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Wave phenomena
analysis and numerics

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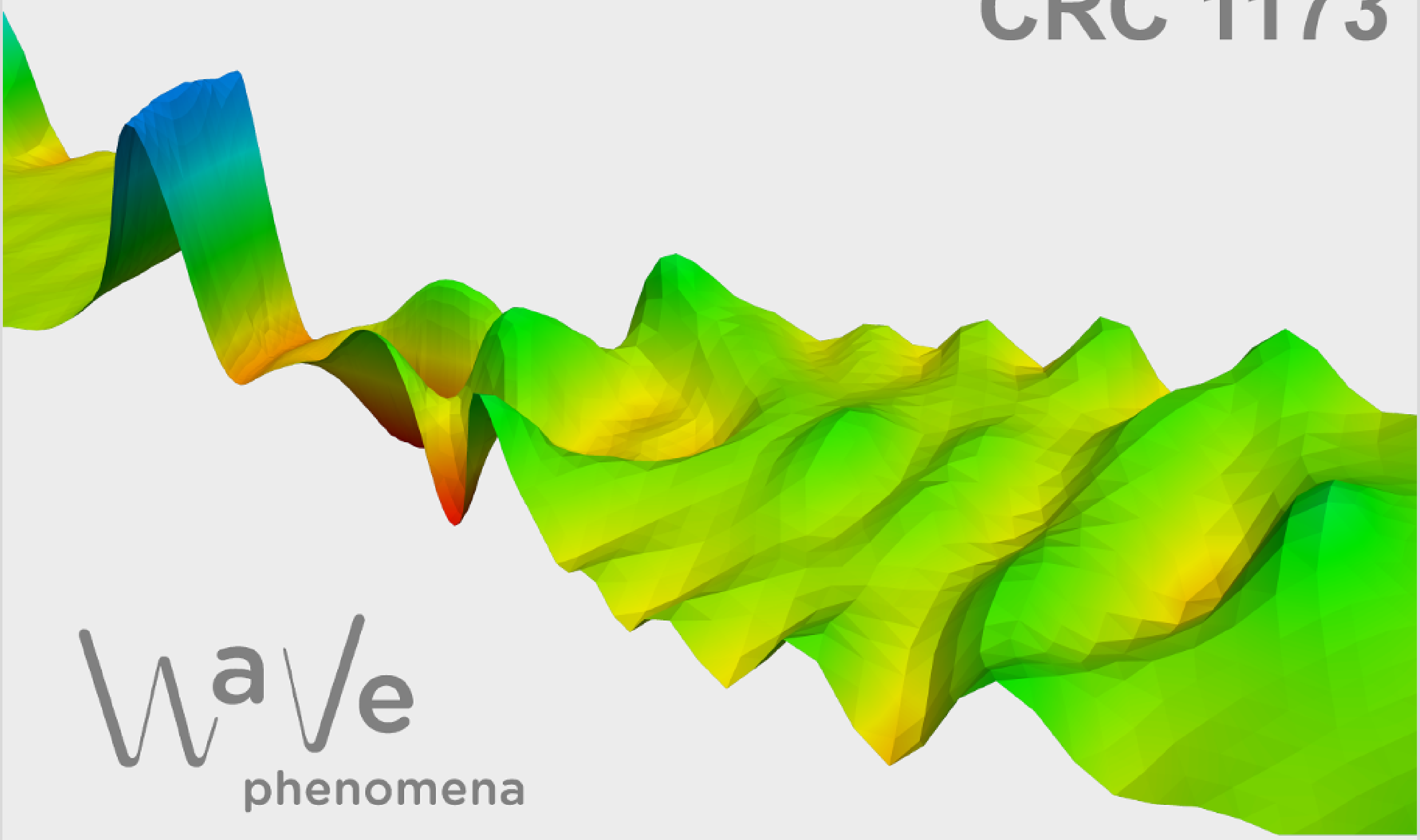

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ON ECHOES IN MAGNETOHYDRODYNAMICS WITH MAGNETIC DISSIPATION

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ABSTRACT. We study the long time asymptotic behavior of the inviscid magnetohydrodynamic equations with magnetic dissipation near a combination of Couette flow and a constant magnetic field. Here we show that there exist nearby explicit global in time low frequency solutions, which we call waves. Moreover, the linearized problem around these waves exhibits resonances under high frequency perturbations, called echoes, which result in norm inflation Gevrey regularity and infinite time blow-up in Sobolev regularity.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

In this article we consider the two-dimensional magnetohydrodynamic (MHD) equations with magnetic resistivity $\kappa > 0$ but without viscosity

$$
\begin{aligned}
&\partial_t V + V \cdot \nabla V + \nabla p = B \cdot \nabla B, \\
&\partial_t B + V \cdot \nabla B = \kappa \Delta B + B \cdot \nabla V, \\
&\operatorname{div}(B) = \operatorname{div}(V) = 0, \\
&(t, x, y) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R},
\end{aligned} \tag{1}
$$

near the stationary solution

$$
\begin{aligned}
V(t, x, y) &= (y, 0), \\
B(t, x, y) &= (\alpha, 0).
\end{aligned} \tag{2}
$$

The MHD equations are a common model of the evolution of conducting fluids interacting with (electro-)magnetic fields in regimes where the magnetization of the fluid can be neglected. They describe the evolution of the fluid in terms of the fluid velocity V, pressure p and magnetic field B. The constant mass and charge densities are normalized to 1. Here particular examples of applications range from the modeling of solar dynamics to geomagnetism and the earths molten core to using liquid metals in industrial applications or in fusion applications [Dav16].

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A main aim of this article is to analyze the long-time asymptotic behavior of solutions to this coupled system and, in particular, the interaction of instabilities, partial dissipation and the system structure of the equations. Here we note that due to the affine structure of the stationary solution (2), the corresponding linearized problem around this solution decouples in Fourier space and can be shown to be stable in arbitrary Sobolev (or even analytic) regularity, as we prove in Section 2.

Lemma 1. *Let $\alpha \in \mathbb{R}$ be given and consider the linear problem*

$$\begin{aligned} \partial_t V + y\partial_x V + (V_2, 0) &= \alpha \partial_x B, \\ \partial_t B + y\partial_x B - (B_2, 0) &= \kappa \Delta B + \alpha \partial_x V, \\ \mathrm{div}(B) = \mathrm{div}(V) &= 0, \\ (t, x) &\in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}. \end{aligned}$$

Then these equations are stable in H^s for any $s \in \mathbb{R}$ in the sense that there exists a constant $C > 0$ such that for any choice of initial data and all times $t > 0$ it holds that

$$\begin{aligned} &\|(\nabla^\perp \cdot V)(t, x - ty, y)\|_{H^s}^2 + \|(\nabla^\perp \cdot B)(t, x - ty, y)\|_{H^s}^2 \\ &\leq (1 + \kappa^{-2/3})^2 (\|\nabla^\perp \cdot V|_{t=0}\|_{H^s}^2 + \|\nabla^\perp \cdot B|_{t=0}\|_{H^s}^2). \end{aligned}$$

Here $\nabla^\perp \cdot V =: W$ is the vorticity of the fluid and $\nabla^\perp \cdot B =: J$ is the (magnetically induced) current.

In contrast to this to this very strong linear stability result, the stability results for the inviscid nonlinear equations are expected to crucially rely on very high, Gevrey regularity (see Section 2.2 for a definition). More precisely, similarly to the nonlinear Euler equations [DM18, DZ21, Zil21b] or Vlasov-Poisson equations [Bed20, Zil21a, MV11] the nonlinear equations are not a priori expected to not remain close to the linear dynamics due to "resonances" or "echoes"[MWGO68, YOD05], which may lead to unbounded norm inflation of any Sobolev norm. It is the main aim of this article to identify and capture this resonance mechanism for the resistive MHD equations. In particular, we ask to which extent magnetic dissipation can stabilize the dynamics. As we discuss in Section 2.2 the main nonlinear resonance mechanism is expected to be given by the repeated interaction of a high frequency perturbation with an underlying low frequency perturbation of (2). In this article we thus explicitly construct such low frequency nonlinear solutions, called *traveling waves* (a combination of an Alfvén waves and shear dynamics; see Section 2 and Lemma 5 for further discussion).

Lemma 2. *Let $\kappa > 0$ and $\alpha \in \mathbb{R}$ and let $(f_0, g_0) \in \mathbb{R}^2$. Then there exist smooth global in time solutions of the nonlinear, resistive MHD equations (1), which are of the form*

$$\begin{aligned} V(t, x, y) &= (y, 0) + \frac{f(t)}{1 + t^2} \nabla^\perp \sin(x - ty), \\ B(t, x, y) &= (\alpha, 0) + \frac{g(t)}{1 + t^2} \nabla^\perp \sin(x - ty), \end{aligned}$$

with $(f(0), g(0)) = (f_0, g_0)$. Furthermore, for a suitable choice of f_0, g_0 it holds that

$$\begin{aligned} f(t) &\to 2c, \\ g(t) &\to 0, \end{aligned}$$

as $t \to \infty$.

In view of the underlying shear dynamics it is natural to change to coordinates

$$(x - ty, y).$$

In these coordinates the corresponding vorticity $W = \nabla^{\perp} \cdot V$ and current $J = \nabla^{\perp} \cdot B$ read

$$W = -1 + f(t)\cos(x),$$
$$J = 0 - g(t)\sin(x).$$

Unlike the stationary solution (2) these waves have a non-trivial x-dependence. As we discuss in Section 2.2 this x-dependence allows resonances to propagate in frequency and underlies the nonlinear instability of the stationary solution (2). More precisely, we show that the (simplified) *linearized* equations around these waves exhibit the above mentioned *nonlinear* resonance mechanism (in terms of both upper and lower bounds on solutions). In particular, we aim to obtain a precise understanding of the dependence of the resonance mechanism on the resistivity $\kappa > 0$ and the frequency-localization of the initial perturbation. The research on well-posedness and asymptotic behavior of the magnetohydrodynamic equations is a very active field of research and we in particular mention the recent work [Lis20], which considers a related, fully dissipative setting in 3D, as well as the articles [JW22, ZZ22, WZ21, BLW20, FL19, DYZ19, HXY18, LCZL18, WZ17]. More precisely, in [Lis20] Liss studied the nonlinear, fully dissipative, three-dimensional MHD equations around the same stationary solution (2) in a doubly-periodic three-dimensional channel $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$ and established bounds on the Sobolev stability threshold as $\nu = \kappa \downarrow 0$. In contrast, this article considers the 2D setting with partial dissipation $\nu = 0$, $\kappa > 0$ in Gevrey regularity. Similar questions on the stability of systems with partial dissipation in critical spaces are also a subject of active research in other (fluid) systems, such as the Boussinesq equations [CW13, EW15, DWZZ18].

For simplicity of presentation and to simplify the analysis in this article we modify the linearized equations for the vorticity and current perturbations w, j

$$\begin{aligned}
\partial_t w &= \alpha \partial_x j - (2c \sin(x) \partial_y \Delta_t^{-1} w)_{\neq} \\
\partial_t j &= \kappa \Delta_t j + \alpha \partial_x w - 2\partial_x \partial_y^t \Delta_t^{-1} j, \\
\Delta_t &= \partial_x^2 + (\partial_y - t\partial_x)^2,
\end{aligned} \tag{3}$$

and fix the x-averages of w and j, which also fixes the underlying shear flow. Here, for simplicity we have also replaced $f(t), g(t)$ by $2c$ and 0, respectively. In analogy to other fluid systems [BBCZD21, BM15a], a similar structure of the equations can be achieved by considering the coordinates

$$(x - \int_0^t \int V_1 dx dt, \frac{1}{t} \int_0^t \int V_1 dx dt) =: (X, Y),$$

which however makes estimates of Δ_t^{-1} technically more involved and less transparent [Zil17]. In the interest of a clear presentation of the resonance mechanism we hence instead fix $Y = y$ by a small forcing.

Theorem 3. *Let $0 < \alpha < 10$ and $0 < \kappa < 1$ with $\beta := \frac{\kappa}{\alpha^2}$ and $c \leq \min(10^{-3}\beta^{\frac{16}{3}}, 10^{-4})$ be given. Consider the (simplified) linearized equations (3) around the wave of Lemma 2.*

Then there exists a constant C such that for any initial data w_0, j_0 whose Fourier transform satisfies

$$\sum_k \int \exp(C\sqrt{|\xi|})(|\mathcal{F}w_0(k,\xi)|^2 + |\mathcal{F}j_0(k,\xi)|^2) d\xi < \infty$$

the corresponding solution stays regular for all times up to a loss of constant in the sense that for all $t > 0$ it holds that

$$\sum_k \int \exp(\tfrac{C}{2}\sqrt{|\xi|})(|\mathcal{F}w(t,k,\xi)|^2 + |\mathcal{F}j(t,k,\xi)|^2)d\xi < \infty.$$

Moreover, there exists initial data w_0, j_0 and $0 < C^* < C$ such that

$$\sum_k \int \exp(C^*\sqrt{|\xi|})(|\mathcal{F}w_0(k,\xi)|^2 + |\mathcal{F}j_0(k,\xi)|^2)d\xi < \infty,$$

but so that the corresponding solution w, j grows unbounded in Sobolev regularity as $t \to \infty$.

Let us comment on these results:

- As we discuss in Section 2.2 the linearized equations around a traveling wave closely resemble the interaction of high and low-frequency perturbations in the nonlinear equations. These equations thus are intended to serve as slightly simplified model of the nonlinear resonance mechanism. We remark that in the full nonlinear problem the x-averages and hence the underlying shear dynamics change with time and the corresponding change of coordinates has to be controlled. For simplicity and clarity the present model instead fixes this change of coordinates.
- The Fourier integrability with a weight $\exp(C\sqrt{|\xi|})$ corresponds to Gevrey 2 regularity with respect to y. For simplicity of presentation the above results are stated with $L^2(\mathbb{T})$ regularity in x. All results also extend to more general Fourier-weighted spaces, such as H^N for any $N \in \mathbb{N}$ or suitable Gevrey or analytic spaces (see Definitions 8 and 9).
- The stability and norm inflation in Gevrey 2 regularity matches the regularity classes of the (nonlinear) Euler equations. In particular, the magnetic field and magnetic dissipation are shown to not be strong enough to suppress this growth. We remark that our choice of coupling between the size of the magnetic field and magnetic dissipation is made so that both effects are "of the same magnitude" and hence their interaction plays a more crucial role (see Section 2.3 for a discussion).
- These results complement the work of Liss [Lis20] on the Sobolev stability threshold in $3D$ with full dissipation. Indeed, the above derived upper and lower bounds establish Gevrey 2 as the optimal regularity class of the linearized problem in $2D$ with partial dissipation. We expect that as for the Euler [BM15b] or Vlasov-Poisson equations [BMM16] nonlinear stability results match the regularity classes of the linearized problem around appropriate traveling waves.

We further point out that the instability result of Theorem 3 also implies a norm inflation result for the nonlinear problem around each wave in slightly different spaces (see Corollary B.1). In particular in any arbitrarily small analytic neighborhood around the stationary solution (2) there exist nonlinearly unstable solutions (with respect to lower than Gevrey 2 regularity) .

The remainder of the article is structured as follows:

- In Section 2 we discuss the linearized problem around the stationary state (2) and introduce waves as low-frequency solutions of the nonlinear problem.
- In Section 2.2 we discuss the resonance mechanism for a toy model. In particular, we discuss optimal spaces for norm inflation and (in)stability results as well as the time- and frequency-dependence of resonances.
- The main results of this article are contained in Section 4, where we establish upper bounds and lower bounds on the norm inflation.

- The Appendix A contains some auxiliary estimates of a growth factor in Section 4. In the second Appendix B we prove a nonlinear instability result for the traveling waves.

2. Linear Stability, Traveling Waves and Echo Chains

In this section we establish the linear stability of the resistive MHD equations (1)

$$\begin{aligned} \partial_t V + (V\cdot\nabla)V + \nabla p &= (B\cdot\nabla)B, \\ \partial_t B + (V\cdot\nabla)B &= \kappa\Delta B + (B\cdot\nabla)V, \\ \nabla\cdot B = \nabla\cdot V &= 0, \end{aligned} \tag{4}$$

around the stationary solution (2) as stated in Lemma 1. Furthermore, we sketch the nonlinear resonance mechanism underlying the norm inflation result of Theorem 3, which is given by the repeated interaction of high and low frequency perturbations. This mechanism motivates the construction of the traveling wave solutions of Lemma 5 and the corresponding (simplified) linearized equations around these waves, which are studied in the remainder of the article.

In order to simplify notation we may restate the MHD equations with respect to other unknowns. That is, since we consider vector fields in two dimensions and V and B are divergence-free, we may introduce the magnetic potential Φ, magnetic current J and fluid vorticity W by

$$\begin{aligned} J &= \nabla^\perp\cdot B, \\ \Delta\Phi &= J, \\ W &= \nabla^\perp\cdot V. \end{aligned}$$

Under suitable decay assumptions (or asymptotics) in infinity the equations can then equivalently be expressed as

$$\begin{aligned} \partial_t W + (V\cdot\nabla)W &= (B\cdot\nabla)\Delta\Phi, \\ \partial_t\Phi + (V\cdot\nabla)\Phi &= \kappa\Delta\Phi, \end{aligned} \tag{5}$$

or in terms of J:

$$\begin{aligned} \partial_t W + (V\cdot\nabla)W &= (B\cdot\nabla)J, \\ \partial_t J + (V\cdot\nabla)J &= \kappa\Delta J + (B\cdot\nabla)W - 2(\partial_i V\cdot\nabla)\partial_i\Phi. \end{aligned} \tag{6}$$

With these formulations we are now ready to establish the linear stability of the stationary solution (2).

Proof of Lemma 1. Consider the formulation of the MHD equations as (5), then the linearization around $V=(y,0), W=-1, B=(\alpha,0), \Phi=\alpha y$ is given by

$$\begin{aligned} \partial_t W + y\partial_x W &= \alpha\partial_x\Delta\Phi, \\ \partial_t\Phi + y\partial_x\Phi + V_2\alpha &= \kappa\Delta\Phi. \end{aligned}$$

We note that all operators other than $y\partial_x$ are constant coefficient Fourier multipliers. Hence we apply a change of variables

$$(x,y)\mapsto(x-ty,y)$$

to remove this transport term and obtain

$$\begin{aligned} \partial_t w &= \alpha\partial_x\Delta_t\phi, \\ \partial_t\phi &= -\alpha\partial_x\Delta_t^{-1}w + \kappa\Delta_t\phi, \end{aligned}$$

where w,ϕ denote the unknowns with respect to these variables and $\Delta_t = \partial_x^2 + (\partial_y - t\partial_x)^2$. We note that this system decouples in Fourier space and for simplicity of notation express it in terms of the (Fourier transform of the) current $j = \Delta_t \phi$:

$$\partial_t w = ik\alpha j,$$
$$\partial_t j = \frac{2k(kt-\xi)}{k^2+(\xi-kt)^2} j - \kappa(k^2+(\xi-kt)^2)j + ik\alpha w,$$

where $k \in Z$ and $\xi \in \mathbb{R}$ denote the Fourier variables with respect to $x \in \mathbb{T}$ and $y \in \mathbb{R}$, respectively. Here and in the following, with slight abuse of notation, we reuse w and j to refer to the Fourier transforms of the vorticity and current perturbation. For $k = 0$ these equations are trivial and we hence in the following we may assume without loss of generality that $k \neq 0$. Furthermore, we note that the right-hand-side depends on ξ only in terms of $\frac{\xi}{k} - t$. Hence, by shifting time we may further assume that $\xi = 0$.

With this reduction we first note that by anti-symmetry for all $\alpha \in \mathbb{R}$ it holds that

$$\partial_t(|w|^2 + |j|^2)/2 = (\tfrac{2t}{1+t^2} - \kappa k^2(1+t^2))|j|^2.$$

We make a few observations:

- If $\kappa k^2 \geq 1$ the horizontal dissipation is sufficiently strong to absorb growth for all times.
- If $\kappa k^2 \leq 1$ is small, then for sufficiently large times $|t| \geq (k^2\kappa)^{-1/3}$ the right-hand-side is non-positive.
- It thus only remains to estimate the growth on the time interval $|t| \leq (k^2\kappa)^{-1/3}$, where

$$\partial_t(|w|^2 + |j|^2) \leq \frac{4(t)_+}{1+t^2}(|w|^2 + |j|^2).$$

The latter case can be bounded by an application of Gronwall's lemma and after shifting back in time it yields

$$|w(t)|^2 + |j(t)|^2 \leq (1 + (k^2\kappa)^{-2/3})^2(|w(0)|^2 + |j(0)|^2)$$

for all $t > 0$.

□

While the ground state is thus linearly stable in arbitrary Sobolev or even analytic regularity, nonlinear stability poses to be a much more subtle question with stronger regularity requirements.

2.1. **Wave-type Perturbations.** In order to investigate the stability of the MHD equations, it is a common approach to consider wave-type perturbation. Here a classical result considers perturbations around a constant magnetic field and a vanishing velocity field.

Lemma 4 ((c.f. [Alf42, Dav16])). *Consider the ideal MHD equations (i.e. $\kappa = 0$) in three dimensions linearized around a constant magnetic field $B = B_0 e_z$ and vanishing velocity field $V = 0$. Then a particular solution is of the form*

$$B = (B_1(t,z), 0, 0), V = (V_1(t,z), 0, 0)$$

where B_1 and V_1 are solutions of the wave equation

$$\partial_t^2 B_1 - B_0^2 \partial_z^2 B_1 = 0,$$
$$\partial_t^2 V_1 - B_0^2 \partial_z^2 V_1 = 0,$$

The linearized problem thus admits wave-type solutions propagating in the direction e_z of the constant magnetic field and pointing into an orthogonal direction. These solutions are known as *Alfvén waves* [Alf42].

Proof of Lemma 4. We make the ansatz that B and v only depend on t and z and express the linearized equations in terms of the current $J = \nabla \times B$ and vorticity $W = \nabla \times V$. Then the equations reduce to

$$\begin{aligned} \partial_t J &= B_0 \partial_z W, \\ \partial_t W &= B_0 \partial_z J. \end{aligned}$$

These equations are satisfied if both J and W solve a wave equation and are chosen compatibly. More precisely, two linearly independent solutions are given by

$$W = f(z + B_0 t) = J$$

and

$$W = g(z - B_0 t) = -J,$$

where f and g are arbitrary smooth function.

We remark that since $B = B(t,z)$ and $V = V(t,z)$ point into a direction orthogonal to the z-axis, they are divergence-free for all times. Finally, since both functions are independent of x it follows that all nonlinearities $V \cdot \nabla V, B \cdot \nabla V, V \cdot \nabla B, B \cdot \nabla B$ identically vanish, so these are also nonlinear solutions. □

In the following we consider the two-dimensional setting and extend this construction to also include an underlying affine shear flow. We call the resulting solutions *traveling waves* in analogy to dispersive equations and related constructions for fluids and plasmas [DZ21, Bed20, Zil21a, Zil21b, DM18]. As we sketch in Section 2.2 the non-trivial x-dependence of these waves will allow us to capture the main nonlinear norm inflation mechanism in the linearized equations around these waves (as opposed to linearizing around the stationary solutions (2)).

Lemma 5. *Let $\alpha \in \mathbb{R}$ and $\kappa \geq 0$ be given. Then for any choice of parameters $(f(0), g(0)) \in \mathbb{R}^2$ there exists a solution of (6) of the form*

$$\begin{aligned} W &= -1 + f(t)\cos(x - yt) \\ J &= -g(t)\sin(x - yt). \end{aligned} \tag{7}$$

We call such a solution a traveling wave.

We remark that this construction also allows for general profiles $h(t, x - ty)$ in place of $\cos(x - ty)$. This particular choice is made so that for $f(0)$ and $g(0)$ small, such a wave is an initially small, analytic perturbation of the stationary solution (2) and localized at low frequency.

Proof of Lemma 5. For easier reference we note that for this ansatz, we obtain

$$\begin{aligned} V &= (y, 0) + \tfrac{f(t)}{1+t^2}\sin(x - yt)(t, 1) \\ W &= -1 + f(t)\cos(x - yt) \\ B &= (\alpha, 0) + \tfrac{g(t)}{1+t^2}\cos(x - yt)(t, 1) \\ J &= -g(t)\sin(x - yt) \\ \Phi &= \alpha y + \tfrac{g(t)}{1+t^2}\sin(x - yt). \end{aligned}$$

Inserting this into the equation (6) the nonlinearities vanish due to the one-dimensional structure of the waves. Therefore, this ansatz yields a solution if and only if f and g solve the ODE system

$$\begin{aligned} f'(t) &= -\alpha g(t), \\ g'(t) &= -\kappa(1 + t^2)g(t) + \alpha f(t) + \tfrac{2t}{1+t^2} g(t). \end{aligned} \tag{8}$$

Thus by classical ODE theory for any choice of initial data there indeed exists a unique traveling wave solution. □

Given such a traveling wave we are interested in its behavior, in particular for large times, and how it depends on the choices of κ and α.

Lemma 6. *Let $\alpha > 0$ and $\kappa > 0$ then for any choice of initial data the solutions $f(t), g(t)$ of the ODE system (8)*

$$\begin{aligned} f'(t) &= -\alpha g(t), \\ g'(t) &= -\kappa(1+t^2)g(t) + \alpha f(t) + \tfrac{2t}{1+t^2}g(t), \end{aligned} \tag{9}$$

satisfy the following estimates:

$$\begin{cases} |f(t)|^2 + |g(t)|^2 \leq (1+t^2)^2(|f(0)|^2 + |g(0)|^2) & \text{if } 0 < t < \kappa^{-1/3} \\ |f(t)|^2 + |g(t)|^2 \leq |f(\kappa^{-1/3})|^2 + |g(\kappa^{-1/3})|^2 & \text{if } t > \kappa^{-1/3}. \end{cases} \tag{10}$$

Furthermore, for a specific choice of initial data it holds

$$|f(t) - \epsilon| \leq \frac{1}{2}\epsilon,$$

for all $t \geq 4\beta^{-1}$ and

$$|g(t)| \to 0$$

as $t \to \infty$.

Proof of Lemma 6. We first observe that by anti-symmetry of the coefficients it holds that

$$\partial_t(|f|^2 + |g|^2) = 2|g|^2\left(-\kappa(1+t^2) + \tfrac{2t}{1+t^2}\right).$$

In particular, for $t > \kappa^{-1/3}$ the last factor is negative and hence $|f|^2 + |g^2|$ is non-increasing. For times smaller than this, we may derive a first rough bound from the estimate

$$\partial_t(|f|^2 + |g|^2) \leq (|f|^2 + |g|^2)\tfrac{4t}{1+t^2},$$

which yields an algebraic lower and upper bound growth bound. We next turn to the case of special data, due to lower and upper norm bounds (9) is time reversible. Therefore, we can obtain

$$f(t_0) = 1, \qquad g(t_0) = 0$$

for $t_0 = 4\beta^{-1}$. Then we deduce

$$g(t) = \alpha \int_{t_0}^{t} d\tau \ \exp(-\kappa(t - \tau + \tfrac{1}{3}(t^3 - \tau^3)))\tfrac{1+t^2}{1+\tau^2}f(\tau)$$

and thus

$$\begin{aligned} f(t) &= 1 - \alpha\int_{t_0}^{t} d\tau_1 g(\tau_1) \\ &= 1 - \tfrac{\kappa}{\beta}\int_{t_0}^{t} d\tau_1(1+\tau_1^2)\int_{t_0}^{\tau_1} d\tau_2 \ \exp(-\kappa(\tau_1 - \tau_2 + \tfrac{1}{3}(\tau_1^3 - \tau_2^3)))\tfrac{1}{1+\tau_2^2}f(\tau_2) \\ &= 1 - \tfrac{1}{\beta}\int_{t_0}^{t} d\tau_2 \tfrac{1}{1+\tau_2^2}f(\tau_2)(1 - \exp(-\kappa(t - \tau_2 + \tfrac{1}{3}(t^3 - \tau_2^3)))). \end{aligned}$$

This gives the estimate

$$0 \leq 1 - f(t) \leq \tfrac{2}{\beta t_0}, \tag{11}$$

which implies that after time t_0 the value of f satisfies the same bound. Similarly, for g we recall that

$$\partial_s g = (\tfrac{2t}{1+t^2} - \kappa(1+t^2))g(t) + \alpha f(t)$$

and hence for $t_1 = 2\kappa^{-\frac{1}{3}}$ it holds that

$$g(t_1) \leq \alpha\tfrac{t_1^2}{t_0^2}.$$

Furthermore, this implies that for times $t \geq t_1$ it holds that

$$\begin{aligned} g(t) &\leq \alpha \tfrac{t_1^2}{t_0^2} \exp(-\tfrac{\kappa}{3} t^2 (t-t_1)) + \alpha \int_{t_1}^{t} \exp(-\tfrac{\kappa}{3}(1+t^2)(t-\tau)) \\ &\leq \alpha \tfrac{t_1^2}{t_0^2} \exp(-\tfrac{\kappa}{3} t^2 (t-t_1)) + \tfrac{3\alpha}{\frac{\kappa}{3}(1+t)^2}. \end{aligned}$$

Finally, for times $t \gg \kappa^{-\frac{1}{3}}$ we may estimate

$$g \leq \tfrac{4\alpha}{\kappa t^2}. \tag{12}$$

□

2.2. **Paraproducts and an Echo Model.** As mentioned in Section 1 the main mechanism for nonlinear instability is expected to be given by the repeated interaction of high- and low-frequency perturbations of the stationary solution (2). In the following we introduce a model highlighting the role of the traveling waves and discuss what stability and norm inflation estimates can be expected.

For this purpose we note that the nonlinear MHD equations (5) for the perturbations w, ϕ of the groundstate (2) in coordinates $(x-ty, y)$ can be expressed as

$$\begin{aligned} \partial_t w + \nabla^\perp \Delta_t^{-1} w \cdot \nabla w &= \alpha \partial_x \Delta \phi + \nabla^\perp \phi \cdot \nabla \Delta_t \phi, \\ \partial_t \phi + \nabla^\perp \Delta_t^{-1} w \cdot \nabla \phi &= \alpha \partial_x \Delta^{-1} w + \kappa \Delta_t \phi, \\ \Delta_t &= \partial_x^2 + (\partial_y - t\partial_x)^2, \end{aligned} \tag{13}$$

where we used cancellation properties of $\nabla^\perp \cdot \nabla$. The stability result of Lemma 1 considered the linearized problem around the trivial solution $(0, 0)$, which removes all effects of the nonlinearities. In order to incorporate these effects into our model we thus consider the nonlinear equations as a coupled system for the low frequency part of the solution (w_{low}, ϕ_{low}) (defined as the Littlewood-Payley projection to frequencies $< N/2$ for some dyadic scale N) and the high frequency part (w_{hi}, ϕ_{hi}). If we for the moment consider the low frequency part as given then the action of the nonlinearities on the high frequency perturbation of the vorticity can be decomposed as

$$\nabla^\perp \Delta_t^{-1} w_{low} \cdot \nabla w_{hi} + \nabla^\perp \Delta_t^{-1} w_{hi} \cdot \nabla w_{low} + \nabla^\perp \Delta_t^{-1} w_{hi} \cdot \nabla w_{hi}. \tag{14}$$

Here the first term is of transport type and hence unitary in L^2 and we expect $\nabla^\perp \Delta_t^{-1} w_{low}$ to decay sufficiently quickly in time that this term should not yield a large contribution to possible norm inflation. Similarly for the last term we note that both factors are at comparable frequencies and that we by assumption consider a small high frequency perturbation and thus this term is also not expected to have a large impact on the evolution. The main norm inflation mechanism thus is expected to be given by the high frequency velocity perturbation interacting with a non-trivial low frequency vorticity perturbation.

In order to build our toy model we thus focus on this part and formally replace w_{low}, ϕ_{low} by the traveling waves, which are solutions of the nonlinear problem. Furthermore, as a simplification by a similar reasoning as above we also fix the underlying shear flow for our model. Then the equations for the (high frequency part of the) current perturbation
$j = \Delta \phi$ and vorticity perturbation w read

$$\begin{aligned} \partial_t w &= \alpha \partial_x j - (2c \sin(x) \partial_y \Delta_t^{-1} w)_{\neq} \\ \partial_t j &= \kappa \Delta_t j_{\neq} + \alpha \partial_x w - 2\partial_x \partial_y^t \Delta_t^{-1} j, \end{aligned} \tag{15}$$

where we also simplified to $f(t) = 2c, g(t) = 0$.

We note that compared to the linearization around the stationary solution these equations break the decoupling in Fourier space. Indeed taking a Fourier transform and relabeling $j \mapsto -ij$ we arrive at

(16)
$$\partial_t w(k) = -\alpha k j(k) - c\tfrac{\xi}{(k+1)^2}\tfrac{1}{1+(\frac{\xi}{k+1}-t)^2} w(k+1) + c\tfrac{\xi}{(k-1)^2}\tfrac{1}{1+(\frac{\xi}{k-1}-t)^2} w(k-1),$$
$$\partial_t j(k) = (2\tfrac{t-\frac{\xi}{k}}{1+(\frac{\xi}{k}-t)^2} - \kappa k^2(1+(\tfrac{\xi}{k}-t)^2))j(k) + \alpha k w(k).$$

Furthermore, if $t \approx \frac{\xi}{k}$ then the additional term is of size $c\frac{\xi}{(k)^2}$ and hence can possibly lead to a very large change of the dynamics. In reference to the experimental results mentioned in Section 1 we can interpret this as the low frequency and high frequency perturbation resulting in an "echo" around the time $t \approx \frac{\xi}{k}$. For the following toy model we neglect all modes except those at frequency k and $k-1$ and only include the action of the resonant mode k on the non-resonant mode $k-1$.

Lemma 7 (Toy model)**.** *Let c, κ, α be as in Theorem 3 such that $\beta = \frac{\kappa}{\alpha^2} \geq \pi$ and consider the Fourier variables $k \geq 2$ and $\xi \geq 10 \max(\kappa^{-1}, \frac{k^2}{c})$. Then for*

$$t_k := \frac{1}{2}(\frac{\xi}{k+1} + \frac{\xi}{k}) < t < \frac{1}{2}(\frac{\xi}{k} + \frac{\xi}{k-1}) =: t_{k-1}$$

we consider the toy model

(17)
$$\begin{aligned}
\partial_t w(k) &= -\alpha k j(k),\\
\partial_t j(k) &= (2\tfrac{t-\frac{\xi}{k}}{1+(\frac{\xi}{k}-t)^2} - \kappa k^2(1+(\tfrac{\xi}{k}-t)^2))j(k) + \alpha k w(k),\\
\partial_t w(k-1) &= -\alpha(k-1)j(k-1) + c\tfrac{\xi}{k^2}\tfrac{1}{1+(\frac{\xi}{k}-t)^2} w(k),\\
\partial_t j(k-1) &= -\kappa\frac{\xi^2}{k^2} j(k-1) + \alpha(k-1)w(k-1).
\end{aligned}$$

Then for initial data $w(k, t_k) = 1$ and $w(k-1, t_k) = j(k, t_k) = j(k-1, t_k) = 0$ we estimate

$$(|w(k)| + |w(k-1)| + \alpha k|j(k)| + \alpha(k-1)|j(k-1)|)|_{t=t_{k-1}} \leq 2\pi c\frac{\xi}{k^2}.$$

Furthermore, this bound is attained up to a loss of constant in the sense that

$$|w(k-1, t_{k-1})| \geq \tfrac{\pi}{2} c\frac{\xi}{k^2}.$$

Proof of Lemma 7. We perform a shift in time such that $t = \frac{\xi}{k} + s$ and thus $s_0 := -\frac{\xi}{2}\frac{1}{k^2+k} \leq s \leq \frac{\xi}{2}\frac{1}{k^2-k} =: s_1$. Integrating the equations in time, for our choice of initial data we obtain that

$$j(s,k) = \alpha k \int_{s_0}^{t} \tfrac{1+s^2}{1+\tau_2^2} \exp(-\kappa k^2(s - \tau_2 + \tfrac{1}{3}(s^3 - \tau_2^3)))w(k, \tau_2)d\tau_2$$

and thus

$$\begin{aligned}
w(k,s) &= 1 - \alpha k \int j(\tau_1, k)\, d\tau_1\\
&= 1 - \alpha^2 k^2 \int_{s_0}^{s}\int_{s_0}^{\tau_1} \tfrac{1+\tau_1^2}{1+\tau_2^2} \exp(-\kappa k^2(\tau_1 - \tau_2 + \tfrac{1}{3}(\tau_1^3 - \tau_2^3)))w(k, \tau_2)d\tau_2 d\tau_1.
\end{aligned}$$

For the second term we insert $\alpha^2 = \frac{\kappa}{\beta}$ and deduce that

$$\begin{aligned}
&\frac{\kappa}{\beta}k^2 \int_{s_0}^{s}\int_{s_0}^{\tau_1} \tfrac{1+\tau_1^2}{1+\tau_2^2}\exp(-\kappa k^2(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))w(k,\tau_2)d\tau_2 d\tau_1 \\
&\quad = \frac{1}{\beta}\int_{s_0}^{s} \tfrac{1}{1+\tau_2^2}\int_{\tau_2}^{s} \kappa k^2(1+\tau_1^2)\exp(-\kappa k^2(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))w(k,\tau_2)d\tau_1 d\tau_2 \\
&\quad = \frac{1}{\beta}\int_{s_0}^{s} \tfrac{w(k,\tau_2)}{1+\tau_2^2}\left[\exp(-\kappa k^2(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))\right]_{\tau_1=\tau_2}^{\tau_1=s} d\tau_2 \\
&\quad = \frac{1}{\beta}\int_{s_0}^{s} \tfrac{w(k,\tau_2)}{1+\tau_2^2}(1-\exp(-\kappa k^2(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3))))d\tau_2.
\end{aligned}$$

This further yields that

$$w(k,s) = 1 - \frac{1}{\beta}\int_{s_0}^{s} d\tau_2 \tfrac{w(k,\tau_2)}{1+\tau_2^2}(1-\exp(-\kappa k^2(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))). \tag{18}$$

Therefore, if $1 \geq w(k,s) \geq 0$ we obtain

$$|w(k,s)-1| \leq \frac{1}{\beta}\int_{s_0}^{s} \tfrac{1}{1+\tau_2^2}d\tau_2 \leq \tfrac{1}{\beta}(\arctan(s)+\tfrac{\pi}{2}).$$

and by bootstrap this assumption holds for all times if $\beta \geq \pi$. For the current $j(k)$ we similarly estimate

$$\begin{aligned}
&\int_{s_0}^{s_1} \tfrac{1+s^2}{1+\tau_2^2}\exp(-\kappa k^2(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))w(k,\tau_2)d\tau_2 \\
&\quad \leq (\int_{s_0}^{\frac{\xi}{5k^2}} + \int_{\frac{\xi}{5k^2}}^{s_1})\exp(-\kappa k^2(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))d\tau_2 \\
&\quad \leq \tfrac{1}{\kappa k^2}(\exp(-\kappa\xi(\tfrac{1}{3}+3^{-4}\tfrac{\xi^2}{k^4}))+\tfrac{4}{\eta^2}) \leq \tfrac{c}{\kappa k^2},
\end{aligned}$$

which yields

$$\alpha k j(s,k) \leq (\alpha k)^2 \tfrac{c}{\kappa k^2} = \tfrac{c}{\beta}.$$

Concerning the $k-1$ mode we argue similarly and write

$$j(k-1) = \alpha(k-1)\int \exp(\kappa\tfrac{\xi^2}{k^2}(s-\tau))w(k-1)d\tau$$

and

$$\begin{aligned}
w(k-1) &= c\tfrac{\xi}{k^2}\int \tfrac{1}{1+\tau^2}w(k)d\tau - \alpha(k-1)\int j(k-1)\ d\tau_1 \\
&= c\tfrac{\xi}{k^2}\int \tfrac{1}{1+\tau^2}w(k)d\tau \\
&\quad - \alpha^2(k-1)^2\iint \exp(\kappa\tfrac{\xi^2}{k^2}(\tau_1-\tau_2))w(\tau_2,k-1)d\tau_2 d\tau_1 \\
&= c\tfrac{\xi}{k^2}\int_{s_0}^{s_1} \tfrac{1}{1+\tau^2}w(k)d\tau - \tfrac{\alpha^2k^2(k-1)^2}{\kappa\xi^2}\int d\tau_2 w(\tau_2,k-1).
\end{aligned}$$

Since

$$|\tfrac{\alpha^2k^2(k-1)^2}{\kappa\xi^2}\int d\tau_2| \leq \tfrac{1}{\beta\frac{\xi}{k^2}}$$

we deduce by bootstrap that

$$|w(k-1)| \leq 2\pi c\tfrac{\xi}{k^2}$$

and thus

$$\begin{aligned}|w(k-1)-\pi c\tfrac{\xi}{k^2}| &\leq c\tfrac{\xi}{k^2}\int_{\tau\notin[s_0,s_1]}\tfrac{1}{1+\tau^2}d\tau \\ &\quad + c\tfrac{\xi}{k^2}\tfrac{1}{\beta}\int_{s_0}^{s_1}\tfrac{1}{1+\tau^2}(\arctan(\tau)-\tfrac{\pi}{2})d\tau + \tfrac{1}{\beta\frac{\xi}{k^2}}2\pi c\tfrac{\xi}{k^2} \\ &\leq \pi c\tfrac{\xi}{k^2}(\tfrac{2}{\pi}\tfrac{k^2}{\xi}+\tfrac{\pi}{2\beta}+\tfrac{2}{\beta\frac{\xi}{k^2}}) \\ &\leq \tfrac{\pi}{2}c\tfrac{\xi}{k^2}\end{aligned}$$

and

$$\begin{aligned}\alpha(k-1)j(k-1) &\leq 2\pi c\tfrac{\xi}{k^2}(\alpha(k-1))^2\int \exp(\kappa\tfrac{\xi^2}{k^2}(s-\tau))d\tau \\ &= 2\pi c\tfrac{\xi}{k^2}(\alpha(k-1))^2\tfrac{1}{\kappa\frac{\xi^2}{k^2}} \\ &= \pi c\tfrac{1}{\beta\xi} \ll \pi c\tfrac{\xi}{k^2}.\end{aligned}$$

□

Based on this model we may thus expect that a repeated interaction or *chain of resonances* starting at k_0

$$k_0 \mapsto k_0-1 \mapsto \dots \mapsto 1$$

results in a possible growth

$$|w(1,t_1)| \geq |w(k_0,t_{k_0})|\prod_{k=1}^{k_0} C'(1+c\frac{\xi}{k^2}),$$

where $C' = C'(\beta)$. Choosing $k_0 \approx \sqrt{C'c\xi}$ to maximize this product and using Stirling's approximation formula we may estimate this growth by an exponential factor:

$$\prod_{k=1}^{k_0} C'c\frac{\xi}{k^2} = \frac{(C'c\xi)^{k_0}}{(k_0!)^2} \approx \exp(\sqrt{C'c\xi})$$

This suggests that stability can only be expected if the initial decays in Fourier space with such a rate, which is naturally expressed in terms of Gevrey spaces.

Definition 8. Let $s \geq 1$, then a function $u \in L^2(\mathbb{T}\times\mathbb{R})$ belongs to the *Gevrey class* $\mathcal{G}_s$ if its Fourier transform satisfies

$$\sum_k \int \exp(C|\xi|^{1/s})|\mathcal{F}(u)(k,\xi)|^2 d\xi < \infty$$

for some constant $C > 0$.

In view of the more prominent role of the frequency with respect to y and for simplicity of notation this definition only includes $|\xi|^{1/s}$ as opposed to $(|k|+|\xi|)^{1/s}$ in the exponent. All results in this article also extend to more general Fourier weighted spaces X (see Definition 9) with respect to x with norms

$$\sum_k \int \exp(C|\xi|^{1/s})\lambda_k|\mathcal{F}(u)(k,\xi)|^2 d\xi.$$

We remark that any Gevrey function is also an element of H^N for any $N \in \mathbb{N}$ and that Gevrey classes are nested with the strongest constraint, $s = 1$, corresponding to analytic regularity with respect to y.

As the main result of this article and as summarized in Theorem 3 we show that the above heuristic model's prediction is indeed accurate and that the optimal regularity class for the (simplified) linearized MHD equations around a traveling wave are given by Gevrey 2.

2.3. **Magnetic Dissipation, Coupling and the Influence of β.** In the preceding proof we have seen that the interaction of interaction of $w(k)$ and $j(k)$ is determined by the combination of the action of the underlying magnetic field of size α and magnetic resistivity $\kappa > 0$ through the parameter

$$\beta = \tfrac{\kappa}{\alpha^2}.$$

More precisely, we recall that ignoring the influence of neighboring modes $w(k)$ and $j(k)$ are solutions of a coupled system:

$$\begin{aligned}
\partial_t w(k) &= -\alpha k j(k), \\
\partial_t j(k) &= \big(2\tfrac{t-\frac{\xi}{k}}{1+(\frac{\xi}{k}-t)^2} - \kappa k^2(1+(\tfrac{\xi}{k}-t)^2)\big)j(k) + \alpha k w(k).
\end{aligned}$$

Hence starting with data $w(k, s_0) = 1, j(k, s_0) = 0$ three different mechanisms interact to determine the size of $w(k, s)$:

- The vorticity $w(k, s)$ by means of the constant magnetic field generates a current perturbation $j(k, s)$.
- The current perturbation $j(k, s)$ is damped by the magnetic resistivity.
- The current $j(k, s)$ in turn by means of the constant magnetic field acts on the vorticity and damps it.

In this system several interesting regimes may arise, which are distinguished by the parameter β.

In the limit of infinite dissipation, $\beta \to \infty$, the current is rapidly damped and the system hence formally reduces to the Euler equations

$$\begin{aligned}
\partial_t w(k) &= 0, \\
j(k) &= 0,
\end{aligned}$$

where $w(k, s)$ remains constant in time.

As the opposite extremal case, if $\beta \downarrow 0$ we obtain the inviscid MHD equations and the system

$$\begin{aligned}
\partial_s w(k) &= -\alpha k j(k), \\
\partial_s j(k) &= 2\tfrac{s}{1+s^2} j(k) + \alpha k w(k).
\end{aligned}$$

Hence at least for $|s|$ large this suggests that

$$w(k) \approx c_1(1+s)\cos(\alpha k s), \quad j(k) \approx c_1(1+s)\sin(\alpha k s).$$

In particular, in stark contrast to the Euler equations (i.e. $\alpha = 0$) for the inviscid MHD equations with a magnetic field the vorticity $w(k)$ and current perturbations $j(k)$ cannot be expected to remain close to 1 and 0, respectively.

This article considers the regime $0 < \beta < \infty$, where the interaction of both extremal phenomena results in behavior which is qualitatively different from both limiting cases. Indeed, recall that by a repeated application of Duhamel's formula $w(k, s)$ satisfies the integral equation (18):

$$w(k, s) = 1 - \frac{1}{\beta}\int_{s_0}^{s} \tfrac{w(k,\tau_2)}{1+\tau_2^2}(1 - \exp(-\kappa k^2(s - \tau_2 + \tfrac{1}{3}(s^3 - \tau_2^3))))d\tau_2$$

Hence, as a first case which we also discussed in the toy model of Lemma 7, if we restrict to $\beta \geq \pi$ then the integral term is bounded and small

$$\frac{1}{\beta}\int_{s_0}^{s} d\tau_2 \tfrac{1}{1+\tau_2^2} \leq 1.$$

Hence, for large β the integral term can be treated as a perturbation and $w(k, s)$ remains comparable to 1 uniformly in s and thus close to the Euler case. However, unlike for the Euler equations the evolution of the current remains non-trivial.

If instead $0 < \beta < \pi$ we obtain different behaviour depending on the dissipation κk^2, the size of the magnetic field and the frequencies considered, whose interaction determines the behavior of the solution. More precisely, considering the integrand

$$\frac{1}{1+\tau_2^2}(1-\exp(-\kappa k^2(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))),$$

we observe that for $\kappa k^2 \gg 1$ large the magnetic dissipation is very strong and hence the integrand is well-approximated by $\frac{1}{1+\tau_2^2}$. In particular, this suggests that for these s it holds that

$$\begin{aligned} w(k,s) &\approx 1-\frac{1}{\beta}\int_{s_0}^{s} d\tau_2 \tfrac{w(k,\tau_2)}{1+\tau_2^2}, \\ \Leftrightarrow\ \partial_s w(k,s) &\approx -\tfrac{1}{\beta}\tfrac{1}{1+s^2} w(k,s), \\ \Leftrightarrow\quad w(k,s) &\approx \exp(-\tfrac{1}{\beta}(\arctan(s)+\tfrac{\pi}{2}))w(k,s_0), \end{aligned}$$

and hence $w(k,s)$ might decay by a factor comparable to $\exp(-\frac{\pi}{\beta})$.

If instead $\kappa k^2 \leq 1$ is small, different effects interact and involve the following natural time scales:

- Mixing enhanced magnetic dissipation becomes relevant on time scales $(\kappa k^2)^{-1/3} \gg 1$.
- The resonant interval I^k is of size about $\frac{\xi}{k^2}$.
- Within this resonant interval most of the L^1 norm of $\frac{1}{1+\tau_2^2}$ is achieved on a much smaller sub-interval of size about 1.

Hence, for times $|s| < s^* \ll (\kappa k^2)^{-1/3}$ which are small compared to the disspation time scale the integrand is small and we may therefore expect that

$$w(k,s) \approx 1$$

remains constant. If we instead consider very large times $|s| \gg (\kappa k^2)^{-1/3} \gg s^*$ in view of the exponential factor and the decay of $\frac{1}{1+\tau_s^2}$ the size of $w(k,s)$ should largely be determined by the action of the time interval $(-s^*, s^*)$, that is

$$\begin{aligned} w(k,s) &\approx 1-\frac{1}{\beta}\int_{-s^*}^{s^*} \tfrac{1}{1+\tau_2^2} d\tau_2 \\ &\approx 1-\tfrac{\pi}{\beta}, \end{aligned}$$

provided such such s exist, that is if the size $\frac{\xi}{k^2}$ of I^k is much bigger than the dissipative time scale. In particular, the size of $w(k,s)$ transitions from being close to 1 for $|s| < s^*$ to being very far from 1 for $|s| \gg (\kappa k^2)^{-1/3}$ and further needs to be controlled on intermediate time scales. These different regimes all have to be considered in the upper and lower bounds of Section 4 and we in particular need to control the size of $w(k,s)$ in order to estimate the resulting norm inflation due to resonances. For this purpose we estimate $w(k,s)$ in terms of a growth factor L such that

$$|w(k,s)| \leq L w(k,s_0),$$

as we discuss in Appendix A. For our upper bounds we will require that $cL \ll 1$ is sufficiently small to control back-coupling estimates.

3. Stability for Small and Large Times

In this section we establish some general estimates on the (simplified) linearized MHD equations (16). We note that these equations decouple with respect to ξ. In the following we hence treat ξ as an arbitrary but fixed parameter of the equations and consider (16) as an evolution equation for the sequences $w(\cdot,\xi,t)$ and $j(\cdot,\xi,t)$. As mentioned following the statement of Theorem 3 in addition to $\ell^2(\mathbb{Z})$ all proofs in the remainder of the article hold for a rather general family of weighted spaces:

Definition 9. Consider a weight function $\lambda_l > 0$ such that

$$\sup_l \tfrac{\lambda_{l\pm 1}}{\lambda_l} =: \hat{\lambda} < 10.$$

Then we define the Hilbert space X associated to this weight function as the set of all sequences $u : \mathbb{Z} \to \mathbb{C}$ such that $(u_l \lambda_l)_l \in \ell^2$.

This definition for instance includes ℓ^2 ($\lambda_l = 1$), (Fourier transforms of) Sobolev spaces H^s ($\lambda_l = 1 + C|l|^{2s}$ with $C > 0$ sufficiently small) or Gevrey regular or analytic functions with a suitable radius of convergence.

As sketched in Section 2.2 for a given frequency $\xi \in \mathbb{R}$ we expect the norm inflation for evolution by (16) to be concentrated around times $t_k \approx \frac{\xi}{k}$ for suitable $k \in \mathbb{Z}$. In particular, if the time is too large, $t > 2\xi$, there exists no such k and we expect the evolution to be stable. Similarly, if t is small also the size of the resonance predicted by the toy model is small and we again expect the evolution to be stable. The results of this section show that this heuristic is indeed valid and establish stability for "small" and "large" times. The essential difficulty in proving Theorem 3 thus lies in control the effects of resonances in the remaining time intervals, which are studied in Section 4. In the following we will often write L^∞_t as the supremum norm till time t.

Lemma 10 (Large time)**.** *Consider the equation* (16) *on the time interval* $(2\xi, \infty)$*. Then the possible norm inflation is controlled uniformly in time*

$$\|w, j\|_X(t) \leq \tfrac{1}{1-4c} \tfrac{1}{1-2c\hat{\lambda}} \|w, j\|_X(2\xi),$$

where $\hat{\lambda} = \max_l \frac{\lambda_l}{\lambda_{l\pm 1}}$ *is as in Definition 9.*

Proof. Let $\hat{w}(k) = |(w(k), j(k))|$. Then we infer

$$\tfrac{1}{2}\partial_t \hat{w}^2(k) \leq (a(k-1)w(k-1) - a(k+1)w(k+1))w(k) + b(k)j(k)^2,$$

where we introduced the short-hand notation a, b for the coefficient functions. Since $b(t,k) \leq 0$ for $t \geq 2\xi$, we further deduce that

$$\begin{aligned}
&\tfrac{1}{2}\partial_t \hat{w}^2(k) \leq c\tfrac{\xi}{1+(t-\xi)^2}(\hat{w}(k+1) + \hat{w}(k-1) + 2\hat{w}(k))\hat{w}(k) \\
\rightsquigarrow\ & \hat{w}^2(k,t) < \hat{w}^2(k, 2\xi) + 2c(|\hat{w}^2(k)|_{L^\infty_t} + \tfrac{1}{2}|\hat{w}^2(k+1)|_{L^\infty_t} + \tfrac{1}{2}|\hat{w}^2(k-1)|_{L^\infty_t}).
\end{aligned}$$

Hence by a bootstrap argument we control

$$\hat{w}^2(k,t) \leq \tfrac{1}{1-4c} \sum_l (2c)^{|k-l|} \hat{w}^2(l, 2\xi).$$

Summing this estimate with the weight λ_k then concludes the proof:

$$\begin{aligned}
\|w, j\|_X(t) &\leq \tfrac{1}{1-4c} \sqrt{\sum_k \lambda_k \sum_l (2c)^{|k-l|} \hat{w}^2(l, 2\xi)} \\
&\leq \tfrac{1}{1-4c} \sqrt{\sum_l \lambda_l \hat{w}^2(l, 2\xi) \sum_k (2c\hat{\lambda})^{|k-l|}} \\
&\leq \tfrac{1}{1-4c} \tfrac{1}{1-2c\hat{\lambda}} \|w, j\|_X(2\xi)
\end{aligned}$$

□

Thus it suffices to study the evolution for times $t < 2\xi$. In view of the estimates of Section 2.2 it here is convenient to partition $(0, 2\xi)$ into intervals where $t \approx \frac{\xi}{k}$ for some $k \in \mathbb{Z}$.

Definition 11. Let $\xi > 0$ be given, then for any $k \in \mathbb{N}$ we define

$$t_k = \frac{1}{2}(\tfrac{\xi}{k+1} + \tfrac{\xi}{k}) \text{ if } k > 0,$$
$$t_0 = 2\xi.$$

We further define the time intervals $I^k = (t_k, t_{k-1})$, for $\xi < 0$ we define t_k analogously for $-k \in \mathbb{N}$.

Note that

$$t_k < \tfrac{\xi}{k} < t_{k-1}$$

and

$$t_{k-1} - t_k = \frac{1}{2}\left(\tfrac{\xi}{k+1} - \tfrac{\xi}{k-1}\right) = \tfrac{\xi}{k^2-1}.$$

Hence I_k is an interval containing the time of resonance $\frac{\xi}{k}$ and is of size about $\frac{\xi}{k^2}$.

The next lemma provides a very rough energy-based estimate, which will allow us to control the evolution for small times and frequencies. That is, we show that it is easy to obtain a energy estimate with ξt in the exponent. If the time t or the frequency ξ are small this rough estimate is sufficient. However, for Gevrey 2 norm estimates it will be necessary to improve this control to a $C\sqrt{\xi}$ term in the exponent in subsequent estimates. Furthermore, we remark that also the magnetic part needs to be handled adequately, since it may give an additional growth by $\exp(\frac{4}{3}\kappa^{-\frac{1}{2}})$.

Lemma 12 (Rough estimate)**.** *Consider a solution of* (16)*, then for fixed ξ and for all times $t > 0$ it holds that*

$$\|w, j\|_X(t) \leq \exp(\tfrac{4}{3}\kappa^{-\frac{1}{2}}) \exp((1 + \hat{\lambda})c\xi t)\|w, j\|_X(0).$$

Proof. We define $\hat{w}(k) = |w, j|(k)$, then

$$\tfrac{1}{2}\partial_t \hat{w}^2(k) = (a(k+1)w(k+1) - a(k-1)w(k-1))w(k) + b(k)j(k)^2$$

with $b(k,t) = (2\frac{t-\frac{\xi}{k}}{1+(\frac{\xi}{k}-t)^2} - \kappa k^2(1 + (\frac{\xi}{k} - t)^2))$. We further define the define the growth factor

$$M(k,t) = \begin{cases} 1 & \text{if } t - \frac{\xi}{k} \leq 0 \quad \text{or} \quad \kappa k^2 \geq 1, \\ \frac{1}{1+(\frac{\xi}{k}-t)^2} & \text{if } 0 \leq t - \frac{\xi}{k} \leq (\frac{2}{\kappa k^2})^{\frac{1}{3}} \quad \text{and} \quad \kappa k^2 \leq 1, \\ \frac{1}{1+(\frac{2}{\kappa k^2})^{\frac{1}{3}}} & \text{if } (\frac{2}{\kappa k^2})^{\frac{1}{3}} \leq t - \frac{\xi}{k} \text{ and } \quad \kappa k^2 \leq 1. \end{cases}$$

We note that this weight satisfies $b(k,t) + \frac{M'}{M}(k,t) \leq 0$. Hence, defining the energy

$$E = (\prod_l M(l,t))^2 \sum_k \lambda_k \hat{w}(k,t)^2,$$

we deduce that

$$\begin{aligned} \tfrac{1}{2}\partial_t E &\leq \left(\prod_l M(l,t)\right)^2 \sum_k \lambda_k \left(a(k+1)\hat{w}(k+1,t) + a(k-1)\hat{w}(k-1,t)\right)\hat{w}(k) \\ &\leq (\prod_l M(l,t))^2 \sum_k (\lambda_k a(k) + \tfrac{a(k-1)\lambda_{k-1} + a(k+1)\lambda_{k+1}}{2})\hat{w}^2(k,t) \\ &= (1 + \hat{\lambda})c\xi E. \end{aligned}$$

Applying Gronwall's inequality thus yields

$$E(t) \leq \exp(2(1 + \hat{\lambda})c\xi t)E(0).$$

This in turn leads to the estimate

$$\begin{aligned}\|w,j\|_X &\leq \exp((1+\hat{\lambda})c\xi t)\prod_l |M(l,t)|^{-1}\|w_0,j_0\|_X\\ &\leq \exp((1+\hat{\lambda})c\xi t)\prod_{l=0}^{\kappa^{-\frac{1}{2}}}(1+(\tfrac{2}{\kappa l^2})^{\frac{1}{3}})\|w_0,j_0\|_X.\end{aligned}$$

Finally, we can use Stirling's approximation of the factorial, which results in the desired estimate:

$$\|w,j\|_X(t) \leq \exp(\tfrac{4}{3}\kappa^{-\frac{1}{2}})\exp((1+\hat{\lambda})c\xi t)\|w_0,j_0\|_X.$$

□

In the following we establish upper and lower bounds for small times. Here we use that for modes k such that $\frac{\xi}{k^2}$ is small nay possible resonance will not produce large enough norm inflation and the evolution can hence be treated perturbatively. More precisely, we consider the evolution on the time interval

$$I = [0, \tfrac{\xi}{2}(\tfrac{1}{k_0}+\tfrac{1}{k_0-1})]$$

for fixed k_0 to be determined later. For this purpose we introduce the parameter $\eta_0 := \frac{\xi}{k_0^2}$ which later will be chosen as $\eta_0 \approx \frac{1}{10c}$.

Lemma 13. *Let w, j be a solution of (16), define $d := c^{-1}$ and let ξ, k_0 be such that $\eta_0 \leq d^2$. Then for all times $0 \leq t \leq t_{k_0}$ it holds that*

$$\begin{aligned}\|w(t),j(t)\|_X^2 &\leq \exp(2(1+\hat{\lambda})\max(c\eta_0,1)\sqrt{\xi\eta_0})\|w_0,j_0\|_X^2\\ &\leq \exp(C\sqrt{\xi})\|w_0,j_0\|_X^2.\end{aligned}$$

Furthermore, suppose that $k_0 \geq \kappa^{-\frac{1}{2}}$ and $10d \leq \frac{\xi}{k_0^2} \leq \frac{1}{100c^2}$, then for the initial data $w(k,0)=\delta_{k_0,k}$ and $j(k,0)=0$ we obtain that

$$\begin{aligned}w(k_0,t_{k_0}) &\geq \tfrac{1}{2}\max(1, w(k,t_{k_0}), j(k,t_{k_0}))\\ j(k_0,t_{k_0}) &\leq \tfrac{1}{\alpha_{k_0}\xi\eta_0}.\end{aligned}$$

Proof. Computing the time derivative, we obtain

$$\tfrac{1}{2}\partial_t\|w,j\|_X^2 = \sum_l (a(l+1)w(l+1)+a(l-1)w(l-1))\lambda_l w(l) + b(l)\lambda_l j(l)^2,$$

where the coefficient functions satisfy

$$\begin{aligned}a(l) &\leq \begin{cases} c\eta_0 & l \geq k_0\\ 4c\frac{1}{1+\eta_0} & l \leq k_0\end{cases}\\ &\leq \max(c\eta_0, 4c),\\ b(l) &\leq 1.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\partial_t\|w,j\|_X^2 &\leq 2(1+\hat{\lambda})\max(c\eta_0,1)\|w,j\|_X^2,\\ \|w,j\|_X^2(t) &\leq \exp(2(1+\hat{\lambda})\max(c\eta_0,1)t)\|w_0,j_0\|_X^2,\\ \|w,j\|_X^2(t_{k_0}) &\leq \exp(2(1+\hat{\lambda})\max(c\eta_0,1)\sqrt{\xi\eta_0})\|w_0,j_0\|_X^2.\end{aligned}$$

To prove lower bounds on the norm inflation we further need to show that for $w(k,0)=\delta_{k_0,k}$ and $j(k,0)=0$, the mode $w(k_0,t_{k_0})$ will stay the largest mode. Therefore, we introduce the short-hand notation

$$\hat{w}(k,t) = |w,j|(k,t)$$

and have to estimate the growth of $\hat{w}(\cdot,t)$. Since on the interval $[0,t_{k_0}]$ it holds that $b(k)\leq 0$ as $k_0\geq \kappa^{-\frac{1}{2}}$, we obtain the system

$$\begin{aligned}\partial_t\hat{w}(k)&\leq a(k+1)\hat{w}(k+1)+a(k-1)\hat{w}(k-1)\\ \hat{w}(k,0)&=\delta_{k,k_0}.\end{aligned}$$

Let $\sqrt{\xi\pi}=l_0\geq l\geq k_0$ to be fixed later. We want to prove by induction that

$$\begin{aligned}\hat{w}(m,t_{l-1})&\leq 6\pi c\eta_0(2c)^{|m-k_0|}\\ \hat{w}(l,t_{l-1})&\leq 4(2c)^{|l-k_0|}\\ \hat{w}(n,t_{l-1})&\leq 2(2c)^{|m-k_0|}\end{aligned}\tag{19}$$

for all $m>l>n$.

Induction start:

We integrate a in time to estimate

$$\begin{aligned}\int_0^{t_{l_0-1}}a(k)&=c\tfrac{\xi}{k^2}\int_0^{t_{l_0-1}}\tfrac{1}{1+(\frac{\xi}{k}-t)^2}\\ &\leq\begin{cases}\pi c\frac{\xi}{k^2} & k>l_0\\ c & k\leq l_0\end{cases}\\ &\leq c.\end{aligned}$$

Thus we obtain that

$$\hat{w}(k,t_{l_0-1})<\delta_{k_0,k}+c(|\hat{w}(k+1)|_{L_t^\infty}+|\hat{w}(k-1)|_{L_t^\infty}),$$

which by a bootstrap argument yields that

$$\hat{w}(k,t_{l_0-1})\leq\tfrac{1}{1-2c}(2c)^{|k_0-k|}$$

for all k which satisfy (19).

Induction step: We fix l and we assume that (19) holds for all $\tilde{l}$ with $l_0\geq\tilde{l}\geq l+1\geq k_0+1$ and then prove that it holds also for l. We here argue by bootstrap. That is, we show that the estimate (19) at least holds up until a time t^* with $t_l\leq t^*\leq t_{l-1}$ and that the maximal time with this property is given by $t^*=t_{l-1}$. For $n<l$ we estimate

$$\begin{aligned}\hat{w}(n,t_{l-1})&\leq\delta_{n,k_0}+\int_0^{t_{l-1}}a(n\pm1,\tau)w(n\pm1,\tau)\\ &\leq\delta_{n,k_0}+c(4(2c)^{|n+1-k_0|}+2(2c)^{|n-1-k_0|})\\ &<2(2c)^{|n-k_0|}.\end{aligned}$$

To estimate the l mode we estimate the integral between t_l and t_{l-1} to deduce

$$\begin{aligned}\hat{w}(l,t_{l-1})&\leq\hat{w}(l,t_l)+\int_{t_l}^{t_{l-1}}a(l\pm1,\tau)w(l\pm1,\tau)\\ &\leq 2(2c)^{l-k_0}+6\pi c\eta_0(2c)^{l+1-k_0}+2c(2c)^{|l-1-k_0|}\\ &\leq 4(2c)^{l-k_0}.\end{aligned}$$

For $m>l$ we split the integrals as

$$\begin{aligned}\hat{w}(m,t_{l-1})\leq{}&\hat{w}(m,t_{m-1})+\int_{t_{m-1}}^{t_l}a(m+1,t)\hat{w}(m+1,\tau)\\ &+\int_{t_{m-1}}^{t_{m-2}}a(m-1,t)\hat{w}(m-1,\tau)+\int_{t_{m-2}}^{t_l}a(m-1,t)\hat{w}(m-1,\tau)\\ \leq{}&4(2c)^{m-k_0}+12\pi c^2\eta_0(2c)^{|m-k_0|}+4\pi\eta_0(2c)^{m-k_0}+6\pi c\eta_0(2c)^{m-k_0}\\ \leq{}&6\pi\eta_0(2c)^{m-k_0}.\end{aligned}$$

So we finally deduce that

$$\hat{w}(k,t_{k_0}) \leq (2c)^{|k-k_0|}\begin{cases} 4 & k \leq k_0+1 \\ 6\pi\eta_0 & k > k_0+1 \end{cases}$$

Thus we established an upper bound for all modes, the next step is to show that for w indeed the k_0 mode is one of the largest modes. Therefore, we estimate $j(k_0)$ by

$$j(k_0,t) = \alpha k_0 \int_0^t d\tau \, \tfrac{1+(\frac{\xi}{k}_0-t)^2}{1+(\frac{\xi}{k}_0-\tau)^2} \exp\left(-\kappa k_0^2(t-\tau+\tfrac{1}{3}((\tfrac{\xi}{k_0}-t)^3-(\tfrac{\xi}{k_0}-\tau)^3)\right) w(k_0,\tau)$$

and hence obtain that

$$\begin{aligned} j(t) &\leq \alpha k_0 \int \exp(-\kappa k_0^2\eta^2(t-\tau) \\ &\leq \tfrac{1}{\sqrt{\beta\kappa\xi\eta_0^3}} = \tfrac{1}{\alpha_{k_0}\xi\eta_0} \end{aligned}$$

and

$$\begin{aligned} \alpha k_0 \int j(k_0,\tau_2)d\tau_1 &= \tfrac{\kappa k_0^2}{\beta}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \, \tfrac{1+(\frac{\xi}{k}_0-\tau_1)^2}{1+(\frac{\xi}{k}_0-\tau_2)^2} \\ &\quad \times \exp\left(-\kappa k_0^2(\tau_1-\tau_2+\tfrac{1}{3}((\tfrac{\xi}{k}_0-\tau_1)^3-(\tfrac{\xi}{k}_0-\tau_2)^3)\right) \\ &\leq \tfrac{1}{\beta}\int d\tau_1 \tfrac{1}{1+(\frac{\xi}{k_0}-\tau_1)^2} \\ &\leq \tfrac{4}{\beta\eta_0}. \end{aligned}$$

With this we conclude that

$$\begin{aligned} |w(k_0,t_{k_0})-1| &\leq \int a(k_0+1)w(k_0+1) + a(k_0-1)w(k_0-1) + \alpha k_0 j(t) \\ &\leq 16\pi\eta_0 c^2 + 8c^2 + \tfrac{4}{\beta\eta_0} \leq \tfrac{1}{2}, \end{aligned}$$

which in turn yields

$$|w(k_0,t_{k_0})| \geq \tfrac{1}{2} \geq \max_{k\neq k_0}(\hat{w}(k,t_{k_0}), |j(k_0,t_{k_0})|).$$

□

4. Resonances and Norm Inflation

Having discussed the evolution for small times and large times in Section 3 it remains to discuss the evolution on the interval

$$(t_{k_0}, 2\xi) = \bigcup_{1\leq k\leq k_0} I^k$$

with I^k as in Definition 11.

Based on the heuristics of the toy model of Section 2.2 our aim here is to establish both upper lower and upper bounds on the norm inflation on each resonant interval I^k, where the resonant mode $w(k)$ can possibly lead to a large growth of its neighboring modes $w(k\pm 1)$. In order to simplify notation we introduce the growth factor

$$L = L(\alpha,\kappa,k),$$

which estimates the maximal growth of $w(k)$ due to its interaction with the current $j(k)$, see Appendix A. In particular, we show that $L=1$ if $\beta \geq \pi$ and if $\beta < \pi$ we obtain an estimate $L = L(\alpha,\kappa,k) \leq \sqrt{c}$. We define M and M_n as

$$\begin{aligned} M &= \sum_m 10^{-|m|}(w + \tfrac{1}{\alpha_{k_m}} j)(k_m, s_0) \\ M_n &= \sum_m 10^{-|m-n|+\chi}(w + \tfrac{1}{\alpha_{k_m}} j)(k_m, \tilde{s}_0) \end{aligned}$$

where $\chi = -|\mathrm{sgn}(m) - \mathrm{sgn}(n)|$. We note that

$$\sum_{l\neq 0} (\tfrac{3}{\eta})^{|k_l - k_0|} M_l \leq \tfrac{3}{\eta} M.$$

With these notations the main results of this section are summarized in the following theorem:

Theorem 14. *Let $c \leq \min(10^{-3}\beta^{\frac{16}{3}}, 10^{-4})$, $\xi \geq 10\kappa^{-1}(1+\beta^{-1})$ and $\eta = \frac{\xi}{k^2} \geq 10d$ and $t_k = \frac{\xi}{2}(\frac{1}{k} + \frac{1}{k+1})$, then it holds that*

$$\|w, j\|_X(t_{k-1}) \leq 18\pi L\hat{\lambda}(c\eta)^\gamma \|w, j\|_X(t_k).$$

Furthermore, let $\kappa_k \min(\beta, 1) \geq \frac{1}{c}$ and $\beta \geq \frac{1}{5}$

$$w(k, t_k) \geq \tfrac{1}{2} \max(w(l, t_k), j(l, t_k)). \tag{20}$$

Then $w(k-1, t_{k-1})$ satisfies (20) with k replaced by $k-1$ and

$$|w(k-1, t_{k-1})| \geq \min(\beta, \pi)(c\eta)^\gamma w(k, t_k).$$

To prove the estimates of Theorem 14 it is convenient to rescale $\tilde{j}(k) = \alpha_k j(k)$ in (16) to obtain

$$\begin{aligned}
\partial_t w(k) &= -\tilde{j}(k) \\
&\quad - c\tfrac{\xi}{(k+1)^2} \tfrac{1}{1+(\frac{\xi}{k+1}-t)^2} w(k+1) \\
&\quad + c\tfrac{\xi}{(k-1)^2} \tfrac{1}{1+(\frac{\xi}{k-1}-t)^2} w(k-1) \\
\partial_t \tilde{j}(k) &= (2\tfrac{t-\frac{\xi}{k}}{1+(\frac{\xi}{k}-t)^2} - \kappa k^2(1+(t-\tfrac{\xi}{k}^2))\tilde{j}(k) + \tfrac{\kappa k^2}{\beta} w(k),
\end{aligned}$$

where we used that $\kappa = \beta\alpha^2$. With respect to these unknowns the norm on our space X changes slightly

$$\begin{aligned}
\|w, j\|_X^2 &= \sum \lambda_k (w^2(k) + \tfrac{\beta}{\kappa k^2} \tilde{j}^2(k)) \\
&=: \|w, \tilde{j}\|_{\tilde{X}}^2.
\end{aligned}$$

In the following sections, with slight abuse of notation we omit writing the tilde symbols both for j and X.

Given a choice of time interval I_{k_0}, considering k_0 as arbitrary but fixed (and unrelated to k_0 of Section 3) we further introduce the relative frequencies

$$k_n := k_0 + n,$$

where $n \in \mathbb{Z}_{>-k_0}$ and also shift our time variable

$$t = \tfrac{\xi}{k_0} + s.$$

Introducing the coefficient functions

$$\begin{aligned}
a(k) &= c\eta \tfrac{k_0^2}{(k)^2} \tfrac{1}{1+(\eta\frac{k_0(k_0-k)}{k} - s)^2}, \\
b(k) &= 2\tfrac{(s-\eta\frac{k_0(k_0-k)}{k})}{1+(\eta\frac{k_0(k_0-k)}{k}-s)^2} - \kappa_k(1+(\eta\tfrac{(k_0-k)k_0}{k} - s)^2),
\end{aligned} \tag{21}$$

the system (16) then reads

$$\begin{aligned}
\partial_s w(k) &= -j(k) \\
&\quad - a(k+1)w(k+1) \\
&\quad + a(k-1)w(k-1), \\
\partial_t j(k) &= \tfrac{\kappa_k}{\beta} w(k) + b(k)j(k).
\end{aligned} \tag{22}$$

For later reference, we note that the coefficient function a satisfies the following estimates:

$$\begin{aligned} a(k_0) &= c\eta\tfrac{1}{1+s^2}, \\ a(k_{\pm 1}) &\leq 4\tfrac{c}{\eta}, \\ a(k_n) &\leq \tfrac{c}{\eta}, \end{aligned} \tag{23}$$

for all $|n| \geq 2$.

Finally, in view of cancellations of $-a(k-1)$ and $+a(k+1)$ on any given time interval I^k it is convenient to work with the unknowns

$$u_1 = w(k_0), u_2 = w(k_1) - w(k_{-1}), u_3 = w(k_1) + w(k_{-1}).$$

We then consider (22) as a forced system for these three modes (and a separate equation for all other modes):

$$\begin{aligned} \partial_s \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ j(k_0) \end{pmatrix} &= \begin{pmatrix} 0 & -a_1 & a_2 & -1 \\ 2c\eta\frac{1}{1+s^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\kappa_k}{\beta} & 0 & 0 & \frac{2s}{1+s^2} - \kappa_k(1+s^2) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ j(k_0) \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ -a(k_{\pm 2})w(k_{\pm 2}) \mp j(k_{\pm 1}) \\ \mp a(k_{\pm 2})w(k_{\pm 2}) - j(k_{\pm 1}) \\ 0 \end{pmatrix} \end{aligned} \tag{24}$$

where $a_1 = \frac{1}{2}(a(k_1) + a(k_{-1}))$ and $a_2 = \frac{1}{2}(a(k_1) - a(k_{-1}))$.

The analysis of this system is split into multiple subsections, where we also split the time interval I^k as

$$I^{k_0} = [s_0, -d] \cup [-d, d] \cup [d, s_1] =: I_1 \cup I_2 \cup I_3,$$

where $s_0 = -\frac{\eta}{2}\frac{k_0-1}{k_0}$ and $s_1 = \frac{\eta}{2}\frac{k_0+1}{k_0}$. Similarly to the setting of the Euler equations [DZ21] here the interaction between growth and decay of various modes interacts to determine the over all norm inflation.

4.1. **Proof of Theorem 3.**

Before proceeding to the proof of Theorem 14, in this subsection we discuss how it can be used to establish Theorem 3. We split the proof into two auxiliary theorems.

Theorem 15 (technical statement)**.** *Let $c \leq \min(10^{-3}\beta^{\frac{16}{3}}, 10^{-4})$, $\xi \geq 10\kappa^{-1}(1+\beta^{-1})$ and $\frac{\xi}{k^2} \geq 10d$. Then there exists exists a constant $C = C(\kappa, \alpha, c)$ such that for a fixed ξ we obtain*

$$\|w, j\|_X(t, \xi) \leq \exp(C\sqrt{\xi})\|w, j\|_X(0, \xi).$$

Furthermore, let $\xi \geq 10^4\frac{d^2}{\beta\kappa}$, $\beta \geq \frac{1}{5}$, $k_0 \approx \frac{c}{10}\sqrt{\xi}$ and $k_1 \approx \frac{4}{\sqrt{\beta\kappa}}$, then there exists a constant $C^* = C^*(\kappa, \alpha, c)$ such that for initial data $w(k, 0) = \delta_{k_0,k}$ and $j(k, 0) = 0$ we obtain

$$w(k_1, t) \geq \exp(\tilde{C}\sqrt{\xi}).$$

for $t \in [t_{k_1} - 1, t_{k_1} + 1]$.

Proof of Theorem 15. For fixed ξ, t and $k_0 =: 10d\sqrt{\xi}$ we consider $w(\cdot, \xi, t)$ as an element in X. On X we define the operator $S_{\tau_1,\tau_2} : X \to X$ as the solution operator of (16) on $[\tau_1, \tau_2]$, i.e.

$$\begin{aligned} S_{\tau_1,\tau_2}[w(\cdot, \xi, \tau_1)] &= w(\cdot, \xi, \tau_2) \\ S_{\tau_1,\tau_2} \circ S_{\tau_2,\tau_3} &= S_{\tau_1,\tau_3}. \end{aligned}$$

By Lemma 13, Theorem 14 and Lemma 10 this S then satisfies the following norm estimates:

$$\begin{aligned} \|S_{0,t_{k_0+1}}\|_{X\to X} &= \exp(C_1\sqrt{\xi}), \\ \|S_{t_k,t_{k-1}}\|_{X\to X} &= 3\pi c(\tfrac{\xi}{k^2})^\gamma, \\ \|S_{t_1,t}\|_{X\to X} &= 2\frac{1}{1-c\sqrt{\hat{\lambda}}}. \end{aligned}$$

Combining these estimates with Stirling's approximation formula we thus obtain the desired upper bound:

$$\begin{aligned} \|S_{0,t}\|_{X\to X} &\leq 2\exp(C_1\sqrt{\xi})\prod_{k=1}^{k_0} 3\pi c(\tfrac{\xi}{k^2})^\gamma \\ &\leq \exp(C\sqrt{\xi}). \end{aligned}$$

Concerning the lower bound, we use first use Lemma 13 and then Theorem 14 to deduce that

$$\begin{aligned} w(k_0, t_{k_0}) &\geq \tfrac{1}{2} \\ w(k-1, t_{k-1}) &\geq (c\tfrac{\xi}{k^2})^\gamma \min(\beta,\pi) w(k,t_k) \end{aligned}$$

for $\sqrt{\frac{c}{10}\xi} \approx k_0 \geq k \geq k_1 \approx \frac{4}{\sqrt{\beta\kappa}}$. Thus, by again using Stirling's approximation, we conclude that

$$\begin{aligned} w(k_1, t_{k_1}) &\geq \tfrac{1}{2}\prod_{k=k_1}^{k_2} (c\tfrac{\xi}{k^2})^\gamma \min(\beta,\pi) \\ &\approx \exp(\tilde{C}\sqrt{\xi}). \end{aligned}$$

□

Theorem 16 (Stability and blow-up)**.** *Let $c \leq \min(10^{-3}\beta^{\frac{16}{3}}, 10^{-4})$ and w, j be a solution to (16) , then there exists a constant $C = C(\kappa,\alpha,c)$ such that for all $C_1 > C$ and initial data which satisfy*

$$\int \exp(C_1\sqrt{\xi})\|w_0, j_0\|_X^2(\xi)\, d\xi < \infty,$$

the solution remains Gevrey 2 regular in the sense that

$$\sup_t \int \exp(C_2\sqrt{\xi})\|w,j\|_X^2(\xi,t)\, d\xi \leq \tilde{C}\int \exp(C_1\sqrt{\xi})\|w_0, j_0\|_X^2(\xi)\, d\xi,$$

where $C_2 = C_1 - C$ and $\tilde{C} > 0$ is a universal constant.

Furthermore, additionally suppose that $\beta \geq \frac{1}{5}$, then there exist a constant $0 < C^* < C$ and initial data w_0, j_0 which satisfy

$$\int \exp(C^*\sqrt{\xi})\|w_0, j_0\|_X^2(\xi)\, d\xi < \infty,$$

such that for a subsequence $k_{n,1}$ the solution diverges in L^2:

$$\|w(\cdot, t_{k_{n,1}})\|_{L^2\ell^2} \to \infty.$$

Proof of Theorem 16. The first part follows directly from Theorem 15. For the second part we fix $\xi_1 = 10^4\frac{d^2}{\beta\kappa}$ and define the sequence $\xi_n = n\xi_1$ with the associated $k_0^{\xi_n} \approx \frac{c}{10}\sqrt{\xi_n}$ and $k_1 \approx \frac{4}{\sqrt{\beta\kappa}}$. Note that the starting mode $k_0^{\xi_n}$ is ξ_n-dependent, but the final mode k_1 is independent of ξ_n. Furthermore, let $z_n(\xi)$ be a function in $C^\infty \cap L^2$, such that

$$\begin{aligned} \operatorname{supp} z_n(\cdot) &\subset [\xi_n - 1, \xi_n + 1] \\ \int z_n(\xi)^2\, d\xi &= 1. \end{aligned}$$

We then define the initial data

$$w(k,\xi,0)=\sum_{n=1}^{\infty}\tfrac{1}{n}z_n(\xi)\exp(-\tfrac{1}{2}C^*\sqrt{\xi})\delta_{k_{\xi_n,0},k}.$$

We observe that it satisfies the estimates

$$\|w(\cdot,\xi,0)\|_{l^2}^2=\sum_{n=1}^{\infty}\tfrac{1}{n^2}z_n(\xi)^2\exp(-C^*\sqrt{\xi}),$$
$$\int\exp(C^*\sqrt{\xi})\|w(\cdot,\xi,0)\|_{l^2}^2\,d\xi=2\sum\tfrac{1}{n^2}=\tfrac{\pi^2}{3}.$$

Furthermore, by the norm inflation results for each mode at each time $t_{k_{n,1}}$ we obtain that

$$\|w(k_{n,1},\xi,t_{k_{n,1}})\|_{l^2}\geq\tfrac{9}{10}\tfrac{1}{n^2}z(\xi,n)\exp((\tilde{C}-C^*)\sqrt{\xi}),$$

and integrating in ξ we conclude that

$$\|w(\cdot,t_{k_{n,1}})\|_{L^2l^2}\geq\tfrac{9}{10}\tfrac{1}{n^2}\exp((\tilde{C}-C^*)\sqrt{\xi_n})\to\infty.$$

□

4.2. **Asymptotic Behavior on the Intervals I_1 and I_3.** In this section we consider the equation (24) on the outer intervals $I_1=[s_0,-d]$ and $I_3=[d,s_1]$. Since a lot of calculations are similar on both intervals we in general write the interval as $[\tilde{s}_0,\tilde{s}_1]$, where we only need to distinguish the two cases on a few occasions and in the statement of the conclusions. For the interval I_1 we prove the following proposition:

Proposition 17 (Interval I_1)**.** *Let $c<\max(10^{-4},10^{-1}\beta)$ and $\xi\geq 10\max(\kappa^{-1}(1+\beta^{-1}),k_0^2d)$. Then for a solution of (24) on the interval I_1 the following estimates hold at the time d:*

$$\begin{aligned}
|u_1(d)|&\leq 2M(c\eta)^{-\gamma_2},\\
|u_2(d)|&\leq 2M(c\eta)^{\gamma_1},\\
|u_3|(-d)&\leq 2M_1,\\
|w(k_n,-d)|&\leq 2M_n,\\
|j|(k_0,-d)&\leq \tfrac{c}{\beta}(c\eta)^{-\gamma_2}M\inf(c,\kappa_{k_0}c^{-2}),\\
|j|(k_{\pm1},-d)&\leq \tfrac{4}{\beta\eta^2}M,\\
|j|(k_n,-d)&\leq \tfrac{4}{\beta\eta^2}M_n.
\end{aligned}$$

If we additionally assume that

$$w(k_0,t_{k_0})\geq\tfrac{1}{2}\sup_l(w(l,t_l),j(l,t_l)),\tag{25}$$

we obtain that

$$\begin{aligned}
|u_1(-d)-(c\eta)^{-\gamma_2}u_1(s_0)|&=50cu_1(\tilde{s}_0)(c\eta)^{-\gamma_2}\\
|u_2(-d)|&\leq 50cu_1(\tilde{s}_0)(c\eta)^{\gamma_1},\\
|u_3|(-d),|w|(k_m,-d)&\leq 2|u|(s_0)\qquad \textit{for } |m|\geq 2,\\
|j|(k_m,-d)&\leq \tfrac{4}{\eta}|u|(s_0)\qquad \textit{for } |m|\geq 1,\\
|j|(k_0,-d)&\leq \tfrac{2c^2}{\beta}|u|(s_0)(c\eta)^{-\gamma_2}.
\end{aligned}\tag{26}$$

The proof of this proposition is split into several lemmas and concludes at the end of this subsection. For the interval I_3 in a first step we only establish asymptotic estimates. The final conclusion for interval I_3 will be postponed to the proof of Theorem 14. On both intervals I_1 and I_2 the interaction of u_1 and

u_2 is the main effect to be analyzed. Therefore, we consider the equations for u_1 and u_2 as an inhomogeneous linear system

$$\partial_s \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{c}{\eta} \\ 2c\eta\frac{1}{s^2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + F, \tag{27}$$

where F is a force term. Equation (27) with $F = 0$ has a explicit homogeneous solution and we aim to show that (24) can be treated as a perturbation. In the following we denote $\tilde{u}$ as the homogeneous solution of (27). Furthermore, we split the forcing as

$$F =: F_{all} = F_{3mode} + F_j + F_{u_3} + F_{j(k_0\pm1)} + F_{\tilde{w}}$$

where we define

$$F_{3mode} = (\tfrac{c}{\eta} - a_1)e_1 u_2 - 2c\eta\tfrac{1}{s^2(s^2+1)}e_2 u_1,$$

as the 3 mode forcing

$$F_j = -e_1 j(k_0)$$

as the k_0-th current forcing and

$$F_{u_3} + F_{j(k_0\pm1)} + F_{\tilde{w}} = e_1 a_2 u_3 \mp e_2 j(k_{\pm1}) - e_2 a(k_{\pm2}) w(k_{\pm2})$$

as the forcings due to u_3, $j(k_0 \pm 1)$ and $\tilde{w}$, respectively. The corresponding $R[F_*]$ are the called r changes. We also define $\gamma = \sqrt{1 - 8c^2}$ and $\gamma_1 = \frac{1}{2}(1 + \gamma)$ and $\gamma_2 = \frac{1}{2}(1 - \gamma)$ and note the following equalities:

$$\begin{aligned} \gamma_1\gamma_2 &= 2c^2, \\ \gamma_1 + \gamma_2 &= 1, \\ \gamma &= 1 + \mathcal{O}(c^2), \\ \gamma_1 &= 1 + \mathcal{O}(c^2), \\ \gamma_2 &= \tfrac{1}{\gamma_1}2c^2 = 2c^2 + \mathcal{O}(c^4). \end{aligned}$$

Lemma 18. *Consider (27) with $F = 0$, then the solution is given by*

$$\tilde{u}(s) = S(s)r$$

with

$$S(s) = \begin{pmatrix} |\frac{s}{\eta}|^{\gamma_1} & |\frac{s}{\eta}|^{\gamma_2} \\ -\frac{\gamma_1}{c}\frac{s}{\eta}|\frac{s}{\eta}|^{\gamma_1-2} & -\frac{\gamma_2}{c}\frac{s}{\eta}|\frac{s}{\eta}|^{\gamma_2-2} \end{pmatrix},$$

and $r = S^{-1}(\tilde{s}_0)\tilde{u}(s_0)$.

Furthermore, we define the operator S^* as

$$S^*(s) = \begin{pmatrix} |\frac{s}{\eta}|^{\gamma_1} & |\frac{s}{\eta}|^{\gamma_2} \\ \frac{\gamma_1}{c}|\frac{s}{\eta}|^{\gamma_1-1} & \frac{\gamma_2}{c}|\frac{s}{\eta}|^{\gamma_2-1} \end{pmatrix},$$

which gives the estimate

$$|S(s)r| \leq S^* r \qquad \forall r \in (\mathbb{R}_+)^2$$

The inverse of S can be computed as

$$\begin{aligned} S^{-1}(s) &= \operatorname{sgn}(s)c\gamma^{-1} \begin{pmatrix} -\frac{\gamma_2}{c}\frac{s}{\eta}|\frac{s}{\eta}|^{\gamma_2-2} & -|\frac{s}{\eta}|^{\gamma_2} \\ \frac{\gamma_1}{c}\frac{s}{\eta}|\frac{s}{\eta}|^{\gamma_1-2} & |\frac{s}{\eta}|^{\gamma_1} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\gamma_2}{\gamma}|\frac{s}{\eta}|^{\gamma_2-1} & -\frac{c}{\gamma}\frac{s}{\eta}|\frac{s}{\eta}|^{\gamma_2-1} \\ \frac{\gamma_1}{\gamma}|\frac{s}{\eta}|^{\gamma_1-1} & \frac{c}{\gamma}\frac{s}{\eta}|\frac{s}{\eta}|^{\gamma_1-1} \end{pmatrix}. \end{aligned}$$

Lemma 19. *Let u_1, u_2 be a solution to (27) with given $F = (F_1, F_2)$, then for*

$$R_1[F] = (1 + 10c^2)2c^2\eta^{1-\gamma_2} \int_{\tilde{s}_0}^{s} \tau^{\gamma_2 - 1} F_1(\tau)\ d\tau + c\eta^{-\gamma_2} \int_{\tilde{s}_0}^{s} \tau^{\gamma_2} F_2(\tau)\ d\tau,$$

$$R_2[F] = (1 + 10c^2)\eta^{1-\gamma_1} \int_{\tilde{s}_0}^{s} \tau^{\gamma_1 - 1} F_1(\tau)\ d\tau + c\eta^{-\gamma_1} \int_{\tilde{s}_0}^{s} \tau^{\gamma_1} F_2(\tau)\ d\tau,$$

we estimate

$$|u - \tilde{u}| \leq S^*(s) R[F].$$

Proof. Since S has an inverse, we write

$$u = S(s) r(s)$$

and our aim is to control the evolution of $r(s)$. Therefore, we calculate

$$\begin{aligned} |\partial_s r| &= S^{-1} F \\ |\partial_s r_1| &\leq 2c^2 |\tfrac{s}{\eta}|^{\gamma_2 - 1} F_1 + c|\tfrac{s}{\eta}|^{\gamma_2} F_2 \\ |\partial_s r_2| &\leq |\tfrac{s}{\eta}|^{\gamma_1 - 1} F_1 + c|\tfrac{s}{\eta}|^{\gamma_1} F_2 \end{aligned}$$

and so

$$\begin{aligned} |r_1(s) - r_1(d)| &\leq 2c^2(1 + 10c^2)\eta^{1-\gamma_2} \int \tau^{\gamma_2 - 1} F_1(\tau) + c\eta^{-\gamma_2} \int \tau^{\gamma_2} F_2(\tau) \\ |r_2(s) - r_2(d)| &\leq (1 + 10c^2)\eta^{1-\gamma_1} \int \tau^{\gamma_1 - 1} F_1(\tau) + c\eta^{-\gamma_1} \int \tau^{\gamma_1} F_2(\tau) \end{aligned}$$

□

In the following we always assume that there exists $c_1, c_2, \tilde{c}_1, \tilde{c}_2 \geq 0$ such that

$$\begin{aligned} |u| &\leq S^*(s) C(s) \\ C_1(s) &= c_1 + \tilde{c}_1 (\tfrac{s}{\eta})^{-\gamma} \\ C_2(s) &= c_2 + \tilde{c}_2 (\tfrac{s}{\eta})^{\gamma} \end{aligned} \tag{28}$$

on a maximal interval $[\tilde{s}_0, s^*]$. We will establish some estimates on the R_i depending on c_i and $\tilde{c}_i$ and then we will determine specific c_i and $\tilde{c}_i$ such that we prove that the maximal s^* will be greater than $\tilde{s}_1$. Later it will be sufficient to choose $\tilde{c}_1 = 0$ on I_1 and $\tilde{c}_2 = 0$ on I_3. We thus deduce

$$\begin{aligned} |u_1(s)| &\leq (c_1 + \tilde{c}_2)|\tfrac{s}{\eta}|^{\gamma_1} + (\tilde{c}_1 + c_2)|\tfrac{s}{\eta}|^{\gamma_2} \\ |u_2(s)| &\leq (\tfrac{\gamma_1}{c} c_1 + \tfrac{\gamma_2}{c} \tilde{c}_2)|\tfrac{s}{\eta}|^{\gamma_1 - 1} + (\tfrac{\gamma_1}{c} \tilde{c}_1 + \tfrac{\gamma_2}{c} c_2)|\tfrac{s}{\eta}|^{\gamma_2 - 1} \\ &\leq c_1^* |\tfrac{s}{\eta}|^{\gamma_1 - 1} + c_2^* |\tfrac{s}{\eta}|^{\gamma_2 - 1}. \end{aligned}$$

where $c_1^* = \frac{\gamma_1}{c} c_1 + \frac{\gamma_2}{c} \tilde{c}_2$ and $c_2^* = \frac{\gamma_1}{c} \tilde{c}_1 + \frac{\gamma_2}{c} c_2$. For sake of simplicity we will often omit absolute values for the estimates.

Lemma 20 (3 mode forcing estimate)**.** *Let $u(s) = S(s) r(s)$ be a solution of (24) on $[\tilde{s}_0, s^*]$, such that $|u(s)| \leq S^*(s) C(s)$, then we estimate*

$$\begin{aligned} R_1[F_{3mode}] &\leq 20c^2 c_1 + (20 + c^4 (\tfrac{s \wedge \tilde{s}_0}{\eta})^{-\gamma}) \tilde{c}_1 + (20c^2 + c^4 (\tfrac{s \wedge \tilde{s}_0}{\eta})^{-\gamma}) c_2 + 20c^4 \tilde{c}_2 \\ R_2[F_{3mode}] &\leq 20 (\tfrac{s \vee \tilde{s}_0}{\eta})^{\gamma} (c_1 + 2c^2 \tilde{c}_2) + 20(\tilde{c}_1 + c^2 c_2). \end{aligned}$$

Proof. We have the forcing term

$$F_{3mode} = (\tfrac{c}{\eta} - a_1) e_1 u_2 - 2c\eta \tfrac{1}{s^2(s^2+1)} e_2 u_1.$$

Therefore, we estimate

$$\begin{aligned}R_1[e_2 2c\eta \tfrac{1}{s^2(1+s^2)}u_1] &\leq 2c^2\eta^{\gamma_1}\int_{\tilde{s}_0}^{s}\tau^{\gamma_2-4}((c_1+\tilde{c}_2)(\tfrac{\tau}{\eta})^{\gamma_1}+(\tilde{c}_1+c_2)(\tfrac{\tau}{\eta})^{\gamma_2})\\ &\leq c^4(c_1+\tilde{c}_2)+c^4(\tfrac{s\wedge\tilde{s}_0}{\eta})^{-\gamma}(\tilde{c}_1+c_2)\\ R_2[e_2 2c\eta \tfrac{1}{s^2(1+s^2)}u_1] &= 2c^2\eta^{\gamma_2}\int_{\tilde{s}_0}^{s}\tau^{\gamma_1-4}((c_1+\tilde{c}_2)(\tfrac{\tau}{\eta})^{\gamma_1}+(\tilde{c}_1+c_2)(\tfrac{\tau}{\eta})^{\gamma_2})\\ &\leq 2c^3\eta^{-\gamma}(c_1+\tilde{c}_2)+c^4(\tilde{c}_1+c_2).\end{aligned}$$

By Taylor formula we obtain $|c\frac{1}{\eta}-a_1|\leq 18c\frac{|s|}{\eta^2}$ and so

$$\begin{aligned}R_1[(c\tfrac{1}{\eta}-a_1)u_2e_1] &\leq (1+10c^2)c^2\eta^{1-\gamma_2}\int\tau^{\gamma_2-1}18c\tfrac{\tau}{\eta^2}((\tfrac{\tau}{\eta})^{\gamma_1-1}c_1^*+(\tfrac{\tau}{\eta})^{\gamma_2-1}c_2^*)\\ &\leq 20c^3c_1^*+20cc_2^*\\ &\leq 20c^2c_1+20\tilde{c}_1+20c^2c_2+20c^4\tilde{c}_2,\\ R_2[(c\tfrac{1}{\eta}-a_1)u_2e_1] &= (1+10c^2)\eta^{1-\gamma_1}\int\tau^{\gamma_1-1}18c\tfrac{\tau}{\eta^2}((\tfrac{\tau}{\eta})^{\gamma_1-1}c_1^*+c(\tfrac{\tau}{\eta})^{\gamma_2-1}c_2^*))\\ &\leq 20c|\tfrac{s\vee\tilde{s}_0}{\eta}|^{\gamma}c_1^*+20cc_2^*\\ &\leq 20|\tfrac{s\vee\tilde{s}_0}{\eta}|^{\gamma}(c_1+2c^2\tilde{c}_2)+20\tilde{c}_1+20c^2c_2.\end{aligned}$$

□

Lemma 21 (k_0-th current estimate). *Let $u(s)=S(s)r(s)$ be a solution of (24) on $[\tilde{s}_0,s^*]$ such that $|u(s)|\leq S^*(s)C(s)$, then we estimate*

$$\begin{aligned}R_1[F_j] &\leq \tfrac{c^3}{\beta}(c_1+\tilde{c}_2)+\tfrac{c^3}{\beta}(\tfrac{s\wedge s_0}{\eta})^{-\gamma}(\tilde{c}_1+c_2)+\begin{cases}\frac{4c^{2+\gamma_1}}{\kappa_{k_0}\eta^{1+\gamma_2}}j(k_0,\tilde{s}_0) & \text{on } I_1\\ \eta^{\gamma_1}\frac{c^{4+\gamma_1}}{\kappa_{k_0}}j(k_0,\tilde{s}_0) & \text{on } I_3\end{cases}\\ R_2[F_j] &\leq \min(\tfrac{1}{\beta}\tfrac{1}{2c^{1+\gamma}}\eta^{-\gamma},\tfrac{c}{\beta}(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma})(c_1+\tilde{c}_2)+\tfrac{1}{\beta}c(c_2+\tilde{c}_1)\\ &\quad+\tfrac{1}{\beta}c(c_2+\tilde{c}_1)+\begin{cases}4\frac{c^{\gamma_2}}{\kappa_{k_0}\eta^{1+\gamma_1}}j(k_0,\tilde{s}_0) & \text{on } I_1\\ \eta^{\gamma_2}\frac{c^{2+\gamma_2}}{\kappa_{k_0}}j(k_0,\tilde{s}_0) & \text{on } I_3\end{cases}.\end{aligned}$$

Furthermore, on I_1 we estimate

$$\begin{aligned}|j(k_0,\tilde{s}_1)| &\leq \tfrac{2d^2}{\eta^2}\exp(-\tfrac{\kappa}{2^5}\xi\eta^2)j(k_0,\tilde{s}_0)\\ &\quad+\tfrac{c^2}{\beta}\big((c_1+\tilde{c}_2)(\tfrac{d}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\tfrac{d}{\eta})^{\gamma_2}\big)\end{aligned}$$

and on I_3

$$\begin{aligned}|j(k_0,\tilde{s}_1)| &\leq c^2\eta^2\exp(-\kappa_{k_0}\eta^3)j(k_0,\tilde{s}_0)\\ &\quad+2\tfrac{16^2}{\beta}\tfrac{1}{\eta^2}(c_1+c_2+\tilde{c}_1+\tilde{c}_2)\end{aligned}$$

Proof. The equation

$$\partial_s j(k_0)=(\tfrac{2s}{1+s^2}-\kappa_{k_0}(1+s^2))j(k_0)+u_1$$

leads to

$$\begin{aligned}j(k_0) &= \tfrac{1+s^2}{1+s_0^2}\exp(-\kappa_{k_0}(s-s_0+\tfrac{1}{3}(s^3-s_0^3)))j(k_0,\tilde{s}_0)\\ &\quad+\tfrac{\kappa_k}{\beta}\int_{s_0}^{s}d\tau_2\,\tfrac{1+s^2}{1+\tau_2^2}\exp(-\kappa_{k_0}(s-\tau_2+\tfrac{1}{3}(s^3-\tau_2^3)))u_1(\tau_2)\\ &= j_1+j_2.\end{aligned}$$

Therefore, we estimate

$$\begin{aligned}
R_1[F_{j_2}] &= \tfrac{\kappa_{k_0}}{\beta}c^2\eta^{1-\gamma_2}\int_{s_0}^{s} d\tau_1 \int_{s_0}^{\tau_1} d\tau_2\ \tau_1^{\gamma_2-1}\tfrac{1+\tau_1^2}{1+\tau_2^2}\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))u_1(\tau_2)\\
&= \tfrac{\kappa_{k_0}}{\beta}c^2\eta^{1-\gamma_2}\int_{s_0}^{s} d\tau_1 \int_{s_0}^{\tau_1} d\tau_2\ \tau_1^{\gamma_2-1}\tfrac{1+\tau_1^2}{1+\tau_2^2}\\
&\qquad\cdot\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))((c_1+\tilde{c}_2)(\tfrac{\tau_2}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\tfrac{\tau_2}{\eta})^{\gamma_2})\\
&\leq \tfrac{1}{\beta}c^2\eta^{1-\gamma_2}\int_{s_0}^{s} d\tau_2\ \frac{((c_1+\tilde{c}_2)(\frac{\tau_2}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\frac{\tau_2}{\eta})^{\gamma_2})\tau_2^{-\gamma_1}}{1+\tau_2^2}\\
&\qquad\cdot\int_{\tau_2}^{s} d\tau_1\kappa_{k_0}(1+\tau_1^2)\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))\\
&= \tfrac{1}{\beta}c^2\eta^{1-\gamma_2}\int_{s_0}^{s} d\tau_2\ ((c_1+\tilde{c}_2)(\tfrac{\tau_2}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\tfrac{\tau_2}{\eta})^{\gamma_2})\tfrac{\tau_2^{-\gamma_1}}{1+\tau_2^2}\\
&\qquad\cdot\Big[-\exp(-\kappa_k(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))\Big]_{\tau_1=\tau_2}^{\tau_1=s}\\
&\leq \tfrac{1}{\beta}c^2\eta^{1-\gamma_2}\int_{s_0}^{s} d\tau_2\ (c_1+\tilde{c}_2)\eta^{-\gamma_1}\tau_2^{-2}+(c_2+\tilde{c}_1)\eta^{-\gamma_2}\tau_2^{-\gamma-2}\\
&\leq (c_1+\tilde{c}_2)\tfrac{1}{\beta}c^2\eta^{1-\gamma_2-\gamma_1}[-\tau^{-1}]_{s_0}^{s}+(c_2+\tilde{c}_1)\tfrac{1}{\beta}c^2\eta^{\gamma}[-\tau^{-\gamma-1}]_{s_0}^{s}\\
&\leq \tfrac{c^3}{\beta}(c_1+\tilde{c}_2)+\tfrac{c^3}{\beta}(\tfrac{s\wedge s_0}{\eta})^{-\gamma}(\tilde{c}_1+c_2)
\end{aligned}$$

and

$$\begin{aligned}
R_2[F_{j_2}] &= \tfrac{\kappa_{k_0}}{\beta}\eta^{1-\gamma_1}\int_{s_0}^{s} d\tau_1 \int_{s_0}^{\tau_1} d\tau_2\ \tau_1^{\gamma_1-1}\tfrac{1+\tau_1^2}{1+\tau_2^2}\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))u_1(\tau_2)\\
&= \tfrac{\kappa_{k_0}}{\beta}\eta^{1-\gamma_1}\int_{s_0}^{s} d\tau_1 \int_{s_0}^{\tau_1} d\tau_2\ \tau_1^{-\gamma_2}\tfrac{1+\tau_1^2}{1+\tau_2^2}\\
&\qquad\cdot\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))((c_1+\tilde{c}_2)(\tfrac{\tau_2}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\tfrac{\tau_2}{\eta})^{\gamma_2})\\
&\leq \tfrac{1}{\beta}\eta^{\gamma_2}\int_{s_0}^{s} d\tau_2\ \frac{((c_1+\tilde{c}_2)(\frac{\tau_2}{\eta})^{\gamma_1}+(c_2+\tilde{c}_1)(\frac{\tau_2}{\eta})^{\gamma_2})\tau_2^{-\gamma_2}}{1+\tau_2^2}\\
&\qquad\cdot\int_{s_0}^{\tau_2} d\tau_1\ \kappa_k(1+\tau_1^2)\exp(-\kappa_k(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))\\
&= \tfrac{1}{\beta}\int_{s_0}^{s} d\tau_2\ ((c_1+\tilde{c}_2)\tau_2^{\gamma-2}\eta^{-\gamma}+(c_2+\tilde{c}_1)\tau_2^{-2}\eta^{-\gamma_2})\\
&\qquad\cdot[\exp(-\kappa_{k_0}(\tau_1-\tau_2+\tfrac{1}{3}(\tau_1^3-\tau_2^3)))]_{\tau_1=\tau_2}^{\tau_1=s}\\
&\leq \tfrac{1}{\beta}\int_{s_0}^{s} d\tau_2\ ((c_1+\tilde{c}_2)\tau_2^{\gamma-2}\eta^{-\gamma}+(c_2+\tilde{c}_1)\tau_2^{-2}\eta^{-\gamma_2}).
\end{aligned}$$

We note that for the first term we obtain

$$\tfrac{1}{\beta}\int_{s_0}^{s} d\tau_2\ \tau_2^{\gamma-2}\eta^{-\gamma}\leq \min(\tfrac{c}{\beta}(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma},\tfrac{1}{\beta c}(c\eta)^{-\gamma}),$$

since we can either integrate it directly or first pull out s^γ and then integrate. Finally, we obtain the following estimate

$$R_1[F_{j_2}]\leq \min(\tfrac{1}{\beta}\tfrac{1}{2c^{1+\gamma}}\eta^{-\gamma},\tfrac{c}{\beta}(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma})(c_1+\tilde{c}_2)+\tfrac{1}{\beta}c(c_2+\tilde{c}_1).$$

On I_1 we estimate the $j(k_0)$ influence by

$$\begin{aligned}
R_1[F_{j_1}] &= c^2\eta^{1-\gamma_2}j(s_0)\int \tau^{\gamma_2-1}\tfrac{1+\tau^2}{1+s_0^2}\exp(-\kappa_{k_0}(\tau-s_0+\tfrac{1}{3}(\tau^3-s_0^3)))\\
&\leq 4c^{2+\gamma_1}\eta^{-1-\gamma_2}j(s_0)\int(1+\tau^2)\exp(-\kappa_{k_0}(\tau-s_0+\tfrac{1}{3}(\tau^3-s_0^3)))\\
&\leq \tfrac{4c^{2+\gamma_1}}{\kappa_{k_0}\eta^{1+\gamma_2}}j(s_0)
\end{aligned}$$

and

$$\begin{aligned} R_2[F_{j2}] &= \eta^{1-\gamma_1} j(s_0) \int \tau^{\gamma_1 - 1} \tfrac{1+\tau^2}{1+s_0^2} \exp(-\kappa_{k_0}(\tau - s_0 + \tfrac{1}{3}(\tau^3 - s_0^3))) \\ &= 4\eta^{-1-\gamma_1} c^{\gamma_2} j(s_0) \int (1+\tau^2) \exp(-\kappa_{k_0}(\tau - s_0 + \tfrac{1}{3}(\tau^3 - s_0^3))) \\ &= 4\tfrac{c^{\gamma_2}}{\kappa_{k_0}\eta^{1+\gamma_1}} j(s_0). \end{aligned}$$

We estimate $j(k_0)$ by

$$\begin{aligned} j(k_0, s) = &\tfrac{1+s^2}{1+\tilde{s}_0^2} \exp(-\kappa_{k_0}(s - \tilde{s}_0 + \tfrac{1}{3}(s^3 - \tilde{s}_0^3))) j(k_0.\tilde{s}_0) \\ &+ \tfrac{\kappa_{k_0}}{\beta} \int_{\tilde{s}_0}^{s} d\tau \, \tfrac{1+s^2}{1+\tau^2} \exp(-\kappa_{k_0}(s - \tau + \tfrac{1}{3}(s^3 - \tau^3))) u_1(\tau) \\ &\leq \tfrac{4d^2}{\eta^2} \exp(-\tfrac{\kappa}{2^5}\xi\eta^2) j(k_0, \tilde{s}_0) + \tfrac{c^2}{\beta}\big((c_1 + \tilde{c}_2)(\tfrac{d}{\eta})^{\gamma_1} + (c_2 + \tilde{c}_1)(\tfrac{d}{\eta})^{\gamma_2}\big). \end{aligned}$$

On I_3 we estimate the $j(k_0)$ influence by

$$\begin{aligned} R_1[F_{j_1}] &= c^2\eta^{1-\gamma_2} \int \tau^{\gamma_2 - 1} \tfrac{1+\tau^2}{1+s_0^2} \exp(-\kappa_{k_0}(\tau - s_0 + \tfrac{1}{3}(\tau^3 - s_0^3))) j(s_0) \\ &\leq c^{4+\gamma_1}\eta^{\gamma_1} \int (1+\tau^2) \exp(-\kappa_{k_0}(\tau - s_0 + \tfrac{1}{3}(\tau^3 - s_0^3))) j(s_0) \\ &\leq \eta^{\gamma_1} \tfrac{c^{4+\gamma_1}}{\kappa_{k_0}} j(s_0) \end{aligned}$$

and

$$\begin{aligned} R_2[F_{j_2}] &= \eta^{1-\gamma_1} \int \tau^{\gamma_1 - 1} \tfrac{1+\tau^2}{1+s_0^2} \exp(-\kappa_{k_0}(\tau - s_0 + \tfrac{1}{3}(\tau^3 - s_0^3))) j(s_0) \\ &= \eta^{\gamma_2} c^{2+\gamma_2} \int (1+\tau^2) \exp(-\kappa_{k_0}(\tau - s_0 + \tfrac{1}{3}(\tau^3 - s_0^3))) j(s_0) \\ &= \eta^{\gamma_2} \tfrac{c^{2+\gamma_2}}{\kappa_{k_0}} j(s_0). \end{aligned}$$

Next we want to estimate the evolution of $j(k_0)$

$$\begin{aligned} j(k_0, \tilde{s}_1) = &\tfrac{1+\tilde{s}_1^2}{1+d^2} \exp(-\kappa_{k_0}(\tilde{s}_1 - d + \tfrac{1}{3}(\tilde{s}_1^3 - d^3))) j(k_0, \tilde{s}_0) \\ &+ \tfrac{\kappa_{k_0}}{\beta} \int_{d}^{\tilde{s}_1} d\tau_2 \, \tfrac{1+s^2}{1+\tau_2^2} \exp(-\kappa_k(s - \tau_2 + \tfrac{1}{3}(s^3 - \tau_2^3)))((c_1 + \tilde{c}_2)(\tfrac{\tau_2}{\eta})^{\gamma_1} + (c_2 + \tilde{c}_1)(\tfrac{\tau_2}{\eta})^{\gamma_2}). \end{aligned}$$

Therefore, we deduce

$$\begin{aligned} &\tfrac{\kappa_{k_0}}{\beta} \int_{d}^{\tilde{s}_1} d\tau_2 \, \tfrac{1+s^2}{1+\tau_2^2} \exp(-\kappa_{k_0}(s - \tau_2 + \tfrac{1}{3}(s^3 - \tau_2^3)))(\tfrac{\tau_2}{\eta})^{\gamma_i} \\ &\qquad \leq \tfrac{\kappa_{k_0}}{\beta} \left(\int_{d}^{\frac{1}{2}\tilde{s}_1} + \int_{\frac{1}{2}\tilde{s}_1}^{\tilde{s}_1} \right) \tfrac{1+s^2}{1+\tau_2^2} \exp(-\kappa_{k_0}(s - \tau_2 + \tfrac{1}{3}(s^3 - \tau_2^3)))(\tfrac{\tau_2}{\eta})^{\gamma_i} \\ &\qquad \leq \tfrac{\kappa_{k_0}}{\beta} \eta^{-\gamma_i} \tfrac{c^{1-\gamma_i}}{\gamma_i} (1+\eta^2) \exp(-\tfrac{\kappa_{k_0}}{25}\eta^3) + \tfrac{\kappa_{k_0}}{\beta} \tfrac{2^4}{\kappa_{k_0}\eta^2} \\ &\qquad \leq \tfrac{2^5}{\beta} \tfrac{1}{\eta^2}, \end{aligned}$$

which leads to

$$\begin{aligned} |j(k_0, \tilde{s}_1)| \leq\, &c^2 \kappa_{k_0} \eta^2 \exp(-\kappa_{k_0}\eta^3) j(k_0, \tilde{s}_0) \\ &+ \tfrac{2^5}{\beta} \tfrac{1}{\eta^2} (c_1 + c_2 + \tilde{c}_1 + \tilde{c}_2). \end{aligned}$$

□

Lemma 22 (Forcing estimate)**.** *Let $u(s) = S(s)r(s)$ be a solution of (24) on $[\tilde{s}_0, s^*]$ such that $|u(s)| \leq S^*(s)C(s)$. We define for $|n| \geq 2$*

$$\begin{aligned} \tilde{w}(n) = 2 &\sum_{|m| \geq 2} (2c)^{|m-n|+\chi} (w + \tfrac{4}{\kappa\xi} j)(k_m, \tilde{s}_0) \\ &+ (2c)^{||m|-2|} c(2c_1^* + \tfrac{1}{c^2} c_2^*) \\ &+ (2c)^{|m|-1} (u_3(\tilde{s}_0) + \tfrac{2}{\kappa\xi\eta}(j(k_{\pm 1}, \tilde{s}_0)) \end{aligned}$$

where $\chi = \chi(m,n) = -|sgn(m) - sgn(n)|$*. Then we estimate*

$$\begin{aligned} R_1[F_{\tilde w}] &= 2c^2(\tilde w(2) + \tilde w(-2)) \\ R_2[F_{\tilde w}] &= c^2(\tilde w(2) + \tilde w(-2))(\tfrac{s\vee \tilde s_0}{\eta})^{\gamma_1} \end{aligned}$$

and

$$\begin{aligned} R_1[F_{j(k_{\pm 1})}] &= \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1}, \tilde s_0) + \tfrac{2c}{\beta\kappa\xi}(\tilde w(1) + c_1^* + c_2^*) \\ R_2[F_{j(k_{\pm 1})}] &= \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1}, \tilde s_0)(\tfrac{s\vee\tilde s_0}{\eta})^{\gamma_1} + \tfrac{c}{\beta\kappa\xi}(\tilde w(1) + c_1^* + c_2^*)(\tfrac{s\vee\tilde s_0}{\eta})^{\gamma_1} \end{aligned}$$

and

$$\begin{aligned} R_1[F_{u_3}] &= 2c\tilde w(1) \\ R_2[F_{u_3}] &= 2c\tilde w(1)(\tfrac{s\vee\tilde s_0}{\eta})^{\gamma_1}. \end{aligned}$$

Furthermore, we estimate

$$\begin{aligned} |w(k_n, s)| &\le \tilde w(n) && |n| \ge 2 \\ |u_3| &\le \tilde w(1) = \tilde w(-1) \\ |j(k_n)| &\le 2e^{-\frac12 \kappa\xi\eta(s-\tilde s_0)} j(k_n, \tilde s_0) + 4\tfrac{1}{\beta\eta^2}\tilde w(n). \end{aligned}$$

Proof. To estimate $w(k_n, s)$ we without loss of generality assume that $n \ge 2$. We begin with the case $n \ge 3$, where we deduce that

$$\begin{aligned} \partial_s w(k_n) &= a(k_{n+1})w(k_{n+1}) - a(k_{n-1})w(k_{n-1}) - j(k_n) \\ &\le \tfrac{c}{\eta}(\tilde w(n-1) + \tilde w(n+1)) + 2e^{-\kappa\xi\eta(s-\tilde s_0)} j(k_0+n, \tilde s_0) + 4\tfrac{1}{\beta\eta^2}\tilde w(n). \end{aligned}$$

We estimate

$$2\int e^{-\frac12\kappa\xi\eta(\tau-\tilde s_0)} j(k_0+n, \tilde s_0) \le 4\tfrac{1}{\kappa\xi\eta} j(k_n, \tilde s_0).$$

Thus integrating $\partial_s w(k_0+n)$ over time yields

$$\begin{aligned} w(k_n) &\le w(k_n, \tilde s_0) + c(\tilde w(n-1) + \tilde w(n+1)) + 4\tfrac{1}{\kappa\xi\eta} j(k_0+n, \tilde s_0) + \tfrac{4}{\beta\kappa\xi}\tilde w(n) \\ &< \tilde w(n). \end{aligned}$$

For the case $n = 2$ we deduce

$$\begin{aligned} \partial_s w(k_2) &= a(k_3)w(k_3) - a(k_1)\tfrac12(u_3 + u_2) - j(k_2) \\ &\le \tfrac{c}{\eta}(\tilde w(3) + 2\tilde w(1) + 2c_1^*(\tfrac{s}{\eta})^{\gamma_1-1} + 2c_2^*(\tfrac{s}{\eta})^{\gamma_2-1}) + 2e^{-\frac12\kappa\eta\xi(s-\tilde s_0)} j(k_2, \tilde s_0) + 4\tfrac{1}{\beta\kappa\xi\eta}\tilde w(2) \\ w(k_2) &\le w(k_2, \tilde s_0) + \tfrac{4}{\kappa\xi\eta} j(k_2, \tilde s_0) + c(\tilde w(3) + 2\tilde w(1) + 2c_1^* + \tfrac{1}{c^2}c_2^*) + 4\tfrac{1}{\beta\kappa\xi}\tilde w(2) \\ &< \tilde w(2). \end{aligned}$$

We estimate u_3 by

$$\begin{aligned} \partial_s u_3 &= a(k+2)w(k_2) - a(k-2)w(k_{-2}) - j(k_1) + j(k_{-1}) \\ &\le \tfrac{2c}{\eta}(w(2) + w(-2)) + 2e^{-\frac12\kappa\xi\eta(s-\tilde s_0)} j(k_{\pm 1}, \tilde s_0) + \tfrac{4}{\beta\kappa\xi\eta}(\tilde w(1) + c_1^* + c_2^*) \\ |u_3| &\le |u_3(\tilde s_0)| + c(\tilde w(2) + \tilde w(-2)) + \tfrac{4}{\kappa\xi\eta}(j(k_{\pm 1}, \tilde s_0) + \tfrac{1}{\beta}\tilde w(1) + \tfrac{1}{\beta}c_1^* + \tfrac{1}{\beta}c_2^*) \\ &\le \tilde w(1). \end{aligned}$$

Non-resonant j will often be estimated similarly. Therefore we will use the following notation frequently. We estimate $j(k_n)$ for $n \ge 2$ by writing $\hat s = s - \frac{k_0(k_0-k)}{k+1}\eta$ and $\hat\tau = \tau - \frac{k_0(k_0-k)}{k+1}\eta$

$$\partial_s j(k_n) = -\kappa_{k_n}(1+\hat s^2) j(k_n) + 2\tfrac{\hat s}{1+\hat s^2} j(k_n) + \tfrac{1}{\beta}\kappa_{k_n} w(k_n)$$

which gives

$$\begin{aligned} j(k_n) &\leq \tfrac{1+\hat{s}^2}{1+\hat{\tilde{s}}_0^2} e^{-\kappa_{k_n}((\hat{s}-\hat{\tilde{s}}_0+\frac{1}{3}(\hat{s}^3-\hat{\tilde{s}}_0^3))} j(k_n,\tilde{s}_0) \\ &\quad + \tfrac{1}{\beta}\kappa_{k_n}\int d\tau \ \tfrac{1+\hat{s}^2}{1+\hat{\tau}^2} e^{-\kappa_{k_n}((\hat{s}-\hat{\tau}+\frac{1}{3}(\hat{s}^3-\hat{\tau}^3))}\tilde{w}(n) \end{aligned}$$

For $\tilde{s}_0 \leq \tau \leq s \leq \tilde{s}_1$ we obtain

$$\begin{aligned} \kappa_{k_n}(\hat{s}-\hat{\tau}+\tfrac{1}{3}(\hat{s}^3-\hat{\tau}^3)) &= \kappa_{k_n}\tfrac{1}{3}(s-\tau)(\hat{s}^2+\hat{s}\hat{\tau}+\hat{\tau}^2+1) \\ &\geq \tfrac{1}{2}\kappa\max(k_n^2,k_0^2)\eta^2(s-\tau) \\ \tfrac{1+\hat{s}^2}{1+\hat{\tau}} &\leq 2. \end{aligned}$$

So we infer

$$\begin{aligned} j(k_n) &\leq 2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_n,\tilde{s}_0) \\ &\quad + 2\kappa_{k_n}\tfrac{1}{\beta}\int d\tau \ e^{-\frac{1}{2}\kappa\xi\eta(s-\tau)}\tilde{w}(n) \\ &\leq 2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_n,\tilde{s}_0) + \tfrac{4}{\beta\eta^2}\tilde{w}(n). \end{aligned}$$

We next turn to the estimate of $j(k_{\pm 1})$, where we without loss of generality consider $j(k_1)$. With the equation

$$\partial_s j(k_1) = (2\tfrac{\hat{s}}{1+\hat{s}^2} - \kappa_{k_1}(1+\hat{s}^2))j(k_1) + \tfrac{\kappa_{k_1}}{2\beta}\kappa_{k_1}(u_3+u_2)$$

we estimate

$$\begin{aligned} j(k_1) &\leq 2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_1,\tilde{s}_0) \\ &\quad + \tfrac{\kappa_{k_1}}{2\beta}\int d\tau \ e^{-\frac{1}{2}\kappa\xi\eta(s-\tau)}(\tilde{w}(1) + c_1^*(\tfrac{\tau}{\eta})^{\gamma_1-1} + c_2^*(\tfrac{\tau}{\eta})^{\gamma_2-1}) \\ &\leq 2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_1,\tilde{s}_0) + \tfrac{2}{\beta\eta^2}(\tilde{w}(1)+c_1^*+c_2^*). \end{aligned}$$

Given these estimates, we next consider the effects on $R[\cdot]$ by forcing:

$$\begin{aligned} F_{\tilde{w}} &= e_1(a(k_2)w(k_2) + a(k_{-2})w(k_{-2}) \\ &\leq e_1\tfrac{c}{\eta}\tilde{w}(2) + e_1\tfrac{c}{\eta}\tilde{w}(-2). \end{aligned}$$

For constant e_2 functions we estimate

$$\begin{aligned} R_1[e_2] &\leq \tfrac{c}{2}\eta \\ R_2[e_2] &\leq \tfrac{c}{3}\eta(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma_1}. \end{aligned}$$

Therefore, we can control $F_{\tilde{w}}$ by

$$\begin{aligned} R_1[F_{\tilde{w}}] &= c^2(\tilde{w}(2)+\tilde{w}(-2)) \\ R_2[F_{\tilde{w}}] &= c^2(\tilde{w}(2)+\tilde{w}(-2))(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma_1}. \end{aligned}$$

For $F_{j(k_{\pm 1})}$ we use

$$\begin{aligned} F_{j(k_{\pm 1})} &= -e_2 j(k_{\pm 1}) \\ &\leq e_2(2e^{-\frac{1}{2}\kappa\xi\eta(s-\tilde{s}_0)} j(k_{\pm 1},\tilde{s}_0) + \tfrac{2}{\beta\kappa\xi\eta}(\tilde{w}(1)+c_1^*+c_2^*)), \end{aligned}$$

to estimate

$$\begin{aligned} R_1[F_{j(k_{\pm 1})}] &= \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1},\tilde{s}_0) + \tfrac{2c}{\beta\kappa\xi}(\tilde{w}(1)+c_1^*+c_2^*) \\ R_2[F_{j(k_{\pm 1})}] &= \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1},\tilde{s}_0)(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma_1} + \tfrac{c}{\beta\kappa\xi}(\tilde{w}(1)+c_1^*+c_2^*)(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma_1}. \end{aligned}$$

Furthermore, for F_{u_2} we estimate

$$\begin{aligned} F_{u_3} &= e_1 a_2 u_3 \\ &\leq e_1\tfrac{c}{\eta}\tilde{w}(1) \end{aligned}$$

and

$$\begin{aligned} R_1[e_1] &\leq \eta \\ R_2[e_1] &\leq \eta(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma_1}. \end{aligned}$$

to deduce

$$\begin{aligned} R_1[F_{u_3}] &\leq c\tilde{w}(1) \\ R_2[F_{u_3}] &\leq c\tilde{w}(1)(\tfrac{s\vee\tilde{s}_0}{\eta})^{\gamma_1}. \end{aligned}$$

□

Proof of Proposition 17. For the interval I_1 we have $\tilde{s}_0 = s_0$, $\tilde{s}_1 = -d$. The initial data of r can be calculated by $r(\tilde{s}_0) = S^{-1}(s_0)u(s_0)$ and so

$$\begin{aligned} r_1(\tilde{s}_0) &= -\tfrac{\gamma_2}{\gamma}(\tfrac{k_0}{2(k_0+1)})^{\gamma_2-1}u_1(\tilde{s}_0) + \tfrac{c}{\gamma}(\tfrac{k_0}{2(k_0+1)})^{\gamma_2}u_2(\tilde{s}_0) \\ &\approx -4c^2u_1(\tilde{s}_0) + cu_2(\tilde{s}_0), \\ r_2(\tilde{s}_0) &= \tfrac{\gamma_1}{\gamma}(\tfrac{k_0}{2(k_0+1)})^{\gamma_1-1}u_1(\tilde{s}_0) - \tfrac{c}{\gamma}(\tfrac{k_0}{2(k_0+1)})^{\gamma_1}u_2(\tilde{s}_0) \\ &\approx u_1(\tilde{s}_0) - \tfrac{c}{2}u_2(\tilde{s}_0). \end{aligned}$$

For other initial data we define

$$\begin{aligned} N = &\sum_{|m|\geq 2}(2c)^{|m|}(w + \tfrac{8}{\kappa\eta\xi}j)(k_m,\tilde{s}_0) \\ &+ 2c(u_3(\tilde{s}_0) + \tfrac{8}{\kappa\eta\xi}j(k_{\pm 1},\tilde{s}_0)) \\ &+ \tfrac{2c}{\xi\kappa}j(k_0,s_0), \end{aligned}$$

to bound the impact of the less important terms in the following bootstrap. Let $C(s)$ be defined by the terms

$$\begin{aligned} c_1 &= 45c^2u_1(s_0) + 2cu_2(s_0) + 2N, \\ \tilde{c}_1 &= 2\tfrac{c^3}{\beta\vee 1}c_2, \\ c_2 &= 2u_1(s_0) + 45cu_2(s_0) + 2N, \\ \tilde{c}_2 &= 0. \end{aligned}$$

AS $c_1 > r_1(\tilde{s}_0)$ and $c_2 > r_2(\tilde{s}_0)$ and we have a smooth solution, the estimate $|u| \leq S^*(s)C(s)$ holds at least for a small time. Let s^* be the maximal time such that $|u| \leq S^*(s)C(s)$. We then aim to show that necessarily $s^* \geq -d$, since otherwise the estimate improves, which contradicts the maximality. By Lemma 20, Lemma 21 and Lemma 22 we estimate

$$\begin{aligned} R_1[F_{all}] &= R_1[F_{3mode}] + R_1[F_j] + R_1[F_{\tilde{w}}] + R_1[F_{j(k_0\pm 1)}] + R_1[F_{u_3}] \\ &= 20c^2c_1 + (20 + c^4(\tfrac{s}{\eta})^{-\gamma})\tilde{c}_1 + (20c^2 + c^4(\tfrac{s}{\eta})^{-\gamma})c_2 \\ &+ \tfrac{c^3}{\beta}c_1 + \tfrac{c^3}{\beta}(\tfrac{s}{\eta})^{-\gamma}(\tilde{c}_1 + c_2) + \tfrac{4c^{2+\gamma_1}}{\kappa_{k_0}\eta^{1+\gamma_2}}j(k_0,s_0) \\ &+ 2c^2(\tilde{w}(2) + \tilde{w}(-2)) \\ &+ \tfrac{2c}{\kappa\xi\eta}j(k_{\pm 1},\tilde{s}_0) + \tfrac{2c}{\beta\kappa\xi}(\tilde{w}(1) + c_1^* + c_2^*) \\ &+ 2c\tilde{w}(1) \\ &< 21c^2c_1 + \tilde{c}_1(21 + \tfrac{c^3}{\beta}(\tfrac{s}{\eta})^{-\gamma}) + c_2(21c^2 + \tfrac{c^3}{\beta}(\tfrac{s}{\eta})^{-\gamma}) + N \end{aligned}$$

and

$$\begin{aligned}
R_2[F_{all}] &= R_2[F_{3mode}] + R_2[F_j] + R_2[F_{\tilde{w}}] + R_2[F_{j(k_0\pm 1)}] + R_2[F_{u_3}] \\
&= 20c_1 + 20\tilde{c}_1 + 20c^2 c_2 \\
&+ \tfrac{c}{\beta} c_1 + \tfrac{c}{\beta}(c_2 + \tilde{c}_1) + 4\tfrac{c^{\gamma_2}}{\kappa_{k_0}\eta^{1+\gamma_1}} j(k_0, s_0) \\
&+ c^2(\tilde{w}(2) + \tilde{w}(-2)) \\
&+ \tfrac{2c}{\kappa\xi\eta} j(k_{\pm 1}, \tilde{s}_0) + \tfrac{c}{\beta\kappa\xi}(\tilde{w}(1) + c_1^* + c_2^*) \\
&+ 2c\tilde{w}(1) \\
&< 21c_1 + 21\tilde{c}_1 + \tfrac{c}{\beta\wedge 1} c_2 + N.
\end{aligned}$$

We split R_1 as

$$R_1[all] = R_1[all][1] + R_1[all][(\tfrac{s}{\eta})^\gamma],$$

into the part with and without a $(\frac{s}{\eta})^\gamma$ term, respectively. We then estimate

$$\begin{aligned}
r_1(\tilde{s}_0) + R_1[all][1] &< r_1(\tilde{s}_0) + 21c^2 c_1 + 21\tilde{c}_1 + 21c^2 c_2 + N, \\
R_1[all][(\tfrac{s}{\eta})^\gamma] &< \tfrac{c^3}{1\wedge\beta}\tilde{c}_1 + \tfrac{c^3}{1\wedge\beta} c_2, \\
r_2(\tilde{s}_0) + R_2[all][1] &< r_2(\tilde{s}_0) + 21c_1 + 21\tilde{c}_1 + 2\tfrac{c}{1\wedge\beta} c_2 + N,
\end{aligned}$$

and thus we conclude the bootstrap that

$$\begin{aligned}
r_1(\tilde{s}_0) + R_1[all][1] &< c_1 \\
R_1[all][(\tfrac{s}{\eta})^\gamma] &< \tilde{c}_1 \\
r_2(\tilde{s}_0) + R_2[all][1] &< c_2.
\end{aligned}$$

We can therefore extend the estimates past the time s^*, which contradicts the maximally. Therefore, we obtain that for all times $s \leq -d$ it holds that

$$|u(s)| \leq S^*(s)C(s),$$

which yields the upper bound

$$\begin{aligned}
|u(-d)| &\leq \begin{pmatrix} (c\eta)^{-\gamma_1} c_1 + (\tilde{c}_1 + c_2)(c\eta)^{-\gamma_2} \\ \frac{1}{c}(c\eta)^{1-\gamma_1} c_1 + (\frac{1}{c}\tilde{c}_1 + cc_2)(c\eta)^{1-\gamma_2} \end{pmatrix} \\
&\leq 2M \begin{pmatrix} (c\eta)^{-\gamma_2} \\ (c\eta)^{1-\gamma_2} \end{pmatrix}.
\end{aligned}$$

We next aim to establish an estimate on $\tilde{w}(n)$. For this purpose we note that

$$\begin{aligned}
c(2c_1^* + \tfrac{1}{c^2} c_2^*) &\approx 2c_1 + \tfrac{1}{c^2}\tilde{c}_1 + 2c_2 \\
&\approx 2c_1 + 2c_2 \\
&\leq 4u_1(s_0) + 100cu_2(s_0) + 3N
\end{aligned}$$

and

$$\begin{aligned}
\tilde{w}(n) &\leq 2\sum_{|m|\geq 2} (2c)^{|m-n|+\chi}(w + \tfrac{4}{\kappa\xi\eta} j)(k_m, \tilde{s}_0) \\
&+ (2c)^{||n|-2|} c(2c_1^* + \tfrac{1}{c^2} c_2^*) \\
&+ (2c)^{|n|-1}(u_3(\tilde{s}_0) + \tfrac{2}{\kappa\xi\eta} j(k_{\pm 1}, \tilde{s}_0)).
\end{aligned}$$

We hence deduce that

$$\begin{aligned}
\tilde{w}(n) &\leq 2\sum_{|m|\geq 2} (2c)^{|m-n|+\chi}(w + \tfrac{4}{\kappa\xi\eta} j)(k_m, \tilde{s}_0) \\
&+ (2c)^{||n|-2|}(4u_1(s_0) + 100cu_2(s_0) + 3N) \\
&+ (2c)^{|n|-1}(u_3(\tilde{s}_0) + \tfrac{2}{\kappa\xi\eta} j(\tilde{s}_0, k_{\pm 1})) \\
&\leq 2M_n,
\end{aligned}$$

when $\chi = -|\mathrm{sgn}(m) - \mathrm{sgn}(n)|$. To prove (26) under the condition (25) we estimate

$$\begin{aligned}|u(-d) - \tilde{u}(-d)| &\leq S(-d)R[all]\\ &\leq u_1(\tilde{s}_0)\begin{pmatrix}(c\eta)^{-\gamma_2}\\ 5c(c\eta)^{\gamma_1}\end{pmatrix}.\end{aligned}$$

Furthermore, we use

$$\begin{aligned}\tilde{u}(-d) &= \begin{pmatrix}(c\eta)^{-\gamma_1} & (c\eta)^{-\gamma_2}\\ -\frac{\gamma_1}{2c}(c\eta)^{1-\gamma_1} & -\frac{\gamma_2}{2c}(c\eta)^{1-\gamma_2}\end{pmatrix}\begin{pmatrix}4c^2u_1(\tilde{s}_0) - 2cu_2(\tilde{s}_0)\\ u_1(\tilde{s}_0) + cu_2(\tilde{s}_0)\end{pmatrix}\\ &\approx u_1(\tilde{s}_0)\begin{pmatrix}(c\eta)^{-\gamma_2}\\ O(c)(c\eta)^{\gamma_1}\end{pmatrix}\end{aligned}$$

and thus

$$\begin{aligned}|u_1(-d) - (c\eta)^{-\gamma_2}u_1(\tilde{s}_0)| &= 10cu_1(\tilde{s}_0)(c\eta)^{-\gamma_2}\\ |u_2(-d)| &\leq 10cu_1(\tilde{s}_0)(c\eta)^{\gamma_1}.\end{aligned}$$

The remaining terms can be estimated by

$$\begin{aligned}M &\leq \tfrac{1}{1-10^{-1}}u_1(\tilde{s}_0),\\ M_n &\leq \tfrac{4}{1-10^{-1}}u_1(\tilde{s}_0).\end{aligned}$$

□

4.3. **The Resonance and Upper Bounds in I_2.** The bounds on the evolution of (24) on the interval $I_2 = [-d, d]$ are summarized in the following proposition:

Proposition 23. *Let $c \leq \min((8\pi)^{-\frac{4}{3}}\beta^{\frac{16}{3}}, 10^{-4})$. Consider a solution of (24) on the interval $I = [s_0, d]$, then it holds that*

$$\begin{aligned}|u_1(d)| &\leq 3(c\eta)^{-\gamma_2}LM,\\ |u_2(d)| &\leq 7\pi(c\eta)^{\gamma_1}LM,\\ |u_3(d)| &\leq 7\pi(\tfrac{5}{\eta})^2(c\eta)^{\gamma_1}LM + 2M_1,\\ |w(k_n, d)| &\leq 7\pi(\tfrac{5}{\eta})^{|n|-1}(c\eta)^{\gamma_1}LM + 2M_n,\\ |j(k_n, d)| &\leq \tfrac{4}{\beta\eta^2}(7\pi(\tfrac{5}{\eta})^{|n|-1}(c\eta)^{\gamma_1}LM + 2M_n),\\ |j(k_0, d)| &\leq \tfrac{4}{\beta}\min(\kappa_{k_0}\pi d^2, 1)(c\eta)^{-\gamma_2}LM.\end{aligned}$$

For interval I_2 we are mostly concerned with the interaction between $j(k_0)$ and u_1 and in particular the growth this induces for u_2. Therefore, consider the ODE system

$$\begin{aligned}\partial_s u_1 &= -j(k_0) + F\\ \partial_s j(k_0) &= \tfrac{\kappa_{k_0}}{\beta}u_1 + (\tfrac{2s}{1+s^2} - \kappa_{k_0}(1+s^2))j(k_0),\end{aligned} \tag{29}$$

our aim is to bound the growth of $j(k_0)$ and u_1 by a factor. Let $U(\tau, s)$ be the solution of (29) with initial data $u_1(\tau) = 1$ and $j(\tau) = 0$ and L as the constant which satisfies

$$|U(\tau, s)| \leq L = L(\beta, \kappa, k). \tag{30}$$

With the restriction

$$c \leq (8\pi)^{-\frac{4}{3}}\beta^{\frac{16}{3}},$$

L is estimated by the following two cases, if $\beta \geq \pi$ we obtain $L = 1$ and if $\beta < \pi$ we obtain a $L = L(\alpha, \kappa, k) \leq \sqrt{c}$. A proof and more specific bounds can be found in Appendix A and for simplicity of presentation we here only consider two cases.

Lemma 24. *Let u_1 be a solution of (29) on $[-d,d]$such that (30) holds, then we estimate*

$$\tfrac{1}{L}|u_1| \leq u_1(-d) + \int_{-d}^{s} |F(\tau)|\, d\tau + |j(-d)| \int \tfrac{1+\tau^2}{1+d^2} \exp(-\kappa_k(\tau + d + \tfrac{1}{3}(s^3+d^3))).$$

Proof. We may without loss of generality restrict to the case $j(-d)=0$, since we can choose $\tilde{F} = F + \frac{1+s^2}{1+d^2}\exp(-\kappa_k(\tau+d+\frac{1}{3}(s^3+d^3)))j(-d)$. By Duhamel' principle the equation (29) is solved by

$$u_1(s) = U(-d,s)u_1(-d) + \int U(\tau,s)F(\tau)$$

which yields the desired bound. ☐

Proof of Proposition 23. With Proposition 17 we estimate until time $-d$

$$\begin{aligned}
|u_1|(-d) &\leq 2M(c\eta)^{-\gamma_2}\\
|u_2|(-d) &\leq 2M(c\eta)^{\gamma_1},\\
|u_3|(-d) &\leq 2M_1,\\
|w(k_n,-d)| &\leq 2M_n,\\
|j|(k_0,-d) &\leq \tfrac{c}{\beta}M(c\eta)^{-\gamma_2}\min(\kappa_{k_0}c^{-2},1),\\
|j|(k_{\pm1},-d) &\leq \tfrac{4}{\beta\eta^2}M,\\
|j|(k_n,-d) &\leq \tfrac{4}{\beta\eta^2}M_n.
\end{aligned}$$

We next aim to prove by a bootstrap that

$$\begin{aligned}
|u_1| &\leq 3L(c\eta)^{-\gamma_2}M,\\
|u_2| &\leq 7\pi L(c\eta)^{\gamma_1}M,\\
|u_3| &\leq 15\pi L(\tfrac{5}{\eta})^2(c\eta)^{\gamma_1}M + 2M_1,\\
|w(k_n)| &\leq 7\pi L(\tfrac{5}{\eta})^{|n|-1}(c\eta)^{\gamma_1}M + 2M_n,\\
\int j(k_{\pm1}) &\leq 7\pi L\tfrac{3d}{\beta\eta^2}(c\eta)^{\gamma_1}M,\\
\int j(k_n) &< \tfrac{2}{\beta}\tfrac{d}{\eta^2}(7\pi L\tfrac{5}{\eta}(c\eta)^{\gamma_1}M + 2M_n).
\end{aligned}$$

To estimate u_1 we use Lemma 24 to deduce

$$\begin{aligned}
|u_1| &\leq L\left(u_1(-d) + \int a_1u_2 + a_2u_3 + \tfrac{1+s^2}{1+d^2}\exp(-\kappa_{k_0}(s-\tau+\tfrac{1}{3}(s^3-\tau^3)))j(-d)\right)\\
&\leq 2L(c\eta)^{-\gamma_2}M + L\tfrac{1}{\eta}(1+\eta^{-2})(7\pi L(c\eta)^{\gamma_1}M + 2M_1)\\
&+ \tfrac{L}{1+d^2}\min(\tfrac{1}{\kappa_{k_0}}, d^3)j(k_0,-d)\\
&\leq 2L(c\eta)^{-\gamma_2}M + L\tfrac{1}{\eta}(1+\eta^{-2})(7\pi L(c\eta)^{\gamma_1}M + 2M_1) + L\tfrac{c}{\beta}(c\eta)^{-\gamma_2}M\\
&< 3L(c\eta)^{-\gamma_2}M,
\end{aligned}$$

where we used that $\frac{40}{\eta}M_1 \leq \frac{1}{10}M(c\eta)^{-\gamma_2}$ since $\eta \geq \frac{1}{10c}$ and that $7\pi Lc < \frac{1}{2}$. We estimate u_2 by

$$\begin{aligned}
|u_2| &\leq 2(c\eta)^{\gamma_1}M + \int 2c\eta\tfrac{1}{1+s^2}u_1 + a(k_{\pm2})w(k_{\pm2}) + j(k_{\pm1})\\
&\leq 2(c\eta)^{\gamma_1}M + 2\pi c\eta|u_1|_{L^\infty_s} + \tfrac{4}{\eta}|w(k_{\pm2})|_{L^\infty_s} + \int j(k_{\pm2})\\
&< 7\pi L(c\eta)^{\gamma_1}M.
\end{aligned}$$

In order to control u_3, we integrate $\partial_s u_3$ in time, which yields

$$\begin{aligned}|u_3| &\le |u_3|(-d) + \int a(k_{\pm 2})w(k_{\pm 2}) + j(k_{\pm 1}) \\ &\le 2M_1 + \tfrac{5}{\eta}(7\pi L\tfrac{4}{\eta}(c\eta)^{\gamma_1}M + 2M_2) + 7\pi L\tfrac{3d}{\beta\eta^2}(c\eta)^{\gamma_1}M \\ &< 8\pi L(\tfrac{5}{\eta})^2(c\eta)^{\gamma_1}M + 2M_1.\end{aligned}$$

For $w(k_n)$ we first consider $|n| \ge 3$. By integrating $\partial_s w(k_n)$ we deduce

$$\begin{aligned}|w(k_n)| &\le |w(k_n,-d)| + \tfrac{2}{\eta}(|w(k_{n+1})|_{L_s^\infty} + |w(k_{n-1})|_{L_s^\infty}) + \int j(k_n) \\ &< 7\pi L(\tfrac{5}{\eta})^{|n|-1}(c\eta)^{\gamma_1}M + 2M_n.\end{aligned}$$

For the cases $n = \pm 2$ we similarly conclude that

$$\begin{aligned}|w(k_{\pm 2})| &\le |w(k_{\pm 2},-d)| + \tfrac{2}{\eta}(|u_2|_{L_s^\infty} + |u_3|_{L_s^\infty} + |w(k_{\pm 3})|_{L_s^\infty}) \\ &< 7\pi L\tfrac{5}{\eta}(c\eta)^{\gamma_1}M + 2M_{\pm 2}.\end{aligned}$$

For the estimates on the current j we argue similarly as on the interval I_1 and introduce $\hat{s}$ as the shifted time coordinates. To estimate $j(k_{\pm 1})$ we integrate $\partial_s j(k_{\pm 1})$ in time:

$$\begin{aligned}j(k_{\pm 1}) &= 2\exp(-\tfrac{1}{2}\kappa\xi\eta(s+d))j(k_{\pm 1},-d) \\ &\quad + \tfrac{\kappa_{k_{\pm 1}}}{\beta}\int \tfrac{1+\hat{s}^2}{1+\hat{\tau}^2}\exp(-\kappa_{k_{\pm 1}}(\hat{s}-\hat{\tau}+\tfrac{1}{3}(\hat{s}^3-\hat{\tau}^3)))(u_2 \pm u_3).\end{aligned}$$

The impact of $j(k_{\pm 1})$ is bounded by

$$\begin{aligned}\int j(k_{\pm 1}) &\le \tfrac{4}{\kappa\xi\eta}j(k_{\pm 1},-d) + \tfrac{2d}{\beta\eta^2}(|u_2|_{L_s^\infty} + |u_3|_{L_s^\infty}) \\ &\le 7\pi L\tfrac{3d}{\beta\eta^2}(c\eta)^{\gamma_1}M\end{aligned}$$

and hence yields the estimate

$$\begin{aligned}j(k_{\pm 1}) &\le 2\exp(-\tfrac{1}{2}d\kappa\xi\eta)j(k_{\pm 1},-d) + \tfrac{4}{\beta\eta^2}(|u_3|_{L_s^\infty} + |u_2|_{L_s^\infty}) \\ &< \tfrac{4}{\beta\eta^2}(7\pi L(c\eta)^{\gamma_1}M + 2M_1).\end{aligned}$$

By integrating we thus obtain the following estimate for $j(k_n)$:

$$\begin{aligned}j(k_n) &= 2\exp(-\kappa\xi\eta(s+d))j(k_n,-d) \\ &\quad + \tfrac{\kappa_{k_n}}{\beta}\int \tfrac{1+\hat{s}^2}{1+\hat{\tau}^2}\exp(-\kappa_{k_n}(\hat{s}-\hat{\tau}+\tfrac{1}{3}(\hat{s}^3-\hat{\tau}^3)))w(k_n),\end{aligned}$$

which leads to

$$\begin{aligned}\int j(k_n) &= \tfrac{1}{\kappa\xi\eta}j(k_n,-d) + \tfrac{1}{\beta}\tfrac{d}{\eta^2}|w(k_n)|_{L_s^\infty} \\ &\le \tfrac{2}{\beta}\tfrac{d}{\eta^2}(7\pi L\tfrac{4}{\eta}(c\eta)^{\gamma_1}M + 2M_n)\end{aligned}$$

and

$$\begin{aligned}j(k_n) &\le 2\exp(-\kappa\xi d\eta)j(k_n,-d) + \tfrac{4}{\beta\eta^2}|w(k_n)|_{L_s^\infty} \\ &< \tfrac{4}{\beta\eta^2}(7\pi L(\tfrac{5}{\eta})^{|n|-1}(c\eta)^{\gamma_1}M + 2M_n).\end{aligned}$$

We estimate $j(k_0)$ by integrating

$$\begin{aligned}j(k_0) &= \tfrac{1+s^2}{1+d^2}\exp(-\kappa_{k_0}(s+d+\tfrac{1}{3}(s^3+d^3))j(k_0,-d) \\ &\quad + \tfrac{\kappa_{k_0}}{\beta}\int \tfrac{1+s^2}{1+\tau^2}\exp(-\kappa_{k_0}(s-\tau+\tfrac{1}{3}(s^3-\tau^3)))u_1(\tau).\end{aligned}$$

The second term can be estimated by

$$\begin{aligned}&\tfrac{\kappa_{k_0}}{\beta}\int \tfrac{1+s^2}{1+\tau^2}\exp(-\kappa_{k_0}(s-\tau+\tfrac{1}{3}(s^3-\tau^3)))u_1(\tau) \\ &\quad \le \tfrac{1}{\beta}\min(\kappa_{k_0}\pi d^2, 1)|u_1|_{L_s^\infty}\end{aligned}$$

and thus

$$\begin{aligned} j(k_0,d) &= \exp(-\kappa_{k_0}(2d+\tfrac{2}{3}d^3)j(k_0,-d) \\ &\quad + \tfrac{1}{\beta}\min(\kappa_{k_0}\pi d^2,1)|u_1|_{L^\infty_s} \\ &\le \exp(-\kappa_{k_0}(2d+\tfrac{2}{3}d^3)\tfrac{c^2}{\beta}M(c\eta)^{-\gamma_2}\min(\kappa_{k_0},1) \\ &\quad + \tfrac{3LM}{\beta}\min(\kappa_{k_0}\pi d^2,1)(c\eta)^{-\gamma_2} \\ &\quad < \tfrac{4LM}{\beta}\min(\kappa_{k_0}\pi d^2,1)(c\eta)^{-\gamma_2}. \end{aligned}$$

□

4.4. **The Echo and Lower Bounds in the Interval I_2.** In this section we establish the echo mechanism on the interval I_2, i.e. our aim is to show that the mode u_1 induces growth of the u_2 mode. For this echo mechanism we need the additional assumption

$$\kappa k_0^2 \min(\beta,1) > \tfrac{1}{c}. \tag{31}$$

As shown in Subsection 2.3, this is not only a technical assumption. When k_0 is too small, the u_1 term can become negative due to the action of j and hence negate the growth of u_2 and we could even obtain $u_2(d)\approx 0$. We will use initial data of the form

$$\begin{aligned} u_1(-d) &= 1, \\ u_2(-d) &\le 50c^2\eta, \\ |j|(k_0,-d) &\le \tfrac{2c^2}{\beta}, \\ |w|(k,-d), |u_3|(-d) &\le 5(c\eta)^{\gamma_2}, \\ |j|(k,-d) &\le \tfrac{20}{\eta}(c\eta)^{\gamma_2}. \end{aligned} \tag{32}$$

Which corresponds to the echoes on I_1 normalized in terms of $u(-d)$. We will prove that u closely matches the following asymptotics:

$$\begin{aligned} \tilde u_1 &= \exp(-\tfrac{1}{\beta}(\tan^{-1}(s)+\tan^{-1}(d))), \\ \tilde u_2 &= u_2(-d) + 2c\eta\beta(1-\exp(-\tfrac{1}{\beta}(\tan^{-1}(s)+\tan^{-1}(d)))). \end{aligned}$$

Proposition 25. *Consider a solution of* (24) *with initial data* (32)*, then the following estimates hold:*

$$\begin{aligned} |u_1(d)-\tilde u_1(d)| &= 12\pi c, \\ |u_2(d)-\tilde u_2(d)| &\le 24\pi c^2\eta, \\ w(k_n,d), u_3(d) &\le 6(c\eta)^{\gamma_2}, \\ j(k_n,d) &\le \tfrac{25}{\beta\eta^2}(c\eta)^{\gamma_2}, \\ j(k_0,d) &\le \tfrac{2}{\beta}. \end{aligned} \tag{33}$$

In the following it is convenient to introduce the *good unknown*:

$$g(s) = (1+s^2)j - \tfrac{u_1}{\beta},$$

In terms of g our equations then read

$$\partial_s u_1 = -\tfrac{1}{\beta}\tfrac{1}{1+s^2}u_1 - a_1u_2 + a_2u_3 - \tfrac{1}{1+s^2}g$$

and

$$\begin{aligned}\partial_s g &= 2sj(k_0) + (1+s^2)\partial_s j(k_0) - \tfrac{1}{\beta}\partial_s u_1 \\ &= \tfrac{2s}{1+s^2}(g + \tfrac{1}{\beta}u_1) \\ &- \kappa_{k_0}(1+s^2)g + \tfrac{2s}{1+s^2}(g + \tfrac{1}{\beta}u_1) \\ &+ \tfrac{1}{\beta^2}\tfrac{1}{1+s^2}u_1 + \tfrac{1}{\beta}a_1 u_2 - \tfrac{1}{\beta}a_2 u_3 + \tfrac{1}{\beta}\tfrac{1}{1+s^2}g \\ &= (\tfrac{4s+\frac{1}{\beta}}{1+s^2} - \kappa_{k_0}(1+s^2))g \\ &+ \tfrac{1}{\beta}\tfrac{4s+\frac{1}{\beta}}{1+s^2}u_1 + \tfrac{1}{\beta}a_1 u_2 - \tfrac{1}{\beta}a_2 u_3.\end{aligned}$$

Therefore, (24) can be equivalently expressed as

$$\partial_s \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ g \end{pmatrix} = \begin{pmatrix} -\frac{1}{\beta}\frac{1}{1+s^2} & -a_1 & a_2 & -\frac{1}{1+s^2} \\ 2c\eta\frac{1}{1+s^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\beta}\frac{4s+\frac{1}{\beta}}{1+s^2} & \frac{1}{\beta}a_1 & \frac{1}{\beta}a_2 & \frac{4s+\frac{1}{\beta}}{1+s^2} - \kappa_k(1+s^2) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ g \end{pmatrix} + \begin{pmatrix} 0 \\ a(k\pm 2)w(k\pm 2) - j(k\pm 1) \\ \pm a(k\pm 2)w(k\pm 2) \mp j(k\pm 1) \\ 0 \end{pmatrix}. \tag{34}$$

The homogeneous system with respect to (34) is given by

$$\partial_s \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\beta}\frac{1}{1+s^2} & 0 \\ 2c\eta\frac{1}{1+s^2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} \tag{35}$$

with the explicit solution

$$\begin{aligned}\tilde{u}_1 &= \exp(-\tfrac{1}{\beta}(\tan^{-1}(s) + \tan^{-1}(d)))u_1(-d) \\ \tilde{u}_2 &= u_2(-d) + 2c\eta\beta(1 - \exp(-\tfrac{1}{\beta}(\tan^{-1}(s) + \tan^{-1}(d))))u_1(-d).\end{aligned} \tag{36}$$

In the following, we prove that the solution of (34) can be treated as a perturbation of (36). Note that we can approximate

$$\beta(1 - \exp(-\tfrac{1}{\beta}(\tan^{-1}(s) + \tan^{-1}(d)))) \approx \min(\beta, (\tan^{-1}(s) + \tan^{-1}(d))),$$

where "≈" in this case corresponds to the explicit bounds

$$\tfrac{1}{2}\min(\beta, \cdot) \leq \beta(1 - \exp(-\tfrac{1}{\beta}\cdot)) \leq \min(\beta, \cdot).$$

Proof of Proposition 25. We want to show by a bootstrap that

$$\begin{aligned}|u_1 - \tilde{u}_1| &\leq c_1 = 12\pi c \\ |u_2 - \tilde{u}_2| &\leq c_2 = (2\pi+1)c\eta c_1 \\ |u_3|, |w(k_n)| &\leq 6(c\eta)^{\gamma_2} \\ \int j(k_n) &\leq \tfrac{13d}{\beta\eta^2}(c\eta)^{\gamma_2} \\ \int j(s, k_{\pm 1}) &\leq \tfrac{10\pi}{\beta\eta}.\end{aligned} \tag{37}$$

Let s^* be the maximal time such that (37) holds. We assume that $s^* \leq d$ and show that this leads to a contradiction by improving (37). The estimates of $j(k_n)$ for $n \neq 0$ are done similarly as in Proposition 23 and we hence omit them here. First, we estimate g:

$$g_0(s) = \tfrac{(1+s^2)^2}{(1+d^2)^2}\exp(\tfrac{1}{\beta}(\tan^{-1}(s)+\tan^{-1}(d)) - \kappa_{k_0}(s+d+\tfrac{1}{3}(s^3+d^3)))g(-d),$$

$$g(s)-g_0(s) = \tfrac{1}{\beta}\int \tfrac{(1+s^2)^2}{(1+\tau^2)^2}\exp\left(\tfrac{1}{\beta}(\tan^{-1}(s)-\tan^{-1}(\tau)) - \kappa_{k_0}(s-\tau+\tfrac{1}{3}(s^3-\tau^3))\right)$$
$$\left(\tfrac{4\tau+\frac{1}{\beta}}{1+\tau^2}u_1(\tau) + \tfrac{1}{\beta}a_2u_2(\tau) - \tfrac{1}{\beta}a_3u_3(\tau)\right)\, d\tau.$$

Next we estimate the size of the perturbations by

$$\begin{aligned}
&\int d\tau_2\ \frac{\exp(-\frac{1}{\beta}(\tan^{-1}(s)-\tan^{-1}(\tau_2)))}{1+\tau_2^2}(g-g_0)(\tau_2)\\
&= \tfrac{1}{\beta}\int d\tau_2 \int d\tau_1\ \exp\left(-\tfrac{1}{\beta}(\tan^{-1}(s)-\tan^{-1}(\tau_1)) - \kappa_{k_0}(\tau_2-\tau_1+\tfrac{1}{3}(\tau_2^3-\tau_1^3))\right)\\
&\quad\cdot \tfrac{1+\tau_2^2}{(1+\tau_1^2)^2}\left(\tfrac{4\tau_1+\frac{1}{\beta}}{1+\tau_1^2}u_1(\tau_1) + \tfrac{1}{\beta}a_2u_2(\tau_1) - \tfrac{1}{\beta}a_3u_3(\tau_1)\right)\\
&\le \tfrac{2}{\beta\kappa_{k_0}}\int d\tau_1\ \exp(-\tfrac{1}{\beta}(\tan^{-1}(s)-\tan^{-1}(\tau_1)))\\
&\quad\cdot \tfrac{1}{(1+\tau_1^2)^2}\left(\tfrac{4\tau_1+\frac{1}{\beta}}{1+\tau_1^2}u_1(\tau_1) + \tfrac{1}{\beta}a_2u_2(\tau_1) - \tfrac{1}{\beta}a_3u_3(\tau_1)\right)\\
&\le \tfrac{2}{\kappa_{k_0}\beta}(2|u_1|_{L^\infty_s} + \tfrac{4c}{\eta}(|u_2|_{L^\infty_s} + |u_3|_{L^\infty_s}))
\end{aligned}$$

and

$$\begin{aligned}
&\int \frac{\exp(-\frac{1}{\beta}(\tan^{-1}(s)-\tan^{-1}(\tau_2)))}{1+\tau_2^2} g_0(\tau_2)\ d\tau_2\\
&= c^4\exp(-\tfrac{1}{\beta}(\tan^{-1}(s)+\tan^{-1}(d)))g(-d)\\
&\quad\cdot\int (1+\tau_2^2)\exp(-\kappa_{k_0}(\tau_2+d+\tfrac{1}{3}(\tau_2^3+d^3)))g_0(-d)\\
&= \tfrac{c^4}{\kappa_{k_0}}\exp(-\tfrac{1}{\beta}(\tan^{-1}(s)+\tan^{-1}(d)))\tfrac{3}{\beta}g(-d) \le \tfrac{3}{\beta}\tfrac{c^4}{\kappa_{k_0}},
\end{aligned}$$

where we used (32) to obtain $g(-d)\le\frac{3}{\beta}$. To estimate u_1 we look at the difference to the homogeneous system,

$$\partial_s(u_1-\tilde{u}_1) = -\tfrac{1}{\beta}\tfrac{1}{1+s^2}(u_1-\tilde{u}_1) - a_1u_2 + a_3u_3 - \tfrac{1}{1+s^2}g$$

which leads after integrating to

$$\begin{aligned}
|u_1-\tilde{u}_1| &\le \tfrac{4}{\eta}(|u_2|_{L^\infty_s} + |u_3|_{L^\infty_s}) + \tfrac{1}{\kappa_{k_0}}(\tfrac{2}{\beta}|u_1|_{L^\infty_s} + \tfrac{4c}{\eta\beta}(|u_2|_{L^\infty_s} + |u_3|_{L^\infty_s})) + \tfrac{3}{\beta}\tfrac{c^4}{\kappa_{k_0}}\\
&\le 8c\min(\beta,\pi) + 4(2\pi+1)cc_1 + \tfrac{4}{\kappa_{k_0}\beta}(1+c_1) + \tfrac{5}{\eta}6(c\eta)^{\gamma_2} < c_1
\end{aligned}$$

since $\kappa k_0^2 \ge \frac{1}{\beta c}$. We estimate $u_2-\tilde{u}_2$ by

$$\partial_s(u_2-\tilde{u}_2) = 2c\eta\tfrac{1}{1+s^2}(u_1-\tilde{u}_1) + a(k_{\pm 2})w(k_{\pm 2}) - j(k_{\pm 1})$$

which implies by integrating in s, that

$$\begin{aligned}
|u_2-\tilde{u}_2| &\le 2\pi c\eta c_1 + \tfrac{2}{\eta}|w(k_{\pm 2})|_{L^\infty_s} + \int j(k_{\pm 1})\\
&\le 2\pi c\eta c_1 + (12+10\pi\tfrac{1}{\beta})c^{\gamma_2}\eta^{-\gamma_1}\\
&< (2\pi+1)c\eta c_1.
\end{aligned}$$

Next we estimate $w(k_n)$ for $|n|\ge 3$. We remark that the estimates for u_3 and $w_{k_{\pm 2}}$ are similar and hence we omit them. By integrating over the derivative we deduce

$$\begin{aligned}
w(k_n,-d) &\le w(k_n,d) + \tfrac{2}{\eta}(|w(k_{n+1})|_{L^\infty_s} + |w(k_{n-1})|_{L^\infty_s}) + \int j(k_n)\\
&< 6(c\eta)^{\gamma_2}.
\end{aligned}$$

So the bootstrap is concluded. It is left to estimate $j(k_0)$. We write

$$\begin{aligned}\partial_s j(k_0) &= \tfrac{\kappa_{k_0}}{\beta} u_1 + \left(\tfrac{2s}{1+s^2} - \kappa_{k_0}(1+s^2)\right) j(k_0)\\ &\leq \tfrac{\kappa_{k_0}}{\beta} u_1 - \tfrac{8}{9}\kappa_{k_0} j(k_0)\end{aligned}$$

where in the second line we used (31). By integrating, we obtain

$$\begin{aligned}j(k_0, s) \leq{}& \exp(-\tfrac{8}{9}\kappa_{k_0}(s+d)) j(k_0, -d)\\ &+ \tfrac{\kappa_{k_0}}{\beta}\int_{-d}^{s} d\tau \exp(-\tfrac{8}{9}\kappa_{k_0}(s-\tau)) u_1(\tau)\end{aligned}$$

which leads to

$$\begin{aligned}j(k_0, d) &\leq \exp(-2d\tfrac{8}{9}\kappa_{k_0}) j(k_0, -d)\\ &\quad + \tfrac{9}{8}\tfrac{1}{\beta}|u_1|_{L^\infty_s}\\ &\leq \tfrac{2}{\beta}.\end{aligned}$$

□

4.5. **Proof of Theorem 14.** In Subsections 4.2, 4.3 and 4.4 we proved lower and upper bounds until the time $s = d$. Furthermore, in Subsection 4.2 we already showed the asymptotic behavior on the interval I_3. In this subsection we need to combine the results of these subsections to obtain the final lower and upper bounds for the complete interval I^k. This will be achieved in two steps: first we conclude the bootstrap on I_3, afterwards we show that all terms result in the desired estimates.

Proof of Theorem 14. Following we proceed similarly as in the proof of Proposition 17, just for I_3. In particular we use the tools from Subsection 4.2. We thus need to prove the missing estimate on $[d, s_1]$. Let $r_i(d)$ be the initial data of $r(s)$. We define the c_i terms by

$$\begin{aligned}c_1 &= 2(r_1(d) + (21c^2 + 2\tfrac{c^3}{\beta}(c\eta)^\gamma) r_2(s_0) + N + N_j)\\ \tilde{c}_1 &= 0\\ c_2 &= \tfrac{1}{1-2\frac{c}{\beta}}(r_2(d) + N + N_j)\\ \tilde{c}_2 &= 22c_1 + \tfrac{c}{\beta}\tilde{c}_2.\end{aligned} \tag{38}$$

and

$$\begin{aligned}N &= 2c\tfrac{1}{\kappa\xi\eta^{\gamma_2}} j(k_0 \pm 1, d) + 2cu_3(d)\\ &\quad + 2\sum_{|m|\geq 2}(2c)^{|m|}(w + \tfrac{4}{\kappa\eta\xi} j)(k_m, d)\\ N_j &= 4(c\eta)^{\gamma_2}\tfrac{c^2}{\kappa_{k_0}} j(k_0, d).\end{aligned}$$

We prove by bootstrap that

$$|u|(s) \leq S^*(s) C(s). \tag{39}$$

Since $c_i \geq r_i(d)$ this estimate holds locally, and we again let s^* be the maximal time such that (39) holds. We assume that $s^* \leq s_1$ and improve the estimate, which gives a contradiction and thus proves that (39) holds on $[d, s_1]$. For the

R_i we obtain with the Lemmas 20, 21, 22 that

$$\begin{aligned} R_1[F_{all}] &= R_1[F_{3mode}] + R_1[F_j] + R_1[F_{\tilde{w}}] + R_1[F_{j(k_0\pm1)}] + R_1[F_{u_3}] \\ &\leq 20c^2c_1 + 20c^4\tilde{c}_2 + (20c^2 + c^4(c\eta)^\gamma)c_2 \\ &+ \tfrac{c^3}{\beta}(c_1+\tilde{c}_2) + \tfrac{c^3}{\beta}(c\eta)^\gamma c_2 + (c\eta)^{\gamma_1}\tfrac{4c^4}{\kappa_{k_0}} j(k_0,d) \\ &+ 2c^2(\tilde{w}(2)+\tilde{w}(-2)) \\ &+ \tfrac{2c}{\kappa\xi\eta} j(k_{\pm1},d) + \tfrac{2c}{\beta\kappa\xi}(\tilde{w}(1)+c_1^*+c_2^*) \\ &+ 2c\tilde{w}(1) \\ &\leq 21c^2c_1 + 2\tfrac{c^3}{\beta}\tilde{c}_2 + (21c^2 + \tfrac{c^3}{\beta}(c\eta)^\gamma)c_2 + N + c^2(c\eta)^\gamma N_j, \end{aligned}$$

and

$$\begin{aligned} R_2[F_{all}] &= R_2[F_{3mode}] + R_2[F_j] + R_2[F_{\tilde{w}}] + R_2[F_{j(k_0\pm1)}] + R_2[F_{u_3}] \\ &\leq 20(\tfrac{s}{\eta})^\gamma(c_1 + 2c^2\tilde{c}_2) + 20c^2c_2 \\ &+ \tfrac{c}{\beta}(\tfrac{s}{\eta})^\gamma(c_1+\tilde{c}_2) + \tfrac{c}{\beta}c_2 + (c\eta)^{\gamma_2}\tfrac{c^2}{\kappa_{k_0}} j(k_0,d)) \\ &+ c^2(\tilde{w}(2)+\tilde{w}(-2))(\tfrac{s}{\eta})^{\gamma_1} \\ &+ \tfrac{2c}{\kappa\xi\eta} j(k_{\pm1},d)(\tfrac{s}{\eta})^{\gamma_1} + \tfrac{c}{\beta\kappa\xi}(\tilde{w}(1)+c_1^*+c_2^*)(\tfrac{s}{\eta})^{\gamma_1} \\ &+ 2c\tilde{w}(1)(\tfrac{s}{\eta})^{\gamma_1} \\ &\leq 21(\tfrac{s}{\eta})^\gamma c_1 + 2\tfrac{c}{\beta}c_2 + \tfrac{c}{\beta}(\tfrac{s}{\eta})^\gamma\tilde{c}_2 + N(\tfrac{s}{\eta})^\gamma + N_j. \end{aligned}$$

Therefore, we deduce that

$$\begin{aligned} r_1(s_0) + R_1[all][1] &\leq r_1(d) + 21c^2(c_1+\tilde{c}_2) + (21c^2 + \tfrac{c^3}{\beta}(c\eta)^\gamma)c_2 \\ &\quad + N + c^2(c\eta)^\gamma N_j < c_1, \\ r_2(s_0) + R_2[all][1] &\leq r_2(d) + 2\tfrac{c}{\beta}c_2 + N + N_j < c_2, \\ R_2[all][(\tfrac{s}{\eta})^\gamma] &< 20c_1 + 2c^2\tilde{c}_2 < \tilde{c}_2. \end{aligned}$$

This concludes the bootstrap and we estimated $|u|(s) \leq S^*(s)C(s)$ for $s \leq s_1$. To finish the proof of the theorem we need to establish the norm estimate at the final time. With Proposition 23 we obtain the folloiwng bounds:

$$\begin{aligned} |u_1|(d) &\leq 3(c\eta)^{-\gamma_2}LM, \\ |u_2|(d) &\leq 7\pi(c\eta)^{\gamma_1}LM, \\ |u_3|(d) &\leq 7\pi(\tfrac{5}{\eta})^2(c\eta)^{\gamma_1}LM + 2M_1, \\ |w(k_n,d)| &\leq 7\pi(\tfrac{5}{\eta})^{|n|-1}(c\eta)^{\gamma_1}LM + 2M_n, \\ j(k_n,d) &\leq \tfrac{4}{\eta^2\beta}(7\pi(\tfrac{4}{\eta})^{|n|-1}(c\eta)^{\gamma_1}LM + 2M_n), \\ j(k_0,d) &\leq \tfrac{4LM}{\beta}\min(\kappa_{k_0}\pi d^2, 1)(c\eta)^{-\gamma_2}. \end{aligned}$$

This in turn yields

$$\begin{aligned} N &= 2c\tfrac{1}{\kappa\xi\eta^{\gamma_2}} j(k_0\pm1,d) + 2cu_3(d) \\ &\quad + 2\sum_{|m|\geq2}(2c)^{|m-k_0|}(w + \tfrac{8}{\kappa\eta\xi n^2}j)(k_m,d) \\ &\quad \leq c(c\eta)^\gamma LM, \\ N_j &\leq 4(c\eta)^{\gamma_2}\tfrac{c^2}{\kappa_{k_0}}\tfrac{4LM}{\beta}\min(\kappa_{k_0}\pi d^2,1)(c\eta)^{-\gamma_2} \\ &\quad \leq \tfrac{16}{\beta}\min(\pi, \tfrac{c^2}{\kappa_{k_0}})LM. \end{aligned}$$

Using these bounds, we consider Afterwards, we estimate

$$\begin{aligned} r(d) &= S^{-1}(d)u(d) \\ &= -2c\gamma^{-1}\begin{pmatrix} -\frac{\gamma_2}{2c}|c\eta|^{-\gamma_2+1} & -|c\eta|^{-\gamma_2} \\ \frac{\gamma_1}{2c}|c\eta|^{-\gamma_1+1} & |c\eta|^{-\gamma_1} \end{pmatrix} u(d) \\ \begin{pmatrix} |r_1| \\ |r_2| \end{pmatrix}(d) &\leq LM \begin{pmatrix} 15\pi c(c\eta)^{\gamma} \\ 4 \end{pmatrix} \end{aligned}$$

and hence deduce that

$$\begin{aligned} c_1 &= (c\eta)^{\gamma}(30\pi c + 30\tfrac{c^2}{\beta} + c)LM \\ &\leq 31\pi c(c\eta)^{\gamma} LM \\ c_2 &= (5 + \tfrac{16\pi}{\beta})LM. \end{aligned}$$

This implies the estimate

$$\begin{aligned} u(s_1) &\leq S^*(s_1)C(s_1) \\ &\leq LM \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2c} & 2c \end{pmatrix} \begin{pmatrix} 31\pi c(c\eta)^{\gamma} \\ (5 + \frac{16\pi}{\beta}) \end{pmatrix} \\ &\leq LM(c\eta)^{\gamma} \begin{pmatrix} 16c + (5 + \frac{16\pi}{\beta})(c\eta)^{-\gamma} \\ 16\pi \end{pmatrix}, \end{aligned}$$

where we used that $(c\eta)^{-\gamma}\frac{1}{\kappa_{k_0}} = (\frac{k_0^2}{c\xi})^{\gamma}\frac{1}{\kappa k_0^2} = \frac{1}{(c\xi)^{\gamma}\kappa k_0^{2\gamma_2}} \ll \beta c$. For $\tilde{w}(n)$ we obtain

$$\begin{aligned} \tilde{w}(n) &= 2\sum_{|m|\geq 2}(2c)^{|m-n|+\chi}(w + \tfrac{4}{\kappa\eta\xi}j)(k_m, d) \\ &\quad + (2c)^{||n|-2|}c(c_1^* + \tfrac{1}{c^2}c_2^*) \\ &\quad + (2c)^{|c|-1}u_3(d) \\ &\leq L(2c)^{|n|}M + M_n \\ u_3(n) &\leq L(2c)^{|n|+2}M + M_1. \end{aligned}$$

Furthermore, by integrating over $\partial_s j(k_n)$ we obtain

$$\begin{aligned} j(k_n, s_1) &\leq L\tfrac{5}{\kappa\xi\eta}((2c)^{|n|}M + M_n), \\ j(k_{\pm 1}, s_1) &\leq L\tfrac{5}{\kappa\xi\eta}((2c)^{|n|+2}M + M_1). \end{aligned}$$

In order to estimate $j(k_0)$ we use Lemma 21:

$$\begin{aligned} |j(k_0, s_1)| &\leq Lc^2\eta^2\exp(-\kappa_{k_0}\eta^3)j(k_0, d) \\ &\quad + 2\tfrac{16^2}{\beta}\tfrac{1}{\eta^2}(c_1 + c_2 + \tilde{c}_1 + \tilde{c}_2) \\ &\leq 3\pi\tfrac{\kappa_{k_0}}{\beta}\eta^2\exp(-\kappa_{k_0}\eta^3)(\tfrac{d}{\eta})^{\gamma_2}M \\ &\quad + L4\tfrac{16^2}{\beta}\tfrac{1}{\eta^2}2M(c\eta)^{-\gamma_2} \\ &\leq \tfrac{2^{11}}{\beta}\tfrac{1}{\eta^2}M(c\eta)^{-\gamma_2}. \end{aligned}$$

We further estimate

$$\begin{aligned}
M^2 &= \sum_{m,n\geq 1} 10^{-m-n}(w+\tfrac{1}{\alpha_{k_n}}j)(k_n)(w+\tfrac{1}{\alpha_{k_m}}j)(k_m)\\
&\leq \tfrac{2}{1-10^{-1}}\sum_{n\geq 1} 10^{-n}(w^2+\tfrac{1}{\alpha_{k_n}^2}j^2)(k_n)\\
&\leq \tfrac{2}{1-10^{-1}}\tfrac{1}{\lambda_{k_0}}\sum_{n\geq 1}(10^{-n}\tfrac{\lambda_{k_0}}{\lambda_{k_n}})\lambda_{k_n}(w^2+\tfrac{1}{\alpha_{k_n}^2}j^2)(k_n)\\
&\leq \tfrac{2}{1-10^{-1}}\tfrac{1}{\lambda_{k_0}}\|w,j\|_X(s_0)^2\\
\sum_{|n|\geq 1}\lambda_{k_n}M_n^2 &= \sum_n \lambda_{k_n}\sum_{m,l} 10^{-|m-l|-|l-n|-\chi_l-\chi_m}(w+\tfrac{1}{\alpha_{k_m}}j)^2(k_m)\\
&\leq \tfrac{2}{1-10^{-1}}\sum_n \lambda_{k_n}\sum_m 10^{-|m-n|-\chi_m}(w^2+\tfrac{1}{\alpha_{k_m}^2}j^2)(k_m)\\
&\leq \tfrac{2}{1-10^{-1}}\sum_m \lambda_{k_m}(w+\tfrac{1}{\alpha_{k_m}}j)^2(k_m)\sum_n 10^{-|m-n|-\chi_m}\tfrac{\lambda_{k_n}}{\lambda_{k_m}}\\
&\leq \tfrac{2\hat\lambda^2}{1-10^{-1}}\|w,j\|_X^2.
\end{aligned}$$

Combining these bounds we infer the norm estimate

$$\begin{aligned}
\|w,j\|_X^2(s_1) &\leq 16\pi L^2 M^2(c\eta)^{2\gamma}(\lambda_{k_{\pm 1}}+\lambda_{k_0}(16\pi+5(c\eta)^{-2\gamma})^2)\\
&\quad+\sum_{|n|\geq 1} L^2\lambda_{k_n}(10^{-|n|}M+M_n)^2\\
&= M^2(c\eta)^{-2\gamma}(\lambda_{k_{\pm 1}^2}(2c)^2+\lambda_{k_0}^2+2\sum_{|n|\geq 1}\lambda_{k_n}^2 10^{-2|n|}+L^2\sum_{|n|\geq 1}\lambda_{k_n}M_n^2\\
&\leq L^2(\hat\lambda(16\pi)^2+3\hat\lambda^2)(c\eta)^{2\gamma}\|w,j\|_X^2(s_0).
\end{aligned}$$

This finally allows us to complete the proof of the upper bound and obtain that

$$\|w,j\|_X(s_1)\leq 18\pi L\hat\lambda(c\eta)^\gamma\|w,j\|_X(s_0).$$

To prove the lower bound we use Proposition 17 and Proposition 25 and obtain that at time $s=d$ it holds that

$$\begin{aligned}
|u_1(d)-\exp(-\tfrac{\pi}{\beta})(c\eta)^{\gamma_2}| &= O(c)\\
|u_2(d)-2\beta(1-\exp(-\tfrac{\pi}{\beta}))(c\eta)^{\gamma_1}| &\leq O(c)\\
w(k_n,d),u_3(d) &\leq 6\\
j(k_n,d) &\leq \tfrac{10\pi}{\beta\eta}\tfrac{1}{\eta}\\
j(k_0,d) &\leq \tfrac{2}{\beta}.
\end{aligned}$$

We calculate $\tilde u_2$ by

$$\begin{aligned}
\tilde u_2(s_1) &= (0\ 1)S(s_1)S^{-1}(d)u(d)\\
&\approx (\tfrac{1}{2c}\ 2c)2c\begin{pmatrix}-c(c\eta)^{\gamma_1} & -(c\eta)^{-\gamma_2}\\ \tfrac{1}{2c}(c\eta)^{\gamma_2} & (c\eta)^{-\gamma_1}\end{pmatrix}u(d)\\
&\approx(-c(c\eta)^{\gamma_1}+2c(c\eta)^{\gamma_2})u_1(d)+(-(c\eta)^{-\gamma_2}+4c^2(c\eta)^{-\gamma_1})u_2(d)\\
&\approx c(c\eta)^{\gamma_1}u_1(d)+(c\eta)^{\gamma_2}u_2(d)\\
&\approx 2(c\eta)^{\gamma_1}\beta(1-\exp(-\tfrac{\pi}{\beta}))u_1(-d)\\
&\approx 2(c\eta)^{\gamma}\beta(1-\exp(-\tfrac{\pi}{\beta}))u_1(s_0).
\end{aligned}$$

The difference $u_2 - \tilde{u}_2$ is estimated by

$$\begin{aligned}|u_2 - \tilde{u}_2| &\leq (0\ 1)S^*(s_1)R[F]\\ &\leq \tfrac{1}{2c}R_1[F] + 2cR_2[F]\\ &\leq (c\eta)^{\gamma_2}u_2(d) + O(c)\\ &= 2(c\eta)^{\gamma}\beta(1-\exp(-\tfrac{\pi}{\beta}))u_1(s_0) + O(c).\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}M &\leq \tfrac{1}{1-10^{-1}}u_1(\tilde{s}_0)\\ M_n &\leq \tfrac{4}{1-10^{-1}}u_1(\tilde{s}_0).\end{aligned}$$

So we finally obtain since $\beta \geq \frac{1}{5}$

$$\begin{aligned}w(k_{-1}, t_{k_{-1}}) &\approx 2(c\eta)^{\gamma}\beta(1-\exp(-\tfrac{\pi}{\beta}))u_1(s_0)\\ &\geq \tfrac{1}{2}\max_l(w(k_l, t_{k_l}),\end{aligned}$$

which gives

$$w(k_{-1}, t_{k_{-1}}) \geq (c\eta)^{\gamma}\min(\beta, \pi)w(k_{-1}, t_{k_{-1}}).$$

□

In this article we have studied the asymptotic (in)stability of the magnetohydrodynamic equations with a shear, a constant magnetic field and magnetic dissipation. Here multiple effects compete to determine the long time behavior of solutions:

- Echoes in the inviscid fluid equations may lead to large norm inflation.
- The underlying magnetic field leads to an exchange between kinetic and magnetic energy. In particular, for large magnetic fields oscillation my diminish norm inflation.
- Magnetic dissipation may stabilize the flow. Hence, a priori, it is not clear whether stability requires Gevrey regularity (as for the Euler equations) or Sobolev regularity (as for the fully dissipative problem) and how the evolution depends on the size of the magnetic field α and on the resistivity κ.

As the main result of this article we show that the balance between these effects is parametrized by the parameter $\beta = \frac{\kappa}{\alpha^2} > 0$ and that the behavior for finite, positive β strongly differs from both the fully non-dissipative case and the large dissipation limit (which reduces to the Euler equations). In particular, we show that in this regime the magnetic dissipation is not strong enough to stabilize the evolution in Sobolev regularity and establish Gevrey regularity as optimal both in terms of upper and lower bounds. It remains an interesting problem for future research to determine the optimal stability classes for other partial dissipation regimes and to study the inviscid limit $\kappa \downarrow 0$.

Appendix A. Estimating the Growth Factor

In Section 4.3 we observe the evolution of (24) on the interval $I_2 = [-d, d]$. Here we observe the interaction between j and u_1

$$\begin{aligned}\partial_s u_1 &= -j\\ \partial_s j &= \tfrac{K}{\beta}u_1 + (\tfrac{2s}{1+s^2} - K(1+s^2))j,\end{aligned} \tag{40}$$

with κ_k replaced by K for simplicity In particular we bound the growth of u_1 by a factor. Let $U(\tau, s)$ be the solution of (40) with initial data $u_1(\tau) = 1$ and $j(\tau) = 0$. We show that

- $|U(\tau, s)| \leq 1$ for $\beta \geq \frac{\pi}{2}$
- $|U(\tau, s)| \leq L = L(\beta, K)$ for $\beta < \frac{\pi}{2}$.

With the restriction

$$c \leq (8\pi)^{\frac{4}{3}} \beta^{\frac{16}{3}}. \tag{41}$$

we obtain

$$L(\beta, K) = \begin{cases} 1 & 1 \leq K \\ \sqrt{d} & \frac{1}{2} c^{\frac{3}{4}} \leq K \leq 1 \\ 2(1 + \frac{\pi}{\beta}) & \frac{2\pi}{\beta} c^3 \leq K \leq \frac{1}{2} c^{\frac{3}{4}} \\ 1 & K \leq \frac{2\pi}{\beta} c^3 \end{cases} \tag{42}$$

We note that (41) is not optimal, in the sense that Section 4.3 we need $Lc << 1$ and we could optimize the $\frac{1}{2}c^{\frac{3}{4}}$ term to obtain a larger L but better (41). However, this would yield a lot dependencies which would make the final theorem more technical to state. The most important part of this estimates is to verify that β can be very small if c is chosen small enough. First we do an energy estimate, let

$$E = u_1^2 + \frac{\beta}{K} j$$

which leads to

$$\frac{1}{2} \partial_s E \leq \left(\frac{2s}{1+s^2} - K(1+s^2)\right)_+ E.$$

Therefore, we obtain for $K \geq 1$ that $\partial_s E \leq 0$, which proves our first estimate. Furthermore, we infer for $K \leq 1$

$$E(s) \leq E(\tau) \begin{cases} 1 & s \leq 0 \\ (1+s^2)^2 & 0 \leq s \leq (\frac{K}{2})^{\frac{-}{1}3} \\ 4(K)^{-\frac{4}{3}} & (\frac{K}{2})^{-\frac{1}{3}} \leq s, \end{cases}$$

We conclude

$$u_1(s) \leq \begin{cases} 1 & s \leq 0 \\ 1 + s^2 & 0 \leq s \leq (\frac{K}{2})^{\frac{1}{3}} \\ 2(K)^{-\frac{2}{3}} & (\frac{K}{2})^{\frac{1}{3}} \leq s \end{cases}$$

which proves (42) for $\frac{1}{2}c^{\frac{3}{4}} \leq K \leq 1$. For small K we need to make a different ansatz. We write j as,

$$j(s) = \frac{K}{\beta} \int_{-d}^{s} \frac{1+s^2}{1+\tau^2} \exp(-K(s - \tau + \tfrac{1}{3}(s^3 - \tau^3))) u(\tau) \, d\tau$$

and so

$$\begin{aligned} u(s) - 1 &= -\frac{K}{\beta} \iint_{-d \leq \tau_1 \leq \tau_2 \leq s} d(\tau_1, \tau_2) \frac{1+\tau_2^2}{1+\tau_1^2} \exp(-K(\tau_2 - \tau_1 + \tfrac{1}{3}(\tau_2^3 - \tau_1^3))) u(\tau_1) \\ &= -\frac{1}{\beta} \int_{-d \leq \tau_1 \leq s} d\tau_1 \; u(\tau_1) \frac{1}{1+\tau_1^2} [\exp(-K(\tau_2 - \tau_1 + \tfrac{1}{3}(\tau_2^3 - \tau_1^3)))]_{\tau_2 = \tau_1}^{\tau_2 = s} \\ &= -\frac{1}{\beta} \int_{-d \leq \tau_1 \leq s} d\tau_1 \; u(\tau_1) \frac{1}{1+\tau_1^2} (1 - \exp(-K(s - \tau_1 + \tfrac{1}{3}(s^3 - \tau_1^3)))). \end{aligned}$$

Now we exploit that u is decreasing till the smallest time such that $u(s) = 0$. This holds, since if u is positive, then j is positive and so $\partial_s u = -j \leq 0$. Therefore, we bound

$$\frac{1}{\beta} \int_{-d \leq \tau_1 \leq s} d\tau_1 \; u(\tau_1) \frac{1}{1+\tau_1^2} (1 - \exp(-K(s - \tau_1 + \tfrac{1}{3}(s^3 - \tau_1^3))))$$

by 1 to deduce $0 \leq u(s) \leq 1$. Let s be positive, then we estimate

$$\begin{aligned}
&\tfrac{1}{\beta} \int_{-d \leq \tau_1 \leq s} d\tau_1 \tfrac{1}{1+\tau_1^2}(1-\exp(-K(s-\tau_1+\tfrac{1}{3}(s^3-\tau_1^3)))) \\
&\qquad = \tfrac{1}{\beta} \int_{-d \leq \tau_1 \leq -s} d\tau_1 \tfrac{1}{1+\tau_1^2}(1-\exp(-K(s-\tau_1+\tfrac{1}{3}(s^3-\tau_1^3)))) \\
&\qquad + \tfrac{1}{\beta} \int_{-s \leq \tau_1 \leq s} d\tau_1 \tfrac{1}{1+\tau_1^2}(1-\exp(-K(s-\tau_1+\tfrac{1}{3}(s^3-\tau_1^3)))) \\
&\qquad \leq \tfrac{1}{\beta s} + \tfrac{\pi}{\beta}(1-\exp(-K(2s+\tfrac{2}{3}s^3))) \\
&\qquad \leq \tfrac{1}{\beta s} + \tfrac{\pi}{\beta}K(2s+\tfrac{2}{3}s^3) \\
&\qquad \leq \tfrac{1}{\beta s} + \tfrac{\pi}{\beta}Ks^3.
\end{aligned}$$

This term is less than zero if $\frac{2}{\beta} \leq s \leq (\frac{\beta}{2\pi K})^{\frac{1}{3}}$. We choose $s = \min((\frac{\beta}{2\pi K})^{\frac{1}{3}}, d)$ maximal. When $s = d$, then $(\frac{\beta}{2\pi K})^{\frac{1}{3}} \geq d$ which is satisfied if $K \leq \frac{2\pi}{\beta}c^3$ and so we obtain the last estimate of (42). Now we need to prove the case if $\frac{2\pi}{\beta}c^3 \leq K \leq \frac{1}{2}c^{\frac{3}{4}}$, with the previous calculation we obtain for $s_1 = (\frac{\beta}{2\pi K})^{\frac{1}{3}}$, that $0 \leq u(s_1) \leq 1$. Then for $s \geq s_1$ we have

$$\begin{aligned}
u(s) - 1 &= \tfrac{1}{\beta} \int_{-d \leq \tau_1 \leq s} d\tau \ u(\tau) \tfrac{1}{1+\tau^2}(1-\exp(-K(s-\tau+\tfrac{1}{3}(s^3-\tau^3)))) \\
|u(s)-1| &\leq \tfrac{1}{\beta} \int_{-d \leq \tau \leq s_1} d\tau \ \tfrac{1}{1+\tau^2} + \tfrac{1}{\beta} \int_{t_1 \leq \tau_1 \leq s} d\tau \ u(\tau) \tfrac{1}{1+\tau^2} \\
&\leq \tfrac{\pi}{\beta} + \tfrac{1}{\beta s_1} |u|_{L^\infty_s}.
\end{aligned}$$

Due to $K \leq \frac{1}{2}c^{\frac{4}{3}}$ and (41) we obtain $s_1\beta = (\frac{\beta^4}{2\pi K})^{\frac{1}{3}} \geq 2$ and so

$$\begin{aligned}
|u(s)| &\leq \tfrac{1}{1-\frac{1}{\beta s_1}}(1+\tfrac{\pi}{\beta}) \\
&\leq 2(1+\tfrac{\pi}{\beta}).
\end{aligned}$$

Appendix B. Nonlinear Instability of Waves

In this appendix we consider the nonlinear instability of the traveling waves.

(43)

$$\begin{aligned}
&\partial_t w + (v\nabla w)_{\neq} = \alpha \partial_x j + (b\nabla j)_{\neq} - (2c\sin(x)\partial_y \Delta_t^{-1} w)_{\neq} \\
&\partial_t j + (v\nabla j)_{\neq} = \kappa \Delta_t j + \alpha \partial_x w + (b\nabla w)_{\neq} - 2\partial_x \partial_y^t \Delta_t^{-1} j - (2(\partial_i v \nabla)\partial_i \Delta^{-1} j)_{\neq},
\end{aligned}$$

For brevity of notation let us denote the Gevrey 2 norm with constant C by

$$\|(w,j)\|_{\mathcal{G}_C}^2 = \int \sum_k \exp(C\sqrt{|\xi|})|\mathcal{F}(w,j)|^2 d\xi.$$

Then the norm inflation result of Theorem 3 further implies the nonlinear instability of any non-trivial traveling wave for C sufficiently small.

Corollary B.1. *Let $0 < c < \min(10^{-4}, 10^{-3}\frac{\kappa}{\alpha^2})$ be given and consider a traveling wave as in Lemma 2 and let $0 < C_2 < C_*$ where $C_* = C_*(c)$ is as in Theorem 3. Then the nonlinear evolution equations around the traveling wave are unstable for small initial data in $\mathcal{G}_{C_2}$ in the sense that for any $0 < C_1 < C_2$, $\epsilon > 0$ and $N > 1$ there exists initial data with*

$$\|(w_0, j_0)\|_{\mathcal{G}_{C_2}} < \epsilon$$

but such that for some time $T > 0$ it holds that

$$\|(w,j)|_{t=T}\|_{\mathcal{G}_{C_2}} \geq N\|(w_0,j_0)\|_{\mathcal{G}_{C_1}}.$$

We stress that this results considers the instability of the traveling waves and that the space with respect to which instability is established depends on the size c of the wave. A nonlinear instability result for the underlying stationary state (2) in the spirit of [DM18, Bed20, DZ21] further requires that the size c of the traveling is comparable to ϵ.

Proof of Corollary B.1. We argue by contradiction. Thus suppose that the nonlinear solution is uniformly controlled in $\mathcal{G}_{C_1}$ for all times:

$$\sup_{t>0} \|(w,j)\|_{\mathcal{G}_{C_1}} \leq D\epsilon.$$

for some constant $D > 0$. Given this a priori control of regularity we may consider the nonlinear equations as a forced linear problem

$$\partial_t(w,j) + L(w,j) = F$$

where L is the linear operator considered throughout this article and F is the quadratic nonlinearity. If we denote by $S(t,\tau)$ the solution operator associated to L it then follows that for any $T > 0$

$$(w,j)_{t=T} = S(T,0)(w_0,j_0) + \int_0^T S(T,\tau)F(\tau)d\tau.$$

By the norm inflation results of Theorem 3 for any $C_2 < C_*$ there exists initial data and a time $T > 0$ such that

$$\|S(T,0)(w_0,j_0)\|_{L^2} \geq N\|(w_0,j_0)\|_{\mathcal{G}_{C_2}}. \tag{44}$$

Since this estimate is linear after multiplication with a factor we may assume that this initial data also has size smaller than ϵ. On the other hand, by the results of Section 3 and of Theorem 3 for any fixed time T, $S(T,\tau)$ is uniformly bounded as a map from L^2 to L^2. More precisely, we recall that $S(T,\tau)$ decouples with respect to the frequency ξ in y.

- For ξ with $|\xi| \gg T^2$ by the results of Section 3 the time interval $(0,T)$ is considered "small time" and hence $S(T,\tau)$ is bounded uniformly.
- If instead $|\xi| \leq T^2$ then Theorem 3 provides an upper bound of the operator norm by $\exp(C\sqrt{\xi}) \leq \exp(CT)$.

Thus there exists an extremely large constant E (depending on T) such that

$$\|\int_0^T S(T,\tau)F(\tau)d\tau\|_{L^2} \leq E\int_0^T \|F(\tau)\|_{L^2}d\tau.$$

Finally, we note that by assumption

$$\|F(\tau)\|_{L^2} \leq D^2\epsilon^2.$$

Hence, choosing $\epsilon \ll \frac{1}{ED^2NT}$ the Duhamel integral can be treated as a perturbation of (44), which concludes the proof. □

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