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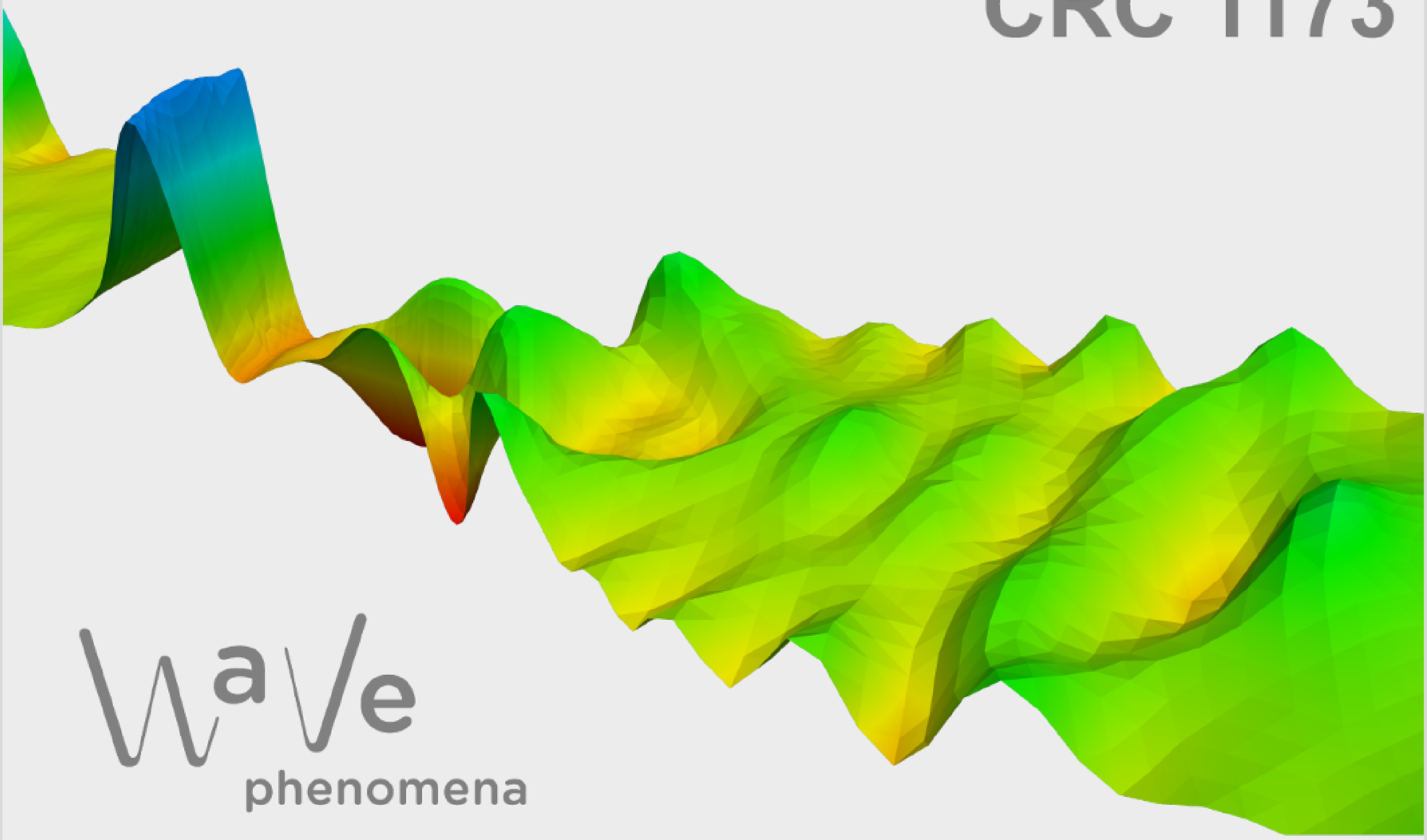

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IMPROVED ERROR ESTIMATES FOR LOW-REGULARITY INTEGRATORS USING SPACE-TIME BOUNDS

MAXIMILIAN RUFF

Abstract. We prove optimal convergence rates for certain low-regularity integrators applied to the one-dimensional periodic nonlinear Schrödinger and wave equations under the assumption of H^1 solutions. For the Schrödinger equation we analyze the exponential-type scheme proposed by Ostermann and Schratz in 2018, whereas in the wave case we treat the corrected Lie splitting proposed by Li, Schratz, and Zivcovich in 2023. We show that the integrators converge with their full order of one and two, respectively. In this situation only fractional convergence rates were previously known. The crucial ingredients in the proofs are known space-time bounds for the solutions to the corresponding linear problems. More precisely, in the Schrödinger case we use the L^4 Strichartz inequality, and for the wave equation a null form estimate. To our knowledge, this is the first time that a null form estimate is exploited in numerical analysis. We apply the estimates for continuous time, thus avoiding potential losses resulting from discrete-time estimates.

1. Introduction

Due to their importance as a model problem in mathematical physics, the nonlinear Schrödinger and wave equations have been intensively studied in the past decades, both analytically and numerically. In this work we study their numerical time integration in the one-dimensional cases. We treat the semilinear Schrödinger equation

$$\begin{aligned} \mathrm{i}\partial_t u + \partial_x^2 u &= \mu |u|^2 u, \quad (t,x) \in [0,T] \times \mathbb{T}, \\ u(0) &= u_0 \in H^1(\mathbb{T}), \end{aligned} \tag{1.1}$$

where we allow for both signs $\mu \in \{\pm 1\}$. Our second problem is the semilinear wave equation

$$\begin{aligned} \partial_t^2 u - \partial_x^2 u &= g(u), \quad (t,x) \in [0,T] \times \mathbb{T}, \\ u(0) &= u_0 \in H^1(\mathbb{T}), \\ \partial_t u(0) &= v_0 \in L^2(\mathbb{T}), \end{aligned} \tag{1.2}$$

with a general nonlinearity $g \in C^2(\mathbb{R}, \mathbb{R})$. Our regularity assumptions on the initial data are natural in view of the energy conservation laws.

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For the time discretization of dispersive or hyperbolic equations such as (1.1) and (1.2), *low-regularity integrators* have recently gained a lot of attention in the literature. See, e.g., [2, 3, 4, 9, 11, 12, 13, 14] for some important contributions. In low-regularity settings, these tailor-made integrators can outperform more classical schemes (such as splitting methods [10] or classical exponential integrators [5, 6]) thanks to an improved local error structure which requires less regularity.

In the present paper we analyze two known schemes of this type, the first-order integrator for (1.1) from [13] and the second-order scheme for (1.2) from [9]. We show that they converge with their full order in particular situations where previously only fractional convergence orders were known. The general outline of proof is in both cases the same. We first derive a suitable representation of the local error. This has the property that in a second step, the sum of the local error terms can be optimally estimated exploiting an equation-specific space-time inequality for the solution u. Here we only use the estimates for continuous time, since discrete-time estimates often involve a loss, see [11, 12, 16]. The proof of the error bound is then completed in a classical way by a discrete Gronwall argument. Hence, our proof strategy is very flexible and could possibly be adapted to show error bounds also for other equations and integrators. In this work we only analyze the temporal semi-discretization, but expect that an extension to a fully discrete setting is possible.

1.1. **The Schrödinger case.**

In the seminal paper [13], a low-regularity integrator was proposed for the time integration of the nonlinear Schrödinger equation (1.1) (and also its higher-dimensional versions). The scheme computes approximation $u_n \approx u(n\tau)$ via

$$u_{n+1} = \Phi_\tau(u_n) := e^{\mathrm{i}\tau\partial_x^2}\Big(u_n - \mathrm{i}\tau\mu(u_n)^2\varphi_1(-2\mathrm{i}\tau\partial_x^2)\bar{u}_n\Big). \tag{1.3}$$

The operator $\varphi_1(-2\mathrm{i}\tau\partial_x^2)$ can be defined in Fourier space or using the functional calculus for $\varphi_1(z) = (e^z - 1)/z$. For our purposes, the definition via the integral representation

$$\varphi_1(-2\mathrm{i}\tau\partial_x^2)f := \frac{1}{\tau}\int_0^\tau e^{-2\mathrm{i}s\partial_x^2} f \,\mathrm{d}s \tag{1.4}$$

for $f \in L^2(\mathbb{T})$ is convenient. The authors in [13] proved a general convergence result which in the one-dimensional case reads as follows.

Theorem 1.1 ([13]). *Let $r > 1/2$ and $\gamma \in (0, 1]$. Assume that the solution u to* (1.1) *satisfies $u(t) \in H^{r+\gamma}(\mathbb{T})$ for all $t \in [0, T]$. Then there are a constant $C > 0$ and a maximum step size $\tau_0 > 0$ such that the approximations u_n obtained by* (1.3) *satisfy the error bound*

$$\|u(n\tau) - u_n\|_{H^r(\mathbb{T})} \le C\tau^\gamma$$

for all $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}_0$ with $n\tau \le T$. The numbers C and τ_0 only depend on T and $\|u\|_{L^\infty([0,T],H^{r+\gamma}(\mathbb{T}))}$.

We will later make use of Theorem 1.1 since it provides an a-priori bound in L^∞ for the numerical solution u_n if τ is small enough. The condition $r > 1/2$ in Theorem 1.1 arises from the use of the algebra property of the

Sobolev space $H^r(\mathbb{T})$. The space $L^2(\mathbb{T})$ does not have this property, so it is a natural question if Theorem 1.1 still holds if $r=0$ and $\gamma=1$. This was addressed in the follow-up work [11], where the problem (1.1) was considered with spatial domain $\mathbb{R}^d$, $d\in\{1,2,3\}$. The difficulty is that the local error of the scheme (1.3) is roughly of the form

$$\tau^2|\partial_x u|^2 u,$$

cf. p.731 of [11]. Estimating this is L^2 for fixed times would require that $u\in W^{1,4}$, which is not covered by the assumption $u\in H^1$. It is however known that solutions to dispersive equations such as (1.1) enjoy better integrability properties in space if we also involve integration in the time variable. This is formalized using *Strichartz estimates*, which control mixed space-time L^pL^q norms of solutions to linear dispersive equations in terms of the data, cf. Chapter 2.3 of [17]. In [11], the authors proved discrete-time Strichartz estimates and used them to show fractional convergence rates (strictly between 1/2 and 1 depending on the dimension) in L^2 for a frequency-filtered variant of (1.3). In the case $d=1$, the convergence rate was 5/6. In the subsequent paper [12], the authors analyzed the problem (1.1) on the torus $\mathbb{T}$. They introduced *discrete Bourgain spaces* and used them to prove a convergence rate of almost 7/8 for a significantly refined frequency-filtered variant of (1.3).

The reason why in [11, 12] the optimal first-order convergence could not be reached is that the discrete-time Strichartz and Bourgain space estimates only hold for frequency localized functions. Moreover, they contain a multiplicative loss depending on $K\tau^{1/2}$, where K is the largest frequency and τ denotes the time step-size. The continuous-time Strichartz estimates however do not suffer from those disadvantages. In this work we extend Theorem 1.1 to the case $r=0$ and $\gamma=1$ with optimal first-order convergence. In contrast to [11, 12], we do not use frequency filtering and discrete-time Strichartz or Bourgain space estimates. Instead, we derive an error representation which allows us to apply the continuous-time periodic Strichartz estimate

$$\|e^{\mathrm{i}t\partial_x^2}f\|_{L^4([0,T]\times\mathbb{T})}\lesssim_T\|f\|_{L^2(\mathbb{T})}. \tag{1.5}$$

A proof of (1.5) can be found in Theorem 1 and the subsequent remark of [18] or Proposition 2.1 of [1]. The idea to use continuous-time Strichartz estimates to control the local error goes back to [7].

Our convergence result in L^2 reads as follows. Its proof is carried out in Section 2.

Theorem 1.2. *Assume that the solution u to* (1.1) *satisfies $u(t)\in H^1(\mathbb{T})$ for all $t\in[0,T]$. Then there are a constant $C>0$ and a maximum step size $\tau_0>0$ such that the approximations u_n obtained by* (1.3) *satisfy the error bound*

$$\|u(n\tau)-u_n\|_{L^2(\mathbb{T})}\le C\tau$$

for all $\tau\in(0,\tau_0]$ and $n\in\mathbb{N}_0$ with $n\tau\le T$. The numbers C and τ_0 only depend on T and $\|u\|_{L^\infty([0,T],H^1(\mathbb{T}))}$.

We comment on possible extensions of Theorem 1.2 to higher dimensions. The embedding $H^1\hookrightarrow L^\infty$ as well as the estimate (1.5), which are both crucially exploited in the proof of Theorem 1.2, are then wrong, in general.

In two dimensions, they however both only require an arbitrary small amount of extra regularity (see Proposition 3.6 of [1] for the 2D version of (1.5)). Therefore, it is possible to extend Theorem 1.2 to the 2D case under the slightly stronger regularity assumption $u \in H^{1+\varepsilon}$ for some $\varepsilon > 0$. One could also stick to the H^1 assumption if one considers a suitably filtered variant of (1.3) and lowers the convergence rate by ε. The three-dimensional case seems to be more difficult and we do not know how the optimal result then looks like. The situation becomes easier if the torus $\mathbb{T}^d$ is replaced by the full space $\mathbb{R}^d$, since then a wider range of Strichartz estimates becomes applicable, cf. Chapter 2.3 of [17]. It seems also possible to extend our analysis to the symmetric two-step variant of (1.3) that was recently proposed in [4].

1.2. **The wave case.** For the nonlinear wave equation (1.2), the authors in [9] proposed a low-regularity integrator which was called the *corrected Lie splitting*. It computes approximations $(u_n, v_n) \approx (u(n\tau), \partial_t u(n\tau))$ via

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = e^{\tau A}\left[\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \tau \begin{pmatrix} 0 \\ g(u_n) \end{pmatrix} + \tau^2 \varphi_2(-2\tau A)\begin{pmatrix} -g(u_n) \\ g'(u_n)v_n \end{pmatrix}\right], \tag{1.6}$$

with wave operator $A(u,v) = (v, \partial_x^2 u)$. The operator $\varphi_2(-2\tau A)$ is defined by the integral representation

$$\varphi_2(-2\tau A)w \coloneqq \frac{1}{\tau^2}\int_0^\tau (\tau - s)e^{-2sA} w \,\mathrm{d}s \tag{1.7}$$

for $w \in H^1 \times L^2$. Similar as in the Schrödinger case, one could equivalently use the functional calculus for $\varphi_2(z) = (e^z - z - 1)/z^2$. In [9], under a Lipschitz condition on the nonlinearity g, it was shown that the scheme (1.6) converges with order 2 in $H^1 \times L^2$ under the regularity assumption $(u, \partial_t u) \in H^{1+d/4} \times H^{d/4}$ for spatial dimensions $d \in \{1, 2, 3\}$. The reason for this additional regularity requirement is that the main part of the local error is roughly of the form

$$\|(\partial_t u)^2 - \nabla u \cdot \nabla u\|_{L^2(\mathbb{T}^d)}, \tag{1.8}$$

cf. equation (2.26) of [9]. This term was then estimated (for fixed times) using the triangle inequality and the Sobolev embedding $H^{d/4}(\mathbb{T}^d) \hookrightarrow L^4(\mathbb{T}^d)$. For the one-dimensional case $d = 1$, the authors in [9] also gave a convergence result under the weaker regularity assumption $(u, \partial_t u) \in H^1 \times L^2$. Using an interpolation argument, it was shown that the scheme (1.6) converges almost with order 4/3 in $H^1 \times L^2$. However, the numerical experiments in [9] suggested that the convergence is of order 2 also in this case.

Here, we give a rigorous proof of this second-order convergence. In contrast to the Schrödinger case, the 1D wave equation does not exhibit dispersive behavior. Instead, the idea is to exploit that the expression (1.8) contains a so-called *null form* which allows for improved space-time bounds compared to the above fixed-time approach. Such null form estimates are widely used in the analysis of nonlinear wave equations, cf. [8] or pp.292 of [17]. They rely on cancellation of parallel interactions (where waves move together) in the bilinear expression in (1.8). In the one-dimensional case one has the following estimate. If ϕ solves the linear inhomogeneous wave equation $\partial_t^2\phi - \partial_x^2\phi = F$

on $[0,T]\times\mathbb{T}$, then one has the inequality

$$\begin{aligned}&\|(\partial_t\phi)^2-(\partial_x\phi)^2\|_{L^2([0,T]\times\mathbb{T})}\\&\lesssim_T \|\partial_x\phi(0)\|^2_{L^2(\mathbb{T})}+\|\partial_t\phi(0)\|^2_{L^2(\mathbb{T})}+\|F\|^2_{L^1([0,T],L^2(\mathbb{T}))}.\end{aligned}\tag{1.9}$$

Note that the right-hand side of (1.9) only contains the L^2 norm of $\partial_{t,x}\phi(0)$ instead of the L^4 norm that would result from the triangle inequality approach. If we replace $\mathbb{T}$ with $\mathbb{R}$, estimate (1.9) can be found in (1.8) of [8] (in a simplified form) or in (6.29) of [17]. For convenience, we give a direct proof of (1.9) on $\mathbb{T}$ based on d'Alembert's formula at the beginning of Section 3.

With the help of estimate (1.9), we are able to show the following improved error bound for the corrected Lie splitting (1.6). The proof is given in Section 3. To our knowledge, this is the first time that a null form estimate like (1.9) is used in numerical analysis.

Theorem 1.3. *Assume that the solution u to* (1.2) *satisfies $(u(t),\partial_t u(t))\in H^1(\mathbb{T})\times L^2(\mathbb{T})$ for all $t\in[0,T]$. Then there are a constant $C>0$ and a maximum step size $\tau_0>0$ such that the approximations (u_n,v_n) obtained by* (1.6) *satisfy the error bound*

$$\|u(n\tau)-u_n\|_{H^1(\mathbb{T})}+\|\partial_t u(n\tau)-v_n\|_{L^2(\mathbb{T})}\le C\tau^2$$

for all $\tau\in(0,\tau_0]$ and $n\in\mathbb{N}_0$ with $n\tau\le T$. The numbers C and τ_0 only depend on g, T, $\|u\|_{L^\infty([0,T],H^1(\mathbb{T}))}$, and $\|\partial_t u\|_{L^\infty([0,T],L^2(\mathbb{T}))}$.

The higher-dimensional versions of the null form estimate (1.9) require more regularity, cf. [8] or inequality (6.29) of [17]. In two dimensions, they could possibly still be used to show an analogue of Theorem 1.3 with a convergence rate greater than one under a suitable growth condition on g. Very recently, convergence rates for a Strang splitting scheme for the 3D semilinear wave equation with power nonlinearity under the assumption $(u,\partial_t u)\in H^1\times L^2$ were obtained in [15]. We do not know whether in this situation it is possible to show higher rates by using a low-regularity integrator instead.

Notation. We use the notation $A\lesssim_\gamma B$ if there is a constant $c>0$ (depending on quantities γ) such that $A\le cB$. If it is clear from the context, we often abbreviate $H^1=H^1(\mathbb{T})$ as well as $L^p=L^p(\mathbb{T})$. For a step size $\tau>0$ and a number $n\in\mathbb{N}_0$, the discrete times are denoted by $t_n:=n\tau$.

2. Proof of the result for the nonlinear Schrödinger equation

In this section we prove Theorem 1.2. We start by converting the linear estimate (1.5) into a bound for the solution u to the nonlinear problem (1.1).

Assumption 2.1. There exists a time $T>0$ and a solution $u\in C([0,T],H^1)\cap C^1([0,T],H^{-1})$ to the nonlinear Schrödinger equation (1.1) with bound

$$M:=\|u\|_{L^\infty([0,T],H^1)}.$$

Corollary 2.2. *Let u, T, and M be given by Assumption 2.1. Then we have the estimate*

$$\|\partial_x u\|_{L^4([0,T]\times\mathbb{T})}\lesssim_{M,T} 1.$$

Proof. We apply estimate (1.5) to Duhamel's formula

$$u(t) = e^{\mathrm{i}t\partial_x^2}u_0 - \mathrm{i}\mu\int_0^t e^{\mathrm{i}(t-s)\partial_x^2}(|u|^2u)(s)\,\mathrm{d}s.$$

Using also Minkowski's inequality and Sobolev's embedding $H^1 \hookrightarrow L^\infty$, we get

$$\begin{aligned}\|\partial_x u\|_{L^4([0,T]\times\mathbb{T})} &\lesssim_T \|\partial_x u_0\|_{L^2} + \int_0^T \|\partial_x(|u|^2u)(s)\|_{L^2}\,\mathrm{d}s\\ &\lesssim_M 1 + \|u\|^2_{L^2([0,T],L^\infty)}\|\partial_x u\|_{L^\infty([0,T],L^2)} \lesssim_{M,T} 1. \qquad \square\end{aligned}$$

We now give a representation of the local error of the low-regularity integrator (1.3). The calculations are inspired by the ones in Section 3 of [14]. But compared to there and also [13], we do not insert the approximation $u(s) \approx e^{\mathrm{i}s\partial_x^2}u_0$ at first. This makes it easier for us to apply Corollary 2.2 in the subsequent Lemma 2.4.

Lemma 2.3. *Let u and T be given by Assumption 2.1. Then for $\tau \in (0,T]$, the local error of* (1.3) *is given by*

$$u(\tau) - u_1 = \mu\int_0^\tau\int_0^s e^{\mathrm{i}(\tau-\sigma)\partial_x^2}D(\sigma,s)\,\mathrm{d}\sigma\,\mathrm{d}s.$$

Here we define

$$D(\sigma,s) = D_1(\sigma,s) + D_2(\sigma,s) + D_3(\sigma,s)$$

with

$$\begin{aligned}D_1(\sigma,s) &\coloneqq \mu u(\sigma)^2\Big(e^{2\mathrm{i}(\sigma-s)\partial_x^2}(|u|^2\bar u)(\sigma) - 2|u(\sigma)|^2e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar u(\sigma)\Big),\\ D_2(\sigma,s) &\coloneqq -2(\partial_x u(\sigma))^2e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar u(\sigma),\\ D_3(\sigma,s) &\coloneqq -4u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar u(\sigma).\end{aligned}$$

Proof. By (1.4), we have

$$\tau\varphi_1(-2\mathrm{i}\tau\partial_x^2)\bar u_0 = \int_0^\tau e^{-2\mathrm{i}s\partial_x^2}\bar u_0\,\mathrm{d}s.$$

Hence, we get by Duhamel's formula and the fundamental theorem of calculus

$$\begin{aligned}u(\tau) - u_1 &= -\mathrm{i}\mu e^{\mathrm{i}\tau\partial_x^2}\int_0^\tau\Big(e^{-\mathrm{i}s\partial_x^2}(u^2\bar u)(s) - u_0^2e^{-2\mathrm{i}s\partial_x^2}\bar u_0\Big)\,\mathrm{d}s\\ &= \mu e^{\mathrm{i}\tau\partial_x^2}\int_0^\tau(N(s,s) - N(0,s))\,\mathrm{d}s = \mu e^{\mathrm{i}\tau\partial_x^2}\int_0^\tau\int_0^s\partial_1N(\sigma,s)\,\mathrm{d}\sigma\,\mathrm{d}s.\end{aligned}$$

Here, the function $N(\cdot,s) \in C^1([0,\tau],H^{-1}(\mathbb{T}))$ is defined as

$$N(\sigma,s) \coloneqq -\mathrm{i}e^{-\mathrm{i}\sigma\partial_x^2}\Big(u(\sigma)^2e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar u(\sigma)\Big).$$

Using the product rule and the differential equation (1.1), we compute the derivative as

$$\begin{aligned}\partial_1N(\sigma,s) &= e^{-\mathrm{i}\sigma\partial_x^2}\Big[-\partial_x^2\Big(u(\sigma)^2e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar u(\sigma)\Big) - 2\mathrm{i}u(\sigma)\partial_t u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar u(\sigma)\\ &\qquad + 2u(\sigma)^2e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar u(\sigma) - \mathrm{i}u(\sigma)^2e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_t\bar u(\sigma)\Big]\\ &= e^{-\mathrm{i}\sigma\partial_x^2}\Big[-2\partial_x^2u(\sigma)u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar u(\sigma) - 2(\partial_x u(\sigma))^2e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar u(\sigma)\end{aligned}$$

$$
\begin{aligned}
&- 4u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma) - u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma)\\
&+ 2u(\sigma)\partial_x^2 u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(\sigma) - 2\mu u(\sigma)(|u|^2u)(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(\sigma)\\
&+ 2u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma) - u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma)\\
&+ \mu u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}(|u|^2\bar{u})(\sigma)\Big]\\
= {}& e^{-\mathrm{i}\sigma\partial_x^2}\Big[- 2(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(\sigma)\\
&- 4u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma)\\
&+ \mu u(\sigma)^2\Big(- 2|u(\sigma)|^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(\sigma) + e^{2\mathrm{i}(\sigma-s)\partial_x^2}(|u|^2\bar{u})(\sigma)\Big)\Big]\\
= {}& e^{-\mathrm{i}\sigma\partial_x^2}\Big[D_1(\sigma,s) + D_2(\sigma,s) + D_3(\sigma,s)\Big],
\end{aligned}
$$

where we exploit the cancellation of all second-order partial derivatives. The derivative is well-defined in $H^{-1}(\mathbb{T})$ since in 1D we can use the embedding $L^1 \hookrightarrow H^{-1}$ and that the multiplication by an H^1 function is a continuous operator on H^{-1}. □

In the next step we bound the sum of the local errors terms, where we will crucially exploit Corollary 2.2 as well as the dual of estimate (1.5).

Lemma 2.4. *Let u, T, and M be given by Assumption 2.1. Then we can bound the sum of local errors of* (1.3) *by*

$$
\Big\|\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2}\Big(u(t_{k+1}) - \Phi_\tau(u(t_k))\Big)\Big\|_{L^2} \lesssim_{M,T} \tau,
$$

for all $\tau \in (0,T]$ and $n \in \mathbb{N}_0$ with $n\tau \le T$.

Proof. By Lemma 2.3 with $u(t_k + \cdot)$ instead of u,

$$
\begin{aligned}
&\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2}\Big(u(t_{k+1}) - \Phi_\tau(u(t_k))\Big)\\
&= \mu\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2}\int_0^\tau\int_0^s e^{-\mathrm{i}\sigma\partial_x^2}D(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s.
\end{aligned}
$$

We now use the decomposition $D = D_1 + D_2 + D_3$ from Lemma 2.3. For the first term we even get

$$
\Big\|\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2}\int_0^\tau\int_0^s e^{-\mathrm{i}\sigma\partial_x^2}D_1(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s\Big\|_{H^1} \lesssim_{M,T} n\tau^2 \lesssim_T \tau,
$$

using the algebra property of H^1 in 1D. The second term is controlled by

$$
\begin{aligned}
&\Big\|\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2}\int_0^\tau\int_0^s e^{-\mathrm{i}\sigma\partial_x^2}D_2(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s\Big\|_{L^2}\\
&\le \sum_{k=0}^{n-1}\int_0^\tau\int_0^s \|D_2(t_k+\sigma, t_k+s)\|_{L^2}\,\mathrm{d}\sigma\,\mathrm{d}s\\
&\lesssim \sum_{k=0}^{n-1}\int_0^\tau\int_0^\tau \|(\partial_x u(t_k+\sigma))^2\|_{L^2}\|e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(t_k+\sigma)\|_{L^\infty}\,\mathrm{d}\sigma\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \tau \sum_{k=0}^{n-1} \int_0^\tau \|\partial_x u(t_k+\sigma)\|_{L^4}^2 \|u(t_k+\sigma)\|_{H^1} \,\mathrm{d}\sigma \lesssim_M \tau \|\partial_x u\|^2_{L^2([0,T],L^4)} \\
&\lesssim_T \tau \|\partial_x u\|^2_{L^4([0,T]\times\mathbb{T})} \lesssim_{M,T} \tau.
\end{aligned}
$$

Here we used Hölder's inequality, the Sobolev embedding $H^1 \hookrightarrow L^\infty$, and Corollary 2.2. The term involving D_3 is first rewritten as

$$
\begin{aligned}
&\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^s e^{-\mathrm{i}\sigma\partial_x^2} D_3(t_k+\sigma, t_k+s) \,\mathrm{d}\sigma \,\mathrm{d}s \\
&= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} D_3(\sigma, s) \,\mathrm{d}\sigma \,\mathrm{d}s \\
&= \int_0^{n\tau} e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} \int_\sigma^{\lceil \frac{\sigma}{\tau}\rceil \tau} D_3(\sigma,s) \,\mathrm{d}s \,\mathrm{d}\sigma,
\end{aligned}
$$

where the application of Fubini's theorem is justified since the double integral converges absolutely in H^{-1}. We next apply the dual of the periodic Strichartz estimate (1.5), which reads

$$
\Big\| \int_0^T e^{-\mathrm{i}t\partial_x^2} F(t) \,\mathrm{d}t \Big\|_{L^2} \lesssim_T \|F\|_{L^{\frac{4}{3}}([0,T]\times\mathbb{T})}.
$$

We infer that

$$
\begin{aligned}
&\Big\| \int_0^{n\tau} e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} \int_\sigma^{\lceil \frac{\sigma}{\tau}\rceil \tau} D_3(\sigma,s) \,\mathrm{d}s \,\mathrm{d}\sigma \Big\|_{L^2} \\
&\lesssim_T \Big\| \sigma \mapsto \int_\sigma^{\lceil \frac{\sigma}{\tau}\rceil \tau} D_3(\sigma,s) \,\mathrm{d}s \Big\|_{L^{\frac{4}{3}}([0,T]\times\mathbb{T})} \\
&\lesssim \Big\| \sigma \mapsto \|u(\sigma)\|_{L^\infty} \|\partial_x u(\sigma)\|_{L^4} \int_\sigma^{\lceil \frac{\sigma}{\tau}\rceil \tau} \|e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x \bar{u}(\sigma)\|_{L^2} \,\mathrm{d}s \Big\|_{L^{\frac{4}{3}}([0,T])} \\
&\lesssim_M \tau \|\partial_x u\|_{L^{\frac{4}{3}}([0,T],L^4)} \lesssim_T \tau \|\partial_x u\|_{L^4([0,T]\times\mathbb{T})} \lesssim_{M,T} \tau,
\end{aligned}
$$

using again Hölder's inequality, the Sobolev embedding $H^1 \hookrightarrow L^\infty$, and Corollary 2.2. □

We can now finish the proof of the global error bound in a classical way with the help of the discrete Gronwall lemma.

Proof of Theorem 1.2. We define the error

$$
e_n \coloneqq u(t_n) - u_n.
$$

We get the recursion formula

$$
\begin{aligned}
e_{n+1} &= u(t_{n+1}) - \Phi_\tau(u(t_n)) + \Phi_\tau(u(t_n)) - \Phi_\tau(u_n) \\
&= u(t_{n+1}) - \Phi_\tau(u(t_n)) + e^{\mathrm{i}\tau\partial_x^2} e_n \\
&\quad - \mathrm{i}\tau\mu e^{\mathrm{i}\tau\partial_x^2} \Big((u(t_n))^2 \varphi_1(-2\mathrm{i}\tau\partial_x^2)\bar{u}(t_n) - (u_n)^2 \varphi_1(-2\mathrm{i}\tau\partial_x^2)\bar{u}_n \Big).
\end{aligned}
$$

This formula implies that

$$
e_n = \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2} \Big(u(t_{k+1}) - \Phi_\tau(u(t_k)) \Big)
$$

$$- \mathrm{i}\tau\mu \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2}\Big((u(t_k))^2\varphi_1(-2\mathrm{i}\tau\partial_x^2)\bar{u}(t_k) - (u_k)^2\varphi_1(-2\mathrm{i}\tau\partial_x^2)\bar{u}_k\Big),$$

exploiting that $e_0 = 0$. From Lemma 2.4 and standard estimates we infer that

$$\|e_n\|_{L^2} \lesssim_{M,T} \tau + \tau \sum_{k=0}^{n-1}(1 + \|e_k\|^2_{H^{\frac{3}{4}}})\|e_k\|_{L^2}$$

using the Sobolev embedding $H^{3/4} \hookrightarrow L^\infty$ and the representation $u_k = u(t_k) - e_k$. By Theorem 1.1 with $r = 3/4$ and $\gamma = 1/4$, we find $\tau_0 > 0$ depending only on M and T, such that

$$\|e_n\|_{H^{\frac{3}{4}}} \leq 1$$

for all $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}_0$ with $n\tau \leq T$. For such τ and n we thus derive that

$$\|e_n\|_{L^2} \lesssim_{M,T} \tau + \tau \sum_{k=0}^{n-1} \|e_k\|_{L^2}.$$

By the discrete Gronwall inequality, we can conclude that

$$\|e_n\|_{L^2} \lesssim_{M,T} \tau. \qquad \square$$

3. Proof of the result for the nonlinear wave equation

In this section we carry out the proof of Theorem 1.3. Our first goal is to show estimate (1.9). Therefore we define the bilinear form

$$Q(\phi, \psi) \coloneqq \partial_t\phi\partial_t\psi - \partial_x\phi\partial_x\psi.$$

As a preparatory step, we treat the homogeneous problem.

Lemma 3.1. *Let ϕ and ψ solve the homogeneous wave equations*

$$\partial_t^2\phi - \partial_x^2\phi = 0, \quad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1$$
$$\partial_t^2\psi - \partial_x^2\psi = 0, \quad \psi(0) = \psi_0, \quad \partial_t\psi(0) = \psi_1$$

with Cauchy data ϕ_0, $\psi_0 \in H^1(\mathbb{T})$ and ϕ_1, $\psi_1 \in L^2(\mathbb{T})$. We then have the estimate

$$\|Q(\phi, \psi)\|_{L^2(\mathbb{T}\times\mathbb{T})} \lesssim (\|\partial_x\phi_0\|_{L^2} + \|\phi_1\|_{L^2})(\|\partial_x\psi_0\|_{L^2} + \|\psi_1\|_{L^2}).$$

Proof. We define

$$v_\phi = \frac{1}{2}(\partial_x\phi_0 + \phi_1), \quad w_\phi = \frac{1}{2}(\partial_x\phi_0 - \phi_1),$$

and similarly for ψ. By d'Alembert's formula, we can then write

$$\partial_t\phi(t, x) = v_\phi(x + t) - w_\phi(x - t), \quad \partial_x\phi(t, x) = v_\phi(x + t) + w_\phi(x - t).$$

We compute

$$\begin{aligned} Q(\phi, \psi)(t, x) &= (v_\phi(x + t) - w_\phi(x - t))(v_\psi(x + t) - w_\psi(x - t)) \\ &\quad - (v_\phi(x + t) + w_\phi(x - t))(v_\psi(x + t) + w_\psi(x - t)) \\ &= -2v_\phi(x + t)w_\psi(x - t) - 2w_\phi(x - t)v_\psi(x + t). \end{aligned}$$

Note that the "parallel interactions" (where one has twice "$x + t$" or twice "$x - t$") are canceled and only the "transverse interactions" (where one has once "$x + t$" and once "$x - t$") remain. See p.293 of [17] for further explanations of this phenomenon that also apply to the higher dimensional cases. We can now obtain the desired estimate in the following way. By integral substitutions $x - t = y$ and $y + 2t = s$, it follows that

$$\int_{\mathbb{T}} \int_{\mathbb{T}} |v(x+t)w(x-t)|^2 \,\mathrm{d}x \,\mathrm{d}t = \|v\|^2_{L^2(\mathbb{T})} \|w\|^2_{L^2(\mathbb{T})}$$

for general functions v, w. Hence,

$$\begin{aligned}\|Q(\phi,\psi)\|_{L^2(\mathbb{T}\times\mathbb{T})} &\lesssim \|v_\phi\|_{L^2}\|w_\psi\|_{L^2} + \|w_\phi\|_{L^2}\|v_\psi\|_{L^2} \\ &\lesssim (\|\partial_x \phi_0\|_{L^2} + \|\phi_1\|_{L^2})(\|\partial_x \psi_0\|_{L^2} + \|\psi_1\|_{L^2}). \qquad \square\end{aligned}$$

Now we give the proof of (1.9).

Proposition 3.2. *Let $T > 0$ and ϕ solve the inhomogeneous wave equation*

$$\partial_t^2 \phi - \partial_x^2 \phi = F, \quad \phi(0) = \phi_0, \quad \partial_t \phi(0) = \phi_1 \tag{3.1}$$

on $[0,T] \times \mathbb{T}$ with data $\phi_0 \in H^1(\mathbb{T})$, $\phi_1 \in L^2(\mathbb{T})$, and $F \in L^1([0,T], L^2(\mathbb{T}))$. Then we have the inequality

$$\|Q(\phi,\phi)\|_{L^2([0,T]\times\mathbb{T})} \lesssim_T \|\partial_x \phi_0\|^2_{L^2} + \|\phi_1\|^2_{L^2} + \|F\|^2_{L^1([0,T],L^2)}.$$

Proof. We decompose $\phi = \phi_{\text{hom}} + \phi_{\text{inh}}$, where ϕ_{hom} solves (3.1) with $F = 0$ and ϕ_{inh} solves (3.1) with $\phi_0 = \phi_1 = 0$. The estimate for $Q(\phi_{\text{hom}}, \phi_{\text{hom}})$ follows directly from Lemma 3.1 and the periodicity of ϕ_{hom} in time. To treat the inhomogeneous part, for almost all $s \in [0,T]$, we define ϕ^s to be the solution to the homogeneous equation

$$\partial_t^2 \phi^s - \partial_x^2 \phi^s = 0, \quad \phi^s(s) = 0, \quad \partial_t \phi^s(s) = F(s).$$

By Duhamel's formula, ϕ_{inh} is then given by

$$\phi_{\text{inh}}(t) = \int_0^t \phi^s(t) \,\mathrm{d}s.$$

It follows that we can express the bilinear term as

$$Q(\phi_{\text{inh}}, \phi_{\text{inh}})(t) = \int_0^t \int_0^t Q(\phi^s, \phi^r)(t) \,\mathrm{d}s \,\mathrm{d}r.$$

From Minkowski's inequality, Lemma 3.1, and the energy equality we thus get

$$\begin{aligned}\|Q(\phi_{\text{inh}}, \phi_{\text{inh}})\|_{L^2([0,T]\times\mathbb{T})} &\le \int_0^T \int_0^T \|Q(\phi^s,\phi^r)\|_{L^2([0,T]\times\mathbb{T})} \,\mathrm{d}s \,\mathrm{d}r \\ &\lesssim \int_0^T \int_0^T (\|\partial_x \phi^s(0)\|_{L^2} + \|\partial_t \phi^s(0)\|_{L^2}) \\ &\qquad \cdot (\|\partial_x \phi^r(0)\|_{L^2} + \|\partial_t \phi^r(0)\|_{L^2}) \,\mathrm{d}s \,\mathrm{d}r \\ &\lesssim \|F\|^2_{L^1([0,T],L^2)}.\end{aligned}$$

The mixed term $Q(\phi_{\text{hom}}, \phi_{\text{inh}})$ is treated similarly. $\square$

We now turn to the nonlinear wave equation (1.2). Here it is convenient to work with the first-order reformulation. With the definitions

$$U := \begin{pmatrix} u \\ v \end{pmatrix} \mathrel{\widehat{=}} \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \; A := \begin{pmatrix} 0 & I \\ \partial_x^2 & 0 \end{pmatrix}, \; G(U) := \begin{pmatrix} 0 \\ g(u) \end{pmatrix}, \; U_0 := \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

we obtain the equivalent differential equation

$$\begin{aligned} \partial_t U(t) &= AU(t) + G(U(t)), \quad t \in [0,T], \\ U(0) &= U_0. \end{aligned} \tag{3.2}$$

Assumption 3.3. There exists a time $T > 0$ and a solution $U = (u, \partial_t u) \in C([0,T], H^1 \times L^2) \cap C^1([0,T], L^2 \times H^{-1})$ to the nonlinear wave equation (3.2) with bound

$$M := \|U\|_{L^\infty([0,T],H^1\times L^2)}.$$

Since the nonlinearity g belongs to $C^2(\mathbb{R},\mathbb{R})$, we can find an increasing function L such that the bound

$$|g(z)| + |g'(z)| + |g''(z)| \le L(|z|) \tag{3.3}$$

holds for all $z \in \mathbb{R}$. In the following, we suppress the dependency on the function L from (3.3) in the $\lesssim$ notation. We now apply Proposition 3.2 to the solution u to the nonlinear problem (1.2).

Corollary 3.4. *Let u, T, and M be given by Assumption 3.3. Then we have the estimate*

$$\|(\partial_t u)^2 - (\partial_x u)^2\|_{L^2([0,T]\times\mathbb{T})} \lesssim_{M,T} 1.$$

Proof. By Proposition 3.2, we only need that

$$\|g(u)\|_{L^1([0,T],L^2)} \lesssim_{M,T} 1,$$

and this follows from (3.3), Hölder's inequality, and the Sobolev embedding $H^1 \hookrightarrow L^\infty$. $\square$

We now give a brief derivation of the corrected Lie splitting (1.6) proposed in [9]. It is based of the Lie splitting approximation for (3.2), which is a formally first-order scheme given by

$$U_{n+1}^{\mathrm{Lie}} = e^{\tau A}[U_n^{\mathrm{Lie}} + \tau G(U_n^{\mathrm{Lie}})]. \tag{3.4}$$

By the Duhamel formulation of (3.2), the fundamental theorem of calculus, and Fubini's theorem, the local error of (3.4) can be represented as

$$U(\tau) - U_1^{\mathrm{Lie}} = e^{\tau A}\int_0^\tau (\tau - s)e^{-sA} H(U(s))\,\mathrm{d}s. \tag{3.5}$$

Here we use the definition

$$H(U(s)) := e^{sA}\frac{\mathrm{d}}{\mathrm{d}s}\Big[e^{-sA}G(U(s))\Big] = \begin{pmatrix} -g(u(s)) \\ g'(u(s))\partial_t u(s) \end{pmatrix}.$$

Similar as in the Schrödinger case, we do not insert the approximation $U(s) \approx e^{sA}U_0$ (which was used in [9]) in order to create better conditions for applying Corollary 3.4 later.

The crucial observation which allows the construction of the low-regularity integrator is that $H(U)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}H(U(s)) = -AH(U(s)) + B(U(s)) \tag{3.6}$$

in $L^2(\mathbb{T}) \times H^{-1}(\mathbb{T})$, where the remainder

$$B(U) := \begin{pmatrix} 0 \\ g''(u)[(\partial_t u)^2 - (\partial_x u)^2] + g'(u)g(u) \end{pmatrix} \tag{3.7}$$

only contains first-order derivatives of u. We plug the Duhamel approximation $H(U(s)) \approx e^{-sA}H(U_0)$ for (3.6) into (3.5) and exploit (1.7) to get

$$U(\tau) - U_1^{\mathrm{Lie}} \approx e^{\tau A} \int_0^\tau (\tau - s)e^{-2sA} H(U_0)\,\mathrm{d}s = \tau^2 e^{\tau A}\varphi_2(-2\tau A)H(U_0).$$

Adding this term on the Lie splitting (3.4) gives the formally second-order *corrected Lie splitting*

$$U_{n+1} = \Psi_\tau(U_n) \coloneqq e^{\tau A}[U_n + \tau G(U_n) + \tau^2\varphi_2(-2\tau A)H(U_n)], \tag{3.8}$$

which corresponds to (1.6). From this derivation we immediately get the following representation of the local error. A related formula was derived in Lemma 6.2 of [16] in the 3D case.

Lemma 3.5. *Let U and T be given by Assumption* (3.3). *Then the local error of the corrected Lie splitting* (3.8) *is given by*

$$U(\tau) - U_1 = e^{\tau A} \int_0^\tau (\tau - s)e^{-2sA} \int_0^s e^{\sigma A} B(U(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}s$$

for all $\tau \in (0, T]$.

Proof. Follows directly from (3.5) and the Duhamel formulation of (3.6). □

We can now bound the sum of local errors with the help of Corollary 3.4.

Lemma 3.6. *Let $U = (u, \partial_t u)$, T, and M be given by Assumption 3.3. Then we can bound the sum of local errors of* (3.8) *by*

$$\Big\| \sum_{k=0}^{n-1} e^{(n-k-1)\tau A}\Big(U(t_{k+1}) - \Psi_\tau(U(t_k))\Big)\Big\|_{H^1\times L^2} \lesssim_{M,T} \tau^2,$$

for all $\tau \in (0, T]$ and $n \in \mathbb{N}_0$ with $n\tau \le T$.

Proof. By the triangle inequality and Lemma 3.5 with $U(t_k + \cdot)$ instead of U,

$$\begin{aligned}
&\Big\| \sum_{k=0}^{n-1} e^{(n-k-1)\tau A}\Big(U(t_{k+1}) - \Psi_\tau(U(t_k))\Big)\Big\|_{H^1\times L^2} \\
&\lesssim_T \tau^2 \sum_{k=0}^{n-1} \int_0^\tau \|B(U(t_k+\sigma))\|_{H^1\times L^2}\,\mathrm{d}\sigma \le \tau^2 \|B(U)\|_{L^1([0,T],H^1\times L^2)}.
\end{aligned}$$

We next insert the definition (3.7) of B and apply (3.3) and finally Corollary 3.4 to obtain

$$\begin{aligned}
\|B(U)\|_{L^1([0,T],H^1\times L^2)} &= \|g''(u)[(\partial_t u)^2 - (\partial_x u)^2] + g'(u)g(u)\|_{L^1([0,T],L^2)} \\
&\lesssim_{M,T} \|(\partial_t u)^2 - (\partial_x u)^2\|_{L^2([0,T]\times\mathbb{T})} + 1 \lesssim_{M,T} 1.
\end{aligned}$$ □

Similar as in the Schrödinger case, we conclude the proof of the global error bound using the discrete Gronwall lemma.

Proof of Theorem 1.3. We proceed similar as in the proof of Theorem 1.2. For the error

$$E_n \coloneqq U(t_n) - U_n$$

we get the formula

$$\begin{aligned}
E_n = \sum_{k=0}^{n-1} e^{(n-k-1)\tau A}\Big(U(t_{k+1}) - \Psi_\tau(U(t_k))\Big) \\
+ \tau \sum_{k=0}^{n-1} e^{(n-k)\tau A}\Big(G(U(t_k)) - G(U_k)\Big) \\
+ \tau^2 \varphi_2(-2\tau A) \sum_{k=0}^{n-1} e^{(n-k)\tau A}\Big(H(U(t_k)) - H(U_k)\Big).
\end{aligned}$$

From Lemma 3.6, (3.3), and standard estimates, we infer that

$$\|E_n\|_{H^1\times L^2} \leq c\tau^2 + \tau \sum_{k=0}^{n-1} K(\|E_k\|_{H^1\times L^2})\|E_k\|_{H^1\times L^2}$$

with a constant $c > 0$ and an increasing function K, both depending on M, T, and L. We define the maximum step size

$$\tau_0 \coloneqq (ce^{K(1)T})^{-\frac{1}{2}}.$$

Using the discrete Gronwall lemma, we obtain via induction on n that

$$\|E_n\|_{H^1\times L^2} \leq c\tau^2 e^{K(1)T} \leq 1$$

for all $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}_0$ with $n\tau \leq T$. □

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