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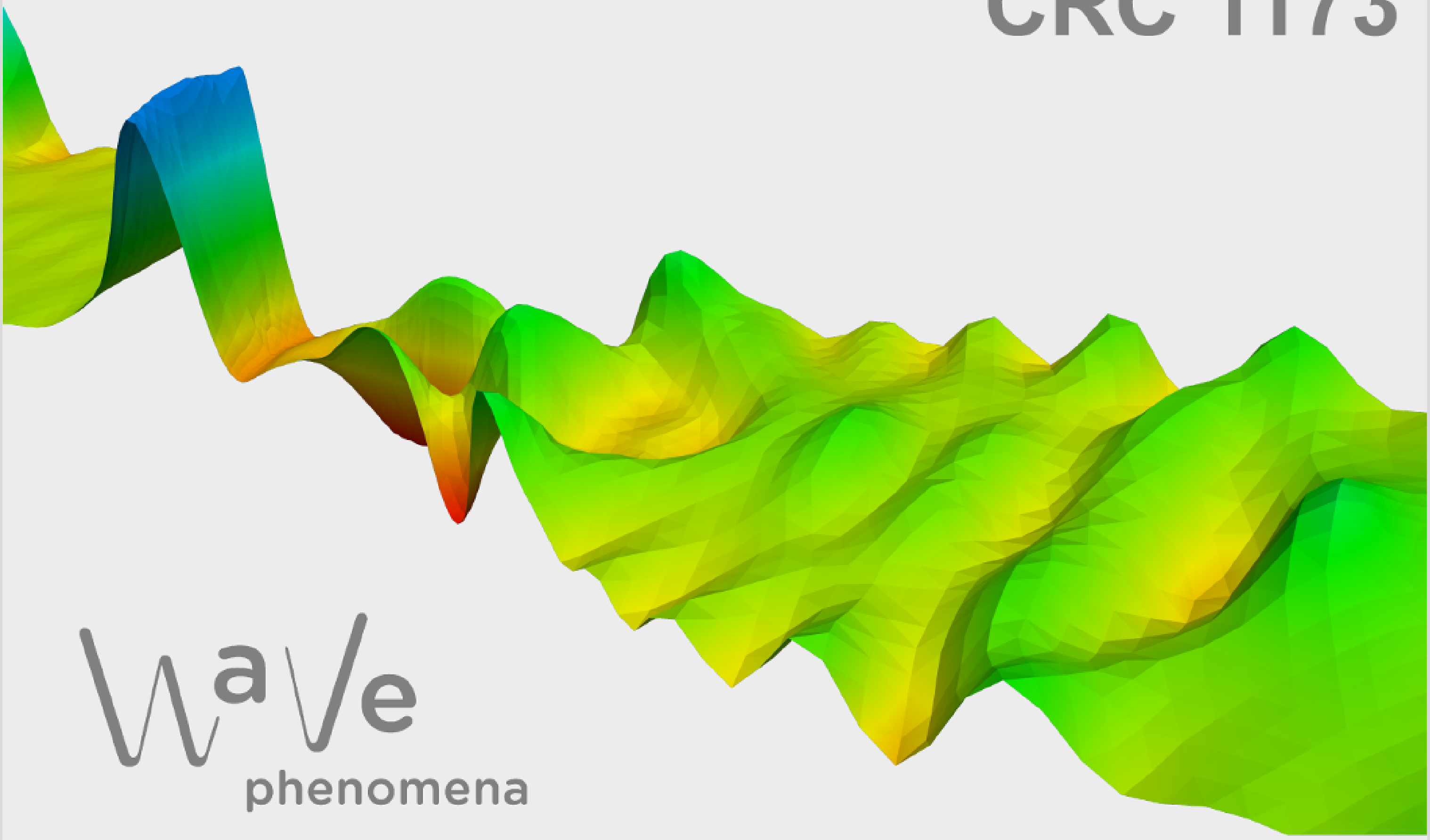

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IMPROVED DECOUPLING FOR THE MOMENT CURVE IN THREE DIMENSIONS

ROBERT SCHIPPA

Abstract. By quantifying a bilinear decoupling iteration of the moment curve in three dimensions due to Guo–Li–Yung–Zorin-Kranich, we show a logarithmic improvement of the decoupling constant at critical exponent.

1. Introduction

Let $\mathcal{E}f(x,t) = \int_{[0,1]} e^{i(x.\xi + t|\xi|^2)} f(\xi) d\xi$ denote the Fourier extension operator of the parabola. Let $\delta \in \mathbb{N}^{-1}$. Bourgain–Demeter [2] proved the decoupling inequality:

$$\|\mathcal{E}g\|_{L^6(B)} \leq C_\varepsilon \delta^{-\varepsilon} \big(\sum_{J \in \mathcal{P}(\delta)} \|\mathcal{E}_J g\|^2_{L^6(w_B)} \big)^{\frac{1}{2}}.$$

In the above display $B \subseteq \mathbb{R}^2$ denotes a square with side length δ^{-2} and center c and $\mathcal{P}(\delta)$ a partition of $[0,1]$ into intervals of length δ. Moreover, $w_B(x) = (1 + |x-c|/(2R))^{100}$ is a polynomial weight decaying away from B.

Let $D_2(\delta)$ denote the smallest constant such that the decoupling inequality holds:

$$\|\mathcal{E}g\|_{L^6(B)} \leq D_2(\delta) \big(\sum_{J \in \mathcal{P}(\delta)} \|\mathcal{E}_J g\|^2_{L^6(w_B)} \big)^{\frac{1}{2}}.$$

Li [8, Theorem 1.1] observed how the double exponential bound

$$D_2(\delta) \leq A^{A^{\frac{1}{\varepsilon}}} \delta^{-\varepsilon} \tag{1}$$

allows one to sharpen the decoupling constant to

$$D_2(\delta) \leq \exp\Big(C \frac{\log \delta^{-1}}{\log(\log \delta^{-1})}\Big).$$

This recovered the bound proved for discrete restriction by Bourgain [1, Proposition 2.36] via a divisor counting argument. By a Gauss sum argument, Bourgain [1, Remark 2, p. 118] showed moreover that

$$D_2(\delta) \gtrsim \log(\delta^{-1})^{1/6}.$$

Li [8] proved (1) via the bilinear approach. More recently, Guth–Maldague–Wang [7] improved (1) to

$$D_2(\delta) \leq \log(\delta^{-1})^c \tag{2}$$

for some (possibly large) constant c. Subsequently, Guo–Li–Yung [5] improved the discrete restriction constant to $C_\varepsilon \log(\delta^{-1})^{2+\varepsilon}$ for $\varepsilon > 0$. The approach in [7] differs from Li [8] as it relies less on induction-on-scales and uses instead a high-low decomposition. However, the high-low method becomes more involved in higher dimensions, whereas the bilinear approach as carried out in higher dimensions by Guo–Li–Yung–Zorin-Kranich [6] (see also [4]) seems more tractable. Li [8] firstly quantified decoupling via the bilinear approach for the paraboloid in $1+1$ dimensions.

Key words and phrases. moment curve, decoupling, Vinogradov mean value theorem.

In this note we turn to the moment curve in three dimensions. Let $\Gamma_3(t) = (t, t^2, t^3)$ denote the moment curve mapping in three dimensions. For an interval $J \subseteq [0,1]$ with center c_J, let $\mathcal{U}_J$ be the parallelepiped of dimensions $|J| \times |J|^2 \times |J|^3$, whose center is $\Gamma_3(c_J)$ and sides are parallel to $\Gamma_3'(c_J)$, $\Gamma_3''(c_J)$, $\Gamma_3^{(3)}(c_J)$.

We define the linear decoupling constant for the three-dimensional moment curve for $\delta \in \mathbb{N}^{-1}$ as smallest constant, which is monotone decreasing in δ, such that:

$$\| \sum_{J \in \mathcal{P}(\delta)} f_J \|_{L^{12}(\mathbb{R}^3)} \leq D_3(\delta) \big(\sum_{J \in \mathcal{P}(\delta)} \| f_J \|^2_{L^{12}(\mathbb{R}^3)} \big)^{1/2}.$$

Above $\mathcal{P}(\delta)$ denotes a partition up to points of I into intervals of length δ, and $f_J \in \mathcal{S}(\mathbb{R}^3)$ with Fourier support in $\mathcal{U}_J$. Bourgain–Demeter–Guth [3] proved that $D(\delta) \leq C_\varepsilon \delta^{-\varepsilon}$ for any $\varepsilon > 0$ in any dimension, which yields as corollary the Vinogradov mean value theorem. The argument in [3] relies on multilinear Kakeya estimates.

More recently, Guo *et al.* [6] found a shorter proof of decoupling for moment curves, which relies on bilinear arguments and induction on dimension. We observe that in one dimension, the decoupling inequality for the moment curve reduces to Plancherel's theorem with $D_1(\delta) = 1$ and in two dimensions the logarithmic loss due to Guth–Maldague–Wang [7] is at disposal. In the present note we quantify the bilinear iteration from [6] for the moment curve in three dimensions and use improved decoupling inequalities without $\delta^{-\varepsilon}$-loss in lower dimensions to show the following:

Theorem 1.1 (Improved decoupling for the moment curve in three dimensions)**.** *There is $0 < \delta_0 < 1$ and $C > 0$ such that for $0 < \delta < \delta_0$, we have the following bound for the decoupling constant of the moment curve in three dimensions:*

$$D_3(\delta) \leq \exp\Big(C \frac{\log \delta^{-1}}{\log(\log(\delta^{-1}))}\Big). \tag{3}$$

In the proof we see how losing additional logarithmic factors in the bilinear approach still allows us to show an estimate like in (1), which implies (3) after optimizing in ε.

As mentioned above, the decoupling result for the moment curve yielded Vinogradov's mean-value theorem. In the present instance the decoupling result in Theorem 3.1 yields a logarithmic improvement on the number of simultaneous solutions to the diophantine equations

$$\begin{cases} \sum_{i=1}^6 x_i &= \sum_{i=1}^6 y_i, \\ \sum_{i=1}^6 x_i^2 &= \sum_{i=1}^6 y_i^2, \\ \sum_{i=1}^6 x_i^3 &= \sum_{i=1}^6 y_i^3. \end{cases} \tag{4}$$

For $1 \leq x_i, y_i \leq N$ we denote the number of integer solutions to (4) by $J(N)$. We have the following corollary to Theorem 1.1:

Corollary 1.2. *For N sufficiently large, there is $C > 0$ such that the following estimate holds:*

$$J(N) \leq \exp\big(\frac{C \log(N)}{\log(\log(N))} \big) N^6.$$

Proof. As the well-known argument goes, we write with $e(x) = \exp(2\pi i x)$:

$$J(N) = \int_{[0,1]^3} \big| \sum_{j=1}^N e(jx_1 + j^2 x_2 + j^3 x_3) \big|^{12} dx_1 dx_2 dx_3.$$

By change of variables, we find

$$J(N) = N^{-6} \int_{[0,N]\times[0,N^2]\times[0,N^3]} \big|\sum_{j=1}^{N} e(x.\Gamma_3(j/N))\big|^{12} dx_1 dx_2 dx_3.$$

Now we use periodicity in x_1 with period N and in x_2 with period N^2 to write

$$J(N) \lesssim N^{-9} \int_{[0,N^3]^3} \big|\sum_{j=1}^{N} e(x.\Gamma_3(j/N))\big|^{12} dx_1 dx_2 dx_3.$$

Let $f_j = \exp(x.\Gamma_3(j/N))\psi(x)$ with $\psi \in \mathcal{S}(\mathbb{R}^3)$ be a Schwartz function such that $\text{supp}(\hat{\psi}) \subseteq N^{-3}$ and $|\psi(x)| \sim 1$ on $B(0,N^3)$. By the above we have

$$J(N) \lesssim N^{-9} \int_{[0,N^3]^3} \big|\sum_{j=1}^{N} f_j(x)\big|^{12} dx_1 dx_2 dx_3.$$

Applying Theorem 1.1 for N large enough gives

$$J(N) \lesssim N^{-9} \exp\big(\frac{C\log(N)}{\log(\log(N))}\big)\big(\sum_{j=1}^{N} \|f_j\|^2_{L^{12}(w_B)}\big)^6.$$

Since $|f_j(x)| = 1$ and $\|f_j\|_{L^{12}(w_B)} \lesssim (N^9)^{\frac{1}{12}}$, we find

$$J(N) \lesssim \exp\big(\frac{C\log(N)}{\log(\log(N))}\big) N^6.$$

The claim follows from choosing C slightly larger. $\square$

Outline of the paper. In Section 2 we introduce notations, explain the bilinear reduction, and give an overview of the constants coming up in the iteration. In Section 3 we recall the improved decoupling results in two dimensions and show stability. In Section 4 we carry out the decoupling iteration using asymmetric decoupling constants.

2. Preliminaries

2.1. **Notations.** For $k \in \mathbb{N}$ let $\Gamma_k : [0,1] \to \mathbb{R}^k$ denote the moment curve in $\mathbb{R}^k$. Let $\delta \in \mathbb{N}^{-1} = \{\frac{1}{n} : n \in \mathbb{N}\}$. For a closed interval $[a,b] = I \subseteq [0,1]$ with $|I|\delta^{-1} \in \mathbb{N}$ we denote by $\mathcal{P}(I,\delta)$ the decomposition into closed intervals of length δ: $I = \bigcup_{j=0}^{N-1} [a + j\delta, a + (j+1)\delta]$ with $N\delta = |I|$. If $I = 1$, we let $\mathcal{P}(I,\delta) = \mathcal{P}(\delta)$. Above we defined for an interval $J \subseteq [0,1]$ with center c_J, the parallelepiped $\mathcal{U}_J$ of dimensions $3|J| \times 3|J|^2 \times 3|J|^3$, whose center is $\Gamma(c_J)$ and sides are parallel to $\Gamma_3'(c_J)$, $\Gamma_3''(c_J)$, $\Gamma_3^{(3)}(c_J)$. More generally, we define for a curve $\gamma : [0,1] \to \mathbb{R}^k$ the parallelepiped $\mathcal{U}_{J,\gamma}$ with center c_J of dimensions $3|J| \times 3|J|^2 \times \ldots \times 3|J|^k$ into directions $\partial\gamma(c_J), \ldots, \partial^k\gamma(c_J)$.

In the following, for an interval J and curve γ, let $\mathcal{U}^o_{J,\gamma}$ denote the parallelepiped centered at the origin, which is dual to $\mathcal{U}_{J,\gamma}$, that is

$$\mathcal{U}^o_{I,\gamma} = \{x \in \mathbb{R}^k : \big|\langle x, \partial^i\gamma(c_J)\rangle\big| \leq \frac{1}{3}|J|^{-i}, \quad 1 \leq i \leq k\}.$$

This is a parallelepiped of size $\sim |J|^{-1} \times |J|^{-2} \times |J|^{-3}$. We define a bump function adapted to $\mathcal{U}^o_I$ by

$$\phi_I(x) = |\mathcal{U}^o_I|^{-1} \inf\{t \geq 1 : x/t \in \mathcal{U}^o_I\}^{-10k}.$$

This is L^1-normalized as can be seen from anisotropic dilation: $\int_{\mathbb{R}^k} \phi_I(x) dx \leq C_{4,k}$. For $k \in \mathbb{N}$, we define the critical decoupling exponent for the moment curve Γ_k as $p_k = k(k+1)$. We define $D_k(\delta)$ as monotone decreasing in δ (this means if δ^{-1}

becomes larger, the decoupling constant is also supposed to become larger) and smallest constant, which satisfies

$$\Big\| \sum_{J \in \mathcal{P}(\delta)} f_J \Big\|_{L^{p_k}(\mathbb{R}^k)} \leq D_k(\delta) \big(\sum_{J \in \mathcal{P}(\delta)} \|f_J\|^2_{L^{p_k}(\mathbb{R}^k)} \big)^{\frac{1}{2}}.$$

2.2. **Bilinear reduction, and uncertainty principle.** We define the bilinear decoupling constant $B_k(\delta)$ as smallest constant decreasing in δ such that

$$\big(\int_{\mathbb{R}^k} \big| \sum_{J_1 \in \mathcal{P}(I_1,\delta)} f_{J_1} \big|^{p_k/2} \big| \sum_{J_2 \in \mathcal{P}(I_2,\delta)} f_{J_2} \big|^{p_k/2} \big)^{1/p_k} \leq B_k(\delta) \big(\sum_{J_1 \in \mathcal{P}(I_1,\delta)} \|f_{J_1}\|^2_{L^{p_k}} \big)^{1/4} \big(\sum_{J_2 \in \mathcal{P}(I_2,\delta)} \|f_{J_2}\|^2_{L^{p_k}} \big)^{1/4}.$$

In the above display, we consider intervals $I_i \subseteq [0,1]$, $i = 1,2$ with $\text{dist}(I_1, I_2) \geq \frac{1}{4}$ and $|I_i|\delta \in \mathbb{N}$.

We have the following linear-to-bilinear reduction:

Lemma 2.1 (Bilinear reduction, [6, Lemma 2.2])**.** *If $\delta = 2^{-M}$, then*

$$D_k(\delta) \leq C \big(1 + \sum_{n=2}^{M} B_k(2^{-M+n-2})^2 \big)^{1/2}.$$

The proof of the above lemma is based on a Whitney decomposition and affine rescaling, which is already very important for the linear decoupling:

Lemma 2.2 (Affine rescaling, [6, Lemma 2.3])**.** *Let $I \in \mathcal{P}(2^{-n})$ for some integer $n \geq 0$. For any $\delta \in (0, 2^{-n})$ and any tuple of functions $(f_J)_{J \in \mathcal{P}(I,\delta)}$ with supp$(\hat{f}_J) \subseteq \mathcal{U}_J$ for all J, the following holds:*

$$\|f_I\|_{L^{p_k}(\mathbb{R}^k)} \leq D_k(2^n \delta) \big(\sum_{J \in \mathcal{P}(I,\delta)} \|f_J\|^2_{L^{p_k}} \big)^{1/2}. \tag{5}$$

We obtain submultiplicativity as a consequence:

Lemma 2.3. *We have for δ, σ, $\delta/\sigma \in \mathbb{N}^{-1}$:*

$$D(\delta) \leq D(\sigma) D(\delta/30\sigma). \tag{6}$$

Proof. We can suppose that $\sigma \leq \frac{1}{10}$ because for $\sigma \in [\frac{1}{10}, 0)$ we have by monotonicity trivially

$$D(\delta) \leq D(\delta/10\sigma).$$

We partition $\mathcal{P}(\delta)$ into collections indexed by $\tilde{J} \in \mathcal{P}(10\sigma)$ such that $\mathcal{U}_J \subseteq \mathcal{U}_{\tilde{J}}$ and write $J \sim \tilde{J}$ (it suffices to take J as a child of $\tilde{J}$ or as a child of a neighbour), and we write $f_{\tilde{J}} = \sum_{J \sim \tilde{J}} f_J$. Then we can apply decoupling at 10σ to find

$$\begin{aligned} \Big\| \sum_{\tilde{J}} f_{\tilde{J}} \Big\|_{L^{p_k}(\mathbb{R}^k)} &\leq D_k(10\sigma) \big(\sum_{\tilde{J}} \|f_{\tilde{J}}\|^2_{L^{p_k}(\mathbb{R}^k)} \big)^{\frac{1}{2}} \\ &\leq D_k(10\sigma) D_k(\delta/30\sigma) \big(\sum_{J \in \mathcal{P}(\delta)} \|f_J\|^2_{L^{p_k}(\mathbb{R}^k)} \big)^{\frac{1}{2}}, \end{aligned}$$

where we used affine rescaling in the second step and that J is either a child of $\tilde{J}$ or a child of a neighbouring interval.

□

In the iteration to estimate $D_k(\delta)$, we use monotonicity of $B_k(\delta)$ to write

$$D_k(\delta) \leq C_1 \log(\delta^{-1}) B_k(\delta). \tag{7}$$

The reason we do not resort to the slightly sharper argument of broad-narrow reduction, which is used by Li [8], is that the unit distance separation of the intervals simplifies the forthcoming arguments, and we are losing logarithmic factors in the iteration anyway.

We also use the following instance of uncertainty prinicple:

Lemma 2.4 (Uncertainty principle (see [6, Lemma 3.3])). *For $p \in [1,\infty)$ and $J \subseteq [0,1]$ we have*

$$|g_J|^p \leq C_p(|g_J|^p * \phi_J),$$

for every g_J with supp$(\hat{g}_J) \subseteq C'\mathcal{U}_J$.

Record the following trivial bound due to Cauchy-Schwarz:

$$D_k(\delta) \leq \delta^{-\frac{1}{2}}. \tag{8}$$

2.3. **Overview of constants.** In the following we denote by C_i, $i = 0,1,\dots,10$ fixed (possibly very large) constants, which will be defined in the course of the argument.

- C_1 is used in the linear-to-bilinear reduction (7),
- C_2, C_3 are used to record constants in the arguments involving lower dimensional decoupling, c denotes the exponent in the logarithmic loss for the $\ell^2 L^6$-decoupling,
- C_4 depends on the L^1-norm of an essentially L^1-normalized function (see Lemma 4.2),
- C_5 is a constant, which comes up in the key iteration to lower the scale,
- C_6 comes from an application of the triangle inequality to lower the scale once to ν (see (21)), d is a related exponent,
- $N = N(\varepsilon, d)$ will later in the proof denote the number of iterations to lower the scale,
- C_7, C_8 are absolute constants used to record intermediate estimates for $D(\delta)$ after carrying out the decoupling iteration (see Lemmas 4.6, 4.7).

3. Decoupling in one and two dimensions

In this section we argue how the improved decoupling result by Guth–Maldague–Wang extends to the family of curves presently considered. Some of the arguments are already contained in [7, Appendix], but we opt to give the details. Then we shall see how we can use lower dimensional decoupling in bilinear expressions. Their improved decoupling result is formulated for normalized curves as follows:

Theorem 3.1 ([7, Appendix]). *Let $\gamma : [0,1] \to \mathbb{R}^2$, $\gamma(t) = (t, h(t))$ be a curve such that $h \in C^2([-1,2])$ with $h(0) = h'(0) = 0$ and $\frac{1}{2} \leq h''(t) \leq 2$. Then, there are $C, c > 0$ such that for $(f_J)_{J \in \mathcal{P}_\delta}$ with supp$(\hat{f}_J) \subseteq \mathcal{U}_{J,\gamma}$ we have:*

$$\Big\| \sum_{J \in \mathcal{P}_\delta} f_J \Big\|_{L^6(\mathbb{R}^2)} \leq C (\log(\delta^{-1}))^c \Big(\sum_J \|f_J\|^2_{L^6(\mathbb{R}^2)} \Big)^{\frac{1}{2}}.$$

In the following we want to argue that the result extends to more general curves $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in C^5$ with

$$\|\gamma\|_{C^5} \leq D_3 < \infty \text{ and } 0 < D_1 \leq |\gamma'(t) \wedge \gamma''(t)| \leq D_2 < \infty. \tag{9}$$

Proposition 3.2 (Stability of improved decoupling). *Suppose $\gamma \in C^5$ satisfies (9), and let $(f_J)_{J \in \mathcal{P}_\delta(I)}$ with supp$(\hat{f}_J) \subseteq C'\mathcal{U}_{J,\gamma}$. Then, there is $C(\underline{D}, C')$ such that*

$$\Big\| \sum_J f_J \Big\|_{L^6(\mathbb{R}^2)} \leq C (\log(\delta^{-1}))^c \Big(\sum_J \|f_J\|^2_{L^6} \Big)^{\frac{1}{2}}. \tag{10}$$

Proof. In the first step we reduce the curves γ to $(t, h(t))$ by finite decomposition, rotation, and translation, which only depends on $\underline{D}$: For any point $\gamma(t_*)$ we can achieve by rotation and translation that $\gamma(t_*) = 0$, $\dot{\gamma}(t_*) = (c, 0)$ for some $c > 0$, and $\ddot{\gamma}(t_*) > 0$. By the implicit function theorem we obtain a reparametrization $t = g(s)$ such that $\gamma_1(g(s)) = s$. The interval on which the reparametrization exists depends on c and $\|\gamma\|_{C^2}$. c is bounded from above by D_3 and from below by using the torsion:

$$\begin{vmatrix} c & \ddot{\gamma}_1(t_*) \\ 0 & \ddot{\gamma}_2(t_*) \end{vmatrix} = c|\ddot{\gamma}_2(t_*)| \geq D_1 \Rightarrow c \geq \frac{D_1}{D_3}.$$

This means we find finitely many curves $\tilde{\gamma}(s) = (s, h(s))$ with $h(0) = h'(0) = 0$ and $0 < D_1' \leq h''(s) \leq D_2' < \infty$ with $D_i' = D_i'(\underline{D})$. We can compare the rectangles $\tilde{\gamma}$ and γ by noting that from $\tilde{\gamma}(s) = \gamma(g(s))$ follows:

$$\dot{\tilde{\gamma}}(s) = \dot{\gamma}(g(s))g'(s), \quad \ddot{\tilde{\gamma}}(s) = \ddot{\gamma}(g(s))(g'(s))^2 + \dot{\gamma}(g(s))g''(s).$$

The bilipschitz comparability of rectangles follows then from $g'(s) \sim_{\underline{D}} 1$ and $|g''(s)| \leq \kappa(\underline{D})$. In s parametrization, the rectangles $C'\mathcal{U}_{J,\gamma}$ become centered at $\gamma(t_J) = \tilde{\gamma}(s_J)$ and can be contained in rectangles of length $C''\delta \times C''\delta^2$ in the directions $\dot{\tilde{\gamma}}(s_J)$, $\ddot{\tilde{\gamma}}(s_J)$. For this reason we observe $\text{supp}(\hat{f}_J) \subseteq \mathcal{C}''\mathcal{U}_{J,\tilde{\gamma}}$.

We turn to normalization of h. We subdivide $[0, 1]$ into intervals I_s of length s. A Taylor expansion of γ at the center t_c gives

$$\gamma(t) = \gamma(t_c) + \gamma'(t_c)(t - t_c) + \gamma''(t_c)\frac{(t - t_c)^2}{2} + O((t - t_c)^3).$$

Since $|\dot{\gamma}(t_c) \wedge \ddot{\gamma}(t_c)| = |h''(t_c)| \neq 0$, there is an anisotropic dilation $D = \text{diag}(d_1, d_2)$ such that after translation

$$\tilde{\gamma}(t) = te_1 + \frac{t^2}{2}e_2 + G(t)t^3 e_2.$$

The representation $G(t)t^3 e_2$ with $G \in C^2$ for the third order remainder term in the Taylor expansion (after dilation) follows from the integral representation of the remainder:

$$R_3(t) = \int_0^t \frac{\gamma^{(3)}(s)}{6}(t - s)^3 ds.$$

We obtain

$$G(t) = \int_0^t \frac{\gamma^{(3)}(s)}{6}(1 - \frac{s}{t})^3 ds = t\int_0^1 \frac{\gamma^{(3)}(ts')}{6}(1 - s')^3 ds'$$

and for $\gamma \in C^5$ we find $G \in C^2$ and $\|G\|_{C^2} \leq \kappa(\underline{D})$. Moreover,

$$\tilde{\gamma}''(t) = e_2 + (G'(t)t^3 + 3G(t)t^2)'e_2 = e_2 + (G''(t)t^3 + 6t^2G'(t) + 6G(t)t)e_2.$$

Clearly, $|G''(t)t^3 + 6t^2G'(t) + 6G(t)t| = O_{\underline{D}}(s)$ and choosing s small enough only depending on $\underline{D}$, we finish the decomposition into curves of the kind $\gamma(t) = (t, h(t))$ with $h(0) = h'(0) = 0$ and $\frac{1}{2} \leq h''(t) \leq 2$. Now we consider the decoupling of $(t, h(t))$ with $\text{supp}\hat{f}_J \subseteq C\mathcal{U}_{J,\gamma}$ and shall prove that

$$\|\sum_{J \in \mathcal{P}(\delta)} f_J\|_{L^6(\mathbb{R}^2)} \leq \tilde{C}(C, C')(\log(\delta^{-1}))^c \big(\sum_J \|f_J\|^2_{L^6(\mathbb{R}^2)}\big)^{\frac{1}{2}}$$

with C like in Theorem 3.1. First, we observe that Theorem 3.1 applies with $\tilde{C} = C$ for $C' \leq 1$. We turn to $C' \geq 1$: The minor technical issue is that the blocks $\mathcal{U}_{J,\gamma}$ are overlapping more often than in the original collection. We observe that these blocks are in the $\tilde{\delta}$-neighbourhood for $\tilde{\delta} = 10^{10}(C')^2\delta$. So we can apply decoupling for $\tilde{\delta}$, but the decomposition into $\mathcal{U}_{\tilde{J},\gamma}$ for $\tilde{J} \in \mathcal{P}_{\tilde{\delta}}$ is too coarse. For $\tilde{J}$ we choose a collection $\mathcal{J}$ of intervals $J \subseteq \tilde{J}$ such that $\sum_{\tilde{J}} f_{\tilde{J}} = \sum_J f_J$ and find:

$$\| \sum_{\tilde{J} \in \mathcal{P}(\tilde{\delta})} f_{\tilde{J}} \|_{L^6(\mathbb{R}^2)} \leq C \log((10^{10}(C')^2\delta)^{-1})^c \big(\sum_{\tilde{J} \in \mathcal{P}(\tilde{\delta})} \|f_{\tilde{J}}\|^2_{L^6(\mathbb{R}^2)} \big)^{\frac{1}{2}}.$$

Since $\#\{J \subseteq \tilde{J}\} = O((C')^2)$, an application of Cauchy-Schwarz finishes the proof:

$$\| \sum_{J \in \mathcal{U}_J} f_J \|_{L^6(\mathbb{R}^2)} \leq \tilde{C}(C, C')(\log(\delta^{-1}))^c \big(\sum_J \|f_J\|^2_{L^6(\mathbb{R}^2)} \big)^{\frac{1}{2}}.$$

□

We summarize uniform decoupling inequalities for families of curves: Suppose $\ell \in \{1, 2\}$ and $\gamma : [0, 1] \to \mathbb{R}^\ell$ is a curve such that

$$\|\gamma\|_{C^5} \leq D_3 \text{ and for any } t \in [0, 1] : \, D_1 \leq \big| \bigwedge_{i=1}^{\ell} \partial^i \gamma(t) \big| \leq D_2. \tag{11}$$

Proposition 3.3 (Decoupling for curves with torsion for $d = 1, 2$)**.** *Suppose that $\ell \in \{1, 2\}$, and $\gamma : [0, 1] \to \mathbb{R}^\ell$ is a curve satisfying* (11)*. Then, for any $C > 0$, any $\delta \in (0, 1)$, and any tuple of functions $(f_J)_{J \in \mathcal{P}(\delta)}$ with supp$(\hat{f}_J) \subseteq C\mathcal{U}_{J,\gamma}$ for any J, the following inequality holds:*

$$\| \sum_{J \in \mathcal{P}(\delta)} f_J \|_{L^{p_\ell}(\mathbb{R}^\ell)} \leq C'_\ell(C, \underline{D}, \delta) \big(\sum_{J \in \mathcal{P}(\delta)} \|f_J\|^2_{L^{p_\ell}(\mathbb{R}^\ell)} \big)^{1/2} \tag{12}$$

with

$$C'_\ell(C, \underline{D}, \delta) = \begin{cases} C'(C, \underline{D}), & \ell = 1, \\ C'(C, \underline{D})(\log(\delta^{-1}))^c, & \ell = 2. \end{cases}$$

Proof. For $\ell = 1$ this is obvious, for $\ell = 2$ this is Proposition 3.2. □

Corollary 3.4. *Under the assumptions of Proposition 3.3, for every ball $B \subseteq \mathbb{R}^\ell$ of radius $\delta^{-\ell}$, we have*

$$\fint_B \big| \sum_{J \in \mathcal{P}(\delta)} f_J \big|^{p_\ell} \leq C'_\ell(C, \underline{D}, \delta) \big(\sum_{J \in \mathcal{P}(\delta)} \|f_J\|^2_{L^{p_\ell}(\phi_B)} \big)^{p_\ell/2}. \tag{13}$$

In the above display $\phi_B(x) = |B|^{-1}(1 + \delta^\ell \mathit{dist}(x, B))^{-30}$ denotes an L^1-normalized bump function adapted to B, and $\fint_B$ denotes the average integral.

Proof. We apply Proposition 3.3 to functions $f_J \psi_B$, where ψ_B is a Schwartz function such that $|\psi_B| \gtrsim 1$ on B and supp$(\hat{\psi}_B) \subseteq B(0, \delta^\ell)$. For B centered at the origin, it suffices to consider $\psi_B(x) = \delta^{-\ell^\ell} \int_{\mathbb{R}^3} e^{ix.\xi} a(\delta^{-\ell}\xi) d\xi$ with $a \in C^\infty_c(\mathbb{R}^3)$ a radially decreasing function satisfying $a(0) = 1$, $a \geq 0$ and having support in $B(0, c)$ for c small enough. The general case follows from translation. Then

$$\begin{aligned} \fint_B \big| \sum_J f_J \big|^{p_\ell} &\lesssim \fint_{\mathbb{R}^3} \big| \sum_J f_J \frac{\psi_B}{|B|^{1/p_\ell}} \big|^{p_\ell} \lesssim \int_{\mathbb{R}^3} \big| \sum_J f_J \frac{\psi_B}{|B|^{1/p_\ell}} \big|^{p_\ell} \\ &\leq C'(C, \underline{D}) C_\ell(\delta) \big(\sum_J \| \frac{f_J \psi_B}{|B|^{1/p_\ell}} \|^2_{L^{p_\ell}} \big)^{p_\ell/2} \end{aligned}$$

with

$$C_\ell(\delta) = \begin{cases} 1, & \ell = 1, \\ (\log(\delta^{-1}))^c, & \ell = 2. \end{cases}$$

To conclude the proof, we need to argue that

$$\sup_x \frac{|\psi_B(x)|^{p_\ell}}{|B|\phi_B(x)} \lesssim 1.$$

This follows from the rapid decay of ψ_B away from B (which is by our definition of ψ_B faster than any polynomial). Let $R = \delta^{-\ell}$ and suppose again B is centered at the origin. For $\text{dist}(x, B) \leq R$ we have $|\psi_B(x)|^{p_\ell} \leq C$, $|B|\phi_B(x) \geq 1$. For $\text{dist}(x, B) \geq R$, which means $x \geq 2R$, we can estimate

$$\psi_B(x) = \psi_1(R^{-1}x) \leq C_N(1 + R^{-1}x)^{-N}.$$

Therefore, $(\psi_1(R^{-1}x))^{p_\ell} \leq C_N(R^{-1}x)^{-N\cdot p_\ell}$ and $|B|\phi_B(x) \sim (R^{-1}x)^{-30}$. Choosing N large enough yields an acceptable contribution. □

Lemma 3.5 (Lower degree decoupling (see [6, Lemma 3.5.])). *Let $\ell \in \{1, 2\}$. Let $\delta \in (0, 1)$ and $(f_K)_{K\in\mathcal{P}(\delta)}$ be a tuple of functions so that supp$\hat{f}_K \subseteq \mathcal{U}_K$ for every K. If $0 \leq a \leq (3 - \ell + 1)b/\ell$, then for any pair of intervals $I \in \mathcal{P}(\delta^a)$, $I' \in \mathcal{P}(\delta^b)$ with dist$(I, I') \geq 1/4$, we obtain for $\ell = 1$:*

$$\int_{\mathbb{R}^3} (|f_I|^2 * \phi_I)(|f_{I'}|^{10} * \phi_{I'}) \leq C_1 \sum_{J\in\mathcal{P}(I,\delta^{3b})} \int_{\mathbb{R}^3} (|f_J|^2 * \phi_J)(|f_{I'}|^{10} * \phi_{I'}), \tag{14}$$

and for $\ell = 2$:

$$\int_{\mathbb{R}^3} (|f_I|^6 * \phi_I)(|f_{I'}|^6 * \phi_{I'}) \leq C_2(\log(\delta^{-b}))^c \Big(\sum_{J\in\mathcal{P}(I,\delta^b)} \Big(\int_{\mathbb{R}^3} \big(|f_J|^6 * \phi_J\big)\big(|f_{I'}|^6 * \phi_{I'}\big)\Big)^{\frac{1}{3}}\Big)^3. \tag{15}$$

We need the following transversality observation:

Lemma 3.6 ([6, Lemma 3.5]). *Let $\Gamma_k(t) = (t, t^2, \ldots, t^k) : [0, 1] \to \mathbb{R}^k$. For any integers $0 \leq \ell \leq k$ and any $\xi_1, \xi_2 \in \mathbb{R}$, we have*

$$|\partial^1\Gamma_k(\xi_1) \wedge \ldots \wedge \partial^\ell\Gamma_k(\xi_1) \wedge \partial^1\Gamma_k(\xi_2) \wedge \ldots \wedge \partial^{k-\ell}\Gamma_k(\xi_2)| \gtrsim_{k,\ell} |\xi_1 - \xi_2|^{\ell(k-\ell)}.$$

Now we are ready to prove Lemma 3.5. We repeat the argument from [6] for convenience. By quantifying the decoupling constants, we can improve the $\delta^{-\varepsilon}$ bound from [6] as claimed.

Proof of Lemma 3.5. Denote $b' = (3 - \ell + 1)b/\ell$ and $k = 3$. Fix $\xi' \in I'$, let $V^m(\xi') = \text{span}(\partial^1\Gamma_k(\xi'), \ldots, \partial^m\Gamma_k(\xi'))$ be the tangent space for $m \in \{1, 2, 3\}$, and let $\hat{H} = \mathbb{R}^k/V^{k-\ell}(\xi')$ be the quotient space. Let $P : \mathbb{R}^k \to \hat{H}$ be the projection onto $\hat{H}$. For every $\xi \in I$, we have by Lemma 3.6 that

$$|\partial^1(P \circ \Gamma_k)(\xi) \wedge \ldots \wedge \partial^\ell(P \circ \Gamma_k)(\xi)| \gtrsim 1.$$

Moreover, $P(\mathcal{U}_J) \subseteq C'\mathcal{U}_{J,P\circ\Gamma}$. Let $H = V^{k-\ell}(\xi')^\perp$ be the orthogonal complement in $\mathbb{R}^k$ so that $\hat{H}$ is its Pontryagin dual. Since the Fourier support of the restriction $f_J\big|_{H+z}$ to almost every translated copy is contained in $P(\text{supp}(\hat{f}_J))$ and $P(\mathcal{U}_J) \subseteq C'\mathcal{U}_{J,P\circ\Gamma}$, we can apply lower dimensional decoupling inequalities. We write by Fubini's theorem

$$\int_{\mathbb{R}^k} \big(|f_I|^{p_\ell} * \phi_I\big)\big(|f_{I'}|^{p_k-p_\ell} * \phi_{I'}\big) = \int_{z\in\mathbb{R}^k} \fint_{B_H(z,\delta^{-b'\ell})} \big(|f_I|^{p_\ell} * \phi_I\big)\big(|f_{I'}|^{p_k-p_\ell} * \phi_{I'}\big), \tag{16}$$

where $B_H(z, \delta^{-b'\ell})$ is the ℓ-dimensional ball with radius $\delta^{-b'\ell}$ centered at z inside the affine subspace $H + z$. Since $B_H(0, \delta^{-b'\ell}) = B_H(0, \delta^{-(k-\ell+1)b}) \subseteq C'\mathcal{U}^o_{I'}$, we have

$$\sup_{x\in B_H(z,\delta^{-b'\ell})} \big(|f_{I'}|^{p_k-p_\ell} * \phi_{I'}\big)(x) \lesssim \big(|f_{I'}|^{p_k-p_\ell} * \phi_{I'}\big)(z).$$

This allows us to continue to write

$$(16) \lesssim \int_{z \in \mathbb{R}^k} \big(\fint_{B_H(z,\delta^{-b'\ell})} |f_I|^{p_\ell} * \phi_I \big) \big(|f_{I'}|^{p_k - p_\ell} * \phi_{I'} \big)(z).$$

Above $*_H$ denotes convolution along H. Now we can use lower dimensional decoupling with $\delta^{b'}$ in place of δ in Corollary 3.4:

$$\leq C'_\ell(C, \underline{D}, \delta) \int_{z'} \phi_I(z - z') \big(\sum_{J \in \mathcal{P}(I,\delta^{b'})} \|f_J\|^2_{L^{p_\ell}(\phi_{B_H}(z',\delta^{-b'\ell}))} \big)^{p_\ell/2}.$$

Taking the p_ℓth root we find for (16):

$$\begin{aligned}(16)^{1/p_\ell} &\leq C'_\ell(C, \underline{D}, \delta^{b'}) \big(\int_{z,z' \in \mathbb{R}^3} \big(|f_{I'}|^{p_k - p_\ell} * \phi_{I'} \big)(z) \\ &\quad \times \phi_I(z - z') \big(\sum_{J \in \mathcal{P}(I,\delta^{b'})} \|f_J\|^2_{L^{p_\ell}(z' + H, \phi_{B_H}(z',\delta^{-b'\ell}))} \big)^{p_\ell/2} \big)^{1/p_\ell} \\ &\leq C'_\ell(C, \underline{D}, \delta^{b'}) \big(\sum_{J \in \mathcal{P}(I,\delta^{b'})} \big(\int_{z,z' \in \mathbb{R}^3} \big(|f_{I'}|^{p_k - p_\ell} * \phi_{I'} \big)(z) \\ &\quad \times \phi_I(z - z') \|f_J\|^{p_\ell}_{L^{p_\ell}(\phi_{B_H(z',\delta^{-b'\ell})})} \big)^{2/p_\ell} \big)^{\frac{1}{2}}.\end{aligned}$$

The last estimate follows from Minkowski's inequality since $2 \leq p_\ell$. The double integral inside the brackets can be written as

$$\begin{aligned}&\int_{\mathbb{R}^k} \big(|f_{I'}|^{p_k - p_\ell} * \phi_{I'} \big) \big(\phi_I * |f_J|^{p_\ell} *_H \phi_{B_H(0,\delta^{-b'\ell})} \big) \\ &= \int_{\mathbb{R}^k} (|f_{I'}|^{p_k - p_\ell} * \phi_I *_H \phi_{B_H(0,\delta^{-b'\ell})})(|f_J|^{p_\ell} * \phi_I) \\ &\lesssim \int_{\mathbb{R}^k} \big(|f_{I'}|^{p_k - p_\ell} * \phi_{I'} \big) \big(|f_J|^{p_\ell} * \phi_I \big),\end{aligned}$$

which follows again by $B_H(0, \delta^{-b'\ell}) \subseteq C\mathcal{U}^o_{I',\gamma}$. Using the uncertainty principle and $\mathcal{U}^o_I \subseteq C\mathcal{U}^o_J$, we find

$$|f_J|^{p_\ell} * \phi_I \lesssim |f_J|^{p_\ell} * \phi_J * \phi_I \lesssim |f_J|^{p_\ell} * \phi_J,$$

and the proof is complete. □

4. Proof of Theorem 1.1

4.1. **Asymmetric decoupling constant.** In the following we define asymmetric decoupling constants, which effectively allow us to lower the scale by using lower-dimensional decoupling stated in the previous section. We consider two intervals I, I' of size $|I| = \delta^a$ and $|I'| = \delta^b$, $a, b \in [0, 1]$, which are separated at unit distance. Following [6], we define bilinear decoupling constants as smallest constants, which satisfy the following:

$$\int_{\mathbb{R}^3} (|f_I|^6 * \phi_I)(|f_{I'}|^6 * \phi_{I'}) \leq M^{12}_{6,a,b}(\delta) \big(\sum_{J \in \mathcal{P}(I,\delta)} \|f_J\|^2_{L^{12}} \big)^3 \big(\sum_{J' \in \mathcal{P}(I',\delta)} \|f_{J'}\|^2_{L^{12}} \big)^3.$$

Secondly, we define

$$\int_{\mathbb{R}^3} (|f_I|^2 * \phi_I)(|f_{I'}|^{10} * \phi_{I'}) \leq M^{12}_{2,a,b}(\delta) \big(\sum_{J \in \mathcal{P}(I,\delta)} \|f_J\|^2_{L^{12}} \big) \big(\sum_{J' \in \mathcal{P}(I',\delta)} \|f_{J'}\|^2_{L^{12}} \big)^5.$$

We have the following as consequence of (14) and (15):

Lemma 4.1 (Lower dimensional decoupling)**.** *Let $a,b \in [0,1]$ such that $0 \leq a \leq 3b$. Then*

$$M_{2,a,b}(\delta) \leq C_2 M_{2,3b,b}(\delta). \tag{17}$$

If $0 \leq a \leq b$, then the following estimate holds for some $c \in \mathbb{N}$:

$$M_{6,a,b}(\delta) \leq C_3 (\log(\delta^{-b}))^c M_{6,b,b}(\delta). \tag{18}$$

The following is straight-forward from Hölder's inequality and parabolic rescaling (cf. [6, Lemma 4.1]):

Lemma 4.2 (Hölder's inequality I)**.** *Let $a,b \in [0,1]$. Then*

$$M_{2,a,b}(\delta) \leq C_4 M_{6,a,b}(\delta)^{1/3} D(\delta/\delta^b)^{2/3}. \tag{19}$$

Proof. We apply Hölder's inequality to find:

$$\begin{aligned} |f_I|^2 * \phi_I &\leq C_4 (|f_I|^6 * \phi_I)^{\frac{1}{3}}, \\ |f_{I'}|^{10} * \phi_{I'} &\leq (|f_{I'}|^6 * \phi_{I'})^{1/3} (|f_{I'}|^{12} * \phi_{I'})^{2/3}. \end{aligned}$$

The constant in the first estimate does not depend on the scale due to L^1-normalization of ϕ_I:

$$\begin{aligned} \int |f_I|^2(y) \phi_I(x-y) dy &= \int |f_I|^2(y) \phi_I(x-y)^{\frac{1}{3}} \phi_I(x-y)^{\frac{2}{3}} dy \\ &\leq \big(\int |f_I|^6 \phi_I(x-y) dy \big)^{\frac{1}{3}} \big(\int \phi_I(x-y) dy \big)^{\frac{2}{3}} \\ &= C_4 (|f_I|^6 * \phi_I)^{\frac{1}{3}}. \end{aligned}$$

By the above and another application of Hölder's inequality we find:

$$\begin{aligned} \int_{\mathbb{R}^3} (|f_I|^2 * \phi_I)(|f_{I'}|^{10} * \phi_{I'}) &\leq C_4 \int_{\mathbb{R}^3} (|f_I|^6 * \phi_I)^{1/3} (|f_{I'}|^6 * \phi_{I'})^{1/3} (|f_{I'}|^{12} * \phi_{I'})^{2/3} \\ &\leq C_4 \big(\int_{\mathbb{R}^3} (|f_I|^6 * \phi_I)(|f_{I'}|^6 * \phi_{I'}) \big)^{1/3} \big(\int_{\mathbb{R}^3} (|f_{I'}|^{12} * \phi_{I'}) \big)^{2/3} \\ &\leq C_4 \big(\int_{\mathbb{R}^3} (|f_I|^6 * \phi_I)(|f_{I'}|^6 * \phi_{I'}) \big)^{1/3} \big(\int_{\mathbb{R}^3} (|f_{I'}|^{12} * \phi_{I'}) \big)^{2/3} \end{aligned}$$

From this estimate and parabolic rescaling, (19) is immediate. □

Another application of Hölder's inequality gives the following (again [6, Lemma 4.2]):

Lemma 4.3 (Hölder's inequality II)**.** *Let $a,b \in [0,1]$. Then*

$$M_{6,a,b}(\delta) \leq M_{2,a,b}(\delta)^{1/2} M_{2,b,a}(\delta)^{1/2}. \tag{20}$$

Proof. By two applications of the Cauchy-Schwarz inequality we find

$$\begin{aligned} \int_{\mathbb{R}^3} (|f_I|^6 * \phi_I)(|f_{I'}|^6 * \phi_{I'}) &\leq \int_{\mathbb{R}^3} (|f_I|^2 * \phi_I)^{1/2} (|f_{I'}|^{10} * \phi_{I'})^{1/2} \cdot (|f_I|^{10} * \phi_I)^{1/2} (|f_{I'}|^2 * \phi_{I'})^{1/2} \\ &\leq \big(\int_{\mathbb{R}^3} (|f_I|^2 * \phi_I)(|f_{I'}|^{10} * \phi_{I'}) \big)^{1/2} \big(\int_{\mathbb{R}^3} (|f_I|^{10} * \phi_I)(|f_{I'}|^2 * \phi_{I'}) \big)^{1/2}. \end{aligned}$$

From this estimate (20) is immediate. □

4.2. **The decoupling iteration.** By Lemma 4.1, 4.2, and 4.3, we have the following key iteration step:

Lemma 4.4 (Iteration step for the moment curve)**.** *Let $a, b \in [0,1]$ and $0 < a \leq 3b$. We find*

$$M_{2,a,b}(\delta) \leq C_5 M_{2,3b,3b}^{1/3}(\delta) \log(\delta^{-3b})^c D(\delta/\delta^b)^{2/3}.$$

Proof. By successive applications of the aforementioned lemmas, we find

$$\begin{aligned} M_{2,a,b}(\delta) &\leq C_2 M_{2,3b,b}(\delta) \leq C_2 C_4 M_{6,3b,b}(\delta)^{1/3} D(\delta/\delta^b)^{2/3} \\ &\leq C_2 C_3 C_4 M_{6,3b,3b}^{1/3}(\delta) \log(\delta^{-3b})^c D(\delta/\delta^b)^{2/3} \\ &\leq \underbrace{C_2 C_3 C_4}_{C_5} M_{2,3b,3b}^{1/3}(\delta) \log(\delta^{-3b})^c D(\delta/\delta^b)^{2/3}. \end{aligned}$$

□

To make the iteration effective, we initially divide the unit size intervals I, I' considered in $B(\delta)$ into ν^{-1} smaller intervals, and then use the previously established iteration. Let $\nu = \delta^b$. We choose ν such that $\nu = \delta^{1/3^N}$ such that in N iterations of Lemma 4.4 we reach the scale δ, where decoupling becomes trivial. We use the estimate

$$M_{6,0,0}(\delta) \leq C_6 \nu^{-\frac{1}{2}} M_{6,b,b}(\delta) \tag{21}$$

due to the Cauchy-Schwarz inequality:

$$\begin{aligned} &\Big(\int \big(\big| \sum_{J \in \mathcal{P}(I,\delta^b)} f_J \big|^6 * \phi_I \big) \big(\big| \sum_{J' \in \mathcal{P}(I',\delta^b)} f_{J'} \big|^6 * \phi_{I'} \big) \Big)^{\frac{1}{6}} \\ &\leq \sum_{\substack{J \in \mathcal{P}(I,\delta^b), \\ J' \in \mathcal{P}(I',\delta^b)}} \Big(\int \big(|f_J|^6 * \phi_I \big) \big(|f_{J'}|^6 * \phi_{I'} \big) \Big)^{1/6} \\ &\leq M_{6,b,b}^2(\delta) \sum_{\substack{J \in \mathcal{P}(I,\delta^b), \\ J' \in \mathcal{P}(I',\delta^b)}} \Big(\sum_{K \in \mathcal{P}(J,\delta)} \|f_K\|_{L^{12}(\mathbb{R}^3)}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K' \in \mathcal{P}(J',\delta)} \|f_{K'}\|_{L^{12}(\mathbb{R}^3)}^2 \Big)^{\frac{1}{2}} \\ &\leq M_{6,b,b}^2(\delta) C_6^2 \nu^{-1} \Big(\sum_{K \in \mathcal{P}(I,\delta)} \|f_K\|_{L^{12}(\mathbb{R}^3)}^2 \Big)^{\frac{1}{2}} \Big(\sum_{K' \in \mathcal{P}(I',\delta)} \|f_{K'}\|_{L^{12}(\mathbb{R}^3)}^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

By the decoupling result of Bourgain–Demeter–Guth [3] we have

$$D_3(\delta) \leq C_\varepsilon \delta^{-\varepsilon}, \tag{22}$$

which gives:

Lemma 4.5. *Let $N \in \mathbb{N}$. Suppose that $\delta \in 2^{\mathbb{Z}}$ and $\delta^{-\frac{1}{3^N}} \in \mathbb{N}$. Then the following estimate holds:*

$$D(\delta) \leq C_7 \delta^{\frac{\varepsilon}{3^N}\left(1 + \frac{2N}{3} - \frac{1}{2\varepsilon}\right)} \log(\delta^{-1})^{3c} C_\varepsilon^{1 - \frac{1}{3^N}} \delta^{-\varepsilon}. \tag{23}$$

Proof. We find iterating Lemma 4.4 N times:

$$M_{2,b,b}(\delta) \leq C_5^2 M_{6,3^N b, 3^N b}^{1/3^N} \log(\delta^{-1})^{2c} \prod_{j=0}^{N-1} D(\delta/\delta^{3^j b})^{(2/3) \cdot 1/3^j}. \tag{24}$$

From the bilinear reduction, we have (here we use $\delta \in 2^{\mathbb{Z}}$)

$$D_3(\delta) \leq C_1 \log(\delta^{-1}) M_{6,0,0}(\delta).$$

We reduce the scale in $M_{6,0,0}(\delta)$ to ν by (21) such that

$$D_3(\delta) \leq C_1 C_6 \nu^{-\frac{1}{2}} \log(\delta^{-1}) M_{6,b,b}(\delta).$$

Now we plug in (24) to find the following recursive estimate for the linear decoupling constant:

$$D_3(\delta) \leq \underbrace{C_1 C_5^2 C_6}_{C_7} \delta^{-\frac{b}{2}} \log(\delta^{-1})^{3c} \prod_{j=0}^{N-1} D(\delta/\nu^{3^j})^{\frac{2}{3}\cdot\frac{1}{3^j}}.$$

By (22), we find

$$\begin{aligned} D(\delta) &\leq C_7 \delta^{-\frac{b}{2}} \log(\delta^{-1})^{3c} \prod_{j=0}^{N-1} (C_\varepsilon \delta^{-\varepsilon(1-3^{j-N})})^{\frac{2}{3}\cdot\frac{1}{3^j}} \\ &= C_7 \delta^{-\frac{b}{2}} \log(\delta^{-1})^{3c} C_\varepsilon^{1-\frac{1}{3^N}} \delta^{-\varepsilon(1-\frac{1}{3^N})} \delta^{\varepsilon \frac{2N}{3\cdot 3^N}} \\ &= C_7 \delta^{\frac{\varepsilon}{3^N}(1+\frac{2N}{3}-\frac{1}{2\varepsilon})} \log(\delta^{-1})^{3c} C_\varepsilon^{1-\frac{1}{3^N}} \delta^{-\varepsilon}. \end{aligned}$$

□

In the next step, we choose $N = N(\varepsilon)$, which simplifies the above expression for $\delta \in 2^{\mathbb{Z}}$ and $\delta^{-\frac{1}{3^N}} \in \mathbb{N}$.

Lemma 4.6. *Let $0 < \varepsilon < \varepsilon_0 = \varepsilon_0(c)$, and $N \in \mathbb{N}$ such that*

$$1 + \frac{2N}{3} - \frac{1}{2\varepsilon} \in [\frac{2}{3}, 2]. \qquad (25)$$

For $\delta \in (\delta_n)_{n=n_0}^\infty$ with $\delta_n = 2^{-n3^{10N}}$, $n_0 = n_0(c)$, we have the following:

$$D_3(\delta) \leq C_7 C_\varepsilon^{1-\frac{1}{3^N}} \delta^{-\varepsilon}.$$

Proof. With the assumptions of Lemma 4.5 satisfied, we find by (24)

$$D_3(\delta) \leq C_7 \delta^{\frac{\varepsilon}{3^N}\left(1+\frac{2N}{3}-\frac{1}{2\varepsilon}\right)} \log(\delta^{-1})^{3c} C_\varepsilon^{1-\frac{1}{3^N}} \delta^{-\varepsilon}.$$

By (25) this simplifies to

$$D_3(\delta) \leq C_7 \log(\delta^{-1})^{3c} \delta^{\frac{2\varepsilon}{3^N\cdot 3}} C_\varepsilon^{1-\frac{1}{3^N}} \delta^{-\varepsilon}.$$

Since $\delta = 2^{-n\cdot 3^{10N}}$, we show that for $n \geq n_0(c)$ and $0 < \varepsilon < \varepsilon_0$

$$\log(\delta^{-1})^{3c} \delta^{\frac{2\varepsilon}{3^N\cdot 3}} \leq 1.$$

First we note that

$$\delta^{\frac{2\varepsilon}{3\cdot 3^N}} \leq \delta^{\frac{1}{3^{2N}}} \leq 2^{-n\cdot 3^{8N}}.$$

Here we use $\varepsilon \sim \frac{1}{N}$, and for $0 < \varepsilon < \varepsilon_0$, N becomes large enough to argue like in the above estimate. Moreover,

$$\log(\delta^{-1})^{3c} \leq n^{3c} 3^{30Nc} \log(2)^{3c} \leq n^{3c} 3^{30Nc}.$$

First, we see that

$$2^{\log(3)30Nc} \leq 2^{\frac{n}{2}3^{8N}}$$

by choosing $0 < \varepsilon < \varepsilon_0(c)$ small enough such that $30Nc\log(3) \leq 3^{8N}/2$ (since N becomes large enough such that the inequality holds). Secondly, we can choose $n \geq n_0(c)$ large enough such that

$$3\log_2(n)c \leq \frac{n}{2} \Rightarrow 2^{\log_2(n)3c} \leq 2^{\frac{n3^{8N}}{2}}.$$

Then we arrive at the claim

$$D_3(\delta) \leq C_7 C_\varepsilon^{1-\frac{1}{3^N}} \delta^{-\varepsilon}.$$

□

We use submultiplicativity to extend this estimate to all $\delta \in \mathbb{N}^{-1}$:

Lemma 4.7. *Let $0 < \varepsilon < \varepsilon_0 = \varepsilon_0(c)$ and $n_0 = n_0(c)$ such that Lemma 4.6 is valid. Then we find for all $\delta \in \mathbb{N}^{-1}$*

$$D_3(\delta) \leq C_8 2^{n_0 \cdot 3^{\frac{a}{\varepsilon}}} C_\varepsilon^{1-a/\varepsilon} \delta^{-\varepsilon} \tag{26}$$

for some a.

Proof. Let N be like in (25) and $\delta \in (\delta_n)_{n=n_0}^\infty = (2^{-n \cdot 3^{10N}})_{n=n_0}^\infty$. If $\delta \in (\delta_{n_0}, 1] \in \mathbb{N}^{-1}$, we use the trivial estimate

$$D_3(\delta) \leq \delta^{-1/2} \leq 2^{\frac{n_0}{2} \cdot 3^{10N}}.$$

If $\delta \in (\delta_{n+1}, \delta_n]$ for $n \geq n_0$, then submultiplicativity and Lemma 4.6 imply

$$\begin{aligned} D_3(\delta) &\leq D_3(\delta_{n+1}) \leq D_3(\delta_n) D_3(\delta_{n+1}/30\delta_n) \leq (C_7 C_\varepsilon^{1-1/3^N} \delta_n^{-\varepsilon})(30(\delta_n/\delta_{n+1}))^{1/2} \\ &= 30^{1/2} C_7 C_0^{1/2} 2^{\frac{1}{2} \cdot 3^{10N}} C_\varepsilon^{1-1/3^N} \delta^{-\varepsilon}. \end{aligned}$$

Taking the two estimates together gives

$$D(\delta) \leq C_8 2^{n_0 \cdot 3^{10N}} C_\varepsilon^{1-1/3^N} \delta^{-\varepsilon}.$$

This estimate holds for all $\delta \in \mathbb{N}^{-1}$. Now we simplify by monotonicity in N. By the choice of N, we have $3^N \leq 3^{a/\varepsilon}$ for some a and $\varepsilon < \varepsilon_0(c)$. We obtain

$$D(\delta) \leq C_8 2^{n_0 \cdot 3^{10a/\varepsilon}} C_\varepsilon^{1-\frac{1}{3^{a/\varepsilon}}} \delta^{-\varepsilon}.$$

□

We bootstrap this bound to find the following:

Lemma 4.8. *There is $\varepsilon_0 = \varepsilon_0(C_8, c)$ such that for all $0 < \varepsilon < \varepsilon_0(c)$ and $\delta \in \mathbb{N}^{-1}$, we have*

$$D_3(\delta) \leq 2^{3^{100a/\varepsilon}} \delta^{-\varepsilon}.$$

Proof. Let $P(C, \lambda)$ be the statement that $D(\delta) \leq C\delta^{-\lambda}$ for all $\delta \in \mathbb{N}^{-1}$. Lemma 4.7 implies that for $\varepsilon \in (0, \varepsilon_0(c))$ and $n_0 = n_0(c)$:

$$P(C_\varepsilon, \varepsilon) \Rightarrow P(C_8 \cdot 2^{n_0 \cdot 3^{10a/\varepsilon}} C_\varepsilon^{1-1/3^{a/\varepsilon}}, \varepsilon).$$

After M iterations of the above implication, we obtain

$$P(C_\varepsilon, \varepsilon) \Rightarrow P((C_8 \cdot 2^{n_0 \cdot 3^{10a/\varepsilon}})^{\sum_{j=0}^{M-1}(1-1/3^{a/\varepsilon})^j} C_\varepsilon^{(1-1/3^{a/\varepsilon})^M}, \varepsilon).$$

We can take limits

$$C_\varepsilon^{(1-1/3^{a/\varepsilon})^M} \to_{M \to \infty} 1, \quad \sum_{j=0}^{M-1} (1 - 1/3^{a/\varepsilon})^j \to_{M \to \infty} 3^{a/\varepsilon}.$$

Hence, letting $M \to \infty$, we obtain

$$P(C_8^{3^{a/\varepsilon}} \cdot 2^{n_0 \cdot 3^{11a/\varepsilon}}, \varepsilon).$$

By choosing $0 < \varepsilon < \varepsilon_0(C_8, n_0(c))$ we find for all $\delta \in \mathbb{N}^{-1}$

$$D(\delta) \leq C_8^{3^{a/\varepsilon}} 2^{n_0 \cdot 3^{11a/\varepsilon}} \delta^{-\varepsilon} \leq 2^{3^{100a/\varepsilon}} \delta^{-\varepsilon}.$$

This finishes the proof. □

In the following we fix $\varepsilon_0 = \varepsilon_0(C_8, c)$ and a such that Lemma 4.8 is valid.

4.3. **Proof of Theorem 1.1.** We can write for $0 < \varepsilon < \varepsilon_0$

$$D(\delta) \leq A^{A^{1/\varepsilon}} \delta^{-\varepsilon} \tag{27}$$

for some $A = A(a)$. It suffices to prove (3) with exponentials and logarithms based on A.

Proof of Theorem 1.1. We optimize (27) by choosing $\varepsilon = \varepsilon(\delta)$. Let

$$B = \log_A(1/\delta) > 1, \quad \eta = \log_A(B) - \log_A \log_A(B), \quad \varepsilon = 1/\eta. \tag{28}$$

This leads to the first constraint

$$\delta < A^{-1}. \tag{29}$$

The constraint on ε_0 translates to

$$\varepsilon = \frac{1}{\eta} \leq \varepsilon_0 \Rightarrow \frac{1}{\varepsilon_0} \leq \log_A(B/\log_A(B)) \leq \log_A(B) = \log_A(\log_A(1/\delta)).$$

This gives the condition on δ:

$$\delta < (A^{A^{1/\varepsilon_0}})^{-1} = \delta_0.$$

It is straight-forward by (28) that

$$A^{1/\varepsilon} \leq \varepsilon \log_A(1/\delta).$$

For this reason we obtain

$$A^{A^{1/\varepsilon}} \delta^{-\varepsilon} \leq 2 \exp_A(\varepsilon \log_A(1/\delta)) \leq 2 \exp_A\Big(\frac{2 \log_A(1/\delta)}{\log_A \log_A(1/\delta)}\Big).$$

□

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