilde{\psi}_1''(\rho)-2\rho\widetilde{\psi}_1'(\rho)\\
&+\frac{(d-1)}{\rho^3}\left(\rho\widetilde{\psi}_1(\rho)+\rho^2\widetilde{\psi}_1'(\rho)\right)\eta_\epsilon'(\rho\widetilde{\psi}_1(\rho)-\frac{3(d-1)}{\rho^3}\eta_\epsilon(\rho\widetilde{\psi}_1(\rho)).
\end{aligned}
$$

Using that $\widetilde{\psi}_1$ solves

$$
(1-\rho^2)u''(\rho)+\left(\frac{d+1}{\rho}-4\rho\right)u'(\rho)-2u(\rho)+\frac{d-1}{\rho^3}\eta_\epsilon(\rho\, u(\rho))=0
$$

we notice that $\widetilde{\psi}_2=\widetilde{\psi}_1+\rho\,\widetilde{\psi}_1'$ solves the following linear second order ODE

$$
(1-\rho^2)w''(\rho)+\left(\frac{n-1}{\rho}-6\rho\right)w'(\rho)-6w(\rho)+\frac{n-3}{\rho^2}\eta_\epsilon'(\rho\,\widetilde{\psi}_1)w(\rho)=0. \tag{3.21}
$$

As we will see later on, this is precisely the eigenvalue equation of the linearization of (2.3) around $\mathbf{\Psi}_\epsilon$ to the eigenvalue $\lambda=1$ (i.e. one also obtains this equation by differentiating (3.20) with respect to T). But for now it is enough to notice that this is a linear second order ODE with a regular singularity at infinity. Therefore, we can apply the method of Frobenius, [37], p.119, Theorem 4.5, to obtain a fundamental system of this equation for large enough $\rho>0$

$$
w_1(\rho)=\rho^{-3}\,h_1(\rho^{-1}),\quad w_2(\rho)=\rho^{-2}h_2(\rho^{-1})+c\log(\rho)\rho^{-3}h_1(\rho^{-1}) \tag{3.22}
$$

where h_1 and h_2 are analytic around zero with $\lim_{\rho\to\infty}h_i(\rho^{-1})=1$. Since $\widetilde{\psi}_2$ can be written as a linear combination of w_1 and w_2 for large ρ one immediately obtains the desired decay. □

By now reversing the coordinate transformations we are ready to prove the first main result of this paper.

Proof of Theorem 1.2. Let $d\geq 3$. Set $n:=d+2$ and let $(s,k)\in\mathbb{R}\times\mathbb{N}$ satisfy (2.20). Take δ^* and $\epsilon^*(\delta^*)$ from Proposition 3.7. For $\epsilon\in\mathbb{R}$ with $|\epsilon|\leq\epsilon^*$ let $\mathbf{\Psi}_\epsilon=(\psi_{\epsilon,1},\psi_{\epsilon,2})\in C_r^\infty(\mathbb{R}^n)\times C_r^\infty(\mathbb{R}^n)$ be the function obtained in Corollary 3.8. Then $\psi_{\epsilon,1}$ solves

$$
(-\Delta+\Lambda^2+3\Lambda+2)\psi_{\epsilon,1}=N_\epsilon(\psi_{\epsilon,1}) \tag{3.23}
$$

and the radial representative $\widetilde{\psi}_{\epsilon,1}$ of $\psi_{\epsilon,1}$ satisfies the nonlinear ODE (3.13). Now we define

$$f_\epsilon(\rho) := \rho\widetilde{\psi}_{\epsilon,1}(\rho) = \rho\widetilde{\psi}_{0,1}(\rho) + \rho\widetilde{\phi}_{\epsilon,1}(\rho)$$

where $\widetilde{\phi}_{\epsilon,1}$ denotes the radial representative of the first component of $\mathbf{\Phi}_\epsilon$. By setting $\phi_\epsilon(\rho) := \rho\widetilde{\phi}_{\epsilon,1}$ we have $f_\epsilon = f_0 + \phi_\epsilon$ and f_ϵ solves

$$(1-\rho^2)f''(\rho) + \left(\frac{d-1}{\rho} - 2\rho\right) f'(\rho) + \frac{d-1}{\rho^2} w_\epsilon(f(\rho))w_\epsilon'(f(\rho)) = 0,$$

for all $\rho \in (0,\infty)$. Hence,

$$u_\epsilon^T(t,r) := f_\epsilon\left(\frac{r}{T-t}\right)$$

provides a solution to Eq. (1.7). For the derivative we obtain

$$|\partial_r u_\epsilon^T(t,0)| = (T-t)^{-1}|\widetilde{\psi}_{\epsilon,1}(0)|$$

for $0 \le t < T$. In view of Lemma 3.5 and our previous assumptions we infer that

$$|\widetilde{\phi}_{\epsilon,1}(0)| < \widetilde{\psi}_{0,1}(0),$$

and thus $\widetilde{\psi}_{\epsilon,1}(0) \neq 0$. In particular, the derivative of u_ϵ^T blows up as $t \to T^-$. Finally the Lipschitz estimates follow from Proposition 3.7 and Sobolev embedding and the decay properties for f_ϵ are an obvious consequence of Proposition 3.9 □

4. Stability analysis

The aim of this section is to analyze the stability properties of the blowup solution $u_\epsilon^T \in C^\infty([0,T)\times[0,\infty))$ constructed in Theorem 1.2. Again, the starting point will be Eq. (2.1) for which we now consider the Cauchy problem

$$\partial_t^2 v - \Delta_x v = \frac{n-3}{|x|^3}\left(|x|v - w_\epsilon(|x|v)w_\epsilon'(|x|v)\right) \tag{4.1}$$

where the initial data $v(0,x) = v_{0,\epsilon}(x)$ and $\partial_t v(0,x) = v_{1,\epsilon}(x)$, $x \in \mathbb{R}^n$ are given by

$$(v_{0,\epsilon}, v_{1,\epsilon}) := (v_\epsilon^1(0,\cdot), \partial_t v_\epsilon^1(0,\cdot)) + (\varphi_0, \varphi_1) \tag{4.2}$$

and v_ϵ^1 denotes the corresponding blowup solution v_ϵ^T with blowup time $T = 1$,

$$v_\epsilon^T(t,x) = \frac{1}{T-t}\psi_\epsilon\left(\frac{x}{T-t}\right), \quad \psi_\epsilon(\xi) = |\xi|^{-1}f_\epsilon(|\xi|).$$

By rewriting the problem as a first order system in similarity coordinates, Eq. (4.1) transforms into Eq. (2.3), see Section 2. By construction, the blowup solution v_ϵ^T corresponds to the static solution $\mathbf{\Psi}_\epsilon$ of Eq. (2.3),

$$\mathbf{\Psi}_\epsilon = \begin{pmatrix}\psi_{\epsilon,1}\\ \psi_{\epsilon,2}\end{pmatrix}, \text{ with } \psi_{\epsilon,1} = \psi_\epsilon \text{ and } \psi_{\epsilon,2} = \psi_{\epsilon,1} + \Lambda\psi_{\epsilon,1}. \tag{4.3}$$

Consequently, the initial value problem (4.1)-(4.2) can be rephrased as

$$\begin{cases} \partial_\tau \Psi(\tau) = \widetilde{\mathbf{L}}\Psi(\tau) + \mathbf{N}_\epsilon(\Psi(\tau)), \\ \Psi(0) = \mathbf{\Psi}_\epsilon^{\mathbf{T}} + \varphi^{\mathbf{T}}, \end{cases} \tag{4.4}$$

for $\mathbf{\Psi}_\epsilon^{\mathbf{T}} := (T\psi_{\epsilon,1}(T\cdot), T^2\psi_{\epsilon,2}(T\cdot))$ and $\varphi^{\mathbf{T}} := (T\varphi_0(T\cdot), T^2\varphi_1(T\cdot))$ the perturbation of the original initial data. We remark that $\widetilde{\mathbf{L}}$ is still the wave operator in similarity coordinates as we introduced it in (2.4) and the nonlinearity $\mathbf{N}_\epsilon$ is defined as in Eq. (2.5).
Assuming that the solution to Eq. (4.4) will be a small perturbation of $\mathbf{\Psi}_\epsilon$, we insert the ansatz $\Psi(\tau) = \mathbf{\Psi}_\epsilon + \Phi_\epsilon(\tau)$ and get an evolution equation for the (time-dependent) perturbation Φ_ϵ,

$$\partial_\tau \Phi_\epsilon(\tau) = \widetilde{\mathbf{L}}\Phi_\epsilon(\tau) + \mathbf{N}_\epsilon(\mathbf{\Psi}_\epsilon + \Phi_\epsilon(\tau)) - \mathbf{N}_\epsilon(\mathbf{\Psi}_\epsilon).$$

A Taylor expansion of $\mathbf{N}_\epsilon$ around $\mathbf{\Psi}_\epsilon$ now leads us to the central evolution equation of this section

$$\begin{cases} \partial_\tau \Phi_\epsilon(\tau) = \widetilde{\mathbf{L}}_\epsilon\Phi_\epsilon(\tau) + \widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\tau)), \\ \Phi_\epsilon(0) = \varphi^{\mathbf{T}} + \mathbf{\Psi}_\epsilon^{\mathbf{T}} - \mathbf{\Psi}_\epsilon. \end{cases} \tag{4.5}$$

The linear part $\widetilde{\mathbf{L}}_\epsilon := \widetilde{\mathbf{L}} + \mathbf{L}'_\epsilon$ consists of the wave operator $\widetilde{\mathbf{L}}$ and of the potential $\mathbf{L}'_\epsilon$ which is given by

$$\mathbf{L}'_\epsilon \mathbf{u} = \begin{pmatrix} 0 \\ V_\epsilon(\psi_{\epsilon,1})\, u_1 \end{pmatrix} \qquad \text{for} \qquad V_\epsilon(\psi_{\epsilon,1})(\xi) = \frac{n-3}{|\xi|^2}\eta'_\epsilon(|\xi|\psi_{\epsilon,1}(\xi)). \tag{4.6}$$

The remaining nonlinearity $\widehat{\mathbf{N}}_\epsilon$ is of the form

$$\widehat{\mathbf{N}}_\epsilon(\mathbf{u}) = \begin{pmatrix} 0 \\ \widehat{N}_\epsilon(u_1) \end{pmatrix}, \tag{4.7}$$

with

$$\widehat{N}_\epsilon(u_1)(\xi) = \frac{n-3}{|\xi|^3}\left(\eta_\epsilon(|\xi|(\psi_{\epsilon,1}(\xi) + u_1(\xi))) - \eta_\epsilon(|\xi|\psi_{\epsilon,1}(\xi)) - \eta'_\epsilon(|\xi|\psi_{\epsilon,1}(\xi))|\xi|u_1(\xi)\right). \tag{4.8}$$

Notice that the only trace of the parameter T is in the initial data and that the here defined linearized operator $\widetilde{\mathbf{L}}_\epsilon$ coincides for $\epsilon = 0$ with the one we introduced in Section 2.4.
To now find a solution to Eq. (4.5) we take the semigroup approach.

4.1. **Perturbations of the free evolution for small parameters.** The free evolution generated by $\widetilde{\mathbf{L}}$ is well understood, see the results summarized in Section 2.4. For the linearized flow, the following properties of the perturbation $\mathbf{L}'_\epsilon$ are crucial. For the rest of the paper let ϵ^* denote the constant determined by Proposition 3.6.

Proposition 4.1. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy* (2.20). *For every $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \epsilon^*$ the operator $\mathbf{L}'_\epsilon : \mathcal{H}_r^{s,k} \to \mathcal{H}_r^{s,k}$ is compact. Furthermore, the family of operators $\mathbf{L}'_\epsilon$ is Lipschitz continuous in ϵ, i.e., the inequality*

$$\|\mathbf{L}'_\epsilon \mathbf{u} - \mathbf{L}'_\kappa \mathbf{u}\|_{s,k} \lesssim |\epsilon - \kappa|\, \|\mathbf{u}\|_{s,k}$$

holds for all $\mathbf{u} \in \mathcal{H}_r^{s,k}$ and all $|\epsilon|, |\kappa| \leq \epsilon^*$.

Proof. Due to the decay of the blowup solution, see Proposition 3.9, $V_\epsilon(\psi_{\epsilon,1})$ satisfies the assumptions of Lemma 5.1 in [23] from which the compactness of $\mathbf{L}'_\epsilon$ follows. For the Lipschitz estimates, we write $V_\epsilon(\psi_{\epsilon,1})$ with (3.2) in the following way

$$V_\epsilon(\psi_{\epsilon,1})(\xi) = (n-3)\psi_{\epsilon,1}(\xi)^2 \int_0^1 x \int_0^1 \eta_\epsilon'''(|\xi|\psi_{\epsilon,1}(\xi)xy)dy\, dx \tag{4.9}$$

and conclude by applying the Schauder estimate from Proposition A.1 using the Lipschitz continuity of $\psi_{\epsilon,1}$.

□

By the bounded perturbation theorem we therefore get the following statement, where we first equip $\widetilde{\mathbf{L}}_\epsilon$ with the domain $\mathcal{D}(\widetilde{\mathbf{L}}_\epsilon) = \mathcal{D}(\widetilde{\mathbf{L}})$, see Proposition 2.5.

Proposition 4.2. *Let $n \geq 5$ and $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy (2.20). For every $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \epsilon^*$ the operator $\widetilde{\mathbf{L}}_\epsilon$ is closable and its closure $\mathbf{L}_\epsilon = \mathbf{L}+\mathbf{L}'_\epsilon$ with $\mathcal{D}(\mathbf{L}_\epsilon) = \mathcal{D}(\mathbf{L}) = \mathcal{D}(\mathbf{L}_0)$ generates a strongly continuous semigroup $(\mathbf{S}_\epsilon(\tau))_{\tau\geq 0}$ of bounded operators on $\mathcal{H}^{s,k}_r$. Furthermore,*

$$\|\mathbf{L}_\epsilon \mathbf{u} - \mathbf{L}_\kappa \mathbf{u}\|_{s,k} \lesssim |\epsilon-\kappa|\, \|\mathbf{u}\|_{s,k}$$

holds for all $\mathbf{u} \in \mathcal{H}^{s,k}_r$ and all $|\epsilon|, |\kappa| \leq \epsilon^*$.

As in the case $\epsilon = 0$ we do not expect decay of the semigroup due to an exponential instability induced by the time translation symmetry of the original problem. More precisely, we have the following result.

Lemma 4.3. *Let $n \geq 5$, $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy (2.20) and $|\epsilon| \leq \epsilon^*$. We then have*

$$\mathbf{L}_\epsilon\, \mathbf{g}_\epsilon = \mathbf{g}_\epsilon, \tag{4.10}$$

where $\mathbf{g}_\epsilon \in C^\infty_r(\mathbb{R}^n) \times C^\infty_r(\mathbb{R}^n) \cap \mathcal{D}(\mathbf{L}_\epsilon)$ is defined as

$$\mathbf{g}_\epsilon := \begin{pmatrix} g_{\epsilon,1} \\ g_{\epsilon,2} \end{pmatrix} := \begin{pmatrix} \psi_{\epsilon,1} + \Lambda\psi_{\epsilon,1} \\ 2\psi_{\epsilon,1} + 3\Lambda\psi_{\epsilon,1} + \Lambda^2\psi_{\epsilon,1} \end{pmatrix}.$$

Proof. We first show that $\mathbf{g}_\epsilon$ lies in the domain of the operator $\mathbf{L}_\epsilon$. Since the domain of $\mathbf{L}_\epsilon$ coincides with the one of $\mathbf{L}_0$ we can apply Lemma 4.5 from [24]. Namely, we have to show that the two components of $\mathbf{g}_\epsilon$ satisfy for every multi-index $\beta \in \mathbb{N}_0^n$ an estimate of the form

$$|\partial^\beta g_{\epsilon,1}(\xi)| \lesssim \langle\xi\rangle^{s-\frac{n}{2}-|\beta|-1} \quad \text{and} \quad |\partial^\beta g_{\epsilon,2}(\xi)| \lesssim \langle\xi\rangle^{s-\frac{n}{2}-|\beta|-2}. \tag{4.11}$$

Recalling Proposition 3.9 and its proof we infer that the radial representative of $g_{\epsilon,1}$ satisfies $\widetilde{g}_{\epsilon,1} = \widetilde{\psi}_{\epsilon,2}$ and this implies

$$|\partial^\beta g_{\epsilon,1}(\xi)| \lesssim_\beta \langle\xi\rangle^{-2-|\beta|}$$

for every $\beta \in \mathbb{N}_0^n$ and $\xi \in \mathbb{R}^n$, which yields the first bound in Eq. (4.11) since $s > \frac{n}{2}-1$. For the second component we note that according to the proof of Proposition 3.9

$$\widetilde{g}_{\epsilon,1}(\rho) = a_1\, w_1(\rho) + a_2\, w_2(\rho)$$

for large $\rho > 0$ where $a_1, a_2 \in \mathbb{C}$ and w_1 and w_2 are given by

$$w_1(\rho) = \rho^{-3}\, h_1(\rho^{-1}), \quad w_2(\rho) = \rho^{-2} h_2(\rho^{-1}) + c\log(\rho)\rho^{-3} h_1(\rho^{-1})$$

with functions h_1 and h_2 which are analytic around zero. Now, we use the fact that

$$g_{\epsilon,2} = 2g_{\epsilon,1} + \Lambda g_{\epsilon,1}$$

and observe that the first term of w_2 (the one with the least decay) cancels so that we obtain

$$|\widetilde{g}_{\epsilon,2}^{(\ell)}(\rho)| \lesssim \log(\rho)\rho^{-3-\ell}$$

for every $\ell \in \mathbb{N}_0$ and every large enough $\rho > 0$. Since the logarithm grows slower than any polynomial and from the fact that s is strictly larger than $\frac{n}{2} - 1$ we infer that the second bound in (4.11) holds and we conclude $\mathbf{g}_\epsilon \in \mathbf{L}_\epsilon$.

Now we show that $\mathbf{g}_\epsilon$ is indeed an eigenfunction of $\mathbf{L}_\epsilon$ to the eigenvalue $\lambda = 1$. We first note that by similar reasoning as in Proposition 3.9 we actually have $\mathbf{L}_\epsilon \mathbf{g}_\epsilon = \widetilde{\mathbf{L}}_\epsilon \mathbf{g}_\epsilon$. Eq. (4.10) yields

$$\Delta g_{\epsilon,1} - \Lambda^2 g_{\epsilon,1} - 5\Lambda g_{\epsilon,1} - 6 g_{\epsilon,1} + V_\epsilon(\psi_{\epsilon,1}) g_{\epsilon,1} = 0$$

so that its radial representative $\widetilde{g}_{\epsilon,1}$ has to solve (3.21). But this is true by definition, since $\widetilde{g}_{\epsilon,1} = \widetilde{\psi}_{\epsilon,2} = \widetilde{\psi}_{\epsilon,1} + \rho\, \widetilde{\psi}'_{\epsilon,1}$. The claim then follows by the definition of $\widetilde{g}_{\epsilon,2}$.

□

For $\epsilon \neq 0$, the potential is not available in closed form which is however crucial for a direct investigation of the spectral problem and the exclusion of other unstable spectral points. Hence, we resort to perturbative arguments in order to characterize the spectral properties of $\mathbf{L}_\epsilon$.

Proposition 4.4. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (2.20). For every arbitrary but fixed $0 < \omega_0 < \widetilde{\omega}$ where $\widetilde{\omega}$ is the constant from Proposition 2.5, there exists $\epsilon^{**} > 0$ such that for every $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \epsilon^{**}$*

$$\sigma(\mathbf{L}_\epsilon) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -\omega_0\} \cup \{1\} \tag{4.12}$$

holds, $\lambda = 1$ is a simple eigenvalue of $\mathbf{L}_\epsilon$ and its eigenspace is spanned by $\mathbf{g}_\epsilon$. Furthermore, for the Riesz projection

$$\mathbf{P}_\epsilon := \frac{1}{2\pi i} \int_{\partial B_{1/2}(1)} \mathbf{R}_{\mathbf{L}_\epsilon}(\lambda)\, d\lambda \tag{4.13}$$

we have $\operatorname{ran}(\mathbf{P}_\epsilon) = \ker(\mathbf{I} - \mathbf{P}_\epsilon) = \langle \mathbf{g}_\epsilon \rangle$.

Proof. We first show

$$\sigma(\mathbf{L}_\epsilon) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq -\omega_0\} \subset B_{1/2}(1),$$

where $B_{1/2}(1)$ denotes the open ball in $\mathbb{C}$ of radius $r = \frac{1}{2}$ centered at 1, and that there exists a constant $C_{\omega_0} > 0$ such that the resolvent satisfies

$$\|\mathbf{R}_{\mathbf{L}_\epsilon}(\lambda)\| \leq C_{\omega_0} \tag{4.14}$$

for all $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$ and all $|\epsilon| \leq \epsilon^{**}$. We argue similarly to [30], Proposition 3.2.

Let $|\epsilon| \leq \epsilon^*$. To see that $\overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$ belongs to the resolvent set of $\mathbf{L}_\epsilon$, we notice that due to Proposition 2.5 every $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus \{1\}$ belongs to the resolvent set of $\mathbf{L}_0$. Therefore, the identity

$$\lambda - \mathbf{L}_\epsilon = (\mathbf{I} - (\mathbf{L}_\epsilon - \mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda))\,(\lambda - \mathbf{L}_0) \tag{4.15}$$

holds and it follows that $\lambda - \mathbf{L}_\epsilon$ is bounded invertible if and only if this is true for $\mathbf{I} - (\mathbf{L}_\epsilon - \mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda)$. We will prove the invertibility of the latter one by showing that the Neumann series of $(\mathbf{L}_\epsilon - \mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda)$ converges. For this we split the resolvent in the following way

$$\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{f} = \mathbf{R}_{\mathbf{L}_0}(\lambda)(\mathbf{I} - \mathbf{P}_0)\mathbf{f} + \mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{P}_0\mathbf{f} \tag{4.16}$$

where $\mathbf{P}_0$ is the Riesz projection from Proposition 2.5. From the decay of the semigroup $(\mathbf{S}_0(\tau))_{\tau\in\mathbb{N}}$ on the stable subspace $\operatorname{ran}(\mathbf{I} - \mathbf{P}_0)$, see (2.19), we obtain for every $0 < \omega < \widetilde{\omega}$ the following bound

$$\|\mathbf{S}_0(\tau)\,(\mathbf{I} - \mathbf{P}_0)\,\mathbf{u}\|_{s,k} \lesssim e^{-\omega\,\tau}\,\|(\mathbf{I} - \mathbf{P}_0)\,\mathbf{u}\|_{s,k}\,.$$

Since the restriction of $(\mathbf{S}_0(\tau))_{\tau\geq 0}$ to $\operatorname{ran}(\mathbf{I} - \mathbf{P}_0)$ equals the semigroup of the restriction of the operator $\mathbf{L}_0$ to $\operatorname{ran}(\mathbf{I} - \mathbf{P}_0)$ we can apply [21], p.55, Theorem 1.10 for every $0 < \omega < \widetilde{\omega}$ to infer the existence of a constant $M_\omega \geq 1$ such that

$$\|\mathbf{R}_{\mathbf{L}_0}(\lambda)(\mathbf{I} - \mathbf{P}_0)\| \leq \frac{M_\omega}{\operatorname{Re}\lambda + \omega} \tag{4.17}$$

holds for every $\lambda \in \mathbb{H}_{-\omega}$. Hence, (by choosing for example $\omega = \frac{\omega_0+\widetilde{\omega}}{2}$ in Eq. (4.17)) there exists a constant $C_{\omega_0} > 0$ such that

$$\|\mathbf{R}_{\mathbf{L}_0}(\lambda)(\mathbf{I} - \mathbf{P}_0)\| \leq C_{\omega_0}$$

holds for all $\lambda \in \overline{\mathbb{H}_{-\omega_0}}$. For the second term in Eq. (4.16) we notice that $\ker(\mathbf{I} - \mathbf{L}_0) = \operatorname{ran}(\mathbf{P}_0) = \langle \mathbf{g}_0 \rangle$ so that there exists a unique $\mathbf{f}_0 \in \operatorname{ran}(\mathbf{P}_0)$ with

$$\mathbf{P}_0\mathbf{f} = \langle \mathbf{f}_0, \mathbf{f}\rangle_{s,k}\,\mathbf{g}_0 \tag{4.18}$$

for every $\mathbf{f} \in \mathcal{H}^{s,k}_r$. Since $\mathbf{g}_0$ is an eigenvector of $\mathbf{L}_0$ to the eigenvalue $\lambda = 1$ we obtain

$$\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{P}_0\mathbf{f} = (\lambda - 1)^{-1}\langle \mathbf{f}_0, \mathbf{f}\rangle_{s,k}\,\mathbf{g}_0. \tag{4.19}$$

From these equations we now infer from (4.16) the existence of a constant $C_{\omega_0} > 0$ with

$$\|\mathbf{R}_{\mathbf{L}_0}(\lambda)\| \leq C_{\omega_0}$$

for every $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$. In sight of Proposition 4.2 we therefore get

$$\|(\mathbf{L}_\epsilon - \mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda)\| \lesssim |\epsilon|$$

for every $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \setminus B_{1/2}(1)$ and every $|\epsilon| \leq \epsilon^*$. We now choose $\epsilon^{**} \leq \epsilon^*$ small enough so that $\|(\mathbf{L}_\epsilon - \mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda)\| < 1$. Hence, we conclude from (4.15) that λ belongs to the resolvent set of $\mathbf{L}_\epsilon$. Furthermore, we obtain from the same equation that the resolvent has the following representation

$$\mathbf{R}_{\mathbf{L}_\epsilon}(\lambda) = \mathbf{R}_{\mathbf{L}_0}(\lambda)\left(\mathbf{I} - (\mathbf{L}_\epsilon - \mathbf{L}_0)\mathbf{R}_{\mathbf{L}_0}(\lambda)\right)^{-1}$$

from which Eq. (4.14) follows.

We will now conclude that the spectrum of $\mathbf{L}_\epsilon$ is contained in a left half-plane except for the unstable eigenvalue $\lambda = 1$. From our previous considerations we know that $\partial B_{1/2}(1)$ belongs to the resolvent set of $\mathbf{L}_\epsilon$. Therefore, the Riesz projection $\mathbf{P}_\epsilon$ from (4.13) is well-defined. We furthermore obtain from the resolvent identity, the Lipschitz continuity of $\mathbf{L}_\epsilon$ and the uniform bound for the resolvent for every $\lambda \in \partial B_{1/2}(1)$,

$$\|\mathbf{R}_{\mathbf{L}_\epsilon}(\lambda) - \mathbf{R}_{\mathbf{L}_\kappa}(\lambda)\| \leq \|\mathbf{R}_{\mathbf{L}_\epsilon}(\lambda)\|\,\|\mathbf{L}_\epsilon - \mathbf{L}_\kappa\|\,\|\mathbf{R}_{\mathbf{L}_\kappa}(\lambda)\| \lesssim |\epsilon - \kappa|.$$

Hence $\mathbf{R}_{\mathbf{L}_\epsilon}$ is Lipschitz continuous with respect to the parameter ϵ and therefore the same holds true for the Riesz projection $\mathbf{P}_\epsilon$. Therefore, we know from [26], p.34, Lemma 4.10 that there holds for every $|\epsilon| \leq \epsilon^{**}$,

$$\dim \operatorname{ran}(\mathbf{P}_\epsilon) = \dim \operatorname{ran}(\mathbf{P}_0) = 1.$$

But since we have the inclusion $\langle \mathbf{g}_\epsilon \rangle \subset \ker(\mathbf{I} - \mathbf{L}_\epsilon) \subset \operatorname{ran}(\mathbf{P}_\epsilon)$ equality must hold due to dimensional reasons and there can also not exist any other spectral points of $\mathbf{L}_\epsilon$ in $B_{1/2}(1)$. □

Now we are able to characterize the linearized evolution generated by $\mathbf{L}_\epsilon$.

Proposition 4.5. *Let $n \geq 5$ and $(s, k) \in \mathbb{R} \times \mathbb{N}$ satisfy (2.20). Then, for every arbitrary but fixed $0 < \omega_0 < \widetilde{\omega}$, where $\widetilde{\omega}$ is the constant from Proposition 2.5, and every $|\epsilon| \leq \epsilon^{**}$, where ϵ^{**} is the constant from Proposition 4.4, the projection $\mathbf{P}_\epsilon$ commutes with the respective semigroup $\mathbf{S}_\epsilon$. In particular,*

$$\mathbf{P}_\epsilon \mathbf{S}_\epsilon(\tau) = \mathbf{S}_\epsilon(\tau)\mathbf{P}_\epsilon = e^\tau \mathbf{P}_\epsilon \tag{4.20}$$

for all $\tau \geq 0$. We furthermore have

$$\|\mathbf{S}_\epsilon(\tau)(\mathbf{I} - \mathbf{P}_\epsilon)\mathbf{u}\|_{s,k} \lesssim e^{-\omega_0 \tau} \|(\mathbf{I} - \mathbf{P}_\epsilon)\mathbf{u}\|_{s,k} \tag{4.21}$$

as well as

$$\|\mathbf{S}_\epsilon(\tau)(\mathbf{I} - \mathbf{P}_\epsilon) - \mathbf{S}_\kappa(\tau)(\mathbf{I} - \mathbf{P}_\kappa)\| \lesssim e^{-\omega_0 \tau}|\epsilon - \kappa| \tag{4.22}$$

for all $\mathbf{u} \in \mathcal{H}^{s,k}_r, \tau \geq 0$ and $|\epsilon|, |\kappa| \leq \epsilon^{**}$.

Proof. Since the semigroup $\mathbf{S}_\epsilon(\tau)$ commutes with its generator $\mathbf{L}_\epsilon$ it also commutes with its corresponding resolvent $\mathbf{R}_{\mathbf{L}_\epsilon}$ and therefore also with the projection $\mathbf{P}_\epsilon$. Eq. (4.20) follows from the fact that $\mathbf{S}_\epsilon(\tau)\mathbf{P}_\epsilon\mathbf{f}$ as well as $e^\tau \mathbf{P}_\epsilon \mathbf{f}$ are solutions of the uniquely solvable Cauchy problem

$$\begin{cases} \partial_\tau \mathbf{u}(\tau) = \mathbf{L}_\epsilon \mathbf{u}(\tau), \\ \quad \mathbf{u}(0) = \mathbf{P}_\epsilon \mathbf{f}. \end{cases}$$

Eq. (4.21) follows from [30] Theorem A.1 provided that one can show the existence of a constant $M_{\omega_0} > 0$ with

$$\|\mathbf{R}_{\mathbf{L}_\epsilon}(\lambda)\,(\mathbf{I} - \mathbf{P}_\epsilon)\| \leq M_{\omega_0} \tag{4.23}$$

for all $\lambda \in \overline{\mathbb{H}_{-\omega_0}}$ and $|\epsilon| \leq \epsilon^{**}$.
From Proposition 4.4 we already know that there exists a constant M_{ω_0} such that

$$\|\mathbf{R}_{\mathbf{L}_\epsilon}(\lambda)\| \leq M_{\omega_0}$$

holds for every $\lambda \in \overline{\mathbb{H}_{-\omega_0}} \backslash B_{1/2}(1)$ and $|\epsilon| \leq \epsilon^{**}$. Now we know that $\mathbf{R}_{\mathbf{L}_\epsilon}(\lambda)\,(\mathbf{I} - \mathbf{P}_\epsilon)$ is an analytic map in $\mathbb{H}_{-\omega_0}$ and since the resolvent of the restriction of $\mathbf{L}_\epsilon$ to the range of $\mathbf{I} - \mathbf{P}_\epsilon$ coincides with $\mathbf{R}_{\mathbf{L}_\epsilon}(\lambda)\,(\mathbf{I} - \mathbf{P}_\epsilon)$ we obtain (4.23) also for every λ lying in the compact ball $\overline{B_{1/2}(1)}$.
To prove Eq. (4.22) we argue similarly to [17], Lemma 4.9. By taking $\mathbf{u} \in \mathcal{D}(\mathbf{L}_\epsilon)$ and then observing

$$\partial_\tau \mathbf{S}_\epsilon(\tau)(\mathbf{I} - \mathbf{P}_\epsilon)\mathbf{u} = \mathbf{L}_\epsilon \mathbf{S}_\epsilon(\tau)(\mathbf{I} - \mathbf{P}_\epsilon)\mathbf{u} = \mathbf{L}_\epsilon(\mathbf{I} - \mathbf{P}_\epsilon)\mathbf{S}_\epsilon(\tau)(\mathbf{I} - \mathbf{P}_\epsilon)\mathbf{u}$$

yields

$$\begin{aligned}&\partial_\tau\left[\mathbf{S}_\epsilon(\tau)(\mathbf{I}-\mathbf{P}_\epsilon)\mathbf{u}-\mathbf{S}_\kappa(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\right]\\ =&\mathbf{L}_\epsilon(\mathbf{I}-\mathbf{P}_\epsilon)\left[\mathbf{S}_\epsilon(\tau)(\mathbf{I}-\mathbf{P}_\epsilon)\mathbf{u}-\mathbf{S}_\kappa(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\right]+\left[\mathbf{L}_\epsilon(\mathbf{I}-\mathbf{P}_\epsilon)-\mathbf{L}_\kappa(\mathbf{I}-\mathbf{P}_\kappa)\right]\mathbf{S}_\kappa(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}.\end{aligned}$$

Consequently, the function

$$\Phi_{\epsilon,\kappa}(\tau):=\frac{\mathbf{S}_\epsilon(\tau)(\mathbf{I}-\mathbf{P}_\epsilon)\mathbf{u}-\mathbf{S}_\kappa(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}}{|\epsilon-\kappa|}$$

satisfies the inhomogeneous equation

$$\partial_\tau\Phi_{\epsilon,\kappa}(\tau)=\mathbf{L}_\epsilon(\mathbf{I}-\mathbf{P}_\epsilon)\Phi_{\epsilon,\kappa}(\tau)+\frac{\mathbf{L}_\epsilon(\mathbf{I}-\mathbf{P}_\epsilon)-\mathbf{L}_\kappa(\mathbf{I}-\mathbf{P}_\kappa)}{|\epsilon-\kappa|}\mathbf{S}_\kappa(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u} \tag{4.24}$$

with initial data $\Phi_{\epsilon,\kappa}(0)=\frac{(\mathbf{I}-\mathbf{P}_\epsilon)-(\mathbf{I}-\mathbf{P}_\kappa)}{|\epsilon-\kappa|}\mathbf{u}$. By Duhamel's principle the integral equation to (4.24) is given by

$$\begin{aligned}\Phi_{\epsilon,\kappa}(\tau)=&\,\mathbf{S}_\epsilon(\tau)(\mathbf{I}-\mathbf{P}_\epsilon)\frac{\mathbf{P}_\kappa-\mathbf{P}_\epsilon}{|\epsilon-\kappa|}\mathbf{u}\\ &+\int_0^\tau\mathbf{S}_\epsilon(\tau-\tau')(\mathbf{I}-\mathbf{P}_\epsilon)\frac{\mathbf{L}_\epsilon(\mathbf{I}-\mathbf{P}_\epsilon)-\mathbf{L}_\kappa(\mathbf{I}-\mathbf{P}_\kappa)}{|\epsilon-\kappa|}\mathbf{S}_\kappa(\tau')(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{u}\,d\tau'.\end{aligned}\tag{4.25}$$

The Lipschitz continuity of $\mathbf{L}_\epsilon$ and $\mathbf{P}_\epsilon$ now implies

$$\left\|\frac{\mathbf{L}_\epsilon(\mathbf{I}-\mathbf{P}_\epsilon)-\mathbf{L}_\kappa(\mathbf{I}-\mathbf{P}_\kappa)}{|\epsilon-\kappa|}\right\|\lesssim 1,\quad\text{for all}\quad |\epsilon|,|\kappa|\le\epsilon^{**}.$$

Therefore, we get from Duhamel's formula (4.25) and the just proven decay of the semigroup on the stable subspace (4.21)

$$\|\Phi_{\epsilon,\kappa}(\tau)\|_{s,k}\lesssim(1+\tau)e^{-\frac{\omega_0+\widetilde{\omega}}{2}\tau}\|\mathbf{u}\|_{s,k}\lesssim e^{-\omega_0\tau}\|\mathbf{u}\|_{s,k}$$

for all $\mathbf{u}\in\mathcal{D}(\mathbf{L}_\epsilon)$. Due to the density of $\mathcal{D}(\mathbf{L}_\epsilon)$ in $\mathcal{H}^{s,k}_r$ this result now extends to all of $\mathcal{H}^{s,k}_r$ so that we have proven (4.22) and therefore the Proposition.

□

4.2. **The abstract nonlinear Cauchy problem.** In the following, for $(s,k)\in\mathbb{R}\times\mathbb{N}$ satisfying (2.20) we fix $\omega:=\widetilde{\omega}/2$, where $\widetilde{\omega}$ is the constant from Proposition 2.5 and denote by $\overline{\epsilon}:=\epsilon^{**}(\omega)$ the constant associated via Proposition 4.4. First, we state Lipschitz estimates for the nonlinearity $\widehat{\mathbf{N}}_\epsilon$ defined in Eq. (4.7).

Lemma 4.6. *Let $n\ge 5$ and $(s,k)\in\mathbb{R}\times\mathbb{N}$ satisfy (2.20). Then, for any $\epsilon\in\mathbb{R}$ with $|\epsilon|\le\overline{\epsilon}$, $\widehat{\mathbf{N}}_\epsilon:\mathcal{H}^{s,k}_r\to\mathcal{H}^{s,k+1}_r$ and the estimate*

$$\left\|\widehat{\mathbf{N}}_\epsilon(\mathbf{u})-\widehat{\mathbf{N}}_\kappa(\mathbf{v})\right\|_{s,k+1}\lesssim\left(\|\mathbf{u}\|_{s,k}+\|\mathbf{v}\|_{s,k}\right)\|\mathbf{u}-\mathbf{v}\|_{s,k}+\left(\|\mathbf{u}\|^2_{s,k}+\|\mathbf{v}\|^2_{s,k}\right)|\epsilon-\kappa| \tag{4.26}$$

holds for all $|\epsilon|,|\kappa|\le\overline{\epsilon}$ and all $\mathbf{u},\mathbf{v}\in\mathcal{B}_\delta\subset\mathcal{H}^{s,k}_r$ for $0<\delta\le 1$.

Proof. The proof is analogous to the proof of Lemma 3.2 using the Schauder estimate from Proposition A.1.

□

We now focus on the existence and uniqueness of solutions to the Cauchy problem

$$\begin{cases} \partial_\tau \Phi_\epsilon(\tau) = \widetilde{\mathbf{L}}_\epsilon(\Phi_\epsilon(\tau)) + \widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\tau)), & \tau \in (0,\infty), \\ \Phi_\epsilon(0) = \mathbf{u} & \mathbf{u} \in \mathcal{H}^{s,k}_r, \end{cases} \tag{4.27}$$

for $\epsilon \in \mathbb{R}$, $|\epsilon| \leq \overline{\epsilon}$ by considering the corresponding integral equation

$$\Phi_\epsilon(\tau) = \mathbf{S}_\epsilon(\tau)\mathbf{u} + \int_0^\tau \mathbf{S}_\epsilon(\tau-\tau')\widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\tau'))\,d\tau' \quad \text{for all} \quad \tau \geq 0 \quad \text{and} \quad \mathbf{u} \in \mathcal{H}^{s,k}_r. \tag{4.28}$$

We introduce the Banach space

$$\mathcal{X} := \{\Phi \in C([0,\infty), \mathcal{H}^{s,k}_r) : \|\Phi\|_{\mathcal{X}} := \sup_{\tau>0} e^{\omega\tau} \|\Phi(\tau)\|_{s,k} < \infty\},$$

as well as

$$\mathcal{X}_\delta := \{\Phi \in \mathcal{X} : \|\Phi\|_{\mathcal{X}} \leq \delta\} = \{\Phi \in C([0,\infty), \mathcal{H}^{s,k}_r) : \|\Phi(\tau)\|_{s,k} \leq \delta\, e^{-\omega\tau}, \forall\, \tau > 0\}.$$

Following the standard approach, we introduce the correction term

$$\mathbf{C}(\Phi,\epsilon,\mathbf{u}) := \mathbf{P}_\epsilon \left(\mathbf{u} + \int_0^\infty e^{-\tau'}\, \widehat{\mathbf{N}}_\epsilon(\Phi(\tau'))\,d\tau'\right) \tag{4.29}$$

to suppress the exponential growth of the semigroup on the unstable subspace. Consequently, we consider the fixed-point problem

$$\Phi(\tau) = \mathbf{K}(\Phi,\epsilon,\mathbf{u})(\tau), \tag{4.30}$$

where $\mathbf{K}(\Phi,\epsilon,\mathbf{u})$ is defined as

$$\mathbf{K}(\Phi,\epsilon,\mathbf{u})(\tau) := \mathbf{S}_\epsilon(\tau)\left[\mathbf{u} - \mathbf{C}(\Phi,\epsilon,\mathbf{u})\right] + \int_0^\tau \mathbf{S}_\epsilon(\tau-\tau')\widehat{\mathbf{N}}_\epsilon(\Phi(\tau'))\,d\tau'. \tag{4.31}$$

This modification stabilizes the evolution as the following result shows:

Proposition 4.7. *Let $n \geq 5$, $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy (2.20). There are constants $0 < \delta_0 < 1$ and $C_0 > 1$ such that for all $0 < \delta \leq \delta_0$, $C \geq C_0$, all $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \overline{\epsilon}$ and all $\mathbf{u} \in \mathcal{H}^{s,k}_r$ with $\|\mathbf{u}\|_{s,k} \leq \frac{\delta}{C}$ there exists a unique function $\Phi_\epsilon(\mathbf{u}) \in \mathcal{X}_\delta$ such that (4.30) holds for all $\tau \geq 0$. Furthermore, the solution map $(\mathbf{u},\epsilon) \mapsto \Phi_\epsilon(\mathbf{u})$ is Lipschitz continuous, i.e.*

$$\|\Phi_\epsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}} \lesssim \|\mathbf{u}-\mathbf{v}\|_{s,k} + |\epsilon-\kappa|$$

for all $\mathbf{u},\mathbf{v} \in \mathcal{H}^{s,k}_r$ with $\|\mathbf{u}\|_{s,k}, \|\mathbf{v}\|_{s,k} \leq \frac{\delta}{C}$ and all $|\epsilon|, |\kappa| \leq \overline{\epsilon}$.

Proof. First, we show that the map $\mathbf{K}_{(\mathbf{u},\epsilon)}(\Phi) := \mathbf{K}(\Phi,\epsilon,\mathbf{u})$ is a well-defined contraction on $\mathcal{X}_\delta$ for all sufficiently large $C > 1$, sufficiently small $\delta > 0$ and all $\mathbf{u} \in \mathcal{H}^{s,k}_r$ with $\|\mathbf{u}\|_{s,k} \leq \frac{\delta}{C}$. For this we will take $\Phi \in \mathcal{X}_\delta$ and $\tau \geq 0$ and notice that we can write $\mathbf{K}_{(\mathbf{u},\epsilon)}$ in the following way

$$\begin{aligned} &\mathbf{K}_{(\mathbf{u},\epsilon)}(\Phi)(\tau) \\ ={}& \mathbf{S}_\epsilon(\tau)(\mathbf{I}-\mathbf{P}_\epsilon)\mathbf{u} - \int_\tau^\infty e^{\tau-\tau'}\,\mathbf{P}_\epsilon\,\widehat{\mathbf{N}}_\epsilon(\Phi(\tau'))\,d\tau' + \int_0^\tau \mathbf{S}_\epsilon(\tau-\tau')(\mathbf{I}-\mathbf{P}_\epsilon)\widehat{\mathbf{N}}_\epsilon(\Phi(\tau'))\,d\tau'. \end{aligned} \tag{4.32}$$

From this we obtain with Proposition 4.5 and Lemma 4.6

$$\left\|\mathbf{K}_{(\mathbf{u},\epsilon)}(\Phi)(\tau)\right\|_{s,k} \lesssim \frac{\delta}{C}e^{-\omega\tau} + \delta^2 e^{-2\omega\tau} + \delta^2 e^{-\omega\tau}(e^{-\omega\tau}+1) \lesssim \left(\frac{1}{C}+\delta\right)\delta e^{-\omega\tau}$$

so that we have

$$\left\|\mathbf{K}_{(\mathbf{u},\epsilon)}(\Phi)(\tau)\right\|_{s,k} \le \delta e^{-\omega\tau}$$

if we choose $C \ge C_0$ and $0 < \delta \le \delta_0$ with C_0 sufficiently large and $\delta_0 > 0$ sufficiently small. Since the continuity of the mapping $\tau \mapsto \mathbf{K}_{(\mathbf{u},\epsilon)}(\Phi)(\tau)$ follows from the definition and dominated convergence we conclude that $\mathbf{K}_{(\mathbf{u},\epsilon)} : \mathcal{X}_\delta \to \mathcal{X}_\delta$ is well-defined.
To show that $\mathbf{K}_{(\mathbf{u},\epsilon)}$ is a contraction on $\mathcal{X}_\delta$ (for potentially even smaller $\delta > 0$) we take $\Phi, \Psi \in \mathcal{X}_\delta$ and calculate for every $\tau \ge 0$ by using again the representation of $\mathbf{K}_{(\mathbf{u},\epsilon)}$ from Eq. (4.32)

$$\begin{aligned}
&\left\|\mathbf{K}_{(\mathbf{u},\epsilon)}(\Phi)(\tau) - \mathbf{K}_{(\mathbf{u},\epsilon)}(\Psi)(\tau)\right\|_{s,k} \\
&\lesssim \int_\tau^\infty e^{\tau-\tau'}\|\widehat{\mathbf{N}}_\epsilon(\Phi(\tau')) - \widehat{\mathbf{N}}_\epsilon(\Psi(\tau'))\|_{s,k}\,d\tau' + \int_0^\tau e^{-\omega(\tau-\tau')}\|\widehat{\mathbf{N}}_\epsilon(\Phi(\tau')) - \widehat{\mathbf{N}}_\epsilon(\Psi(\tau'))\|_{s,k}\,d\tau' \\
&\lesssim \left(\delta e^{-2\omega\tau} + \delta e^{-\omega\tau}(e^{-\omega\tau}+1)\right)\|\Phi-\Psi\|_{\mathcal{X}}.
\end{aligned}$$

If we now choose $\delta_0 > 0$ sufficiently small we get

$$\left\|\mathbf{K}_{(\mathbf{u},\epsilon)}(\Phi) - \mathbf{K}_{(\mathbf{u},\epsilon)}(\Psi)\right\|_{\mathcal{X}} \le \frac{1}{2}\|\Phi-\Psi\|_{\mathcal{X}},$$

for all $0 < \delta \le \delta_0$, $C \ge C_0$ and all $\epsilon \in \mathbb{R}$ with $|\epsilon| \le \overline{\epsilon}$. The existence of a unique solution in $\mathcal{X}_\delta$ now follows by application of the contraction mapping principle.
What is now left to show is the Lipschitz continuity of the solution map $(\mathbf{u},\epsilon) \mapsto \Phi_\epsilon(\mathbf{u}) \in \mathcal{X}_\delta$. For this we take $(\mathbf{u},\epsilon), (\mathbf{v},\kappa) \in \mathcal{B}_{\frac{\delta}{C}} \times [-\overline{\epsilon}, \overline{\epsilon}]$ and obtain by the previous considerations functions $\Phi_\epsilon(\mathbf{u}), \Phi_\kappa(\mathbf{v}) \in \mathcal{X}_\delta$ solving

$$\Phi_\epsilon(\mathbf{u})(\tau) = \mathbf{K}(\Phi_\epsilon(\mathbf{u})(\tau), \epsilon, \mathbf{u})(\tau) \quad \text{and} \quad \Phi_\kappa(\mathbf{v})(\tau) = \mathbf{K}(\Phi_\kappa(\mathbf{v})(\tau), \kappa, \mathbf{v})(\tau) \quad \forall\, \tau \ge 0.$$

We now show that

$$\|\mathbf{K}(\Phi_\epsilon(\mathbf{u}), \epsilon, \mathbf{u}) - \mathbf{K}(\Phi_\kappa(\mathbf{v}), \kappa, \mathbf{v})\|_{\mathcal{X}} \lesssim \|\mathbf{u}-\mathbf{v}\|_{s,k} + |\epsilon-\kappa|.$$

For this we take $\tau \ge 0$ and estimate the terms in (4.32) separately. For the first term we simply get from (4.22)

$$\|\mathbf{S}_\epsilon(\tau)(\mathbf{I}-\mathbf{P}_\epsilon)\mathbf{u} - \mathbf{S}_\kappa(\tau)(\mathbf{I}-\mathbf{P}_\kappa)\mathbf{v}\| \lesssim \frac{\delta}{C}e^{-\omega\tau}|\epsilon-\kappa| + e^{-\omega\tau}\|\mathbf{u}-\mathbf{v}\|_{s,k}.$$

For the second term we apply Lemma 4.6 and the Lipschitz continuity of $\mathbf{P}_\epsilon$ to get

$$\begin{aligned}
&\int_\tau^\infty e^{\tau-\tau'}\|\mathbf{P}_\epsilon\,\widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\mathbf{u})(\tau')) - \mathbf{P}_\kappa\,\widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau'))\|_{s,k}\,d\tau' \\
&\lesssim |\epsilon-\kappa|\int_\tau^\infty e^{\tau-\tau'}\|\widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau'))\|_{s,k}\,d\tau' + \int_\tau^\infty e^{\tau-\tau'}\|\widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\mathbf{u})(\tau')) - \widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau'))\|_{s,k}\,d\tau' \\
&\lesssim |\epsilon-\kappa|\,\delta^2\,e^{-2\omega\tau} + \delta\,e^{-2\omega\tau}\,\|\Phi_\epsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}}
\end{aligned}$$

and for the last term we similarly get

$$
\begin{aligned}
&\int_0^\tau \left\| \mathbf{S}_\epsilon(\tau-\tau')(\mathbf{I}-\mathbf{P}_\epsilon)\widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\mathbf{u})(\tau')) - \mathbf{S}_\kappa(\tau-\tau')(\mathbf{I}-\mathbf{P}_\kappa)\widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau')) \right\|_{s,k} d\tau' \\
&\lesssim |\epsilon-\kappa| \int_0^\tau e^{-\omega\,(\tau-\tau')} \|\widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau'))\|_{s,k}\, d\tau' \\
&\quad + \int_0^\tau e^{-\omega\,(\tau-\tau')} \|\widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\mathbf{u})(\tau')) - \widehat{\mathbf{N}}_\kappa(\Phi_\kappa(\mathbf{v})(\tau'))\|_{s,k}\, d\tau' \\
&\lesssim |\epsilon-\kappa|\, \delta^2 e^{-\omega\,\tau} + |\epsilon-\kappa|\, \delta\, e^{-\omega\,\tau}\, \|\Phi_\epsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}} .
\end{aligned}
$$

With that we obtain

$$
\begin{aligned}
\|\Phi_\epsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}} &= \|\mathbf{K}(\Phi_\epsilon(\mathbf{u}),\epsilon,\mathbf{u}) - \mathbf{K}(\Phi_\kappa(\mathbf{v}),\kappa,\mathbf{v})\|_{\mathcal{X}} \\
&\lesssim |\epsilon-\kappa| + \|\mathbf{u}-\mathbf{v}\|_{s,k} + \delta\, \|\Phi_\epsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}} .
\end{aligned}
$$

For small enough $\delta > 0$ we get $\|\Phi_\epsilon(\mathbf{u}) - \Phi_\kappa(\mathbf{v})\|_{\mathcal{X}} \lesssim |\epsilon-\kappa| + \|\mathbf{u}-\mathbf{v}\|_{s,k}$ as desired.

□

Now we will go back to considering the specific form of the initial data of the Cauchy problem (4.5). For this, we define the following initial data operator

$$
\mathbf{U}_\epsilon(\mathbf{v},T) := \mathbf{v}^{\mathbf{T}} + \mathbf{\Psi}_\epsilon^{\mathbf{T}} - \mathbf{\Psi}_\epsilon := \begin{pmatrix} T v_1(T\cdot) \\ T^2 v_2(T\cdot) \end{pmatrix} + \begin{pmatrix} T\psi_{\epsilon,1}(T\cdot) - \psi_{\epsilon,1} \\ T^2\psi_{\epsilon,2}(T\cdot) - \psi_{\epsilon,2} \end{pmatrix}. \tag{4.33}
$$

Lemma 4.8. *Let $n \geq 5$ and $(s,k) \in \mathbb{R}\times\mathbb{N}$ satisfy* (2.20). *Let $0 < \delta \leq \frac{1}{2}$. For every $\epsilon \in \mathbb{R}$, $|\epsilon| \leq \overline{\epsilon}$ the map*

$$
T \mapsto \mathbf{U}_\epsilon(\mathbf{v},T) : [1-\delta, 1+\delta] \to \mathcal{H}_r^{s,k}
$$

is continuous for $\mathbf{v} \in \mathcal{H}_r^{s,k}$. Furthermore, for every $T \in [\frac{1}{2}, \frac{3}{2}]$ the initial data operator can be written as

$$
\mathbf{U}_\epsilon(\mathbf{v},T) = \mathbf{v}^T + (T-1)\mathbf{g}_\epsilon + \mathbf{R}_\epsilon(T), \tag{4.34}
$$

and there exists a constant $M_\epsilon > 0$ such that

$$
\|\mathbf{R}_\epsilon(T)\| \leq M_\epsilon |T-1|^2.
$$

Proof. Eq. (4.34) follows from Taylor's Theorem applied to the map $[\frac{1}{2}, \frac{3}{2}] \to \mathcal{H}_r^{s,k}, T \mapsto \Psi_\epsilon^T$ using the fact that

$$
\partial_T \begin{pmatrix} T\psi_{\epsilon,1}(T\cdot) \\ T^2\psi_{\epsilon,2}(T\cdot) \end{pmatrix} \Bigg|_{T=1} = \mathbf{g}_\epsilon.
$$

The components $R_{\epsilon,1}(T)$ and $R_{\epsilon,2}(T)$ of the remainder term $\mathbf{R}_\epsilon(T)$ satisfy

$$
\|R_{\epsilon,i}(T)\|_{\dot{H}^{s-(i-1)}\cap \dot{H}^{k-(i-1)}(\mathbb{R}^n)} \lesssim (T-1)^2 \sum_{j=0}^{2} \left\|\Lambda^j \psi_{\epsilon,i}\right\|_{\dot{H}^{s-(i-1)}\cap \dot{H}^{k-(i-1)}(\mathbb{R}^n)} \quad \text{for} \quad i = 1,2.
$$

The norms on the right hand side are finite, due to the decay of $\psi_{\epsilon,i}$ described in Proposition 3.9. This determines the constant $M_\epsilon > 0$. The continuity follows by a standard argument, see for example [24], Lemma 8.2. □

Now, we are in the position to prove the central result of this section.

Theorem 4.9. *Let $n \geq 5$ and $(s,k) \in \mathbb{R} \times \mathbb{N}$ satisfy (2.20). For any $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \overline{\epsilon}$, there are constants $0 < \delta_\epsilon < 1$ and $C_\epsilon > 1$ such that for all $0 < \delta \leq \delta_\epsilon$ and all $C \geq C_\epsilon$ the following statement holds: If $\mathbf{v} \in \mathcal{H}^{s,k}_r$ is real-valued with $\|\mathbf{v}\|_{s,k} \leq \frac{\delta}{C^2}$ then there exists a $T_\epsilon = T_\epsilon(\mathbf{v}) \in [1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$ and a unique solution $\Phi_\epsilon \in C([0,\infty); \mathcal{H}^{s,k}_r)$ satisfying*

$$\Phi_\epsilon(\tau) = \mathbf{S}_\epsilon(\tau)\mathbf{U}_\epsilon(\mathbf{v}, T_\epsilon) + \int_0^\tau \mathbf{S}_\epsilon(\tau - \tau')\widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\tau'))\, d\tau' \quad \textit{for all} \quad \tau \geq 0. \tag{4.35}$$

Furthermore,

$$\|\Phi_\epsilon(\tau)\|_{s,k} \leq \delta e^{-\omega\tau}, \quad \forall \tau \geq 0.$$

Proof. Let $0 < \delta \leq \delta_0$ and $C \geq C_0 \geq 1$ with δ_0 and C_0 as in Proposition 4.7. Let $|\epsilon| \leq \overline{\epsilon}$ and $\mathbf{v} \in \mathcal{H}^{s,k}_r$ with $\|\mathbf{v}\|_{s,k} \leq \frac{\delta}{C^2}$. Then we obtain from Lemma 4.8

$$\begin{aligned}\|\mathbf{U}_\epsilon(\mathbf{v}, T)\|_{s,k} &\lesssim \left\|\mathbf{v}^T\right\|_{s,k} + |T-1| \, \|\mathbf{g}_\epsilon\|_{s,k} + \|\mathbf{R}_\epsilon(T)\|_{s,k} \\ &\lesssim \frac{\delta}{C^2} + \frac{\delta}{C}L_\epsilon + \frac{\delta^2}{C^2}M_\epsilon\end{aligned}$$

for every $T \in [1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$ where $L_\epsilon, M_\epsilon > 0$ are some constants depending on ϵ. If we now choose δ sufficiently small and C sufficiently large we obtain for every $T \in [1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$ from Proposition 4.7 the existence of a unique $\Phi_\epsilon = \Phi_\epsilon(\mathbf{v}, T) \in \mathcal{X}_\delta$ which solves

$$\Phi_\epsilon(\tau) = \mathbf{S}_\epsilon(\tau)\left[\mathbf{U}_\epsilon(\mathbf{v}, T) - \mathbf{C}(\Phi_\epsilon, \epsilon, \mathbf{U}_\epsilon(\mathbf{v}, T))\right] + \int_0^\tau \mathbf{S}_\epsilon(\tau - \tau')\widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\tau'))\, d\tau'. \tag{4.36}$$

Since $\mathbf{C}$ takes values in $\operatorname{ran} \mathbf{P}_\epsilon = \langle \mathbf{g}_\epsilon \rangle$ it is enough to show, given $\mathbf{v}$, the existence of a T such that

$$\langle \mathbf{C}(\Phi_\epsilon(\mathbf{v}, T), \epsilon, \mathbf{U}_\epsilon(\mathbf{v}, T)), \mathbf{g}_\epsilon \rangle_{s,k} = 0. \tag{4.37}$$

Due to Lemma 4.8 and the definition of $\mathbf{C}$ this equation reads as

$$0 = \langle \mathbf{P}_\epsilon \mathbf{v}^T, \mathbf{g}_\epsilon \rangle_{s,k} + (T-1)\|\mathbf{g}_\epsilon\|^2_{s,k} + \langle \mathbf{P}_\epsilon \mathbf{R}_\epsilon(T), \mathbf{g}_\epsilon \rangle_{s,k} + \langle \mathbf{P}_\epsilon \int_0^\infty e^{-\tau'} \widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\tau'))\, d\tau', \mathbf{g}_\epsilon \rangle_{s,k},$$

which can be written as a fixed-point equation for $T \in [1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$,

$$T = 1 + \langle \mathbf{P}_\epsilon \mathbf{v}^T, \widehat{\mathbf{g}}_\epsilon \rangle_{s,k} + \langle \mathbf{P}_\epsilon \mathbf{R}_\epsilon(T), \widehat{\mathbf{g}}_\epsilon \rangle_{s,k} + \langle \mathbf{P}_\epsilon \int_0^\infty e^{-\tau'} \widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\tau'))\, d\tau', \widehat{\mathbf{g}}_\epsilon \rangle_{s,k}, \tag{4.38}$$

where we have set $\widehat{\mathbf{g}}_\epsilon = \mathbf{g}_\epsilon / \|\mathbf{g}_\epsilon\|^2_{s,k}$. Now we obtain from the assumptions on $\mathbf{v}$, the fact that Φ_ϵ belongs to $\mathcal{X}_\delta$ and Lemma 4.8 as well as Lemma 4.6 the following estimate

$$\begin{aligned}&|\langle \mathbf{P}_\epsilon \mathbf{v}^T, \widehat{\mathbf{g}}_\epsilon \rangle_{s,k}| + |\langle \mathbf{P}_\epsilon \mathbf{R}_\epsilon(T), \widehat{\mathbf{g}}_\epsilon \rangle_{s,k}| + |\langle \mathbf{P}_\epsilon \int_0^\infty e^{-\tau'} \widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\tau'))\, d\tau', \widehat{\mathbf{g}}_\epsilon \rangle_{s,k}| \\ \lesssim& \frac{\delta}{C^2}L_\epsilon + \frac{\delta^2}{C^2}M_\epsilon + \delta^2 N_\epsilon\end{aligned}$$

for again some constants L_ϵ, M_ϵ and $N_\epsilon > 0$. If we now choose $C \geq C_\epsilon$ and $0 < \delta < \delta_\epsilon$ with $C_\epsilon > 1$ sufficiently large and $\delta_\epsilon < 1$ sufficiently small we get that the right-hand side of (4.38) is a continuous mapping from $[1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$ into itself so that we obtain by the fixed-point theorem of Brouwer a $T_\epsilon \in \left[1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}\right]$ such that equation (4.37) is fulfilled. We therefore conclude that the corresponding solution $\Phi_\epsilon(\mathbf{v}, T_\epsilon)$ is a solution to (4.35). The claimed uniqueness follows along the lines of the proof of Theorem 5.4 in [25].

□

Now we will show the regularity of the just constructed solution.

Proposition 4.10. *Let* $\mathbf{v}$ *satisfy the assumptions of Theorem 4.9. If* $\mathbf{v} \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ *then the solution* Φ_ϵ *of Eq.* (4.35) *guaranteed by Theorem 4.9 is smooth. More precisely,* $\Phi_\epsilon(\tau)(\xi) = (\phi_{\epsilon,1}(\tau,\xi), \phi_{\epsilon,2}(\tau,\xi))$ *with* $\phi_{\epsilon,i} \in C^\infty([0,\infty) \times \mathbb{R}^n)$*. Furthermore, the components satisfy*

$$\begin{pmatrix} \partial_\tau \phi_{\epsilon,1}(\tau,\xi) \\ \partial_\tau \phi_{\epsilon,2}(\tau,\xi) \end{pmatrix} = \begin{pmatrix} \phi_{\epsilon,2}(\tau,\xi) - \Lambda\, \phi_{\epsilon,1}(\tau,\xi) - \phi_{\epsilon,1}(\tau,\xi) \\ \Delta_\xi \phi_{\epsilon,1}(\tau,\xi) - \Lambda\, \phi_{\epsilon,2}(\tau,\xi) - 2\phi_{\epsilon,2}(\tau,\xi) + \hat{N}_\varepsilon(\phi_{\epsilon,1}(\tau,\cdot))(\xi) \end{pmatrix} \tag{4.39}$$

for all $\xi \in \mathbb{R}^d$ *and all* $\tau \geq 0$*, and*

$$\begin{pmatrix} \phi_{\epsilon,1}(0,\cdot) \\ \phi_{\epsilon,2}(0,\cdot) \end{pmatrix} = \begin{pmatrix} T_\epsilon v_1(T_\epsilon \cdot) + T_\epsilon \psi_{\epsilon,1}(T_\epsilon \cdot) - \psi_{\epsilon,1} \\ T_\epsilon^2 v_2(T_\epsilon \cdot) + T_\epsilon^2 \psi_{\epsilon,2}(T_\epsilon \cdot) - \psi_{\epsilon,2} \end{pmatrix}. \tag{4.40}$$

Proof. Assume that all of the constants are chosen such that Theorem 4.9 is fulfilled. Then there exists a solution $\Phi_\epsilon = \Phi_\epsilon(\mathbf{v}, T_\epsilon) \in \mathcal{X}_\delta$ satisfying

$$\Phi_\epsilon(\tau) = \mathbf{S}_\epsilon(\tau)\mathbf{U}_{\epsilon,T_\epsilon}(\mathbf{v}) + \int_0^\tau \mathbf{S}_\epsilon(\tau - \tau')\widehat{\mathbf{N}}_\epsilon(\Phi_\epsilon(\tau'))\, d\tau' \quad \text{for all} \quad \tau \geq 0.$$

We will now show via an inductive argument that $\Phi_\epsilon(\tau)$ belongs to $\mathcal{H}^{s,\ell}$ for every $\ell \geq k$ and every $\tau \geq 0$. We already know from Lemma 4.6 that $\widehat{\mathbf{N}}_\epsilon$ maps $\mathcal{H}^{s,\ell}$ into $\mathcal{H}^{s,\ell+1}$ and that the initial data operator $\mathbf{U}_{\epsilon,T_\epsilon}(\mathbf{v})$ belongs to $\mathcal{H}^{s,\ell}$ for every $\ell \geq k$. Since the restriction of the semigroups are equal to the restricted semigroups, see Lemma C.1 from [24], we can already inductively conclude that $\Phi_\epsilon(\tau)$ belongs to $\mathcal{H}^{s,\ell}$ for every $\ell \geq k$ and every $\tau \geq 0$. From the Sobolev embedding (2.13) we can therefore conclude that $\Phi_\epsilon(\tau)$ belongs to $C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n)$ for every $\tau \geq 0$.
Now we show that $\mathbf{U}_\epsilon(\mathbf{v}, T) \in \mathcal{D}(\mathbf{L}_\epsilon)$ for $\mathbf{v} \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ and $T \in [\frac{1}{2}, \frac{3}{2}]$. This is not entirely obvious as the blowup solution is not explicit. Set $\hat{\psi}_1 := T\psi_{\epsilon,1}(T\cdot) - \psi_{\epsilon,1}$, $\hat{\psi}_2 := T^2\psi_{\epsilon,2}(T\cdot) - \psi_{\epsilon,2}$. We verify that for every $\beta \in \mathbb{N}_0^n$ we have

$$|\partial^\beta \hat{\psi}_1(\xi)| \lesssim \langle\xi\rangle^{s-\frac{n}{2}-|\beta|-1} \quad \text{and} \quad |\partial^\beta \hat{\psi}_2(\xi)| \lesssim \langle\xi\rangle^{s-\frac{n}{2}-|\beta|-2} \tag{4.41}$$

and apply Lemma 4.5 from [24] as in the proof of Lemma 4.3. Let $\tilde{\psi}_{\epsilon,i}$, $i \in \{1,2\}$ denote the radial representative of $\psi_{\epsilon,i}$. We start with the second component: As outlined in the proof of Proposition 3.9 we have

$$\tilde{\psi}_{\epsilon,2} = c_2 w_1 + \tilde{c}_2 w_2,$$

with w_i given in Eq. (3.22) and $c_2, \tilde{c}_2 \in \mathbb{C}$. Now, for large $\rho > 0$,

$$\tilde{\psi}_{\epsilon,2}\,(\rho) = c_2\rho^{-3}h_1(\rho^{-1}) + \tilde{c}_2\rho^{-2}h_2(\rho^{-1}) + \tilde{c}\rho^{-3}\log(\rho)h_1(\rho^{-1}) \tag{4.42}$$

for some $\tilde{c} \in \mathbb{C}$ with h_1, h_2 analytic around zero and $h_i(0) = 1$, $i \in \{1,2\}$. By the scaling behavior of the second term, we find that the bad behavior cancels and thus

$$|T^2\tilde{\psi}_{\epsilon,2}(T\rho) - \tilde{\psi}_{\epsilon,2}(\rho)| \lesssim_T \rho^{-3}\log(\rho),$$

which implies the bound for $\hat{\psi}_2$ in the case β equal to zero since we assume $s > \frac{n}{2} - 1$. For higher derivatives, the analogous bounds follow using the analyticity of h.

For the first component we set $f_\varepsilon(\rho) = \rho\tilde{\psi}_{\epsilon,1}(\rho)$ and infer with Proposition 3.9 that there are constants $c_1, \tilde{c}_1 \in \mathbb{R}$ such that

$$\lim_{\rho\to\infty} f_\varepsilon(\rho) = c_1, \quad \lim_{\rho\to\infty} \rho^2 f_\varepsilon'(\rho) = \tilde{c}_1.$$

Thus, $v_\varepsilon(y) := f_\varepsilon(\frac{1}{y})$ can be extended to a continuously differentiable function $v_\varepsilon \in C^1[0,1]$. By Taylor's theorem, $v_\varepsilon(y) = c_1 - \tilde{c}_1 y + o(y)$, for $y > 0$ close to zero and thus $f_\varepsilon(\rho) = c_1 - \tilde{c}_1\rho^{-1} + o(\rho^{-1})$. Consequently,

$$|T\tilde{\psi}_{\epsilon,1}(T\rho) - \tilde{\psi}_{\epsilon,1}(\rho)| = \rho^{-1}|f_\varepsilon(T\rho) - f_\varepsilon(\rho)| \lesssim_T \rho^{-2}$$

for large values of ρ, which implies the bound for $\hat{\psi}_1$ in the case β equal to zero.
For the case $|\beta| = 1$ we calculate for every $j \in \{1, \dots, n\}$ using the crucial fact $f_\epsilon'(\rho) = \widetilde{\psi}_{\epsilon,2}(\rho)$

$$\begin{aligned}
|\partial^{e_j}\widehat{\psi}_1(\xi)| &= |T^2\partial^{e_j}\psi_{\epsilon,1}(T\xi) - \partial^{e_j}\psi_{\epsilon,1}(\xi)| = \frac{|\xi_j|}{|\xi|}|T^2\widetilde{\psi}_{\epsilon,1}'(T|\xi|) - \widetilde{\psi}_{\epsilon,1}'(|\xi|)| \\
&= \frac{|\xi_j|}{|\xi|^3}|T|\xi|\widetilde{\psi}_{\epsilon,2}(T|\xi|) - |\xi|\widetilde{\psi}_{\epsilon,2}(|\xi|) + f_\epsilon(|\xi|) - f_\epsilon(T|\xi|)| \\
&= \frac{|\xi_j|}{|\xi|^3}|T|\xi|\widetilde{\psi}_{\epsilon,2}(T|\xi|) - |\xi|\widetilde{\psi}_{\epsilon,2}(|\xi|) + |\xi|\widehat{\psi}_1(\xi)| \lesssim_T |\xi|^{-3}.
\end{aligned}$$

The last inequality follows from the decay of $\widetilde{\psi}_{\epsilon,2}$ (here one does not need the cancellation of the worst behaving term) and the above shown decay for $\widehat{\psi}_1$ in the case $|\beta| = 1$.
The decay for the higher derivatives can now be shown via the representation (4.42) of $\widetilde{\psi}_{\epsilon,2}$ and via induction on $\widehat{\psi}_1$ so that we can conclude that $\mathbf{U}_\epsilon(\mathbf{v}, T)$ belongs to the domain of $\mathbf{L}_\epsilon$. Hence, we invoke [[32], p.189, Theorem 1.6] to infer that $\Phi_\epsilon \in C^1([0,\infty), \mathcal{H}_r^{s,k})$ is a classical solution of the operator equation and the claimed regularity follows form Sobolev embedding, the spatial regularity proved above and the Theorem of Schwarz as formulated in [33], p. 235, Theorem 9.41. In particular, the components of satisfy Eq. (4.39) by definition of the operators involved. □

Now we are finally able to prove our main stability results, Theorem 1.6 and Theorem 1.3.

Proof of Theorem 1.6. Under the assumption stated in Theorem 1.6 choose $\omega = \widetilde{\omega}/2$ and $0 < \overline{\epsilon}$, depending on ω, as at the beginning of Section 4.2
For $\epsilon \in \mathbb{R}$ with $|\epsilon| \leq \overline{\epsilon}$ let $\delta = \delta_\epsilon$ and $C = C_\epsilon$ denote the constants from Theorem 4.9. Let $(\varphi_0, \varphi_1) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ be radial, real-valued, functions satisfying

$$\|(\varphi_0, \varphi_1)\|_{\dot{H}^s\cap\dot{H}^k(\mathbb{R}^n)\times\dot{H}^{s-1}\cap\dot{H}^{k-1}(\mathbb{R}^n)} < \frac{\delta}{C^2}.$$

Then, by Theorem 4.9 and Proposition 4.10 there is a $T = T_\epsilon \in [1 - \frac{\delta}{C}, 1 + \frac{\delta}{C}]$ and unique radial functions $(\varphi_{\epsilon,1}, \varphi_{\epsilon,2}) \in C^\infty([0,\infty) \times \mathbb{R}^n) \times C^\infty([0,\infty) \times \mathbb{R}^n)$ solving the initial value problem Eq. (4.39) - (4.40). Moreover,

$$\|(\varphi_{\epsilon,1}(\tau, \cdot), \varphi_{\epsilon,2}(\tau, \cdot))\|_{s,k} \leq \delta e^{-\omega\tau},$$

for all $\tau \geq 0$. We set

$$v(t,x) := v_\epsilon^T(t,x) + \frac{1}{T-t}\varphi_{\epsilon,1}\left(\log\left(\frac{T}{T-t}\right), \frac{x}{T-t}\right).$$

By construction, $v \in C^\infty([0,T)\times\mathbb{R}^n)$ satisfies Eq. (2.1) and

$$(v(0,\cdot),\partial_t v(0,\cdot)) = (v^1_\epsilon(0,\cdot),\partial_t v^1_\epsilon(0,\cdot)) + (\varphi_0,\varphi_1).$$

Moreover for $r \in [s,k]$,

$$\|\varphi_{\epsilon,1}(-\log(T-t)+\log T,\cdot)\|_{\dot H^r(\mathbb{R}^n)} \lesssim \|\Phi_\varepsilon(\tau)\|_{s,k} \lesssim \delta(T-t)^\omega$$

and

$$\begin{aligned}\|(\partial_0+\Lambda+1)\varphi_{\epsilon,1}(-\log(T-t)+\log T,\cdot)\|_{\dot H^{r-1}(\mathbb{R}^n)} &= \|\varphi_{\epsilon,2}(\log T-\log(T-t),\cdot)\|_{\dot H^{r-1}(\mathbb{R}^n)}\\ &\lesssim \|\Phi_\varepsilon(\tau)\|_{s,k} \lesssim \delta(T-t)^\omega\end{aligned}$$

by definition and Theorem 4.9.

□

Proof of Theorem 1.3. By the assumptions of Theorem 1.3 the initial data are of the form

$$U_0(x) = U^1_\epsilon(0,x) + xv_0(|x|), \quad U_1(x) = \partial_t U^1_\epsilon(0,x) + xv_1(|x|)$$

Consequently, $v_0, v_1 \in C^\infty_e[0,\infty)$ and by setting

$$\varphi_j(y) := v_j(|y|)$$

for $y \in \mathbb{R}^{d+2}$, we obtain radially symmetric, real-valued functions $(\varphi_0,\varphi_1) \in \mathcal{S}(\mathbb{R}^{d+2}) \times \mathcal{S}(\mathbb{R}^{d+2})$. By Proposition A.5 and Remark A.6 of [23] there exists a constant $C > 0$ such that

$$\|(\varphi_0,\varphi_1)\|_{\dot H^s\cap\dot H^k(\mathbb{R}^{d+2})\times\dot H^{s-1}\cap\dot H^{k-1}(\mathbb{R}^{d+2})} \leq C\,\|(\nu_0,\nu_1)\|_{\dot H^s\cap\dot H^k(\mathbb{R}^d,\mathbb{R}^d)\times\dot H^{s-1}\cap\dot H^{k-1}(\mathbb{R}^d,\mathbb{R}^d)}\,. \tag{4.43}$$

If the Sobolev exponents (s,k) satisfy condition (1.10), then (1.22) holds for $n := d+2$. Let $\omega,\overline{\epsilon},\delta,M > 0$ be the constants from Theorem 1.6. By setting $M_0 := CM$ and requiring

$$\|(\nu_0,\nu_1)\|_{\dot H^s\cap\dot H^k(\mathbb{R}^d,\mathbb{R}^d)\times\dot H^{s-1}\cap\dot H^{k-1}(\mathbb{R}^d,\mathbb{R}^d)} \leq \frac{\delta}{M_0},$$

we find that (φ_0,φ_1) satisfy the assumption of Theorem 1.6. Hence, there is a $T \in [1-\delta,1+\delta]$ and a unique radial solution $v \in C^\infty([0,T)\times\mathbb{R}^{d+2})$ to (1.14). If we set $v(t,\cdot) = \tilde v(t,|\cdot|)$ then $\tilde v$ solves Eq. (1.13) for $t \in [0,T)$ and can be written as

$$\tilde v(t,|\cdot|) = \frac{1}{|\cdot|}f_\epsilon\left(\frac{|\cdot|}{T-t}\right) + \frac{1}{T-t}\tilde\varphi\left(\log\left(\frac{T}{T-t}\right),\frac{|\cdot|}{T-t}\right)$$

for $\tilde\varphi(t,\cdot) \in C^\infty_e[0,\infty)$ satisfying

$$\begin{aligned}&\|\tilde\varphi(-\log(T-t)+\log T,|\cdot|)\|_{\dot H^r(\mathbb{R}^{d+2})}\\ &\qquad + \|(\partial_0+\Lambda+1)\tilde\varphi(-\log(T-t)+\log T,|\cdot|)\|_{\dot H^{r-1}(\mathbb{R}^{d+2})} \lesssim \delta(T-t)^\omega\end{aligned} \tag{4.44}$$

for all $r \in [s,k]$. We define for $x \in \mathbb{R}^d$, $U(t,x) := x\tilde v(t,|x|) \in C^\infty([0,T)\times\mathbb{R}^d,\mathbb{R}^d)$ and find that U can be written as

$$U(t,x) = U_\epsilon(t,x) + \nu\left(t,\frac{x}{T-t}\right),$$

where ν is a co-rotational function defined via $\nu(t,x) = x\,\tilde\varphi(-\log(T-t)+\log T,|x|)$. The inequality from (4.44) now implies (1.12) by applying Proposition A.5 and Remark A.6 from [23] and the local uniform convergence follows immediately from Sobolev embedding.

Appendix A. Schauder-type estimates for parameter depending nonlinear operators

Here we will show an adaptation of a Schauder-type estimate whose original form can be found in [24], p.26, Proposition A.1.

Proposition A.1. *Let $n \geq 5$ and $\epsilon_0 > 0$. For $\epsilon \in \mathbb{R}$, $|\epsilon| \leq \epsilon_0$ let $F_\epsilon \in C^\infty(\mathbb{R})$ be a family of even functions such that for all $\ell \in \mathbb{N}_0$ there exists a constant $C_\ell \geq 0$ such that*

$$|F_\epsilon^{(\ell)}(x) - F_\kappa^{(\ell)}(y)| \leq C_\ell \, (|\epsilon - \kappa| + |x - y|) \tag{A.1}$$

holds for all $x, y \in \mathbb{R}$ and all $|\epsilon|, |\kappa| \leq \epsilon_0$. Then, for every $s \in \mathbb{R}$ and $k \in \mathbb{N}$ that satisfy

$$\frac{n}{2} - 1 < s \leq \frac{n}{2} - 1 + \frac{1}{2(n-1)}, \quad k > n \tag{A.2}$$

we have

$$\|u_1 u_2 u_3 \, (F_\epsilon(|\cdot|v) - F_\kappa(|\cdot|v))\|_{\dot{H}^{s-1} \cap \dot{H}^k(\mathbb{R}^n)} \lesssim |\epsilon - \kappa| \prod_{i=1}^{3} \|u_i\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \sum_{j=0}^{k} \|v\|^{2j}_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \tag{A.3}$$

as well as

$$\begin{aligned} &\|u_1 u_2 u_3 \, (F_\epsilon(|\cdot|v_1) - F_\epsilon(|\cdot|v_2))\|_{\dot{H}^{s-1} \cap \dot{H}^k(\mathbb{R}^n)} \\ &\lesssim \prod_{i=1}^{3} \|u_i\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} P(\|v_1\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)}, \|v_2\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)}) \, \|v_1 - v_2\|_{\dot{H}^s \cap \dot{H}^k(\mathbb{R}^n)} \end{aligned} \tag{A.4}$$

for all $|\epsilon|, |\kappa| \leq 1$ and all $u_1, u_2, u_3, v, v_1, v_2 \in \dot{H}^s_r(\mathbb{R}^n) \cap \dot{H}^k_r(\mathbb{R}^n)$ where v, v_1 and v_2 are real-valued and P is a polynomial of degree $\leq 2k+1$.

Proof. We will start with the case where u_1, u_2, u_3, v, v_1 and v_2 belong to $C^\infty_{c,r}(\mathbb{R}^n)$. By repeated application of the fundamental theorem of calculus, see for example [31], there exists for every ϵ a function $G_\epsilon \in C^\infty[0, \infty)$ with $G_\epsilon(x^2) = F_\epsilon(x)$. By the assumptions on F_ϵ we obtain for every $\ell \in \mathbb{N}_0$ the existence of a constant $\widetilde{C}_\ell \geq 0$ such that

$$|G_\epsilon^{(\ell)}(x^2) - G_\kappa^{(\ell)}(y^2)| \leq \widetilde{C}_\ell \, (|\epsilon - \kappa| + |x - y|) \, .$$

Now we choose s, k according to (A.2) and note that by Lemma 2.11 it is enough to bound the $\dot{H}^{\lfloor s-1 \rfloor} \cap \dot{H}^k$-norm of the respective left-hand side of (A.3) and (A.4). Therefore, we can reduce the analysis to estimating

$$\partial^\alpha \left(u_1 u_2 u_3 \left(G_\epsilon(|\cdot|^2 v^2) - G_\kappa(|\cdot|^2 v^2) \right) \right) \tag{A.5}$$

and

$$\partial^\alpha \left(u_1 u_2 u_3 \left(G_\epsilon(|\cdot|^2 v_1^2) - G_\epsilon(|\cdot|^2 v_2^2) \right) \right) \tag{A.6}$$

in $L^2(\mathbb{R}^n)$ for $|\alpha| \in \{\lfloor s-1 \rfloor, k\}$. For the first term we find that after applying the Leibniz rule and (A.1) it suffices to prove a suitable bound for

$$I(x) := x^\gamma \, \partial^{\alpha_1} u_1 \, \partial^{\alpha_2} u_2 \, \partial^{\alpha_3} u_3 \prod_{j=1}^{2\ell} \partial^{\beta_j} v, \tag{A.7}$$

where

$$\sum_{i=1}^{3}|\alpha_i|+\sum_{j=1}^{2\ell}|\beta_j|+2\ell-|\gamma|=|\alpha| \tag{A.8}$$

with the condition that $\ell\le|\alpha|$ and $|\gamma|\le 2\ell$. This corresponds exactly to the situation considered in [24], Proposition A.1 and arguing along these lines we infer that

$$\|I\|_{L^2(\mathbb{R}^n)}\lesssim\prod_{i=1}^{3}\|u_i\|_{\dot H^s\cap\dot H^k(\mathbb{R}^n)}\prod_{j=1}^{2\ell}\|v\|_{\dot H^s\cap\dot H^k(\mathbb{R}^n)} \tag{A.9}$$

for $|\alpha|\in\{\lfloor s-1\rfloor,k\}$. To handle the expression in (A.6) we prove the bound

$$\|J\|_{L^2(\mathbb{R}^n)}\lesssim\prod_{i=1}^{3}\|u_i\|_{\dot H^s\cap\dot H^k(\mathbb{R}^n)}\prod_{j=1}^{2\ell+1}\|v_j\|_{\dot H^s\cap\dot H^k(\mathbb{R}^n)} \tag{A.10}$$

for

$$J(x):=x^\gamma\,\partial^{\alpha_1}u_1\,\partial^{\alpha_2}u_2\,\partial^{\alpha_3}u_3\prod_{j=1}^{2\ell+1}\partial^{\beta_j}v_j \tag{A.11}$$

with functions $u_1,u_2,u_3,v_1,\dots,v_{2\ell+1}\in C^\infty_{c,r}(\mathbb{R}^n)$, and indices
$\ell\le|\alpha|$, $\alpha_1,\alpha_2,\alpha_3,\beta_1,\dots,\beta_{2\ell+1},\gamma\in\mathbb{N}_0^n$ with $|\gamma|\le 2\ell+1$ satisfying

$$|\alpha_1|+|\alpha_2|+|\alpha_3|+\sum_{j=1}^{2\ell+1}|\beta_j|+2\ell+1-|\gamma|=|\alpha|.$$

We will start with the case $|\alpha|=\lfloor s\rfloor-1$. Here we define for $i=2,3$ and $j=1,\dots,2\ell+1$

$$a_i:=|\alpha_i|+\frac{2\ell+1+s-|\alpha|}{2\ell+3}\quad\text{and}\quad b_j:=|\beta_j|+\frac{2\ell+1+s-|\alpha|}{2\ell+3}$$

and get

$$\|J\|_{L^2(\mathbb{R}^n)}\lesssim\left\||\cdot|^{|\alpha_1|-s}\partial^{\alpha_1}u_1\right\|_{L^2(\mathbb{R}^n)}\prod_{i=2}^{3}\||\cdot|^{a_i}\partial^{\alpha_i}u_i\|_{L^\infty(\mathbb{R}^n)}\prod_{j=1}^{2\ell+1}\left\||\cdot|^{b_j}\partial^{\beta_j}v_j\right\|_{L^\infty(\mathbb{R}^n)}$$

due to $|\gamma|=|\alpha_1|+a_2+a_3+\sum_{j=1}^{2\ell+1}b_j-s$. Since $0\le s-|\alpha_1|<\frac{n}{2}$ we can use Hardy's inequality ([28], p. 243, Theorem 9.5) for the first term and for the rest of the terms we use the generalized Strauss inequality (2.17) to obtain (A.10). We are allowed to use these inequalities since we have $0<a_i,b_j<\frac{n-1}{2}$ as well as

$$s\le\frac{n}{2}-a_i+|\alpha_i|<\frac{n}{2}\quad\text{and}\quad s\le\frac{n}{2}-b_j+|\beta_j|<\frac{n}{2}. \tag{A.12}$$

Since we are in the case where $|\alpha|$ is strictly smaller than s we immediately obtain $0<a_i,b_j$. For the upper bound on a_i and b_j one has

$$a_i=|\alpha_i|+1+\frac{s-\lfloor\frac{n}{2}\rfloor}{2\ell+3}\le\frac{n}{2}-1+\frac{s-\lfloor\frac{n}{2}\rfloor}{2\ell+3}\le\frac{n}{2}-1<\frac{n-1}{2}$$

and the same holds true for b_j. Since a_i and b_j are strictly larger than $|\alpha_i|$ and $|\beta_j|$ respectively the upper bound from (A.12) is immediate. The most restrictive inequality (for s) is $s\le\frac{n}{2}-a_i+|\alpha_i|$. This inequality is equivalent to $s+1+\frac{s-\lfloor\frac{n}{2}\rfloor}{2\ell+3}\le\frac{n}{2}$ which can be reformulated

as $2(\ell+2)s+2\ell+3 \leq (\ell+\frac{3}{2})n+\lfloor\frac{n}{2}\rfloor$. By making a case distinction between odd and even n and using the fact that $0 \leq \ell \leq \lfloor s \rfloor - 1$ this inequality holds for $s \leq \frac{n}{2} - 1 + \frac{1}{2n-2}$.
The case $|\alpha| = k$ will be divided into two sub-cases. First, we assume that the highest derivative in (A.11) is of order at least $\frac{n}{2} - 1$. Without loss of generality we assume that this derivative is α_1. We now split the L^2-norm of J into the unit ball and its complement. For $|x| \leq 1$ we get for $a_1 := \min\{1, k - |\alpha_1|\}$

$$|J(x)| \lesssim |x|^{-a_1}\, |\partial^{\alpha_1} u_1|\, |\partial^{\alpha_2} u_2|\, |\partial^{\alpha_3} u_3| \prod_{j=1}^{2\ell+1} |\partial^{\beta_j} v_j|.$$

For the first term we use Hardy's inequality and for the rest of the terms we want to use Lemma 2.12 to estimate

$$\|\partial^{\alpha_i} u_i\|_{L^\infty(\mathbb{R}^n)} \lesssim \|u_i\|_{\dot{H}^s\cap \dot{H}^k(\mathbb{R}^n)} \quad \text{and} \quad \left\|\partial^{\beta_j} v_j\right\|_{L^\infty(\mathbb{R}^n)} \lesssim \|v_j\|_{\dot{H}^s\cap \dot{H}^k(\mathbb{R}^n)}$$

for every $i = 2,3$ and $j = 1, \ldots, 2\ell+1$. To see this we consider first the case where $|\alpha_1|$ is strictly smaller than $k - \frac{n}{2}$. Since α_1 is by assumption the highest occuring derivative we also have $|\alpha_i|, |\beta_j| < k - \frac{n}{2}$ so that we can use the embedding. Is $|\alpha_1|$ greater or equal than $k - \frac{n}{2}$ we use the fact that there can only fall at most $k - |\alpha_1|$ derivatives onto u_i and v_j and also that we have chosen $k > n$ so that we obtain

$$|\alpha_i|, |\beta_j| \leq k - |\alpha_1| \leq \frac{n}{2} < k - \frac{n}{2}.$$

Now for the complement of the unit ball we first of all define

$$a_i := \frac{1}{2} + \min\{1, |\alpha_i|\}, \quad b_j := \frac{1}{2} + \min\{1, |\beta_j|\}$$

for every $i = 2,3$ and $j = 1, \ldots, 2\ell+1$. Since we now have $|\gamma| \leq a_2 + a_3 + \sum_{j=1}^{2\ell+1} b_j - a_1$ due to $|\gamma| \leq \ell + \sum_{j=1}^{2\ell+1} \min\{1, |\beta_j|\}$ we can estimate

$$|J(x)| \lesssim |x|^{-a_1}\, |\partial^{\alpha_1} u_1|\, |x|^{a_2} |\partial^{\alpha_2} u_2|\, |x|^{a_3} |\partial^{\alpha_3} u_3| \prod_{j=1}^{2\ell+1} |x|^{b_j} |\partial^{\beta_j} v_j|$$

for all $|x| \geq 1$. The L^2-norm of the first term gets again estimated by Hardy's inequality and then by the $\dot{H}^s \cap \dot{H}^k$-norm of u_1 since we have $s \leq a_1 + |\alpha_1| \leq k$. We estimate the other terms in the $L^\infty-$norm by (2.17). We can apply this Lemma due to the fact that we again have $0 < a_i, b_j < \frac{n-1}{2}$. The $\dot{H}^s \cap \dot{H}^k$-estimate then follows from $s \leq \frac{n}{2} - a_i + |\alpha_i| \leq k$ where the upper bound can be seen by making a case distinction for $|\alpha_1|$ strictly greater than $k - \frac{n-3}{2}$ or smaller than that term. Over all we have finished the proof in the case where one derivative is at least of order $\frac{n}{2} - 1$.
We now assume that all of the derivatives in (A.11) are of order strictly smaller than $\frac{n}{2} - 1$, which implies $|\alpha_i|, |\beta_j| \leq \frac{n-3}{2}$ for all $i = 1,2,3$ and $j = 1, \ldots, 2\ell+1$. On the unit ball we can immediately estimate

$$|J(x)| \lesssim |x|^{|\alpha_1| - \frac{n-1}{2}}\, |\partial^{\alpha_1} u_1|\, |\partial^{\alpha_2} u_2|\, |\partial^{\alpha_3} u_3| \prod_{j=1}^{2\ell+1} |\partial^{\beta_j} v_j|$$

and obtain (A.10) with Hardy's inequality and the embedding from Lemma 2.12.
For the complement of the unit ball we define $\tilde{a}_i := |\alpha_i| + \frac{3}{4}, \tilde{b}_j := |\beta_j| + \frac{3}{4}$ and estimate

$$|J(x)| \lesssim |x|^{|\alpha_1| - \frac{n-1}{2}} |\partial^{\alpha_1} u_1| \, |x|^{\tilde{a}_2} |\partial^{\alpha_2} u_2| \, |x|^{\tilde{a}_3} |\partial^{\alpha_3} u_3| \prod_{j=1}^{2\ell+1} |x|^{\tilde{b}_j} |\partial^{\beta_j} v_j|$$

for all $|x| \geq 1$ since we have $|\gamma| \leq |\alpha_1| - \frac{n-1}{2} + \tilde{a}_2 + \tilde{a}_3 + \sum_{j=1}^{2\ell+1} \tilde{b}_j$. The claim now follows by again applying Hardy's inequality and (2.17) due to the fact that we again have $0 < \tilde{a}_i, \tilde{b}_j < \frac{n-1}{2}$ as well as $s \leq \frac{n}{2} - \frac{3}{4} \leq k$. We have therefore shown both inequalities in the case where all the functions belong to $C^\infty_{c,r}(\mathbb{R}^n)$.
The general case now follows via a density argument and the $L^\infty-$embedding of the Sobolev spaces. We will only give the details for the first inequality, the second one can be handled in a similar manner. Take $u_1, u_2, u_3, v \in \dot{H}^s_r(\mathbb{R}^n) \cap \dot{H}^k_r(\mathbb{R}^n)$. Then there exist sequences $(u_{1,j})_{j\in\mathbb{N}}, (u_{2,j})_{j\in\mathbb{N}}, (u_{3,j})_{j\in\mathbb{N}}$ and $(v_j)_{j\in\mathbb{N}}$ in $C^\infty_{c,r}(\mathbb{R}^n)$ which converge to u_1, u_2, u_3 and v in the $\dot{H}^s(\mathbb{R}^n) \cap \dot{H}^k(\mathbb{R}^n)-$norm, respectively. Eqns. (A.3) and (A.4) imply that

$$\left(u_{1,j} u_{2,j} u_{3,j} \left(F_\epsilon(|\cdot|v_j) - F_\kappa(|\cdot|v_j)\right)\right)_{j\in\mathbb{N}}$$

forms a Cauchy sequence in $\dot{H}^{s-1}(\mathbb{R}^n) \cap \dot{H}^k(\mathbb{R}^n)$. Due to the $L^\infty-$embedding one can easily show that this sequence converges pointwise to $u_1 u_2 u_3 (F_\epsilon(|\cdot|v) - F_\kappa(|\cdot|v)$ and therefore also in the $\dot{H}^{s-1} \cap \dot{H}^k(\mathbb{R}^n)-$norm. With this observation (A.3) follows for arbitrary $u_1, u_2, u_3, v \in \dot{H}^s_r(\mathbb{R}^n) \cap \dot{H}^k_r(\mathbb{R}^n)$.

□

Appendix B. Proof of Lemma 2.1

Proof. To prove this Lemma we will argue along the lines of [25], p.12, Lemma 2.2. We take a radial cut-off function $\chi \in C^\infty_{c,r}(\mathbb{R}^n)$ with $0 \leq \chi \leq 1$ satisfying

$$\chi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

Then we consider the sequence $(f_j)_{j\in\mathbb{N}} \subset C^\infty_{c,r}(\mathbb{R}^n)$ for $f_j(x) := f(x)\chi(\frac{x}{j})$ for $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$.
Now we take an arbitrary $s \geq 0$ with $s > \frac{n}{2} - k$. We note that we have by the homogeneous Sobolev embedding (see [36], p. 335)

$$\|u\|_{\dot{W}^{t,q}(\mathbb{R}^n)} \lesssim \|u\|_{\dot{W}^{\ell,p}(\mathbb{R}^n)} \tag{B.1}$$

for every $u \in C^\infty_c(\mathbb{R}^n)$ whenever $1 < p \leq q < \infty$ and $t, \ell \geq 0$ obey the scaling condition $\ell - \frac{n}{p} = t - \frac{n}{q}$. Here, the space $\dot{W}^{s,p}(\mathbb{R}^n)$ for $s \geq 0$ and $1 \leq p < \infty$ is given by the completion of $C^\infty_c(\mathbb{R}^n)$ under the norm

$$\|u\|_{\dot{W}^{s,p}(\mathbb{R}^n)} := \left\|\mathcal{F}^{-1}[|\cdot|^s \mathcal{F}u]\right\|_{L^p(\mathbb{R}^n)}$$

and it coincides with the one we introduced in section 2.1 for $p = 2$ due to Plancherel. Furthermore, if $s \in \mathbb{N}_0$ is a non-negative integer we also have a representation via partial derivatives in the sense that

$$\|u\|_{\dot{W}^{s,p}(\mathbb{R}^n)} \simeq \sum_{|\beta|=s} \left\|\partial^\beta u\right\|_{L^p(\mathbb{R}^n)} \tag{B.2}$$

holds for every $u \in C_c^\infty(\mathbb{R}^n)$.
If we now choose $\ell \in \mathbb{N}_0$ with $s \leq \ell < s + \frac{n}{2}$ (note that we have $n \geq 2$) we obtain from (B.1)

$$\|f_j\|_{\dot{H}^s(\mathbb{R}^n)} \lesssim \|f_j\|_{\dot{W}^{\ell,p}(\mathbb{R}^n)}\,, \tag{B.3}$$

where p is defined via $\frac{1}{p} = \frac{1}{2} + \frac{\ell - s}{n}$ and fulfills $\frac{1}{2} \leq \frac{1}{p} < 1$ by the choice of ℓ. We now show that $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $\dot{W}_r^{\ell,p}(\mathbb{R}^n)$. Using the Leibniz rule it is enough to show

$$\left\|\partial^\gamma f\, \partial^{\beta-\gamma}\left(\chi_j - \chi_i\right)\right\|_{L^p(B_{i,j})} \to 0 \quad \text{as} \quad i, j \to \infty$$

for all multi-indices $\beta, \gamma \in \mathbb{N}_0^n$ satisfying $|\beta| = \ell$ and $\gamma \leq \beta$ and $B_{i,j} := B_{2\max\{i,j\}} \backslash B_{\min\{i,j\}}$.
For $\gamma = \beta$ we have

$$\left\|\partial^\beta f\ (\chi_j - \chi_i)\right\|_{L^p(B_{i,j})} \lesssim \int_{B_{i,j}} |\partial^\beta f(x)|^p\, dx \lesssim \int_{\min\{i,j\}}^{2\max\{i,j\}} r^{-p|\beta|-pk+n-1} dr$$

and the last integral converges towards 0 as i and j approach infinity due to the assumptions on p and β.
For $\gamma \neq \beta$ we have $|\gamma| < |\beta|$ and get

$$\left\|\partial^\gamma f\, \partial^{\beta-\gamma}\left(\chi_j - \chi_i\right)\right\|_{L^p(B_{i,j})} = \left\|\partial^\gamma f\left(j^{|\gamma|-|\beta|}\, \partial^{\beta-\gamma}\chi(\cdot/j) - i^{|\gamma|-|\beta|}\, \partial^{\beta-\gamma}\chi(\cdot/i)\right)\right\|_{L^p(B_{i,j})}.$$

Since both terms from above can be treated analogously we will only consider the first one:

$$\begin{aligned} j^{|\gamma|-|\beta|}\left\|\partial^\gamma f\, \partial^{\beta-\gamma}\chi(\cdot/j)\right\|_{L^p(B_{i,j})} &\lesssim j^{|\gamma|-|\beta|}\, \|\partial^\gamma f\|_{L^p(B_{2j}\backslash B_1)} \\ \lesssim j^{|\gamma|-|\beta|}\left(\int_{B_{2j}\backslash B_1} |\partial^\gamma f(x)|^p\, dx\right)^{\frac{1}{p}} &\lesssim j^{|\gamma|-|\beta|}\left(\int_1^{2j} r^{-p|\gamma|-pk+n-1} dr\right)^{\frac{1}{p}}. \end{aligned}$$

For $\frac{n}{p} \neq |\gamma| + k$ we obtain

$$j^{|\gamma|-|\beta|}\left(\int_1^{2j} r^{-p|\gamma|-pk+n-1} dr\right)^{\frac{1}{p}} \lesssim j^{|\gamma|-|\beta|}\left(j^{-p|\gamma|-pk+n} - 1\right)^{\frac{1}{p}} \lesssim j^{\frac{n}{p}-|\beta|-k}$$

and for $\frac{n}{p} = |\gamma| + k$

$$j^{|\gamma|-|\beta|}\left(\int_1^{2j} r^{-p|\gamma|-pk+n-1} dr\right)^{\frac{1}{p}} = j^{|\gamma|-|\beta|} \log(2j)^{\frac{1}{p}}.$$

Since both of these terms converge due to the assumptions on s and the fact that we are in the case $|\gamma| < |\beta|$ to 0 as j goes to infinity we have shown that $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $\dot{W}_r^{\ell,p}(\mathbb{R}^n)$.
By (B.3) we obtain that this sequence is also Cauchy in $\dot{H}_r^s(\mathbb{R}^n)$. Therefore, we have shown that $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $\dot{H}_r^s(\mathbb{R}^n)$ for every $s \geq 0$ with $s > \frac{n}{2} - k$. For every such s, we therefore infer the existence of a function $g \in \dot{H}_r^{s_1}(\mathbb{R}^n) \cap \dot{H}_r^{s_2}(\mathbb{R}^n)$ with $s_1 \leq s < s_2$ where s_1 is strictly smaller than $\frac{n}{2}$ (here one needs that k is strictly greater than 0) and s_2 is strictly greater than $\frac{n}{2}$, so that $(f_j)_{j\in\mathbb{N}}$ converges to g in the $\dot{H}^{s_1} \cap \dot{H}^{s_2}$−norm. By (2.12)

we know that $(f_j)_{j\in\mathbb{N}}$ must therefore also converge to g in $L^\infty(\mathbb{R}^n)$ (in particular pointwise) so that we conclude $f = g \in \dot{H}^{s_1}_r(\mathbb{R}^n) \cap \dot{H}^{s_2}_r(\mathbb{R}^n) \subset \dot{H}^s_r(\mathbb{R}^n)$ which finishes the proof.

□

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