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Wave phenomena
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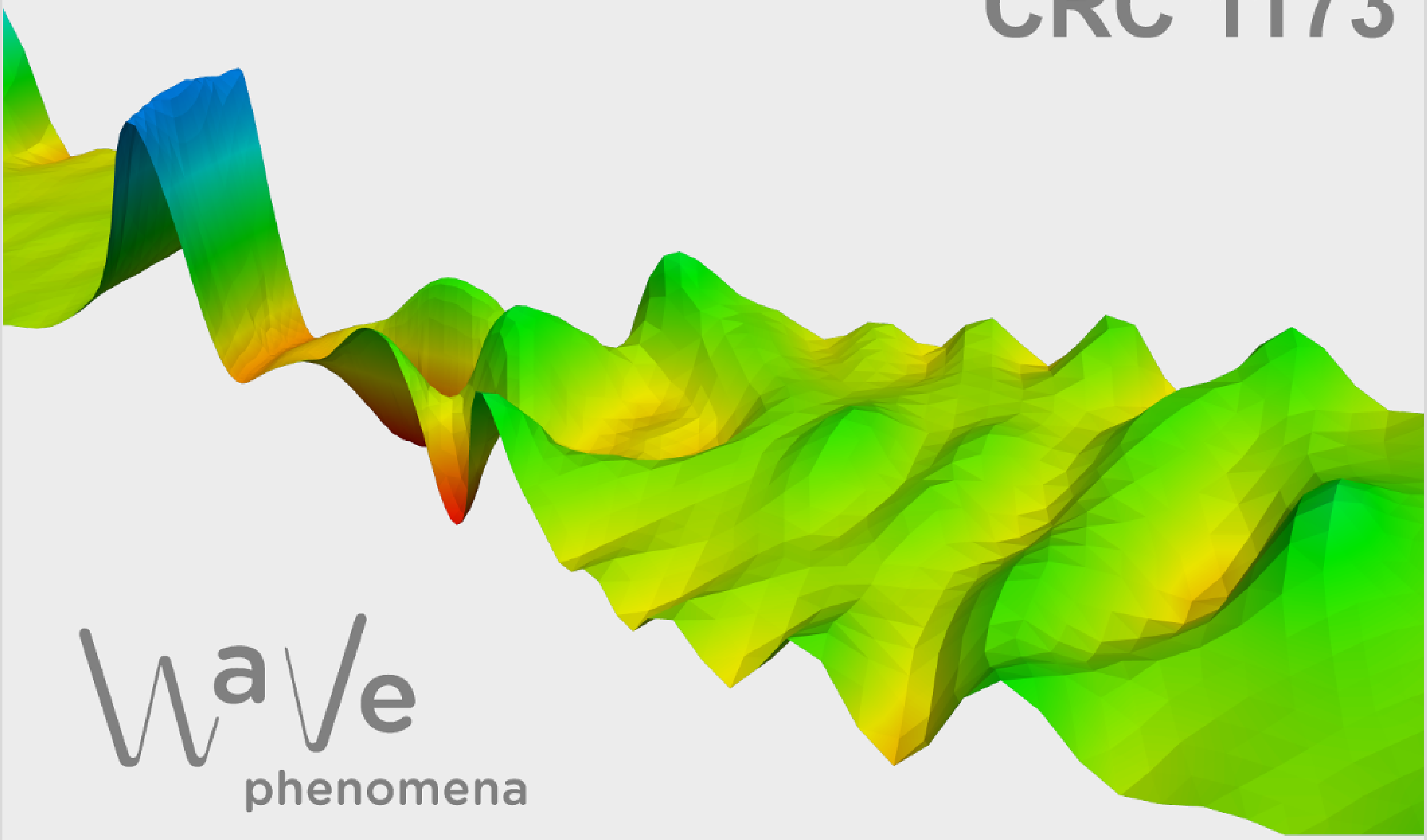

Wave phenomena

ERROR ANALYSIS OF DGTD FOR LINEAR MAXWELL EQUATIONS WITH INHOMOGENEOUS INTERFACE CONDITIONS

BENJAMIN DÖRICH, JULIAN DÖRNER, AND MARLIS HOCHBRUCK

ABSTRACT. In the present paper we consider linear and isotropic Maxwell equations with inhomogeneous interface conditions. We discretize the problem with the discontinuous Galerkin method in space and with the leapfrog scheme in time. An analytical setting is provided in which we show wellposedness of the problem, derive stability estimates, and exploit this in the error analysis to prove rigorous error bounds for both the spatial and full discretization. The theoretical findings are confirmed with numerical experiments.

1. INTRODUCTION

Graphene is a monolayer of carbon atoms arranged in a hexagonal lattice and has demonstrated exceptional thermal, electrical and optical properties, as well as structural robustness. This makes it an attractive candidate for reinforcement in materials, as well as for applications in organic electronics and optoelectronics. There is significant interest in both the scientific community and industrial sectors about graphene. In recent years, numerous other 2D materials have emerged, with the term referring to crystalline solids that have a reduced thickness, usually consisting of a single or only a few atomic layers. This unique characteristic gives them exceptional properties. Another important class of composite 2D materials are the semiconducting Transition Metal Dichalcogenides (TMDCs) such as MoS_2 or WS_2 which also come in single layers, albeit with a more complicated unit cell than graphene. These materials have a wide range of applications, including catalysis, spintronics, and optoelectronics, cf. we refer to the reviews [25, 27]. Numerical simulations are vital for studying such materials.

The optical properties of such materials can be studied by depositing a sheet on a thin dielectric layer on top of a metal plate and exciting it with light pulses. The interaction between the pulses and the material is described by Maxwell equations coupled to quantum mechanical models, see, e.g., [2, 19, 24]. A simpler way of modelling the interaction of the 2D material in the Maxwell equations is to use conductivity surfaces or current sheets. Here it is assumed that the material sheet has zero thickness, thus is truly two-dimensional, and the constitutive equation

$$\boldsymbol{J}_{\mathrm{surf}}(\omega) = \sigma_{\mathrm{surf}}(\omega)\boldsymbol{E}(\omega)$$

holds along the plane of the material in the frequency domain, where σ_{surf} describes the surface conductivity of the 2D material. We refer to Chapter 1 in [7] for details about the modelling.

As a first step towards the full model, we consider linear and isotropic time-dependent Maxwell equations on the cuboidal domain Q composed of two cuboids Q_- and Q_+ with a common interface $F_{\mathrm{int}} = \overline{Q_-} \cap \overline{Q_+}$, cf. Figure 1. We assume

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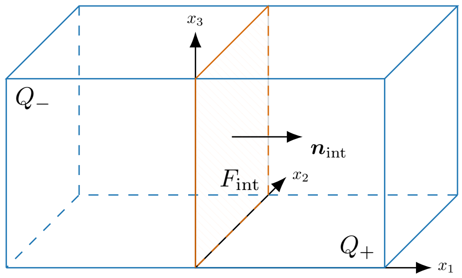

FIGURE 1. Sketch of the domain Q.

that the surface current $\boldsymbol{J}_{\mathrm{surf}}$ supported on F_{int} is a given function. After rescaling, we assume, without loss of generality, that

$$Q = (-1,1) \times (0,1)^2, \; Q_- = (-1,0) \times (0,1)^2, \; Q_+ = (0,1) \times (0,1)^2.$$

The governing equations read

$$\partial_t \boldsymbol{H}_\pm = -\mu_\pm^{-1} \operatorname{curl} \boldsymbol{E}_\pm, \qquad \operatorname{div}\Big(\mu_\pm \boldsymbol{H}_\pm\Big) = 0, \tag{1.1a}$$

$$\partial_t \boldsymbol{E}_\pm = \varepsilon_\pm^{-1} \operatorname{curl} \boldsymbol{H}_\pm - \varepsilon_\pm^{-1} \boldsymbol{J}_\pm, \qquad \operatorname{div}\Big(\varepsilon_\pm \boldsymbol{E}_\pm\Big) = \rho_\pm, \tag{1.1b}$$

on $Q_\pm$, for $t \geq 0$. We denote by $f_\pm = f\big|_{Q_\pm}$ the restriction of a function $f \in L^2(Q)$. Here, for $x \in Q_\pm$, $\boldsymbol{E}(t,x), \boldsymbol{H}(t,x) \in \mathbb{R}^3$ denote the electric and magnetic field, $\boldsymbol{J}(t,x) \in \mathbb{R}^3$ the volume current density and $\rho(t,x) \in \mathbb{R}$ the charge density, respectively. We assume that the material parameters $\mu_\pm, \varepsilon_\pm \geq \delta > 0$ are constant on $Q_\pm$. The equations are equipped with perfectly conduction boundary conditions

$$\mu \boldsymbol{H} \cdot \boldsymbol{\nu} = 0, \qquad \boldsymbol{E} \times \boldsymbol{\nu} = 0, \tag{1.2}$$

on ∂Q, for $t \geq 0$ with outer unit normal vector $\boldsymbol{\nu}$, see, e.g., [4, Sec. I.4.2.4]. At the interface F_{int}, the conditions

$$[\![\mu \boldsymbol{H} \cdot \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}} = 0, \qquad [\![\varepsilon \boldsymbol{E} \cdot \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}} = \rho_{\mathrm{surf}}, \tag{1.3a}$$

$$[\![\boldsymbol{H} \times \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}} = \boldsymbol{J}_{\mathrm{surf}}, \qquad [\![\boldsymbol{E} \times \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}} = 0, \tag{1.3b}$$

hold for $t \geq 0$, where $\boldsymbol{n}_{\mathrm{int}}$ denotes the inner unit normal vector on F_{int} pointing from Q_- to Q_+ and $[\![f]\!]_{F_{\mathrm{int}}} = f_+\big|_{F_{\mathrm{int}}} - f_-\big|_{F_{\mathrm{int}}}$ denotes the jump on F_{int} whenever the functions $f_\pm$ admit well-defined traces on the interface. Note, that the surface current $\boldsymbol{J}_{\mathrm{surf}}(t,x) = \big(0, J_{\mathrm{surf},2}(t,x), J_{\mathrm{surf},3}(t,x)\big) \in \mathbb{R}^3$ has no components perpendicular to F_{int}. By $\rho_{\mathrm{surf}}(t,x) \in \mathbb{R}$ we denote the surface charge density, see, e.g., for details [4, Sec. I.4.2.2].

Discretization. The discontinuous Galerkin (dG) time-domain method is a well establish method for Maxwell equations, see, e.g., [14]. We briefly recall the construction of the dG space discretization and refer to Section 3 for details.

Assume that $\mathcal{T}_h$ is the union of suitable meshes for $Q_\pm$ with elements K and matching faces at F_{int}. The set of all element faces F is denoted by $\mathcal{F}_h$. The broken polynomial space of degree at most $k \geq 1$ is defined as

$$\mathbb{P}_3^k(\mathcal{T}_h) = \big\{\, v_h \in L^2(Q) \;\big|\; v_h|_K \in \mathbb{P}_3^k(K) \text{ for all } K \in \mathcal{T}_h \,\big\}, \tag{1.4a}$$

which are polynomials on every element K and, in general, discontinuous across element faces F. The vector valued ansatz space for the magnetic and electric field is given by the broken finite element space of degree k defined as

$$V_h = \mathbb{P}_3^k(\mathcal{T}_h)^3. \tag{1.4b}$$

The dG method is a non-conforming method in the sense that a function $\boldsymbol{U}_h \in V_h$ does not admit a curl on the whole domain Q, i.e., $V_h \not\subset H(\mathrm{curl}, Q)$. Therefore, we need to introduce a discretized curl operator acting on V_h. One way to do so is given by means of the central flux discretization, see, e.g., [14]. The notation is based on [18]. We define the discrete operator $\mathrm{curl}_h : V_h \to V_h$ such that for all $\boldsymbol{\phi}_h \in V_h$ it holds

$$\begin{aligned} \int_Q \mathrm{curl}_h \boldsymbol{U}_h \cdot \boldsymbol{\phi}_h \,\mathrm{d}x = & \sum_{K\in\mathcal{T}_h} \int_K \mathrm{curl}\, \boldsymbol{U}_h \cdot \boldsymbol{\phi}_h \,\mathrm{d}x \\ & - \sum_{F\in\mathcal{F}_h} \int_F [\![\boldsymbol{U}_h \times \boldsymbol{n}_F]\!]_F \cdot \{\!\{\boldsymbol{\phi}_h\}\!\}_F \,\mathrm{d}s. \end{aligned} \tag{1.5}$$

Here, $[\![\boldsymbol{U}_h \times \boldsymbol{n}_F]\!]_F$ denotes the tangential jump of $\boldsymbol{U}_h$ and $\{\!\{\boldsymbol{\phi}_h\}\!\}_F$ a weighted average of $\boldsymbol{\phi}_h$ on F defined below (3.1a). The first sum on the right hand side of (1.5) acts locally on single elements, thus decoupling the action of the curl, while the second sum couples neighboring elements through tangential jumps. The coupling terms are referred to as numerical fluxes and they penalize non-zero tangential jumps across faces. We recall that functions $\boldsymbol{H} \in H(\mathrm{curl}, Q)$ have zero tangential jumps across faces, i.e.,

$$[\![\boldsymbol{H} \times \boldsymbol{n}_F]\!]_F = 0.$$

By $\mathrm{curl}_{h,0} : V_h \to V_h$ we denote a discrete operator that additionally enforces homogenous tangential boundary conditions of the electric field (1.2).

In order to incorporate the surface current $\boldsymbol{J}_{\mathrm{surf}}$, we follow the idea of (1.5), but instead of penalizing zero tangential jumps, we apply the inhomogeneous interface condition (1.3b) for all faces $F \in \mathcal{F}_h^{\circ}$ with $F \subset F_{\mathrm{int}}$, i.e.,

$$[\![\boldsymbol{H} \times \boldsymbol{n}_F]\!]_F = \boldsymbol{J}_{\mathrm{surf}}\big|_F.$$

Equation (1.5) motivates to define an extension $\boldsymbol{J}_{\mathrm{surf},h} \in V_h$ via

$$\int_Q \boldsymbol{J}_{\mathrm{surf},h} \cdot \boldsymbol{\phi}_h \,\mathrm{d}x = \sum_{F\in\mathcal{F}_h, F\subset F_{\mathrm{int}}} \int_F \boldsymbol{J}_{\mathrm{surf}} \cdot \{\!\{\boldsymbol{\phi}_h\}\!\} \,\mathrm{d}s \tag{1.6}$$

for all $\boldsymbol{\phi}_h \in V_h$. Note that $\boldsymbol{J}_{\mathrm{surf}}$ is defined only on the interface F_{int} whereas $\boldsymbol{J}_{\mathrm{surf},h}$ is defined on the whole domain Q, with support only on elements adjacent to F_{int}. We end up with the following spatially discrete system of differential equations

$$\partial_t \boldsymbol{H}_h(t) = -\mu^{-1} \mathrm{curl}_{h,0}\, \boldsymbol{E}_h(t), \tag{1.7a}$$

$$\partial_t \boldsymbol{E}_h(t) = \varepsilon^{-1} \mathrm{curl}_h\, \boldsymbol{H}_h(t) - \boldsymbol{J}_h(t) - \boldsymbol{J}_{\mathrm{surf},h}(t), \tag{1.7b}$$

for $t \geq 0$. Here, the inhomogeneous interface conditions (1.3b) are incorporated by $\boldsymbol{J}_{\mathrm{surf},h}$, that acts like an artificial current on the evolution of the electric field. We refer to Section 3 for the precise definitions.

We integrate the spatially discrete system in time by the second order leapfrog method. Let $\tau > 0$ be the time step size and $t_n = n\tau$ for $n \in \mathbb{N}$. The fully discrete scheme then reads

$$\begin{aligned} \boldsymbol{H}_h^{n+1/2} - \boldsymbol{H}_h^n &= -\frac{\tau}{2}\mu^{-1} \mathrm{curl}_{h,0}\, \boldsymbol{E}_h^n, \\ \boldsymbol{E}_h^{n+1} - \boldsymbol{E}_h^n &= \tau\varepsilon^{-1} \mathrm{curl}_h\, \boldsymbol{H}_h^{n+1/2} - \frac{\tau}{2}(\boldsymbol{J}_h^n + \boldsymbol{J}_h^{n+1}) - \frac{\tau}{2}(\boldsymbol{J}_{\mathrm{surf},h}^n + \boldsymbol{J}_{\mathrm{surf},h}^{n+1}), \\ \boldsymbol{H}_h^{n+1} - \boldsymbol{H}_h^{n+1/2} &= -\frac{\tau}{2}\mu^{-1} \mathrm{curl}_{h,0}\, \boldsymbol{E}_h^{n+1}, \end{aligned}$$

for $n \geq 0$, starting from appropriate initial values $\left(\boldsymbol{H}_h^0, \boldsymbol{E}_h^0\right) \in V_h^2$. Other time integration schemes can be applied to (1.7) as well.

It is well known that the explicit leapfrog scheme exhibits a step size restriction, which is also known as the Courant-Friedrichs-Lewy (CFL) condition. The scheme is only stable for time step sizes $\tau < \tau_{\mathrm{CFL}}$, with $\tau_{\mathrm{CFL}} \sim h_{\min}$, where $h_{\min}$ denotes the diameter of the smallest element $K \in \mathcal{T}_h$.

Contributions of the paper. The challenges associated with interface problems have been thoroughly investigated from both analytical and numerical perspectives, albeit within a geometric framework different from the one specified earlier. In that context, it is assumed that a positive distance exists between the interface and the domain boundary. For example, wellposedness and regularity of quasilinear Maxwell equations for such a geometric setting is found in [23]. From a numerical point of view, finite element methods have been explored in [3] concerning elliptic and parabolic problems, and in [5, 6] for hyperbolic equations. However, these results are not applicable to the problem described in (1.1) to (1.3).

In a recent study by Dörich and Zerulla [10], a different technique is employed to establish wellposedness and regularity for the model problem (1.1) to (1.3). From a numerical perspective, the discontinuous Galerkin time-domain method was successfully applied to an interface problem concerned with Graphene sheets in the above mentioned setting, see, e.g., [28, 29]. There, the focus is on the physical modelling of such sheets. Their excellent numerical results motivate a thorough mathematical error analysis.

In this paper, we provide a mathematical framework that is suitable for both, analysis and numerics of the problem at hand. We prove wellposedness and stability for the governing equations building up on the techniques in [10]. Transferring the ideas from analysis, a rigorous spatial and full discretization error analysis is provided for the numerical scheme. Under suitable regularity conditions on the exact solution, we prove that the error of the scheme is of second order in time and of kth order in space with respect to the L^2-norm, i.e.,

$$\|\big(\boldsymbol{H}, \boldsymbol{E}\big)(t_n) - (\boldsymbol{H}_h^n, \boldsymbol{E}_h^n)\|_{L^2(Q)^3 \times L^2(Q)^3} \leq C(\tau^2 + h^k), \quad 0 \leq t_n \leq T.$$

Furthermore, the results are underpinned by several numerical examples showing the sharpness of the estimates with respect to spatial regularity.

Note, that the results are consistent with the case where the surface current vanishes, i.e., $\boldsymbol{J}_{\mathrm{surf}} = 0$. However, new techniques are required for $\boldsymbol{J}_{\mathrm{surf}} \neq 0$. We address the problems in brief.

One of the challenges is that by the interface condition (1.3b), the state-space $H(\mathrm{curl}, Q) \times H_0(\mathrm{curl}, Q)$, typically used for the evolution of linear Maxwell equations, is no longer suitable for the problem described by (1.1) to (1.3). Circumventing this, we enlarge the state-space with functions $\boldsymbol{V} \in L^2(Q)^3$ that only possess a weak curl on each sub-cuboid, i.e., $\boldsymbol{V}_\pm \in H(\mathrm{curl}, Q_\pm)$. This causes several problems both from an analytical and numerical perspective. Analytically, C^0-semigroup techniques are no longer applicable and numerically, we must handle a non-consistent discretization. Motivated by the treatment of inhomogeneous Dirichlet boundary conditions, we modify the problem that we can treat it in a standard way. Nonetheless, since the interface F_{int} intersects with the boundary, special care is necessary to treat the perfectly conducting boundary conditions (1.2) correctly in the modified problem. For this, sophisticated techniques from [10] are essential to both analysis and numerics.

Structure of the paper. In Section 2, we first introduce a suitable analytical framework for the problem described by (1.1) to (1.3). We proceed by presenting the main result of this section concerning an existence and stability result for the analytical problem. With the strategy of proof outlined, we introduce an important extension result that is later used frequently. The section is closed with the proof of the main result.

Section 3 is concerned with space discretization. We first provide a standard description of the discontinuous Galerkin method and point out in detail how inhomogeneous interface problems are treated. Proceeding that, the main result of this section provides an error bound on the semi-discretization. The section carries on with the discussion of an important extension of the semi-discrete scheme utilizing a nodal interpolation on the interface. The section again closes with the remaining proofs.

Section 3 is concerned with space discretization. We provide a detailed description of the discontinuous Galerkin method and point out in detail how inhomogeneous interface problems are treated. We present the main result of this section consisting of error bounds for the semi-discretization as well as an important extension of the semi-discrete scheme utilizing a nodal interpolation on the interface. The section again closes with the remaining proofs.

The main result of Section 4 is concerned with an error bound on the full discretization. We first prove stability of the scheme and provide afterwards the proof of the main result.

In Section 5, we provide three different numerical experiments that confirm our theoretical findings.

2. Wellposedness

General setting and notation. The volume charge density ρ is determined by the volume current $\boldsymbol{J}$ through

$$\rho_\pm(t) = \rho_\pm(0) + \int_0^t \operatorname{div} \boldsymbol{J}_\pm(s)\, \mathrm{d}s \tag{2.1}$$

on $Q_\pm$, for $t \geq 0$. Equation (2.1) is called the continuity relation for electricity. It is well known that the divergence conditions (1.1a) and (1.1b) and the magnetic boundary condition in (1.2) hold, if they are valid for $t = 0$ and (2.1) holds. We refer to [4, Sec. I.4.1.2] for details.

A similar relation exists for the surface charge density ρ_{surf}. It is determined by the volume current $\boldsymbol{J}$ and the surface current $\boldsymbol{J}_{\text{surf}}$ through

$$\rho_{\text{surf}}(t) = \rho_{\text{surf}}(0) + \int_0^t \operatorname{div}_{F_{\text{int}}} \boldsymbol{J}_{\text{surf}}(s) - [\![\boldsymbol{J}(s) \cdot \boldsymbol{n}_{\text{int}}]\!]_{F_{\text{int}}}\, \mathrm{d}s \tag{2.2}$$

on F_{int}, for $t \geq 0$, where $\operatorname{div}_{F_{\text{int}}}$ denotes the two-dimensional divergence on F_{int}. It is shown in [23, Lemma 8.1] that equations (1.3a) are valid for $t \geq 0$ if they are valid for $t = 0$ and (2.2) holds. Thus, it remains to solve the curl-equations in (1.1) subject to the boundary conditions (1.2) and the tangential interface conditions (1.3b).

The speed of light is denoted with $c_\pm = (\mu_\pm \varepsilon_\pm)^{-1/2}$ and we use the notation

$$\eta_\infty = \max\{\eta_-, \eta_+\}, \quad \eta \in \{\varepsilon, \mu, c\}$$

for piecewise defined constants. We employ the weighted inner L^2-products

$$(\cdot,\cdot)_\mu = (\mu\cdot,\cdot)_{L^2(Q)}, \quad (\cdot,\cdot)_\varepsilon = (\varepsilon\cdot,\cdot)_{L^2(Q)}, \quad (\cdot,\cdot)_{\mu\times\varepsilon} = (\cdot,\cdot)_\mu + (\cdot,\cdot)_\varepsilon \tag{2.3}$$

and their induced norms $\|\cdot\|_\mu$, $\|\cdot\|_\varepsilon$ and $\|\cdot\|_{\mu\times\varepsilon}$. Note that they are equivalent to the standard $L^2(Q)$-norm.

The spaces $H^s(Q)$ for $s \in \mathbb{R}$ denote fractional Sobolev spaces. We write $\|\cdot\|_{H^s(Q)}$ and $|\cdot|_{H^s(Q)}$ for their associated norms and semi-norms and refer to [11] for details. Additionally, we write $H^s(\Gamma)$ for Sobolev spaces on the boundary $\Gamma = \partial Q$.

We introduce the space of functions that exhibit a weak variation curl

$$H(\operatorname{curl}, Q) = \left\{ \boldsymbol{V} \in L^2(Q)^3 \;\middle|\; \operatorname{curl} \boldsymbol{V} \in L^2(Q)^3 \right\},$$

as well as the subspace that contains functions with a vanishing tangential trace

$$H_0(\operatorname{curl}, Q) = \left\{ \boldsymbol{V} \in H(\operatorname{curl}, Q) \;\middle|\; \boldsymbol{V} \times \boldsymbol{\nu}|_\Gamma = 0 \right\}.$$

Since we deal with solutions that lag regularity across the interface F_{int}, we employ the notion of piecewise spaces. For $s \in \mathbb{R}$ we denote the piecewise Sobolev spaces

$$PH^s(Q) = \left\{ v \in L^2(Q) \;\middle|\; v_\pm \in H^s(Q_\pm) \right\}$$

and we define analogously

$$PH(\operatorname{curl}, Q) = \left\{ \boldsymbol{V} \in L^2(Q)^3 \;\middle|\; \operatorname{curl} \boldsymbol{V}_\pm \in L^2(Q_\pm)^3 \right\}.$$

Note that we use the same symbol for both curl-operators. The following connection between $H(\operatorname{curl}, Q)$ and $PH(\operatorname{curl}, Q)$ is a simple corollary of Green's formula, see for example [13, Thm. 2.11].

Corollary 2.1. *Let $\boldsymbol{V} \in PH(\operatorname{curl}, Q)$. It holds $\boldsymbol{V} \in H(\operatorname{curl}, Q)$ if and only if*

$$[\![\boldsymbol{V} \times \boldsymbol{n}_{\text{int}}]\!]_{F_{\text{int}}} = 0, \quad \text{in } H^{-1/2}(F_{\text{int}}).$$

We define the Maxwell operators acting on the magnetic and electric field respectively as

$$\widehat{\mathcal{C}}_{\mathrm{H}} : D(\widehat{\mathcal{C}}_{\mathrm{H}}) = PH(\operatorname{curl}, Q) \to L^2(Q)^3,\ \boldsymbol{H} \mapsto \varepsilon^{-1} \operatorname{curl} \boldsymbol{H}, \tag{2.4a}$$

$$\mathcal{C}_{\mathrm{E}} : D(\mathcal{C}_{\mathrm{E}}) = H_0(\operatorname{curl}, Q) \to L^2(Q)^3,\quad \boldsymbol{E} \mapsto \mu^{-1} \operatorname{curl} \boldsymbol{E}. \tag{2.4b}$$

The operator acting on the combined field $\boldsymbol{u} = \big(\boldsymbol{H}, \boldsymbol{E}\big)$ is defined as

$$\widehat{\mathcal{C}} : D(\widehat{\mathcal{C}}) = D(\widehat{\mathcal{C}}_{\mathrm{H}}) \times D(\mathcal{C}_{\mathrm{E}}) \to L^2(Q)^6,\ \widehat{\mathcal{C}} = \begin{pmatrix} 0 & -\mathcal{C}_{\mathrm{E}} \\ \widehat{\mathcal{C}}_{\mathrm{H}} & 0 \end{pmatrix}. \tag{2.4c}$$

We emphasize that $H(\operatorname{curl}, Q) \subset PH(\operatorname{curl}, Q)$ and define the restricted operators

$$\mathcal{C}_{\mathrm{H}} : D(\mathcal{C}_{\mathrm{H}}) = H(\operatorname{curl}, Q) \to L^2(Q)^3, \qquad \mathcal{C}_{\mathrm{H}} = \widehat{\mathcal{C}}_{\mathrm{H}}|_{D(\mathcal{C}_{\mathrm{H}})}, \tag{2.4d}$$

$$\mathcal{C} : D(\mathcal{C}) = H(\operatorname{curl}, Q) \times H_0(\operatorname{curl}, Q) \to L^2(Q)^6, \qquad \mathcal{C} = \widehat{\mathcal{C}}|_{D(\mathcal{C})}. \tag{2.4e}$$

Note that operators with a hat are always associated with piecewise domains. We stick to this notation throughout the paper.

The Maxwell equations now read: seek $\big(\boldsymbol{H}(t), \boldsymbol{E}(t)\big) \in D(\widehat{\mathcal{C}}_{\mathrm{H}}) \times D(\mathcal{C}_{\mathrm{E}})$ such that

$$\partial_t \boldsymbol{H} = -\mathcal{C}_{\mathrm{E}}\, \boldsymbol{E} \qquad \text{in } [0, T] \times Q, \tag{2.5a}$$

$$\partial_t \boldsymbol{E} = \widehat{\mathcal{C}}_{\mathrm{H}}\, \boldsymbol{H} - \varepsilon^{-1} \boldsymbol{J} \qquad \text{in } [0, T] \times Q, \tag{2.5b}$$

$$\boldsymbol{H}(0) = \boldsymbol{H}^0, \quad \boldsymbol{E}(0) = \boldsymbol{E}^0 \qquad \text{in } Q, \tag{2.5c}$$

$$[\![\boldsymbol{H} \times \boldsymbol{n}_{\text{int}}]\!]_{F_{\text{int}}} = \boldsymbol{J}_{\text{surf}} \qquad \text{on } [0, T] \times F_{\text{int}}, \tag{2.5d}$$

Surface current. The surface current $\boldsymbol{J}_{\text{surf}}$, as an tangential trace of the magnetic field $\boldsymbol{H}$, needs to satisfy boundary conditions prescribed by the relation (1.2) and (2.5d). We want to motivate them in the following. Thus, let $\Gamma = \partial Q$. We define the sub-facets of Γ as

$$\Gamma_1 = \left\{ (x_1, x_2, x_3) \in \overline{Q} \;\middle|\; x_1 \in \{-1, 1\} \right\}, \tag{2.6a}$$

$$\Gamma_j = \left\{ (x_1, x_2, x_3) \in \overline{Q} \;\middle|\; x_j \in \{0, 1\} \right\}, \qquad \text{for } j \in \{2, 3\}. \tag{2.6b}$$

Furthermore, let $S = (0, 1)^2$. We identify F_{int} with S in the following and define

$$S_2 = \left\{ (x_2, x_3) \in \overline{S} \;\middle|\; x_2 \in \{0, 1\} \right\},\ S_3 = \left\{ (x_2, x_3) \in \overline{S} \;\middle|\; x_3 \in \{0, 1\} \right\}. \tag{2.6c}$$

Note that the lower index in (2.6) indicates which axis is fixed.

We assume that $\boldsymbol{H} = (H_1, H_2, H_3)$ is sufficiently smooth such that we can carry out the following formal calculations. From (2.5d), we obtain that

$$J_{\text{surf},2} = [\![H_3]\!]_{F_{\text{int}}}, \quad J_{\text{surf},3} = -[\![H_2]\!]_{F_{\text{int}}}, \qquad \text{on } F_{\text{int}}. \tag{2.7}$$

Thus, the perfectly conducting boundary conditions (1.2) of $\boldsymbol{H}$, prescribe the homogenous boundary conditions

$$J_{\text{surf},2}|_{S_3} = 0, \quad J_{\text{surf},3}|_{S_2} = 0.$$

This consideration determines half of the boundary conditions of the surface current. The other half is determined as follows. In order to proof that there exist regular solutions, we additionally require the following compatibility condition

$$\widehat{\boldsymbol{\mathcal{C}}}(\boldsymbol{H}, \boldsymbol{E}) = \big(-\boldsymbol{\mathcal{C}}_{\mathrm{E}}\, \boldsymbol{E}, \widehat{\boldsymbol{\mathcal{C}}}_{\mathrm{H}}\, \boldsymbol{H}\big) \in D(\widehat{\boldsymbol{\mathcal{C}}}).$$

Therefore, we obtain that $\operatorname{curl} \boldsymbol{H} \in H_0(\operatorname{curl}, Q)$ and thus $\operatorname{curl} \boldsymbol{H} \times \nu = 0$ on Γ. Since we assumed that $\boldsymbol{H}$ is smooth, this can be written as

$$\partial_2 H_3 - \partial_3 H_2 = 0 \quad \text{on } \Gamma_2 \cup \Gamma_3.$$

Together with (1.2) and (2.7) we obtain the homogenous Neumann boundary conditions on the remaining boundary, i.e.,

$$\partial_2 J_{\text{surf},2}|_{S_3} = 0, \quad \partial_3 J_{\text{surf},3}|_{S_2} = 0, \quad \text{on } F_{\text{int}}.$$

We can include both boundary conditions in the following spaces

$$D(-\Delta_2) = \big\{\, u \in H^2(S) \;\big|\; u|_{S_2} = 0, \partial_2 u|_{S_3} = 0 \,\big\}, \tag{2.8a}$$

$$D(-\Delta_3) = \big\{\, u \in H^2(S) \;\big|\; u|_{S_3} = 0, \partial_1 u|_{S_2} = 0 \,\big\}. \tag{2.8b}$$

Let $j \in \{2, 3\}$ in the following. The operators $(-\Delta_j) : D(-\Delta_j) \to L^2(S)$ are self-adjoint and positive, see, e.g., [15, 26]. This allows to define fractional powers $(-\Delta_j)^\gamma : D(-\Delta_j)^\gamma \to L^2(S)$ for $\gamma \in \mathbb{R}$.

We require in the following that $J_{\text{surf},2}(t)$ and $J_{\text{surf},3}(t)$ are elements in certain powers of $D(-\Delta_3)$ and $D(-\Delta_2)$ respectively. This, however, seems to be an unnatural choice compared to the usual Sobolev setting. The following remark closes this gap.

Remark 2.2. The fractional domains $D(-\Delta_j)^\gamma$ are associated with certain fractional Sobolev spaces via interpolation. We briefly state the relations below and refer to [10, Rem. 2.2] for details. With equivalent norms, it holds

$$D(-\Delta_2)^{1/2} = \big\{\, u \in H^1(S) \;\big|\; u|_{S_2} = 0 \,\big\},$$
$$D(-\Delta_3)^{1/2} = \big\{\, u \in H^1(S) \;\big|\; u|_{S_3} = 0 \,\big\}.$$

Let $\epsilon > 0$. It holds

$$\big\{\, u \in H^{1/2+\epsilon}(S) \;\big|\; u|_{S_2} = 0 \,\big\} \subset D(-\Delta_2)^{1/4} \subset H^{1/2}(S),$$
$$\big\{\, u \in H^{1/2+\epsilon}(S) \;\big|\; u|_{S_3} = 0 \,\big\} \subset D(-\Delta_3)^{1/4} \subset H^{1/2}(S).$$

Therefore, the space $D(-\Delta_j)^{1/4}$ generalizes Dirichlet traces for functions in the space $H^{1/2}(S)$. A similar interpretation holds for the Neumann traces, i.e.

$$\{\, u \in H^{3/2+\epsilon}(S) \mid u|_{S_2} = 0, \partial_2 u|_{S_3} = 0 \,\} \subset D(-\Delta_2)^{3/4} \subset H^{3/2}(S),$$
$$\{\, u \in H^{3/2+\epsilon}(S) \mid u|_{S_3} = 0, \partial_2 u|_{S_2} = 0 \,\} \subset D(-\Delta_3)^{3/4} \subset H^{3/2}(S).$$

Motivated by those inclusions, we introduce for $\gamma \in \mathbb{R}$ the abbreviation

$$\mathcal{X}_j^\gamma = D(-\Delta_j)^{\gamma/2}.$$

The exponent of $\mathcal{X}_j$ agrees with the associated Sobolev regularity.

Main result. Our first main result states existence, uniqueness and stability under appropriate regularity assumptions in weak variational curl spaces. This is a variation of the statements in [10].

Theorem 2.3. *If* $\boldsymbol{u}^0 \in D(\widehat{\boldsymbol{\mathcal{C}}})$, $\boldsymbol{J} \in C^0\big([0,T], D(\boldsymbol{\mathcal{C}}_{\mathrm{H}})\big) + C^1\big([0,T], L^2(Q)^3\big)$, *and*

$$(J_{\mathrm{surf},2}, J_{\mathrm{surf},3}) \in C^1\big([0,T], \mathcal{X}_3^{1/2} \times \mathcal{X}_2^{1/2}\big) \cap C^2\big([0,T], \mathcal{X}_3^{-1/2} \times \mathcal{X}_2^{-1/2}\big),$$

then there exists a unique solution

$$\boldsymbol{u} = \big(\boldsymbol{H}, \boldsymbol{E}\big) \in C^0\big([0,T], D(\widehat{\boldsymbol{\mathcal{C}}})\big) \cap C^1\big([0,T], L^2(Q)^6\big)$$

of (2.5)*. Furthermore, for all* $t \in [0,T]$ *it holds*

$$\begin{aligned} \|\boldsymbol{u}(t)\|_{\mu\times\varepsilon} \lesssim & \|\boldsymbol{u}^0\|_{\mu\times\varepsilon} + \|\boldsymbol{J}_{\mathrm{surf}}(0)\|_{L^2(F_{\mathrm{int}})^3} + \|\boldsymbol{J}_{\mathrm{surf}}(t)\|_{L^2(F_{\mathrm{int}})^3} \\ & + \int_0^t \|\boldsymbol{J}(s)\|_{L^2(Q)^3} \,\mathrm{d}s + \int_0^t \|\partial_t \boldsymbol{J}_{\mathrm{surf}}(s)\|_{L^2(F_{\mathrm{int}})^3} \,\mathrm{d}s \\ & + \int_0^t \Big\| \big(J_{\mathrm{surf},2}(s), J_{\mathrm{surf},3}(s)\big) \Big\|_{(-\Delta_3)^{1/4}\times(-\Delta_2)^{1/4}} \,\mathrm{d}s, \end{aligned} \tag{2.11}$$

with a constant which is independent of $\boldsymbol{J}, \boldsymbol{J}_{\mathrm{surf}}$ *and* $\boldsymbol{u}$.

Strategy of proof. It is well known that the operator $\boldsymbol{\mathcal{C}} : D(\boldsymbol{\mathcal{C}}) \to L^2(Q)^6$ is the generator of a unitary C^0-semigroup, whereas the operator $\widehat{\boldsymbol{\mathcal{C}}} : D(\widehat{\boldsymbol{\mathcal{C}}}) \to L^2(Q)^6$ does not inherit any good properties, see, e.g., [10, Rem. 2.1]. Thus, we aim to construct an extension $\boldsymbol{J_H} \in PH(\mathrm{curl}, Q)$ that exhibits the correct normal jump on F_{int}, i.e., $[\![\boldsymbol{J_H} \times \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}} = \boldsymbol{J}_{\mathrm{surf}}$. This enables us to introduce a shifted magnetic field $\tilde{\boldsymbol{H}} = \boldsymbol{H} - \boldsymbol{J_H}$ with a vanishing tangential jump $[\![\tilde{\boldsymbol{H}} \times \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}} = 0$. Therefore, we are interested in solving the following shifted system: seek $\big(\tilde{\boldsymbol{H}}(t), \boldsymbol{E}(t)\big) \in D(\boldsymbol{\mathcal{C}}_{\mathrm{H}}) \times D(\boldsymbol{\mathcal{C}}_{\mathrm{H}})$ such that

$$\partial_t \tilde{\boldsymbol{H}} = -\boldsymbol{\mathcal{C}}_{\mathrm{E}}\, \boldsymbol{E} - \tilde{\boldsymbol{J}}_1 \qquad \text{in } [0,T]\times Q, \tag{2.12a}$$

$$\partial_t \boldsymbol{E} = \boldsymbol{\mathcal{C}}_{\mathrm{H}}\, \tilde{\boldsymbol{H}} - \tilde{\boldsymbol{J}}_2 \qquad \text{in } [0,T]\times Q, \tag{2.12b}$$

$$\tilde{\boldsymbol{H}}(0) = \tilde{\boldsymbol{H}}^0, \quad \boldsymbol{E}(0) = \boldsymbol{E}^0 \qquad \text{in } Q, \tag{2.12c}$$

with $\tilde{\boldsymbol{J}}_1 = -\partial_t \boldsymbol{J_H}$, $\tilde{\boldsymbol{J}}_2 = -\varepsilon^{-1}\boldsymbol{J} + \varepsilon^{-1}\,\mathrm{curl}\,\boldsymbol{J_H}$ and $\tilde{\boldsymbol{H}}^0 = \boldsymbol{H}^0 - \boldsymbol{J_H}(0)$. The shifted problem (2.12) allows us to use C^0-semigroup theory to show existence, uniqueness and stability. However, we need to make sure that the extension $\boldsymbol{J_H}$ is sufficiently regular in time and space to meet the requirements necessary for (2.12) to be a wellposed system. This will boil down to certain regularity conditions for the surface current $\boldsymbol{J}_{\mathrm{surf}}$. Additionally, $\boldsymbol{J_H}$ needs to satisfy certain boundary conditions.

Extension. The extension $\boldsymbol{J_H}$ is constructed in two steps. We first extend scalar valued functions from the interface to the whole domain, cf. Lemma 2.4. The combination of scalar extension then yields the vector valued extension $\boldsymbol{J_H}$ in Theorem 2.5.

Lemma 2.4. *There exists a bounded linear operator* $\Phi_j : \mathcal{X}_j^{-1/2} \to L^2(Q)$ *with the following properties.*

1. *The operator* $\Phi_j|_{\mathcal{X}_j^{1/2}} : \mathcal{X}_j^{1/2} \to PH^1(Q)$ *is bounded.*
2. *The operator* $\Phi_j|_{\mathcal{X}_j^{3/2}} : \mathcal{X}_j^{3/2} \to PH^2(Q)$ *is bounded.*
3. *For* $v \in \mathcal{X}_j^{1/2}$ *it holds* $[\![\Phi_j(v)]\!]_{F_{\mathrm{int}}} = v$ *on* F_{int}.
4. *For* $v \in \mathcal{X}_j^{1/2}$ *it holds* $\Phi_j(v) = 0$ *on* $\Gamma_j \cup \Gamma_1$.

Proof. Let $\chi\colon[-1,1]\to[0,1]$ be a smooth cut-off function with $\operatorname{supp}\chi\subset[-3/4,3/4]$ and $\chi=1$ on $[-1/2,1/2]$. For $v\in\mathcal{X}_j^{-1/2}$ and $x\in Q$ we define

$$\Phi_j(v)(x)=\begin{cases}\frac{1}{2}\chi(x_1)(-\Delta_j)^{1/4}\Big(e^{-x_1(-\Delta_j)^{1/2}}(-\Delta_j)^{-1/4}v\Big)(x_2,x_3) & \text{for } x_1>0,\\ -\Phi_j(v)(-x_1,x_2,x_3) & \text{for } x_1<0.\end{cases}$$

This defines a linear and bounded map.

The proof of (1) and (2) follows along the lines of [10, Lem. 4.1]. Alternatively, we give a short proof of (1) in Appendix A.

We proceed with the proof of (4) and show the claim first for $v\in D(-\Delta_j)^{3/4}$. Without loss of generality, we assume $j=2$. Since $v\in D(-\Delta_2)^{3/4}$, we conclude with (2) that $\Phi_2(v)\in PH^2(Q)$. The space $H^2(Q_\pm)$ embeds continuously into $C^0(\overline{Q_\pm})$. Therefore, we obtain

$$\Phi_2(v)(x_1,0,\cdot)=\Phi_2(x_1,1,\cdot)=0$$

since by construction, for $x_1\in(-1,1)\setminus\{0\}$ it holds $\Phi_2(v)(x_1,\cdot)\in D(-\Delta_2)$. This shows that $\|\Phi_2(v)\|_{L^2(\Gamma_2)}=0$.

Furthermore, due to the cut-off function, it holds $\|\Phi_2(v)\|_{L^2(\Gamma_1)}=0$. The density of $D(-\Delta_2)^{3/4}$ in $D(-\Delta_2)^{1/2}$ proves the claim for $v\in D(-\Delta_2)^{1/4}$.

It remains to prove (3). With a similar argument as above, we can take the classical limit in the first component and obtain by construction

$$[\![\Phi_j(v)]\!]_{F_{\text{int}}}=\Phi_j(v)(0^+,\cdot,\cdot)-\Phi_j(v)(0^-,\cdot,\cdot)=\frac{1}{2}v-(-\frac{1}{2}v)=v$$

on F_{int}. This proves the statement. $\square$

With this we can define the desired extension $\boldsymbol{J_H}$ which satisfies the properties outlined above.

Theorem 2.5. *Let* $\boldsymbol{J}_{\text{surf}}=\big(0,J_{\text{surf},2},J_{\text{surf},3}\big)$ *and*

$$(J_{\text{surf},2},J_{\text{surf},3})\in C^1\big([0,T],\mathcal{X}_3^{1/2}\times\mathcal{X}_2^{1/2}\big)\cap C^2\big([0,T],\mathcal{X}_3^{-1/2}\times\mathcal{X}_2^{-1/2}\big).$$

Then, there exists a function

$$\boldsymbol{J_H}\in C^1\big([0,T],PH^1(Q)^3\big)\cap C^2\big([0,T],L^2(Q)^3\big)$$

such that for $t\in[0,T]$ *it holds*

(1) $[\![\boldsymbol{J_H}(t)\times\boldsymbol{n}_{\text{int}}]\!]_{F_{\text{int}}}=\boldsymbol{J}_{\text{surf}}(t)$ *on* F_{int}*,*
(2) $\boldsymbol{J_H}(t)\cdot\nu=0$ *on* ∂Q*.*

Furthermore, the following estimates hold

$$\begin{aligned}\|\boldsymbol{J_H}(t)\|_{L^2(Q)^3}&\lesssim\|\boldsymbol{J}_{\text{surf}}(t)\|_{L^2(S)^3},\\ \|\partial_t\boldsymbol{J_H}(t)\|_{L^2(Q)^3}&\lesssim\|\partial_t\boldsymbol{J}_{\text{surf}}(t)\|_{L^2(S)^3},\\ \|\boldsymbol{J_H}(t)\|_{PH^1(Q)^3}&\lesssim\|(J_{\text{surf},2}(t),J_{\text{surf},3}(t))\|_{(-\Delta_3)^{1/2}\times(-\Delta_2)^{1/4}},\end{aligned}$$

with constants independent of $\boldsymbol{J}_{\text{surf}}$*.*

Proof. Define $\boldsymbol{J_H}(t)=\Big(0,-\Phi_2\big(J_{\text{surf},3}(t)\big),\Phi_3\big(J_{\text{surf},2}(t)\big)\Big)$. The statement is an application of Lemma 2.4. $\square$

Proof of main result. We write (2.12) in the compact form: seek $\tilde{\boldsymbol{u}}(t)\in D(\boldsymbol{\mathcal{C}})$ such that

$$\partial_t\tilde{\boldsymbol{u}}(t)=\boldsymbol{\mathcal{C}}\,\tilde{\boldsymbol{u}}(t)+\tilde{\boldsymbol{j}}(t),\ \text{ for } t\in[0,T], \tag{2.13a}$$

$$\tilde{\boldsymbol{u}}(0)=\tilde{\boldsymbol{u}}^0, \tag{2.13b}$$

with $\tilde{\boldsymbol{u}}(t) = \big(\tilde{\boldsymbol{H}}(t), \boldsymbol{E}(t)\big)$, $\tilde{\boldsymbol{j}}(t) = \big(-\partial_t \boldsymbol{J}_{\boldsymbol{H}}(t), -\varepsilon^{-1}\boldsymbol{J}(t) + \varepsilon^{-1} \operatorname{curl} \boldsymbol{J}_{\boldsymbol{H}}(t)\big)$, $\boldsymbol{j}_{\boldsymbol{H}}(t) = \big(\boldsymbol{J}_{\boldsymbol{H}}(t), 0\big)$ and $\tilde{\boldsymbol{u}}^0 = \boldsymbol{u}^0 - \boldsymbol{j}_{\boldsymbol{H}}(0)$. This problem fits into the framework of Cauchy problems and standard theorems for existence and stability can be applied.

Lemma 2.6. *Let $\boldsymbol{u}^0 \in D(\widehat{\boldsymbol{\mathcal{C}}})$ and $\boldsymbol{J}_{\text{surf}} = \big(0, J_{\text{surf},2}, J_{\text{surf},3}\big)$ with*

$$(J_{\text{surf},2}, J_{\text{surf},3}) \in C^1\big([0,T], \mathcal{X}_3^{1/2} \times \mathcal{X}_2^{1/2}\big) \cap C^2\big([0,T], \mathcal{X}_3^{-1/2} \times \mathcal{X}_2^{-1/2}\big).$$

Furthermore, let

$$\boldsymbol{J} \in C^0\big([0,T], D(\boldsymbol{\mathcal{C}}_{\mathrm{H}})\big) + C^1\big([0,T], L^2(Q)^3\big).$$

Then, there exists a unique solution

$$\tilde{\boldsymbol{u}} = \big(\tilde{\boldsymbol{H}}, \boldsymbol{E}\big) \in C^0\big([0,T], D(\boldsymbol{\mathcal{C}})\big) \cap C^1\big([0,T], L^2(Q)^6\big)$$

of (2.12) given by

$$\tilde{\boldsymbol{u}}(t) = e^{t\boldsymbol{\mathcal{C}}}\tilde{\boldsymbol{u}}^0 + \int_0^t e^{(t-s)\boldsymbol{\mathcal{C}}}\tilde{\boldsymbol{j}}(s)\,\mathrm{d}s. \tag{2.14}$$

Proof. In view of standard results for Cauchy problems, cf. [22, Thm. 4.2.4, Cor. 4.2.5], we need to check the two conditions

$$\tilde{\boldsymbol{u}}^0 \in D(\boldsymbol{\mathcal{C}}), \quad \tilde{\boldsymbol{j}} \in C^0\big([0,T]; D(\boldsymbol{\mathcal{C}})\big) + C^1\big([0,T]; L^2(Q)^6\big).$$

By construction $\tilde{\boldsymbol{u}}^0 \in D(\widehat{\boldsymbol{\mathcal{C}}})$. Furthermore, by Theorem 2.5 it holds

$$[\![\tilde{\boldsymbol{u}}^0 \times \boldsymbol{n}_{\text{int}}]\!]_{F_{\text{int}}} = [\![\boldsymbol{u}^0 \times \boldsymbol{n}_{\text{int}}]\!]_{F_{\text{int}}} - [\![\boldsymbol{j}_{\boldsymbol{H}}(0) \times \boldsymbol{n}_{\text{int}}]\!]_{F_{\text{int}}} = 0.$$

Therefore, we conclude with Corollary 2.1 that $\tilde{\boldsymbol{u}}^0 \in D(\boldsymbol{\mathcal{C}})$. Again by Theorem 2.5, we see that

$$\varepsilon^{-1} \operatorname{curl} \boldsymbol{J}_{\boldsymbol{H}}, \partial_t \boldsymbol{J}_{\boldsymbol{H}} \in C^1\big([0,T]; L^2(Q)^3\big).$$

This proves the claim together with the assumption on $\boldsymbol{J}$. ☐

We are now able to proof the first main result.

Proof of Theorem 2.3. A straight forward calculation shows that $\boldsymbol{u} = \tilde{\boldsymbol{u}} + \boldsymbol{j}_{\boldsymbol{H}}$ solves (2.5). This solution is unique as a consequence of the uniqueness in Lemma 2.6.

It remains to prove stability. Taking norms in (2.14), we obtain

$$\begin{aligned}
\|\tilde{\boldsymbol{u}}(t)\|_{\mu\times\varepsilon} &\le \|\boldsymbol{u}^0\|_{\mu\times\varepsilon} + \|\boldsymbol{J}_{\boldsymbol{H}}(0)\|_{\mu} \\
&\quad + \int_0^t \big\|\varepsilon^{-1}\boldsymbol{J}(s)\big\|_{\varepsilon} + \big\|\varepsilon^{-1} \operatorname{curl} \boldsymbol{J}_{\boldsymbol{H}}(s)\big\|_{\varepsilon} + \big\|\partial_t \boldsymbol{J}_{\boldsymbol{H}}(s)\big\|_{\mu}\,\mathrm{d}s \\
&\le \|\boldsymbol{u}^0\|_{\mu\times\varepsilon} + \sqrt{\mu_\infty}\|\boldsymbol{J}_{\boldsymbol{H}}(0)\|_{L^2(Q)^3} \\
&\quad + \frac{1}{\sqrt{\delta}}\int_0^t \|\boldsymbol{J}(s)\|_{L^2(Q)^3}\,\mathrm{d}s + \sqrt{\mu_\infty}\int_0^t \|\partial_t \boldsymbol{J}_{\boldsymbol{H}}(s)\|_{L^2(Q)^3}\,\mathrm{d}s \\
&\quad + \frac{1}{\sqrt{\delta}}\int_0^t \big\|\operatorname{curl} \boldsymbol{J}_{\boldsymbol{H}}(s)\big\|_{L^2(Q)^3}\,\mathrm{d}s.
\end{aligned}$$

By Theorem 2.5, we estimate further

$$\begin{aligned}
\|\tilde{\boldsymbol{u}}(t)\|_{\mu\times\varepsilon} &\lesssim \|\boldsymbol{u}^0\|_{\mu\times\varepsilon} + \|\boldsymbol{J}_{\text{surf}}(0)\|_{L^2(F_{\text{int}})^3} \\
&\quad + \int_0^t \|\boldsymbol{J}(s)\|_{L^2(Q)^3}\,\mathrm{d}s + \int_0^t \|\partial_t \boldsymbol{J}_{\text{surf}}(s)\|_{L^2(F_{\text{int}})^3}\,\mathrm{d}s \\
&\quad + \int_0^t \Big\|\big(J_{\text{surf},2}(s), J_{\text{surf},3}(s)\big)\Big\|_{(-\Delta_3)^{1/4}\times(-\Delta_2)^{1/4}}\,\mathrm{d}s.
\end{aligned}$$

The claim follows with $\|\boldsymbol{u}(t)\|_{\mu\times\varepsilon} \le \|\tilde{\boldsymbol{u}}(t)\|_{\mu\times\varepsilon} + \|\boldsymbol{j}_{\boldsymbol{H}}(t)\|_{\mu\times\varepsilon}$. ☐

3. Spatial discretization

In this chapter, we introduce a concrete space discretization and derive rigorously the discrete curl in (1.5) and the discrete extension in (1.6). We first present our two main results involving rigorous error bounds for the spatially discrete scheme and an extended scheme. The chapter then proceeds with a spatially discrete analogue of the stability bound Theorem 2.3 and is concluded with the proofs of the main results.

Discrete setting. We denote with $\mathcal{T}_h$ matching simplicial meshes of the domain Q, generated by a reference element $\widehat{K}$. The subscript h indicates the mesh size defined as $h = \max_{K\in\mathcal{T}_h} h_K$, where h_K denotes the diameter of a mesh element K. Furthermore, we assume that the mesh sequence is shape regular in the sense of [12, Def. 11.2]. Thus, there exists $\sigma > 0$ independent of h such that $h_K \leq \sigma\rho_K$, where ρ_K denotes the diameter of the largest inscribing ball of K.

We collect the faces F of all mesh elements in the set $\mathcal{F}_h = \mathcal{F}_h^\circ \cup \mathcal{F}_h^\partial$, where $\mathcal{F}_h^\circ$ denotes the set of all faces in the interior of Q and $\mathcal{F}_h^\partial$ the set of all faces on the boundary ∂Q. Refer to [12, Def. 8.10] for a precise definition of mesh faces.

The outer unit normal vector of K is denoted by $\boldsymbol{n}_K$. Every interior face $F \in \mathcal{F}_h^\circ$ intersects two elements $K_{F,l}$ and $K_{F,r}$. The order of the elements is arbitrary but fixed. We choose the unit normal $\boldsymbol{n}_F$ to F pointing from $K_{F,l}$ to $K_{F,r}$. For boundary faces $F \in \mathcal{F}_h^\partial$, we choose the unit normal $\boldsymbol{n}_F$ to F as the outer unit normal vector $\boldsymbol{n}_K$ of the associated element K.

Let F be an interior face and $v : Q \to \mathbb{R}$ be a function that admits a well-defined trace on F. The weighted average of v on the face F is defined as

$$\{\!\{v\}\!\}_F^\omega = \frac{\omega_{K_{F,l}}(v|_{K_{F,l}})\big|_F + \omega_{K_{F,r}}(v|_{K_{F,r}})\big|_F}{\omega_{K_{F,l}} + \omega_{K_{F,r}}}, \tag{3.1a}$$

where $\omega : Q \to (0,\infty)$ denotes a positive weight function that is piecewise constant, i.e., $\omega|_K \equiv \omega_K$ for all $K \in \mathcal{T}_h$. Analogously, we define the jump of v on F as

$$[\![v]\!]_F = (v|_{K_{F,r}})\big|_F - (v|_{K_{F,l}})\big|_F. \tag{3.1b}$$

For vector fields, both definitions hold component-wise.

The following assumption is necessary to resolve the interface conditions (1.3).

Assumption 3.1. *We assume that every element $K \in \mathcal{T}_h$ lies completely on one side of the interface F_{int}, i.e.,*

$$K \cap F_{\mathrm{int}} = \emptyset, \quad \text{for all } K \in \mathcal{T}_h.$$

Furthermore, we assume that the unit normal $\boldsymbol{n}_F$ for every face $F \in \mathcal{F}_h^\circ$ with $F \subset F_{\mathrm{int}}$ points in the same direction as $\boldsymbol{n}_{\mathrm{int}}$, i.e.,

$$\boldsymbol{n}_F \cdot \boldsymbol{n}_{\mathrm{int}} = 1, \quad \text{for all } F \in \mathcal{F}_h^\circ \text{ with } F \subset F_{\mathrm{int}}.$$

The set of all faces $F \in \mathcal{F}_h^\circ$ with $F \subset F_{\mathrm{int}}$ is denoted by $\mathcal{F}_h^{\mathrm{int}}$.

Similar to the definition of the broken polynomial space (1.4a), we introduce for $s \geq 0$ the broken Sobolev space on $\mathcal{T}_h$ defined by

$$H^s(\mathcal{T}_h) = \{\, v \in L^2(Q) \mid u|_K \in H^s(K) \text{ for all } K \in \mathcal{T}_h \,\}. \tag{3.2a}$$

The piecewise semi-norm on $H^s(\mathcal{T}_h)$ is denoted by $|\cdot|_{H^s(\mathcal{T}_h)}$ and we define

$$\|\cdot\|^2_{H^s(\mathcal{T}_h)} = \|\cdot\|^2_{L^2(Q)} + |\cdot|^2_{H^s(\mathcal{T}_h)}. \tag{3.2b}$$

In the following, slightly more regularity of the solution is assumed such that it admits classical traces on element faces. Therefore, we define the spaces

$$\widehat{V}_*^{\boldsymbol{H}} = D(\widehat{\boldsymbol{\mathcal{C}}}_{\mathrm{H}}) \cap H^1(\mathcal{T}_h), \quad V_*^{\boldsymbol{E}} = D(\boldsymbol{\mathcal{C}}_{\mathrm{E}}) \cap H^1(\mathcal{T}_h), \quad \widehat{V}_* = \widehat{V}_*^{\boldsymbol{H}} \times V_*^{\boldsymbol{E}} \tag{3.3a}$$

and the restricted spaces

$$V_*^{\boldsymbol{H}} = D(\mathcal{C}_{\mathrm{H}}) \cap H^1(\mathcal{T}_h), \quad V_* = V_*^{\boldsymbol{H}} \times V_*^{\boldsymbol{E}}. \tag{3.3b}$$

Since functions of the approximation space V_h, defined in (1.4a), do not admit a well-defined curl, we introduce the following spaces containing both the analytical solution and the approximation

$$\widehat{V}_{*,h}^{\boldsymbol{H}} = \widehat{V}_*^{\boldsymbol{H}} + V_h, \quad V_{*,h}^{\boldsymbol{E}} = V_*^{\boldsymbol{E}} + V_h, \widehat{V}_{*,h} = \widehat{V}_{*,h}^{\boldsymbol{H}} \times V_{*,h}^{\boldsymbol{E}}, \tag{3.4a}$$

and similarly

$$V_{*,h}^{\boldsymbol{H}} = V_*^{\boldsymbol{H}} + V_h, \quad V_{*,h} = V_{*,h}^{\boldsymbol{H}} \times V_{*,h}^{\boldsymbol{E}}. \tag{3.4b}$$

Remark 3.2. Note that the results are not specific to matching simplicial meshes but are also valid for quadrilateral meshes and general meshes as described in [8, Sec. 1.2]. We omit the details for the sake of presentation.

Spatial discretization. As motivated in the introduction with (1.6), we define the discrete lift operator

$$\mathfrak{L}_{\mathrm{int}} : L^2(F_{\mathrm{int}})^3 \to V_h, \quad (\mathfrak{L}_{\mathrm{int}} \boldsymbol{V}, \boldsymbol{\phi}_h)_\varepsilon = -\sum_{F \in \mathcal{F}_h^{\mathrm{int}}} (\boldsymbol{V}|_F, \{\!\!\{\boldsymbol{\phi}_h\}\!\!\}_F^{\overline{\mu c}}) \tag{3.5a}$$

for $\boldsymbol{\phi}_h \in V_h$, and the discrete magnetic Maxwell operator $\widehat{\mathfrak{C}}_{\mathrm{H}} : \widehat{V}_{*,h}^{\boldsymbol{H}} \to V_h$

$$\begin{aligned}(\widehat{\mathfrak{C}}_{\mathrm{H}} \boldsymbol{H}, \boldsymbol{\phi}_h)_\varepsilon = &\sum_{K \in \mathcal{T}_h} (\boldsymbol{H}, \operatorname{curl} \boldsymbol{\phi}_h)_K \\ &- \sum_{F \in \mathcal{F}_h^\partial} (\boldsymbol{H} \times \boldsymbol{n}_F, \boldsymbol{\phi}_h)_F - \sum_{F \in \mathcal{F}_h^\circ} (\{\!\!\{\boldsymbol{H}\}\!\!\}_F^{\mu c}, [\![\boldsymbol{\phi}_h]\!]_F \times \boldsymbol{n}_F)_F.\end{aligned} \tag{3.5b}$$

Analogously, we define the electric Maxwell operator $\mathfrak{C}_{\mathrm{E}} : V_{*,h}^{\boldsymbol{E}} \to V_h$ for $\boldsymbol{E} \in V_{*,h}^{\boldsymbol{E}}$ and $\boldsymbol{\psi}_h \in V_h$ by

$$(\mathfrak{C}_{\mathrm{E}} \boldsymbol{E}, \boldsymbol{\psi}_h)_\mu = \sum_{K \in \mathcal{T}_h} (\boldsymbol{E}, \operatorname{curl} \boldsymbol{\psi}_h)_K - \sum_{F \in \mathcal{F}_h^\circ} (\{\!\!\{\boldsymbol{E}\}\!\!\}_F^{\varepsilon c}, [\![\boldsymbol{\psi}_h]\!]_F \times \boldsymbol{n}_F)_F. \tag{3.5c}$$

This definition incorporates the perfectly conducting boundary condition for the electric field. The discrete operator acting on the combined field is defined as

$$\mathfrak{C} : \widehat{V}_{*,h} \to V_h^2, \quad \mathfrak{C} = \begin{pmatrix} 0 & -\mathfrak{C}_{\mathrm{E}} \\ \widehat{\mathfrak{C}}_{\mathrm{H}} & 0 \end{pmatrix}. \tag{3.5d}$$

Analogously to (2.4d), we define the restrictions

$$\mathfrak{C}_{\mathrm{H}} : V_{*,h}^{\boldsymbol{H}} \to V_h, \qquad \mathfrak{C}_{\mathrm{H}} = \widehat{\mathfrak{C}}_{\mathrm{H}}|_{V_{*,h}^{\boldsymbol{H}}}, \tag{3.5e}$$

$$\mathfrak{C} : V_{*,h} \to V_h^2, \qquad \mathfrak{C} = \widehat{\mathfrak{C}}|_{V_{*,h}}. \tag{3.5f}$$

The semi-discrete problem now reads: seek $\big(\boldsymbol{H}_h(t), \boldsymbol{E}_h(t)\big) \in V_h^2$ such that

$$\partial_t \boldsymbol{H}_h(t) = -\mathfrak{C}_{\mathrm{E}} \boldsymbol{E}_h(t) \qquad \text{for } t \in [0,T], \tag{3.6a}$$

$$\partial_t \boldsymbol{E}_h(t) = \widehat{\mathfrak{C}}_{\mathrm{H}} \boldsymbol{H}_h(t) - \boldsymbol{J}_h(t) - \boldsymbol{J}_{\mathrm{surf},h}(t) \qquad \text{for } t \in [0,T], \tag{3.6b}$$

$$\boldsymbol{H}_h(0) = \boldsymbol{H}_h^0, \quad \boldsymbol{E}_h(0) = \boldsymbol{E}_h^0, \tag{3.6c}$$

where $\boldsymbol{J}_{\mathrm{surf},h} = \mathfrak{L}_{\mathrm{int}} \boldsymbol{J}_{\mathrm{surf}}$, $\boldsymbol{H}_h^0 = \Pi_h \boldsymbol{H}^0$, $\boldsymbol{E}_h^0 = \Pi_h \boldsymbol{E}^0$ and $\boldsymbol{J}_h = \Pi_h \varepsilon^{-1} \boldsymbol{J}$. We denote with $\Pi_h : L^2(Q) \to \mathbb{P}_3^k(\mathcal{T}_h)$ the broken L^2-orthogonal projection defined by

$$(v - \Pi_h v, \phi_h)_{L^2(Q)} = 0 \qquad \text{for all } \phi_h \in \mathbb{P}_k^3(\mathcal{T}_h). \tag{3.7}$$

For typical properties of this projection, compare [12, Sec. 18.4] or [8, Sec. 1.4.4].

The second main result gives an error bound on the spatially discrete solution $\boldsymbol{u}_h = \big(\boldsymbol{H}_h, \boldsymbol{E}_h\big)$ of (3.6). For a sufficiently regular problem, we obtain convergence in the mesh parameter h. The proof is given below.

Theorem 3.3. *Let the solution* $\boldsymbol{u} = \big(\boldsymbol{H}, \boldsymbol{E}\big)$ *of* (2.5) *satisfy*

$$\boldsymbol{u} \in C^0([0,T], \widehat{V}_* \cap H^{1+s}(\mathcal{T}_h)^6) \cap C^1([0,T], L^2(Q)^6), \tag{3.8}$$

with $s \geq 0$. *Furthermore, let Assumption* 3.1 *hold. Then, the appropriation* $\boldsymbol{u}_h = \big(\boldsymbol{H}_h, \boldsymbol{E}_h\big)$ *defined in* (3.6) *satisfies*

$$\|\boldsymbol{u}(t) - \boldsymbol{u}_h(t)\|_{\mu\times\varepsilon} \leq Ch^{r_*}, \qquad 0 \leq t \leq T,$$

with a constant $C > 0$ *independent of* h. *Here,* $r_* = \min\{s, k\}$ *with* k *denoting the polynomial degree of the approximation space defined in* (1.4a).

Note, that this agrees with the results obtained for the special case $\boldsymbol{J}_{\mathrm{surf}} = 0$, see, e.g., [14].

Nodal interpolation. The calculation of the lift operator (3.5a) involves the evaluation of integrals over mesh faces. In practice, those integrals are approximated by quadrature formulas. This can be quite expensive, since the evaluation may be required at every time step. Moreover, if the surface current depends on the solution itself, i.e., $\boldsymbol{J}_{\mathrm{surf}} = \boldsymbol{J}_{\mathrm{surf}}(\boldsymbol{E})$. The calculation of the lift operator would cause evaluations of the finite element functions at every quadrature point which is quite expensive. In such cases, nodal discontinuous Galerkin methods are attractive since they allow for a fast evaluation of integrals and functions, see, e.g., [8, App. 2] for a detailed discussion. In the following, we construct a scheme that makes use of nodal interpolation of $\boldsymbol{J}_{\mathrm{surf}}$ and provide error bounds.

We specify the construction from Section 3 and choose $\mathcal{N}_k = \dim \mathbb{P}_3^k$ nodes $\Sigma_{\widehat{K}} = \{\sigma_{\widehat{K},1}, \dots, \sigma_{\widehat{K},\mathcal{N}_k}\}$ in the closure of the reference element $\widehat{K}$. Then, the Lagrange polynomials, defined by $\theta_{K,i}(\sigma_{K,j}) = \delta_{ij}$ for $i,j \in \{1,\dots,\mathcal{N}_k\}$, form a basis of $\mathbb{P}_k^3(K)$. Thus, we can define for $\ell > 3/2$ the local interpolation operator

$$\mathcal{I}_K^h : H^\ell(K) \to \mathbb{P}_k^3(K), \quad \mathcal{I}_K^h v = \sum_{j=1}^{\mathcal{N}_k} v|_K(\sigma_{K,i})\theta_{K,i}$$

and hence, the global interpolation operator by restriction, i.e.,

$$\mathcal{I}^h : H^\ell(\mathcal{T}_h) \to \mathbb{P}_k^3(\mathcal{T}_h), \quad \mathcal{I}^h v|_K = \mathcal{I}_K^h v, \quad \text{for } K \in \mathcal{T}_h.$$

Note, that the interpolation operator acts component-wise for vector fields.

The surface current $\boldsymbol{J}_{\mathrm{surf}}$ is only supported on the interface F_{int} and hence we construct an interpolation operator on the sub-mesh $\mathcal{F}_h^{\mathrm{int}}$. Therefore, we need the following two assumptions.

Assumption 3.4 ([12, Ass. 20.1])**.** *Let* $\widehat{F}$ *be a face of the reference element* $\widehat{K}$ *and denote with* $\Sigma_{\widehat{F}}$ *the nodes that are located on* $\widehat{F}$, *i.e.,* $\Sigma_{\widehat{F}} = \Sigma_{\widehat{K}} \cap \widehat{F}$. *We assume that for any* $p \in \mathbb{P}_3^k(\widehat{K})$ *it holds* $p|_{\widehat{F}} \equiv 0$ *if and only if* $p(\sigma) = 0$ *for all* $\sigma \in \Sigma_{\widehat{F}}$.

We also need to make sure how the nodes of neighboring elements come in contact with each other.

Assumption 3.5 ([12, Ass. 20.3])**.** *For any face* $F \in \mathcal{F}_h^\circ$ *it holds*

$$\Sigma_{K_{F,l}} \cap F = \Sigma_{K_{F,r}} \cap F =: \Sigma_F.$$

We write again $\Sigma_F = \{\sigma_{F,1}, \dots, \sigma_{F,\mathfrak{N}_k}\}$.

Remark 3.6. These assumptions ensure that the triple $(F, \mathbb{P}_2^k(F), \Sigma_F)_{F\in\mathcal{F}_h^{\mathrm{int}}}$ is again a finite element for F_{int} in the sense of [12, Def. 5.2], see [12, Lem. 20.2] for details. Note that the usual $\mathbb{P}_k$ and $\mathbb{Q}_k$ nodal Lagrange elements satisfy both assumptions, see [12, Sec. 20.2] for details.

Given Assumptions 3.4 and 3.5, we are able to define for $\kappa > 1$ the local interpolation operator

$$\mathfrak{I}_F^h : H^\kappa(F) \to \mathbb{P}_2^k(F), \quad \mathfrak{I}_F^h v = \sum_{j=1}^{\mathfrak{N}_k} v|_F(\sigma_{F,j})\theta_{F,j}$$

and the global interpolation operator

$$\mathfrak{I}^h : H^\kappa(\mathcal{F}_h^{\mathrm{int}}) \to \mathbb{P}_2^k(\mathcal{F}_h^{\mathrm{int}}), \quad \mathfrak{I}^h v|_F = \mathfrak{I}_F^h v, \quad \text{for } F \in \mathcal{F}_h^{\mathrm{int}}.$$

The problem now reads: seek $\big(\check{\boldsymbol{H}}(t), \check{\boldsymbol{E}}(t)\big) \in V_h^2$ such that

$$\partial_t \check{\boldsymbol{H}}_h(t) = -\,\mathfrak{C}_{\mathrm{E}}\, \check{\boldsymbol{E}}_h(t) \qquad \text{for } t \in [0,T], \tag{3.9a}$$

$$\partial_t \check{\boldsymbol{E}}_h(t) = \widehat{\mathfrak{C}}_{\mathrm{H}}\, \check{\boldsymbol{H}}_h(t) - \boldsymbol{J}_h(t) - \check{\boldsymbol{J}}_{\mathrm{surf},h}(t) \qquad \text{for } t \in [0,T], \tag{3.9b}$$

$$\check{\boldsymbol{H}}_h(0) = \boldsymbol{H}_h^0, \quad \check{\boldsymbol{E}}_h(0) = \boldsymbol{E}_h^0, \tag{3.9c}$$

with $\check{\boldsymbol{J}}_h = \mathfrak{L}_{\mathrm{int}}\mathfrak{I}^h \boldsymbol{J}_{\mathrm{surf}}$. Note, that the semi-discrete solutions of (3.6) and (3.9) only differ in the fact that we use nodal interpolation under the lift operator. Our third main result is concerned with the error introduced by this additional approximation. The proof is given below.

Theorem 3.7. *Let Assumptions 3.1, 3.4, and 3.5 hold and further let the solution $\boldsymbol{u} = \big(\boldsymbol{H}, \boldsymbol{E}\big)$ of (2.5) satisfy*

$$\boldsymbol{u} \in C^0([0,T], \widehat{V}_* \cap H^{1+s}(\mathcal{T}_h)^6) \cap C^1([0,T], L^2(Q)^6), \tag{3.10}$$

with $s > 1/2$. For the approximations $\boldsymbol{u}_h$ defined in (3.6) and $\check{\boldsymbol{u}}_h$ defined in (3.9) it holds

$$\|\boldsymbol{u}_h(t) - \check{\boldsymbol{u}}_h(t)\|_{\mu\times\varepsilon} \le Ch^{\min\{s,k+1/2\}}$$

with a constant $C > 0$ which is independent of h. Here, k denotes the polynomial degree of the approximation space (1.4a).

The following corollary follows immediately from Theorems 3.3 and 3.7.

Corollary 3.8. *Under the assumptions of Theorem 3.7 it holds*

$$\|\boldsymbol{u}(t) - \check{\boldsymbol{u}}_h(t)\|_{\mu\times\varepsilon} \le Ch^{r_*},$$

a constant $C > 0$ which is independent of h. Here, $r_* = \min\{s,k\}$ with k denoting the polynomial degree of the approximation space (1.4a).

Stability. We proceed by proving a discrete analogue to the stability bound (2.11).

The broken L^2-projection Π_h, defined in (3.7), has the following piecewise approximation properties, see, e.g., [12, Sec. 18.4].

Lemma 3.9. *For all $K \in \mathcal{T}_h$ and all $v \in H^{1+s}(K)$ with $s \ge 0$ it holds*

$$\|v - \Pi_h v\|_{L^2(K)} \le Ch_K^{r_*+1}|v|_{H^{r_*+1}(K)}, \tag{3.11a}$$

$$\|v - \Pi_h v\|_{L^2(F)} \le Ch_K^{r_*+1/2}|v|_{H^{r_*+1}(K)}, \tag{3.11b}$$

with constants $C > 0$ that are independent of h_K. Here, $r_* = \min\{s,k\}$ with k denoting the polynomial degree of the approximation space (1.4a).

The following lemma shows an important relation between the Maxwell operators (2.4) and their discrete counterparts (3.5).

Lemma 3.10. (1) *The operators $\mathfrak{C}_{\mathrm{H}}, \mathfrak{C}_{\mathrm{E}}$ are consistent, i.e., for $\boldsymbol{u} = (\boldsymbol{H}, \boldsymbol{E}) \in V_*$ it holds*

$$\Pi_h\, \mathcal{C}_{\mathrm{H}}\, \boldsymbol{H} = \mathfrak{C}_{\mathrm{H}}\, \boldsymbol{H},$$
$$\Pi_h\, \mathcal{C}_{\mathrm{E}}\, \boldsymbol{E} = \mathfrak{C}_{\mathrm{E}}\, \boldsymbol{E}.$$

(2) *The operator* $\widehat{\mathfrak{C}}_{\mathrm{H}}$ *is non-consistent, i.e., for* $\tilde{\boldsymbol{H}} \in \widehat{V}_*^{\boldsymbol{H}}$ *it holds*

$$\Pi_h \widehat{\mathcal{C}}_{\mathrm{H}} \tilde{\boldsymbol{H}} = \widehat{\mathfrak{C}}_{\mathrm{H}} \tilde{\boldsymbol{H}} - \mathfrak{L}_{\mathrm{int}}([\![\tilde{\boldsymbol{H}} \times \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}}).$$

The result (1) is stated in [18, Sec. 2.3]. Thus, we only prove (2) involving the new domain special to the inhomogeneous interface problem.

Proof. Let $\boldsymbol{\phi}_h \in V_h$. With integration by parts, we obtain

$$\begin{aligned}(\widehat{\mathcal{C}}_{\mathrm{H}} \tilde{\boldsymbol{H}}, \boldsymbol{\phi}_h)_\varepsilon = &\sum_{K \in \mathcal{T}_h} (\tilde{\boldsymbol{H}}, \operatorname{curl} \boldsymbol{\phi}_h)_K - \sum_{F \in \mathcal{F}_h^\partial} (\tilde{\boldsymbol{H}} \times \boldsymbol{n}_F, \boldsymbol{\phi}_h)_F \\ &+ \sum_{F \in \mathcal{F}_h^\circ} ([\![\tilde{\boldsymbol{H}}]\!]_F \times \boldsymbol{n}_F, \{\!\{\boldsymbol{\phi}_h\}\!\}^{\overline{\mu c}})_F \\ &- \sum_{F \in \mathcal{F}_h^\circ} (\{\!\{\tilde{\boldsymbol{H}}\}\!\}_F^{\mu c}, [\![\boldsymbol{\phi}_h]\!]_F \times \boldsymbol{n}_F)_F.\end{aligned}$$

Thus, with definitions (3.5a) and (3.5b), we see that

$$(\widehat{\mathcal{C}}_{\mathrm{H}} \tilde{\boldsymbol{H}}, \boldsymbol{\phi}_h)_\varepsilon = (\widehat{\mathfrak{C}}_{\mathrm{H}} \tilde{\boldsymbol{H}}, \boldsymbol{\phi}_h)_\varepsilon - (\mathfrak{L}_{\mathrm{int}}\big([\![\tilde{\boldsymbol{H}}]\!]_{F_{\mathrm{int}}} \times \boldsymbol{n}_{\mathrm{int}}\big), \boldsymbol{\phi}_h)_\varepsilon.$$

This proves the statement since $(\widehat{\mathcal{C}}_{\mathrm{H}} \tilde{\boldsymbol{H}}, \boldsymbol{\phi}_h)_\varepsilon = (\Pi_h \widehat{\mathcal{C}}_{\mathrm{H}} \tilde{\boldsymbol{H}}, \boldsymbol{\phi}_h)_\varepsilon$ by definition of the projection (3.7). □

Lemma 3.11. *Let* $\boldsymbol{u} \in \widehat{V}_{*,h} \cap H^{1+s}(\mathcal{T}_h)^6$ *for* $s \geq 0$. *It holds*

$$\|\widehat{\mathfrak{C}}(\boldsymbol{u} - \Pi_h \boldsymbol{u})\|_{\mu \times \varepsilon} \leq C h^{r_*} |\boldsymbol{u}|_{H^{r_*+1}(\mathcal{T}_h)^6}$$

with a constant $C > 0$ *which is independent of* h *and* $\boldsymbol{u}$. *Here,* $r_* = \min\{s, k\}$ *and* k *denotes the polynomial degree of the approximation space* (1.4a).

A proof of this statement is included in [18, eq. (5.5)]. We emphasize that all estimates there hold since they are local to every element $K \in \mathcal{T}_h$ and, thus, do not depend on the domain $D(\widehat{\mathcal{C}})$.

The following Lemma is essential for the wellposedness of the semi-discrete problem. A proof is provided in in [18, Lem. 2.2].

Lemma 3.12. *The operator* $\widehat{\mathfrak{C}}$ *is skew-adjoint on* V_h^2 *with respect to the inner product* $(\cdot,\cdot)_{\mu \times \varepsilon}$, *i.e., for* $\boldsymbol{u}_h, \boldsymbol{v}_h \in V_h^2$ *it holds*

$$(\widehat{\mathfrak{C}}\, \boldsymbol{u}_h, \boldsymbol{v}_h)_{\mu \times \varepsilon} = -(\boldsymbol{u}_h, \widehat{\mathfrak{C}}\, \boldsymbol{v}_h)_{\mu \times \varepsilon}.$$

We infer from the skew-adjointness that $\widehat{\mathfrak{C}}$ is a generator of a unitary C^0-semigroup on V_h^2. Therefore, the semi-discrete problem (3.6) has a unique solution $\boldsymbol{u}_h(t) = \big(\boldsymbol{H}_h(t), \boldsymbol{E}_h(t)\big) \in V_h^2$ given by the variation-of-constants formula

$$\boldsymbol{u}_h(t) = e^{t\,\widehat{\mathfrak{C}}} \boldsymbol{u}_h^0 + \int_0^t e^{(t-s)\,\widehat{\mathfrak{C}}} \big(\boldsymbol{j}_h(s) + \boldsymbol{j}_{\mathrm{surf},h}(s)\big) \mathrm{d}s,$$

with $\boldsymbol{u}_h^0 = \big(\boldsymbol{H}_h^0, \boldsymbol{E}_h^0\big)$, $\boldsymbol{j}_h = (0, -\boldsymbol{J}_h)$ and $\boldsymbol{j}_{\mathrm{surf},h} = (0, -\boldsymbol{J}_{\mathrm{surf},h})$.

The following stability bound holds true for the semi-discrete problem. We emphasize, that is an discrete analogue to (2.11).

Theorem 3.13. *Under Assumption* 3.1 *and the assumptions of Theorem* 2.3, *the numerical solution* $\boldsymbol{u}_h = \big(\boldsymbol{H}_h, \boldsymbol{E}_H\big)$ *of* (3.6) *is stable, i.e., for* $t \in [0,T]$ *it holds*

$$\begin{aligned}\|\boldsymbol{u}_h(t)\|_{\mu \times \varepsilon} \lesssim\ &\|\boldsymbol{u}^0\|_{\mu \times \varepsilon} + \|\boldsymbol{J}_{\mathrm{surf}}(0)\|_{L^2(F_{\mathrm{int}})^3} + \|\boldsymbol{J}_{\mathrm{surf}}(t)\|_{L^2(F_{\mathrm{int}})^3} \\ &+ \int_0^t \|\boldsymbol{J}(s)\|_{L^2(Q)^3}\, \mathrm{d}s + \int_0^t \|\partial_t \boldsymbol{J}_{\mathrm{surf}}(s)\|_{L^2(F_{\mathrm{int}})^3}\, \mathrm{d}s \\ &+ \int_0^t \Big\| \big(J_{\mathrm{surf},2}(s), J_{\mathrm{surf},3}(s)\big) \Big\|_{(-\Delta_3)^{1/4} \times (-\Delta_2)^{1/4}} \mathrm{d}s,\end{aligned}$$

with a constant which is independent of h *and* $\boldsymbol{u}$.

Proof. We proceed similar to the proof of Theorem 2.3 and introduce a shifted semi-discrete solution $\tilde{\boldsymbol{u}}_h(t) = \boldsymbol{u}_h(t) - \Pi_h \boldsymbol{j}_{\boldsymbol{H}}(t)$, where $\boldsymbol{j}_{\boldsymbol{H}}(t) = \big(\boldsymbol{J}_{\boldsymbol{H}}(t), 0\big)$ denotes the extension of Theorem 2.5. Thus, the shifted solution solves

$$\begin{aligned}\partial_t \tilde{\boldsymbol{u}}_h(t) &= \widehat{\mathfrak{C}}\, \boldsymbol{u}_h(t) + \boldsymbol{j}_h(t) + \boldsymbol{j}_{\mathrm{surf},h}(t) - \Pi_h \partial_t \boldsymbol{j}_{\boldsymbol{H}}(t)\\ &= \widehat{\mathfrak{C}}\, \tilde{\boldsymbol{u}}_h(t) + \widehat{\mathfrak{C}}\, \Pi_h \boldsymbol{j}_{\boldsymbol{H}}(t) + \boldsymbol{j}_h(t) + \boldsymbol{j}_{\mathrm{surf},h}(t) - \Pi_h \partial_t \boldsymbol{j}_{\boldsymbol{H}}(t)\end{aligned}$$

By Lemma 3.10 it holds

$$\Pi_h \widehat{\mathcal{C}}\, \boldsymbol{j}_{\boldsymbol{H}}(t) = \widehat{\mathfrak{C}}\, \boldsymbol{j}_{\boldsymbol{H}}(t) - \Big(0, \mathfrak{L}_{\mathrm{int}}\big([\![\boldsymbol{J}_{\boldsymbol{H}}(t) \times \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}}\big)\Big) = \widehat{\mathfrak{C}}\, \boldsymbol{j}_{\boldsymbol{H}}(t) + \boldsymbol{j}_{\mathrm{surf},h}(t).$$

Therefore, we obtain $\partial_t \tilde{\boldsymbol{u}}_h(t) = \widehat{\mathfrak{C}}\, \tilde{\boldsymbol{u}}_h(t) + \tilde{\boldsymbol{r}}_h(t)$, with

$$\tilde{\boldsymbol{r}}_h(t) = -\widehat{\mathfrak{C}}\big(I - \Pi_h\big)\boldsymbol{j}_{\boldsymbol{H}}(t) + \Pi_h\big(\boldsymbol{j}(t) + \widehat{\mathcal{C}}\, \boldsymbol{j}_{\boldsymbol{H}}(t) - \partial_t \boldsymbol{j}_{\boldsymbol{H}}(t)\big).$$

We emphasize that $\tilde{\boldsymbol{r}}_h(t) \in V_h^2$ and thus write the solution by means of the variations-of-constants formula as

$$\tilde{\boldsymbol{u}}_h(t) = e^{t\,\widehat{\mathfrak{C}}}\big(\boldsymbol{u}_h^0 - \Pi_h \boldsymbol{j}_{\boldsymbol{H}}(0)\big) + \int_0^t e^{(t-s)\,\widehat{\mathfrak{C}}}\, \tilde{\boldsymbol{r}}_h(s)\, \mathrm{d}s.$$

Furthermore, since $\widehat{\mathfrak{C}}$ generates a unitary C^0-semigroup on V_h^2, we obtain that

$$\|\tilde{\boldsymbol{u}}_h(t)\|_{\mu\times\varepsilon} \le \|\boldsymbol{u}^0\|_{\mu\times\varepsilon} + \|\boldsymbol{j}_{\boldsymbol{H}}(0)\|_{\mu\times\varepsilon} + \int_0^t \|\tilde{\boldsymbol{r}}_h(s)\|_{\mu\times\varepsilon}\, \mathrm{d}s$$

It remains to bound $\|\tilde{\boldsymbol{r}}_h(s)\|_{\mu\times\varepsilon}$. By Theorem 2.5, it holds $\boldsymbol{J}_{\boldsymbol{H}}(s) \in PH^1(Q)^3$ and thus, by Lemma 3.11 with $s = 0$, we conclude that

$$\|\widehat{\mathfrak{C}}\big(I - \Pi_h\big)\boldsymbol{j}_{\boldsymbol{H}}(s)\|_{\mu\times\varepsilon} \le C|\boldsymbol{J}_{\boldsymbol{H}}(s)|_{H^1(\mathcal{T}_h)^3} = C|\boldsymbol{J}_{\boldsymbol{H}}(s)|_{PH^1(Q)^3}.$$

The right-hand side can be further estimated with Theorem 2.5, and we obtain

$$\|\widehat{\mathfrak{C}}\big(I - \Pi_h\big)\boldsymbol{j}_{\boldsymbol{H}}(s)\|_{\mu\times\varepsilon} \le C\|(J_{\mathrm{surf},2}(s), J_{\mathrm{surf},3}(s))\|_{(-\Delta_3)^{1/4}\times(-\Delta_2)^{1/4}}.$$

The remaining parts of $\tilde{\boldsymbol{r}}_h(s)$ can be bounded analogously by Theorem 2.5. This proves the claim similar to Theorem 2.3. ☐

Error analysis. We proceed by proving the main error bounds of this section.

Proof of Theorem 3.3. We define the error $\boldsymbol{e}(t) = \boldsymbol{u}(t) - \boldsymbol{u}_h(t)$, where $\boldsymbol{u}(t)$ denotes the solution of (2.5) and $\boldsymbol{u}_h(t)$ denotes the semi-discrete solution of (3.6). We split the error into $\boldsymbol{e}(t) = \boldsymbol{e}_\Pi(t) - \boldsymbol{e}_h(t)$ with

$$\boldsymbol{e}_\Pi(t) = \boldsymbol{u}(t) - \Pi_h \boldsymbol{u}(t), \tag{3.12a}$$

$$\boldsymbol{e}_h(t) = \boldsymbol{u}_h(t) - \Pi_h \boldsymbol{u}(t). \tag{3.12b}$$

Thus, $\boldsymbol{e}_\Pi(t)$ denotes the best approximation error and and $\boldsymbol{e}_h(t)$ the dG-error. By (2.5) and Lemma 3.10, it holds

$$\partial_t \Pi_h \boldsymbol{u}(t) = \Pi_h\big(\widehat{\mathcal{C}}\, \boldsymbol{u}(t) + \boldsymbol{j}\big) = \widehat{\mathfrak{C}}\, \boldsymbol{u}(t) + \boldsymbol{j}_h(t) + \boldsymbol{j}_{\mathrm{surf},h}(t). \tag{3.13}$$

Since $\boldsymbol{u}_h(t)$ solves (3.6), i.e.,

$$\partial_t \boldsymbol{u}_h(t) = \widehat{\mathfrak{C}}\, \boldsymbol{u}_h(t) + \boldsymbol{j}_h(t) + \boldsymbol{j}_{\mathrm{surf},h}(t),\ t \in [0,T], \quad \boldsymbol{u}_h(0) = \Pi_h \boldsymbol{u}^0,$$

we see that the dG-error solves the initial value problem

$$\partial_t \boldsymbol{e}_h(t) = \widehat{\mathfrak{C}}\, \boldsymbol{e}_h(t) + \boldsymbol{d}_\pi(t),\ t \in [0,T], \quad \boldsymbol{e}_h(0) = 0, \tag{3.14}$$

with the defect $\boldsymbol{d}_\pi(t) = -\widehat{\mathfrak{C}}\, \boldsymbol{e}_\Pi(t)$. We can write the solution of (3.14) with the variation-of-constants formula and obtain

$$\boldsymbol{e}_h(t) = \int_0^t e^{(t-s)\,\widehat{\mathfrak{C}}}\, \boldsymbol{d}_\pi(s)\, \mathrm{d}s.$$

Since $\widehat{\mathfrak{C}}$ is the generator of a unitary C^0-semigroup on V_h^2, we conclude with Lemma 3.11 that

$$\|\boldsymbol{e}_h(t)\|_{\mu\times\varepsilon} \leq \int_0^t \|\boldsymbol{d}_\pi(s)\|_{\mu\times\varepsilon}\, \mathrm{d}s \leq Ch^{r_*}\int_0^t |\boldsymbol{u}(s)|_{H^{r_*+1}(\mathcal{T}_h)^6}\, \mathrm{d}s$$

with a constant independent of h and $\boldsymbol{u}$. Together with the approximation properties of Lemma 3.9, we obtain

$$\begin{aligned}\|\boldsymbol{e}(t)\|_{\mu\times\varepsilon} &\leq \|\boldsymbol{e}_\Pi(t)\|_{\mu\times\varepsilon} + \|\boldsymbol{e}_h(t)\|_{\mu\times\varepsilon}\\ &\leq \widetilde{C}h^{r_*+1}|\boldsymbol{u}(t)|_{H^{r_*+1}(\mathcal{T}_h)^6} + Ch_*^r\int_0^t |\boldsymbol{u}(s)|_{H^{r_*+1}(\mathcal{T}_h)^6}\, \mathrm{d}s,\end{aligned}$$

which proves the claim. □

Remark 3.14. Note, that the stability result of Theorem 3.13 is not used in the proof of Theorem 3.3. The reason for that is the fact that the lifted surface current appears in (3.13) due to Lemma 3.10 (2). Thus, there is no contribution of the surface current in the defect. This, on the other hand, assumes that the lifted surface current can be calculated exactly which is not feasible in practice, cf. Section 3.

The following section deals with errors introduced due to nodal interpolation on the interface.

Interpolation error. The following local estimates for the nodal interpolation hold, compare for example [12, Thm. 11.13].

Lemma 3.15. *For all $K \in \mathcal{T}_h$ and all $v \in H^{1+s}(K)$ with $s > 1/2$ it holds*

$$\|v - \mathcal{I}^h v\|_{L^2(K)} \leq Ch_K^{r_*+1}|v|_{H^{r_*+1}(K)} \tag{3.15}$$

with a constant $C > 0$ which is independent of h_K. Here, $r_* = \min\{s,k\}$ and k denotes the polynomial degree of the approximation space (1.4a).

In order to obtain approximation properties for the local interpolation operator $\mathfrak{I}^h$ on the sub-mesh, we need to ensure that $\mathcal{F}_h^{\text{int}}$ does not degenerate, i.e., that the sub-mesh is again shape regular. Recall the following notation. For $F \in \mathcal{F}_h^{\text{int}}$, we denote with h_F the largest diameter of F and with ρ_F the diameter of the largest inscribing ball of F. It is clear from the definition that $h_F \leq h_K$. Furthermore, [20, Thm. 10, (10)] shows that $\rho_K \leq \rho_F$, i.e., the diameter of the largest inscribing ball of K is always less or equal to the diameter of the largest inscribing ball of F. Therefore,

$$h_K \leq \sigma\rho_K \implies h_F \leq \sigma\rho_F,$$

i.e., the sub-mesh $\mathcal{F}_h^{\text{int}}$ inherits the shape regularity from $\mathcal{T}_h$. We interfere again from [12, Thm. 11.13] the following approximation properties.

Lemma 3.16. *For all $F \in \mathcal{F}_h^{\text{int}}$ and all $w \in H^{1+s}(F)$ with $s > 0$ it holds*

$$\|w - \mathfrak{I}^h w\|_{L^2(F)} \leq Ch_F^{r_*+1}|w|_{H^{r_*+1}(F)} \tag{3.16}$$

with a constant $C > 0$ which is independent of h_F. Here, $r_* = \min\{s,k\}$ and k denotes the polynomial degree of the approximation space (1.4a).

Similar to Lemma 3.11, we obtain an approximation result under the discrete lift operator.

Lemma 3.17. *Let $\boldsymbol{V} \in H^{1+s}(\mathcal{F}_h^{\text{int}})^3$ with $s > 0$. Under Assumption 3.1, it holds*

$$\|\mathfrak{L}_{\text{int}}\big(\boldsymbol{V} - \mathfrak{I}^h\boldsymbol{V}\big)\|_\varepsilon \leq Ch^{r_*+1/2}|\boldsymbol{V}|_{H^{r_*+1}(\mathcal{F}_h^{\text{int}})^3}$$

with a constant $C > 0$ which is independent of h and $\boldsymbol{V}$. Here, $r_* = \min\{s,k\}$ and the polynomial degree of approximation space (1.4a) *is denoted with k.*

Proof. Let $\boldsymbol{\phi}_h \in V_h$. By the defintion of the discrete lift operator (3.5a) and the Cauchy-Schwarz inequality it holds

$$|(\mathfrak{L}_{\mathrm{int}}(\boldsymbol{V} - \mathfrak{I}^h \boldsymbol{V}), \boldsymbol{\phi}_h)_\varepsilon| \leq \Big(\sum_{F \in \mathcal{F}_h^{\mathrm{int}}} \omega_F^{-1} \| \boldsymbol{V} - \mathfrak{I}^h \boldsymbol{V} \|_{L^2(F)}^2 \Big)^{1/2} \cdot \Big(\sum_{F \in \mathcal{F}_h^{\mathrm{int}}} \omega_F \| \{\!\{ \boldsymbol{\phi}_h \}\!\}^{\varepsilon c} \|_{L^2(F)}^2 \Big)^{1/2} \tag{3.17}$$

with the weight $\omega_F = \min\{h_{K_{F,l}}, h_{K_{F,r}}\}$. By Lemma 3.16, we obtain the estimate

$$\omega_F^{-1} \| \boldsymbol{V} - \mathfrak{I}^h \boldsymbol{V} \|_{L^2(F)}^2 \leq C^2 \omega_F^{-1} h_F^{2r_*+2} |\boldsymbol{V}|_{H^{r_*+1}(F)}^2 \tag{3.18}$$

Since by definition $h_F \leq \max\{h_{K_{F,l}}, h_{K_{F,r}}\}$, we obtain with the shape regularity that

$$h_F \leq \max\{h_{K_{F,l}}, h_{K_{F,r}}\} \leq \sigma \rho_F \leq \sigma \min\{h_{K_{F,l}}, h_{K_{F,r}}\} \leq \sigma \omega_F. \tag{3.19}$$

Therefore, we lose one h_F in (3.18) due to the weight ω_F and end up with the estimate

$$\omega_F^{-1} \| \boldsymbol{V} - \mathfrak{I}^h \boldsymbol{V} \|_{L^2(F)}^2 \leq C^2 \sigma h_F^{2r_*+1} |\boldsymbol{V}|_{H^{r_*+1}(F)}^2$$

With the discrete trace inequality [12, Lem. 12.8], we further estimate

$$\| \{\!\{ \boldsymbol{\phi}_h \}\!\}^{\varepsilon c} \|_{L^2(F)}^2 \leq 2C^2 c_\infty^2 \mu_\infty \big(h_{K_{F,l}}^{-1} \| \boldsymbol{\phi}_h|_{K_{F,l}} \|_{\varepsilon, K_{F,l}}^2 + h_{K_{F,r}}^{-1} \| \boldsymbol{\phi}_h|_{K_{F,r}} \|_{\varepsilon, K_{F,r}}^2 \big) \tag{3.20}$$

with a constant $C > 0$ which is independent of $F, K_{F,l}$ and $K_{F,r}$, but depends on the polynomial degree k.

Multiplication of (3.20) with ω_F proves the statement together with (3.18) and (3.17). ☐

With Lemma 3.17, we have all ingredients to prove the second main result of this section.

Proof of Theorem 3.7. Since $\boldsymbol{H} \in C^0([0,T], \widehat{V}_*^{\boldsymbol{H}} \cap H^{1+s}(\mathcal{T}_h)^3)$ with $s > 1/2$, we conclude by [12, Thm. 3.10] that

$$\boldsymbol{J}_{\mathrm{surf}} = [\![\boldsymbol{H} \times \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}} \in C^0([0,T], H^{1+\kappa}(\mathcal{F}_h^{\mathrm{int}})^3) \tag{3.21}$$

with $\kappa = s - 1/2 > 0$.

We write (3.9) in vector form, i.e., $\check{\boldsymbol{u}}_h(t) = \big(\check{\boldsymbol{H}}_h(t), \check{\boldsymbol{E}}_h(t)\big)$ such that

$$\partial_t \check{\boldsymbol{u}}_h(t) = \widehat{\mathfrak{C}}\, \check{\boldsymbol{u}}_h(t) + \boldsymbol{j}_h(t) + \check{\boldsymbol{j}}_{\mathrm{surf},h}(t), \text{ for } t \in [0,T], \quad \check{\boldsymbol{u}}_h(0) = \boldsymbol{u}_h^0, \tag{3.22}$$

with $\boldsymbol{j}_h = \big(0, -\boldsymbol{J}_h\big)$, $\check{\boldsymbol{j}}_{\mathrm{surf},h} = \big(0, -\check{\boldsymbol{J}}_{\mathrm{surf},h}\big)$ and $\check{\boldsymbol{u}}_h^0 = \big(\check{\boldsymbol{H}}_h^0, \check{\boldsymbol{E}}_h^0\big)$.

Writing $\check{\boldsymbol{e}}_h(t) = \boldsymbol{u}_h(t) - \check{\boldsymbol{u}}_h(t)$ and subtracting (3.6) and (3.22), we obtain

$$\partial_t \check{\boldsymbol{e}}_h(t) = \widehat{\mathfrak{C}}\, \check{\boldsymbol{e}}_h(t) + \check{\boldsymbol{d}}_h(t), \text{ for } t \in [0,T], \quad \check{\boldsymbol{e}}_h(0) = 0, \tag{3.23}$$

with a defect $\check{\boldsymbol{d}}_h(t) = \boldsymbol{j}_{\mathrm{surf},h}(t) - \check{\boldsymbol{j}}_{\mathrm{surf},h}(t)$.

We can write the solution of (3.23) with the variations-of-constants formula and obtain with Lemma 3.17 the estimate

$$\| \check{\boldsymbol{e}}_h(t) \|_{\mu \times \varepsilon} \leq C h^{\min\{\kappa, k\} + 1/2} \int_0^t |\boldsymbol{J}_{\mathrm{surf}}(s)|_{H^{1+\kappa}(\mathcal{F}_h^{\mathrm{int}})^3} \, \mathrm{d}s.$$

This proves the claim since $\kappa + 1/2 = s$. ☐

4. Full discretization

In time, we discretize (3.6) with the explicit leapfrog scheme with step size $\tau > 0$ and set $t_n = n\tau$ for $n \in \mathbb{N}$. The fully discrete scheme reads

$$\boldsymbol{H}_h^{n+1/2} - \boldsymbol{H}_h^n = -\frac{\tau}{2}\,\mathfrak{C}_{\mathrm{E}}\,\boldsymbol{E}_h^n, \tag{4.1a}$$

$$\boldsymbol{E}_h^{n+1} - \boldsymbol{E}_h^n = \tau\,\widehat{\mathfrak{C}}_{\mathrm{H}}\,\boldsymbol{H}_h^{n+1/2} - \frac{\tau}{2}(\boldsymbol{J}_h^n + \boldsymbol{J}_h^{n+1}) - \frac{\tau}{2}(\boldsymbol{J}_{\mathrm{surf},h}^n + \boldsymbol{J}_{\mathrm{surf},h}^{n+1}), \tag{4.1b}$$

$$\boldsymbol{H}_h^{n+1} - \boldsymbol{H}_h^{n+1/2} = -\frac{\tau}{2}\,\mathfrak{C}_{\mathrm{E}}\,\boldsymbol{E}_h^{n+1}, \tag{4.1c}$$

for $n \geq 0$ and $\boldsymbol{H}_h^0 = \Pi_h \boldsymbol{H}^0$, $\boldsymbol{E}_h^0 = \Pi_h \boldsymbol{E}^0$. It is well-known that the leapfrog scheme is stable if for some $\theta \in (0,1)$, the CFL condition

$$\tau < \tau_{\mathrm{CFL}} = \frac{2\theta}{\|\widehat{\mathfrak{C}}_{\mathrm{H}}\,\mathfrak{C}_{\mathrm{E}}\|_\varepsilon} \lesssim \theta \min_{K\in\mathcal{T}_h} h_K \tag{4.2}$$

is satisfied. Here, $\|\cdot\|_\varepsilon$ denotes the induced operator norm, cf. (2.3). For more details on the constant within the CFL condition, we refer to [18, eq. (2.35)].

The main result of this section is the following bound on the full discretization error.

Theorem 4.1. *Let Assumption 3.1 hold and further let the solution $\boldsymbol{u} = (\boldsymbol{H}, \boldsymbol{E})$ of (2.5) satisfy*

$$\boldsymbol{u} \in C^0([0,T], \widehat{V}_* \cap H^{1+s}(\mathcal{T}_h)^6) \cap C^3([0,T], L^2(Q)^6), \tag{4.3}$$

with $s \geq 0$ and assume that the CFL condition (4.2) holds. Then, the approximations $\boldsymbol{u}_h^n = (\boldsymbol{H}_h^n, \boldsymbol{E}_h^n)$ defined in (4.1) with approximation space (1.4) satisfies

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n\|_{\mu\times\varepsilon} \leq C(h^{r_*} + \tau^2), \qquad 0 \leq t_n \leq T.$$

Here, $r_* = \min\{s, k\}$ and $C > 0$ is a constant which is independent of h and τ.

Remark 4.2. Note, that the interface condition does not induce an additional step size restriction, since the CFL condition (4.2) coincides with that for problems on the full domain Q.

Remark 4.3. We note that in order to prove the regularity assumptions on $\boldsymbol{u}$ in Theorem 4.1 certain compatibility conditions have to be satisfied at the initial time. Assuming that the solution is sufficiently smooth, we obtain from (1.3b)

$$\begin{aligned}
\partial_t \boldsymbol{J}_{\mathrm{surf}} &= [\![\partial_t \boldsymbol{H} \times \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}} = -[\![\mu^{-1}\operatorname{curl}\boldsymbol{E} \times \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}},\\
\partial_t^2 \boldsymbol{J}_{\mathrm{surf}} &= -[\![\mu^{-1}\operatorname{curl}\partial_t\boldsymbol{E} \times \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}} = -[\![\mu^{-1}\operatorname{curl}\varepsilon^{-1}\operatorname{curl}\boldsymbol{H} \times \boldsymbol{n}_{\mathrm{int}}]\!]_{F_{\mathrm{int}}}.
\end{aligned}$$

The reader should refer to [10, Thm. 2.4-2.6] for a thorough treatment.

Our analysis is inspired by [18], where the locally implicit method for linear Maxwell equations is considered. With the discrete Maxwell operators from (3.5) and $\boldsymbol{u}_h^n = (\boldsymbol{H}_h^n, \boldsymbol{E}_h^n)$, we write (4.1) in the following one-step formulation

$$\widehat{\mathfrak{R}}_- \boldsymbol{u}_h^{n+1} = \widehat{\mathfrak{R}}_+ \boldsymbol{u}_h^n + \frac{\tau}{2}(\boldsymbol{j}_h^{n+1} + \boldsymbol{j}_h^n) + \frac{\tau}{2}(\boldsymbol{j}_{\mathrm{surf},h}^{n+1} + \boldsymbol{j}_{\mathrm{surf},h}^n) \tag{4.4a}$$

for $n \geq 0$ with operators

$$\widehat{\mathfrak{R}}_\pm : V_h^2 \to V_h^2, \quad \widehat{\mathfrak{R}}_\pm = I \pm \frac{\tau}{2}\,\widehat{\mathfrak{C}} - \frac{\tau^2}{4}\,\widehat{\mathfrak{D}} \tag{4.4b}$$

and perturbation operator

$$\widehat{\mathfrak{D}} : V_h^2 \to V_h^2, \quad \widehat{\mathfrak{D}} = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{\mathfrak{C}}_{\mathrm{H}}\,\mathfrak{C}_{\mathrm{E}} \end{pmatrix}. \tag{4.4c}$$

For $\widehat{\mathfrak{D}} = 0$, the scheme (4.4) is equivalent to the Crank–Nicolson method and thus, one can interpret the leapfrog scheme as perturbation of it. We further use the operator $\widehat{\mathfrak{R}} : V_h^2 \to V_h^2$, defined as $\widehat{\mathfrak{R}} = \widehat{\mathfrak{R}}_-^{-1} \widehat{\mathfrak{R}}_+$.

Note, that for the special choice of $\mathcal{C}^i_{\boldsymbol{H}} = 0$ and $\mathcal{C}^i_{\boldsymbol{E}} = 0$ in [18, eq. (2.34)], one obtains the leapfrog scheme on the whole spatial domain. Thus, we can use bounds on the operators in (4.4) from that work.

Stability. The following theorem provides stability for the fully discrete scheme and is a discrete analogue of the bound provided in Theorem 2.3 for the exact solution.

Theorem 4.4. *Assume that the CFL condition (4.2), Assumption 3.1, and the assumptions of Theorem 2.3 are satisfied. Then, the fully discrete scheme (4.1) is stable, i.e., for all $n \geq 0$ it holds*

$$\begin{aligned}
\|\boldsymbol{u}_h^n\|_{\mu\times\varepsilon} \lesssim \|\boldsymbol{u}^0\|_{\mu\times\varepsilon} &+ \|\boldsymbol{J}_{\mathrm{surf}}(t_0)\|_{L^2(F_{\mathrm{int}})^3} + \|\boldsymbol{J}_{\mathrm{surf}}(t_n)\|_{L^2(F_{\mathrm{int}})^3} \\
&+ \frac{\tau}{2}\sum_{\ell=0}^{n-1} \|\boldsymbol{J}(t_{\ell+1}) + \boldsymbol{J}(t_\ell)\|_{L^2(Q)^3} \\
&+ \int_{t_0}^{t_n} \|\partial_t \boldsymbol{J}_{\mathrm{surf}}(s)\|_{L^2(F_{\mathrm{int}})^3}\,\mathrm{d}s \\
&+ \frac{\tau}{2}\sum_{\ell=0}^{n-1} \Big\| \big(J_{\mathrm{surf},2}(t_{\ell+1}) + J_{\mathrm{surf},2}(t_\ell), \\
&\qquad\qquad J_{\mathrm{surf},3}(t_{\ell+1}) + J_{\mathrm{surf},3}(t_\ell)\big)\Big\|_{(-\Delta_3)^{1/4}\times(-\Delta_2)^{1/4}},
\end{aligned}$$

with a constant which is independent of h, τ and $\boldsymbol{u}$.

Proof. The proof relies on the same arguments as in Theorems 2.3 and 3.13. Hence, we introduce the shifted field

$$\check{\boldsymbol{u}}_h^n = \boldsymbol{u}_h^n - \Pi_h \boldsymbol{j}_{\boldsymbol{H}}^n, \tag{4.5}$$

where $\boldsymbol{j}_{\boldsymbol{H}}^n = \big(\boldsymbol{J}_{\boldsymbol{H}}^n, 0\big)$ denotes the extension from Theorem 2.5. The shifted variables satisfy the recursion

$$\begin{aligned}
\widehat{\mathfrak{R}}_- \check{\boldsymbol{u}}_h^{n+1} = \widehat{\mathfrak{R}}_+ \check{\boldsymbol{u}}_h^n &+ \frac{\tau}{2}\big(\boldsymbol{j}_h^{n+1} + \boldsymbol{j}_h^n\big) + \frac{\tau}{2}\big(\boldsymbol{j}_{\mathrm{surf},h}^{n+1} + \boldsymbol{j}_{\mathrm{surf},h}^n\big) \\
&+ \widehat{\mathfrak{R}}_+ \Pi_h \boldsymbol{j}_{\boldsymbol{H}}^n - \widehat{\mathfrak{R}}_- \Pi_h \boldsymbol{j}_{\boldsymbol{H}}^{n+1}.
\end{aligned}\tag{4.6}$$

Next, we study the action of $\widehat{\mathfrak{R}}_\pm$ on $\Pi_h \boldsymbol{j}_{\boldsymbol{H}}^\ell$ for $\ell \geq 0$. Since the second component of $\boldsymbol{j}_{\boldsymbol{H}}^\ell$ is zero, it holds $\widehat{\mathfrak{D}}\,\Pi_h \boldsymbol{j}_{\boldsymbol{H}}^\ell = 0$. Thus, (4.4b) yields

$$\begin{aligned}
\widehat{\mathfrak{R}}_\pm \Pi_h \boldsymbol{j}_{\boldsymbol{H}}^\ell &= \Pi_h \boldsymbol{j}_{\boldsymbol{H}}^\ell \pm \frac{\tau}{2}\,\widehat{\mathfrak{C}}\,\Pi_h \boldsymbol{j}_{\boldsymbol{H}}^\ell \\
&= \Pi_h \boldsymbol{j}_{\boldsymbol{H}}^\ell \pm \frac{\tau}{2}\,\widehat{\mathfrak{C}}(\Pi_h - I)\boldsymbol{j}_{\boldsymbol{H}}^\ell \pm \frac{\tau}{2}\Pi_h\,\widehat{\mathcal{C}}\,\boldsymbol{j}_{\boldsymbol{H}}^\ell \mp \frac{\tau}{2}\boldsymbol{j}_{\mathrm{surf},h}^\ell.
\end{aligned}$$

Here, the second identity follows from Theorem 2.5 and Lemma 3.10. Inserting this into (4.6) leads to

$$\widehat{\mathfrak{R}}_- \check{\boldsymbol{u}}_h^{n+1} = \widehat{\mathfrak{R}}_+ \check{\boldsymbol{u}}_h^n + \check{\boldsymbol{r}}_h^n \tag{4.7}$$

with the remaining terms

$$\begin{aligned}
\check{\boldsymbol{r}}_h^n = \frac{\tau}{2}&\big(\boldsymbol{j}_h^{n+1} + \boldsymbol{j}_h^n\big) \\
&- \Pi_h\big(\boldsymbol{j}_{\boldsymbol{H}}^{n+1} - \boldsymbol{j}_{\boldsymbol{H}}^n\big) \\
&+ \frac{\tau}{2}\Pi_h\,\widehat{\mathcal{C}}\big(\boldsymbol{j}_{\boldsymbol{H}}^{n+1} + \boldsymbol{j}_{\boldsymbol{H}}^n\big) + \frac{\tau}{2}\,\widehat{\mathfrak{C}}\big(\Pi_h - I\big)\big(\boldsymbol{j}_{\boldsymbol{H}}^{n+1} + \boldsymbol{j}_{\boldsymbol{H}}^n\big).
\end{aligned}$$

Solving the recursion (4.7), we obtain

$$\check{\boldsymbol{u}}_h^n = \widehat{\mathfrak{R}}^n \check{\boldsymbol{u}}_h^0 + \sum_{\ell=0}^{n-1} \widehat{\mathfrak{R}}^{n-1-\ell} \widehat{\mathfrak{R}}_-^{-1} \check{\boldsymbol{r}}_h^\ell.$$

By [9, Lem. 11.14] we have

$$\|\widehat{\mathfrak{R}}_-^{-1}\|_{\mu\times\varepsilon} \le \sqrt{1+\theta^2+\theta^4} \tag{4.8a}$$

and by [18, Lem. 4.2] it holds

$$\|\widehat{\mathfrak{R}}^m\|_{\mu\times\varepsilon} \le C_{\text{stb}} = (1-\theta^2)^{-1/2}, \quad m = 0, 1, \dots. \tag{4.8b}$$

Together with (4.5) we infer

$$\|\boldsymbol{u}_h^n\|_{\mu\times\varepsilon} \le C_{\text{stb}}\Big(\|\boldsymbol{u}_h^0\|_{\mu\times\varepsilon} + \|\Pi_h \boldsymbol{j}_{\boldsymbol{H}}^0\|_{\mu\times\varepsilon} + \sqrt{3}\sum_{\ell=0}^{n-1}\|\check{\boldsymbol{r}}_h^\ell\|_{\mu\times\varepsilon}\Big) + \|\Pi_h \boldsymbol{j}_{\boldsymbol{H}}^n\|_{\mu\times\varepsilon}.$$

The remainders $\|\check{\boldsymbol{r}}_h^\ell\|_{\mu\times\varepsilon}$ are bounded in the same way as in the proof of Theorem 3.13. □

Error analysis. As usual, we split the full discretization error into

$$\boldsymbol{u}^n - \boldsymbol{u}_h^n = \boldsymbol{u}^n - \Pi_h \boldsymbol{u}^n + \Pi_h \boldsymbol{u}^n - \boldsymbol{u}_h^n = \boldsymbol{e}_\Pi^n + \boldsymbol{e}_h^n. \tag{4.9}$$

Here, $\boldsymbol{e}_\Pi^n = \boldsymbol{u}^n - \Pi_h \boldsymbol{u}^n$ is the best approximation error and $\boldsymbol{e}_h^n = \Pi_h \boldsymbol{u}^n - \boldsymbol{u}_h^n$ is the dG-leapfrog-error at time t_n. Since the best approximation error is covered by projection results, cf. Lemma 3.9, we determine the defect $\boldsymbol{d}^n$ by inserting the projected exact solution into the scheme (4.4a), i.e.,

$$\widehat{\mathfrak{R}}_- \Pi_h \boldsymbol{u}^{n+1} = \widehat{\mathfrak{R}}_+ \Pi_h \boldsymbol{u}^n + \frac{\tau}{2}\big(\boldsymbol{j}_h^{n+1} + \boldsymbol{j}_h^n\big) + \frac{\tau}{2}\big(\boldsymbol{j}_{\text{surf},h}^{n+1} + \boldsymbol{j}_{\text{surf},h}^n\big) - \boldsymbol{d}^n. \tag{4.10}$$

This allows us to infer the error recursion for the dG-leapfrog scheme.

Lemma 4.5. *Let the assumptions of Theorem 4.1 be satisfied. Then, the dG-leapfrog-error $\boldsymbol{e}_h^n$ defined in (4.9) satisfies the error recursion*

$$\widehat{\mathfrak{R}}_- \boldsymbol{e}_h^{n+1} = \widehat{\mathfrak{R}}_+ \boldsymbol{e}_h^n + \boldsymbol{d}^n, \qquad \boldsymbol{d}^n = \boldsymbol{d}_\Pi^n + \boldsymbol{\delta}^n + \big(\widehat{\mathfrak{R}}_- - \widehat{\mathfrak{R}}_+\big)\boldsymbol{d}_h^n \tag{4.11a}$$

with

$$\boldsymbol{d}_\Pi^n = -\frac{\tau}{2}\,\widehat{\mathfrak{C}}\big(I-\Pi_h\big)\big(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n\big) - \frac{\tau^2}{4}\,\widehat{\mathfrak{D}}\big(I-\Pi_h\big)\big(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\big), \tag{4.11b}$$

$$\boldsymbol{\delta}^n = \tau^2 \Pi_h \int_{t_n}^{t_{n+1}} \frac{(s-t_n)(t_{n+1}-s)}{2\tau^2}\partial_t^3 \boldsymbol{u}(s)\,\mathrm{d}s, \tag{4.11c}$$

$$\boldsymbol{d}_h^n = -\frac{\tau}{4}\begin{pmatrix}\Pi_h\big(\partial_t \boldsymbol{H}^{n+1} - \partial_t \boldsymbol{H}^n\big)\\ 0\end{pmatrix}. \tag{4.11d}$$

Proof. With the fundamental theorem of calculus and the error estimate of the trapazoidal rule, we obtain

$$\begin{aligned}\boldsymbol{u}^{n+1} - \boldsymbol{u}^n = {}& \frac{\tau}{2}\widehat{\mathcal{C}}\big(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n\big) + \frac{\tau}{2}\big(\boldsymbol{j}^{n+1}+\boldsymbol{j}^n\big)\\ &- \tau^2\int_{t_n}^{t_{n+1}} \frac{(s-t_n)(t_{n+1}-s)}{2\tau^2}\partial_t^3 \boldsymbol{u}(s)\,\mathrm{d}s.\end{aligned}$$

Projecting both sides onto V_h^2 and using Lemma 3.10, we infer that

$$\Pi_h \boldsymbol{u}^{n+1} - \frac{\tau}{2}\,\widehat{\mathfrak{C}}\,\boldsymbol{u}^{n+1} = \Pi_h \boldsymbol{u}^n + \frac{\tau}{2}\,\widehat{\mathfrak{C}}\,\boldsymbol{u}^n + \frac{\tau}{2}\big(\boldsymbol{j}_h^{n+1}+\boldsymbol{j}_h^n\big) + \frac{\tau}{2}\big(\boldsymbol{j}_{\text{surf},h}^{n+1} + \boldsymbol{j}_{\text{surf},h}^n\big) - \boldsymbol{\delta}^n.$$

Writing $\widehat{\mathfrak{C}} = \widehat{\mathfrak{C}}\,\Pi_h + \widehat{\mathfrak{C}}(I-\Pi_h)$ and comparing with (4.10) gives

$$\boldsymbol{d}^n = \boldsymbol{\delta}^n - \frac{\tau}{2}\,\widehat{\mathfrak{C}}\Big(I-\Pi_h\Big)\big(\boldsymbol{u}^{n+1}+\boldsymbol{u}^n\big) + \frac{\tau^2}{4}\,\widehat{\mathfrak{D}}\,\Pi_h\big(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\big).$$

Moreover, as in [18, eq. (5.21)], we can write

$$\begin{aligned}\frac{\tau^2}{4}\,\widehat{\mathfrak{D}}\,\Pi_h\big(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\big) = &-\frac{\tau^2}{4}\,\widehat{\mathfrak{D}}\Big(I-\Pi_h\Big)\big(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\big)\\ &-\frac{\tau}{4}\Big(\widehat{\mathfrak{R}}_- - \widehat{\mathfrak{R}}_+\Big)\begin{pmatrix}\Pi_h\big(\partial_t\boldsymbol{H}^{n+1}-\partial_t\boldsymbol{H}^n\big)\\ 0\end{pmatrix}.\end{aligned}$$

This proves the claim. □

With this representation of the defect, we are able prove the main result of this section.

Proof of Theorem 4.1. The proof proceeds in three steps and makes use of a generic constant C which is independent of h and τ. First, we bound the projection error $\boldsymbol{e}_\Pi^n$. Second, we solve the error recursion for the dG-leapfrog-error and estimate the defects separately. The claim then follows in a third step via the application of the triangle inequality.

Lemma 3.9 directly implies

$$\|\boldsymbol{e}_\Pi^n\|_{\mu\times\varepsilon} = \|\boldsymbol{u}^n - \Pi_h\boldsymbol{u}^n\|_{\mu\times\varepsilon} \le Ch^{r_*+1}|\boldsymbol{u}^n|_{H^{r_*+1}(\mathcal{T}_h)}. \tag{4.12}$$

With Lemma 4.5, the error recursion (4.11a), and the discrete variation-of-constants formula we conclude

$$\boldsymbol{e}_h^n = \sum_{\ell=0}^{n-1}\widehat{\mathfrak{R}}^{n-1-\ell}\widehat{\mathfrak{R}}_-^{-1}\boldsymbol{d}_\Pi^\ell + \sum_{\ell=0}^{n-1}\widehat{\mathfrak{R}}^{n-1-\ell}\widehat{\mathfrak{R}}_-^{-1}\boldsymbol{\delta}^\ell + \sum_{\ell=0}^{n-1}\widehat{\mathfrak{R}}^{n-1-\ell}\big(I-\widehat{\mathfrak{R}}\big)\boldsymbol{d}_h^\ell.$$

To bound the first term on the right-hand side we use Lemma 3.11 to see

$$\|\boldsymbol{d}_\Pi^n\|_{\mu\times\varepsilon} \le Ch^{r_*}\frac{\tau}{2}\Big(|\boldsymbol{u}^{n+1}+\boldsymbol{u}^n|_{H^{r_*+1}(\mathcal{T}_h)^6} + |\boldsymbol{E}^{n+1}-\boldsymbol{E}^n|_{H^{r_*+1}(\mathcal{T}_h)^3}\Big).$$

Utilizing (4.8) shows

$$\sum_{\ell=0}^{n-1}\|\widehat{\mathfrak{R}}^{n-1-\ell}\widehat{\mathfrak{R}}_-^{-1}\boldsymbol{d}_\Pi^\ell\|_{\mu\times\varepsilon} \le C_{\mathrm{stb}}\sqrt{3}\sum_{\ell=0}^{n-1}\frac{\tau}{2}\|\boldsymbol{d}_\Pi^\ell\|_{\mu\times\varepsilon} \le Ch^{r_*}.$$

For the second term, we obtain

$$\sum_{\ell=0}^{n-1}\|\widehat{\mathfrak{R}}^{n-1-\ell}\widehat{\mathfrak{R}}_-^{-1}\boldsymbol{\delta}^\ell\|_{\mu\times\varepsilon} \le C\tau^2\int_{t_0}^{t_n}\|\partial_t^3\boldsymbol{u}(s)\|_{\mu\times\varepsilon}\,\mathrm{d}s.$$

Hence, it remains to bound the third term. Using summation-by-parts, we infer

$$\sum_{\ell=0}^{n-1}\widehat{\mathfrak{R}}^{n-1-\ell}\big(I-\widehat{\mathfrak{R}}\big)\boldsymbol{d}_h^\ell = -\widehat{\mathfrak{R}}^n\boldsymbol{d}_h^0 + \boldsymbol{d}_h^{n-1} + \sum_{\ell=0}^{n-2}\widehat{\mathfrak{R}}^{n-1-\ell}\big(\boldsymbol{d}_h^{\ell+1}-\boldsymbol{d}_h^\ell\big).$$

We estimate all terms separately. Again, using (4.11d) and the fundamental theorem of calculus implies

$$\|\widehat{\mathfrak{R}}^n\boldsymbol{d}_h^0\|_{\mu\times\varepsilon} \le C\tau\int_{t_0}^{t_1}\|\partial_t^2\boldsymbol{H}(s)\|_\mu\,\mathrm{d}s \le C\tau^2\max_{s\in[t_0,t_1]}\|\partial_t^2\boldsymbol{H}(s)\|_\mu$$

and

$$\|\boldsymbol{d}_h^n\|_{\mu\times\varepsilon} \le C\tau^2\max_{s\in[t_{n-1},t_n]}\|\partial_t^2\boldsymbol{H}(s)\|_\mu.$$

For the third sum, the fundamental theorem is used twice in order to exploit the difference of the defects. We obtain

$$\frac{\tau}{4}\Pi_h\big((\partial_t\boldsymbol{H}^{\ell+2} - 2\partial_t\boldsymbol{H}^{\ell-1} + \partial_t\boldsymbol{H}^\ell\big) = \frac{\tau^2}{4}\int_{t_\ell}^{t_{\ell+2}}\Big(1-\frac{|t_{\ell+1}-s|}{\tau}\Big)\Pi_h\partial_t^3\boldsymbol{H}(s)\,\mathrm{d}s.$$

Thus, we end up with the bound

$$\sum_{\ell=0}^{n-2} \|\widehat{\mathfrak{R}}^{n-1-\ell}(\boldsymbol{d}_h^{\ell+1} - \boldsymbol{d}_h^{\ell})\|_{\mu\times\varepsilon} \leq C\tau^2 \int_{t_1}^{t_{n-1}} \|\partial_t^3 \boldsymbol{H}(s)\|_\mu \,\mathrm{d}s.$$

Combining all estimates, we have shown that

$$\|\boldsymbol{e}_h^n\|_{\mu\times\varepsilon} \leq C(h^{r_*} + \tau^2).$$

Together with (4.12) this proves the claim. □

5. Numerical experiments

In this section, we present several numerical experiments that underline our theoretical findings. The software was build with the Maxwell toolbox `TiMaxdG`[1] which is build upon the finite element library `deal.II`[2] [1]. The full software with executables for reproduction purposes can be found under

`https://gitlab.kit.edu/kit/ianm/ag-numerik/projects/b10/dgtd-interface-problem`.

All experiments are conducted in transverse electric (TE) polarization, i.e., $H_1 = H_2 = E_3 = 0$, to reduce the computational effort, with material parameters $\mu_\pm = \varepsilon_\pm = 1$. Thus, we solve the system

$$\partial_t H_{3,\pm} = -\partial_1 E_{2,\pm} + \partial_2 E_{1,\pm}, \quad \text{in } Q_\pm, \tag{5.1a}$$

$$\partial_t E_{1,\pm} = \partial_2 H_{3,\pm} - J_{1,\pm}, \quad \text{in } Q_\pm, \tag{5.1b}$$

$$\partial_t E_{2,\pm} = -\partial_1 H_{3,\pm} - J_{2,\pm}, \quad \text{in } Q_\pm, \tag{5.1c}$$

$$H_3(0) = H_3^0,\ E_1(0) = E_1^0,\ E_2(0) = E_2^0, \quad \text{in } Q, \tag{5.1d}$$

$$[\![H_3]\!]_{F_{\mathrm{int}}} = J_{1,\mathrm{surf}}, \quad \text{on } F_{\mathrm{int}}. \tag{5.1e}$$

Cavity solution. In this experiment, we use the well-known cavity solution, cf. [16, Sec. 6], to construct a regular reference solution of the interface problem (2.5). On each cuboid Q_-, Q_+ we make the ansatz

$$H_{3,\pm}(x_1,x_2,t) = \frac{1}{\omega^\pm}(k_1^\pm A_2 - k_2 A_1^\pm)\cos(k_2 x_2)\cos(k_1^\pm(x_1+1))\sin(\omega^\pm t), \tag{5.2a}$$

$$E_{1,\pm}(x_1,x_2,t) = -A_1^\pm \sin(k_2 x_2)\sin(k_1^\pm(x_1+1))\cos(\omega^\pm t), \tag{5.2b}$$

$$E_{2,\pm}(x_1,x_2,t) = -A_2 \cos(k_2 x_2)\sin(k_1^\pm(x_1+1))\cos(\omega^\pm t), \tag{5.2c}$$

with spatial wave numbers

$$k_1^\pm = \frac{\pi k^\pm}{2}, \quad k_2 = \pi m, \quad k^\pm, m \in \mathbb{N}, \tag{5.2d}$$

temporal wave numbers

$$\omega^\pm = \sqrt{\left(k_1^\pm\right)^2 + k_2^2}, \tag{5.2e}$$

and amplitudes

$$A_1^\pm = -A_2 \frac{k_2}{k_1^\pm}, \quad A_2 \in \mathbb{R}. \tag{5.2f}$$

We choose the data such that

$$J_{\mathrm{surf}}(x_2,t) = \lim_{x_1\to 0^+} H_{3,+}(x_1,x_2,t) - \lim_{x_1\to 0^-} H_{3,-}(x_1,x_2,t), \tag{5.3a}$$

$$J_{1,\pm} = J_{2,\pm} = 0. \tag{5.3b}$$

[1] `https://gitlab.kit.edu/kit/ianm/ag-numerik/projects/dg-maxwell/timaxdg`
[2] `https://www.dealii.org`

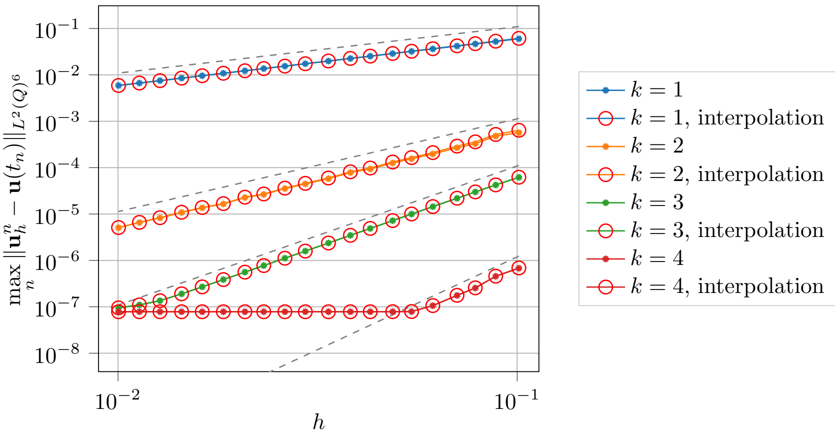

FIGURE 2. Error in the L^2-norm of the cavity solution (5.2) plotted against the mesh size for a fixed time-step width $\tau = 1 \cdot 10^{-4}$. The dotted lines correspond to the lift defined in (3.5a), and the circled lines to the interpolation in (3.9).

The remaining constants are chosen such that $J_{\text{surf}} \neq 0$. In our simulation we used the specific values

$$k^- = 2, \qquad k^+ = 4, \qquad m = 1, \qquad A_2 = 1.$$

We chose a mesh sequence of 20 meshes with mesh sizes in the range of $1 \cdot 10^{-1}$ and $1 \cdot 10^{-2}$, a fixed time-step width $\tau = 1 \cdot 10^{-4}$, and polynomial degrees between one and four. For each mesh size, we calculated two different numerical solutions differing in the treatment of the surface current (5.3a). One series of simulations is done with the lifting defined in (3.5a) and one series is done with the interpolation of the surface current described in (3.9). At several time steps, we calculated the L^2-error against the reference solution (5.2). Figure 2 depicts the different mesh sizes on the x-axis and the maximal L^2-error obtained on the y-axis. We observe for k-th order ansatz polynomials k-th order spatial convergence until a plateau is reached where the error of the time discretization dominates. This agrees with both, Theorem 3.3 and Corollary 3.8 Additionally, Figure 2 shows that the interpolation of the surface current leads to the same spatial error. This is expected since the surface current (5.3a) is smooth.

Low regularity surface current. The aim of this experiment is to show the effect of spatial regularity of surfaces currents on the spatial convergence order. We follow the ideas in [17] and construct for $\alpha \geq 0$ trigonometric polynomials

$$f_\alpha(x) = \sum_{j=-M/2+1}^{M/2} \nu_{\alpha,j} e^{ijx}, \qquad x \in [-\pi, \pi], \qquad M = 2^m, m \in \mathbb{N}, \tag{5.4}$$

with coefficients

$$\begin{aligned} \nu_{\alpha,0} &= \nu_{M/2} = 0, \\ \nu_{\alpha,j} &= \frac{i \cdot r_j}{(1 + j^2)^{\frac{1}{2}(\frac{1}{2}+\alpha)}} && \text{for } j = 1, \dots, \frac{M}{2} - 1, \\ \nu_{\alpha,j} &= -\nu_{\alpha,j+M/2} && \text{for } j = -\frac{M}{2} + 1, \dots, -1. \end{aligned}$$

The factors $(r_j)_{j=1}^{M/2-1}$ are uniformly sampled numbers from the interval $[-1, 1]$. In the limit $M \to \infty$, the sequence of trigonometric polynomials converges to a

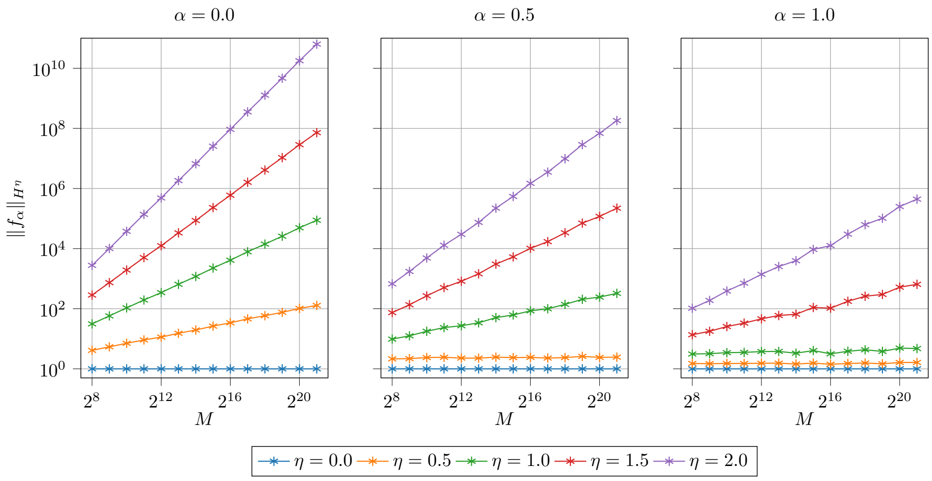

FIGURE 3. Illustration of discrete regularity: the norm $\|f_\alpha\|_{H^\eta}$, for f_α given in (5.4), for different values of α and η plotted against the number M of Fourier modes.

function in the Sobolev space $H^\alpha_{\mathrm{per}}(-\pi,\pi)$ with norm

$$\|g\|_\alpha^2 = 2\pi \sum_{j\in\mathbb{Z}} (1+j^2)^\alpha |\widehat{g}_j|^2, \quad g(x) = \sum_{j\in\mathbb{Z}} \widehat{g}_j e^{ijx}.$$

The coefficients are chosen in such a way that the trace of f_α vanishes on the boundary $\{-\pi,\pi\}$ and, thus, satisfy homogenous Dirichlet boundary conditions. By construction, the norm $\|f_\alpha\|_\eta$ is bounded uniformly in M for $\eta \le \alpha$ and grows otherwise. Figure 3 demonstrates this behavior. The surface current is defined as

$$J_{\mathrm{surf}}(t,x_2) = \frac{1}{\|f_\alpha\|_0} f_\alpha(2\pi x_2 - \pi)\sin(\pi t)^2$$

for $x_2 \in [0,1]$ and $t \in [0,T]$.

In our experiment, we chose $M = 2^m$ for $m = 22$ and a series of regularity coefficients $\alpha \in [0,4]$. For every α, we calculated a reference solution on fine mesh with polynomial degree $k_{\mathrm{ref}} = 3$, mesh size $h_{\mathrm{ref}} = 1\cdot 10^{-3}$, and step size $\tau_{\mathrm{ref}} = 5\cdot 10^{-5}$. We then compared the L^2-error of a sequence of solutions on 8 different meshes with mesh sizes in the range between $1\cdot 10^{-1}$ and $5\cdot 10^{-3}$ at the end time $T=1$ against the reference solution and estimated the order of convergence (EOC). The experiment was performed for first and second order polynomials with the fixed time step size $\tau = 2.5\cdot 10^{-4}$.

Figure 4 shows the dependence of the convergence order on the spatial regularity of the surface current. For $\alpha < 1/2$ little to no convergence is observed. The order then grows linearly for $\alpha \in [0.5, 1.5]$ until it stagnates.

This agrees with Theorem 3.3 provided that one can improve on the results in [10, Sec. 2] to solutions with piecewise regularity $PH^s(Q)$ for $s = \min\{\alpha + 1/2, 2\}$, $\alpha > 1/2$. In particular, looking at the proofs one would expect that the regularity requirements on $\boldsymbol{J}_{\mathrm{surf}}$ can be reduced by one order of Sobolev regularity.

Polynomial solution. In this example, we investigate the temporal errors by constructing a polynomial solution which does not create spatial errors. We construct a solution that is polynomial in space in order to isolate the error introduced by time discretization. The ansatz polynomials in space are given by

$$q_-(x_1) = 2 + x_1, \quad q_+(x_1) = -1 + x_1, \quad r(x_2) = x_2(1-x_2) \tag{5.5a}$$

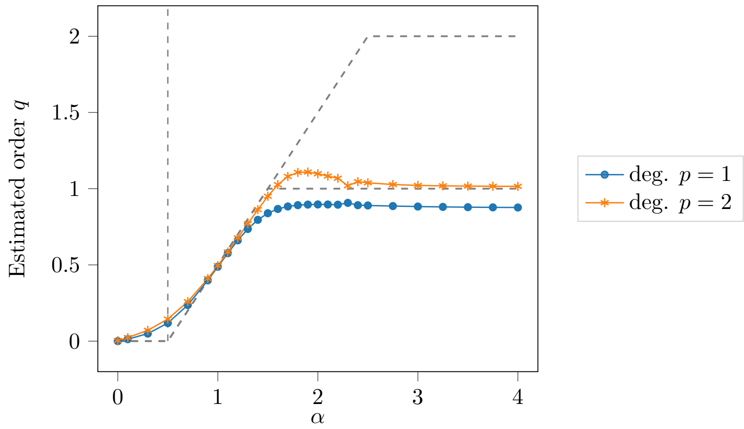

FIGURE 4. Plot of the estimated order of convergence against the discrete regularity parameter α. For each α, we computed approximations using 8 different mesh sizes between $1 \cdot 10^{-1}$ and $5 \cdot 10^{-3}$ with a fixed time step size $\tau = 2.5 \cdot 10^{-4}$. They are then compared to a reference solution computed with $k_{\mathrm{ref}} = 3$, $h_{\mathrm{ref}} = 1 \cdot 10^{-3}$ and $\tau_{\mathrm{ref}} = 5 \cdot 10^{-5}$.

and the ansatz function in time by

$$p(t) = \sin(2\pi t). \tag{5.5b}$$

We define on $Q_\pm$ the fields

$$H_{3,\pm}(x_1, x_2, t) = p(t)q_\pm(x_1)r'(x_2), \tag{5.5c}$$

$$E_{1,\pm}(x_1, x_2, t) = p'(t)q_\pm(x_1)r(x_2), \tag{5.5d}$$

$$E_{2,\pm}(x_1, x_2, t) = 0 \tag{5.5e}$$

and define the surface current again as

$$J_{\mathrm{surf}}(x_2, t) = \lim_{x_1 \to 0^+} H_{3,+}(x_1, x_2, t) - \lim_{x_1 \to 0^-} H_{3,-}(x_1, x_2, t). \tag{5.5f}$$

Additionally, we define the volume current on $Q_\pm$ as

$$J_{1,\pm}(x_1, x_2, t) = p(t)q_\pm(x_1)r''(x_2), \tag{5.5g}$$

$$J_{2,\pm}(x_1, x_2, t) = -p(t)q'_\pm(x_1)r'(x_2). \tag{5.5h}$$

We chose a mesh sequence with 5 different mesh sizes between $5 \cdot 10^{-1}$ and $5 \cdot 10^{-2}$. For every mesh in the sequence, we compared the L^2-error between the reference solution and the numerical scheme at several time steps for a total of 40 different time step sizes in the range between $1 \cdot 10^{-1}$ and $1 \cdot 10^{-4}$. Throughout all calculations, we used a polynomial degree of 3 in order to discretize the reference solution exactly in space. Figure 5 shows on the x-axis the time stepsize τ and the maximal L^2-error on y-axis. The method converges with second order in time if the CFL condition (4.2) is satisfied.

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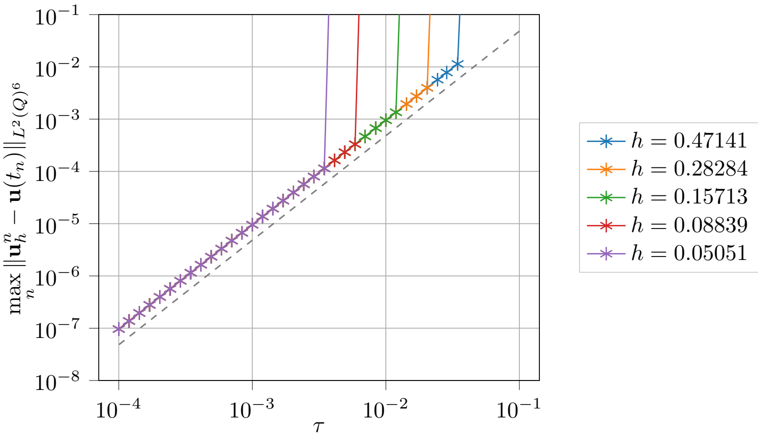

FIGURE 5. Error in the L^2-norm of the polynomial solution (5.5) plotted against the time step size τ for 5 different mesh sizes between $5 \cdot 10^{-1}$ and $5 \cdot 10^{-2}$ using elements of order $k = 3$.

APPENDIX A. EXTENSION

Proof of Lemma 2.4 (1). Let $v \in \mathcal{X}_j^{1/4}$. It holds

$$\begin{aligned}|\Phi_j(v)|^2_{H^1(Q_2)} &= \int_0^1 \|\partial_{x_1}\Phi_j(v)(x_1,\cdot)\|^2_{L^2(S)}\,\mathrm{d}x_1 + \int_0^1 |\Phi_j(v)(x_1,\cdot)|^2_{H^1(S)}\,\mathrm{d}x_1\\ &\le \int_0^1 \|\partial_{x_1}\Phi_j(v)(x_1,\cdot)\|^2_{L^2(S)}\,\mathrm{d}x_1 + C\int_0^1 \|\Phi_j(v)(x_1,\cdot)\|^2_{(-\Delta_j)^{1/2}}\,\mathrm{d}x_1,\end{aligned}$$

where we used that the $H^1(S)$-norm is equivalent to the graph norm of $(-\Delta_j)^{1/2}$, see, e.g., Remark 2.2. For $x_1 > 0$, we obtain

$$\partial_{x_1}\Phi_j(v)(x) = \frac{1}{2}\big(\chi'(x_1) - \chi(x_1)(-\Delta_j)^{1/2}\big)\Big(e^{-x_1(-\Delta_j)^{1/2}}v\Big)(x_2,x_3).$$

Hence, it holds

$$|\Phi_j(v)|^2_{H^1(Q_2)} \le C\int_0^1 \|e^{-x_1(-\Delta_j)^{1/2}}v\|^2_{(-\Delta_j)^{1/2}}\,\mathrm{d}x_1.$$

Proposition 6.2 in [21] implies the estimate

$$\int_0^1 \|(-\Delta_j)^{1/2}e^{-x_1(-\Delta_j)^{1/2}}v\|^2_{L^2(S)}\,\mathrm{d}x_1 \le C\|v\|_{(-\Delta_j)^{1/4}}.$$

We therefore concluded that (1) holds. □

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