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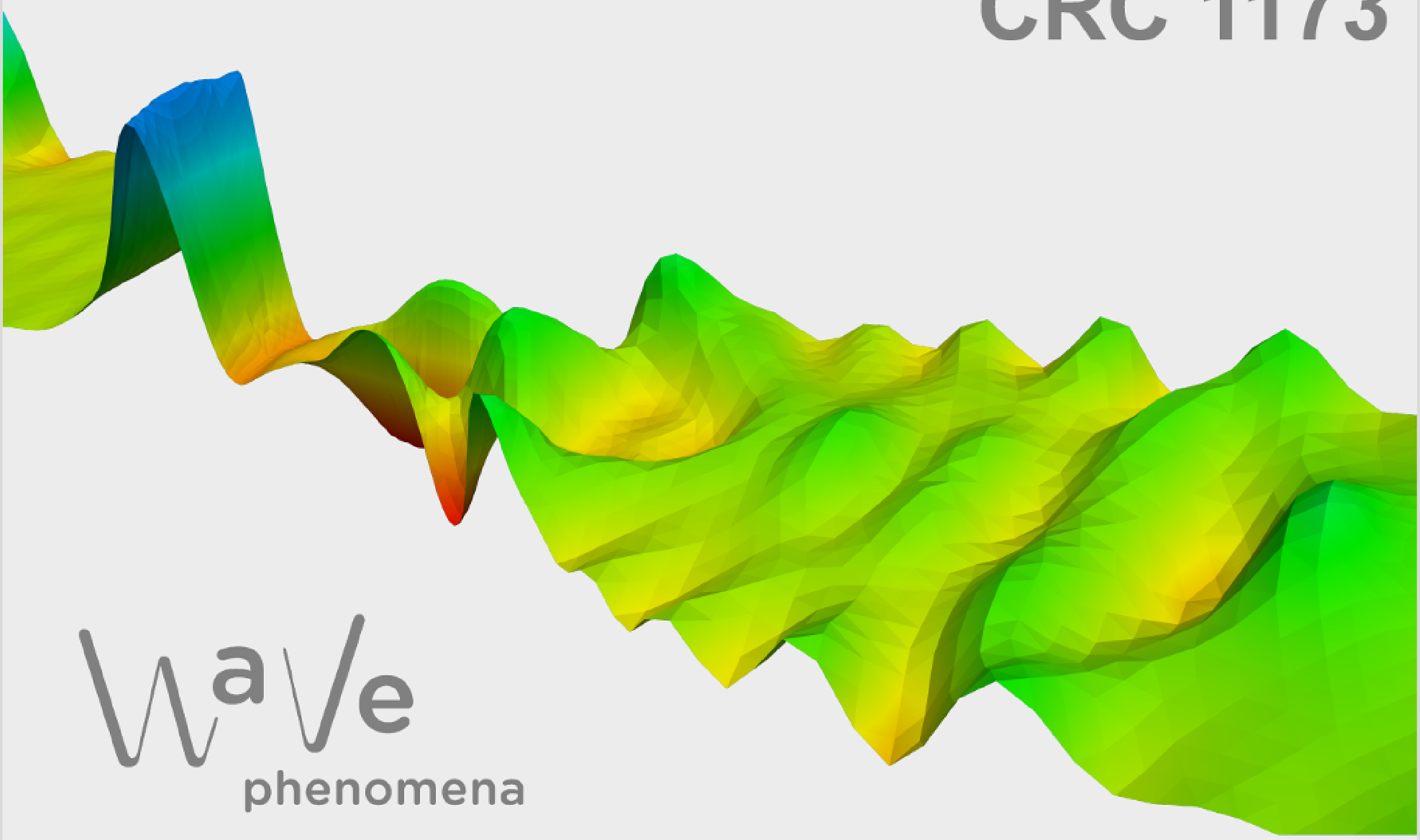

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WELL-POSEDNESS FOR THE KDV HIERARCHY

FRIEDRICH KLAUS, HERBERT KOCH, AND BAOPING LIU

Abstract. We prove a version of wellposedness for all equations of the KdV hierarchy in H^{-1}. Ingredients are

(1) The Miura map which allows to define the Gardner hierarchy through the generating function of the energies so that the Nth Gardner equation is equivalent to the Nth KdV equation.
(2) A rigorous relation between the generating functions of the energies and the KdV resp. Gardner Hamiltonians.
(3) Kato smoothing estimates for weak solutions and approximate flows.

Contents

1. Introduction

The Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} - 6uu_x = 0 \tag{1.1}$$

is a fascinating and basic object in diverse areas: It is a generic asymptotic equation for propagating waves, and it has a deep algebraic structure visible in the existence of an infinite sequence of formally conserved energies H_n^{KdV}. The KdV equation is a member of the KdV hierarchy: It is the Hamiltonian equation with the Hamiltonian function

$$H_1^{\text{KdV}}(u) = \frac{1}{2}\int u_x^2 + 2u^3 dx$$

with respect to the Gardner Poisson structure (defined in (2.14)).

The study of rough initial data is related to weaker assumptions on the frequency localization for the validity of the asymptotic equation, whereas the study of higher order equations is related, or more precisely a necessary ingredient, to larger time scales.

Wellposedness for the KdV equation itself has been an active and stimulating area for the last three decades. Wellposedness results on H^s spaces have been proven by Kato [32] (local in $H^s(\mathbb{R})$, $s > 3/2$ and global in $H^2(\mathbb{R})$) by energy methods. Bourgain [4] introduced the $X^{s,b}$ space and showed global wellposedness in $L^2(\mathbb{R})$ and $L^2(\mathbb{S})$. Kenig, Ponce and Vega [34] proved sharp bilinear estimate in $X^{s,b}$ space and thus showed local wellposedness of KdV in $H^s(\mathbb{R}), s > -\frac{3}{4}$ and $H^s(\mathbb{S}), s > -\frac{1}{2}$. The results were extended to global wellposedness by Colliander, Keel, Staffilani, Takaoka and Tao [11], via the construction of almost conserved quantities. The local existence in $H^{-\frac{3}{4}}(\mathbb{R})$ was shown by Christ, Colliander, Tao

[9] and global existence in $H^{-3/4}(\mathbb{R})$ was proved independently by Guo [23] and Kishimoto [41]. The last author used modified energies to establish local in time apriori bounds in $H^s(\mathbb{R})$ for $s \geq -\frac{4}{5}$ [44]. Buckmaster and the second author [7] and later Killip, Visan and Zhang [40] and the second author and Tataru [43] proved uniform in time apriori estimates in H^{-1}. The apriori estimates remain true for all higher flows. The estimates in [7] are strong enough to construct weak solutions (which was done in [7] for KdV in the nonperiodic case, with a weaker notion of weak solution through a compactness argument). Finally Killip and Visan [39] proved global wellposedness for initial data in $H^{-1}(\mathbb{R})$ and $H^{-1}(\mathbb{S})$, in the sense that the solution map extends uniquely from Schwartz space to a jointly continuous flow map $\Phi : \mathbb{R} \times H^{-1} \to H^{-1}$. This also provides a new proof of the wellposedness in the periodic case, which was first shown by Kappeler and Topalov [29]. The results in H^{-1} are sharp since Molinet [47, 48] showed that the solution map can not be continuously extended to $H^s, s < -1$ in both periodic and nonperiodic case.

For higher order equations in the KdV hierarchy, Saut [54] proved global existence of persistent solutions of the k-th KdV equation for initial data in $H^k(\mathbb{R})$, but the uniqueness was left open. Kenig, Ponce and Vega [35, 36] studied generalized higher order equation, not necessarily integrable, and showed local wellposedness for initial data in weighted spaces. Pilod [50] showed that all higher equations are ill-posed in any $H^s(\mathbb{R}), s \in \mathbb{R}$, in the sense that data-to-solution map is not C^2 at origin. Grünrock [22] proved local wellposedness for the KdV hierarchy in Fourier–Lebesgue spaces. Kenig and Pilod [33], Guo, Kwak and Kwon [24] showed global wellposedness for general 5th order KdV in energy space $H^2(\mathbb{R})$. Bringmann, Killip and Visan [5] proved global wellposedness for 5th order KdV for inital data in $H^{-1}(\mathbb{R})$ by a new strategy that integrates dispersive effects into the method of commuting flows. In the periodic case, Kappeler and Molnar [30] showed 5th KdV with is C^0 wellposed in $H^s(\mathbb{S})$ if $s \geq 0$, and strongly illposed if $s < 0$, in the sense that data-to-solution map does not admit a continuous extension to $H^s(\mathbb{S}), s < 0$.

Beyond the KdV hierarchy there have been striking new developments at the interface of PDE-techniques and integrable structures: The work of Killip and Visan on the KdV equation [39] introduced a new perspective and powerful technique which has motivated the study of a number of integrable problems: sharp global wellposedness for cubic NLS and mKdV in $H^s(\mathbb{R}), s > -\frac{1}{2}$ [25], for the derivative NLS [27] (which uses crucially the work of Bahouri and Perelman [2] as well as the equicontinuity of Harrop-Griffith, Killip and Visan [26]) and for the Benjamin-Ono hierarchy by Killip, Laurens and Visan [38].

Gerard and coworkers introduced new integrable pdes, the cubic Szëgo [20] being only the first, with striking new ideas like an explicit formula for solutions to the Benjamin-Ono equation [19], which in turn become a crucial element in [38]. Clearly this list of results is incomplete and there are many omissions. Similarly we omit a presentation of consequences, possibly the most important being on random initial data.

We conceive these developments as evidence that beyond the single results a new picture of integrable PDEs seems to be emerging, which is still incomplete and to which we hope to contribute.

The Schrödinger operator

$$L^{\mathrm{KdV}}\phi = -\phi'' + u\phi \tag{1.2}$$

is the Lax operator of the KdV hierarchy. The KdV Hamiltonians are defined as the coefficients of the asymptotic series for the logarithm of the transmission coefficient (see (2.2)) and they all Poisson commute.

The main result of this paper is wellposedness, more precisely

(1) Existence and uniqueness for the Nth Gardner equation (2.16) in $C(\mathbb{R}, H^N(\mathbb{X}))$, $\mathbb{X} = \mathbb{R}$ or $\mathbb{X} = \mathbb{S}^1$ for initial data $u_0 \in H^N(\mathbb{X}))$ which has a continuous extension to $H^{N-1} \ni w_0 \to w \in C(\mathbb{R}; H^{N-1})$, see Theorem 2.14. Analogous results for the Nth KdV equation (2.15) in spaces with one derivative less are an immediate consequence.
(2) In the case $\mathbb{X} = \mathbb{R}$, Kato smoothing estimates and tightness of weak solutions with initial data in $L^2(\mathbb{R})$.
(3) Uniqueness for a class of weak solutions to the Nth KdV equation with initial data in $L^2(\mathbb{R})$.

The following theorem collects some of these statements.

Theorem 1.1. *Let $N \geq 1$. Suppose that*

$$u \in L^\infty(\mathbb{R}; L^2), \quad \partial_x^{N-1} u \in L^2_{loc}(\mathbb{R} \times \mathbb{R}) \tag{1.3}$$

and that it is a weak solution to the Nth KdV equation (2.15)*. Suppose that in addition for every $t_0 \in \mathbb{R}$*

$$\limsup_{x_0 \to \pm\infty} \|u^{(N-1)}\|_{L^2((t_0,t_0+1)\times(x_0-1,x_0+1))} = 0. \tag{1.4}$$

Then $u \in C([0,\infty); L^2(\mathbb{R}))$ and we denote the initial trace by $u_0 := u(0) \in L^2$. The Kato smoothing estimate

$$\sup_t \|u(t)\|_{H^{-1}} + \sup_{x_0} \|\operatorname{sech}(x - \kappa^{2N} t - x_0) u^{(N-1)}\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq c(\|u_0\|_{H^{-1}}) \|u_0\|_{H^{-1}} \tag{1.5}$$

holds for all $\kappa \geq \kappa_0(\|u_0\|_{H^{-1}})$.

Given $u_0 \in L^2(\mathbb{R})$ there is a unique weak solution to the Nth KdV equation which satisfies (1.3) *and* (1.4)*. The map $L^2(\mathbb{R}) \ni u_0 \to u \in C(\mathbb{R}; H^{-1}(\mathbb{R}))$ extends to a continuous map*

$$H^{-1} \ni u_0 \to u \in C(\mathbb{R}; H^{-1})$$

to weak solutions which satisfy in addition (1.4) *and* (1.5)*.*

While the regularity assumption in (1.3) looks inconsistent with the Kato smoothing estimate in (1.5) we need L^2 regularity to get equivalence of weak solutions to the KdV and the Gardner hierarchy, see Theorem 5.8, and the condition $u \in L^\infty L^2$ is used only to prove the equivalence. We prove the theorem by studying the analogous statement for the Gardner hierarchy (Theorem 2.19).

Why should one be interested in higher KdV equations? From applications one would like to explain why the Korteweg-de Vries equation provides a good description of nonlinear waves in the KdV regime. Typical results are consistency results up to a certain time scale for localized well-prepared initial data (see [55, 10] and the references in these papers) - whereas the KdV solitons seem to be relevant in many situations like tsunamis despite the interaction with other waves and despite large time scales and amplitudes.

It is striking that the theory of integrable systems provides a very detailed and geometric picture of the simultaneous dynamics of all the KdV flows. Inverse scattering allows a linearization of the evolutions [14, 56] and hence implies wellposedness of the hierarchy on the Schwartz space. The geometric contents is clearly visible in the relation of the Korteweg-de Vries hierarchy to the diffeomorphism group on $\mathbb{R}^1$ and the torus $\mathbb{S}^1$, more precisely on a central extension, the Virasoro-Bott group (see [37, 13, 12]). Ignoring the topology the tangent space at the identity of the Virasoro-Bott group is given by the pairs $(v\partial, g)$ of vectors fields times $\mathbb{R}$. It is a Lie algebra. The set of Lax operators

$$2a\partial^2 + u$$

can be understood as the dual space of the Virasoro algebra, the Lie algebra which is the tangent space of the Virasoro-Bott group at the identity, on which the Virasoro-Bott group acts by the coadjoint representation. The orbit structure is well understood in the torus case and can be classified in terms of the spectrum of the Lax operator. The Korteweg-de Vries equation can be realized through moment maps and the natural biHamiltonian structure allows to construct a countable sequence of Poisson commuting Hamiltonians. It is tempting to ask whether larger time scales for asymptotic equations can be understood in terms of this striking symmetry of the KdV hierarchy.

Another geometric interpretation of the KdV hierarchy is as flow on restricted Grassmanians [58, 52]. This is the origin of the ubiquitous τ function [53, 46] and the bilinear relation of Hirota [28]. Let $(t_j)_{j\in\mathbb{N}}$ be a sequence with only finitely many nonzero components. We denote by $u(\cdot, t_1, t_2 \dots) \in H^{-1}$ the function resp. distribution obtained from $u_0 \in H^{-1}$ by moving the times t_j along the jth KdV flow. This is well defined since the flows commute. The τ function satisfies

$$\partial_x^2 \ln \tau = u$$

which is defined for the such sequences (t_j) provided $u_0 \in H^{-1}(\mathbb{R})$. If $u \in H^\infty(\mathbb{R}) = \cap H^n(\mathbb{R})$, then

$$\frac{\partial^2}{\partial t_i \partial t_j} \ln \tau$$

is well defined as a differential polynomial (see [16]) and by evaluation for $u \in H^\infty$. It is not difficult to see that it can be integrated and hence a τ function exists in this situation. It seems a natural question whether a τ function can be defined for $u \in H^{-1}$ (as unique continuous extension of a τ function on H^∞) and whether related objects like vertex operators can be defined for $u \in H^\infty$ or even for $u \in H^{-1}$.

In this paper we study more basic questions, however for $u \in H^N$. Crucial points are

(1) rigorous estimates for the difference between the generating function of the KdV Hamiltonians and the partial sums for Sobolev functions
(2) a study of the Miura map

$$w \to u = w_x + 2\tau w + w^2$$

resp. the operator factorization

$$(\partial + \tau + w)(-\partial + \tau + w) = -\partial^2 + u + \tau^2.$$

We completely characterize the global mapping properties. The hardest part is a bound of $\|w\|_{L^2}$ in terms of $\|u\|_{H^{-1}}$ and the distance of the ground state energy of $-\partial^2 + u$ and $-\tau^2$.

(3) The Miura map allows to translate wellposedness questions for the KdV hierarchy to the Gardner hierarchy for w, which has better properties. The Miura map itself and its inverse enter at a number of points.

(4) We prove uniqueness of rough weak solutions for the Gardner equations, and not only continuous extensions of flows on more regular function spaces.

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2. Proof of Theorem 1.1

This section explains the structure and the strategy. We will prove the main theorems using various results from later sections. This will make the structure highly modular, it weakens the dependencies of the later sections and allows to give a coherent presentation of the proof. A large part is the same on the line and on the circle and we denote $\mathbb{X} = \mathbb{S}^1$ or $\mathbb{R}$ if statements are true for both spaces.

2.1. **The generating function of the KdV hierarchy.** We formulate the setting of the Korteweg-de Vries hierarchy and describe our approach. More details and proofs about the structure can be found in Section 3. A central quantity of the scattering theory of L^{KdV} is the transmission coefficient. To avoid technical complications in the discussion, we take Schwartz functions as potentials in this paragraph whenever needed. For z in the upper half plane the equation

$$L^{\mathrm{KdV}}\phi = z^2\phi$$

has a unique solution called left Jost function, normalized by

$$\lim_{x\to-\infty} \phi_l e^{izx} = 1 \tag{2.1}$$

The transmission coefficient is the meromorphic function defined by

$$(T^{KdV})^{-1} = \lim_{x\to\infty} \phi_l e^{izx}$$

on the upper half plane with the poles given by the square root of the eigenvalues of the Schrödinger operator. Its logarithm can be related to an asymptotic series

$$\frac{i}{2}\log T^{KdV}(z) \sim \sum_{n=-1}^{\infty} H_n^{\mathrm{KdV}}(2z)^{-2n-3} \tag{2.2}$$

where

$$\begin{aligned} H_{-1}^{\mathrm{KdV}} &= \frac{1}{2}\int u\,dx, \qquad & H_0^{\mathrm{KdV}} &= \frac{1}{2}\int u^2 dx, \\ H_1^{\mathrm{KdV}} &= \frac{1}{2}\int u_x^2 + 2u^3 dx, \qquad & H_2^{\mathrm{KdV}} &= \frac{1}{2}\int u_{xx}^2 + 10u_x^2 u + 5u^4 dx, \end{aligned} \tag{2.3}$$

and in general

$$H_n^{\rm KdV}(u) = \frac{1}{2}\int |u^{(n)}|^2 dx + O\big((1+\|u\|_{H^{-1}})^{n-1}\|u\|_{H^{-1}}\|u\|_{H^n}^2)\big).$$

The precise meaning of (2.2) is the content of Proposition 2.3 below: We define the difference Hamiltonian as $(2z)^{2N+3}$ times the difference to the partial sum

$$\mathcal{T}_N^{\rm KdV}(z,u) := (2z)^{2N+3}\left(\frac{i}{2}\log T^{\rm KdV}(z) - \sum_{n=-1}^{N} H_n^{\rm KdV}(2z)^{-2n-3}\right). \tag{2.4}$$

The generating function $\mathcal{T}_{-1}^{\rm KdV}$ for the KdV energies plays a central role and we state estimates for it.

We define the microlocal Sobolev spaces for $\tau>0, s\in\mathbb{R}$

$$\|u\|_{H^s_\tau} = \|(\tau^2+|\xi|^2)^{\frac{s}{2}}\hat u\|_{L^2(\mathbb{R})}. \tag{2.5}$$

Lemma 2.1. *Let $d(z^2, S(-\partial^2+u))$ be the distance to the spectrum. Then*

$$|\mathcal{T}_{-1}^{\rm KdV}(z,u)| \lesssim \frac{2}{\operatorname{Im} z|z|}\Big(1+\min\Big\{0,-\log\big(2d(z^2,S(-\partial^2+u))|z|^{-2}\big)\Big\}\Big)\|u\|_{L^2}^2 \tag{2.6}$$

and

(2.7)

$$|\mathcal{T}_{-1}^{\rm KdV}(z)| \lesssim \frac{\tau^2+(\operatorname{Re} z)^2}{(\operatorname{Im} z)^2}\Big(1+\min\Big\{0,-\log\big(2d(z^2,\sigma(-\partial^2+u))|z|^{-2}\big)\Big\}\Big)\|u\|^2_{H^{-1}_\tau}\Big)$$

if $\tau \ge \|u\|_{H^{-1}_\tau}$.

Proof. We recall (see (9.7) in [43])

$$0 \le \operatorname{Im} 2z\mathcal{T}_{-1}^{\rm KdV}(z,u). \tag{2.8}$$

and

$$-\Delta \operatorname{Im}\Big(2z\mathcal{T}_{-1}^{\rm KdV}(z,u)\Big) = 8\pi\sum \kappa_j^2 \delta_{i\kappa_j} \tag{2.9}$$

where the sum runs over the positive numbers so that $-\kappa_j^2$ is an eigenvalue of the Lax operator. The superharmonic function $z\to \operatorname{Im} 2z\mathcal{T}_{-1}^{\rm KdV}(z,u)$ on the upper half plane has a trace at $\operatorname{Re} z=0$ which is a measure (see [43] for the case $u\in H^{-1}$) and by an abuse of notation

$$\frac{1}{2}\|u\|_{L^2}^2 = \frac{1}{\pi}\int \xi \operatorname{Im}\mathcal{T}_{-1}^{\rm KdV}(\xi)d\xi + \sum \frac{8}{3}\kappa_j^3. \tag{2.10}$$

In particular

$$\frac{1}{\pi}\int \xi \operatorname{Im}\mathcal{T}_{-1}^{\rm KdV}(\xi)d\xi \le \frac{1}{2}\|u\|_{L^2}^2$$

and

$$\begin{aligned}\operatorname{Im} 2z\mathcal{T}_{-1}^{\rm KdV}(z,u) = {}& \frac{\operatorname{Im} z}{2\pi}\int \frac{\xi}{|z-\xi|^2}\operatorname{Im} T_{-1}^{\rm KdV}(\xi,u)d\xi \\ &+ \sum 4\kappa_j^2\Big(\log|z+i\kappa_j| - \log(|z-i\kappa_j|\Big)\end{aligned}$$

hence

$$|\operatorname{Im} 2z\mathcal{T}_{-1}^{\rm KdV}| \le \frac{1}{2\pi\operatorname{Im} z}\|u\|_{L^2}^2 + \sum 4\kappa_j^2(\log|1+i\kappa_j/z| - \log|1-i\kappa_j/z|).$$

and we obtain similar gradient bounds. $\operatorname{Re} 2z\mathcal{T}_{-1}^{\mathrm{KdV}}$ is the complex conjugate function which decays at $i\infty$ and hence we obtain (2.6).

The map $\tau \to \|u\|_{H^{-1}_\tau}$ is monotonically decreasing. Given $u \in H^{-1}$ there is a unique τ so that $\tau^{-1/2}\|u\|_{H^{-1}_\tau} = \frac{1}{100}$ which we choose in the sequel. Hence

$$\left| \mathcal{T}_{-1}^{\mathrm{KdV}}(i\tau) - \frac{1}{2}\|u\|^2_{H^{-1}_\tau} \right| \le \frac{1}{100}\|u\|^2_{H^{-1}_\tau}$$

and

$$\operatorname{Re} \mathcal{T}_{-1}^{\mathrm{KdV}}(i\tau) = \frac{1}{4\pi}\int \frac{\xi}{\tau^2+\xi^2} \operatorname{Im} T_{-1}^{\mathrm{KdV}}(\xi) d\xi + \frac{1}{2\tau}\sum 4\kappa_j^2 (\log(|1+\kappa_j/\tau| - \log|1-\kappa_j/\tau|)$$

hence

$$|\operatorname{Re} \mathcal{T}_{-1}^{\mathrm{KdV}}(z)| \le c\frac{\tau^2 + (\operatorname{Re} z)^2}{(\operatorname{Im} z)^2}\Big(1 + \min\Big\{0, -\log\big(2d(z^2, S(-\partial^2+u))|z|^{-2}\big)\Big\}\Big)\|u\|^2_{H^{-1}_\tau}.$$

Estimate (2.7) follows in the same fashion as (2.6) followed from the analogous bound of the real part. □

The difference Hamiltonian plays a central role, not only for giving (2.2) a precise meaning.

Definition 2.2. *Let $\tau > 0$. We define*

$$\sigma(u) = \begin{cases} 0 \text{ if } -\partial^2 + u \text{ is p.d.} \\ \text{the square root of the negative of the lowest eigenvalue,} \end{cases}$$

the set of functions

$$U(\tau_0) = \{u \in H^{-1} : \sigma(u) < \tau_0\}$$

and $\mathbb{C}_{\tau_0} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}\backslash(0,\tau_0)i$.

Proposition 2.3. *Let $N \ge -1$. The difference Hamiltonian is defined on $\mathbb{C}_{\tau_0} \times U(\tau_0)$. It is holomorphic in the first component and analytic in the second component. It satisfies*

$$|\mathcal{T}_N^{\mathrm{KdV}}(z,u)| \le C_N \left(\frac{|z|}{\operatorname{Im} z}\right)^{2N+3} (\operatorname{Im} z)^{-2}(1+\|u\|_{H^{-1}})^N \|u\|^2_{H^{N+1}}, \tag{2.11}$$

if $(\operatorname{Im} z)^{-3/2}\|u\|_{L^2} \le \delta < 1$ for an absolute constant δ. Moreover

$$\left\| \frac{\delta}{\delta u}\mathcal{T}_N^{\mathrm{KdV}}(i\tau, u)\right\|_{H^{-N-1}} \le C_N \tau^{-2}(1+\|u\|_{H^{-1}})^N(\|u\|_{H^{N+1}} + \|u\|^2_{H^{N+1}}). \tag{2.12}$$

Remark 2.4. *We can combine (2.6) with (2.11) to obtain bounds for $\mathcal{T}_N^{\mathrm{KdV}}(z,u)$ on $\mathbb{C}_{\tau_0}$. We do not know whether the estimate (2.12) has a meaningful extension to $\mathbb{C}_{\tau_0}$*

Remark 2.5. *Proposition 2.3 has the immediate consequence that for $u \in H^{N+1}$*

$$\lim_{\tau\to\infty} (2i\tau)^2 \mathcal{T}_{N-1}(i\tau, u) = H_N^{\mathrm{KdV}}(u) \tag{2.13}$$

and

$$\limsup_{\tau\to\infty} |\mathcal{T}_N^{\mathrm{KdV}}(i\tau,u)| + |H_N^{\mathrm{KdV}}| \le c\Big(1+\|u\|_{H^{-1}}\Big)^N \|u\|^2_{H^N}.$$

We will later see that $u \in H^N$ suffices for the convergence in (2.13).

The structure of the coefficients H_N^{KdV} is described in Theorem 3.10. The inverse of the transmission coefficient is a holomorphic function on the upper half plane for $u \in \mathcal{S}(\mathbb{R})$.

The Nth equation of the KdV hierarchy is the Hamiltonian equation of the Hamiltonian H_N with respect to the Gardner Poisson bracket

$$\{F, G\}^{\mathrm{Gardner}} = \int \frac{\delta}{\delta u} F(u) \partial \frac{\delta}{\delta u} G(u) dx. \tag{2.14}$$

where $\frac{\delta}{\delta u}$ denotes the variational derivative defined by

$$\int \frac{\delta}{\delta u} F v dx = \frac{d}{dt} F(u + tv)|_{t=0}$$

for $v \in C_c^\infty$, assuming that the right hand side is defined. More explicitly, the Nth KdV equation is

$$u_t = \partial_x \frac{\delta}{\delta u} H_N. \tag{2.15}$$

Our approach crucially depends on the commutation of various flows. The core of this is the fact the Hamiltonians $\mathcal{T}_{-1}^{\mathrm{KdV}}$ Poisson commute (see Lemma 3.5):

Proposition 2.6. *The Hamiltonian $\mathcal{T}_{-1}^{\mathrm{KdV}}(z, u)$ Hamiltonian Poisson commute with respect to the Gardner bracket,*

$$\{\mathcal{T}_{-1}^{\mathrm{KdV}}(z_1, \cdot), \mathcal{T}_{-1}^{\mathrm{KdV}}(z_2, \cdot)\}^{\mathrm{Gardner}} = 0.$$

The formal Hamiltonian vector fields defined by the Hamiltonians H_n^{KdV} are however unbounded on any Sobolev space (the linear part of $\partial \frac{\delta}{\delta u} H_n^{\mathrm{KdV}}$ is $(-1)^n \partial^{2n+1}$), $\{H_n, H_N\}$ is defined for $u \in H^{2n+2N+2}$ and it can be extended to $u \in H^{n+N+1}$ and even slightly beyond that, but certainly not to open subsets of H^{-1}.

The KdV Hamiltonians are defined as limits of Poisson commuting functions, hence they Poisson commute with another and with $T_{-1}(i\tau)$, at least for sufficiently regular functions.

2.2. **Miura map and Gardner hierarchy.** We find it is easier to study well-posedness questions for the Gardner hierarchy, which we define and study here and in Section 3. The Hamiltonians $H_N^{\mathrm{Gardner}}(w, \tau_0)$ with

$$H_0^{\mathrm{Gardner}} = \frac{1}{2} \int w^2 dx, \quad H_1^{\mathrm{Gardner}}(w, \tau_0) = \frac{1}{2} \int w_x^2 + w^4 + 4\tau_0 w^3 dx$$

and the Gardner equations

$$w_t = \partial \frac{\delta}{\delta w} H_N^{\mathrm{Gardner}} \tag{2.16}$$

depend on a spectral parameter τ_0. They are connected to KdV by the remarkable modified Miura map (with $z = i\tau_0$)

$$M(-iz, w) := w_x + w^2 - 2izw \tag{2.17}$$

which has been used by Miura, Gardner and Kruskal to formally derive the Hamiltonians of the KdV hierarchy [45]. A short calculation shows that the inverse is given by

$$u \to \partial_x \log \phi_l + iz.$$

The modified Miura map defines an analytic diffeomorphism for $N \geq 0$ and $z = i\tau_0$

$$M(\tau_0, .) : H^N \to U(\tau_0) \cap H^{N-1}, \tag{2.18}$$

(Lemma 4.2 and Proposition 4.8 in Section 4). The function $w \in C(I;H^N)$ is a weak solution to the Nth Gardner equation (2.16) if and only if $u = M(\tau_0, w) \in C(I, H^{N-1})$ is a weak solution to the Nth KdV equation (see Theorem 5.8).

A short calculation (3.12) shows that for $M(-iz, w(z)) = u$,

$$\mathcal{T}_{-1}(z,u) = \frac{1}{2}\int w^2(z)dx$$

and as a consequence, using the chain rule, we compute

$$(-\partial - 2iz + 2w)\frac{\delta}{\delta u}\mathcal{T}_{-1}(z,u)$$

and we can write the equation for the τ flow as a system of differential equations

$$u_t = \partial_x \frac{\delta}{\delta u}\mathcal{T}_{-1}(i\tau,u) = \partial_x(-\partial + 2\tau + 2w)^{-1}w = \partial_x F(u) \tag{2.19}$$

$$w_x + 2\tau w + w^2 = u, -\partial_x F + 2\tau F + 2wF = w.$$

We call the flow defined by the Hamiltonian $\mathcal{T}_{-1}(i\tau)$ τ-flow. The formulas above easily imply the following (see also Section 4).

Proposition 2.7. *The Hamiltonian $\mathcal{T}^{\mathrm{KdV}}_{-1}(i\tau,u)$ is real for $\tau > \tau_0$. The map to the variational derivative*

$$H^{N-1} \ni u \to \frac{\delta}{\delta u}\mathcal{T}^{\mathrm{KdV}}_{-1}(z,u) \in H^{N+1}$$

for $N \in \mathbb{N}$ is locally Lipschitz and smooth, hence also the Hamiltonian vector field

$$H^{N-1} \ni u \to \partial_x\frac{\delta}{\delta u}\mathcal{T}^{\mathrm{KdV}}_{-1}(z,u) \in H^{N}$$

is locally Lipschitz and smooth.

Proof. Let $\operatorname{Im} z > 0$, either $\operatorname{Re} z \neq 0$ or $z = i\tau$ and $-\partial^2 + u + \tau^2 > 0$. Then the renormalized transmission coefficient is non zero and we define $w(z) = \partial \log \phi_l + iz$. Since

$$\partial w(z) - 2izw(z) + w^2(z) = u$$

and for $n \geq 0$

$$\|w\|_{H^n_{\operatorname{Im} z}} \leq c\|u\|_{H^{n-1}_{\operatorname{Im} z}}$$

with a constant depending on $(\operatorname{Im} z)^{-1/2}\|w(z)\|_{L^2}$. ☐

By the Cauchy-Lipschitz theorem they define a local flow on H^N for $N \geq -1$ by

$$u_t = \partial\frac{\delta}{\delta u}\mathcal{T}^{\mathrm{KdV}}_{-1}(u).$$

The Hamiltonians $\mathcal{T}^{\mathrm{KdV}}_{-1}(i\tau)$ for different τ are conserved under the τ flow as a consequence of Proposition 2.6 and the flows commute with themselves. The KdV Hamiltonians are conserved under the τ flows. The KdV Hamiltonians control the Sobolev norms H^N and hence the τ flows are global in time and preserve higher regularity.

Killip and Visan [39] (see also [1], Chapter 11) introduced the diagonal Green's function (the diagonal of the Green's function of the Lax operator) into the well-posedness question. It is related to w through the factorization

$$-\partial^2 + u + \tau_0^2 = (\partial + w + \tau_0)(-\partial + w + \tau_0).$$

The linear operators on the right hand side can be inverted and one obtain the formula for the integral kernel of the resolvent

$$G(x,y) = \int_{\max\{y,x\}}^{\infty} \exp\Big(\tau_0(x+y-2t) - 2\int_{\max\{x,y\}}^{t} wds - \int_{\min\{x,y\}}^{\max\{x,y\}} wds)\Big)dt.$$

Slightly deviating from the notation of Killip and Visan we define the good variable v via the diagonal Green's function (see Lemma 3.14 and its proof, compare to [1], Chapter 11)

$$\beta = G(i\tau_0, x, x) = \frac{\delta \log T^{\mathrm{KdV}}(i\tau_0)}{\delta u}$$

$$v = \frac{1}{2\tau_0 \beta} - 1.$$

Now u, v, w are related by the following relations (see Lemma 3.2)

$$w = W(\tau_0, v) := \tau_0 v - \frac{1}{2}\partial \log(1+v) \tag{2.20}$$

$$u = M(\tau_0, w) = w_x + w^2 + 2\tau_0 w \tag{2.21}$$

$$u = -\frac{1}{2}\frac{v_{xx}}{v+1} + \frac{3}{4}\frac{v_x^2}{(v+1)^2} + \tau_0^2 v^2 + 2\tau_0^2 v \tag{2.22}$$

For $s > -\frac{1}{2}$

$$W : \mathcal{V}^s := \{v \in H^{s+1} : v > -1\} \ni v \to \tau_0 v - \frac{1}{2}\partial_x \log(1+v) \in H^s \tag{2.23}$$

is an analytic diffeomorphism (Theorem 4.13). With these definitions $u \in C(I, H^{N-1}))$ is a solution to Nth KdV equation (2.15) if and only if $v \in C(I, \mathcal{V}^{N+1})$ is a weak solution to

$$v_t = 2\partial\Big((v+1)\sum_{n=-1}^{N-1} \frac{\delta H_n^{\mathrm{KdV}}}{\delta u}(2i\tau_0)^{2(N-1-n)}(u)\Big). \tag{2.24}$$

if and only if $w = W(v)$ satisfies the Nth Gardner equation (2.16). This is the contents of Theorem 3.8 for smooth solutions and Theorem 5.8 and Theorem 3.12 for weak solutions under weaker resp. different regularity assumptions. We write down the equation for v for $N = 1$

$$v_t = 2\partial\big[(v+1)(u-(2\tau_0)^2)\big] = \partial_x\Big[-v_{xx} + \frac{3}{2}\frac{v_x^2}{v+1} + 2\tau_0^2 v^3 + 6\tau_0^2 v^2\Big]. \tag{2.25}$$

It is remarkable that all the equations (2.24) are differential equations.

The relation between the Nth KdV equation (2.15), the Nth Gardner equation (2.16) and and the Nth 'good variable' equation (2.24) via the diffeomorphisms (2.21) (Miura map M), (2.20) (the map W) and the composition (2.22) extents to more general weak solutions (Theorem 5.8 and Theorem 5.10). This reduces the proof of Theorem 1.1 to a similar statement for the Gardner hierarchy, Theorem 2.19 below.

2.3. **Well-posedness for the KdV equation in $H^{-1}(\mathbb{X})$.** We find it instructive to follow the proof of Killip and Visan [39] in our setup to prepare for the case of the higher equations. We follow the strategy of Killip and Visan and prove wellposedness and the commutation property simultaneously, by approximating by flows defined by the Hamiltonians $(2\tau)^2\mathcal{T}_{N-1}(i\tau)$, motivated by (2.13). Alternatively one can deduce that the flows defined on smooth functions commute, and approximate the initial data to verify commutation of the flows.

Using the previous section we study wellposedness for the Gardner equation for $N \geq 2$ resp. wellposedness for the good variable equation for $N = 1$, similar to Killip and Visan.

There is a difference in the case $N = 1$, for which we cannot define weak solutions to the Gardner equation assuming only $w \in L^\infty(L^2)$ since the nonlinearity contains the term w^3. For $N \geq 2$ this is no issue since $L^\infty H^N \subset L^\infty L^\infty$. Also when $N = 1$ and $\mathbb{X} = \mathbb{R}$ the local smoothing estimates allow to make sense of weak solutions, see Theorem 2.19. In this case $N = 1$ for general geometry $\mathbb{X}$ we can use the equation for the good variable v instead, as Killip and Visan do. We give this argument now.

The approximate KdV flow is defined by the Hamiltonian

$$(2i\tau)^2\mathcal{T}_0(i\tau) = \frac{i(2i\tau)^5}{2}\log T^{\mathrm{KdV}}(i\tau) - \frac{(2i\tau)^4}{2}\int u\,dx - \frac{(2i\tau)^2}{2}\int u^2dx.$$

The functional $\frac{1}{2}\int u^2dx$ generates the translations hence

$$\{v(\tau), (2\tilde\tau)^2\mathcal{T}_0(i\tilde\tau)\} = -\frac{8\tilde\tau^4}{4\tilde\tau^2 - 4\tau^2}\partial_x\frac{v(\tau) - v(\tilde\tau)}{v(\tilde\tau)+1} + (2\tilde\tau)^2\partial_x v(\tau). \tag{2.26}$$

We recall the good variable version of KdV (2.25)

$$\begin{aligned} v_t &= 2\partial_x\Big[(v+1)\Big(-(2\tau)^2\frac{\delta H_{-1}}{\delta u} + \frac{\delta H_0}{\delta u}\Big)\Big] \\ &= \partial_x\Big[-2\tau^2 v + 2(v+1)u\Big]. \end{aligned} \tag{2.27}$$

The building blocks for the proof of wellposedness for KdV - following Killip and Visan - are also building blocks for the higher order KdV equations.

Step 1: Wellposedness of the approximate flow. The approximate flow is the flow of the Hamiltonian vector field $\mathcal{T}_0^{\mathrm{KdV}}$ which we discussed above. The term $\frac{(2\tau)^2}{2}$ generates translations and can be removed by using a moving coordinates.

Step 2: Equicontinuity. Equicontinuity of a set $Q \subset H^{-1}$ can be characterized as

$$Q \text{ is equicontinuous } \iff \lim_{\tau\to\infty}\sup_{u\in Q}|\mathcal{T}_{-1}^{\mathrm{KdV}}(i\tau, u)| = 0. \tag{2.28}$$

This fact is an immediate consequence of

$$\mathcal{T}_{-1}^{\mathrm{KdV}}(i\tau, u) = \frac{1}{2}\int w^2dx$$

with w satisfying (2.21). Since $\mathcal{T}_{-1}^{\mathrm{KdV}}(i\tau, \cdot)$ is preserved under the $\mathcal{T}_1^{\mathrm{KdV}}(i\tau_1, \cdot)$ flow also equicontinuity is preserved along the flow.

Step 3: Bounding the difference vector field in H^{-2} on equicontinuous sets. Let $Q \subset H^{-1}$ be an equicontinuous bounded set. We claim

$$\lim_{\tilde\tau\to\infty}\sup_{u\in Q}\Big\|\big\{v(\tau,u), \mathcal{T}_1^{\mathrm{KdV}}(i\tilde\tau, \cdot)\big\}^{\mathrm{Gardner}}\Big\|_{H^{-2}} = 0. \tag{2.29}$$

More explicitly

$$\begin{aligned}\big\{v(\tau,u),\mathcal{T}_1^{\mathrm{KdV}}(\tilde\tau)\big\} &= \big\{v(\tau,u),-(2\tilde\tau)^2\mathcal{T}_0^{\mathrm{KdV}}-H_1^{\mathrm{KdV}}\big\}\\ &=\partial\Big[\frac{-2\tilde\tau^4}{\tilde\tau^2-\tau^2}\frac{v(\tau)-v(\tilde\tau)}{v(\tilde\tau)+1}+2\tau^2v-(-2\tilde\tau^2v+2(v+1)u\Big]\end{aligned}$$

We do some algebraic manipulations

$$\begin{aligned}\frac{-2\tilde\tau^4}{\tilde\tau^2-\tau^2}\frac{v(\tau)-v(\tilde\tau)}{v(\tilde\tau)+1}+2\tilde\tau^2v &= \frac{\tilde\tau^2}{\tilde\tau^2-\tau^2}\frac{2\tilde\tau^2(v(\tilde\tau)-v(\tau))+2(\tilde\tau^2-\tau^2)v(\tau)(1+v(\tilde\tau))}{v(\tilde\tau)+1}\\ &=\frac{\tilde\tau^2}{\tilde\tau^2-\tau^2}\Big(\frac{2\tilde\tau^2v(\tilde\tau)(v(\tau)+1)}{v(\tilde\tau)+1}-2\tau^2v(\tau)\Big).\end{aligned}$$

Since

$$w=\frac12\partial_x\log(v+1)-\tau v,\qquad u=w_x+2\tau w+w^2$$

with the linearization at $v=0$ resp. $w=0$ (we indicate the linearized variables by a dot)

$$\dot w=\frac12\dot v_x-\tau\dot v,\qquad \dot u=\dot w_x+2\tau\dot w$$

we obtain

$$\tilde\tau^2v(\tilde\tau)\to-u\qquad\text{as }\tilde\tau\to\infty$$

uniformly in H^{-1} for u in bounded sets of H^{-1} and

$$v(\tilde\tau)\to0\qquad\text{as }\tilde\tau\to\infty$$

in H^1 uniformly for u in bounded equicontinuous sets of H^{-1}. Thus, again uniformly on bounded equicontinuous sets in H^{-1}

$$\lim_{\tilde\tau\to\infty}\frac{\tilde\tau^2}{\tilde\tau^2-\tau^2}\Big(\frac{2\tilde\tau^2v(\tilde\tau)(v(\tau)+1)}{v(\tilde\tau)+1}-2\tau^2v(\tau)\Big)=2u(v(\tau)+1)-2\tau^2v(\tau)$$

which is the KdV equation expressed in the good variables $v(\tau)$.

Step 4: The difference flow. Let $\tau_1,\tau_2\ge1$ and consider the difference flow

$$u_t=\partial\frac{\delta}{\delta u}\Big[-(2\tau_2)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_2,\cdot)+(2\tau_1)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_1,\cdot)\Big].$$

In order to keep the notation brief we introduce the formal notation

$$\exp\Big(tJDH\Big)u_0$$

for the solution to the Hamiltonian equations with Hamiltonian H and initial data u_0. By commutativity of the flow

$$\begin{aligned}u(\tau_1,\tau_2,t)&:=\exp\Big(-t\partial\frac{\delta}{\delta u}\big\{(2\tau_2)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_2,\cdot)-(2\tau_1)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_1,\cdot)\big\}\Big)u_0\\ &=\exp\Big(-t\partial(2\tau_2)^2\frac{\delta}{\delta u}\mathcal{T}_0^{\mathrm{KdV}}(i\tau_2,\cdot)\Big)\exp\Big(t\partial(2\tau_1)^2\frac{\delta}{\delta u}\mathcal{T}_0^{\mathrm{KdV}}(i\tau_1,\cdot)\Big)u_0\end{aligned}$$

The set

(2.30)

$$Q=\Big\{\exp(t_1\partial\frac{\delta}{\delta u}\mathcal{T}_0^{\mathrm{KdV}}(i\tau_1,\cdot))\exp(t_2\partial\frac{\delta}{\delta u}\mathcal{T}_0^{\mathrm{KdV}}(i\tau_2,\cdot))u_0:\tau_1,\tau_2>\tau_0,t_1,t_2\in\mathbb{R}\Big\}$$

is equicontinuous in H^{-1} by Step 2. Let $v(\tau_1,\tau_2,t)$ be the corresponding v functions corresponding to $u(\tau_1,\tau_2,t)\in Q$. Then

$$\begin{aligned} v_t &= \{v, -(2\tau_2)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_2,\cdot)+(2\tau_1)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_1,\cdot)\} \\ &= \{v,\mathcal{T}_1^{\mathrm{KdV}}(i\tau_2,\cdot)\}-\{v,\mathcal{T}_1^{\mathrm{KdV}}(i\tau_1,\cdot)\} \end{aligned}$$

and

$$\begin{aligned} \|v(t)-v_0\|_{H^{-2}} &\le \int_0^t \|\{v(s),\mathcal{T}_1^{\mathrm{KdV}}(i\tau_2,\cdot)\}-\{v,\mathcal{T}_1^{\mathrm{KdV}}(i\tau_2,\cdot)\}\|_{H^{-2}}ds \\ &\le t\Big(\sup_{u\in Q}\|\{v,\mathcal{T}_1^{\mathrm{KdV}}(i\tau_2,\cdot)\}\|_{H^{-2}}+\sup_{u\in Q}\|\{v,\mathcal{T}_1^{\mathrm{KdV}}(i\tau_1,\cdot)\}\|_{H^{-2}}\Big)\to 0 \end{aligned}$$

as $\tau_1,\tau_2\to\infty$ by Step 3.

Step 5: Convergence of the approximate flow. We want to prove that $e^{t\partial\frac{\delta}{\delta u}(2\tau)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau,\cdot)}u_0$ is a Cauchy sequence in τ. By commutativity of the flow (and a suggestive abuse of notation)

$$\begin{aligned} u(t,\tau_1,\tau_2) &:= \exp\Big(t\partial\frac{\delta}{\delta u}(2\tau_2)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_2,\cdot)\Big)u_0-\exp\Big(t\partial\frac{\delta}{\delta u}(2\tau_1)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_1,\cdot)\Big)u_0 \\ &= \Big\{\exp\Big(t\partial\frac{\delta}{\delta u}(2\tau_2)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_2,\cdot)\Big)\exp\Big(-t\partial\frac{\delta}{\delta u}(2\tau_1)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_1,\cdot)\Big)-1\Big\} \\ &\quad\times\exp\Big(t\partial\frac{\delta}{\delta u}(2\tau_1)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_1,.)\Big)u_0. \end{aligned}$$

Let $\tau<\tau_1,\tau_2$ by sufficiently large and Q as in (2.30) Let $v(t,\tau_1,\tau_2)$ the v function corresponding to $u(t,\tau_1,\tau_2)$ and $v(t,\tau_1)$ the one corresponding to

$$\exp\Big(-t\partial\frac{\delta}{\delta u}(2\tau_1)^2\mathcal{T}_0^{\mathrm{KdV}}(i\tau_1,.)\Big)u_0\in Q.$$

By Step 4

$$\lim_{\tau_1,\tau_2\to\infty}\|v(t,\tau_1,\tau_2)\|_{H^{-2}}=0.$$

However all functions $u(t,\tau_1,\tau_2)$ are in the fixed equicontinuous set $Q\subset H^{-1}$ (and the corresponding functions v are equicontinuous in H^1), hence

$$\lim_{\tau_1,\tau_2\to\infty}\|v(t,\tau_1,\tau_2)\|_{H^{1}}=0.$$

Thus $v(t,\tau_1)\in H^1$ is a Cauchy in τ_1. It extents to a continuous map

$$H^{-1}\times\mathbb{R}\times(\tau,\infty]\ni(u_0,t,\tau_1)\to u(t,\tau_1)\in H^{-1}$$

resp.

$$H^{-1}\times\mathbb{R}\times(\tau,\infty]\ni(u_0,t,\tau_1)\to v(t,\tau_1)\in H^{1}.$$

We have proven a slightly stronger version of the seminal theorem of Killip and Visan [39].

Theorem 2.8. *Let $Q\subset H^{-1}(\mathbb{X})$ by a equicontinuous bounded subset of $H^{-1}(\mathbb{X})$ and let τ be sufficiently large and $\tau_1>\tau$. Then the approximate flow*

$$u_t=\partial\big(-(2\tau_1)^2\frac{\delta}{\delta u}\mathcal{T}_0(i\tau_1,\cdot)\big)$$

with initial data in Q has a unique global solution $u(t,\tau_1)$ in $L^\infty(\mathbb{R},H^{-1}(\mathbb{X}))$. The set $\{u(t,\tau):u_0\in Q,t\in\mathbb{R},\tau\ge 1\}\subset H^{-1}$ is bounded and equicontinuous. The good variable $v(t,\tau_1)$ converges in $H^1(\mathbb{X})$ uniformly on compact time intervals to a weak *solution of* (2.27)*, the good variable KdV equation, as* $\tilde\tau \to \infty$*. The flow commutes with the* τ *flows. Higher Hamiltonians are preserved.*

The claim on commutation and preservation of Hamiltonians are an immediate consequence of the construction and Proposition 2.6. We can use Theorem 5.10 and Theorem 5.8 to translate this result to the Gardner and KdV equation.

2.4. **The generating function of the Gardner hierarchy.** For $N \ge 2$ we consider the Gardner equations instead of the good variables equation. The starting point is the τ flow for the Gardner hierarchy defined by the generating function of the Gardner hierarchy

$$\mathcal{T}_{-1}^{\mathrm{Gardner}}(z,w,\tau_0) = \frac{1}{8(\tau_0^2+z^2)}\int w^2 - w^2(z)dx$$

where $w(z)$ is defined by the left Jost solution ϕ_l for $-\partial^2+u-z^2$,

$$w(z) = \partial_x \log\phi_l + iz \tag{2.31}$$

or, equivalently, as unique solution to

$$w_x(z) - 2izw(z) + w^2(z) = w_x + 2\tau_0 w + w^2.$$

Proposition 2.9. *Let* $0<\tau_0$*.* $\mathcal{T}_{-1}^{\mathrm{Gardner}}(z,w,\tau_0)$ *is holomorphic in* z *for*

$$\{z : \operatorname{Im} z > 0 \text{ and either } \operatorname{Re} z \neq 0 \text{ or } \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z > \tau_0\}.$$

It satisfies for $\operatorname{Im} z > \tau_0$ *with an implicit constant depending on* $\tau_0^{-1/2}\|w\|_{L^2}$

$$|\mathcal{T}_{-1}^{\mathrm{Gardner}}(z,w,\tau_0)| \lesssim \frac{\tau^2+(\operatorname{Re} z)^2}{(\operatorname{Im} z)^2}\Big(1+\min\Big\{0,-\log\big|\frac{z}{\tau_0}-1\big|\Big\}\Big)\|w\|_{L^2}^2.$$

Let $\tau>\tau_0$*. Then* $\mathcal{T}_{-1}^{\mathrm{Gardner}}(i\tau,w,\tau_0)\in\mathbb{R}$ *and*

$$\frac{\delta}{\delta w}\mathcal{T}_{-1}^{\mathrm{Gardner}}(i\tau,.,\tau_0)(w) = (-\partial+2\tau_0+2w)(-\partial+2\tau+2w(i\tau))^{-1}w(i\tau)-w. \tag{2.32}$$

The Fréchet derivative in direction ϕ *is* $D_\phi\frac{\delta}{\delta w}\mathcal{T}_{-1}^{\mathrm{Gardner}}(i\tau,.,\tau_0) = -A(w)\phi$ *where* $A\phi$ *is given by*

$$4\tau\Big(\frac{1}{v+1}\phi-(-\partial+2\tau_0+2w)(-\partial+2\tau+2w(i\tau))^{-1}\frac{1}{v+1}(\partial+2\tau+2w(i\tau))^{-1}(\partial+2\tau_0+2w)\phi\Big). \tag{2.33}$$

Let $0<\tau_0<\tau_1,\tau_2$*. The functionals* $\mathcal{T}_{-1}^{\mathrm{Gardner}}(i\tau_1,\cdot,\tau_0)$ *and* $\mathcal{T}_{-1}^{\mathrm{Gardner}}(i\tau_2,\cdot,\tau_0)$ *Poisson commute with respect to the Gardner Poisson bracket.*

Proof. The proof relies on the modified Miura map. It suffices to consider $\tau_0=0$. The

$$\|w_x+2w+w^2\|_{H_\tau^{-1}} \le \|w_x+2w+w^2\|_{H^{-1}} \le c(1+\|w\|_{L^2})\|w\|_{L^2}.$$

The first estimate now follows from (2.7).

Let $\tau\ge\tau_0$. By Lemma 4.2 the solution $w(i\tau)\in L^2$ to

$$\partial w(i\tau)+2\tau w(i\tau)+w^2(i\tau) = w_x+2\tau_0 w+w^2$$

is uniquely determined and

$$\|w(i\tau)\|_{L^2} \le c(\tau_0^{-1/2}\|w\|_{L^2})\|w\|_{L^2}.$$

We compute with $w(s) = w + s\phi$

$$\frac{d}{ds} w(i\tau)\Big|_{s=0} = (\partial + 2\tau + 2w(i\tau))^{-1}(\partial + 2\tau_0 + 2w)\phi$$

$$\frac{d}{ds}\frac{1}{2}\int w^2(i\tau) - w^2 dx\Big|_{s=0} = \int \Big[(\partial + 2\tau_0 + 2w)\phi\Big](-\partial + 2\tau + 2w(i\tau))^{-1} w(i\tau) - \phi w dx$$

which implies (2.32)

$$\frac{d^2}{ds^2}\frac{1}{2}\int w^2(i\tau) - w^2 dx\Big|_{s=0} =$$

$$2\tau \int \Big[\phi^2 - \Big\{(\partial + 2\tau + 2w(i\tau))^{-1}(\partial + 2\tau_0 + 2w)\phi\Big\}^2\Big](-\partial + 2\tau + 2w(i\tau))^{-1} 1 dx$$

By (3.16) $(-\partial + 2\tau + 2w(i\tau))^{-1}1 = -\beta$, the negative of the diagonal Green's function and by Lemma 3.2, (3.13) $2\tau\beta = \frac{1}{1+v}$. This implies (2.33). $\square$

We will need various bounds on the variational derivatives. For later use we consider weighted estimates. We call η τ slowly varying (see also Definition 4.5) if

$$|\eta_x| \le \tau\eta, \qquad |\eta^{(j)}| \le c_j \tau^j \eta. \tag{2.34}$$

Proposition 2.10. *Let $N \ge 0$ and $\tau > \tau_0$. The maps*

$$H^N \ni w \to \frac{\delta}{\delta w}\mathcal{T}_{-1}^{\text{Gardner}}(i\tau, w, \tau_0) \in H^{N+1}$$

are smooth and locally Lipschitz continuous. With constants depending on τ, τ_1 and $\tau^{-1/2}\|w\|_{L^2}$ for τ slowly varying η we have

$$\Big\|\eta \frac{\delta}{\delta w}\frac{1}{2}\int w^2(i\tau) - w^2 dx\Big\|_{H^{N+1}} \le c\|\eta w\|_{H^N}, \tag{2.35}$$

if $\eta w \in H^N$ (see (2.33) for the definition of $A(w)$)

$$\Big\|\eta A(w)\phi\Big\|_{H^{N+1}} \le c\Big(\|\eta\phi\|_{H^N} + \|\phi\|_{L^2}\|\eta w\|_{H^N}\Big) \tag{2.36}$$

and

$$\Big\|\cosh A(w)\phi\Big\|_{H^{N+1}} \le c\Big(\|\cosh\phi\|_{H^N} + \|\cosh^2\phi\|_{L^2}\|\operatorname{sech} w\|_{H^N}\Big). \tag{2.37}$$

Here in the sequel we often omit the argument (most often x) of sech^2 and similar functions).

Proof. We rewrite the variational derivative using (2.32) as

$$\begin{aligned}\frac{\delta}{\delta w}\frac{1}{2}\int w^2(i\tau) - w^2 dx &= (-\partial + 2\tau_0 + 2w)(-\partial + 2\tau + 2w(i\tau))^{-1} w(i\tau) - w \\ &= 2(\tau_0 - \tau + w - w(i\tau))(-\partial + 2\tau + 2w(i\tau))^{-1} w(i\tau) + w(i\tau) - w.\end{aligned} \tag{2.38}$$

We write c for generic constants depending only on τ, τ_0 and $\|w\|_{L^2}$. An algebraic manipulation gives

$$\begin{aligned}w(i\tau) - w &= (\partial + 2\tau + w(i\tau))^{-1}(\partial + 2\tau_0 + w)w - w \\ &= 2(\tau_0 - \tau)(\partial + 2\tau + w(i\tau))^{-1} w + (\partial + 2\tau + w(i\tau))^{-1}(w - w(i\tau))w,\end{aligned}$$

hence, for $\tau > 2\tau_0$

$$\|\eta(w(i\tau) - w)\|_{L^\infty} \le c\big(\tau^{1/2} + \tau^{-1/2}\|w(i\tau) - w\|_{L^2}\big)\|\eta w\|_{L^2} \le c\tau^{1/2}\|\eta w\|_{L^2}$$

and

$$\|w(i\tau)-w\|_{H^1_\tau} \le c\big(\tau + \|w(i\tau)-w\|_{L^\infty}\|\eta w\|_{L^2}\big), \quad \|\eta w(i\tau)\|_{H^N_\tau} \le c\|w\|_{H^N_\tau}.$$

By induction

$$\|\eta(w(i\tau)-w)\|_{H^{N+1}_\tau} \le c\tau\|\eta w\|_{H^N_\tau} \tag{2.39}$$

which implies the desired estimate for $w(i\tau)-w$ on the right hand side of (2.38). Moreover by Lemma 4.9

$$\|\eta(-\partial+2\tau+2w(i\tau))^{-1}w(i\tau)\|_{H^{N+1}} \le c(\tau^{-1/2}\|w(i\tau)\|_{L^2})\|\eta w(i\tau)\|_{H^{N-1}}$$

and together with calculus type estimates we arrive at (2.35).

It remains to prove (2.36) and (2.37). The arguments are the same and we focus on (2.36). We use the algebraic manipulation

$$(-\partial+2\tau_0+2w)(-\partial+2\tau+2w(i\tau))^{-1} = 2(\tau_0-\tau+w-w(i\tau))(-\partial+2\tau+2w(i\tau))^{-1}+1$$

resp.

$$(\partial+2\tau+2w(i\tau))^{-1}(\partial+2\tau_0+2w) = (\partial+2\tau+2w(i\tau))^{-1}2(\tau_0-\tau+w-w(i\tau))+1$$

and expand A. We have to prove the bound for

$$(2(\tau_0-\tau+w-w(i\tau))(-\partial+2\tau+2w(i\tau))^{-1}\frac{1}{v+1}(\partial+2\tau+2w(i\tau))^{-1}(\partial+2\tau_0+2w)\phi,$$

in particular, with constants depending on $\tau^{-1/2}\|w\|_{L^2}$, by the linear estimates (4.49) and (4.50)

$$\begin{aligned}
&\|\eta(\partial+2\tau+2w(i\tau))^{-1}(\partial+2\tau_0+2w)\phi\|_{H^N} \\
&\qquad\lesssim \|\eta(\partial+2\tau_0+2w)\phi\|_{H^{N-1}} + \|\eta w\|_{H^{N-1}}\|(\partial+2\tau_0+2w)\phi\|_{H^{-1}} \\
&\qquad\lesssim \|\eta\phi\|_{H^N} + \|\eta w\|_{H^{N-1}}\|\phi\|_{L^2}
\end{aligned}$$

$$\begin{aligned}
&\Big\|\eta(-\partial+2\tau+2w(i\tau))^{-1}\frac{1}{v+1}(\partial+2\tau+2w(i\tau))^{-1}(\partial+2\tau_0+2w)\phi\Big\|_{H^{N+1}} \\
&\quad\lesssim \Big\|\frac{1}{v+1}\Big\|_{L^\infty}(\|\eta\phi\|_{H^N}+\|\eta w\|_{H^N}\|\phi\|_{L^2}) + \Big\|\frac{1}{v+1}\Big\|_{W^{N,\infty}}\|\eta\phi\|_{L^2}
\end{aligned}$$

Similarly we deal with

$$\frac{8\tau}{v+1}(\partial+2\tau+2w(i\tau))^{-1}(\tau_0-\tau+w-w(i\tau)).$$

Finally we obtain (2.36) and similarly (2.37).

We deduce Poisson commutation from the analogous statement for the KdV hierarchy. By Lemma 3.5 $\mathcal{T}^{\mathrm{KdV}}_{-1}(z_j) = \log T_r(z_j)$ Poisson commute with respect to the Gardner and almost with the Magri Poisson bracket. Lemma 3.4 relates the Magri bracket in terms of w with a linear combination of the Gardner and the Magri bracket for KdV. Hence the functionals

$$\frac{1}{2}\int w^2 - w^2(i\tau_j)dx = \mathcal{T}^{\mathrm{KdV}}_{-1}(i\tau_j)(w_x+2\tau_0 w+w^2)$$

Poisson commute with respect the Gardner Poisson structure. The same is then true for $\mathcal{T}^{\mathrm{Gardner}}_{-1}$. □

There is a fundamental nonobvious relation between the generating function of the KdV hierarchy and the Gardner hierarchy.

Lemma 2.11. *The following formula hold for* $\operatorname{Im} z > \tau_0$.

$$\partial_x \frac{\delta \mathcal{T}_{-1}^{\mathrm{KdV}}(z)}{\delta u}\Big|_{u=w_x+2\tau_0 w+w^2} = (\partial + 2\tau_0 + 2w)\partial \frac{\delta \mathcal{T}_{-1}^{\mathrm{Gardner}}(z,.,\tau_0)}{\delta w}. \tag{2.40}$$

Ultimately this property implies the equivalence of the Gardner equations and the KdV equations: w satisfies the Nth Gardner equation if and only if $u = w_x + 2\tau_0 w + w^2$ satisfies the Nth KdV equation, modulo regularity and integrability assumptions.

Proof. We calculate

$$\frac{\delta \mathcal{T}_{-1}^{\mathrm{KdV}}}{\delta u}\Big|_{u=w_x+2\tau w+w^2} = (-\partial + 2 + 2w(z))^{-1} w(z),$$

$$\frac{\delta \mathcal{T}_{-1}^{\mathrm{Gardner}}}{\delta w} = \frac{1}{4(z^2+\tau^2)}\Big[(-\partial + 2\tau + 2w)\frac{\delta \mathcal{T}_{-1}^{\mathrm{KdV}}}{\delta u}\Big|_{u=w_x+2\tau w+w^2} - w\Big],$$

$$(\partial + 2\tau_0 + 2w)\partial w = \partial_x(w_x + 2\tau_0 w + w^2) = u_x$$

and finally, with $\beta = \frac{\delta \mathcal{T}^{\mathrm{KdV}}(z,.)}{\delta u}\big|_{u=w_x+2\tau w+w^2}$

$$\begin{aligned}
4(z^2+\tau^2)&\partial\beta - (\partial + 2\tau + 2w)\partial(-\partial + 2\tau + 2w)\beta + u_x \\
&= (-\partial^3 + 4z^2\partial + 8\tau w\partial + 4\tau w_x + 4w^2\partial + 2\partial w^2 + 2w_{xx} + 2w_x\partial)\beta + u_x \\
&= (-\partial^3 + 4u\partial + 2u_x)\frac{\delta T_{-1}^{\mathrm{KdV}}}{\delta u} + 4z^2\frac{\delta T_{-1}^{\mathrm{KdV}}}{\delta u} + u_x \\
&= 0
\end{aligned}$$

by the Lenard recursion, see Lemma 3.1. □

The Hamiltonians H_N^{Gardner} are formally defined as the coefficients of the asymptotic series

$$\mathcal{T}_{-1}^{\mathrm{Gardner}}(z,w,\tau_0) \sim \sum_{n=0}^{\infty} H_{n,\tau_0}^{\mathrm{Gardner}}(w)(2z)^{-2n}.$$

More precisely the coefficients are defined by iteratively by limits involving a linear combination of $\mathcal{T}_{-1}^{\mathrm{Gardner}}$ evaluated at different points, hence they Poisson commute with one another and the generating function on sufficiently regular functions. The structure of the Hamiltonian equations is given by the following lemma.

Lemma 2.12. *The Nth Gardner Hamiltonian can be written as*

$$\int \frac{1}{2}|w^{(N)}|^2 + \sum_{n_0=0}^{N} \tau_0^{n_0} \sum_{n=3,n+n_0\, even}^{2N+2-n_0} \sum_{\alpha\in A_{N,n+n_0}} c_{\alpha,n,n_0,N} \prod_{j=0}^{n} w^{(\alpha_j)} dx$$

with

$$A_{N,m} = \{|\alpha| + m = 2N+2, \alpha_j \le 2N+2-m\}.$$

The variational derivative can be written as

$$\frac{\delta}{\delta w} H_N^{\mathrm{Gardner}}(w,\tau_0) = (-1)^N w^{(2N+1)} + \sum_{n_0,n,\alpha} \tilde{c}_{\alpha,n,n_0,N} \tau_0^{n_0} \partial^{\alpha_n} \prod_{j=1}^{n-1} w^{(\alpha_j)}.$$

Let η be bounded and τ slowly varying and $\alpha \in A_{n,n+n_0}$. Then

$$\Big\| \eta^2 \partial^{\alpha_n} \prod_{j=1}^{n-1} w^{(\alpha_j)} \Big\|_{H^{-\alpha_n}(\mathbb{X})} \le c\|w\|_{L^2}^{n-2+\frac{\frac{n}{2}+n_0-1-\alpha_n}{N}} \|\eta w\|_{H^N(\mathbb{X})}^{2-\frac{\frac{n}{2}+n_0-1-\alpha_n}{N}}. \tag{2.41}$$

It particular we obtain

$$\Big\| \frac{\delta}{\delta w} H_N^{\text{Gardner}}(w) \Big\|_{H^{-N-1}(\mathbb{X})} \le c(\tau_0, \|w\|_{L^2}) \Big(\|w\|_{H^{N-1}(\mathbb{X})} + \|w\|^2_{H^{N-1}(\mathbb{X})} \Big). \tag{2.42}$$

Since the Hamiltonian flow of $\mathcal{T}_{-1}^{\text{Gardner}}$ preserve higher regularity and H_M^{Gardner} is defined via limits it is preserved for initial data in H^M. Let $w(t)$ be the solution with initial data in H^M. Then $t \to w(t)$ is continuous as a map to H^{M-1}, hence weakly continuous as a map to H^M. Conservation of H_M^{Gardner} implies continuity of $t \to \|w\|_{H^M}$ and hence $w \in C(\mathbb{R}, H^M)$.

We define the difference Hamiltonian for $N \ge 0$ by

$$\mathcal{T}_N^{\text{Gardner}}(z, w, \tau_0) = \frac{(2z)^{2N+2}}{8(\tau_0^2 + z^2)} \int w^2 - w^2(z) dx - \sum_{n=0}^{N} (2z)^{2(N-n)} H_n^{\text{Gardner}}.$$

Its bilinear part is $-\frac{1}{2}\|w^{(N+1)}\|_{H^{-1}_{2\tau}}$. We denote

$$\mathcal{T}_N^{NL}(i\tau, w) = \mathcal{T}_N^{\text{Gardner}}(i\tau, w) + \frac{1}{2}\|w^{(N+1)}\|^2_{H_N^{-1}} \tag{2.43}$$

To shorten the notation we often write $\mathcal{T}_N^{\text{Gardner}}(z)$ instead of $\mathcal{T}_N^{\text{Gardner}}(z, w, \tau_0)$.

Proposition 2.13. *The Gardner difference Hamiltonians $\mathcal{T}_N^{\text{Gardner}}$ are defined in $\{z \in C : \operatorname{Im} z > \tau_0\} \times H^N$. They are holomorphic in z and analytic in w. On the imaginary axis the difference Hamiltonian is real: $\mathcal{T}_N^{\text{Gardner}}(i\tau, w, \tau_0) \in \mathbb{R}$. They satisfy with*

$$c = c(N, (\tau_0/\operatorname{Im} z)^{\frac{1}{2}} \|w\|_{L^2})$$

(2.44)

$$(2\operatorname{Im} z)^2 |\mathcal{T}_N^{\text{Gardner}}(z, w, \tau_0)| \le c \Big(\frac{|z|}{\operatorname{Im} z} \Big)^{2N+3} \Big(\|w\|^2_{H^{N+1}_{\tau_0}} + \|w\|^{2(N+2)}_{L^{2(N+2)}} + \tau_0^{N+2} \|w\|_{L^{N+2}} \Big),$$

A set $Q \subset H^N$ is equicontinuous if and only if

$$\lim_{\tau \to \infty} \sup_{w \in Q} |\mathcal{T}_N^{\text{Gardner}}(i\tau, w, \tau_0)| = 0. \tag{2.45}$$

If $Q \subset H^{N-1}$ is equicontinuous then

$$\Big\| \frac{\delta}{\delta w} \mathcal{T}_N^{\text{Gardner}}(i\tau, w, \tau_0) \Big\|_{H^{-N-2}} \to 0 \quad \text{as } \tau \to \infty \tag{2.46}$$

uniformly in $w \in Q$.

The estimates are proven in Section 7.4. We obtain as an immediate consequence

$$\lim_{\tau \to \infty} (2i\tau)^2 \mathcal{T}_{N-1}^{\text{Gardner}}(i\tau, w, \tau_0) = H_N^{\text{Gardner}}(w, \tau_0)$$

and we consider $-(2\tau)^2 \mathcal{T}_{N-1}$ as approximate Hamiltonian for the Nth Gardner Hamiltonian. Moreover the recursion relation follows

$$\mathcal{T}_N^{\text{Gardner}}(z, w, \tau_0) = -H_N^{\text{Gardner}} + (2z)^2 \mathcal{T}_{N-1}^{\text{Gardner}}(z, w, \tau_0).$$

2.5. **Well-posedness for the Nth KdV equation in H^{N-2}.** In the high regularity regime we obtain the following existence and uniqueness result.

Theorem 2.14. *Suppose that $2 \le N \le M$, $w_0 \in H^M(\mathbb{X})$. There exists a weak solution $w \in C(\mathbb{R}; H^M(\mathbb{X}))$ with equicontinuous orbit in H^M to the Nth Gardner equation which depends continuously on the initial data. The flow map has a continuous extension to a map*

$$H^{N-1} \ni w_0 \to w \in C(\mathbb{R}; H^{N-1}).$$

Let $1 \le N < \tilde{N} \le M+1$. Then the Nth and the $\tilde{N}$th Gardner evolutions commute. They also commute with the evolution of the generating function $\mathcal{T}_{-1}^{\text{Gardner}}(i\tau, .\tau_0)$. The corresponding Hamiltonians are preserved whenever they are defined.

As a consequence of the theorem we obtain wellposedness of the Nth KdV equation with initial data in H^{N-2} by Theorem 5.8 for $N \ge 2$. The case $N = 1$ is covered in Theorem 2.8.

Corollary 2.15. *Suppose that $N \ge 2$. Let $u_0 \in H^{N-1}(\mathbb{X})$. Then there exists a unique weak solution $u \in C(\mathbb{R}; H^{N-1})$. The evolution has a unique continuous extension to a map*

$$H^{N-2} \times \mathbb{R} \ni (u_0, t) \to u(t) \in H^{N-2}.$$

By the Cauchy-Lipschitz theorem (and Proposition 2.10) there exists a unique (local at first) solution to the τ flow (defined by the Hamiltonian $\mathcal{T}_{-1}^{\text{Gardner}}(i\tau, ., \tau_0)$ in H^N which preserves H^N regularity. The flows for different τ commute since the Hamiltonians Poisson commute and $\mathcal{T}_{-1}^{\text{Gardner}}(z, ., \tau_0)$ are preserved under the τ flow. Due to the conservation of $\frac{1}{2}\|w\|_{L^2}^2$ the flow is global.

Proof. Proof of Theorem 2.14 for the Nth Gardner equation with $N \ge 2$ in $C(\mathbb{R}, H^{N-1})$ for initial data in H^{N-1}. The argument does not distinguish between the equation on the line or on the circle. We will construct solutions by induction on N. The induction hypothesis is:

Assumption 2.16. *Let $n < N$. There exist unique weak solutions to the nth Gardner equation for initial data in H^j with $j \ge n-1$ which depends continuously on the initial data. We also assume that the evolutions commute with another and also with the Hamiltonian evolution of $\mathcal{T}_{-1}^{\text{Gardner}}$. The functionals $\mathcal{T}_{-1}^{\text{Gardner}}(i\tau, \cdot, \tau_0)$ are conserved for all Gardner evolutions. The Hamiltonians H_N^{Gardner} is conserved for the $\mathcal{T}_{-1}^{\text{Gardner}}$ evolution on H^N and for the H_n^{Gardner} evolution on H^N if $n \le N+1$.*

Let $Q \in H^{N-1}$ be an equicontinuous set. By (2.45) in Proposition 2.13

$$\lim_{\tau\to\infty} \sup_{w \in Q} |\mathcal{T}_{N-1}^{\text{Gardner}}(i\tau, w, \tau_0)| = 0.$$

By the induction hypothesis $\mathcal{T}_{N-1}^{\text{Gardner}}(i\tau)$ is conserved under the evolution of all the lower Hamiltonians, of $\mathcal{T}_{-1}^{\text{Gardner}}$, and hence also of the difference Hamiltonian $\mathcal{T}_{N-1}^{\text{Gardner}}$. Using (2.45) we see that all orbits starting from Q are again equicontinuous in H^{N-1}.

Let $w_0 \in H^{N-1}$ and, for $\tau > \tau_0$, $w(\tau,t)$ the Hamiltonian evolution of $(2\tau)^2 \mathcal{T}_{N-1}^{\text{Gardner}}$. Then, using commutativity

$$
\begin{aligned}
v(t,\tau_1,\tau_2) :=& \exp\Big(t\partial \frac{\delta}{\delta w}(2\tau_2)^2 \mathcal{T}_{N-1}^{\text{Gardner}}(i\tau_2)\Big) w_0 - \exp\Big(t\partial\frac{\delta}{\delta w}(2\tau_1)^2\mathcal{T}_{N-1}^{\text{Gardner}}(i\tau_1)\Big)w_0 \\
=& \Big\{ \exp\Big(t\partial (2\tau_1)^2 \frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}(i\tau_2)\Big) \exp\Big(-t\partial\frac{\delta}{\delta w}(2\tau_1)^2 \mathcal{T}_{N-1}^{\text{Gardner}}(i\tau_1)\Big) - 1\Big\} w(\tau_1,t)
\end{aligned}
$$

We claim

$$
\exp\Big(t\partial(2\tau_1)^2\frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}(\tau_2)\Big)\exp\Big(-t\partial\frac{\delta}{\delta w}(2\tau_1)^2\mathcal{T}_{N-1}^{\text{Gardner}}(\tau_1)\Big) w \to w \tag{2.47}
$$

as $\tau_1,\tau_2 \to \infty$ in H^{-N-3} uniformly on equicontinuous sets in compact time intervals. Assume the claim for the moment. The orbit $\{w(\tau_1,t) : \tau_1 > \tau_0, t \in \mathbb{R}\}$ is equicontinuous. By the claim $v(\tau_1,\tau_2,t) \to 0$ in H^{-N-3} uniformly on compact time intervals as $\tau_1,\tau_2\to\infty$. Thus $w(\tau,t)$ is Cauchy in H^{-N-3} uniformly on compact time intervals. By equicontinuity it is also Cauchy in H^{N-1} uniformly on compact time intervals. Let $w \in C(\mathbb{R}; H^{N-1})$ be the limit. It is a weak solution to the Nth Gardner equation again by (2.46).

We turn to the proof of the claim (2.47). Let

$$
w(t) = \exp\Big(t\partial(2\tau_2)^2\frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}(\tau_2)\Big)\exp\Big(-t\partial\frac{\delta}{\delta w}(2\tau_1)^2\mathcal{T}_{N-1}^{\text{Gardner}}(\tau_1)\Big)w_0.
$$

Then, by definition and commutativity of the flows

$$
\begin{aligned}
w_t &= \partial\frac{\delta}{\delta w}\Big((2\tau_2)^2\mathcal{T}_{N-1}^{\text{Gardner}}(\tau_2) - (2\tau_1)^2\mathcal{T}_{N-1}^{\text{Gardner}}(\tau_1)\Big) \\
&= \partial\frac{\delta}{\delta w}\Big(-\mathcal{T}_{N}^{\text{Gardner}}(\tau_2) + \mathcal{T}_{N}^{\text{Gardner}}(\tau_1)\Big).
\end{aligned}
$$

By (2.45) and equicontinuity of the orbit the right hand side converges to 0 in H^{-N-3} uniformly on compact time intervals. We obtain more: The evolutions of $(2\tau)^2\mathcal{T}_{N-1}^{\text{Gardner}}$ converges uniformly for compact time intervals and equicontinuous sets of initial data to a weak solution of the Nth Gardner equation.

The higher regularity claim is an immediate consequence: Let $n \ge N$ and suppose that $w_0 \in H^n$. The approximate flow $(2\tau)^2\mathcal{T}_{N-1}^{\text{Gardner}}$ defines a flow in H^n by the induction hypthesis. The quantities $\|w(t)\|_{L^2}^2$ and $H_n^{\text{Gardner}}(w(t))$ are independent of time and hence $\|w\|_{H^n}$ is uniformly bounded for the approximate flow. Hence the solution to the approximate flow (and hence also the limit, the solution to the Nth equation) is uniformly bounded in H^n. Moreover $\mathcal{T}_N^{\text{Gardner}}$ is conserved for the approximate flow hence the solutions to the approximate flow and the limit are equicontinuous, uniformly in t. This implies convergence in H^n.

It remains to prove uniqueness of weak solutions. Let w be a weak solution to the Nth Gardner equation in $C(\mathbb{R}; H^N)$ and let

$$
v(\tau,t) = \exp\big(-t\partial\frac{\delta}{\delta w}(2\tau)^2 T_{N-1}^{\text{Gardner}}(i\tau)\big)w(t).
$$

The orbit $\{v(\tau,t) : \tau > \tau_0, t\in\mathbb{R}\}$ is again an equicontinuous set in H^N.

Proposition 2.17. *Let $v \in C(\mathbb{R}; H^N)$ be as above. Then in a distributional sense*

$$
\partial_t v(\tau,t) = \partial\frac{\delta}{\delta w}\mathcal{T}_N^{\text{Gardner}}(i\tau). \tag{2.48}
$$

We postpone the proof of Proposition 2.17 since we will prove a stronger version in the next subsection.

The right hand side of (2.48) converges uniformly to zero in H^{-N-1} on equicontinuous sets as $\tau \to \infty$. Thus, again using equicontinuity

$$v(\tau, t) \to w_0$$

uniformly in H^N on compact time intervals and w_0 in bounded equicontinuous sets in H^N.

We are now in the position to prove convergence of the approximate flow to the weak solution:

$$\begin{aligned}
&\exp\Big(- t\partial(2\tau)^2 \frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}\Big) w_0 - w(t) \\
&\qquad = \exp\Big(- t\partial(2\tau)^2 \frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}\Big) w_0 - \exp\Big(- t\partial(2\tau)^2 \frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}\Big) v(\tau, t).
\end{aligned}$$

The $\mathcal{T}_{N-1}^{\text{Gardner}}$ flow is uniformly continuous for t in compact time intervals, $\tau > \tau_0$ and arguments in equicontinuous sets. Hence the difference above converges to 0 in H^{N-1} as $\tau \to \infty$, uniformly on compact time intervals. Commutativity of the flows is a consequence of the construction, as is conservation of Hamiltonians. This establishes the induction hypothesis and completes the proof. $\square$

2.6. **The Gardner equation hierarchy with initial data in $L^2(\mathbb{R})$.** We will prove existence and uniqueness of weak solutions to the Gardner hierarchy. The gain of regularity by Kato smoothing is crucial and we restrict to the case of $\mathbb{X} = \mathbb{R}$ for that reason. Theorem 1.1 follows by applying the modified Miura map, more precisely Theorem 5.8. The results for the Gardner hierarchy are cleaner and slightly stronger than those for the KdV hierarchy.

Let I be an interval and let

$$\|f\|_{L^2_u(I\times\mathbb{R})} = \sup\{\|f\|_{L^2(J\times K)} : J \subset I, |J| = |K| = 1\}.$$

Definition 2.18. *We say that w is an element of the Kato smoothing space X_N if*

$$w \in L^\infty L^2, \quad w^{(N)} \in L^2_u$$

and for all $t_0 \in \mathbb{R}$

$$\lim_{x_0 \to \pm\infty} \|w^{(N)}\|_{L^2((t_0,t_0+1)\times(x_0-1,x_0+1))} = 0. \tag{2.49}$$

Theorem 2.19. A. Regularity of weak solutions in X_N. *Suppose that $w \in X_N$ is a weak solution to the Nth τ Gardner equation. Then $w \in C([0,\infty); L^2(\mathbb{R}))$. We define the initial trace of w by $w_0 := w(0) \in L^2(\mathbb{R})$. Then $\|w(t)\|_{L^2} = \|w_0\|_{L^2}$ and the Kato smoothing estimate*

$$\begin{aligned}
&\sup_t \|(1-\tanh(x-\kappa\tau^{2N}t)^{1/2})w(t)\|_{L^2} + \|\operatorname{sech}(x-\kappa\tau^{2N}t)w^{(N)}\|_{L^2(\mathbb{R}\times\mathbb{R})} \\
&\qquad \le c(\|w_0\|_{L^2})\|(1-\tanh(x))^{1/2}u_0\|_{L^2}
\end{aligned} \tag{2.50}$$

holds for all $\kappa \ge \kappa_0$ where κ_0 depends only on N and $\tau^{-1/2}\|w_0\|_{L^2}$.

B. Existence of weak solutions in X_N. *Given $w_0 \in L^2$ there exists a weak solution to the Nth Gardner equation in X_N with $w(0) = w_0$.*

C. Uniqueness of weak solutions in X_N *Weak solutions in X_N are unique. The map*

$$L^2 \ni w_0 \to w \in C(I; L^2) \cap X_N(I)$$

to the weak solution of the Nth flow is continuous for every N and every bounded interval I.
D. Commuting flows. *All Gardner flows and the flows of $\mathcal{T}_{-1}^{\text{Gardner}}(\tau_1, \cdot, \tau)$ commute.*

Proof. It is not obvious that this regularity suffices to define weak solutions. This point is elaborated in Section 5. The proof of Theorem 2.19 implements a variant of the second commuting vector field method of Bringmann, Killip and Visan [6]. Part A on the regularity of weak solutions in X_N and the Kato smoothing estimate will be proven in Subsection 6.2.

It relies on the energy-flux identity of Lemma 3.11 which holds in a distributional sense for weak solutions in X_N(Lemma 6.1),

$$\partial_t w^2 = \partial_x \operatorname{Fl}_N(x)$$

where Fl_N has a very similar structure to the energy density. Here we use typical PDE arguments. It will be convenient to split the Hamiltonian into a quadratic and a higher order part, corresponding to splitting the Nth Gardner equation into a linear and a nonlinear part,

$$H_N^{\text{Gardner}} = \frac{1}{2}\|w^{(N)}\|_{L^2}^2 + H_N^{NL}, \tag{2.51}$$

$$w_t = (-1)^N \partial_x w^{(2N)} + \partial_x \frac{\delta}{\delta w} H_N^{NL} \tag{2.52}$$

Similarly

$$\operatorname{Fl}_N = \operatorname{Fl}_N^L + \operatorname{Fl}_N^{NL} \tag{2.53}$$

with

$$\operatorname{Fl}_N^L = (2N+1)|w^{(N)}|^2 + \sum_{j=1}^N f_{N,j} \partial^{2j} |w^{(N-j)}|^2$$

for some unimportant combinatorical constants $f_{N,j}$. The structure of Fl_N^{NL} is given in Lemma 3.11.

Proposition 2.20. *Let $w \in X_N$ be a weak solution to the Nth Gardner equation. Then $w \in C(\mathbb{R}; L^2)$, $\|w(t)\|_{L^2} = \|w_0\|_{L^2}$ and (we omit the argument $(x - \kappa\tau^{2N} t)/R$ of* sech*)*

$$\begin{aligned}
&\int (1 + \tanh\big((x - \kappa\tau^{2N} t)/R\big) w^2 dx\Big|_{t=0}^{t=T} \\
&\quad + \frac{1}{R}\int_0^T \int \operatorname{sech}^2 \Big[(2N+1)(w^{(N)})^2 + \sum_{j=1}^N f_{N,j} |w^{(N-j)}|^2 \cosh^2 \partial^{2j} \operatorname{sech}^2 + \kappa\tau^{2N} w^2\Big] dxdt \\
&\qquad = \frac{1}{R}\int_0^T \int \operatorname{sech}^2 \operatorname{Fl}_N^{NL} \, dxdt.
\end{aligned} \tag{2.54}$$

The second line is the linear Kato smoothing term. It is equivalent to

$$\frac{1}{R}\int_0^T \|w\|_{H_\tau^N}^2 + \kappa\tau^{2N}\|w\|_{L^2}^2 dt$$

if $\kappa \geq 1$ and R is large, independent of τ, since the middle terms carry a factor R^{-2j} from the differentiation. We bound the nonlinear term by

$$\begin{aligned}\left| \int \operatorname{sech}^2(x/R)\, \mathrm{Fl}_N^{NL}\, dx \right| &\leq c(\tau, \|w\|_{L^2}) \Big(\tau^N \|w\|_{L^2} \Big)^{\frac{2}{2N-1}} \|w\|_{H^N_\tau}^{2\frac{2N-2}{2N-1}} \\ &\leq \varepsilon \|w\|_{H^N_\tau}^2 + c(\varepsilon)\tau^{2N}\|w\|_{L^2}^2.\end{aligned}$$

We choose ε small and then κ large so that we can subsume the third line under the second line. Integration with respect to time implies the Kato smoothing estimate (2.50) for weak solutions.

The next step is precompactness.

Proposition 2.21. *Let $Q \subset L^2$ be a precompact set of initial data and let Q_w be a set of weak solutions with initial data in Q. For all compact intervals the set*

$$\{w(t) : w \in Q_w \text{ and } t \in I\} \subset L^2$$

is precompact.

This is contained in Theorem 6.3. It relies on

(1) Kato smoothing (2.50) and its translated versions which gives tightness to the right,
(2) the modified Miura map and the estimates of Proposition 4.8 applied twice to obtain equicontinuity of the orbit and high frequency Kato smoothing,
(3) a backward Kato smoothing with 'bad' terms controlled by the high frequency estimates of the previous step.

This is quite intricate and the object of Subsection 6.2, Theorem 6.3.

It is an easy consequence that the set of weak solutions in X_N is closed in the following sense: Let w^n be a sequence of weak solutions which converges in $C(I, L^2)$ for every bounded interval I. Estimate (2.50) provides uniform bounds and by precompactness and the high-frequency Kato smoothing estimates there exists a subsequence so that $\partial^N w^{n_j}$ converges in L^2 for every compact subset against a weak solution in X_N.

We prove more than that: The uniform convergence (and hence uniformy continuity) of the flow $(2\tau)^2 \mathcal{T}_{N-1}^{\text{Gardner}}$ to weak solutions to the Nth Gardner flow on precompact sets of initial data. A Kato smoothing estimate for weak solutions $w \in X_N$ to the difference equation

$$w_t = -\partial \frac{\delta}{\delta w} \mathcal{T}_N^{\text{Gardner}}(i\tau, w, \tau_0)$$

plays a central role. Observe that we return to denoting the parameter for the Gardner hierarchy by τ_0.

Proposition 2.22. *Suppose that $w \in L^\infty(\mathbb{R}; L^2)$ with $w^{(N)} \in L^2_{loc}(\mathbb{R}^2)$ is a weak solution to*

$$w_t = -\partial \frac{\delta}{\delta w} \mathcal{T}_N^{\text{Gardner}}(i\tau, w, \tau_0). \tag{2.55}$$

Then $w : \mathbb{R} \to L^2$ is weakly continuous and there exists κ depending only on N so that

$$\|\operatorname{sech}(x - \kappa\tau_0^{2N} t) w^{(N+1)}\|_{L^2(\mathbb{R}; H^{-1}_{2\tau})} + \|\operatorname{sech}(x - \kappa\tau_0^{2N} t) w\|_{L^2(\mathbb{R}^2)} \leq c\|w(0)\|_{L^2} \tag{2.56}$$

and

$$\|\operatorname{sech}(x)(w(t) - w(0))\|_{H^{-N-3}} \leq c\tau^{-\frac{3}{N+1}}(|t|+1)\|w(0)\|_{L^2}.$$

The proposition relies on natural but fairly sharp estimates.

Lemma 2.23. *The following estimates hold for* $R \geq 1$

(2.57)
$$\Big\| \operatorname{sech}^2(x/R) \frac{\delta}{\delta w} \mathcal{T}_N^{\text{Gardner}} \Big\|_{H^{-N-3}} \leq c\tau^{-\frac{3}{2(N+1)}}(1+\|w\|_{L^2})(\|\operatorname{sech}(x/R)w^{(N+1)}\|^2_{H^{-1}_{2\tau}} + \|\operatorname{sech}(x/R)w\|^2_{L^2})$$

and

(2.58)
$$\Big| \int \tanh(x/R)w(x)\partial \frac{\delta}{\delta w}\mathcal{T}_N^{NL}(i\tau)\Big)\, dx \Big| \leq c\tau^{-\frac{1}{(N+1)}}(1+\|w\|^{2N+2}_{L^2})\Big(\|\operatorname{sech}(x/R)w^{(N+1)}\|^2_{H^{-1}_{2\tau}} + \|\operatorname{sech}(x/R)w\|^2_{L^2}\Big).$$

The estimates (2.57) and (2.58) are proven after Proposition 7.13. Estimate (2.58) bounds the nonlinear part of the Kato smoothing estimate for the Hamiltonian vector fields. The linear part leads to an important gain:

$$\int \tanh(x/R)w\partial_x \frac{\delta}{\delta w}\frac{1}{2}\|w^{(N+1)}\|^2_{H^{-1}_{2\tau}}dx + \frac{2N}{R}\|\operatorname{sech}(x/R)w^{(N+1)}\|^2_{H^{-1}_{2\tau}} \leq \frac{c}{R}\|\operatorname{sech}(x/R)w\|^2_{L^2}$$

hence, if κ is chosen sufficiently large so that the nonlinear terms can be controlled,

$$\begin{aligned} \frac{d}{dt}\int \tanh((x-\kappa\tau_0^{2N}t)/R)w^2dx + \frac{N}{R}\|\operatorname{sech}((x-\kappa\tau_0^{2N})/R)w^{(N+1)}\|^2_{H^{-1}_{2\tau}} \\ + \frac{\kappa\tau_0^{2N}}{R}\|\operatorname{sech}((x-\kappa\tau_0^{2N})/R)w\|^2_{L^2} \leq 0. \end{aligned} \tag{2.59}$$

In particular, since $\|w(t)\|_{L^2} = \|w_0\|_{L^2}$, if $R \geq 10$

$$\int_0^1 \|\operatorname{sech}(x/R)v^{(N+1)}(t)\|^2_{H^{-1}_\tau}dt \leq 2\|w_0\|^2_{L^2}. \tag{2.60}$$

This estimate is uniform for w_0 bounded. Moreover by (2.57)

$$\|\operatorname{sech}^2(x)(w(t)-w_0)\|_{H^{-N-3}} \leq c(t+1)\tau^{-\frac{3}{2(N+1)}}\|w_0\|_{L^2} \tag{2.61}$$

and

$$\operatorname{sech}^2(x)w(\tau,t) \to \operatorname{sech}^2(x)w_0 \text{ in } H^{-N-3} \text{ as } \tau \to \infty$$

uniformly on bounded sets. Since $\|w(\tau,t)\|_{L^2} = \|w_0\|_{L^2}$ and $\{w(t) : t \in [-T,T]\}$ is precompact in L^2 we obtain convergence in L^2.

We make this more rigoros by induction on N and we formulate the induction hypothesis

Claim 2.1. *Let* $N \geq 1$. *The approximate Hamiltonian* $(2\tau)^2\mathcal{T}_{N-1}^{\text{Gardner}}$ *defines an evolution which converges to the* N*th Gardner evolution uniformly on compact time intervals for precompact sets of initial data. We call the limit the evolution of the* N*th Gardner equation and denote it by* $\exp(t\partial\frac{\delta}{\delta w}H_N^{\text{Gardner}})$. *All these flows commute and preserve higher regularity and the Hamiltonians* $\mathcal{T}_{-1}^{\text{Gardner}}(i\tau,.,\tau_0)$ *and* H_j^{Gardner}.

We assume the induction hypothesis. By the hypothesis the Hamiltonian flow of $\mathcal{T}_{N-1}^{\text{Gardner}}$ is well defined. We consider a subset $Q \subset H^N$ which is precompact in L^2 and claim that

$$\exp\Big(t(2\tau)^2\partial\frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}\Big)w_0$$

converges uniformly in $L^\infty(I, L^2)$ as $\tau \to \infty$ for $I \subset \mathbb{R}$ bounded and $w_0 \in Q$. By Theorem 2.14 there is a unique weak solution in $C(\mathbb{R}, H^N)$ to the N Gardner equation and also to the difference Hamiltonian. Let $w_0 \in L^2$, $w_0^n \in H^N$ so that $w_0^n \to w_0$ in L^2. Let w^n resp. $w^n(\tau)$ be the corresponding solutions to the Nth Gardner equation resp. the Nth approximate equation. By the induction assumption the limit

$$\lim_{n\to\infty} w^n(\tau, t) = w(\tau)$$

exists in L^2 uniformly on compact time intervals for initial data in Q. By precompactness there exists a subsequence w^n (which we do not relabel) which converges to a weak solution w in L^2. By Theorem 2.14

$$v^n(\tau) = \exp\Big(-t(2\tau)^2\partial\frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}\Big)w^n(t)$$

satisfies

$$v_t^n(\tau) = \partial\frac{\delta}{\delta w}\mathcal{T}_N^{\text{Gardner}}(i\tau)(v^n(\tau)), \qquad v^n(0,\tau) = w_0 \in Q$$

By (2.59) the solutions satisfy

$$\sup\{\|(v^n(\tau))^{(N)}\|_{L^2(I\times K)} : |I|, |K| \le 1\} \le c(\tau)\|w_0\|_{L^2}$$

with a constant depending on τ. By continuity and the induction assumption

$$\lim_{n\to\infty} v^n(\tau, t) = \exp\Big(-t(2\tau)^2\partial\frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}\Big)w(t) =: v(t, \tau)$$

and by Proposition 2.22

$$\|\operatorname{sech}(v(t,\tau) - w_0)\|_{L^2(\mathbb{R})} \le c(1+t)\tau^{-\frac{3}{N+1}}\|w_0\|_{L^2}.$$

By Proposition 2.20 $\|w(t)\|_{L^2} = \|w_0\|_{L^2}$ and by the induction hypthesis $\|v(t)\|_{L^2} = \|w(t)\|_{L^2} = \|w_0\|_{L^2}$. By this observation we can upgrade the convergence to

$$v(t,\tau) \to w_0$$

in L^2 uniformly on compact time intervals for initial data in a precompact set. Now

$$\begin{aligned}
&\exp\Big(t(2\tau)^2\partial\frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}(i\tau)\Big)w_0^n - \exp\Big(t\partial\frac{\delta}{\delta w}H_N^{\text{Gardner}}\Big)w_0^n \\
&\quad= \exp\Big(-t\partial\frac{\delta}{\delta w}\mathcal{T}_N^{\text{Gardner}}\Big)w^n(t) \\
&\qquad - \exp\Big(-t\partial\frac{\delta}{\delta w}\mathcal{T}_N^{\text{Gardner}}\Big)\exp\Big(-t(2\tau)^2\partial\frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}\Big)w_0 \\
&\quad\to 0 \qquad \text{in } L^2
\end{aligned}$$

uniformly on compact time intervals at $\tau \to \infty$ for the sequence v^n and we obtain convergence of the limit to the weak solution from above,

$$\exp\Big(-t(2\tau)^2\partial\frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}\Big)w_0 \to w.$$

We have proven Claim 2.1 for all N and hence part A and B of Theorem 2.19 and part C for the solutions we have constructed.

We claim that

$$L^2(\mathbb{R}) \ni w_0 \to \operatorname{sech}(x-\kappa\tau^{2N})\partial^N \exp\Big(t\partial\frac{\delta}{\delta w}H^N\Big)w_0 \in L^2(\mathbb{R}^2) \tag{2.62}$$

is continuous. Let $w_0^n \to w_0$ in L^2 and let w^n be the corresponding solutions. Then $w_n(t) \to w(t)$ in L^2 uniformly on compact time intervals. By Proposition 2.20 the second and the third line of equality (2.54) are continuous with respect to the initial data. Let $w_0^n \to w_0$ in L^2 and let w^n resp. w be the corresponding solutions, uniformly bounded in X^N, and converging in L^2 uniformly on compact time intervals. Then

$$\begin{aligned}\Big|\int_0^T\int \operatorname{sech}^2 \mathrm{Fl}_N^{NL}(w^n)dxdt - \int_0^T\int \operatorname{sech}^2 \mathrm{Fl}_N^{NL}(w)dxdt\Big|\\ \le c(\varepsilon)\|w^n-w\|_{L^\infty L^2} + \varepsilon\|\operatorname{sech}(\partial^N(w^n-w))\|_{L^2}^2\end{aligned}$$

and hence the quadratic term in (2.54) converges. It is equivalent to a norm and hence a norm of a Hilbert space. Weak convergence and norm convergence imply convergence, and hence convergence of u^n in X^N.

2.7. **Uniqueness of weak solutions.** We turn to proving uniqueness of weak solutions in X_N. Let w be a weak solution and define

$$v(t) = \exp\Big(-t(2\tau)^2\partial\frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\mathrm{Gardner}}(i\tau,.,\tau_0)\Big)w(t). \tag{2.63}$$

Proposition 2.24. *$v \in X_N$ and it is a weak solution to*

$$\partial_t v = \partial\frac{\delta}{\delta w}\mathcal{T}_N^{\mathrm{Gardner}}(i\tau,.,\tau_0).$$

Proof. We approach the claim by a number of small steps. Let

$$v = \exp\Big(-t\partial\frac{\delta}{\delta w}\mathcal{T}_{-1}^{\mathrm{Gardner}}(i\tau,.,\tau_0)\Big)w(t). \tag{2.64}$$

To simplify the notation we denote the Hamiltonian flow of $\mathcal{T}_{-1}^{\mathrm{Gardner}}(i\tau,.,\tau_0)$ by $\Phi_{-1,\tau}(t,.)$.

Lemma 2.25. *Let $w \in X^N$ be a weak solution and $v(t) = \Phi_{-1,\tau}(ct, w(t))$. Then*

$$\operatorname{sech} v \in L^2(I, H^N)$$

for bounded intervals I.

Proof. By the conservation of the L^2 norm $\|v(t)\|_{L^2} = \|w_0\|_{L^2}$. By the definition of a weak solution $w \in X^N(I)$ hence $\operatorname{sech} w \in L^2(I, H^N)$. The Kato smoothing estimates (2.50) provide even bounds, which are not important here. The vector field $\frac{\delta}{\delta w}T_{-1}^{\mathrm{Gardner}}(i\tau,.,\tau_0)$ is locally Lipschitz continuous on

$$Y = \{w \in L^2 : \operatorname{sech} w \in H^N\}$$

by Proposition 2.10. Suppose that $v_0 \in L^2$ and $\operatorname{sech} v_0 \in H^N$. Let $v(t)$ be the solution to the Hamiltonian equation of $\mathcal{T}_{-1}^{\mathrm{Gardner}}$. By estimate (2.35)

$$\|\operatorname{sech} v(t)\|_{H^N} \le \|\operatorname{sech} v(0)\|_{H^N} + c\int_0^t \|\operatorname{sech} v(s)\|_{H^N}ds$$

and by Grönwall we obtain

$$\|\operatorname{sech} v(t)\|_{H^N} \le ce^{ct}\|\operatorname{sech} v_0\|_{H^N}.$$

We obtain similar estimates for translates. Thus $v(t) = \Phi_{-1,\tau}(t, w(t))$ satisfies

$$\|v(t)\|_{L^2} = \|w_0\|_{L^2}, \quad \|\operatorname{sech} v\|_{L^2((0,T),H^N)} \le c(T)\|\operatorname{sech} w\|_{L^2((0,T),H^N)}.$$

□

Lemma 2.26. *We have*

$$v_t = \partial\Big(\frac{\delta}{\delta w} c\mathcal{T}_{-1}^{\text{Gardner}}(i\tau, w(t), \tau) + D\Phi_{-1,\tau}(t, w(t))\frac{\delta}{\delta w} H_N^{\text{Gardner}}(w(t)\Big).$$

Here $D\Phi$ denotes the Fréchet derivative. It is part of the claim that all terms are well defined in a sense described in the proof.

This calculation is a consequence of the chain rule under more regularity. However it can be done at the level of regularity at hand.

Proof. We decompose

$$\begin{aligned} v(t+h) - v(t) \, &= \Big[\Phi_{-1,\tau}(-(t+h), w(t+h) - \Phi_{-1,\tau}(-t, w(t+h)\Big] \\ &\quad + \Big[\Phi_{-1,\tau}(-t, w(t+h)) - \Phi_{-1,\tau}(-t, w(t))\Big] \end{aligned}$$

We divide by h and take the limit $h \to 0$. The first term on the right hand side converges to $-ct\partial \frac{\delta}{\delta w}\mathcal{T}_{-1}^{\text{Gardner}}(v)$ in $L^\infty L^2$. By the definition of the weak solution

$$\begin{aligned} w(t+h) - w(t) \, &= \int_t^{t+h} \partial \frac{\delta}{\delta w} H_N^{\text{Gardner}}(w) \\ &= \int_t^{t+h} (-1)^N w^{((2N+1)} + \sum_{n=3}^{2N+2} \sum_{\alpha=1}^{N+1-n/2} \partial^\alpha f_{n,\alpha}(s) ds \end{aligned}$$

where (we put the dependence on $\|w\|_{L^2}$ and τ_0 into the constants) by Lemma 5.5, (2.41)

$$\|\operatorname{sech}^2 f_n(t)\|_{L^2} \le c\|\operatorname{sech} w(t)\|_{H^N}^{2-\frac{\frac{n}{2}-1-\alpha}{N}}$$

We complement this estimate by Lipschitz bounds for $\Phi_{-1,\tau}$. The flow map $\Phi_{-1,\tau}$ is differentiable at least for more regular initial data. We denote its Fréchet derivative with respect to the initial data as above by $D\Phi_{-1,\tau}$. It is the evolution to the differential equation (see Proposition 2.10, (2.33))

$$\phi_t = -\partial A(t, w)\phi$$

and satisfies

$$\|\operatorname{sech}^2 D\Phi_{-1,\tau}(t, w)w_0\|_{L^2} \le ce^{c|t|}\|\operatorname{sech}^2 w_0\|_{L^2}. \tag{2.65}$$

Suppose that $\operatorname{sech} w \in H^N(\mathbb{R})$. We claim

$$\Big|\int \eta D\Phi_{-1,\tau}(t, w)w_0 dx\Big| \le c\Big(1 + \|\operatorname{sech} w\|_{H^N}\Big)\|\cosh^2 \eta\|_{H^{N+1}}\|\operatorname{sech} w_0\|_{H^{-N-1}}. \tag{2.66}$$

This follows by duality from

$$\|\cosh D\Phi^*_{-1,\tau}(t, w)w_0\|_{H^{N+1}} \le ce^{c|t|}\Big(\|\cosh w_0\|_{H^{N+1}} + \|\operatorname{sech} w\|_{H^N}\|\cosh^2 w_0\|_{L^2}\Big).$$

The adjoint equation is (reversing the time direction as well)

$$\tilde{\phi}_t = -A(w(T-t))\partial_x\tilde{\phi}.$$

Let $\psi = \tilde\phi_x$. It satisfies

$$\psi_t = -\partial A \psi$$

and again

$$\begin{aligned} \|\cosh \psi(t)\|_{H^N} &\le \|\cosh\psi(0)\|_{H^N} + \int_0^t \|\cosh A\psi\|_{H^{N+1}} ds \\ &\le \|\cosh \psi(0)\|_{H^N} + c\int_0^t \|\cosh \psi\|_{H^N} + \|\cosh^2\psi(s)\|_{L^2} \|\operatorname{sech}(s) w(t)\|_{H^N} ds. \end{aligned}$$

Again by Grönwall

$$\|\cosh\psi(t)\|_{H^N} \le c\Big(\|\cosh\psi(0)\|_{H^N} + \|\operatorname{sech} w_0\|_{H^N} \|\cosh^2\psi(0)\|_{L^2}\Big).$$

We interpolate (2.65) and (2.66) and obtain for $0 \le m \le N$

$$\|\operatorname{sech} D\Phi_{-1,\tau}(t,w) w_0\|_{H^{-m-1}} \le ce^{c|t|}(1+\|\operatorname{sech} w\|_{H^N})^{\frac{m}{M}} \|\operatorname{sech} D\Phi_{-1,\tau}(t,w)$$

Together we bound (with M the Hardy-Littlewood maximal function with respect to time)

$$\frac{1}{h}\|\operatorname{sech}(v(t+h)-v(t))\|_{H^{-N}} \le c\Big(M\|\operatorname{sech} v\|_{H^N}^{2-\frac1N} + \Big(1+\|\operatorname{sech} w\|_{H^N}^{1-\frac1N}\Big) M \|\operatorname{sech} v\|_{H^N}.$$

The right hand side is an integrable majorant. The same estimates imply (together with higher differentiability of $D\Phi_{-1,\tau}$ for more regular data) that at Lebesgue points

$$\frac1h [\Phi_{-1,\tau}(-t, w(t+h) - \Phi_{-1,\tau}(-t,w(t)) \to D\Phi_{-1,\tau}(-t) w_t(t)$$

and hence

$$v_t = \partial \frac{\delta}{\delta w} \mathcal{T}_{-1}^{\mathrm{Gardner}}(-t,v) + D\Phi_{-1,\tau}(-t,w)\partial \frac{\delta}{\delta w} H_N^{\mathrm{Gardner}}.$$

$\square$

Lemma 2.27. *Let $w \in C(\mathbb{R}; L^2)$ with $w^{(N)} \in L^2_u$ be a weak solution to the Nth Gardner equation. Then $v(t) = \Phi_{-1,\tau_0}(ct, w(t))$ satisfies*

$$v_t = \partial \frac{\delta}{\delta w}\Big(H_N^{\mathrm{Gardner}} + c\mathcal{T}_{-1}(i\tau, ., \tau_0)\Big)(v) \tag{2.67}$$

Proof. Here we rely on integrability. In view of Lemma 2.26 we have to prove

$$D\Phi_{-1,\tau}(t,w) \partial \frac{\delta}{\delta w} H_N^{\mathrm{Gardner}}(w) = \partial \frac{\delta}{\delta w} H_N^{\mathrm{Gardner}}(i\tau, \Phi_{-1,\tau}(t,w)). \tag{2.68}$$

We begin with the case in which one has two smooth flows, which is the Frobenius theorem. Let X and Y be two C^2 vector fields and $\phi^X(t,\cdot)$ resp $\phi^Y(t,\cdot)$ the corresponding flows. Then the commutator of the vector fields considered as derivations satisfies $[X,Y]=0$ if and only if the $\phi^X(s,\phi^Y(t,\cdot)) = \phi^Y(t,\phi^X(s,\cdot))$. The implication $\Longleftarrow$ requires differentiation. Assume $[X,Y]=0$. We differentiate

$$\begin{aligned} &\frac{d}{ds}(\phi^X(s,\phi^Y(t,\cdot)) - \phi^Y(t,\phi^X(s,\cdot))) \\ &\qquad = X(\phi^X(s,\phi^Y(t,\cdot)) - (D\phi^Y)(t,\phi^X(s,\cdot))X(\phi^X(s,\cdot)) \end{aligned}$$

We claim

$$(D\phi^Y)(t,\cdot)X(\cdot) = X(\phi^Y(t,\cdot)).$$

Assume the claim. Then the right hand side is

$$X(\phi^X(s,\phi^Y(t,\cdot)) - X(\phi^Y(t,\phi^X(s,\cdot))$$

and by Gronwall's inequality, the identity for $s=0$, and local Lipschitz continuity of X, we find $\phi^X(s,\phi^Y(t,\cdot)) = \phi^Y(t,\phi^X(s,\cdot))$. To prove the claim we differentiate with respect to t.

$$\begin{aligned}
&\frac{d}{dt}((D\phi^Y)(t,\cdot)X(\cdot) - X(\phi^Y(t,\cdot))) \\
&\qquad = D(Y(\phi^Y(t,\cdot)))X(\cdot) - DX(\phi^Y(t,\cdot))Y(\phi^Y(t,\cdot)) \\
&\qquad = DY(\phi^Y(t,\cdot))D\phi^Y(t,\cdot)X(\cdot) - DX(\phi^Y(t,\cdot))Y(\phi^Y(t,\cdot)) \\
&\qquad = ((DY)X - (DX)Y)(\phi^Y(t,\cdot)) + DY(\phi^Y(t,\cdot))(D\phi^Y(t,\cdot)X(\cdot) - X(\phi^Y(t,\cdot))) \\
&\qquad = [X,Y](\phi^Y(t,\cdot)) + DY(\phi^Y(t,\cdot))(D\phi^Y(t,\cdot)X(\cdot) - X(\phi^Y(t,\cdot)))
\end{aligned}$$

and the claim follows again by $[X,Y]=0$, Grönwall's inequality, the identity for $s=0$ and local Lipschitz continuity of DY.

We specialize to Hamiltonian vector fields. The Hamiltonians H_1, H_2 Poisson commute if and only if

$$\exp(sJ\nabla H_1)\exp(tJ\nabla H_2) = \exp(tJ\nabla H_2)\exp(sJ\nabla H_1)$$

which holds if and only if H_2 is conserved on orbits of $\exp(tJ\nabla H_1)$. To see this assume that H_1 and H_2 Poisson commute and calculate

$$\frac{d}{dt}H_2(\exp(tJ\nabla H_1) = \{H_2, H_1\} = 0.$$

Moreover, a calculation shows

$$-[J\nabla H_1, J\nabla H_2] = J\nabla\{H_1, H_2\}$$

and the Hamiltonian vector fields commute if the Hamiltonians Poisson commute. Indeed, let w, ϕ be test functions. Then, with $X = J\nabla H_1, Y = J\nabla H_2$,

$$\begin{aligned}
&\langle DX(w)Y(w) - DY(w)X(w), \phi\rangle \\
&\qquad = \frac{d}{ds}\Big|_0 \langle J\nabla H_1(w+sY(w)) - J\nabla H_2(w+sX(w)), \phi\rangle \\
&\qquad = -\frac{d}{ds}\Big|_0 \frac{d}{dt}\Big|_0 H_1(w+sY(w)+tJ\phi) - H_2(w+sX(w)+tJ\phi) \\
&\qquad = -\frac{d}{dt}\Big|_0 \langle \nabla H_1(w+tJ\phi), J\nabla H_2(w)\rangle - \langle \nabla H_2(w+tJ\phi), J\nabla H_1(w)\rangle \\
&\qquad = -\frac{d}{dt}\Big|_0 \langle \nabla H_1(w+tJ\phi), J\nabla H_2(w+tJ\phi)\rangle \\
&\qquad = \langle J\nabla\{H_1, H_2\}(w), \phi\rangle.
\end{aligned}$$

We turn to (2.68), which we will prove for $H = c\mathcal{T}_{-1}^{\mathrm{Gardner}}$. For $\phi \in \mathcal{S}(\mathbb{R})$, we have

$$D\Phi_{-1,\tau}(ct,\phi)\partial\frac{\delta}{\delta w}H_N^{\mathrm{Gardner}}(\phi) = \partial\frac{\delta}{\delta w}H_N(\Phi_{-1,\tau}(ct,\phi) \tag{2.69}$$

from the considerations above for smooth flows. It remains to prove continuity of both sides as maps from $H^N \to H^{-N-1}$ so that the statement holds by approximation. The maps

$$\partial \frac{\delta}{\delta w} H_N^{\text{Gardner}} : H^N \to H^{-N-1}, \quad \text{and}$$
$$\Phi_{-1,\tau} : H^N \to H^N$$

are seen to be continuous, and we are left to show continuity of

$$D\Phi_{-1,\tau}(ct, .) : H^N \to \mathcal{L}(H^{-N-1}, H^{-N-1}).$$

This has been proven in Subsection 2.7. □

We turn to the proof of Proposition 2.24. Let $(\kappa_j)_{0\le j\le N}$ be the unique solution to the Vandermonde system

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2^{-1} & \dots & (N+1)^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{-N} & \dots & (N+1)^{-N} \end{pmatrix} \kappa = \begin{pmatrix} (2i\tau)^{2N} \\ (2i\tau)^{2(N-1)} \\ \vdots \\ (2i\tau)^2 \\ 0 \end{pmatrix}$$

so that with

$$\mathcal{T}_{N-1}^{app}(\sigma, \tau, ., \tau_0) = (2i\tau)^{2N} \mathcal{T}_{-1}^{\text{Gardner}}(i\tau, ., \tau_0) - (2i\tau)^2 \sum_{j=0}^{N} \kappa_j \mathcal{T}_{-1}^{\text{Gardner}}(ij\sigma, ., \tau_0)\Big)$$

$$\lim_{\sigma\to\infty} (2\sigma)^2 \Big(\mathcal{T}_{N-1}^{\text{Gardner}}(i\tau) - \mathcal{T}_{N-1}^{app}(\tau, i\sigma)\Big) = 0$$

for $w \in H^N$ uniformly on bounded and equicontinuous subsets in H^{N-1}. Let $w \in X_N$ be a weak solution and

$$v(t, \tau, \sigma) = \exp\Big(-t\partial \frac{\delta}{\delta w} \mathcal{T}_{N-1}^{app}(i\sigma)\Big) w(t).$$

By the previous argument (which applies in the very same fashion in this situation since the flows $\Phi_{-1,\tau}$ for different τ commute) $v \in X_N$ and it satisfies

$$\partial_t u = \partial \frac{\delta}{\delta w} \Big(H_N^{\text{Gardner}} - \mathcal{T}_{N-1}^{app}\Big)(u). \tag{2.70}$$

By the previous part

$$\lim_{\sigma\to\infty} u(t, \tau, \sigma) = v(t)$$

uniformly on compact time intervals in L^2.

We verify that $v \in X_N$. The linear part of the equation is

$$\dot{u}_t = (-1)^N \partial^{(2N+1)} \dot{u} - \sum_{j=0}^{N} \kappa_j \partial (D^2 + (2j\sigma)^2)^{-1}.$$

and $|\kappa_j| \le c\tau^{2N}$. Together with the nonlinear estimate

$$\int \operatorname{sech}(x) u(\tau, \sigma) \partial \frac{\delta}{\delta w} (H_N^{\text{Gardner}} - \mathcal{T}_{N-1}^{app}(i\tau))(u) dx + 2N \int \operatorname{sech}^2(x) |u^{(N)}|^2$$
$$\le c\tau^{2N} \| \operatorname{sech}(x) w\|_{L^2}^2$$

uniform in σ. Now we can take that limit $\sigma \to \infty$ and obtain the Proposition 2.24. □

As a consequence $v(t) \to w_0$ by Proposition 2.22 uniformly on compact time intervals for w_0 in a precompact set. We decompose

$$\exp\left(-t\partial\frac{\delta}{\delta w}\tau^2\mathcal{T}_{N-1}^{\text{Gardner}}\right)w_0 - w(t) = \exp\left(-t\partial\frac{\delta}{\delta w}\tau^2\mathcal{T}_{N-1}^{\text{Gardner}}\right)w_0 - \exp\left(-t\partial\frac{\delta}{\delta w}\mathcal{T}_{N-1}^{\text{Gardner}}\right)v(t)$$

and the claim follows from uniform convergence of $\exp\left(t\partial\frac{\delta}{\delta w}\tau^2\mathcal{T}_N^{\text{Gardner}}\right)w_0$ on precompact sets which in turn follows from pointwise convergence. □

2.8. **Outline of the paper.** In Section 3 we develop the structure and the formulas for KdV and the Gardner hierarchy. The modified Miura map and estimates for it are the central object in Section 4. Section 5 uses the modified Miura map to verify equivalence of weak solutions to the Nth KdV equation, the Nth Gardner equation and the Nth good variable equation. The object of Section 6 are properties of weak solutions, Kato smoothing and precompactness of orbits. Section 7 gives the asymptotic series a precise meaning. It provides varies multilinear estimates in particular for the difference flow.

3. The KdV and the Gardner hierarchy

The KdV hierarchy is a long studied object and we refer to Babelon et al [1], Faddeev and Takthajan [17], Novikov et al [49], Dickej [15] and Gesztesy and Holden [21]. The KdV Hamiltonians and related quantities are coefficients of asymptotic series, which are typically treated as formal series. We give the asymptotic series a precise meaning. In contrast to the expositions above we insist on working in an H^N setting for the potentials and on sharp error estimates for the difference of partials sums and quantities expanded in asymptotic series. We deduce consequences on the coefficients from properties of the logarithm of the transmission coefficient and related quantities. This allows to use rigorous arguments for potentials in Sobolev spaces, and is central in the approximation of higher flows. The various connections between the KdV hierarchy and the Gardner hierarchy may be an original contribution, including a Lenard recursion without antiderivatives. In the Subsection 3.4 on Schatten class operators and determinants, we also used some elegant arguments from Harrop-Griffiths, Killip and Visan [25].

In what follows we will often omit arguments of functions when they are clear from the context.

3.1. **The KdV hierarchy.** We consider the Korteweg-de Vries equation (KdV) (1.1),

$$u_t = -u_{xxx} + 6uu_x$$

and its hierarchy. The KdV equation (1.1) has the form of a Hamiltonian equation

$$u_t = \partial\frac{\delta H}{\delta u},$$

where $H^{\text{KdV}} = \frac{1}{2}\int u_x^2 + 2u^3 dx$ and $\frac{\delta H}{\delta u}$ is the functional derivative of H defined to be the unique function such that

$$\int \phi(x)\frac{\delta H}{\delta u}(x)\,dx = \frac{d}{dt}\bigg|_{t=0} H(u+t\phi) \tag{3.1}$$

for all test functions $\phi \in \mathcal{S}(\mathbb{R})$. Given a functional of the form

$$H(u) = \int_{\mathbb{R}} h(u, u', u'', ...)dx,$$

one can check easily that

$$\frac{\delta H}{\delta u} = \sum_{i=0}^{\infty} (-\partial)^i \left(\frac{\partial h}{\partial u^{(i)}}(u, u', ...) \right).$$

Whenever the functional can be written as an integral over a differential polynomial the sum on the right-hand side is finite.

The Lax operator for KdV is the Schrödinger operator

$$L\phi = (-\partial_x^2 + u)\phi, \tag{3.2}$$

with potential $u \in H^{-1}$. Let Im $z > 0$ and consider the left and right Jost solutions ϕ_l, ϕ_r of

$$(L - z^2)\phi = -\partial_x^2 \phi + u\phi - z^2\phi = 0, \tag{3.3}$$

with the normalization at $\pm\infty$ [1]

$$\lim_{x \to -\infty} e^{izx + \frac{1}{2iz}\int_0^x u(y)dy} \phi_l(x) = 1, \tag{3.4}$$

in a local L^2 sense, and similarly

$$\lim_{x \to \infty} e^{-izx - \frac{1}{2iz}\int_0^x u(y)dy} \phi_r(x) = 1.$$

The integral term in the normalization is needed to obtain a limit for $u \in H^{-1}$ - it is needed even for $u \in H^N$ for any N. For $u \in H^{-1}$ we need an inessential correction to the normalization described below. One way to see its origin as follows: We write $u = v_x - 2izv$ with $\|v\|_{L^2} = \|e^{2i \operatorname{Re} zx} u\|_{H^{-1}_{2 \operatorname{Im} z}}$. Then

$$L - z^2 = (\partial - iz + v)(-\partial - iz + v) - v^2,$$

and (3.3) is equivalent to the system

$$\psi' = \begin{pmatrix} 0 & -1 \\ v^2 & 2iz - 2v \end{pmatrix} \psi \tag{3.5}$$

with

$$\psi_1 = e^{izx - \int_0^x v dy} \phi_l, \quad \psi_2 = e^{izx - \int_0^x v dy}(-\partial + iz + v)\phi_l$$

The system (3.5) can easily be solved by a Picard iteration from $-\infty$ starting with the constant function $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, see Lemma 4.4. We observe that formally $\int u dx = -2iz \int v dx$. The limit in (3.4) becomes a standard limit if we replace $-\frac{1}{2iz}\int_0^x u$ by $\int_0^x v$. We define the renormalized transmission coefficent $T_r(z)$ on the upper half plane to be the meromorphic function

$$T_r(z) = \frac{-2iz}{W(\phi_l, \phi_r)} = \frac{\lim\limits_{x \to -\infty} \exp\left(izx - \int_0^x v(y)dy\right)\phi_l(x)}{\lim\limits_{x \to \infty} \exp\left(izx - \int_0^x v(y)dy\right)\phi_l(x)}.$$

[1] This deviates slightly from the standard normalization. It has the advantage that it is correct even for $u \in H^N$, $N \geq -1$, in contrast to the standard normalization. It leads to the renormalized transmission coefficient below.

Here, $W(f,g) = f'g - fg'$ is the Wronskian and the enumerator is 1, which we kept to make the expression independent from the normalization. For Schwartz functions the transmission coefficient T is defined using the standard normalization of the Jost solutions without the integral in the exponent. Then

$$T_r = e^{-\frac{1}{2iz}\int_{\mathbb{R}} u dx} T. \tag{3.6}$$

Indeed, for functions $u \in L^1$ the standard left Jost function $\tilde{\phi}_l$ is related to the Jost function ϕ_l defined in (3.3) and (3.4) by $\phi_l = \tilde{\phi}_l \exp(\frac{1}{2iz}\int_{-\infty}^0 u\, dy)$.

None of the factors on the right hand side of (3.6) is defined unless $u \in L^1$ (or $W^{-1,1} \cap H^{-1}$ with a suitable definition of spaces), but the left hand side is defined for $u \in H^{-1}$. On the lower half plane we define $T_r(z) = T_r(-z)^{-1}$ which is the correct choice for (3.17) below. The inverse of the transmission coefficient $(T_r(z,u))^{-1}$ is defined in this way for $u \in H^{-1}$ meromorphic in $z \in \mathbb{C}\backslash\mathbb{R}$ and holomorphic in the upper half plane, with the zeros given by the square roots of the eigenvalues in the upper half plane.

A direct calculation (see the proof of Lemma 3.14 and [49]) shows that

$$\frac{\delta}{\delta u} T_r = \frac{T_r^2}{2iz}\phi_l\phi_r - \frac{T_r}{2iz}, \quad \text{and hence} \quad \frac{\delta \log T_r}{\delta u} = \frac{T_r}{2iz}\phi_l\phi_r - \frac{1}{2iz}, \tag{3.7}$$

respectively for the non-renormalized Transmission coefficient and Jost solutions,

$$\frac{\delta T}{\delta u} = \frac{T^2}{2iz}\tilde{\phi}_l\tilde{\phi}_r, \quad \text{and hence} \quad \frac{\delta \log T}{\delta u} = \frac{T}{2iz}\tilde{\phi}_l\tilde{\phi}_r. \tag{3.8}$$

We specialize (2.4) to $N = -1$

$$\mathcal{T}_{-1}^{\mathrm{KdV}}(z,u) = iz \log T_r(z).$$

An innocent calculation gives a formula which is hard to overestimate: the Lenard recursion.

Lemma 3.1. *Let $u \in H^{-1} \cap L^2_{loc}$. Then the functional derivative of $\mathcal{T}_{-1}^{\mathrm{KdV}}$ satisfies the ODE*

$$(-\partial^3 + 4u\partial + 2u_x)\frac{\delta \mathcal{T}_{-1}^{\mathrm{KdV}}}{\delta u} = 4z^2\partial \frac{\delta \mathcal{T}_{-1}^{\mathrm{KdV}}}{\delta u} - u_x. \tag{3.9}$$

In view of (3.8) and the regularity of the Jost solution $\phi_{l,r} \in H^2_{loc}$ the equation (3.9) can be understood in a distributional sense.

Proof. Consider ϕ_1, ϕ_2 solutions to $(-\partial^2 + u - z^2)\phi_i = 0$. A short calculation reveals

$$\partial^3(\phi_1\phi_2) = 2u_x\phi_1\phi_2 + 4(u - z^2)\partial(\phi_1\phi_2).$$

Hence (3.9) follows from (3.7). □

There are different formulations of the equations of the Korteweg-de Vries hierarchy: the Gardner hierarchy and the good variable hierarchy, which are equivalent under weak conditions. They correspond to taking different coordinates in a large part of the phase space. The different coordinates are based on relations between the Lax operator, resolvent and Jost solutions. We give connections between these variables in the next Lemma.

Lemma 3.2 (Definition of w and v)**.** *Let $u \in H^{-1}$ and let $\phi_l \in H^1_{loc}$ be the left Jost solution with the normalization* (3.4)*. We assume that $-\partial^2 + u + \tau_0^2$ is positive* *semi definite and either* $\operatorname{Re} z \neq 0$ *or* $z \notin i[0,\tau_0]$. *Then* ϕ_l *never vanishes and we define* $w \in L^2$ *(see Lemma 4.4) as*

$$w(x,z,u) = (\log(\phi_l(x,z,u)e^{izx}))'.$$

Then w *satisfies the Ricatti equation*

$$w' - 2izw + w^2 = u. \tag{3.10}$$

It allows to factorize the Lax operator

$$L - z^2 = -\partial^2 + u - z^2 = (\partial - iz + w)(-\partial - iz + w). \tag{3.11}$$

Moreover

$$\mathcal{T}_{-1}^{\mathrm{KdV}}(z,u) = -\frac{1}{2}\int_{\mathbb{R}} w^2(x,z,u)\,dx. \tag{3.12}$$

Let $G(z,x,y)$ *be the integral kernel of the resolvent and* $\beta(z,x) = G(z,x,x)^2$ *its value on the diagonal. We define*

$$v = -\frac{1}{2iz\beta} - 1. \tag{3.13}$$

Then

$$w = -\frac{1}{2}\partial_x \log(v+1) - izv. \tag{3.14}$$

Proof. In Lemma 4.4 we will prove that the map $L^2 \ni w \to u \in H^{-1}$ is a diffeomorphism with the natural restrictions on the spectrum. From

$$\phi_l' = (-iz + w)\phi_l, \qquad \phi_l'' = (w' + (-iz+w)^2)\phi_l = (u - z^2)\phi_l$$

we infer (3.10). The factorization (3.11) is an immediate consequence. $L^1 \subset H^{-1}$ is dense. Then

$$\begin{aligned}
\mathcal{T}_{-1}^{\mathrm{KdV}}(u,z) &= iz\Big(\log T(z) - \frac{1}{2iz}\int u\Big) \\
&= -iz\Big(\lim_{x\to\infty}\log\frac{\tilde\phi_l(x)e^{izx}}{\tilde\phi_l(-x)e^{-izx}} + \frac{1}{2iz}\int u\Big) \\
&= -iz\int w + \frac{1}{2iz}u dx \\
&= -iz\int w + \frac{1}{2iz}(w_x - 2izw + w^2)dx \\
&= -\frac{1}{2}\int w^2 dx.
\end{aligned}$$

The factors in the factorization (3.11) can be inverted independently:

$$(L - z^2)^{-1} = (-\partial - iz + w)^{-1}(\partial - iz + w)^{-1},$$

where we formally write

$$\begin{aligned}
(\partial - iz + w)^{-1}f(t) &= \int_{-\infty}^t e^{iz(t-y) - \int_y^t w\,ds} f(y)\,dy, \\
(-\partial - iz + w)^{-1}f(t) &= \int_t^\infty e^{-iz(t-y) + \int_y^t w\,ds} f(y)\,dy.
\end{aligned}$$

[2] Also we note here for comparison that the renormalized perturbation determinant α as defined in [39] satisfies $\mathcal{T}_{-1}(i\tau,u) = \tau\alpha(\tau,u)$.

Thus,

$$\begin{aligned}(L_z^{-1}f)(x) &= \int_x^\infty e^{iz(t-x)} \exp\Big(\int_x^t -w\Big) \int_{-\infty}^t e^{-iz(t-y)} \exp\Big(\int_y^t -w\Big) f(y)dydt \\ &= \int G(z,x,y)f(y)dy\end{aligned}$$

where

(3.15)

$$G(z,x,y) = \int_{\max\{y,x\}}^\infty \exp\Big(- iz(x+y-2t) - 2\int_{\max\{x,y\}}^t wds - \int_{\min\{x,y\}}^{\max\{x,y\}} wds\Big)dt.$$

is the Green's function and the evaluation at the diagonal gives

$$\begin{aligned}\beta(z,x) = G(z,x,x) &= \int_x^\infty \exp\Big(- 2iz(x-t) - 2\int_x^t wds\Big)dt \\ &= (-\partial - 2iz + 2w)^{-1}(1),\end{aligned}$$

and

$$\beta' + 2iz\beta - 2w\beta = -1. \tag{3.16}$$

We substitute $\beta = -\frac{1}{2iz(v+1)}$ and see

$$2iz = -\frac{v'}{(v+1)^2} + \frac{2iz}{v+1} - 2\frac{w}{v+1}$$

which implies (3.14). □

We note that (3.16) is equivalent to the ODE for the Lenard recursion, (3.9). Indeed, differentiating (3.16) twice and using $\beta'' = -2iz\beta' + 2(w\beta)'$ gives

$$\beta''' + 2iz\beta'' - 2(w\beta)'' = \beta''' + 4z^2\beta' + 4iz(w\beta)' - 2(w\beta)'' = 0.$$

Now since

$$\begin{aligned}4iz(w\beta)' - 2(w\beta)'' &= 4izw\beta' + 4izw'\beta - 2w''\beta - 4w'\beta' - 2w(-2iz\beta' + 2(w\beta)') \\ &= -2(w'' - 2izw' + 2ww')\beta - 4(w' - 2izw + w^2)\beta',\end{aligned}$$

the equivalence follows by $u = w' - 2izw + w^2$.

The KdV Hamiltonians are defined for Schwartz functions u as the coefficients

$$\mathcal{T}_{-1}^{\mathrm{KdV}}(z,u) = iz\log T_r(z,u) \sim \sum_{n=0}^\infty (2z)^{-2n-2} H_n^{\mathrm{KdV}}(u) \tag{3.17}$$

of the formal asymptotic series, or, equivalently

$$\begin{aligned}&\mathcal{T}_1^{\mathrm{KdV}}(z) = iz\log T(z) - \frac12 \int u\,dx \\ &\sim (2z)^{-2}\frac12\int u^2\,dx + (2z)^{-4}\frac12\int u_x^2 + 2u^3\,dx + (2z)^{-6}\frac12\int u_{xx}^2 - 5uu_{xx} + 5u^4\,dx...\end{aligned}$$

We recall for $N \ge -1$ (see (2.4)) ,

$$\mathcal{T}_N^{\mathrm{KdV}}(z,u) := \frac{i}{2}(2z)^{2N+3}\log T_r^{\mathrm{KdV}}(z) - \sum_{n=0}^N (2z)^{2(N-n)} H_n^{\mathrm{KdV}}(u), \tag{3.18}$$

where in the case $N=-1$ the sum is empty. By (3.17) $\mathcal{T}_{N+1}^{\mathrm{KdV}} = (2z)^2\mathcal{T}_N - H_N^{\mathrm{KdV}}$ and in the sense of Proposition 2.3,

$$\mathcal{T}_N^{\mathrm{KdV}}(z,u) \sim \sum_{n=1}^{\infty} (2z)^{-2n} H_{N+n}^{\mathrm{KdV}}(u).$$

3.2. **Poisson structures.** We recall the definition of symplectic forms and Poisson structures on $\mathbb{R}^{2n}$, $\mathbb{C}^n$ and $\mathbb{C}^{2n}$. A real symplectic form is a nondegenerate closed real two form. The most relevant real symplectic form on $\mathbb{C}^n$ is

$$\omega(z_1,z_2) = -\operatorname{Im}\langle z_1, z_2\rangle$$

where we choose the convention that the inner product is complex linear in the first component. Equivalently we write $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_y$ and define

$$\omega((x_1,y_1),(x_2,y_2)) = \langle x_1,y_2\rangle - \langle y_1, x_2\rangle = \langle (x_1,y_1), J^{-1}(x_2,y_2)\rangle$$

where $J = \begin{pmatrix} 0 & 1_{\mathbb{R}^n} \\ -1_{\mathbb{R}^n} & 0 \end{pmatrix}$. It defines a bilinear form on the dual space which we denote by

$$\omega^{-1}((\xi_1,\eta_1),(\xi_2,\eta_1)) = \langle (\xi_1,\eta_1), J(\xi_2,\eta_2)\rangle.$$

The Hamiltonian vector field of the function f is $J\nabla f$.

A complex symplectic form on $\mathbb{C}^{2n}$ is a holomorphic closed 2 form, the most important being (writing $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_\zeta$)

$$\omega((z_1,\zeta_1),(z_2,\zeta_2)) = \sum_{j=1}^n z_1^j \zeta_2^j - z_2^j\zeta_1^j.$$

Obviously the real part of the restriction to the real subspace of the symplectic form is a real symplectic form.

A real (complex) Poisson bracket is a bilinear map mapping a pair of smooth (holomorphic) functions to smooth (holomorphic) functions which satisfies

$$\{f,g\} = -\{g,f\}, \qquad \text{(skew symmetry)}$$
$$\{f,gh\} = \{f,g\}h + \{f,h\}g, \qquad \text{(derivation)}$$
$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0. \qquad \text{(Jacobi identity)}$$

Given a smooth (holomorphic) function H (called Hamiltonian) we define the Hamiltonian vector field by

$$X_H f = \{f,H\}.$$

A real (complex) symplectic form ω defines a Poisson structure on smooth (holomorphic) functions by (where in the Hilbert space case by an abuse of notation we identify derivatives with functional derivatives via duality)

$$\{f,g\} = \omega(X_f, X_g)$$

(ω defines an isomorphism between tangent and cotangent space, and ω^{-1} is the unique induced two form on the cotangent space). The Jacobi identity is a consequence (and it is equivalent to it) of the closedness of the two form ω.

There is no difficulty to extent these notion to infinite dimensional real and complex Hilbert spaces. It is important to note that not every Poisson structure comes from a symplectic form, our most important examples for that being the Gardner Poisson bracket and the Magri Poisson structure [18]:

Definition 3.3. *The Gardner Poisson structure is defined as*

$$\{F,G\}^{\text{Gardner}} = \int \frac{\delta F}{\delta u}\partial_x \frac{\delta G}{\delta u} dx. \tag{3.19}$$

and the Magri Poisson structure

$$\{F,G\}^{\text{Magri}} = \int \frac{\delta F}{\delta u}(-\partial_x^3 + 2(u\partial_x + \partial_x u))\frac{\delta G}{\delta u} dx. \tag{3.20}$$

Any constant skew symmetric bracket, and in particular the Gardner bracket, satisfies the Jacobi identity (see [8]),

$$\{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\} = 0.$$

We argue differently for the Magri Poisson bracket.

Lemma 3.4. *Let $f(w) = F(w_x - 2izw + w^2)$ and $g(w) = G(w_x - 2izw + w^2)$. Then*

$$\int \frac{\delta}{\delta w} f \partial_x \frac{\delta}{\delta w} g dx = \{F,G\}^{\text{Magri}} + (2iz)^2\{F,G\}^{\text{Gardner}}.$$

In particular every linear combination of the Gardner and the Magri Poisson structure satisfies the Jacobi identity.

We compute using the chain rule $\frac{\delta}{\delta w} f(w) = (-\partial - 2iz + 2w)\frac{\delta}{\delta u}F|_{u=w_x-2izw+w^2}$, hence

$$\begin{aligned}
&\int \frac{\delta}{\delta w} f \partial_x \frac{\delta}{\delta w} g dx \\
&\quad = \int (-\partial - 2iz + 2w)\frac{\delta F}{\delta u}\Big|_{u=w_x-2izw+w^2} \partial(-\partial - 2iz + 2w)\frac{\delta G}{\delta u}\Big|_{u=w_x-2izw+w^2} dx \\
&\quad = \int \frac{\delta F}{\delta u}(\partial - 2iz + 2w)\partial(-\partial - 2iz + 2w)\frac{\delta G}{\delta u} dx \\
&\quad = \int \frac{\delta F}{\delta u}\Big((-\partial^3 + 2(\partial(w_x - 2iz + w^2) + (w_x - 2izw + w^2)\partial) + (2iz)^2\Big)\partial\frac{\delta G}{\delta u} dx.
\end{aligned}$$

With this at hand we prove

Lemma 3.5. *The transmission coefficients $T_r(z_1)$ and $T_r(z_2)$ and $\mathcal{T}_{-1}^{\text{KdV}}(z_1)$ and $\mathcal{T}_{-1}^{\text{KdV}}(z_2)$ Poisson commute for every linear combination of the Gardner and the Magri Poisson bracket.*

Proof. This is a direct formal calculation, which we do first for the Gardner bracket. $\int u\,dx$ is a Casimir for the Gardner structure ($\{f, \int u dx\} = 0$ for all functions f and we may ignore it. We use (3.8) to find

$$\begin{aligned}
&-8z_1z_2 \int \frac{\delta T(z_1)}{\delta u}\partial_x\Big(\frac{\delta T(z_2)}{\delta u}\Big)dx = 2T(z_1)^2T(z_2)^2 \int (\phi_l\phi_r)(z_1)\partial_x(\phi_l\phi_r)(z_2)dx \\
&\quad = T(z_1)^2T(z_2)^2 \int (\phi_l\phi_r)(z_1)\partial_x(\phi_l\phi_r)(z_2) - (\phi_l\phi_r)(z_2)\partial_x(\phi_l\phi_r)(z_1)dx \\
&\quad = \frac{T(z_1)^2T(z_2)^2}{z_1^2 - z_2^2} \int \partial_x\Big(W(\phi_l(z_1),\phi_l(z_2))W(\phi_r(z_1),\phi_r(z_2))\Big)dx \\
&\quad = 0,
\end{aligned}$$

where in the second last step the defining equation $-\phi''+u\phi = z^2\phi$ was used. This implies the same statement for $\log T$ resp. $\log T_r$ and also $\mathcal{T}_{-1}^{\mathrm{KdV}}$. From the above, and Lemma 3.1,

$$\int \frac{\delta T(z_1)}{\delta u}(-\partial^3 + 4u\partial + 2u_x)\frac{\delta T(z_2)}{\delta u} dx = 4z_2^2 \int \frac{\delta T(z_1)}{\delta u}\partial_x \frac{\delta T(z_2)}{\delta u} dx = 0.$$

Hence the transmission coefficients Poisson commute also with respect to the Magri structure. We complete the argument for the Magri structure for T_{-1}^{KdV} by

$$\begin{aligned} \{\log T(z,u), \textstyle\int u dx\}^{\mathrm{Magri}} &= 2\int \frac{\delta \log T(z,.)}{\delta u}\partial_x u dx \\ &= 2\lim_{s\to 0}\frac{1}{s}\Big(\log T(z, u+su_x) - \log T(z,u)\Big) \\ &= 2\lim_{s\to 0}\frac{1}{s}\Big(\log T(z, u(.+s)) - \log T(z,u)\Big) = 0 \end{aligned}$$

by translation invariance and regularity of $\mathcal{T}_{-1}^{\mathrm{KdV}}$ on Schwartz functions, which are dense. We obtain Poisson commutation with respect to the Magri structure. □

We want to express Poisson brackets with $\mathcal{T}_{-1}^{\mathrm{KdV}}$, which can be read as evolution equations for a number of quantities. Let

$$(3.21)\qquad \begin{aligned} \mathcal{T}_{-1}^{\mathrm{Gardner}}(z,w,\tau) &= \frac{1}{4z^2+4\tau^2}\Big(\frac{1}{2}\int w^2\, dx + \mathcal{T}_{-1}^{\mathrm{KdV}}(z, w_x + 2\tau w + w^2)\Big) \\ &= \frac{1}{2}\frac{1}{4z^2+4\tau^2}\int w^2 - w^2(z) dx. \end{aligned}$$

Here, $w(z)$ is defined as in (2.31) and equality of the first and the second line is ensured by (3.12).

Lemma 3.6. *The following identities hold for $u \in H^{-1}$*

$$(3.22)\qquad \{u, \mathcal{T}_{-1}^{\mathrm{KdV}}(z)\} = \frac{1}{2}\partial_x \frac{v(z)}{v(z)+1},$$

$$(3.23)\qquad \{v(z_1), \mathcal{T}_{-1}^{\mathrm{KdV}}(z_2,u)\} = \frac{1}{4z_1^2 - 4z_2^2}\partial_x \frac{v(z_1)-v(z_2)}{v(z_2)+1},$$

$$(3.24)\qquad \{w(z_1), \mathcal{T}_{-1}^{\mathrm{KdV}}(z_2,u)\} = \partial_x \frac{\delta}{\delta w}\mathcal{T}_{-1}^{\mathrm{Gardner}}(z_2, w, z_1).$$

At first sight the Poisson bracket as defined in Definition 3.3 only makes sense for functionals. We understand identities like (3.22) as follows though: For any test function we consider $\int v(z_1,x)\phi(x)dx$ as a functional of $u \in H^{-1}$. We ask then that

$$\left\{\int v(z_1,x)\phi(x)dx, \mathcal{T}_{-1}^{\mathrm{KdV}}(z_2)\right\} = -\frac{1}{4z_1^2-4z_2^2}\int \frac{v(z_1,x)-v(z_2,x)}{v(z_2,x)+1}\partial_x\phi dx$$

for all test function ϕ. In the same way we interpret (3.22) and (3.24). Observe that (3.24) is equivalent to Lemma 2.11, for which we give a different proof here.

Proof. By definition of $\mathcal{T}_{-1}^{\mathrm{KdV}}$, the variational derivative of the renormalized transmission coefficient (3.7), the definition of β as diagonal Green's function of Lemma 3.2 and the formula for the Green's function

$$G(x,y) = \frac{T^{\mathrm{KdV}}}{2iz} \times \begin{cases} \phi_l(x)\phi_r(y) & \text{if } x < y \\ \phi_r(x)\phi_l(y) & \text{if } y < x \end{cases}$$

$$\{u, \mathcal{T}_{-1}^{\mathrm{KdV}}(z)\} = \partial_x \frac{\delta \mathcal{T}_{-1}^{\mathrm{KdV}}(z)}{\delta u} = iz\partial_x(\beta + \frac{1}{2iz}) = \frac{1}{2}\partial_x(\frac{v}{v+1})$$

which is (3.22). Using the notation $L_z = -\partial^2 + u - z^2$, we calculate (see also Killip and Visan [39] and [17])

$$\begin{aligned} 2\left\{v(z_1), \mathcal{T}_{-1}^{\mathrm{KdV}}(z_2)\right\} &= -\frac{2iz_2}{2iz_1}\left\{\frac{1}{\beta(z_1)}, \log T_r^{\mathrm{KdV}}(z_2)\right\} \\ &= \frac{z_2}{z_1}\beta^{-2}(z_1)\big\{\beta(z_1), \log T_r^{\mathrm{KdV}}(z_2)\big\} \\ &= -\frac{z_2}{z_1}\beta^{-2}(z_1)\Big(L_{z_1}^{-1}\{u, \log T_r^{\mathrm{KdV}}(z_2)\}L_{z_1}^{-1}\delta_x\Big)(x) \\ &= -\frac{z_2}{z_2}\beta^{-2}(z_1)\int G(z_1,x,y)(\partial_y\beta(z_2,y))G(z_1,y,x)dy. \end{aligned}$$

We use (3.9), and rewrite as operators

$$\begin{aligned} \beta'''(z_2) - 2(u\beta(z_2))' &- 2u\beta'(z_2) + 4z_1^2\beta'(z_2) \\ &= L_{z_1}\beta'(z_2) + \beta'(z_2)L_{z_1} - 2L_{z_1}\beta(z_2)\partial_y + 2\partial_y\beta(z_2)L_{z_1}. \end{aligned}$$

Thus,

$$\begin{aligned} \left\{v(z_1), \mathcal{T}_{-1}^{\mathrm{KdV}}(z_2)\right\} &= -\frac{\frac{z_2}{z_1}}{4z_1^2 - 4z_2^2}\beta(z_1)^{-2}\int G(z_1,x,y)G(z_1,y,x)\Big\{\beta'''(z_2) \\ &\qquad - 2(u\beta(z_2))' - 2u\beta'(z_2) + 4z_2^2\beta'(z_2) + (4z_1^2 - 4z_2^2)\beta'(z_2)\Big\}dy \\ &= \frac{\frac{z_2}{z_1}}{4z_1^2 - 4z_2^2}\beta(z_1)^{-2}(\beta'(z_1)\beta(z_2) - \beta(z_1)\beta'(z_2)) \\ &= \frac{\frac{z_2}{z_1}}{4z_2^2 - 4z_1^2}\partial\frac{\beta(z_2)}{\beta(z_1)} \\ &= \frac{1}{4z_2^2 - 4z_1^2}\partial\frac{v(z_1)+1}{v(z_2)+1} \\ &= \frac{1}{4z_2^2 - 4z_1^2}\partial\frac{v(z_1)-v(z_2)}{v(z_2)+1} \end{aligned}$$

which is (3.23).

For the last identity we compute using $2w = -\partial \log(1+v) - 2izv$ and by identifying functions with multiplication operators

$$\begin{aligned}
2w_t &= -\partial \frac{v_t(z_1)}{v(z_1)+1} - 2iz_1 v_t(z_1)\\
&= \frac{1}{4(z_2^2 - z_1^2)} \partial \Big[(v(z_1)+1)^{-1} \partial (v(z_1)+1) + 2iz_1(v(z_1)+1) \Big] (v(z_2)+1)^{-1}\\
&= \frac{1}{4(z_2^2 - z_1^2)} \partial \Big[\partial + \big[\partial \log(v(z_1)+1)\big] + 2iz_1 + 2iz_1 v(z_1) \Big] (v(z_2)+1)^{-1}\\
&= \frac{1}{4z_1^2 - 4z_2^2} \partial \big(-\partial + 2w - 2iz_1 \big)(v(z_2)+1)^{-1}\\
&= \frac{iz_2}{2z_2^2 - 2z_1^2} \partial \big(-\partial + 2w - 2iz_1 \big) \beta(z_2) = \frac{iz_2}{2z_2^2 - 2z_1^2} \partial \frac{\delta}{\delta w} \log T(z_2, w_x - 2izw + w^2),
\end{aligned}$$

where in the last step we used the chain rule for $f(w) = g(w_x - 2izw + w^2)$. We rewrite

$$\begin{aligned}
\log T(z_2, u) = \log T_r(z_2, u) - \frac{1}{2iz_2} \int u \, dx &= \frac{1}{iz_2} \Big(\mathcal{T}_{-1}^{\mathrm{KdV}}(z_2, u) - \frac{1}{2} \int u \, dx \Big)\\
&= \frac{1}{iz_2} \Big(\mathcal{T}_{-1}^{\mathrm{KdV}}(z_2, u) - \frac{1}{2} \int w^2 \, dx + \text{Casimir} \Big)
\end{aligned}$$

and get

$$\{ w(z_1), \mathcal{T}_{-1}^{\mathrm{KdV}}(z_2) \} = \frac{1}{4z_2^2 - 4z_1^2} \partial \frac{\delta}{\delta w} \Big(\mathcal{T}_{-1}^{\mathrm{KdV}}(z_2, w_x - 2iz_1 w + w^2) - \frac{1}{2} \int w^2 \, dx \Big).$$

which in view of (3.21) is (3.24). ∎

We obtain the hierarchies for the Gardner variable and good variable by expanding the expressions above into asymptotic power series in $(2z)^{-1}$. The generating function $\mathcal{T}_{-1}^{\mathrm{Gardner}}(z, w, i\tau)$ has an asymptotic expansion in z,

$$\mathcal{T}_{-1}^{\mathrm{Gardner}}(z, w, i\tau) \sim \sum_{n=1}^{\infty} (2z)^{-2n} H_{n-1}^{\mathrm{Gardner}}(w, \tau) \tag{3.25}$$

where we call the coefficients H_n^{Gardner} the Gardner Hamiltonians. Then e.g.

$$\begin{aligned}
H_0^{\mathrm{Gardner}} &= \frac{1}{2} \int w^2 dx, \quad H_1^{\mathrm{Gardner}} = \frac{1}{2} \int w_x^2 + w^4 + 4\tau w^3 dx,\\
H_2^{\mathrm{Gardner}} &= \frac{1}{2} \int w_{xx}^2 + 10 w^2 w_x^2 + 2w^6 + 4\tau(5 w w_x^2 + 3w^5) + 24\tau^2 w^4 \, dx.
\end{aligned}$$

In correspondence with KdV we define

$$\mathcal{T}_N^{\mathrm{Gardner}}(z, w, \tau) := (2z)^{2N+2} \mathcal{T}_{-1}^{\mathrm{Gardner}}(z, w, \tau) - \sum_{n=0}^{N} (2z)^{2(N-n)} H_n^{\mathrm{Gardner}}(w, \tau). \tag{3.26}$$

The relation (3.21) can be written as

$$\mathcal{T}_{-1}^{\mathrm{KdV}}(z, w_x + 2\tau w + w^2) = (4z^2 + 4\tau^2) \mathcal{T}_{-1}^{\mathrm{Gardner}}(z, w, \tau) - \frac{1}{2} \|w\|_{L^2}^2.$$

It implies a similar identity for all N,

$$(2z)^2 \mathcal{T}_N^{\mathrm{KdV}}(z, w_x + 2\tau w + w^2) = (2z)^2 \mathcal{T}_{N+1}^{\mathrm{Gardner}}(z, w, \tau) + (2\tau)^2 \mathcal{T}_N^{\mathrm{Gardner}}(z, w, \tau) \tag{3.27}$$

and

$$H_N^{\mathrm{KdV}}(w_x + 2\tau w + w^2) = H_{N+1}^{\mathrm{Gardner}}(w,\tau) + 4\tau^2 H_N^{\mathrm{Gardner}}(w,\tau), \tag{3.28}$$

or, equivalently

$$H_N^{\mathrm{Gardner}}(\tau, w) = (-4\tau^2)^N \frac{1}{2}\|w\|_{L^2}^2 + \sum_{n=0}^{N-1} (-4\tau^2)^{N-n-1} H_n^{\mathrm{KdV}}(w_x + 2\tau w + w^2). \tag{3.29}$$

From the properties of the generating functions we can derive properties of the Hamiltonians. As a first instance, we note:

Corollary 3.7 (Corollary of Lemma 3.5)**.** *The KdV Hamiltonians Poisson commute with respect to the Gardner Poisson structure on the Schwartz space with another and the renormalized transmission coefficient resp.* $\mathcal{T}_{-1}^{\mathrm{KdV}}$.

Proof. This is an immediate consequence of Lemma 3.5 since the Hamiltonians can be defined by limits of Poisson commuting quantities. By definition (3.17) and from (2.11) we see that

$$H_{N+1}^{\mathrm{KdV}} = \lim_{\tau\to\infty} (2i\tau)^2 \mathcal{T}_N^{\mathrm{KdV}}(i\tau, u), \quad \text{if} \quad u \in H^{N+1}.$$

Thus by Lemma 3.5,

$$\left\{H_{N+1}^{\mathrm{KdV}}, T_{-1}^{\mathrm{KdV}}(z)\right\} = \lim_{\tau\to\infty} \{(2i\tau)^2 \mathcal{T}_N^{\mathrm{KdV}}(i\tau,u), T_{-1}^{\mathrm{KdV}}(z)\} = 0.$$

Here in the last equality we argue inductively that the Poisson bracket with all lower index Hamiltonians vanishes, and the first equality holds because

$$\begin{aligned} &\Big|\{H_{N+1}^{\mathrm{KdV}} - 4\tau^2 \mathcal{T}_N^{\mathrm{KdV}}(i\tau,u), \mathcal{T}_{-1}^{\mathrm{KdV}}(z)\}\Big| \\ &\qquad = \Big| \int \frac{\delta \mathcal{T}_{N+1}^{\mathrm{KdV}}(i\tau,u)}{\delta u} \partial \frac{\delta \mathcal{T}_{-1}^{\mathrm{KdV}}(z)}{\delta u}\, dx \Big| \\ &\qquad \le \Big\| \frac{\delta \mathcal{T}_{N+1}^{\mathrm{KdV}}(i\tau,u)}{\delta u} \Big\|_{H^{-N-2}} \Big\| \frac{\delta T_{-1}^{\mathrm{KdV}}(i\tau,u)}{\delta u} \Big\|_{H^{N+3}} \\ &\qquad \lesssim \tau^{-2}\big(\|u\|_{H^{N+2}} + \|u\|_{H^{N+2}}^2\big)\|u\|_{H^{N+1}}, \end{aligned}$$

if $\tau \gtrsim \|u\|_{H^{-1}}^2$, using (2.11). ☐

From Lemma 3.6 we obtain the evolution of the variables $v(z)$ and $w(z)$ when u evolves according to the Nth KdV flow, respectively the flows $\mathcal{T}_N$.

Theorem 3.8. *The following identities hold on Schwartz space*

$$\{w(i\tau), H_N^{\mathrm{KdV}}(u)\} = \partial_x \frac{\delta}{\delta w} H_N^{\mathrm{Gardner}}(w,\tau)$$

$$\{w(i\tau), \mathcal{T}_N^{\mathrm{KdV}}(z,u)\} = \partial_x \frac{\delta}{\delta w} \mathcal{T}_N^{\mathrm{Gardner}}(z,w,\tau)$$

$$\{v(z), H_N^{\mathrm{KdV}}(u)\} = 2\partial_x\Big[(v(z)+1)\frac{\delta}{\delta u}\sum_{j=0}^{N-1}(2z)^{2(N-1-j)} H_j^{\mathrm{KdV}}(u)\Big]$$

$$\{v(z_1), \mathcal{T}_N^{\mathrm{KdV}}(z_2,u)\} = 2\partial_x\Big[(v(z_1)+1)\frac{\delta}{\delta u}\sum_{j=0}^{N-1}(2z_1)^{2(N-1-j)} \mathcal{T}_j^{\mathrm{KdV}}(z_2,u)\Big]$$

Proof. We expand (3.22), (3.23) and (3.24) and compare coefficients of $(2z)^{-3-2n}$. The first two lines follow immediately by definition. For the third and fourth line on the right hand side we write

$$\partial_x \frac{v(z_1)-v(z_2)}{v(z_2)+1} = 2\partial_x\Big((v(z_1)+1)\Big(\frac{1}{2}-\frac{\delta \mathcal{T}^{\mathrm{KdV}}_{-1}(u)}{\delta u}\Big)\Big)$$

and expand both this and $(4z_1^2-4z_2^2)^{-1}$. □

3.3. **Structure of the Hamiltonians.** The Lenard recursion formula (see [51] for how it is connected to Lenard)

$$\partial \frac{\delta}{\delta u} H^{\mathrm{KdV}}_{N+1} = (-\partial^3 + 4u\partial + 2u_x)\frac{\delta}{\delta u} H^{\mathrm{KdV}}_N \tag{3.30}$$

holds in a distributional sense for $u \in H^N$ by (3.9),(3.17) and (2.11). We will see that the Hamiltonians have a very special structure. For that we introduce the notion of differential polynomials.

Definition 3.9. *A differential polynomial in u is a polynomial in u and its derivatives. We say the monomial*

$$\prod_{j=0}^{N} (u^{(j)})^{\alpha_j}$$

has

- *homogeneity (total number of factors) $H = \sum_{j=0}^N \alpha_j$,*
- *weight (total number of derivatives) $M = \sum_{j=1}^N j\alpha_j$ and*
- *for KdV, degree $d^{\mathrm{KdV}} = H + M/2$,*
- *for Gardner, degree $d^{\mathrm{Gardner}} = H + M$.*

We say a differential polynomial has degree n if it is a sum of monomials of degree n.

In Chapter 7.2 (see Lemma 7.4) we prove that the Gardner Hamiltonians are differential polynomials. By the chain rule we have

$$\partial \frac{\delta H_N^{\mathrm{KdV}}}{\delta u}(u) = (\partial + 2\tau + 2w)\partial \frac{\delta H_N^{\mathrm{Gardner}}}{\delta w}(w,\tau)$$

where $u = w_x + 2\tau w + w^2$. On the left hand side every factor of τ carries a factor of w, hence the homogeneity τ has to be less or equal than the homogeneity of w in each monomial in $H_N^{\mathrm{Gardner}}(w,\tau)$, since the variational derivative decreases the homogeneity in w by 1 which can be compensated by the multiplication by w. Similarly for each monomial of the integrand on the left hand side of (3.28) the homogeneity in w is at least as high as the homogeneity in τ. The same is true for the first term on the RHS, hence $H^{\mathrm{Gardner}}_{N-1}$ can be written as integral over multiples of monomials with the homogeneity in u is at least two more than the homogenity in w. The formulas also show that the sum of the homogeneities in τ and w is always even.

Together with (3.28) this allows us to obtain the KdV Hamiltonians directly from the Gardner Hamiltonians by picking the monomials where the homogenity of w is exactly two more than the homogenity of τ, which is positive.

$$\begin{aligned} H_N^{\mathrm{KdV}}(u) &= \lim_{\tau\to\infty} H_N^{\mathrm{KdV}}((u/(2\tau))_x + (u/(2\tau))^2 + 2\tau(u/(2\tau))) \\ &= \lim_{\tau\to\infty} 4\tau^2 H_N^{\mathrm{Gardner}}(u/(2\tau),\tau) \end{aligned}$$

We obtain a new recursion formula for the Gardner Hamiltonians

(3.31)
$$H_N^{\mathrm{Gardner}}(w,\tau) = \lim_{\lambda\to\infty} \lambda^2 H_{N-1}^{\mathrm{Gardner}}((w_x + 2\tau w + w^2)/\lambda, \lambda) - 4\tau^2 H_{N-1}^{\mathrm{Gardner}}(w,\tau).$$

as well as a recursion formula for the KdV Hamiltonians which does not involve taking antiderivatives,

$$\begin{aligned} H_N^{\mathrm{KdV}}(u) &= \lim_{\tau\to\infty} 4\tau^2 H_N^{\mathrm{Gardner}}\Big(\frac{u}{2\tau},\tau\Big) \\ &= \lim_{\tau\to\infty} 4\tau^2 \Big[H_{N-1}^{\mathrm{KdV}}\Big(\frac{u_x}{2\tau} + u + \frac{u^2}{4\tau^2}\Big) - 4\tau^2 H_{N-2}^{\mathrm{KdV}}\Big(\frac{u_x}{2\tau} + u + \frac{u^2}{4\tau^2}\Big) + \dots \Big]. \end{aligned}$$

Starting with $H_0^{\mathrm{Gardner}} = \frac12 \int w^2 dx$ we obtain $H_0^{\mathrm{KdV}} = \frac12 \int u^2 dx$ and

$$H_1^{\mathrm{Gardner}} = \frac12 \int (w_x + 2\tau w + w^2)^2 - 4\tau^2 w^2 dx = \frac12 \int w_x^2 + w^4 + 4\tau w^3 dx.$$

$$H_1^{\mathrm{KdV}} = \frac12 \int u_x^2 + 2u^3 dx$$

$$\begin{aligned} H_2^{\mathrm{Gardner}} &= \frac12 \int (w_{xx} + 2\tau w_x + 2ww_x)^2 + 2(w_x + 2\tau w + w^2)^3 - 4\tau^2(w_x^2 + w^4 + 4\tau w^3) dx \\ &= \frac12 \int w_{xx}^2 + 10w^2 w_x^2 + 20\tau w w_x^2 + 2w^6 + 12\tau w^5 + 24\tau^2 w^4 dx \end{aligned}$$

$$H_2^{\mathrm{KdV}} = \frac12 \int u_{xx}^2 + 10uu_x^2 + 6u^4 dx.$$

We arrive at the following general structure of the Hamiltonians.

Theorem 3.10. *A) H_n^{KdV} can be written as an integral over a sum of homogeneous differential polynomials in u,*

$$H_n^{\mathrm{KdV}} = \int e_n dx = \frac12 \int |u^{(n)}|^2 + \sum_{k=3}^{n+2} \int e_{n,k}\, dx$$

where

$$e_{n,n+2} = \frac{1}{n+2} \binom{2n+2}{n+1} u^{n+2}.$$

The degree of e_n is $d^{\mathrm{KdV}} = n+2$. Hence, $e_{n,k}$ is a sum of products of k factors, each product carrying a total of $2(n+2-k)$ derivatives of order at most $n+2-k$. B) H_n^{Gardner} can we written as an integral over a linear combination of differential monomials

$$H_n^{\mathrm{Gardner}}(\tau,w) = \frac12 \int |w^{(n)}|^2 dx + \sum_{m=0}^{n+1} \sum_{k=3}^{2n+2-m} \int \tau^m e_{n,m,k}^{\mathrm{Gardner}} dx$$

Here $e_{n,m,k}^{\mathrm{Gardner}}$ are differential polynomials of homogeneity $k \geq 3$ in w, of degree $d^{\mathrm{Gardner}} = 2n+2-m$ and a with total number of $2n+2-k-m$ derivatives. No factor contains more than $[n+1-(k+m)/2]$ derivatives. The term of highest homogeneity is

$$e_{n,0,2n+2}^{\mathrm{Gardner}} = \frac{1}{2(2n+1)} \binom{2n+2}{n+1} w^{2n+2}.$$

The homogeneity k of w and homogeneity m of τ in $e_{n,m,k}$ are related by $k \geq m+2$.

Proof. A) Since the functional derivative reduces the number of factors by one but keeps the weight the same, we see that the first statement is equivalent to showing that

$$\frac{\delta H_n^{\mathrm{KdV}}}{\delta u} = (-1)^n u^{(2n)} + \cdots + \frac{1}{2}\binom{2n+2}{n+1} u^{n+1},$$

where the terms in between contain k factors, $1 < k < n+1$, and are all of degree $n+1$. This is done inductively. Clearly, for $n = 0$ the statement holds. On the other hand, from the Lenard recursion,

$$\partial \frac{\delta H_{n+1}^{\mathrm{KdV}}}{\delta u} = (-\partial^3 + 4u\partial + 2u_x)\frac{\delta H_n^{\mathrm{KdV}}}{\delta u}$$

and the fact that derivatives keep the number of factors the same while they add one derivative, we conclude that the linear term gains two derivatives and the $(n+1)$-linear term becomes

$$\frac{1}{2}\binom{2n+2}{n+1}\partial^{-1}(4u\partial + 2u_x)u^{n+1} = \frac{1}{2}\binom{2n+2}{n+1}\frac{4n+6}{n+2}u^{n+2} = \frac{1}{2}\binom{2n+4}{n+2}u^{n+2},$$

by explicitly calculating the binomial coefficient. The degree of the differential polynomial is increased by the operators $-\partial^2, \partial^{-1}(4u\partial+2u_x)$ by one, which determines the monomials in between. The bound on the maximal order of derivatives involved in the monomials can then be reached by a finite time of partial integrations.

B) By (3.29) we have to analyze what happens when we plug in $u = w_x + 2\tau w + w^2$ into the Nth KdV Hamiltonians with $N \leq n-1$. We first prove the degree condition. Consider a monomial in the Nth KdV Hamiltonian. By splitting the sum in $u = w_x + 2\tau w + w^2$, every factor of u becomes a factor of w and gains either a derivative, another factor of w or a factor of τ. Additionally a factor of $(2\tau)^{2(n-1-N)}$ is multiplied. Thus by the degree condition of KdV,

$$\begin{aligned} m + d^{\mathrm{Gardner}} &= 2(n-1-N) + H^{\mathrm{Gardner}} + M^{\mathrm{Gardner}} \\ &= 2(n-1-N) + 2H^{\mathrm{KdV}} + M^{\mathrm{KdV}} \\ &= 2(n-1-N) + 2(N+2) = 2n+2. \end{aligned}$$

The bound on the maximal order of derivatives involved in the monomials can again be reached by a finite time of partial integrations. To obtain the form of the bilinear term we notice that only the part $w_x + 2\tau w$ can contribute to it. Now from the nth KdV Hamiltonian respectively we obtain a contribution of

$$\frac{1}{2}\int \left(w^{(n+1)} + 2\tau w^{(n)}\right)^2 dx = \frac{1}{2}\int \left(w^{(n+1)}\right)^2 + 4\tau^2\left(w^{(n)}\right)^2 dx.$$

This shows that the sum of the bilinear parts in (3.29) is a telescopic sum in which only the highest order term survives, giving the desired form of the bilinear term in the Gardner Hamiltonian. Likewise, the form of the term of highest homogeneity is reached by setting $u = w^2$ in the u^{n+1} summand of the $n-1$th KdV Hamiltonian. □

Before analyzing the form of the good variable equations we state another result concerning the conservation law for the momentum of the Gardner equations. This will be used in proving the local smoothing properties.

Lemma 3.11 (Energy-flux)**.** *There exist differential polynomials which we call fluxes so that*

$$\frac{\partial}{\partial t} w^2 = \partial_x \operatorname{Fl}_N$$

if w satisfies the Nth Gardner equation. The flux can be written as

$$\operatorname{Fl}_N = \sum_{m=0}^{N+1} \sum_{j=3}^{2N+2-m} \tau_0^m \operatorname{Fl}_{m,j,N},$$

where each $F_{m,j,N}$ has homogeneity $j \geq 3$, degree $2N+2-m$ and weight $2N+2-(m+j)$.

Proof. Let h_N be the density of H_N^{Gardner}, given as a differential polynomial. Then

$$\begin{aligned} \partial_x w \frac{\delta}{\delta w} H_N^{\text{Gardner}} &= \partial_x w \sum_j (-1)^j \partial^j \frac{\partial h_N}{\partial w^{(j)}} \\ &= \partial_x h_N + \sum_j (-1)^j \sum_{k_1+k_2=j, k_1 \geq 1} \partial^{k_1} \Big(w'' \partial^{k_2} \frac{\partial h_N}{\partial w^{(j)}} \Big). \end{aligned}$$

We obtain a differential polynomial Fl_N such that

$$\partial_t w^2 = \partial_x \operatorname{Fl}_N \,. \tag{3.32}$$

We decompose Fl_N as a finite sum of differential polynomials $\operatorname{Fl}_N = \sum_l \operatorname{Fl}_N^l$ where Fl_N^l has degree l. Taking derivatives leaves the number of factors invariant and increases the degree by one. On the other hand, we know that

$$w \partial \frac{\delta H_N^{\text{Gardner}}}{\delta w} = \partial_x \sum_{m=0}^{N+1} \sum_{j=3}^{2N+2-m} \tau_0^m \tilde{h}_{m,j,N}$$

is also a sum over homogeneous differential polynomials of homogeneity j and degree $2N+3-m$. Indeed, the functional derivative leaves the number of derivatives constant while decreasing the homogeneity by one, the derivative increases the degree and the former homogeneity is restored by multiplication by w. By identifying terms of like degree we see that

$$\operatorname{Fl}_N = \sum_{m=0}^{N+1} \sum_{j=3}^{2N+2-m} \tau_0^m \tilde{\operatorname{Fl}}_{m,j,N},$$

where each $\operatorname{Fl}_{m,j,N}$ has homogeneity $j \geq 3$, degree $2N+2-m$ and weight $2N+2-(m+j)$. This shows

$$\int (1 - \tanh(\kappa x)) w \partial \frac{\delta H_N^{\text{Gardner}}}{\delta w} dx = \kappa \int \operatorname{sech}^2(\kappa x) \sum_{m=0}^{N+1} \sum_{j=3}^{2N+2-m} \tau_0^m \tilde{\operatorname{Fl}}_{m,j,N} dx.$$

By partial integration, we can reduce to the situation where each factor has less than $[n+1-(m+j)/2]$ derivatives, by making errors where derivatives fall onto the localization factor $\operatorname{sech}(\kappa x)$ and the degree of the homogenous polynomial is decreased, thus being easier to handle. $\square$

We turn to the good variable equations. We have seen that if u solves the Nth KdV equation, then by Theorem 3.8

$$\partial_t v(z) = 2\partial_x\Big[(v(z)+1)\frac{\delta}{\delta u}\sum_{j=0}^{N-1}(2z)^{2(N-1-j)}H_j^{\mathrm{KdV}}(u)\Big]$$

Using the relations between u, w, and v, we can turn this into a single differential equation. By combining (3.10) and (3.14) we see that

$$u = -\frac{1}{2}\frac{v_{xx}}{v+1} + \frac{3}{4}\frac{v_x^2}{(v+1)^2} - 2z^2 v - z^2 v^2. \tag{3.33}$$

To turn the above system into a single ODE we plug (3.33) into the equation. For $N=1$ and $z=i\tau$, we find

$$v_t = \partial_x\Big[-v_{xx} + 6\tau^2 v^2 + 2\tau^2 v^3 + \frac{3}{2}\frac{v_x^2}{v+1}\Big]. \tag{3.34}$$

Hence the time evolution equation for the good variable is a deformation of the Gardner equation! For $N=2$ we obtain the somewhat lengthy equation

$$\begin{aligned} v_t =&\partial_x\Big[v_{xxxx} - 7\tau^2 v_{xx}v^2 - 4\tau^2 v v_x^2 - 14\tau^2 v_{xx}v - 4\tau^2 v_x^2 \\ &\quad + 6\tau^4 v^5 + 30\tau^4 v^4 + 40\tau^4 v^3 \\ &\quad + (v+1)^{-1}\big(-\frac{5}{2}v_{xx}^2 - 5v_{xxx}v_x + 18\tau^2 v_x^2 v + \frac{9}{2}\tau^2 v_x^2 v^2 - 6\tau^2 v_x^2\big) \\ &\quad + (v+1)^{-2}\Big(\frac{25}{2}v_{xx}v_x^2\Big) + (v+1)^{-3}\Big(-\frac{45}{8}v_x^4\Big)\Big]. \end{aligned} \tag{3.35}$$

To prove equivalence of weak solutions we need an understanding of the form of the equation for general N. This is given in the next theorem whose proof can be found in Appendix A.1.

Theorem 3.12. *The Nth equation for v can be written in the form $v_t = \partial_x F_N$, where*

$$F_N = \sum_{n,l,L,d}^{2N-1}(v+1)^{-n}\tau^l f_{N,n,k,d}(v), \tag{3.36}$$

where $f_{N,n,k,d}$ has homogeneity k in v and a total number of derivatives d, and the sum is restricted by

$$\begin{aligned} &0 \le n \le 2N-1, \qquad l+d = 2N, \qquad n+1 \le k \le 2N+1, \\ &\#\{\textit{factors of } \omega \textit{ with at least 1 derivative}\} \ge n+1 \quad \textit{if} \quad n \ge 1. \end{aligned}$$

Moreover, the linear part of the equation is $(-1)^N v^{(2N+1)}$, and $\tau^l f_{N,n,k,d}$ contains no term of the form $v^{n+1}v^{(2N)}$. The number of derivatives d, and l, are always even.

In order to estimate later we also need be able to pull out derivatives. Weak solutions will have regularity $L^\infty H^N$, and localized one more derivative, which means that the single factors of v in nonlinear terms are not allowed to carry more than N derivatives. If $N=2$, the only bad term is

$$(v+1)^{-1}(v_{xxx}v_x),$$

and we can pull out derivatives to rewrite it as

$$\begin{aligned}&(v+1)^{-1}(\partial_x(v_{xx}v_x)-v_{xx}^2)\\&\qquad\qquad=-(v+1)^{-1}(v_{xx}^2)+\partial_x\big((v+1)^{-1}(v_{xx}v_x)\big)+(v+1)^{-2}(v_{xx}v_x^2).\end{aligned}$$

For general N we could have a terms of the form

$$(v+1)^{-n}\prod_{i=1}^{k}v^{(\alpha_i)},$$

where $\sum\alpha_i=d$ and some of the α_i are larger than $d/2$. Note that d is always even, because the number of derivatives on each differential monomial of $\delta H_N^{\mathrm{KdV}}/\delta u$ is even, as can be seen from the Lenard recursion (3.9). Without loss of generality assume $\alpha_1>\cdots>\alpha_k$. We pull out one derivative from α_1. This produces a total derivative of a monomial with $d-1$ derivatives, and a term where we replace α_1 by α_1-1 one some other α_i by α_i+1. We can iterate this until $\alpha_1-\alpha_2=0$, or $\alpha_1-\alpha_2=1$. In the latter case we simply pull out another derivative using $2u^{(\alpha_1)}(u^{(\alpha_1-1)})^{p-1}=\partial(u^{(\alpha_1-1)})^p$. We arrive at

$$\begin{aligned}\prod_{i=1}^{k}v^{(\alpha_i)}=&\sum_{\alpha_1+\cdots+\alpha_k=d,\alpha_i\le d/2}c_{\alpha_1,\ldots,\alpha_k}\prod_{i=1}^{k}v^{(\alpha_i)}\\&+\sum_{l=1}^{\alpha-d/2}\partial^l\sum_{\alpha_1+\cdots+\alpha_k=d-l,\alpha_i\le(d-l)/2}c_{l,\alpha_1,\ldots,\alpha_k}\prod_{i=1}^{k}v^{(\alpha_i)}.\end{aligned}$$

Now we pull the derivatives in front of the factor $(v+1)^{-n}$ as well. The maximal amount of derivatives we have to pull out is $\alpha-d/2=N-1$, hence we will never create a term with too many derivatives. Moreover, because the differential polynomial with prefactor $(1+v)^{-n}$ has at least $n+1$ factors with derivatives, the most derivatives a single factor can have is $2N-n$, hence at most $N-n$ derivatives have to be pulled out if $n\ge1$. This may create up to $N-1$ new factors.

We arrive at the following form.

Lemma 3.13. *The Nth equation for v can be written in the form $v_t=\partial_xF_N$, where*

$$F_N=\sum_{j,l,n,k,d}\partial_x^j\big((v+1)^{-n}\tau^lF_{N,j,l,n,k,d}(v)\big),\tag{3.37}$$

where $F_{N,j,n,k,d}$ has homogeneity k in v, a total number of derivatives $d-j$, and no factor of v carries more than N derivatives. The sum is restricted by the conditions

$$\begin{aligned}&0\le n\le 2N-1,\qquad 0\le j\le N-1,\qquad l+d=2N-j,\\&n+1\le k\le 2N+1\quad\textit{if}\quad n\ge N+1,\\&n+1\le k\le 3N\quad\textit{if}\quad 1\le n\le N.\end{aligned}$$

The linear part of the equation is $(-1)^Nv^{(2N+1)}$.

3.4. **Regularised Fredholm determinants and the Wadati Lax operator.** The importance of these objects is that they characterize $\log T$ and its functional derivatives. We introduce the resolvents $R_\pm=(-iz\pm\partial)^{-1}$ for $\operatorname{Im}z>0$,

$$R_+f(x)=\int_{-\infty}^{x}e^{iz(x-y)}f(y)dy,\quad R_-f(x)=\int_{x}^{\infty}e^{-iz(x-y)}f(y)dy.\tag{3.38}$$

For $q, r \in L^2$ and $z \in \mathbb{C}$ we define the AKNS Lax operator

$$L(q,r) = i \begin{pmatrix} \partial & q \\ -r & -\partial \end{pmatrix}$$

so that

$$L(q,r) - z1 = (L(0,0) - z1)\left(1 + \begin{pmatrix} 0 & R_- q \\ -R_+ r & 0 \end{pmatrix}\right).$$

Unfortunately the operator in the bracket is only Hilbert-Schmidt for $q, r \in L^2$, but not trace class, even for Schwartz functions. For trace class operators K one has the expansion

$$\ln \det(1-K) = \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} K^n \tag{3.39}$$

where $\operatorname{tr} K^n$ is defined for K in the L^n Schatten class. In particular only the first term is problematic for the bracket above. On the other hand, formally at least this trace should be zero due to the off-diagonal block matrix form of the operator. This motivates the use of the renormalized determinant

$$\det{}_2(1+K) = \det(I+K)\exp(-\operatorname{tr} K)$$

for trace class functions, which has a unique extension to Hilbert Schmidt operators K. We refer to Simon [59] for details.

The Lax operator $-\partial^2 + u$ without potential can be factorised as

$$-\partial^2 - z^2 = (\partial + iz)(-\partial + iz).$$

Lemma 3.14. *Suppose that $u \in L^1$. Then*

$$(\partial + iz)^{-1} u (-i\partial + iz)^{-1}$$

is a trace class operator. Moreover, if $u \in H^{-1}$ and $\operatorname{Im} z$ *is sufficiently large then*

$$iz \log \det{}_2(1 + (\partial + iz)^{-1} u(-\partial + iz)^{-1}) = \mathcal{T}_{-1}^{\mathrm{KdV}}(z, u).$$

Proof. We factor

$$(\partial + iz)^{-1} u(-\partial - iz)^{-1} = \Big((\partial + iz)^{-1}|u|^{1/2}\Big)\Big(|u|^{-1/2} u(-\partial + iz)^{-1}\Big)$$

and verify that the factors are Hilbert-Schmidt operators. More precisely let $f \in L^2$. Then $(\partial + iz)^{-1} f$ has the integral kernel

$$k(x,y) = \chi_{x<y} e^{-iz(x-y)} f(y)$$

which has the L^2 norm $(\operatorname{Im} z)^{-1/2}\|f\|_{L^2}$. The same argument applies to $f(-\partial + iz)^{-1}$. $(\partial + iz)^{-1} u(-\partial + iz)^{-1}$ is the product of two Hilbert-Schmidt operators and thus of trace class.

Let

$$\alpha(u) = iz \log \det(1 + (\partial + iz)^{-1} u(-\partial + iz)^{-1})$$

still assuming $u \in L^1$. Then $\alpha(0) = 0$ and, if $u, v \in L^1$

$$\frac{1}{iz}\frac{d}{ds}\alpha(u + sv) = \operatorname{tr}\Big((-\partial^2 - z^2 + u)^{-1} v\Big) = \int G(x,x) v(x) dx$$

where $G(x,x)$ is the diagonal Green's function. We compute

$$\operatorname{tr}\Big((\partial + iz)^{-1} u(-\partial + iz)^{-1}\Big) = \int_{y<x} \exp(2iz(x-y)) u(x) dx dy = \frac{1}{2iz}\int u dx$$

Comparison with (3.8) implies the claim. □

The Wadati Lax operator is defined by

$$L^{\text{Wadati}}(w) = i\begin{pmatrix} \partial & -w \\ w+2\tau & -\partial \end{pmatrix}$$

for $w \in L^2$

Lemma 3.15. *Suppose that $w \in L^2$ and $\operatorname{Im} z > 0$. Then*

$$(L^{\text{Wadati}}(0) - z1)^{-1}\begin{pmatrix} 0 & -iw \\ iw & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(\partial+iz)^{-1}w \\ (-\partial+iz)^{-1}w & 2\tau(-\partial^2-z^2)^{-1}w \end{pmatrix} \tag{3.40}$$

is a Hilbert-Schmidt operator and

(3.41)

$$iz\log\det{}_2\left(1+\begin{pmatrix} 0 & -(\partial+iz)^{-1}w \\ (-\partial+iz)^{-1}w & 2\tau(-\partial^2-z^2)^{-1}w \end{pmatrix}\right) = \mathcal{T}^{\text{KdV}}_{-1}(z, w_x + 2\tau w + w^2).$$

Proof. By the proof of Lemma 3.14 the components are Hilbert-Schmidt operators, and hence the operator is Hilbert-Schmidt. The spectral equation is the system

$$\partial_x \phi^1 = w\phi^2 - iz\phi^1 \qquad \partial_2\phi^2 = -(w+2\tau)\phi^1 - iz\phi^2. \tag{3.42}$$

The matrix $\begin{pmatrix} -iz & 0 \\ 2\tau & iz \end{pmatrix}$ has the eigenvalues $\pm iz$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ -\frac{iz}{2\tau} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We consider the case $\operatorname{Im} z > 0$. The Jost solutions are defined by the normalization (if $w \in L^1$, which we assume for simplicity for the moment)

$$\lim_{x\to-\infty} e^{izx}\phi_l = \begin{pmatrix} 1 \\ -\frac{iz}{2\tau} \end{pmatrix} \qquad \text{resp. } \lim_{x\to\infty} e^{-izx}\phi_r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We define the transmission coefficient by $\big(T^{\text{Wadati}}(z,w)\big) = a^{-1}$ where

$$a := \lim_{x\to\infty} e^{izx}(\phi_l(x))^1$$

which is the same as the Wronskian

$$W(\phi_l,\phi_r) = \phi_l^1\phi_r^2 - \phi_l^2\phi_r^1$$

which does not depend on x.

Again we calculate the variational derivative of $\log a$. Let $\dot\phi_l = \frac{d}{ds}\phi_l(w+sv)|_{s=0}$. It satisfies (compare (3.42))

$$\begin{aligned} \partial_x\dot\phi_l^1 - w\dot\phi_l^2 + iz\dot\phi_l^1 &= v\phi_l^2, \\ \partial_2\dot\phi_l^2 - (w+2\tau)\dot\phi_l^1 - -iz\dot\phi_l^2 &= v\phi_l^1. \end{aligned}$$

The forward fundamental solution is, using the notation $\tilde\phi_{l,r} = (-\phi_{l,r}^2, \phi_{l,r}^1)$

$$T^{\text{Wadati}}\begin{cases} \phi_r(x)\tilde\phi_l(y) - \phi_l(x)\tilde\phi_r(y) & \text{if } x > y \\ 0 & \text{if } x < y \end{cases}$$

hence

$$\lim_{x\to\infty} e^{izx}\dot\phi_l^1(x) = T^{\text{Wadati}}\int(\phi_l^1(x)\phi_r^1(x) - \phi_l^2(x)\phi_r^2(x))v(x)dx.$$

and

$$\frac{\delta}{\delta w} a = T^{\text{Wadati}}(\phi_l^1 \phi_r^1 - \phi_l^2 \phi_r^2). \tag{3.43}$$

Let

$$G(z, w, x, y) = T^{\text{Wadati}}(z, w) \left\{ \begin{array}{ll} \phi_r(x)\tilde{\phi}_l(y) & \text{if } x > y \\ \phi_l(x)\tilde{\phi}_r(y) & \text{if } x < y \end{array} \right\} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

be the integral kernel of the resolvent $(L^{\text{Wadati}} - z)^{-1}$ which, by an abuse of notation, we identify with $L(z, w)^{-1}$ whenever it is defined. Moreover we will suppress arguments and write $G(z, w, x, y) = G(x, y) = G(z, w)$ whenever this is convenient.

We claim

$$\ln \det{}_2 \Big(1 + G(z, w) \begin{pmatrix} 0 & iw \\ -iw & 0 \end{pmatrix}\Big) = - \ln T^{\text{Wadati}}(z, w) + \frac{2}{z} \int w dx \tag{3.44}$$

for which we provide a short conceptional proof by calculating the derivative of the functional determinant with respect to the potential w(see also [57]).

This requires a bit of care. We observe that both sides are identically 0 if $w = 0$. We approximate w by Schwartz functions and replace $R_\pm = (\pm\partial - iz)^{-1}$ by the operators $R_\pm^\sigma = [(1 \pm \frac{1}{\sigma}\partial)(-iz \pm \partial)]^{-1}$ which have convolution kernels

$$k_+^\sigma(x) == \chi_{x>0} \frac{1}{iz + \sigma}(e^{izx} - e^{-\tau x}), \quad k_-^\sigma(x) = \chi_{x<0} \frac{-1}{iz + \sigma}(e^{-izx} - e^{\sigma x}).$$

Then, with the obvious notation

$$\ln \det{}_2 \left(1 + G(z, 0) \begin{pmatrix} 0 & iw \\ -iw & 0 \end{pmatrix}\right) = \lim_{\sigma \to \infty} \ln \det{}_2 \left(1 + G^\sigma(z, 0) \begin{pmatrix} 0 & iw \\ -iw & 0 \end{pmatrix}\right)$$

since the trace of the second summand on the right hand side vanishes, and the operator converges in the Hilbert-Schmidt norm. By the definition of the regularized determinant

$$\begin{aligned} &\ln \det \left(1 + G^\sigma(z, 0) \begin{pmatrix} 0 & iw \\ -iw & 0 \end{pmatrix}\right) \\ &\qquad = \operatorname{tr}\, G^\sigma(z, 0) \begin{pmatrix} 0 & iw \\ -iw & 0 \end{pmatrix} + \ln \det{}_2 \left(1 + G^\sigma(z, 0) \begin{pmatrix} 0 & iw \\ -iw & 0 \end{pmatrix}\right). \end{aligned}$$

We compute

$$\operatorname{tr}\, G^\sigma(z, 0) \begin{pmatrix} 0 & iw \\ -iw & 0 \end{pmatrix} \to \operatorname{tr}\, \big(2\tau(-\partial^2 - z^2)^{-1} w\big) = \frac{\tau}{z} \int w dx \qquad \text{as } \sigma \to \infty$$

since the resolvent converges.

We use the operator identity $1+(t+s)A = (1+tA)(1+(1+tA)^{-1}sA)$ below, and calculate

$$\begin{aligned}
&\frac{d}{ds}\log\det\left(1+G^\sigma(z,0)\begin{pmatrix} 0 & (t+s)iw \\ -(t+s)iw & 0\end{pmatrix}\right)_{s=0} \\
&\quad= \frac{d}{ds}\log\det\left[1+s\left(1+G^\sigma(z,0)\begin{pmatrix} 0 & (t+s)iw \\ -(t+s)iw & 0\end{pmatrix}\right)^{-1}G^\sigma(z,0)\begin{pmatrix} 0 & iw \\ -iw & 0\end{pmatrix}\right]_{s=0} \\
&\quad= \frac{d}{ds}\log\det\left[1+sG^\sigma(z,tw)\begin{pmatrix} 0 & iw \\ -iw & 0\end{pmatrix}\right]_{s=0} = \operatorname{tr}\left[G^\sigma(z,tw)\begin{pmatrix} 0 & iw \\ -iw & 0\end{pmatrix}\right] \\
&\quad= \operatorname{tr}\left[\left(1+G^\sigma(z,0)\begin{pmatrix} 0 & itw \\ -itw & 0\end{pmatrix}\right)^{-1}G^\sigma(z,0)\begin{pmatrix} 0 & iw \\ -iw & 0\end{pmatrix}\right] \\
&\qquad\quad\to \operatorname{tr}\left[\left(1+G(z,0)\begin{pmatrix} 0 & itw \\ -itw & 0\end{pmatrix}\right)^{-1}G(z,0)\begin{pmatrix} 0 & iw \\ -iw & 0\end{pmatrix}\right] \\
&\quad\to \int (\phi_l^1\phi_r^1 - \phi_l^2\phi_r^2)w(x)\,dx
\end{aligned}$$

where $\phi_{l,r}$ are the Jost solutions at tw. For the limit we use that

$$G^\sigma\begin{pmatrix} 0 & iw \\ -iw & 0\end{pmatrix} \to G\begin{pmatrix} 0 & iw \\ -iw & 0\end{pmatrix}$$

in the Hilbert Schmidt norm and, assuming $w \in W^{1,1}$,

$$\operatorname{tr}\, G^\sigma\begin{pmatrix} 0 & iw \\ -iw & 0\end{pmatrix} = \operatorname{tr}\, 2\tau R_+^\sigma R_-^\sigma w$$

and

$$R_+^\sigma R_-^\sigma w \to R_+R_-w$$

in the trace norm. This implies (3.44).

It remains to connect the transmission coefficients for Wadati and KdV. A straight forward calculation gives

$$\begin{aligned}
&\begin{pmatrix} -\partial^2+w^2+2\tau w+w' & 0 \\ 0 & -\partial^2+w^2+2\tau w - w'\end{pmatrix} \\
&\qquad\qquad = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix} -\partial & w \\ -(w+2\tau) & \partial\end{pmatrix}^2\begin{pmatrix} 1 & -1 \\ 1 & 1\end{pmatrix}.
\end{aligned}\tag{3.45}$$

Let ϕ_l be the left Jost function for the Wadati Lax operator. Then, by this calculation, $\phi_l^1+\phi_l^2$ is a multiple of the left Jost solution to the KdV Lax operator. Thus

$$T^{\text{Gardner}}(z,w) = T^{\text{KdV}}(z, w_x+2\tau w+w^2). \tag{3.46}$$

□

This connection is used in Chapter 7 where we derive estimates on the multilinear expansion of the Gardner generating function.

Theorem 3.16. *In particular, the Gardner generating function $\mathcal{T}_{-1}^{\text{Gardner}}$ defined in* (3.21) *can be written in the following forms*

$$\begin{aligned}\mathcal{T}_{-1}^{\text{Gardner}}(z,w,\tau) &= (4z^2+4\tau^2)^{-1}\Big(\mathcal{T}_{-1}^{\text{KdV}}(z,w_x+2\tau w+w^2)+\frac{1}{2}\int w^2dx\Big)\\ &= (4z^2+4\tau^2)^{-1}\frac{1}{2}\int w^2-w^2(z)dx \qquad (3.47)\\ &= \frac{iz}{4z^2+4\tau^2}\log\det{}_2\Big(1+(L^{\text{Wadati}}-z1)^{-1}\begin{pmatrix}0 & w\\ -w & 0\end{pmatrix}\Big).\end{aligned}$$

where $w(z)$ in the second line is defined by

$$w(z)_x-2izw(z)+w(z)^2=w_x+2\tau w+w^2.$$

Proof. The first and second equality in (3.47) are (3.21) and the definition of $w(z)$. Recall (3.6)

$$\mathcal{T}_{-1}^{\text{KdV}}(z,u)=iz\big(\log T(z,u)-\frac{1}{2iz}\int u\big),$$

hence setting $u=w_x+2\tau w+w^2$ in (3.21) gives

$$\mathcal{T}_{-1}^{\text{KdV}}(z,u)+\frac{1}{2}\int w^2dx=iz\log T^{\text{Wadati}}(z,w,\tau)-\tau\int w\,dx,$$

from which the proof of the third equality follows. □

3.5. **Periodic functions.** Here we assume that r and q are 1 periodic. The monodromy matrix plays a crucial role now: Let Ψ be the $\mathbb{R}^{2\times2}$ valued solution to

$$L(z)\Psi=0 \quad \text{on } [0,1] \qquad \Psi(0)=1_{\mathbb{R}^2}$$

We define the monodromy matrix $M=\Psi(1)$. Its eigenvalues are the Floquet exponents. If $\operatorname{Im} z$ is sufficiently large then one of the Floquet exponents is close to $\exp(-iz)$ and its one dimensional eigenspace $N(z)$ is the initial datum for all exponentially growing solutions (we keep z fixed). Transmission coefficient and resolvent (on $\mathbb{R}$) are defined as in the decaying case, this time without normalizing (we could choose a nonzero element of $N(z)$ for the normalization. The transmission coefficient is then independent of this choice). Of course the entries of the resolvent are now 1 periodic (since it is unique). There is also the resolvent for the periodic Lax operator in the 1 periodic problem, which coincides (via natural identifications) with the Lax operator on $\mathbb{R}$. We can again express the transmission coefficient in terms of the 2 regularized Fredholm determinant for the 1 periodic Lax operator. The relations between transmission coefficients, resolvent and α, β and γ remain the same. As a rule of thumb all constructions and formulas remain valid in the periodic case, also in the next section, at least for $|\operatorname{Im} z|$ large.

4. Analytic properties of the Miura map and the map W

The Miura map will play a central role in our analysis. As we have seen, it provides a connection between the KdV and the mKdV hierarchy. It also occurs in the factorization of Schrödinger operators.

Definition 4.1. *The Miura map $M : L^2_{loc}(\mathbb{R}) \to H^{-1}_{loc}(\mathbb{R}), v \mapsto u$ is defined by $u=v_x+v^2$.*

The basic questions:

(1) Can we characterize the range?
(2) If u is in the range, can we characterize the preimage?
(3) Are there interesting classes of functions resp. function spaces so that we can obtain a complete understanding of the mapping properties?

are nontrivial and interesting. Formally

$$L := -\partial_x^2 + u = (\partial_x + v)(-\partial_x + v)$$

iff $u = M(v)$. Since

$$\int (-\partial_x^2 \psi + u\psi)\psi dx = \|(-\partial_x + v)\psi\|_{L^2}^2 \tag{4.1}$$

at least for $\psi \in C_c^2(\mathbb{R})$ we see that u can only be in the range of the Miura map if L is positive semidefinite. Kappeler, Perry, Shubin and Topalov [31] have shown that, if L is positive semidefinite in this sense, then u is in the range of the Miura map, and the preimage is either a point, or homeomorphic to an interval.

If $u = \partial_x v + v^2 \in H^{-1}$ then the spectrum contains $[0, \infty)$. Kappeler, Perry, Shubin and Topalov characterized the range. Checking $u = t\delta_0$ one easily sees that it is in the range if $t \geq 0$ but not if $t < 0$, hence the range is not open. Given $u \in H^{-1}(\mathbb{R})$, the Schrödinger operator is bounded from below and there exists τ_0 depending on $\|u\|_{H^{-1}}$ so that

$$-\partial_x^2 + u + \tau_0^2 : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$$

is positive definite, in the sense that the associated quadratic form is strictly positive. Let $\operatorname{Im} z > 0$. If in addition $u \in L^1$ there exist unique left and right Jost solutions

$$\begin{aligned} -\psi'' + u\psi &= z^2\psi \\ \lim_{x\to-\infty} e^{izx}\psi_l(x) &= 1 \\ \lim_{x\to\infty} e^{-izx}\psi_r(x) &= 1. \end{aligned} \tag{4.2}$$

If $u \in H^{-1}$ and $\operatorname{Im} z = \tau > 0$ we write it as $u = -2izv + v'$ with

$$\|v\|_{L^2} \leq \|u\|_{H^{-1}_{2\tau}}, \quad \|v\|_{L^2} = \|u\|_{H^{-1}_{2\tau}} \quad \text{if } z = i\tau. \tag{4.3}$$

We can replace the normalization by

$$\begin{aligned} \lim_{x\to-\infty} e^{izx - \int_0^x v ds}\psi_l(x) &= 1 \\ \lim_{x\to\infty} e^{-izx + \int_0^x v ds}\psi_r(x) &= 1 \end{aligned} \tag{4.4}$$

which uniquely defines Jost solutions, see Lemma 4.4.

If $z = i\tau$ then ψ_l and ψ_r are real valued, and if in addition $\tau > \tau_0$ then $L + \tau^2$ is positive definite and ψ_l and ψ_r are nonnegative (suppose $\psi_l(x_0) = 0$ and $\psi_l > 0$ on $(-\infty, x_0)$. Then we use $\chi_{(-\infty,x_0)}\psi_l$ as test function and see

$$0 = \langle \psi_l, (-\partial^2 + u + \tau^2)\chi_{(-\infty,x_0)}\psi_l \rangle,$$

and $L + \tau^2$ would not be positive definite, a contradiction). Moreover

$$\lim_{x\to\infty} e^{-\tau x - \int_0^x v}\psi_l(x) > 0, \qquad \lim_{x\to-\infty} e^{\tau x + \int_0^x v}\psi_r(x) > 0.$$

Let $\operatorname{Im} z > 0$ and $w = \partial_x \log \psi_l$. A short calculation shows that

$$M(w - iz) = u - z^2.$$

Motivated by this calculation we define the modified Miura map

$$M_\tau(w) = w_x + w^2 + 2\tau w$$

on $L^2(\mathbb{R})$. An inverse is given by

$$\{u \in H^{-1} : -\partial^2 + u + \tau^2 \text{ is } p.d.\} \ni v \to w = \frac{d}{dx}\log\psi_l - \tau. \tag{4.5}$$

The modified Miura map

$$H^N_\tau \ni w \to u = w_x + 2\tau w + w^2 \in H^{N-1}_\tau$$

for $N \geq 0$ is related to a factorization

$$-\partial^2 + u + \tau^2 = (\partial + w + \tau)(-\partial + w + \tau) \tag{4.6}$$

and, as a consequence $-\partial^2 + u + \tau^2$ is positive definite if u is in the range of the Miura map and (4.1) obtains the form

$$\langle(-\partial^2 + u + \tau^2)\phi, \phi\rangle = \|(-\partial + w + \tau)\phi\|^2_{L^2}. \tag{4.7}$$

Let

$$\mathcal{U}_\tau = \{u \in H^{-1} : -\partial^2 + u + \tau^2 > 0\}. \tag{4.8}$$

The modified Miura map is an analytic diffeomorphism from H^N to $\mathcal{U}_\tau \cap H^{N-1}$, see Theorem 4.13, it relates (weak) solutions of the KdV hierarchy and the Gardner hierarchy (see Theorem 3.16 for smooth solutions and Theorem 5.8 for weak solutions) and precompactness, equicontinuity and tightness for u and w (see Subsection 6.3). The next key lemma characterizes the range of the Miura map in a quantitative fashion.

Lemma 4.2. *The quantities $\|w\|_{L^2}$, $\|w_x + w^2 + 2\tau w\|_{H^{-1}_\tau}$ and the ground state energy of $-\partial^2 + w_x + w^2 + 2\tau w + \tau^2$ are related as follows:*

$$\|w_x + w^2 + 2\tau w\|_{H^{-1}_\tau} \leq (2 + \tau^{-1/2}\|w\|_{L^2})\|w\|_{L^2}. \tag{4.9}$$

For all $\psi \in H^1$

$$\langle(-\partial^2 + w_x + 2\tau w + w^2 + \tau^2)\psi, \psi\rangle \geq \frac{\tau^2}{4} e^{-\frac{1}{\tau}\|w\|^2_{L^2}} \|\psi\|^2_{L^2}, \tag{4.10}$$

Let $w \in L^2_{loc}$ and $-\tau_1^2$ be the infimum of the spectrum of $-\partial^2 + w_x + w^2 + 2\tau w$. Then there exists an absolute constant $C > 0$ so that

$$\|w\|_{L^2} \leq C\Big(1 + \log(\frac{\tau}{\tau - \tau_1})\Big)^{1/2} \|w_x + 2\tau w + w^2\|_{H^{-1}_\tau}. \tag{4.11}$$

We will prove Lemma 4.2 in Subsection 4.2

A central element for the diffeomorphism property is the study of the linear equation

$$\phi \to L_w\phi := \phi_x + 2\tau\phi + 2w\phi =: f. \tag{4.12}$$

The equation $L_w\phi = f$ can explicitly be solved by

$$\phi(x) = \int_{-\infty}^x \exp\Big(-2\tau(x-y) - 2\int_y^x w\,dt\Big) f(y)dy.$$

The two first order operators on the RHS of (4.6) can be inverted separately and we can solve $(-\partial^2 + u + \tau^2)\phi = f$ by

$$\begin{aligned}\phi(x) &= (-\partial + w + \tau)^{-1}(\partial + w + \tau)^{-1} f \\ &= \int_x^\infty \exp\Big(- \tau(t-x) - \int_x^t w ds\Big) \int_{-\infty}^t \exp\Big(- \tau(t-y) - \int_y^t w ds'\Big) f(y) dy dt.\end{aligned}$$

The Green's function can be expressed explicitly by (3.15)

$$G(x,y) = \int_{\max\{y,x\}}^\infty \exp\Big(\tau(x+y-2t) - 2\int_{\max\{x,y\}}^t w ds - \int_{\min\{x,y\}}^{\max\{x,y\}} w ds)\Big) dt$$

and the diagonal Green's function by

$$\beta(x) := G(x,x) = \int_x^\infty \exp\Big(2\tau(x-t) - 2\int_x^t w ds\Big) dt = (-\partial + 2\tau + 2w)^{-1}(1),$$

resp. (3.16)

$$-\beta' + 2\tau\beta + 2w\beta = 1.$$

We define the 'good variable'

$$v := \frac{1}{2\tau\beta} - 1, \tag{4.13}$$

and using (3.16), we calculate

$$\partial_x \log(1+v) = -\frac{\beta'}{\beta} = 2\tau v - 2w,$$

and record the final simple formula

$$2\tau v - \partial_x \log(1+v) = 2w. \tag{4.14}$$

We define for $s > -\frac{1}{2}$

$$\mathcal{V}^s = \{v \in H^{s+1} : v > -1\}. \tag{4.15}$$

Then

$$\mathcal{V}^s \ni v \to \tau v - \frac{1}{2}\partial_x \log(1+v) =: w \in H^s$$

is a diffeomorphism (see Theorem 4.13) which relates weak solutions to the Gardner hierarchy to weak solution for the good variables of Theorem 3.12, and preserves precompactness, tightness and equicontinuity (see Subsection 6.3).

Again linear first order equations resp. operators are central objects: The linearization of (4.14) leads to

$$\partial_x(v/\phi) + 2\tau v = f$$

as well as the equivalent formulation

$$\partial_x \psi + 2\tau(\psi\phi) = f.$$

Remark 4.3. *As discussed above we can define*

$$w(z) := \partial_x \log \phi_l + iz$$

for $\operatorname{Im} z > 0$ *and either* $\operatorname{Re} z \neq 0$ *or if* $\operatorname{Im} z$ *so large enough so that* $-\partial^2 + u - (\operatorname{Im} z)^2$ *is positive definite. We can even allow complex valued potentials* u *for which we assume*

$$\|e^{2i \operatorname{Re} z z x} u\|_{H^{-1}_{2\operatorname{Im} z}} < \frac{1}{4}$$

Then, since

$$w_x - 2izw + w^2 = u,$$

by the triangle inequality and the Sobolev inequality

$$\begin{aligned} \Big| \|e^{2i\operatorname{Re} zx} u\|_{H^{-1}_{2\operatorname{Im} z}} - \|w\|_{L^2} \Big| &\le \|e^{2i\operatorname{Re} zx} w^2\|_{H^{-1}_{2\operatorname{Im} z}} \\ &\le (\operatorname{Im} z)^{-1/2} \|w^2\|_{L^1} \\ &= (\operatorname{Im} z)^{-1/2} \|w\|_{L^2}^2 \end{aligned}$$

the modified complex Miura map defines a diffeomorphism

$$M_z : B^{L^2}_{1/4}(0) \to U \subset H^{-1}$$

with

$$\Big\{ u \in H^{-1} : \|e^{2izx} w\|_{H^{-1}_{2\operatorname{Im} z}} < \frac{1}{4} \Big\} \subset U \subset \Big\{ u \in H^{-1} : \|e^{2izx} w\|_{H^{-1}_{2\operatorname{Im} z}} < \frac{3}{4} \Big\}.$$

It is not difficult to obtain the analogous properties for higher regularity. However the proof of last part of Lemma 4.2 below fails: Suppose z is not purely imaginary. Then $w = -iz(\tanh(-izx - \zeta) - 1)$ for $\zeta \in \mathbb{C}$ satisfies $w_x - 2izw + w^2 = 0$. It is not uniformly bounded on intervals of length 1 and Claim 2 below fails. We do not know whether there is a bound of $\|w(z)\|_{L^2}$ in terms of $\|u\|_{L^2}$ and the distance of z to the spectrum.

This section is organized as follows. In Subsection 4.1 we define Jost solutions, in Subsetion 4.2 we prove Lemma 4.2. The diffeomorphism property is made more precise and proven in Subsection 4.3. Finally we sketch the analogous statements for the relation between the good variable hierarchy and the Gardner hierarchy in Subsection 4.4.

4.1. **Jost solutions.** In this section we define Jost solutions and study some of their properties.

Lemma 4.4. *There exist unique solutions to* (4.2) *with the normalization* (4.4). *The left and the right Jost solutions are linearly dependent iff z^2 is an eigenvalue.*

Proof. Using $u = -2izv + v_x$ we write

$$-\partial_x^2 - 2izv + v_x - z^2 = (\partial - iz + v)(-\partial - iz + v) - v^2.$$

Let

$$\phi_1 = e^{izx - \int_0^x v} \psi, \phi_2 = e^{izx - \int_0^x v} (-\partial - iz + v)\psi.$$

Then

$$\phi' = \begin{pmatrix} 0 & -1 \\ v^2 & 2iz - 2v \end{pmatrix} \phi$$

which yields the fixed point identity

$$\phi_1(t) = 1 - \int_{-\infty}^t \phi_2(y) dy = 1 - \int_{y<x<t} \exp\Big(2iz(x-y) - 2\int_y^x v(s) ds\Big) v^2(y) \phi_1(y) dy dx.$$

If $\|v\|_{L^2(-\infty, x_0)}$ is sufficiently small a contraction mapping argument gives existence of a unique solution on $(-\infty, x_0)$. Since smallness can be achieved by chosing x_0, and since we can solve the initial value problem we obtain a unique solution.

It is obvious that z^2 is an eigenvalue if the left and the right Jost solutions are linearly dependent. Suppose that the Jost solutions are linearly independent. We show that then z^2 is not an eigenvalue. Equation (4.2) is a second order ODE and its space of solutions has dimension 2. It suffices to prove that there exists a solution which is unbounded on the right. There is an explicit formula in terms of ψ_r,

$$\psi(x) = \psi_r(x) \int_{x_0}^{x} (\psi_r(y))^{-2} dy$$

where we choose x_0 large so that ψ_r does not vanish on (x_0, ∞). $\square$

It follows from the construction that both functions $e^{-\frac{1}{2iz}\int_0^x v\,ds}\psi_l$ resp. $e^{\frac{1}{2iz}\int_0^x v\,ds}\psi_r$ depend analytically on z and u.

4.2. **Proof of Lemma 4.2.** Surjectivity of

$$L^2 \ni w \to w_x + 2\tau w + w^2 := u \in \{u \in H^{-1} : -\partial^2 + u + \tau^2 > 0\}$$

follows from Lemma 4.4. The inverse is given by $w = \partial_x \log \psi_l - 2\tau$. Before beginning with the proof seriously we observe that

$$\|f_x + \tau g\|_{H^{-1}_\tau} \le \|f\|_{L^2} + \|g\|_{L^2}$$

and by the Fourier transform

$$\|f\|_{L^2} = \|\tau f + f_x\|_{H^{-1}_\tau}.$$

Given an open interval I we define $H^{-1}(I)$ as the equivalence classes of distributions in $H^{-1}(\mathbb{R})$ defining the same distributions on I with the norm

$$\|f\|_{H^{-1}_\tau(I)} = \inf\{\|\tilde f\|_{H^{-1}_\tau(\mathbb{R})} : f = \tilde f \text{ on } I\}.$$

We observe that

$$\|f_x + \tau g\|_{H^{-1}_\tau(I)} \le \|f\|_{L^2(I)} + \|g\|_{L^2(I)}$$

and every distribution in $H^{-1}(I)$ has a representation of this form,

$$\|f\|_{H^{-1}_\tau} \le \inf\{\|g\|_{L^2(I)} + \|h\|_{L^2(I)} : f = \tau g + h_x\}$$

and the right hand side is equivalent to $\|f\|_{(H^1_{\tau,0}(I))^*}$. We fix an extension

$$H^{-1}((-\infty, 0]) \ni f \to \tilde f \in H^{-1}(\mathbb{R})$$

so that $\tilde f$ is supported in $(-\infty, 1)$ and $\tilde f = f$ if f is supported in $(-\infty, -1)$ so that

$$\|\tilde f\|_{H^{-1}(\mathbb{R})} \le 2\|f\|_{H^{-1}(-\infty,0)}.$$

Given $I = (a, b)$ with $|I| = b - a \ge 2$ we use it to define an extension $H^{-1}(I) \ni f \to \tilde f_I \in H^{-1}(\mathbb{R})$ with a uniformly bounded norm so that $\tilde f_I$ is supported in $(a - 1, b + 1)$, $\tilde f = f$ if $\operatorname{supp} f \subset (a + 1, b - 1)$ and $\|\tilde f\|_{L^2} \le 2\|f\|_{L^2}$, with obvious modifications if $2 > b - a \ge 1$. We write

$$\tilde f = \tau g + \partial g$$

as above and hence $f = \tau g + \partial g$ on (a, b). Then

$$\sup_{I \subset (a,b):|I|=1} \|f\|_{H^{-1}(I)} \sim \sup_{I \subset (a,b):|I|=1} \|g\|_{L^2(I)}. \tag{4.16}$$

In the same fashion we define Sobolev spaces on open intervals.

Proof of Lemma 4.2 . We begin with some preparations. Let $w \in L^2$. Then
$$\|w_x + 2\tau w + w^2\|_{H^{-1}_\tau} \le 2\|w\|_{L^2} + \|w^2\|_{H^{-1}_\tau}.$$
By duality and the consequence of the fundamental theorem of calculus
$$\|f\|^2_{L^\infty} \le \|f\|_{L^2}\|f'\|_{L^2} \le \tau^{-1}\|f\|^2_{H^1_\tau}$$
we have
$$\|f\|_{H^{-1}_\tau} \le \tau^{-1/2}\|f\|_{L^1}. \tag{4.17}$$
Hence
$$\|w^2\|_{H^{-1}_\tau} \le \tau^{-1/2}\|w^2\|_{L^1} \le \tau^{-1/2}\|w\|^2_{L^2}$$
which implies (4.9). To prove (4.10), we recall
$$\langle(-\partial^2 + w_x + 2\tau w + w^2 + \tau^2)\psi, \psi\rangle = \|(-\partial + \tau + w)\psi\|^2_{L^2} \tag{4.18}$$
and we consider (compare (4.12))
$$-\psi_x + \tau\psi + w\psi = f. \tag{4.19}$$
We represent the solution ψ by
$$\psi(x) = \int_x^\infty \exp\Big(\tau(x-y) - \int_x^y w dt\Big) f(y) dy. \tag{4.20}$$
Denote the integral kernel by $g(x, y)$ (where $g(x, y) = 0$ if $y \le x$) and estimate for $x \le y$
$$\tau(x-y) - \int_x^y w dt \le \tau(x-y) + |x-y|^{1/2}\|w\|_{L^2} \le \frac{1}{2}\tau(x-y) + \frac{1}{2\tau}\|w\|^2_{L^2}. \tag{4.21}$$
Then
$$\max\Big\{\sup_x \int \exp(g(x,y))dy, \sup_y \int \exp(g(x,y))dx\Big\} \le \exp\Big(\frac{1}{2\tau}\|w\|^2_{L^2}\Big)\frac{2}{\tau}$$
We bound the integral operator (4.20) using Schur's lemma,
$$\|(-\partial + \tau + w)^{-1}\|_{L^2\to L^2} \le \frac{2}{\tau}\exp\Big(\frac{1}{2\tau}\|w\|^2_{L^2}\Big). \tag{4.22}$$
Together with (4.18) we see that
$$-\partial^2 + w_x + 2\tau w + w^2 + \tau^2 - \frac{\tau^2}{4}\exp(-\frac{1}{\tau}\|w\|^2_{L^2}) \tag{4.23}$$
is positive semi definite. This implies (4.10).

Let $-\tau_1^2$ the minimum of the spectrum of $-\partial^2 + w_x + w^2 + 2\tau w$. We claim that there are absolute constants C and $\varepsilon > 0$ so that
$$\frac{\tau - \tau_1}{\tau} \le \exp\left(C - \varepsilon\frac{\|w\|^2_{L^2}}{\|w_x + 2\tau w + w^2\|^2_{H^{-1}_\tau}}\right). \tag{4.24}$$
The estimate (4.11) is an immediate consequence.

We turn to the proof of (4.24) We will construct an approximate eigenfunction ϕ so that
$$\|(-\partial_x^2 + w_x + (w+\tau)^2)\phi\|_{L^2} \le \delta^2\tau^2\|\phi\|_{L^2}, \tag{4.25}$$
with
$$\delta \le \frac{1}{2}\exp\left(C - \varepsilon\frac{\|w\|^2_{L^2}}{\|w_x + 2\tau w + w^2\|^2_{H^{-1}_\tau}}\right)$$

which implies (4.24).

Recall that with ψ_l the left Jost function of $-\partial^2 + w_x + w^2 + 2\tau w + \tau^2$ one has $w = \partial_x \log \psi_l - \tau$. We choose a point $y \in \mathbb{R}$, and $\eta \in C^\infty$, $\eta(x) = 1$ for $x \le -1$, $\eta(x) = 0$ for $x \ge 1$ and define

$$\phi(x) = \eta(\tau(x-y))\psi_l(x)$$

The main objective is to choose y so that δ above is small if $\|w\|_{L^2}$ is large compared to $\|u\|_{H^{-1}_\tau}$. By scaling it suffices to consider $\tau = 1$. The construction depends on the following two claims.

Claim 1: There exists an absolute constant $C > 1$ so that for all $-\infty \le a < b \le \infty$, if

$$\sup_{x \in J=(a,b)} \|u\|_{H^{-1}(x-1,x+1)} \le \frac{1}{1000} \tag{4.26}$$

so that for every solution w to

$$w_x + 2w + w^2 = u$$

on $(a-1, b+1)$ there exists $y \in \mathbb{R} \cup \{\pm\infty\}$ so that

$$\|w - \tanh(\cdot - y) + 1\|_{L^1(a,b)} \le C + 500 \sup_{|I|=1, I \subset (a-1,b+1)} \|u\|_{H^{-1}(I)}(b-a) \tag{4.27}$$

and

$$\|w - \tanh(\cdot - y) + 1\|_{L^2(a,b)} \le C + 1000\|u\|_{H^{-1}(a-1,b+1)}. \tag{4.28}$$

Here, by an abuse of notation, we denote $\tanh(\cdot \pm \infty) = \pm 1$.

Claim 2: There exists $C > 0$ so that if

$$w_x + 2w + w^2 = u$$

on $(-2, 2)$, then

$$\|w\|_{L^2(-1,1)} \le C(1 + \|u\|_{H^{-1}(-2,2)}) \tag{4.29}$$

We postpone the proof of the claims and deduce (4.24) from the claims by constructing approximate eigenfunctions. On unit size intervals I where $\|u\|_{H^{-1}(I)} > \frac{1}{1000}$, we apply the large data estimate (4.29) and obtain

$$\|w\|_{L^2(I)} \le c\|u\|_{H^{-1}(\tilde I)}$$

for some enlarged interval $\tilde I$. Let

$$A = \Big\{ x \in \mathbb{R} : \text{ there exists } k \in \mathbb{Z} \text{ with } |x-k| < 3 \text{ and } \|u\|_{H^{-1}(k-1,k+1)} > \frac{1}{1000} \Big\}.$$

Then A is an open set which can be written as union of of at most $N = 4 \times 10^6 \|u\|^2_{H^{-1}}$ disjoint open intervals $A = \bigcup_j I_j$ since by construction $\|u\|_{H^{-1}(I_j)} \ge \frac{1}{1000}$. Moreover,

$$\begin{aligned}
\|w\|^2_{L^2(A)} &\le \frac{1}{2} \sum_{k \in A \cap \mathbb{Z}} \|w\|^2_{L^2(k-1,k+1)} \\
&\le c \sum_{k \in A \cap \mathbb{Z}} (1 + \|u\|^2_{H^{-1}(k-2,k+2)}) \\
&\le \tilde c \sum_{k \in A \cap \mathbb{Z}} \|u\|^2_{H^{-1}(k-2,k+2)}
\end{aligned}$$

where the last inequality holds since for every integer $k \in A$ there is a k' with $|k-k'| < 3$ so that the norm of u is large. By the same reason we may drop the first and the last terms in the summation so that

$$\|w\|^2_{L^2(A)} \le C \sum_{k:(k-2,k+2)\in A} \|u\|^2_{H^{-1}(k-2,k+2)} \le \tilde{C}\|u\|^2_{H^{-1}(A)}.$$

If $J=(a-1,b+1)$ is an interval satisfying (4.26) then by Claim 1 there exists y such that

$$\|w-(\tanh(x-y)-1)\|_{L^2((a,b))} \le c(1+\|u\|_{H^{-1}((a-1,b+1))}) \tag{4.30}$$

hence

$$\|w\|_{L^2((a,b))} \le c(1+\|u\|_{H^{-1}((a-1,b+1))}) + 2\sqrt{(\min\{y,b\}-a)_+} \tag{4.31}$$

which gives a bound on the length of the interval $[a,y]$ where $w \sim -2$. Note that in the case $a=-\infty$ (4.30) holds with $y=-\infty$ and (4.31) holds without the second term on the right-hand side.

By construction $\mathbb{R} = A \cup \bigcup_j J_j$ where J_j are the disjoint intervals decomposing the complement of $\mathbb{R}\backslash A$ which satisfy (4.26) and hence (4.30) for some $y=y_j$.

We square and sum the estimate over A and the intervals. The sum of the H^{-1} norms on the right hand sides is bounded by $C\|u\|_{H^{-1}}$ since each constant from Claim 1 comes in a pair with a large H^{-1} norm from Claim 2. Thus

$$\|w\|^2_{L^2} \le c\|u\|^2_{H^{-1}} + 8\sum_j \Big(\min\{y_j,b_j\}-a_j\Big)_+. \tag{4.32}$$

Now either

$$\|w\|^2_{L^2} \le c\|u\|^2_{H^{-1}}$$

which immediately implies (4.24) and (4.11), or at least for one j,

$$(\min\{y_j,b_j\}-a_j)_+ \ge \frac{1}{8\times 10^6 \times \|u\|^2_{H^{-1}}}\left(\|w\|^2_{L^2} - c\|u\|^2_{H^{-1}}\right). \tag{4.33}$$

We fix this j in the sequel.

Write $J=(a,b)$ and without loss of generality assume $a<y<b$, otherwise we still get (4.24) immediately. Then $y-a$ is bounded from below by (4.33). With $\eta = 1$ on $(-\infty,-1)$ supported in $(-\infty,1)$ and $\eta' \in C_c^\infty$ nonnegative we define $\phi(x) = \eta(x-(y-2))\psi_l(x)$. We want to estimate

$$\begin{aligned}\|(-\partial^2+u+1)(\eta\psi_l)\|_{L^2(\mathbb{R})} &= \|-\eta''\psi_l - 2\eta'\psi_l'\|_{L^2}\\ &\le c\|\psi_l\|_{L^2(y-3,y-1)}\\ &\le C\exp\Big(-\frac{\|w\|^2_{L^2}}{16\times 10^6\times\|u\|^2_{H^{-1}}}\Big)\|\psi_l\|_{L^2(a,a+1)}\\ &\le C\exp\Big(-\frac{\|w\|^2_{L^2}}{16\times 10^6\times\|u\|^2_{H^{-1}}}\Big)\|\eta\psi_l\|_{L^2(\mathbb{R})}.\end{aligned} \tag{4.34}$$

The first inequality is an energy inequality for ψ_l on a slightly larger interval. Indeed, since ψ_l solves the Schrödinger equation, it satisfies

$$-(\eta')^2\psi\psi'' + (\eta')^2\psi^2 = -u(\eta')^2\psi^2,$$

and hence

$$\|(\eta'\psi)'\|^2_{L^2} + \|\eta'\psi\|^2_{L^2} \le \|\eta''\psi\|^2_{L^2} + c\|u\|_{H^{-1}(y-3,y-1)}\|\eta'\psi\|^2_{H^1}$$

Using smallness of $\|u\|_{H^{-1}}$ on J, shows the first estimate of (4.34).

For the second inequality, if $a \le x \le y-2$ (which we may assume without loss of generality), we use (4.28)

$$\begin{aligned}\frac{\psi_l(x)}{\psi_l(a)} &= \exp\left(\int_a^x (w(s)+1)ds\right)\\ &= \exp\left(\int_a^x w(s)-\tanh(s-y)+1\,ds-(x-a)+\int_a^x \tanh(s-y)+1\,ds\right)\\ &\le \exp(\|w-\tanh(.-y)+1\|_{L^1(a,x)}-(x-a))\\ &\le \exp(C+(500\sup_{|I|=1, I\subset J}\|u\|_{H^{-1}(I)}-1)(x-a)).\end{aligned}$$

Here we used again that $\|u\|_{H^{-1}}$ is small on J. Together with (4.33), this gives the second inequality of (4.34).

This completes the proof of Lemma 4.2 and it remains to prove the two claims. Claim 1 relies on Claim 2 which we prove first.

Proof of Claim 2: Large data. We may replace u by (an abuse of notation) $u+v_x$ with $u,v\in L^2(I)$ with

$$\|u\|_{L^2(-2,2)}+\|v\|_{L^2(-2,2)} \le 2\|u+v_x\|_{H^{-1}((-2,2))}.$$

Suppose that

$$w_x+2w+w^2 = v_x+u \qquad \text{on } (-2,2). \tag{4.35}$$

We claim

$$\|w\|_{L^2(-1,1)} \le c\Big(1+\|v\|_{L^2(-2,2)}+\|u\|_{L^2(-2,2)}\Big). \tag{4.36}$$

which implies

$$\|w\|_{L^2(-1,1)} \le c\Big(1+\|v_x+u\|_{H^{-1}(-2,2)}\Big).$$

We prove (4.36) with several reductions. The function $w_1 = w+1$ satisfies

$$\partial_x w_1 + w_1^2 = v_x+u+1$$

and, including 1 into u it suffices to prove the bound (4.36) for solutions to

$$w_x+w^2 = v_x+u. \tag{4.37}$$

which we consider from now on. Since

$$\|w\|_{L^2(-1,1)} \le \|w_+\|_{L^2(-1,1)}+\|w_-\|_{L^2(-1,1)}$$

it suffices to prove the following estimate for the positive part w_+ of w_1

$$\|w_+\|_{L^2(0,1)} \le c\Big(1+\|v\|_{L^2(-1,1)}+\|u\|_{L^2(-1,1)}\Big). \tag{4.38}$$

We apply this and the corresponding shifted estimate on $(-2,0)$. The argument for w_- being similar by reversing the x direction. In this way (4.38) implies

$$\|w\|_{L^2(-1,1)} \le \|w_+\|_{L^2(-1,1)}+\|w_-\|_{L^2(-1,1)} \le c\Big(1+\|v\|_{L^2(-2,2)}+\|u\|_{L^2(-2,2)}\Big).$$

We prove the estimate first on smaller intervals, and then deduce (4.38) from the estimates on the smaller intervals. Assume that $0<R\le 1$ and $(-R,R)$ is an interval so that

$$\|v\|_{L^2(-R,R)} \le (2R)^{-1/2} \tag{4.39}$$

so that $\|v\|_{L^1(-R,R)} \le 1$ The Ansatz $w_1 = v + w_2$ leads to

$$\partial_x w_2 + w_2^2 + 2vw_2 = u - v^2$$

and $w_3 = \exp\Big(-2\int_0^x v\Big) w_2$ satisfies

$$\partial_x w_3 + \exp\Big(2\int_0^x v\Big) w_3^2 = e^{-2\int_0^x v}(u - v^2).$$

We set

$$w_4 = w_3 - \int_0^x \exp\Big(-2\int_0^y v\Big)(u(y) - v^2(y))dy$$

which satisfies with $\kappa = e^2(\|u\|_{L^1(-R,R)} + \|v\|^2_{L^2(-R,R)})$

$$\partial_x w_4 + e^{-2}(|w_4| - \kappa)^2_+ \le 0.$$

Thus w_4 cannot have a inner local maximum larger than κ or a inner local minimum less than $-\kappa$. If J is an interval were $w_4 \ge \max(2\kappa, 4e^2/R)$ then

$$\partial_x w_4 \le -\frac{1}{4e^2} w_4^2.$$

Comparison with the general solution to the equation

$$\hat w = \frac{4e^2}{x - c}$$

shows that the length of the interval is at most R. As a consequence, arguing by contradiction,

$$w_4 \le \max\Big\{2\kappa, \frac{4e^2}{R}\Big\} = \max\Big\{2e^2(\|u\|_{L^1(-R,R)} + \|v\|^2_{L^2(-R,R)}), \frac{4e^2}{R}\Big\} \quad \text{on} \quad (0,R)$$

and, by reversing the sign and the x direction

$$w_4 \ge -\max\Big\{2e^2(\|u\|_{L^1(-R,R)} + \|v\|^2_{L^2(-R,R)}), \frac{4e^2}{R}\Big\} \quad \text{on} \quad (-R,0).$$

Retracing the construction we see (and taking into account (4.39))

$$\|w\|_{L^2(0,R)} \lesssim R^{-1/2} + R\|u\|_{L^2(-R,R)} + \|v\|_{L^2(-R,R)}. \tag{4.40}$$

We want to show (4.38). If $\|v\|_{L^2} \lesssim 1$ we choose $R = 1$ and obtain (4.38). Otherwise we choose $0 < x_1 < 1$ so that $\|v\|_{L^2(-x_1,x_1)} = (2R_1)^{-1/2}, R_1 = x_1$ and obtain

$$\|w_+\|_{L^2(0,x_1)} \lesssim R_1\|u\|_{L^2(-x_1,x_1)} + \|v\|_{L^2(-x_1,x_1)}.$$

We choose recursively the points

$$x_0 < x_1 < \dots < x_N < 1 \le x_{N+1}$$

with $x_0 = -x_1$. We will prove

$$\|w_+\|_{L^2(x_{j-1},x_j)} \le c\Big(\|v\|_{L^2(x_{j-2},x_j)} + \|u\|_{L^2(x_{j-2},x_j)}\Big) \tag{4.41}$$

and obtain (4.38) by squareing (4.41) and adding over j. We choose the points so that with $R_j = \frac{x_j - x_{j-1}}{2}$

$$2R_j\|v\|^2_{L^2(x_{j-1},x_j)} = 1 \tag{4.42}$$

or $R_j = \frac12$. This latter case is easier and we assume the identity (4.42) . Then

$$\|w_+\|_{L^2(\frac{x_{j-1}+x_j}{2},x_j)} \lesssim \|u\|_{L^2(x_{j-1},x_j)} + \|v\|_{L^2(x_{j-1},x_j)}.$$

The estimate on the left half of the intervals $(x_{j-1}, x_{j-1}+R_j)$ is more delicate and we distinguish the cases $R_j \le R_{j-1}$ and $R_j \ge R_{j-1}$. To simplify the notation we consider $j=1$ and assume first that $R_2 \le R_1$. Then, by the same argument as above (with slightly worse constants)

$$\|w_+\|_{L^2(x_1,x_1+R_1)} \lesssim \|u\|_{L^2(0,x_1+R_1)} + \|v\|_{L^2(x_0,x_2)}.$$

Now assume that $R_2 > R_1$. As above we estimate on $(0,2x_1)$ and on $(x_1,3x_1)$ together

$$\|w_+\|_{L^2(x_1,3x_1)} \le c\big(R_1^{-\frac12} + R_1\|u\|_{L^2(0,3x_1)} + \|v\|_{L^2(0,3x_1)}\big)$$

with slightly worse constant due to using the same terms several times. We repeat the argument and, as long as $(2^{j+1}-1)x_1 \le x_1 + R_2$, we control w_+ on $((2^j-1)x_1,(2^{j+1}-1)x_1)$ by repeating the argument on $((2^{j-1}-1)x_1,(3\times 2^{j-1}-1)x_1)$ and $((3\times 2^{j-1}-1)x_1),(2^{j+1}-1)x_1)$ and get

$$\begin{aligned}\|w_+\|_{L^2((2^j-1)x_1,(2^{j+1}-1)x_1)} \le{}& c\big((2^{j-1}R_1)^{-\frac12} + 2^{j-1}R_1\|u\|_{L^2((2^{j-1}-1)x_1,(2^{j+1}-1)x_1)} \\ &+ \|v\|_{L^2((2^{j-1}-1)x_1,(2^{j+1}-1)x_1)}\big).\end{aligned}$$

Let J be the maximal natural number so that $(2^{J+1}-1)x_1 \le x_1 + R_2$, and, to simplify the notation assume that we have equality. Then

$$\begin{aligned}\|w_+\|^2_{L^2(x_1,x_2)} &= \sum_{j=1}^{J} \|w_+\|^2_{L^2((2^j-1)x_1,(2^{j+1}-1)x_1)} + \|w_+\|^2_{L^2(x_1+R_2,x_2)} \\ &\lesssim R_1^{-1} + R_2^2\|u\|^2_{L^2(0,x_2)} + \|v\|^2_{L^2(0,x_2)}\end{aligned}$$

We can repeat the process to get the general estimate

$$\|w_+\|^2_{L^2(x_j,x_{j+1})} \lesssim \sum_{k=j}^{j+1}\Big(R_k^{-1} + R_k^2\|u\|^2_{L^2(x_k,x_{k+1})} + \|v\|^2_{L^2(x_k,x_{k+1})}\Big). \tag{4.43}$$

Notice that at most one $R_j = \frac12$, and while $R_j < \frac12$, $(2R_j)^{-1} = \|v\|^2_{L^2(x_{j-1},x_j)}$. We arrive at (4.41) using $R_k < 1$ and can sum up the estimate to obtain

$$\|w_+\|_{L^2(0,1)} \le c(1 + \|u+v_x\|_{H^{-1}(-1,2)}).$$

The reason we get the estimate on $(-1,2)$ is that we could have $x_{N+1} > 1$.

Proof of Claim 1: Small data. Again we replace u by $u+v_x$ with

$$\|u\|_{L^2(a-1,b+1)} + \|v\|_{L^2(a-1,b+1)} \le 2\|u+v_x\|_{H^{-1}(a-1,b+1)}$$

and

$$\sup_{I\subset(a-1,b+1),|I|=2} \|u\|_{L^2(I)} + \|v\|_{L^2(I} \le 2 \sup_{I\subset(a-1,b+1),|I|=2} \|u+v_x\|_{H^{-1}(I)}$$

It is useful to consider the more symmetric formulation with $\omega = w+1-v$ which satisfies

$$\omega_x + \omega^2 + 2v\omega - 1 = u - v^2, \qquad \text{on } I=(a-1,b+1) \tag{4.44}$$

and we recall that we may restrict to $\tau = 1$. If $v_x + u = 0$ it can be solved explicitly and the set of all global solutions are given by

$$\omega = \tanh(x-y)$$

or $\omega = \pm 1$. Suppose that $\omega(y) = 0$ for a point $y \in I$ and let $\dot\omega(x) = \omega(x) - \tanh(x-y)$. It satisfies

$$\partial_x \dot\omega + (\dot\omega + 2\tanh + 2v)\dot\omega = u - v^2 - 2v\tanh$$

hence, if $x > y$ (the argument for $x < y$ being similar)

$$\dot\omega(x) = \int_y^x \exp\Big(\int_s^x -2\tanh(\sigma - y) - 2v(\sigma) - \dot\omega(\sigma))d\sigma\Big)(u(s) - v^2(s) - 2v(s)\tanh(s-y))ds$$

Suppose that $\|\dot\omega\|_{L^\infty(y,x)} < \frac12$ and

$$\|v\|_{L^2(I)} + \|u\|_{L^2(I)} < \frac{1}{1000}$$

on unit sized intervals $I \subset (a,b)$. Then

$$\int_s^x -2\tanh(\sigma - y) - 2v(\sigma) - \dot\omega(\sigma))d\sigma \le 3 - |x-s|,$$

hence, by (4.26)

$$|\dot\omega(x)| \le e^3 \sum_{k=0}^\infty e^{-k}\Big(\|u\|_{L^2((x-k-1,x-k)\cap(a,b))} + 3\|v\|_{L^2((x-k-1,x-k)\cap(a,b)}\Big) \le \frac{8e^3}{1000}\frac14.$$

By a continuity argument

$$\|\dot\omega\|_{L^\infty(a,b)} \le \frac14$$

and

$$|\dot\omega(x)| \le c \sup_{I\subset(a,b),|I|=1}(\|u\|_{L^2(I)} + \|v\|_{L^2(I)})$$

and again by using the L^∞ bound in the exponential

$$\|\dot\omega\|_{L^\infty(x,x+1)} \le e^4 \sum_{k=0}^\infty e^{-k}\Big(\|u\|_{L^2((x-k-1,x-k)\cap(a,b))} + 3\|v\|_{L^2((x-k-1,x-k)\cap(a,b)}\Big),$$

we estimate the L^2 norm of $\dot\omega$ on the unit size interval by the L^∞ norm and apply Schur's lemma to arrive at

$$\|w - \tanh(x-y)\|_{L^2(a,b)} \le \|v\|_{L^2(a,b)} + \|\dot\omega\|_{L^2(a,b)} \le 8e^4(\|v\|_{L^2(a,b)} + \|u\|_{L^2(a,b)}).$$

The two estimates (4.27) and (4.28) are an immediate consequence. We can easily adapt the argument to the case when $|\omega(y)| \le \frac12$ at one point. Suppose that $\omega \ge \frac12$ on $(a-1,b+1)$, the case $\omega \le -\frac12$ being similar. By Claim 2 $\|\omega\|_{L^2(I)} \le \|w\|_{L^2(I)} + 1 \le C$ on unit sized intervals in (a,b). Since ω satisfies equation (4.44)

$$\frac12 \le \omega \le C \quad \text{ on } (a,b)$$

for some universal constant C and hence

$$-\frac12 \le w - v \le C \qquad \text{ on } (a,b).$$

Let $\dot\omega = w - v$. Then

$$\partial_x\dot\omega + \dot\omega^2 + 2\dot\omega + 2v\dot\omega = u - v^2$$

and

$$\dot\omega(x) = \exp\Big(-\int_a^x 2 + \dot\omega + 2v ds\Big)\dot\omega(a) + \int_a^x \exp\Big(-\int_s^x 2 + \dot\omega + 2v d\sigma\Big)(u + v^2)ds$$

Since $\sup_I \|v\|_{L^2(I)} \le \frac{1}{1000}$

$$\int_a^x 2 + \dot\omega + 2v ds \ge (x-a) - \frac{1}{500}$$

and, with a small modification of the previous argument for $a \le x \le b$

$$\dot\omega(x) \le 2Ce^{-(x-a)} + 500 \sup_I \Big(\|u\|_{L^2(I)} + \|v\|_{L^2(I)} \Big)$$

and

$$\|\dot\omega\|_{L^2(a,b)} \le C + 500\|u+v_x\|_{H^{-1}(a-1,b+1)}.$$

Again (4.27) and (4.28) are an immediate consequence. □

4.3. **The diffeomorphism $w \to u$.** Properties of the modified Miura map are collected in the next proposition. All constants will depend on $\tau^{-1}\|w\|^2_{L^2}$ which by Lemma 4.2 is equivalent to having them depend on $\tau^{-1/2}\|u\|_{H^{-1}_\tau}$ and the norm of $(\tau^{-2}(-\partial^2+u)+1)^{-1}$ as operator on L^2. For simplicity we write $c(\tau^{-1/2}\|w\|_{L^2})$. Given τ we call a subset $Q_U \in H^{-1}_\tau$ bounded if there exists C and δ so that

$$\tau^{-1/2}\|u\|_{H^{-1}_\tau} \le C$$

and

$$\| - \phi_{xx} + (u+\tau^2)\phi\|_{L^2} \ge (\delta\tau)^2 \|\phi\|_{L^2}.$$

By Lemma 4.2 a set $Q_W \subset L^2$ is bounded if and only if

$$Q_U = \{ w_x + 2\tau w + w^2 : w \in Q_W\}$$

is bounded. We will use this notation and the notions below. To cover later needs we formulate the next result in larger generality than needed as this point. Let $W^{n,p}_\tau$ be the standard Sobolev space if $N \ge 0$ equipped with the norm

$$\|f\|^p_{W^{N,p}_\tau} = \sum_{j=0}^N \tau^{p(N-n)} \|f^{(j)}\|^p_{L^p}$$

with obvious modifications if $p = \infty$. If $N = -1$ we define

$$\|f\|^p_{W^{-1,p}_\tau(I)} = \inf \Big\{ \|g\|^p_{L^p} + \|h\|^p_{L^p} : f = g_x + \tau h \Big\}.$$

Definition 4.5. *We say that a nonnegative function $\gamma \in C^\infty(\mathbb{R})$ is slowly varying of rate α if*

$$|\gamma'| \le \alpha\gamma$$

and if for every k there exists c_k so that

$$|\gamma^{(k)}(x)| \le c_k \alpha^k \gamma(x).$$

Typically examples are $\gamma = e^{\alpha(x-x_0)}$, $\cosh(\alpha(x-x_0))$ and $\operatorname{sech}(\alpha(x-x_0))$.

The following lemma provides an alternative description of a weighted version of $W^{-1,p}$.

Lemma 4.6. *Let γ be slowly varying of rate $\tau/2$. Let $\gamma g \in W^{-1,p}_\tau$. There exists a unique solution f with $\gamma f \in L^p$ to*

$$f' + \tau f = g$$

which satisfies

$$\frac{2}{5}\|\gamma g\|_{W^{-1,p}_\tau} \le \|\gamma f\|_{L^p} \le 4\|\gamma g\|_{W^{-1,p}_\tau}.$$

Proof. First

$$\gamma g = \tau\gamma f - \frac{\gamma'}{\gamma}(\gamma f) + (\gamma f)' \tag{4.45}$$

and

$$\|\gamma g\|_{W^{-1,p}_\tau} \le \frac{5}{2}\|\gamma f\|_{L^p}.$$

We decompose according to the definition $\gamma g = \tau g_0 + g_1'$ and rewrite the equation above as

$$(\gamma f)' + (\tau - \frac{\gamma'}{\gamma})(\gamma f) = \tau g_0 + g_1'.$$

Then

$$\begin{aligned}
\gamma f(x) &= \gamma(x)\int_{-\infty}^x \exp(-\tau(x-y))\gamma^{-1}(y)(\tau g_0 + g_1')dy \\
&= \tau\gamma(x)\int_{-\infty}^x \exp(-\tau(x-y))\gamma^{-1}(y) g_0 dy \\
&\quad - \gamma(x)\int_{-\infty}^x \exp(-\tau(x-y))\gamma^{-1}(y)\big(\tau + \frac{\gamma'}{\gamma}\big) g_1 dy + g_1.
\end{aligned}$$

and by Young's inequality

$$\|\gamma f\|_{L^p} \le 2\|g_0\|_{L^p} + 4\|g_1\|_{L^p}.$$

Indeed, by using Lipschitz continuity of $\log(\gamma(x))$,

$$\int_{-\infty}^x e^{-\tau(x-y)}\frac{\gamma(x)}{\gamma(y)}|h(y)|\,dy \le \int_{-\infty}^x e^{-\frac{\tau}{2}(x-y)}|h(y)|\,dy,$$

which is estimated in L^p using Young's inequality. □

Definition 4.7. *A subset $Q \subset H^N(\mathbb{R})$ is equicontinuous if and only if*

$$\lim_{h\to 0}\sup_{w\in Q}\|w(\cdot + h) - w\|_{H^N} \to 0. \tag{4.46}$$

A subset $Q \subset H^N(\mathbb{R})$ is called tight, if for every $\varepsilon > 0$ there exists R so that

$$\sup_{w\in Q}\|w\|_{H^N(\mathbb{R}\backslash[-R,R])} < \varepsilon. \tag{4.47}$$

A subset Q is precompact if it is tight and equicontinuous.

We collect the analytic properties of the modified Miura map in the following proposition.

Proposition 4.8. *Let $\tau \ge 1$ and γ slowly varying of rate $\tau/2$. The implicit constants in the sequel are independent of τ but they depend on constants in Definition 4.5*

(1) *Let $u = w_x + 2\tau w + w^2$ and $N \geq 0$. Then following estimates hold:*

$$\begin{aligned} \|\gamma u\|_{H^{N-1}_\tau} &\leq c(\tau^{-1/2}\|w\|_{L^2})\|\gamma w\|_{H^N_\tau} \\ \|\gamma w\|_{H^N_\tau} &\leq c\big(\tau^{-1/2}\|w\|_{L^2}\big)\|\gamma u\|_{H^{N-1}_\tau}. \end{aligned} \tag{4.48}$$

(2) *The map*

$$\Theta : H^N_\tau(\mathbb{R}) \ni w \to w_x + 2\tau w + w^2 \in \{u \in H^{N-1}_\tau : -\partial^2 + u + \tau^2 \ p.d.\}$$

is a diffeomorphism with all (including higher) Fréchet derivatives of Θ and Θ^{-1} bounded by a constant depending only on $\tau^{-N-\frac{1}{2}}\|w\|_{H^N_\tau}$.

(3) *Let $Q_W \subset L^2$ be a bounded subset and Q_U its image under the modified Miura map. Then $Q_U \subset H^{-1}_\tau$ is equicontinuous if and only $Q_W \subset L^2$ is equicontinuous. A set $Q_U \subset H^{-1}$ is equicontinuous if and only for every $\varepsilon > 0$ there exists τ_1 so that for $w \in L^2$ with*

$$w_x + 2\tau_1 w + w^2 \in Q_U$$

we have $\|w\|_{L^2} < \varepsilon$.

(4) *Suppose that $Q_W \subset L^2$ is bounded. Then $Q_U \subset H^{-1}_\tau$ is tight if and only if $Q_W \subset L^2$ is tight.*

(5) *Suppose that $Q_W \subset L^2$ is bounded. Then $Q_U \subset H^{-1}_\tau$ is precompact if and only if $Q_W \subset L^2$ is precompact.*

Proof. We consider the linear equation

$$L_w\psi := \psi_x + 2\tau\psi + 2w\psi = f$$

in considerable detail.

Lemma 4.9. *Let γ be slowly varying of rate τ, let $n \geq 0$, $w \in L^2$ and $\gamma w \in H^n$. Then*

$$\|\gamma L_w\psi\|_{H^{n-1}_\tau} \leq c(1 + \tau^{-1/2}\|w\|_{L^2})\|\gamma\psi\|_{H^n_\tau} + c\tau^{-1/2}\|\psi\|_{L^2}\|\gamma w\|_{H^n_\tau}, \tag{4.49}$$

$$\|\gamma\psi\|_{H^n_\tau} \leq c_n \exp(2\tau^{-1}\|w\|^2_{L^2})\Big(\|\gamma L_w\psi\|_{H^{n-1}_\tau} + \tau^{-1/2}\|\gamma w\|_{H^n_\tau}\|L_w\psi\|_{H^{-1}_\tau}\Big) \tag{4.50}$$

and

$$\|\gamma\psi\|_{H^n_\tau} \leq c_n \exp(2\tau^{-1}\|w\|^2_{L^2})\Big(\|\gamma L_w\psi\|_{H^{n-1}_\tau} + \tau^{-1/2}\|w\|_{H^n_\tau}\|\gamma L_w\psi\|_{H^{-1}_\tau}\Big). \tag{4.51}$$

Proof. We begin with the case $n = 0$ and estimate

$$\begin{aligned} \|\gamma L_w\psi\|_{H^{-1}_\tau} &\leq 2\|\gamma\psi\|_{L^2} + \tau^{-1/2}\|\gamma w\psi\|_{L^1} + \tau^{-1}\|\gamma'\psi\|_{L^2} \\ &\leq (3 + \tau^{-1/2}\|w\|_{L^2})\|\gamma\psi\|_{L^2} \end{aligned}$$

where we used (4.17) and Lemma 4.6. To bound the inverse we consider the representation (4.20)

$$\psi(x) = \int_{-\infty}^x \exp\Big(-2\tau(x-y) - 2\int_y^x w\,dt\Big)f(y)dy,$$

and write the integral kernel as $\exp(g(x,y))$ where as in (4.21) $g(x,y) \leq -\tau(x-y) + \frac{1}{\tau}\|w\|^2_{L^2}$. We again bound

$$\|\gamma(L_w)^{-1}\gamma^{-1}\|_{L^2\to L^2} \leq \frac{1}{\tau}\exp\Big(\tau^{-1}\|w\|^2_{L^2}\Big) \tag{4.52}$$

using Schur's lemma and the obvious estimates

$$\max\Big\{\sup_x \gamma(x)\int_{-\infty}^x \gamma(y)^{-1}\exp\Big(-2\tau(x-y)-2\int_y^x wdt\Big)\,dy,$$
$$\sup_y \gamma^{-1}(y)\int_y^\infty \gamma(x)\exp\Big(-2\tau(x-y)-2\int_y^x wdt\Big)\,dx\Big\}$$
$$\le \frac{1}{\tau}\exp\Big(\frac{1}{\tau}\|w\|_{L^2}^2\Big)$$

We next bound $\|\gamma L_w^{-1}\gamma^{-1}\|_{H_\tau^{-1}\to L^2}$ and write using Lemma 4.6 with 2τ instead of τ

$$\psi' + 2\tau\psi + 2w\psi = 2\tau f + f'$$

so that $\phi = \psi - f$ satisfies

$$\phi' + 2\tau\phi + 2w\phi = -2wf$$

and using (4.52),

$$\|\gamma\phi\|_{L^2} \le c\exp\Big(\tau^{-1}\|w\|_{L^2}^2\Big)\Big(1+\tau^{-1/2}\|w\|_{L^2}\Big)\|\gamma f\|_{L^2}.$$

Before we turn to $n>1$ we collect calculus type estimates in Sobolev spaces.

Lemma 4.10. *Let γ be slowly $\tau/2$ varying. The following estimates hold:*

$$\|\gamma fg\|_{H_\tau^n} \le c_n\Big(\|\gamma f\|_{W_\tau^{n,\infty}}\|g\|_{L^2} + \|f\|_{L^2}\|\gamma g\|_{W_\tau^{n,\infty}}\Big) \tag{4.53}$$

$$\|\gamma f\|_{W_\tau^{n,\infty}} \le \|\gamma f\|_{H_\tau^n}^{\frac12}\|\gamma f\|_{H_\tau^{n+1}}^{\frac12} \le c\tau^{-1/2}\|\gamma f\|_{H_\tau^{n+1}}. \tag{4.54}$$

Proof. (4.53) simply follows from the Leibniz rule if $\gamma = 1$:

$$\|fg\|_{H_\tau^n} \lesssim \sum_{j=0}^n \tau^{n-j}\|(fg)^{(j)}\|_{L^2} \lesssim \sum_{j=0}^n \tau^{n-j}\Big(\|f^{(j)}g\|_{L^2} + \|fg^{(j)}\|_{L^2}\Big)$$
$$\lesssim \sum_{j=0}^n \tau^{n-j}\Big(\|f^{(j)}\|_{L^\infty}\|g\|_{L^2} + \|f\|_{L^2}\|g^{(j)}\|_{L^\infty}\Big)$$
$$\lesssim \|f\|_{W_\tau^{n,\infty}}\|g\|_{L^2} + \|f\|_{L^2}\|g\|_{W_\tau^{n,\infty}}$$

(4.54) is a consequence of the following estimate for $\gamma = 1$

$$\|f\|_{L^\infty} \le \|f\|_{L^2}^{\frac12}\|f'\|_{L^2}^{\frac12}.$$

The case of general γ follows from

$$\|f\|_{W_\gamma^{n,p}}^p \sim \sum_{k\in\mathbb{Z}}\|f\|_{W^{n,p}(\tau(k-1),\tau(k+1))}^p$$

and

$$\|\gamma f\|_{W_\tau^{n,p}(\tau(k-1),\tau(k+1))} \sim \gamma(\tau k)\|f\|_{W_\tau^{n,p}(\tau(k-1),\tau(k+1))}.$$

□

Let $n\ge 1$. Inequalities (4.53) and (4.54) immediately give the forward estimate (4.49):

$$\|\gamma\partial\psi\|_{H_\tau^{n-1}} \le C\|\gamma w\|_{H_\tau^n}$$
$$2\tau\|\gamma\psi\|_{H_\tau^{n-1}} \le 2\|\gamma\psi\|_{H_\tau^n}$$

where the first estimate follows by commuting the derivative with γ and estimating lower order terms, and

$$\begin{aligned}\|\gamma w\psi\|_{H^{n-1}_\tau} &\le c_n\Big(\|\gamma w\|_{W^{n-1,\infty}}\|\psi\|_{L^2}+\|w\|_{L^2}\|\gamma\psi\|_{W^{n-1,\infty}}\Big)\\ &\lesssim \tau^{-1/2}\|\gamma w\|_{H^n}\|\psi\|_{L^2}+\tau^{-1/2}\|w\|_{L^2}\|\gamma\psi\|_{H^n_\tau}\end{aligned}$$

To prove (4.50) consider

$$\psi'+2\tau\psi+2w\psi=f.$$

The case $n=0$ has been proven above. Then, taking $n-1$ derivatives of the equation, and using (4.53)

$$\|\gamma\psi^{(n)}\|_{L^2}\le 2\tau\|\gamma\psi^{(n-1)}\|_{L^2}+\|\gamma f^{(n-1)}\|_{L^2}+c\|\gamma w\psi\|_{H^{n-1}_\tau}.$$

By Lemma 4.10 and Young's inequality

$$\begin{aligned}\|\gamma w\psi\|_{H^{n-1}_\tau} &\le \tau^{-1/2}\|\psi\|_{L^2}\|\gamma w\|_{H^n_\tau}+\|w\|_{L^2}\big(\|\gamma\psi\|_{H^{n-1}_\tau}\|\gamma\psi\|_{H^n_\tau}\big)^{1/2}\\ &\le \tau^{-1/2}\|\psi\|_{L^2}\|\gamma w\|_{H^n_\tau}+2c\|w\|^2_{L^2}\|\gamma\psi\|_{H^{n-1}_\tau}+\frac{1}{2c}\|\gamma\psi\|_{H^n_\tau}.\end{aligned}$$

We subtract $\frac12\|\gamma\psi\|_{H^n_\tau}$ from the combined estimate and iterate. Altogether,

$$\|\gamma\psi\|_{H^n_\tau}\lesssim \|\gamma f\|_{H^{n-1}_\tau}+\tau^n\big(1+\tau^{-1/2}\|w\|_{L^2}\big)^{2n}\|\gamma\psi\|_{L^2}$$

Togehter with the L^2 bounds above this implies (4.50). The estimate (4.51) follows by a small obvious modification of the estimates. □

We return to the proof of (4.48) with $N\ge 1$. We recall

$$L_{\frac{w}{2}}w=w_x+2\tau w+w^2=u$$

and (4.49) implies the first inequality of (4.48). For the second inequality (4.50) we observe

$$\begin{aligned}\|\gamma w\|_{H^N_\tau} &\le \|\partial_x(\gamma w)+2\tau\gamma w\|_{H^{N-1}_\tau}\\ &\le \|\gamma(\partial_x w+2\tau w)\|_{H^{n-1}_\tau}+\|\gamma' w\|_{H^{n-1}_\tau}\\ &\le \|\gamma u\|_{H^{N-1}_\tau}+\|\gamma w^2\|_{H^{N-1}_\tau}+c_n\tau\|\gamma w\|_{H^{n-1}_\tau}\\ &\le \|\gamma u\|_{H^{N-1}_\tau}+c_n\|w\|_{L^2}\Big(\|\gamma w\|_{H^{N-1}_\tau}\|\gamma w\|_{H^N_\tau}\Big)^{1/2}+c_n\tau\|\gamma w\|_{H^{n-1}_\tau}\\ &\le \|\gamma u\|_{H^{N-1}_\tau}+c_n^2\|w\|^2_{L^2}\|\gamma w\|_{H^{N-1}_\tau}+\frac14\|\gamma w\|_{H^N_\tau}+c_n\tau\|\gamma w\|_{H^{n-1}_\tau}\end{aligned}$$

and we complete the argument in the same fashion as for the linear estimate.

To see injectivity we consider for $j=1,2$

$$\partial w_j+2\tau w_j+w_j^2=u_j$$

hence with $w=w_2-w_1$

$$\partial w+2\tau w+(w_1+w_2)w=u_2-u_1$$

and (4.50) provides a bound

$$\|w^2-w^1\|_{L^2}\le c\exp(\tau^{-1}(\|w_1\|^2_{L^2}+\|w_2\|^2_{L^2})\|u_2-u_1\|_{H^{-1}_\tau}.$$

Moreover (4.50) provides an estimate for the Lipschitz norm of the inverse restricted to bounded sets corresponding to w in a ball in L^2. Fréchet differentiability of the inverse is immediate and its differential is given by the inverse of L_w. The inverse function theorem gives Fréchet smoothness of the inverse and bounds can be obtained from differentiating $(L_w)^{-1}$ as in multivariate calculus.

The range of Ξ is $H^{N-1}_\tau \cap \{-\partial^2 + u + \tau^2 > 0\}$. Indeed, one inclusion follows from (4.10). For the other inclusion note that if $u \in H^{-1}$ with $-\partial^2 + u + \tau^2 > 0$, we can define w via its Jost function as $w = \partial_x \log \psi - \tau$. By positivity of $\partial^2 + u + \tau^2$ and with the notation as in (4.11), $\tau^2 - \tau_1^2 > 0$, and hence $w \in L^2$. $w \in H^N_\tau$ then follows from (4.48).

We turn to the proof of Proposition 4.8 (3). The maps Ξ and $(\Xi)^{-1}$ are uniformly Lipschitz on bounded sets satisfying the equivalent conditions in Lemma 4.2, hence Q_u in (3) is equicontinuous iff Q_v is equicontinuous.

It is easily seen that condition (4.46) for $N = 0$ implies tightness in the Fourier space, more precisely $Q \subset H^{-1}$ is equicontinuous if and only if for every $\varepsilon > 0$ there exists R so that

$$\sup_{f \in Q} \| |\xi|^{-1} \hat{f} \|_{L^2(\mathbb{R} \setminus (-R,R))} < \varepsilon$$

which is equivalent to the existence of τ_0 so that

$$\sup_{f \in Q, \tau \geq \tau_0} \|f\|_{H^{-1}_\tau} < \varepsilon.$$

For details we refer to [39].

Together with Lemma 4.2 and Proposition 4.8 (4.48) with $N = 0$ we see that equicontinuity is equivalent to the L^2 norm of preimages of the modified Miura map with parameter τ_1 being small for τ_1 large, which is the second part of the equicontinuity claim of Proposition 4.8 (3).

To prove equivalence of tightness we use the following equivalent characterization.

Lemma 4.11. *A set $Q \subset H^s$ is bounded and tight if and only if for all $\alpha > 0$ there exists an α-slowly varying function $\eta \geq 1$ with $\lim_{x \to \pm\infty} \eta(x) = \infty$ so that*

$$\sup_{w \in Q} \|\eta w\|_{H^s} < \infty.$$

Proof. Let

$$\alpha_n = \sup_{w \in Q} \|w\|_{H^s(\mathbb{R} \setminus [-2^n, 2^n])}.$$

A bounded set Q is tight if and only if $\alpha_n \to 0$. Choose $n_k \geq k$ so that $\alpha_{n_k} \leq \exp^{-2\alpha k}$. It is not hard to define an α- slowly varying function $\eta \geq 1$, monotone in radial direction, so that

$$\eta(\pm 2^{n_k}) = \exp(\alpha k).$$

Then

$$\sup_{w \in Q} \|\eta w\|_{L^2} < \infty.$$

$\square$

We turn to the proof of Proposition (4.8) (4). Suppose that $Q_w \in L^2$ is bounded and tight. Then, if γ is slowly varying with rate $\tau/2$ than by Proposition 4.8 (1)

$$\|\gamma(\partial_x w + 2\tau w + w^2)\|_{H^{-1}_\tau} \sim \|\gamma w\|_{L^2}$$

with implicit constants depending on $\tau^{-1/2}\|w\|_{L^2}$. Finally Proposition 4.8 (5) follows from Proposition 4.8 (3) and (4). $\square$

Let $1 \le \tau < \operatorname{Im} z$. The map

$$L^2 \ni w \to w(z) \in L^2$$

defined by inverting the Miura map resp. solving the equation on the left via the left Jost function

$$w'(z) - 2izw(z) + w^2(z) = w' + 2\tau w - w^2$$

which will be useful at several points.

Theorem 4.12. *Let $1 \le \tau < \tau_1$, $N \in N$ and γ be slowly $\tau/2$ varying. Then*

$$\Big\| \gamma \frac{\delta}{\delta w} \int w^2(i\tau_1, x) dx \Big\|_{H^N} \le c(\|w\|_{L^2}) \|\gamma w\|_{H^N}$$

and the Lipschitz constant of the variational derivative is bounded by

$$\Big\| \gamma \Big(\frac{\delta}{\delta w} \int w^2(i\tau_1, x; w) dx(w_1) - \frac{\delta}{\delta w} \int w^2(i\tau_1, x; w) dx(w_2) \Big) \Big\|_{H^N}$$
$$\le c(\|w_1\|_{L^2}, \|w_2\|_{L^2}) \Big(\|\gamma(w_2 - w_1)\|_{H^N} + \|w_2 - w_1\|_{L^2} (\|\gamma w_2\|_{H^N} + \|\gamma w_1\|_{H^N}) \Big).$$

Proof. We first compute the variational derivative. Let ϕ be a test function. Then

$$\int \frac{\delta}{\delta w} \|w^2(i\tau_1)\|_{L^2}^2 \phi dy = 2 \int (-\partial + 2\tau + 2w)(-\partial + 2\tau_1 + 2w(i\tau_1))^{-1} w(i\tau) \phi dx.$$

The first estimate is a consequence of (4.48). Moreover

$$\begin{aligned} u_2 - u_1 &= w_2' + 2\tau w_2 + w_2^2 - (w_1' + 2\tau w_1 + w_1^2) \\ &= \partial(w^2 - w^1) + 2\tau(w^2 - w^1) + (w^2 + w^1)(w^2 - w^1) \\ &= \partial(w^2(i\tau_1) - w^1(i\tau_1) + 2\tau_1(w^2(i\tau_1) - w^1(i\tau_1) \\ &\qquad + (w^2(i\tau_1) + w^1(i\tau_1))(w^2(i\tau_1) - w^1(i\tau_1)) \end{aligned}$$

and estimate follows by the linear estimates by Lemma 4.9 , (4.49) and (4.50). ☐

4.4. **The good variable hierarchy.** The map

$$v \to \tau v - \frac{v_x}{2(1+v)}$$

is a diffeomorphism between subsets of Banach spaces. It relates the good variable hierarchy and the Gardner hierarchy in a very similar fashion as the modified Miura map related the Gardner hierarchy and the KdV hierarchy. The results in this section complement the previous results. Proofs are similar, technical but to a large extend standard, which was different for Lemma 4.2. We only provide part of the proofs here which we consider nonstandard. The following theorem describes properties of this map.

Theorem 4.13. *(1) If $s > -\frac{1}{2}$, $Q_v \subset \{v \in H^{s+1} : v > -1\}$, $Q_w = \{\tau v - \frac{1}{2}\partial \log(1+v)\} \subset H^s$ the following is equivalent*

- *There exists r so that $\|w\|_{H^s_\tau} \le r$ for all $w \in Q_w$.*
- *There exists R and $\delta > 0$ so that $1 + v \ge \delta$ and $\|v\|_{H^{s+1}_\tau} \le R$*

(2) Let $s \ge -1$. Then following estimates hold for $0 < \varepsilon \le 1$

$$\|\gamma w\|_{H^s_\tau} \le c\Big(\|v\|_{L^\infty}, \|(1+v)^{-1}\|_{L^\infty} \Big) \|\gamma w\|_{H^{s+1}_\tau}$$

$$\|\gamma v\|_{H^{s+1}} \le c(\|w\|_{H_\tau^{-\frac{1}{2}+\varepsilon}}) \|\gamma w\|_{H^s_\tau}.$$

(3) *Let $s > -\frac{1}{2}$. The map*

$$\Theta_v : \{v \in H^{s+1}_\tau(\mathbb{R}) : v > -1\} \ni v \to \tau v - \frac{1}{2}\partial \log(1+v) \in H^s_\tau(\mathbb{R})$$

is a diffeomorphism with all (including higher) Fréchet derivatives of Θ and Θ^{-1} bounded by a constant depending on $\tau^{-s}\|v\|_{H^s_\tau}$ and $\inf v + 1$.

(4) *Suppose that $s > -\frac{1}{2}$ and that the equivalent conditions of 1 hold. Then $Q_w \subset H^s_\tau$ is equicontinuous if and only if $Q_v \subset H^{s+1}_\tau$ is equicontinuous.*

(5) *Suppose that the equivalent condition of* (1) *hold, then Q_w is tight in H^s_τ if and only if Q_v is tight in H^{s+1}_τ.*

(6) *Suppose that the equivalent condition of* (1) *hold. Then $Q_w \subset H^s_\tau$ is precompact if and only if $Q_v \in H^{s+1}_\tau$ is precompact.*

4.4.1. *Proof of Theorem 4.13.*

Proof. We study the map

$$v \to \tau v - \frac{1}{2}\partial_x \log(1+v) =: w$$

with inverse

$$v = \frac{1}{\tau (L_w)^{-1}(1)} - 1.$$

The lower bound $1 + v \geq \delta$ of (1) is equivalent to the upper bound

$$(L_w)^{-1} 1 \leq \delta^{-1}.$$

Let $-\frac{1}{2} < s < \frac{1}{2}$. We combine the embedding $H^{s+1} \subset C^{s+\frac{1}{2}}$ and $x \to \int_y^x w \in H^{s+1}_{loc}$ if $w \in H^s$ with Young's inequality

$$\left| \int_y^x w d\sigma \right| \leq c|x-y|^{s+\frac{1}{2}} \|w\|_{H^s} \leq \tau |x-y| + c\tau^{-\frac{1+2s}{1-2s}} \|w\|_{H^s}^{\frac{2}{1-2s}}.$$

Again by Schur's lemma

$$\|L_w^{-1} f\|_{L^\infty} \leq \frac{1}{\tau} \exp\left(c\tau^{-\frac{1+2s}{1-2s}} \|w\|_{H^s}^{\frac{2}{1-2s}} \right) \|f\|_{L^\infty}$$

which implies the lower bound

$$1 + v \geq \tau \exp(-c\tau^{-\frac{1+2s}{1-2s}} \|w\|_{H^s}^{\frac{2}{1-2s}}\Big). \tag{4.55}$$

Trivially we have

$$\|w\|_{W^{-1,\infty}_\tau} \leq \|v\|_{L^\infty} + \left\| \frac{1}{1+v} \right\|_{L^\infty}, \tag{4.56}$$

We write

$$\log(1+v) = \int_0^1 \frac{1}{1+tv} dtv = \psi^{-1} v$$

with $\psi \geq \delta$. Let $\phi = \log(1+v)$. Then

$$\phi_x + 2\tau\psi\phi = 2w$$

hence, for $1 \leq p \leq \infty$

$$\|\phi\|_{L^p} \leq \frac{2}{\delta\tau} \|w\|_{W^{-1,p}_\tau} \tag{4.57}$$

hence

$$\|v\|_{L^\infty} \leq \exp\Big(\frac{1}{\delta\tau} \|w\|_{W^{-1,\infty}}\Big). \tag{4.58}$$

The bounds (4.55), (4.56) and (4.58) are the basis for the remaining estimates.

□

We simplify the estimate a bit by substituting $w = \tau v - \frac{v_x}{1+v}$. We obtain linear combinations of

$$(\tau + \partial \frac{1}{2(1+\tilde{v})})^{-1} \partial^{1+j_0} (1+v)^{-M} \tau^{2l} \prod_{k=1}^{K} v^{(j_k)}$$

with (denoting by $D = 1 + \sum_{k=0}^{K} j_k$ the total number of derivatives) $2L + D = 2N + 2$.

5. Weak solutions

In this section we study weak solutions to equations of the Nth KdV equation and the Nth Gardner equation. Under weak regularity conditions w is a weak solution to the Nth Gardner equation if and only if $u = w_x + 2\tau w + w^2$ is a weak solution to the Nth KdV equation. This reduces the proof of the main theorem to a study of the weak solutions to the Gardner hierarchy.

5.1. **Calculus estimates in Sobolev spaces.** In almost all sections below we will need estimates of differential monomials in L^p or weighted L^p.

Definition 5.1. *Let $1 \le p \le \infty$, $N \ge 0$, $\tau > 0$ and $I = (a,b)$. We define*

$$W_\tau^{N,p}(I) = \Big\{ f = \sum_{j=0}^{N} \tau^{N-j} \partial^j f_j : f_j \in L^p(I) \Big\}$$

with

$$\|f\|_{W_\tau^{-N,p}(I)} = \inf \Big\{ \Big(\sum_{j=0}^{N} \|f_j\|_{L^p(I)}^p \Big)^{1/p} : f = \sum_{j=0}^{N} \tau^{N-j} \partial^j f_j \Big\}.$$

We define for $N \ge 0$ and $\tau > 0$

$$\|g\|_{W_\tau^{N,p}(I)} = \sum_{j=0}^{N} \tau^{N-j} \|g^{(j)}\|_{L^p(I)}$$

and for $\tau > 0$ and $s \in \mathbb{R}$

$$\|f\|_{H_\tau^s(\mathbb{R})} = \|(\tau^2 + \xi^2)^{s/2} \hat{f}\|_{L^2(\mathbb{R})}$$

and for an interval I

$$\|f\|_{H_\tau^s(I)} = \inf\{ \|\tilde{f}\|_{H_\tau^s}, f = \tilde{f} \text{ on } I \}$$

It is obvious that the restriction to smaller intervals in bounded linear operator of norm 1. There exists a bounded extension operator to functions supported on twice the interval.

Lemma 5.2. *The following norms are equivalent: Suppose that $n \in \mathbb{Z}$, $\tau > 0$ $\tau r \ge 1$ and $1 \le p \le \infty$. Then*

$$\|f\|_{W_\tau^{n,p}(\mathbb{R})} \sim \Big\| \|f\|_{W_\tau^{n,p}((k-1)r,(k+1)r))} \Big\|_{l^p}.$$

Proof. It suffices to verify the claim for $\tau = 1$ and $r = 1$. The case $n \geq 0$ is obvious. Let $n > 0$, $f = \sum_{j=0}^n \partial^j f_j$ with $\sum_{j=0}^n \|f_j\|_{L^p(\mathbb{R})}^p \sim \|f\|^p_{W^{-n,p}(\mathbb{R})}$. Then

$$\big\| \|f\|_{W^{-n,p}(k-1,k+1)} \big\|_{l^p}^p \leq \sum_k \sum_{j=0}^n \|f_j\|^p_{L^p(k-1,k+1)} = 2 \sum_{j=0}^n \|f_j\|^p_{L^p(\mathbb{R})}$$

if $1 \leq p < \infty$ with obvious modifications if $p = \infty$.

For the opposite direction choose $f_{k,j} \in L^p(k-1,k+1)$ so that $f = \sum_{j=0}^n \partial^j f_j$ on $[k-1,k+1]$, $\sum_{j=0}^n \|f_{k,j}\|^p_{L^p(k-1,k+1)} \leq 2\|f\|^p_{W^{-n,p}(k-1,k+1)}$, choose a partition of unity $\sum \eta(x-k) = 1$ with $\operatorname{supp} \eta \subset (-1,1)$, $\eta = 1$ on $(-1/4, 1/4)$. Then

$$f = \sum_{k\in\mathbb{Z}} \eta(x-k) \sum_{j=0}^n \partial^j f_{k,j} = \sum_{k\in\mathbb{Z}} \sum_{j=0}^n \partial^j \eta f_j - \sum_{l=1}^{j-1} \binom{j}{l} \partial^{j-l}(\eta^{(l)}(x-k) f_j).$$

We set

$$f_j = \sum_{k\in\mathbb{Z}} \eta(x-k) f_j - \sum_{l=j+1}^n \binom{l}{j} \eta^{(l-j)}(x-k) f_l$$

and obtain

$$\sum_{j=0}^n \|f_j\|^p_{L^p} \leq c \sum_k \|f\|^p_{L^p(k-1,k+1)}$$

again with obvious modifications if $p = \infty$. □

We turn to an interpolation inequality, for which we provide a proof for completeness.

Lemma 5.3. *Let* $0 \leq j < s$, $2 \leq q \leq \infty$, $2 \leq r < \infty$ *and*

$$\frac{s}{r} = \frac{s-j}{q} + \frac{j}{2}.$$

Then

$$\|f^{(j)}\|^s_{L^r} \leq c \|f\|^{s-j}_{L^q} \|f^{(s)}\|^j_{L^2}.$$

Proof. The lemma relies on three elementary estimates. Suppose that

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{r},\ 2 \leq r \leq \infty,\ 1 \leq p, q \leq \infty$$

Then

$$\|f'\|^2_{L^r} \leq (r-1)\|f\|_{L^p}\|f''\|_{L^q}. \tag{5.1}$$

This follows from

$$\begin{aligned}\int |f'|^r dx &= \int \partial_x(|f'|^{r-2} f' f) dx - (r-1) \int f |f'|^{r-2} f'' dx \\ &\leq (r-1)\|f'\|^{r-2}_{L^r} \|f\|_{L^p} \|f''\|_{L^q}\end{aligned}$$

which implies (5.1). There is a version for fractional derivatives. Let $0 < s < 1$ and $1 \leq p, q \leq \infty$. We define the homogeneous Besov norm

$$\|f\|_{\dot{B}^s_{pq}} = \left(\int_0^\infty \left(h^{-s} \|f(.+h) - f\|_{L^p} \right)^q \frac{dh}{h} \right)^{1/q}$$

with the obvious modification for $q = \infty$. Then $\dot{B}^s_{pq} \subset \dot{B}^2_{p\tilde{q}}$ whenever $\tilde{q} \leq q$ and

$$\|f\|_{\dot{B}^s_{22}} = c \||\xi|^s \hat{f}\|_{L^2} = \|f\|_{\dot{H}^s}.$$

Let $\frac{1}{r} = \frac{1-s}{q} + \frac{s}{p}$ then

$$\|h^{-s} f(.+h) - f\|_{L^r} = \||f(.+h)-f|^{1-s}(h^{-1}|f(.+h)-f(.)|^s\|_{L^r} \le (2\|f\|_{L^q})^{1-s}\|f'\|^s_{L^p}$$

hence

$$\|f\|_{\dot B^s_{r,\infty}} \le 2\|f\|_{L^p}\|f'\|_{L^q}. \tag{5.2}$$

Here we used

$$\|f'\|_{L^r} = \sup_{h>0} h^{-1}\|f(.+h) - f\|_{L^r}.$$

Similarly

$$\begin{aligned} h^{-r} \int & |f(x+h) - f(x)|^{r-2}(f(x+h) - f(x)(f(x+h) - f(x))dx \\ &= h^{-r}\int f(x)\Big(|f(x+h)-f(x)|^{r-2}(f(x+h)-f(x)) \\ &\qquad - |f(x)-f(x-h)|^{r-2}(f(x)-f(x-h)\Big)dx \\ &\le \|f\|_{\dot H^{s-1}}\Big(h^{-1}\|f(.+h)-f\|_{L^r}\Big)^{r-2}\||h|^{-1-s}f(.+h) - 2f(x) + f(.-h)\|_{L^q} \end{aligned}$$

hence, if

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{r}, \quad r \ge 2$$

$$\|f'\|^2_{L^r} \le \|f\|_{\dot B^{1-s}_{q,\infty}}\|f'\|_{\dot B^s_{p,\infty}}. \tag{5.3}$$

Recursively we obtain for $j \le n$, $2 \le p, q, r$ satisfying

$$\frac{n}{r} = \frac{n-j}{p} + \frac{j}{q}$$

$$\|f^{(j)}\|^n_{L^r} \le c\|f\|^{n-j}_{L^p}\|f^{(n)}\|^j_{L^q}. \tag{5.4}$$

Indeed, suppose this estimate holds for n. Let $1 \le j \le n+1$ and

$$\frac{1}{r_j} = \frac{n+1-j}{n+1}\frac{1}{p} + \frac{j}{n}\frac{1}{q}$$

Then

$$\begin{aligned} \|f^{(j)}\|^{nj}_{L^{r_j}} &\lesssim \Big(\|f'\|^{n+1-j}_{L^{r_1}}\|f^{(n+1)}\|^{j-1}_{L^q}\Big)^j \\ &\le c\|f\|^{(n+1-j)(j-1)}_{L^p}\|f^{(j)}\|^{n+1-j}_{L^{r_j}}\|f^{(n+1)}\|^{(j-1)j}_{L^q} \end{aligned}$$

where

$$nj - (n+1-j) = (j-1)(n+1)$$

which implies estimate (5.4) for $s \in \mathbb{N}$. The general case follows by using (5.2) and (5.3) in addition. □

A particular instance is the following. Let $0 \le d \le N$ and $p \ge 2$. Then

$$\|u^{(d)}\|_{L^{\frac{N+2}{1+d}}} \le c\|u\|^{\frac{N-2d}{N}}_{L^{N+2}}\|u^{(N/2)}\|^{\frac{2d}{N}}_{L^2}. \tag{5.5}$$

A weighted variant is

$$\|\operatorname{sech}(x)u^{(d)}\|_{\frac{N+2}{1+d}} \le c\|u\|^{\frac{N-2d}{N}}_{L^{N+2}}\|\operatorname{sech}(x)u\|^{\frac{2d}{N}}_{H^{N/2}}. \tag{5.6}$$

To see this we apply a standard extension argument to deduce from (5.5) on intervals I of length 2

$$\|u^{(d)}\|_{L^{\frac{n*2}{1+d}}(I)} \le c\|u\|_{L^{N+2}(I)}^{\frac{N-2d}{N}} \|u\|_{H^{N/2}(I)}^{\frac{2d}{N}}.$$

We multiply by the weight, take the power $\frac{n+2}{1+d}$, add the intervals to arrive at (5.6).

To proceed, we show the following multilinear estimates.

Lemma 5.4. *If $h = \prod_{j=1}^{N+2-d} u^{(\alpha_j)}$ is a product with a total number of derivatives $d = \sum \alpha_j \le N$ then, if no term carries more than $N/2$ derivatives. Then*

$$\prod_{j=1}^{N+2-d} \|u^{(\alpha_j)}\|_{L^{\frac{N+2}{1+\alpha_j}}} \le c \begin{cases} \|u\|_{L^{N+2}}^{N+2-d-\frac{d}{N}} \|u^{(N/2)}\|_{L^2}^{\frac{d}{N}} \\ \|u\|_{L^{N+2}}^{N+2} + \|u^{(N/2)}\|_{L^2}^2 \\ \|u\|_{L^2}^{N} \|u^{(\frac{N+d}{4})}\|_{L^2}^2 \end{cases} \tag{5.7}$$

and

$$\prod_{j=1}^{N+2-d} \|\operatorname{sech}^{\frac{2(1+\alpha_j)}{N+2}}(x) u^{(\alpha_j)}\|_{L^{\frac{N+2}{1+\alpha_j}}} \le c \begin{cases} \|u\|_{L^2}^{N-d} \|\operatorname{sech}(x)u\|_{L^2}^{1-\frac{d}{N}} \|\operatorname{sech}(x)u\|_{H^{N/2}}^{1+\frac{d}{N}} \\ \|u\|_{L^2}^{N} \|\operatorname{sech}(x)u^{(\frac{N+d}{4})}\|_{H^{\frac{N+d}{4}}}^2. \end{cases} \tag{5.8}$$

If $h = \prod_{j=1}^{N+2-d} u_j^{(\alpha_j)}$ is a product with a total number of derivatives $d = \sum \alpha_j \le N$ then, if no term carries more than $N/2$ derivatives

$$\begin{aligned} \int |h| dx &\le \prod_{j=1}^{N+2-d} \|u_j^{(\alpha_j)}\|_{L^{\frac{N+2}{1+\alpha_j}}} \\ &\le c \prod_{j=1}^{N+2-d} \left(\|u_j\|_{L^{N+2}}^{N+2-d-\frac{2\alpha_j}{N}} \|u_j^{(N/2)}\|_{L^2}^{\frac{2\alpha_j}{N}} \right)^{\frac{1}{N+2-d}} \\ &\le c \prod_{j=1}^{N+2-d} \left(\|u_j\|_{L^{N+2}}^{N+2} + \|u_j^{(N/2)}\|_{L^2}^2 \right)^{\frac{1}{N+2-d}} \\ \|u_j^{(\alpha_j)}\|_{L^{\frac{N+2}{1+\alpha_j}}} &\le c\|u_j\|_{L^2}^{1-\frac{4\alpha_j(N+1)+2N}{(N+2)(N+d)}} \|u_j^{(\frac{N+d}{4})}\|_{L^2}^{\frac{4\alpha_j(N+1)+2N}{(N+2)(N+d)}} \end{aligned} \tag{5.9}$$

and

(5.10)

$$\begin{aligned} \int \operatorname{sech}^2(x)|h| dx &\le \prod_{j=1}^{N+2-d} \|\operatorname{sech}^{\frac{2(1+\alpha_j)}{d+m}}(x) u_j^{(\alpha_j)}\|_{L^{\frac{N+2}{1+\alpha_j}}} \\ &\le c \prod_{j=1}^{N+2-d} \left(\|u_j\|_{L^2}^{N-d} \|\operatorname{sech}(x)u_j\|_{L^2}^{2-\frac{2\alpha_j}{N}} \|\operatorname{sech}(x)u_j^{(N/2)}\|_{H^{N/2}}^{\frac{2\alpha_j}{N}} \right)^{\frac{1}{N+2-d}} \\ \|u_j^{(\alpha_j)}\|_{L^{\frac{N+2}{1+\alpha_j}}} &\le c\|u_j\|_{L^2}^{1-\frac{4\alpha_j(N+1)+2N}{(N+2)(N+d)}} \|\operatorname{sech}(x)u_j^{(\frac{N+d}{4})}\|_{L^2}^{\frac{4\alpha_j(N+1)+2N}{(N+2)(N+d)}}. \end{aligned}$$

Proof. It suffices to prove (5.9) and (5.10). The inequalities (5.7) and (5.8) are immediate consequences. The first inequality in (5.9) is a consequence of Hölder's inequality, for the second estimate we apply lemma 5.3. The third estimate follows by Young's inequality.

To prove the fourth estimate we recall that for $0 < s < 1$
$$f = c_s |x|^{-1+s} * f^{(s)}$$
from which we obtain the fractional Sobolev embedding
$$\|f\|_{L^p} \le c \|u^{(\frac12 - \frac1p)}\|_{L^2}$$
by the Hardy Littlewood Sobolev inequality. Let $d \ge 0$ and $p \ge 2$. Then by the fractional Sobolev embedding
$$\|u^{(d)}\|_{L^p} \le c \|u^{(d+\frac12 - \frac1p)}\|_{L^2}$$
and the interpolation inequality for $0 \le \sigma \le s$
$$\|f\|_{\dot H^\sigma} \le \|f\|_{L^2}^{1-\frac{\sigma}{s}} \|f\|_{\dot H^s}^{\frac{\sigma}{s}}$$
implies the fourth estimate.

We turn to (5.10). Again the first inequality is Hölder's inequality. We continue with the Hardy-Littlewood Sobolev inequality. Again w interpolate on intervals for fixed length, multiply by the weight, square and sum over the intervals to arrive at the second inequality of (5.10). □

5.2. **The nonlinearities of the KdV and the Gardner equations.** We recall the structure of the Nth equation in each of the two hierarchies involved. Each of them can be written as
$$\psi_t - (-1)^N \psi^{(2N+1)} = \partial F_N(\psi),$$
where the structure of F_N is described by Theorem 3.10. For later purposes we want to pull out as many derivatives as possible:

Lemma 5.5. *For KdV, we have,*
$$F_N^{\mathrm{KdV}}(u) = \sum_{K,(j_k)_{0\le k\le K}} c_{K,(j_k)} \partial^{j_0} \prod_{k=1}^K u^{(j_k)},$$
$$2 \le K \le N+1, \qquad K + \frac12 \sum_{k=0}^K j_k = N+1, \quad j_k \le N+1-K-\frac{j_0}{2}, \ \textit{if } j \ge 1$$
and for Gardner,
$$F_N^{\mathrm{Gardner}}(w) = \sum_{K,(j_k)_k,l} c_{(j_k)_k,K,l} \partial^{j_0} \tau^l \prod_{k=1}^K w^{(j_k)},$$
$$2 \le K \le 2N+1, \qquad l + K + \sum_{k=0}^K j_k = 2N+1,$$
$$j_k \le \frac{2N+1-K-l-j_0}{2} \ \textit{if } j \ge 1$$
where $K + l$ is always odd.

Proof. Theorem 3.10 describes the structure of the Hamiltonians. For KdV, $F_N(u)$ is a sum over differential monomials $\prod_{k=1}^K u^{(j_k)}$ with
$$2 \le K \le N+1, \qquad K + \frac12 \sum_{k=1}^K j_k = N+1,$$

and for Gardner, $F_N(w)$ is a sum over $\tau^l \prod_{k=1}^K w^{(j_k)}$ with

$$2 \le K \le 2N+1, \qquad l + K + \sum_{k=1}^K j_k = 2N+1.$$

For KdV and Gardner we reduce the highest number of derivatives falling on one factor until there are at least two factors with the highest number of derivatives. This can be done as follows: Consider a differential monomial $\prod_{k=1}^K u^{(j_k)}$. We order the factors so that $k \to j_k$ decreases monotonically. If $j_1 > j_2 = j_l > j_{l+1}$ we write

$$\prod_{k=1}^K u^{(j_k)} = \frac{1}{l}\partial (u^{(j_2)})^l \prod_{k=l+1}^K u^{(j_k)} - \frac{1}{l}(u^{(j_2)})^l \partial \prod_{k=l+1}^K u^{(j_k)}.$$

$K+l$ being odd follows from the formula

$$\partial_x \frac{\delta H_N^{\mathrm{KdV}}}{\delta u}(w_x + 2\tau w + w^2) = (\partial_x + 2\tau + 2w)\frac{\delta H_N^{\mathrm{Gardner}}}{\delta w}(w).$$

□

The Gardner Hamiltonian H_N^{Gardner} is an integral over $\frac{1}{2}(w^{(N)})^2$ plus a sum of differential monomials

$$\tau^l \prod_{k=1}^K w^{(j_k)}$$

with $K \ge 3$, $l + \sum_k j_k + K = 2N+2$, $l+K$ even, and no term carries more then $2N - l - K$ derivatives. We apply (5.9) of Lemma 5.4:

$$\Big\| \tau^l \prod_{k=1}^K w^{(j_k)} \Big\|_{L^1} \le c\tau^l \|w\|_{L^2}^{K-2} \|w^{(N - \frac{K-2}{4} - \frac{l}{2})}\|_{L^2}^2 \le c(\tau^{-1/2}\|w\|_{L^2})^{K-2}\|w\|^2_{H^N_\tau}.$$

hence

$$\Big| H_N^{\mathrm{Gardner}}(w) - \|w^{(N)}\|_{L^2}^2 \Big| \le c(1+\tau^{-1/2}\|w\|_{L^2})^{2N-1}(\tau^{-1/2}\|w\|_{L^2})\|w\|^2_{H^N_\tau}. \tag{5.11}$$

The same argument shows with $u = w_x + \tau w$

$$\begin{aligned} \Big| H_N^{\mathrm{KdV}} - \|u^{(N)}\|_{L^2}^2 \Big| &\le c(1+\tau^{-1/2}\|w\|_{L^2})^{N-1}(\tau^{-1/2}\|w\|_{L^2})\|w\|^2_{H^{N+1}_\tau} \\ &\le c(1+\tau^{-1/2}\|u\|_{H^{-1}_\tau})^{N-1}(\tau^{-1/2}\|u\|_{H^{-1}_\tau})\|u\|^2_{H^N_\tau}. \end{aligned} \tag{5.12}$$

In the same fashion

$$\Big| \int \frac{\delta H_N^{\mathrm{Gardner}}}{\delta w}\phi dx \Big| \le c(1+\tau^{-1/2}\|w\|_{L^2})^{2N-1}\Big(\tau^{-1/2}\|\phi\|_{L^2}\|w\|^2_{H^N_\tau} + \tau^{-1/2}\|w\|_{L^2}\|w\|_{H^N_\tau}\|\phi\|_{H^N_\tau}\Big).$$

Let γ be slowly varying of rate τ. We can localize this estimate and add up the intervals of length τ^{-1} to obtain

$$\Big\| \gamma^2 \big(\frac{\delta}{\delta w} H_N^{\mathrm{Gardner}} - w^{(2N)}\big)\Big\|_{H^{-N}_\tau} \le c\tau^{-N-\frac{1}{2}}\Big(1+\tau^{-1/2}\|w\|_{L^2}\Big)^{2N-1}\|\gamma w\|^2_{H^N_\tau}. \tag{5.13}$$

We claim the similar estimate

$$\Big\| \gamma^2 \big(\frac{\delta}{\delta u} H_N^{\mathrm{KdV}} - u^{(2N)}\big)\Big\|_{H^{-N}_\tau} \le c\tau^{-N+1/2}\Big(1+\tau^{-1/2}\|u\|_{H^{-1}_\tau}\Big)^{N-1}\|\gamma u\|^2_{H^{N-1}_\tau}. \tag{5.14}$$

By Lemma 5.5 we have to bound a sum of monomials

$$\partial^{j_0} \prod_{k=1}^{K} u^{(j_k)}$$

where $2 \le K \le N+1$, the total number of derivatives being $2(N+1-K)$, with at most half the derivatives on a single factor. We set $u = \partial_x v + \tau v$ so that the total number of derivatives becomes $d = 2(N+1) - K - j_0$ (we count factors τ like derivatives). By Lemma 5.4,

$$\begin{aligned} \tau^{1/2} \Big\| \prod_{k=1}^{K} v^{(1+\alpha_k)} \Big\|_{H_\tau^{-1}} &\le \Big\| \prod_{k=1}^{K} v^{(1+\alpha_k)} \Big\|_{H_\tau^{-1}} \\ &\le c \|v\|_{L^2}^{K-2} \|v^{(N+\frac12 - \frac{K}{4} - \frac{j_0}{2})}\|_{L^2}^2 \\ &\le c\tau^{-j_0 - \frac{K-2}{2}} \|u\|_{H_\tau^{-1}}^{K-2} \|u\|_{H_\tau^{N-1}}^2 \end{aligned}$$

which implies (5.14) is the same fashion as we proved (5.13).

The estimates (5.14) and (5.13) allow to define weak solutions to equations of the KdV and the Gardner hierarchy.

Proposition 5.6. *The following estimates hold for $R \ge 1$ and an interval I of length 1*

(5.15)
$$\Big\| \operatorname{sech}^2(x/R) \partial \frac{\delta}{\delta u} H_N^{\mathrm{KdV}}(u) \Big\|_{H^{-N-2}} \le c(\|u\|_{H^{-1}} \Big(\|\operatorname{sech}^2 u\|_{H^{N-1}} + \|\operatorname{sech} u\|_{H^{N-1}}^2 \Big)$$

(5.16)
$$\Big\| \operatorname{sech}^2(x/R) \partial \frac{\delta}{\delta w} H_N^{\mathrm{Gardner}}(w) \Big\|_{H^{-N-2}} \le c(\|w\|_{L^2}) \Big(\|\operatorname{sech}^2 w\|_{H^{N}} + \|\operatorname{sech} w\|_{H^{N}}^2 \Big)$$

The following estimates hold for $R \ge 1$ and an interval I of length 1

(5.17)
$$\Big\| \operatorname{sech}^2(x/R) \partial \frac{\delta}{\delta u} H_N^{\mathrm{KdV}}(u) \Big\|_{H^{-N-3}} \le c(\|u\|_{H^{-1}}) \Big(\|\operatorname{sech}^2 u\|_{H^{N-2}} + \|\operatorname{sech} u\|_{H^{N-2}}^2 \Big)$$

(5.18)
$$\Big\| \operatorname{sech}^2(x/R) \partial \frac{\delta}{\delta w} H_N^{\mathrm{Gardner}}(w) \Big\|_{H^{-N-2}} \le c(\|w\|_{L^2})(1+\|w\|_{L^\infty}) \Big(\|\operatorname{sech}^2 w\|_{H^{N-1}} + \|\operatorname{sech} w\|_{H^{N-1}}^2 \Big)$$

The proposition is a consequence of Theorem 3.10 and (5.9) resp (5.10) of Lemma 5.4.

The proposition allows to define weak solutions in natural regularity classes.

Definition 5.7. *Let $I = (a,b)$ be an open interval. We call*

$$u \in L^\infty(I, H^{-1}) \qquad \text{with } u^{(N-1)} \in L^2_{loc}(I \times \mathbb{R})$$

a weak solution to the Nth KdV equation if it satisfies the equation in the distributional sense. We call

$$w \in L^\infty(I, L^2) \qquad \text{with } w^{(N)} \in L^2_{loc}(I \times \mathbb{R})$$

a weak solution to the Nth Gardner equation if it satisfies the equation in a distributional sense. We call $u \in L^\infty(H^{N-1}(\mathbb{X}))$ a weak solution to the Nth KdV equation and $w \in L^\infty H^{N-1}$ a weak solution to the Nth Gardner equation if they are weak solutions to th corresponding equation.

We define $L^2_u(I\times\mathbb{R})\subset L^2_{loc}(I\times\mathbb{R})$ by

$$\|f\|_{L^2_u(I\times\mathbb{R}^n)}=\sup_k \|f\|_{L^2(I\times(k,k+1))}.$$

We will relate weak solutions to different Gardner equations to another, and to weak solutions to the KdV equation via the modified Miura map. Since

$$\begin{aligned}\|\partial_x\psi\|_{L^2}^2+\int(u\psi-v\partial_x\psi)\psi dx &\ge \|\psi_x\|_{L^2}^2-\|u\|_{L^2}\|\psi\|_{L^2}^{3/2}\|\psi_x\|_{L^2}^{\frac12}-\|v\|_{L^2}\|\psi\|_{L^2}\|\psi_x\|_{L^2}\\ &\ge -\big(\frac12\|v\|^2+\frac{3}{2^{\frac{10}{3}}}\|u\|_{L^2}^{\frac43}\big)\|\psi\|_{L^2}^2\end{aligned}$$

we see that $u+v_x$ lies in the range of the Miura map if

$$\tau^2>\frac12\sup_{t\in I}\|u(t)\|_{H^{-1}}^2+\frac{3}{2^{\frac{10}{3}}}\sup_{t\in I}\|u(t)\|_{H^{-1}}^{\frac43} \tag{5.19}$$

Suppose that $0<\tau_1<\tau_2$. Then $w+2\tau_1 w+w^2$ is in the range of the τ_2 Miura map.

Theorem 5.8. *Let $1\le\tau_1<\tau_2$, assume $w_1\in L^\infty(I;L^2)$ with $w_1^{(N)}\in L^2_u(I\times\mathbb{R})$ is a weak solutions to the Nth τ_1 Gardner equation. Then $u=\partial_x w_1+2\tau_1 w_1+w_1^2$ satisfies $u\in L^\infty(I;H^{-1})$ and $u^{(N-1)}\in L^2_u(I\times\mathbb{R})$. Moreover it is a weak solution to the Nth KdV equation. Define w_2 by*

$$\partial_x w_2+2\tau_2 w_2+w_2^2=\partial_x w_1+2\tau_1 w_1+w_1^2.$$

It satisfies $w_2\in L^\infty(I;L^2)$ and $w_2^{(N)}\in L^2_u(I\times\mathbb{R})$ if this holds for w_1. Moreover it is a weak solution to the Nth τ_2 Gardner equation if and only if w_1 is a weak solution to the Nth τ_1 Gardner equation.

Suppose that $u\in L^\infty(I;L^2)$ with $u^{(N-1)}\in L^2_u(I\times\mathbb{R})$ is a weak solution to the Nth KdV equation,

$$-\partial^2+u(t)+\tau^2$$

is positive definite uniformly in t (which holds if $\tau>0$ satisfies (5.19)) and w is defined by

$$w_x+2\tau w+w^2=u.$$

Then $w\in L^\infty(\mathbb{R};L^2)$, $w^{(N)}\in L^2_u(I\times\mathbb{R})$. If moreover $w\in L^\infty$ or $w\in L^\infty(I;H^N)$ then w is a weak solution to the Nth Gardner equation.

Proof of Theorem 5.8. **Step 0: The spaces.** Let $w_1\in L^\infty(I;L^2)$. Lemma 4.2 implies

$$u:=\partial_x w_1+2\tau_1 w_1+w_1^2\in L^\infty(I;H^{-1}(\mathbb{R}))$$

and

$$-\partial^2+u(t)+\tau_1^2$$

is uniformly positive definite and hence in the range of the τ_2 modified Miura map and $w_2\in L^\infty(I;L^2)$ if $w_1\in L^\infty(I;L^2)$. Since

$$\|f\|_{L^2_u(I\times\mathbb{R})}\sim\sup_{x_0}\|\operatorname{sech}(\kappa(x-x_0))f\|_{L^2(I\times\mathbb{R})}$$

we can apply Proposition 4.8 with N and $\gamma=\operatorname{sech}(\kappa(x-x_0))$ and $\kappa<\tau$ to see that in addition

$$w_1^{(N)}\in L^2_u(I\times\mathbb{R})\Longleftrightarrow u^{(N-1)}\in L^2_u\Longleftrightarrow w_2^{(N)}\in L^2_u.$$

Step 1: Weak solution to Gardner define weak solutions to KdV. We first prove that the modified Miura map maps weak solutions to Nth Gardner equation to weak solutions to Nth KdV. Suppose that $w \in L^\infty(I; L^2(\mathbb{R}))$ with $w^{(N)} \in L^2_u(I \times \mathbb{R})$ is a weak solution to the Nth τ Gardner equation and $\tau_1 \geq \tau$. We define $w_\varepsilon = J_\varepsilon w := j_\varepsilon * w$, for a mollifier j_ε, and

$$u_\varepsilon = w_{\varepsilon,x} + 2\tau w_\varepsilon + w_\varepsilon^2 \tag{5.20}$$

We assumed that w is a weak solution hence w_ε satisfies

$$\partial_t w_\varepsilon = \partial_x J_\varepsilon \frac{\delta H_N^{\text{Gardner}}}{\delta w}(w)$$

Let $\phi \in C_c^\infty(\mathbb{R})$. Then

$$t \to \int \frac{\delta H_N^{\text{Gardner}}}{\delta w} \phi dx \in L^1(I),$$

hence, since w is assumed to be a weak solution to the Nth Gardner equation

$$(t \to \int w\phi dx) \in W^{1,1}(I)$$

and for all $n > 0$

$$\sup_x \|\partial^n w_\varepsilon(t,x)\|_{W^{1,1}(I)} < \infty.$$

We calculate using the chain rule for functions in $W^{1,1}$

$$\begin{aligned}
\partial_t u_\varepsilon &= \partial_t(\partial_x w_\varepsilon + 2\tau w_\varepsilon + w_\varepsilon^2) \\
&= (\partial + 2\tau + 2w_\varepsilon) J_\varepsilon \partial_t w \\
&= (\partial_x + 2\tau + 2w_\varepsilon)\partial_x J_\varepsilon \frac{\delta H_N^{\text{Gardner}}}{\delta w}(w).
\end{aligned}$$

Since w_ε and u_ε are smooth for almost all $t \in I$, (5.20) implies

$$(\partial_x + 2\tau + 2w_\varepsilon)\partial_x \frac{\delta H_N^{\text{Gardner}}}{\delta w}(w_\varepsilon) = \partial_x \frac{\delta H_N^{\text{KdV}}}{\delta u}(u_\varepsilon).$$

Using this identity and the fact that J_ε commutes with the linear part of the equation, we obtain

$$\partial_t u_\varepsilon - \partial \frac{\delta H_N^{\text{KdV}}}{\delta u}(u_\varepsilon) = (\partial + 2\tau + 2w_\varepsilon)\partial_x \big(J_\varepsilon F_N^{\text{Gardner}}(w) - F_N^{\text{Gardner}}(w_\varepsilon)\big). \tag{5.21}$$

The modified Miura map is continuous as a map from

$$L^\infty(I; L^2) \cap L^2 H_u^N \to L^\infty(I; H^{-1}) \cap L^2 H_u^{N-1}.$$

By (5.14)

$$L^\infty(I; H^{-1}) \cap L^2 H_u^{N-1} \ni u \to \frac{\delta H_N^{\text{KdV}}}{\delta u}(u) \in L^2 H_u^{-N}$$

hence in $L^\infty(H_{loc}^{-N-1})$

$$\partial \frac{\delta H_N^{\text{KdV}}}{\delta u}(u_\varepsilon) \to \partial \frac{\delta H_N^{\text{KdV}}}{\delta u}(u).$$

By construction $J_\varepsilon F_N^{\text{Gardner}} \to F_N^{\text{Gardner}}$ and $F_N^{\text{Gardner}}(w_\varepsilon) \to F_N^{\text{Gardner}}(w)$ in $L^2 H_u^{-N}$ by the bound (5.13). This gives in the sense of distributions

$$(\partial + 2\tau)\partial(J_\varepsilon F_N^{\text{Gardner}}(w) - F_N^{\text{Gardner}}(w_\varepsilon)) \to 0$$

It remains to verify

$$w_\varepsilon \partial_x\big(\big(J_\varepsilon F_N^{\text{Gardner}}(w) - F_N^{\text{Gardner}}(w_\varepsilon)\big) \to 0$$

in a distributional sense. We pull out the derivative and the claim follows from the bound

$$\|\gamma^2(\partial_x w_\varepsilon)(F_N^{\text{Gardner}}(w) - w^{(2N)})\|_{L^1} \le c\tau(1+\tau^{-\frac12}\|w\|_{L^2})^{2N}\|\gamma w\|^2_{H^N_\tau}. \tag{5.22}$$

Again we need do bound differential monomials of $K+1 \ge 3$ factors with $d = 2N+3-K-1-l-\alpha_0$ derivatives with at most 1 or half the derivatives on one factor,

$$\begin{aligned} \tau^l \Big\| w_x \partial_x \prod_{k=1}^K w^{(\alpha_k)} \Big\|_{L^1} &\le c\tau^l \|w\|_{L^2}^{K-1} \|w^{(N+\frac12-\frac{K-1}{4}-\frac{l+\alpha_0}{2})}\|^2_{L^2} \\ &\le \tau^{1-\frac{K-1}{2}-\alpha_0}\|w\|_{L^2}^{K-1}\|w\|^2_{H^N_\tau}. \end{aligned} \tag{5.23}$$

Thus u is a weak solution to the Nth KdV hierarchy.

Step 2: Weak solutions to the Nth Gardner equation with different τ Changing the notation slightly we assume $2 \le \tau \le \tau_1$, that w lies in the Nth Gardner Kato smoothing space and that it is a weak solution of the N th τ Gardner equation, and

$$w_x^{\tau_1} + 2\tau_1 w^{\tau_1} + (w^{\tau_1})^2 = w_x + 2\tau w + w^2.$$

We want to prove that w^{τ_1} is a weak solution to the Nth τ_1 Gardner equation. We regularize the solution as above $w_\varepsilon = j_\varepsilon * w =: J_\varepsilon w$. Again

$$\partial_t w_\varepsilon - \partial \frac{\delta H_N^{\text{Gardner}}}{\delta w}(w_\varepsilon) = \partial\Big(J_\varepsilon F_N^{\text{Gardner}}(w) - F_N^{\text{Gardner}}(w_\varepsilon)\Big) \tag{5.24}$$

Clearly w_ε is smooth in space and by (5.13)

$$\|J_\varepsilon \partial(F_N^{\text{Gardner}} - \partial^{2N} w)\|_{L^1_u(I\times\mathbb{R})} \le c(\tau^{-1/2}\sup_t \|w(t)\|_{L^2})\|w\|^2_{L^2(H^N_{\tau,u})}.$$

Define $w_\varepsilon^{\tau_1}$ by

$$\partial w_\varepsilon^{\tau_1} + 2\tau_1 w_\varepsilon^{\tau^1} + (w_\varepsilon^{\tau_1})^2 = \partial_x w_\varepsilon + 2\tau w_\varepsilon + w_\varepsilon^2 \tag{5.25}$$

We differentiate both sides of the equation with respect to t, use the chain rule for $W^{1,1}$ functions for fixed x, and invert one operator to arrive at

$$\begin{aligned} \partial_t w_\varepsilon^{\tau_1} &= (\partial + 2\tau_1 + 2w_\varepsilon^{\tau_1})^{-1}(\partial + 2\tau + 2w_\varepsilon)\partial_t w_\varepsilon \\ &= \partial_t w_\varepsilon + (\partial + 2\tau_1 + 2w_\varepsilon^{\tau_1})^{-1}(w_\varepsilon - w_\varepsilon^{\tau_1} + \tau - \tau_1)\partial_t w_\varepsilon. \end{aligned}$$

We use the identity

$$\partial \frac{\delta H_N^{\text{KdV}}}{\delta u} = (\partial + 2\tau + 2w)\partial \frac{\delta H^{\text{Gardner}}}{\delta w}$$

twice, once for τ and then for τ_1 to see that

$$\partial \frac{\delta H_{\tau_1,N}^{\text{Gardner}}}{\delta w}(w_\varepsilon^{\tau_1}) = \Big\{1 + (\partial + 2\tau_1 + 2w_\varepsilon^{\tau_1})^{-1}(w_\varepsilon - w_\varepsilon^{\tau_1} + \tau - \tau_1)\Big\}\partial \frac{\delta H_N^{\text{Gardner}}}{\partial w}(w_\varepsilon).$$

Altogether

$$\partial_t w_\varepsilon^{\tau_1} - \partial \frac{\delta}{\delta w} H_{\tau_1,N}^{\text{Gardner}}(w_\varepsilon^{\tau_1}) = \Big\{1 + (\partial + 2\tau_1 + 2w_\varepsilon^{\tau_1})^{-1}(w_\varepsilon - w_\varepsilon^{\tau_1} + \tau - \tau_1)\Big\}\partial\Big(J_\varepsilon F_N^{\text{Gardner}}(w) - F_N^{\text{Gardner}}(w_\varepsilon)\Big)$$

By the continuity of $w \to \frac{\delta H^{\text{Gardner}}}{\delta w}(w)$ the left hand side converges to

$$w_t - \partial \frac{\delta H_N^{\text{Gardner}}}{\delta w}(w)$$

in the sense of distributions. We will show that the right hand side converges to 0 in the sense of distributions which implies that w^{τ_1} is a weak solution to the Nth τ_1 Gardner equation. We immediately turn to the most difficult term

$$(\partial + 2\tau_1 + 2w_\varepsilon^{\tau_1})^{-1}(w_\varepsilon - w_\varepsilon^{\tau_1})\partial\Big\{\Big(J_\varepsilon F_N^{\text{Gardner}}(w) - F_N^{\text{Gardner}}(w_\varepsilon)\Big)$$

which contains no linear term. By the same continuity argument as before it suffices that the map from w to

$$\tau^l(\partial + 2\tau_1 + 2w_1)^{-1}w_2\partial^{j_0+1}\prod_{k=1}^{K} w^{(j_k)}$$

is continuous in w_1, w_2 and w. We want to pull the derivatives $\partial^{(j_0+1)}$ recursively to the left. In the first step we obtain a term were the derivative falls on w_2, and one with the derivative between the integral operator and w_2. We recall

$$(\partial + 2\tau + 2w_1)^{-1}\partial = 1 - (\partial + 2\tau + 2w_1)^{-1}(2\tau + 2w_1).$$

By a repeated application of this argument we obtain a linear combination of terms

$$\tau_1^l \partial^{l_0} \prod_{k=1}^{K_1} w_1^{(j_k)} w_2^{(l_{K_1+1})} \prod_{k=K_1+2}^{K_1+K+1} w^{(j_k)}$$

where

$$l + \sum_{k=0}^{K+K_1+1} j_k + K + K_1 + 1 = 2N + 1$$

and

$$\tau_1^l(\partial + 2\tau + 2w_1)^{-1}\prod_{k=1}^{K_1} w_1^{(j_k)} w_2^{(l_{K_1+1})} \prod_{k=K_1+2}^{K_1+K+1} w^{(j_k)}$$

where

$$\sum_{k=1}^{K+K_1+1} j_k + K + K_1 + 1 = 2N + 2.$$

It suffices to bound the differential monomials in L^1, with $D = \sum_{k=0}^{K_1} l_k + \sum_{k=1}^{K} j_k$ the total number of derivatives and $M = K_1 + 1 + K_0$ the number of factors

$$\Big\| \operatorname{sech}^2(x)\prod_{j=1}^{K_1} w_1^{(l_k)} w_2^{(l_0)} \prod_{j=1}^{K} w^{(j_k)}\Big\|_{L^1} \le c\prod_{j=1}^{3}\Big(\|w_j\|_{L^2}^{M-2}\|\operatorname{sech}(x)w_j\|^2_{H^{D+\frac{M-2}{4}}}\Big)^{\alpha_j}$$

with $\alpha_1 = \frac{K_1}{M}$, $\alpha_2 = \frac{1}{M}$, $\alpha_3 = \frac{K}{M}$. Altogether we arrive at

$$\begin{aligned}&\Big\|\gamma(\partial + 2\tau_1 + 2w_1)^{-1}w_2\partial F_N^{\text{Gardner}}\|_{L^1}\\ &\qquad \le c(\|w_1\|_{L^2}, \|w_2\|_{L^2}, \|w\|_{L^2})\Big(\|\gamma w\|^2_{H^N_\tau} + \|\gamma w_1\|^2_{H^N_{\tau_1}} + \|\gamma w_2\|^2_{H^N_{\tau_1}}\Big).\end{aligned}$$

This completes the proof that if w is a weak solution to the Nth τ Gardner equation then the modified Miura maps define a weak solution to the Nth τ_1 Gardner equation, and vice versa.

Step 3: Weak solutions to KdV define weak solutions to Gardner under the additional assumption $\|w\|_{L^\infty} < \infty$. Now suppose that u is a weak solution to the Nth KdV equation. We want to prove that w is a weak solution to the Gardner equation. Let $u_\varepsilon = J_\varepsilon u$ and define w_ε by the Miura map,

$$\partial_x w_\varepsilon + 2\tau w_\varepsilon + w_\varepsilon^2 = u_\varepsilon.$$

We apply the chain rule and argue as above to obtain

$$\partial_t w_\varepsilon - \partial \frac{\delta H_N^{\text{Gardner}}}{\delta w}(w_\varepsilon) = (\partial + 2\tau + 2w_\varepsilon)^{-1}\partial\big(J_\varepsilon F_N^{\text{KdV}}(u) - F_N^{\text{KdV}}(u_\varepsilon)\big).$$

The Gardner terms on the left hand side are covered by our previous considerations. Only the right-hand side needs a new consideration. Again it suffices to provide bounds for

$$(\partial + 2\tau + 2\tilde{w})^{-1}\partial^{1+j_0}\prod_{k=1}^{K} u^{(j_k)}.$$

As above we pull derivatives to the left, making use of

$$(\partial + 2\tau + 2\tilde{w})^{-1}\partial = 1 - (\partial + 2\tau + 2\tilde{w})^{-1}(2\tau + 2\tilde{w}).$$

We substitute $u = v_x + 2\tau v$ and expand so that we obtain a linear combination of terms, and, by an abuse of notation we write v for $\tilde{w}$,

$$\tau^l \partial^{j_0}\prod_{k=1}^{K} v^{(j_k)} \tag{5.26}$$

where

$$2 \le K, \quad \sum_{k=0}^{K} j_k = 2N + 2 - K - l, \quad j_k \le N$$

and

$$\tau^l(\partial + 2\tau + 2v)^{-1}\prod_{k=1}^{K} v^{(j_k)} \tag{5.27}$$

where

$$2 \le K, \quad \sum_{k=1}^{K} j_k = 2N + 3 - K - l, \quad j_k \le N.$$

The estimate for (5.26) is

$$\|\tau^l \partial^{j_0}\prod_{k=1}^{K} v^{(j_k)}\|_{L^1_u(\mathbb{R})} \le c(\tau^{-1/2}\|v\|_{L^2})\|v\|^2_{H^N_{\tau,u}}$$

and for (5.27)

$$\begin{aligned}\|\tau^l(\partial + 2\tau + 2v)^{-1}\prod_{k=1}^{K} v^{(j_k)}\|_{L^1_u} &\lesssim \tau^{l-1}\Big\|\prod_{k=1}^{K} v^{(j_k)}\Big\|_{L^1_u} \\ &\lesssim c(\tau^{-1/2}\|v\|_{L^2})(1 + \tau^{-1}\|v\|_{L^\infty})\|v\|^2_{H^N_{\tau,u}}.\end{aligned}$$

Indeed the first inequality follows from writing $(\partial+2\tau+2v)^{-1}$ as an integral operator which is bounded as an operator on L^1_u. For the second inequality, note that by (5.9) from Lemma 5.4 we have

$$\tau^{l-1}\Big\|\prod_{k=1}^K v^{(j_k)}\Big\|_{L^1} \lesssim \tau^{l-1}\|v\|_{L^2}^{K-2}\|v^{(N-\frac{K-2}{4}-\frac{l-1}{2})}\|_{L^2}^2 \lesssim (\tau^{-1/2}\|v\|_{L^2})^{K-2}\|v\|^2_{H^N_\tau},$$

in all situations but when $l=0, K=3$ because of $N-\frac{K-2}{4}-\frac{l-1}{2}<N$ and since when $l=0$, then $K\geq 3$ in (5.27). Now assume $j_1+j_2+j_3=2N, j_i\leq N$, and $j_1\geq j_2\geq j_3$. When $j_3=0$ then we can use Hölder's inequality directly, otherwise

$$\big\|v^{(j_1)}v^{(j_2)}v^{(j_3)}\big\|_{L^1} \leq \big\|v^{(j_1)}\big\|_{L^2}\big\|v^{(j_2)}\big\|_{L^2}\big\|v^{(j_3)}\big\|_{L^\infty} \lesssim \|v\|_{L^\infty}\|v^{(N)}\|^2_{L^2},$$

from the Gagliardo-Nirenberg inequality. Performing a weighted summation implies

$$\tau^{-1}\big\|v^{(j_1)}v^{(j_2)}v^{(j_3)}\big\|_{L^1_u} \lesssim \tau^{-1}\|v\|_{L^\infty}\|v\|^2_{H^N_{\tau,u}},$$

which proves the second inequality. □

5.3. **The good variable hierarchy.** The fundamental Sobolev estimate in this setting is

$$\Big\|\prod_{k=1}^K v^{(j_k)}\Big\|_{L^1_u} \leq c\|v\|^{K-2}_{L^\infty}\|v\|^2_{H^\sigma_u} \quad \text{where} \quad 2\sigma=\sum_{j=1}^K j_k. \tag{5.28}$$

Let $\frac12<s\leq 1$, $w\in L^\infty(J,H^{s-1})\cap L^2H^{s+N-1}_u$. We claim that this suffices to define weak solutions to the Nth Gardner equation. We write $w=\partial_x v-\tau v$ and we have to bound

$$\tau^l\partial^{j_0}\prod_{k=1}^K\partial^{j_k}(\partial_x v-\tau v)$$

which can be written as a linear combination of terms (changing the meaning of l, j_0, K)

$$\tau^l\partial^{j_0}\prod_{k=1}^K v^{(j_k)}$$

where

$$l+\sum_{k_0}^K j_k = 2N+1$$

and no term carries more than N derivatives. We obtain a bound

$$\|\prod_{k=1}^K v^{(j_k)}\|_{L^1_u} \leq C\|v\|^{K-2}_{L^\infty}\|v\|^2_{H^D}$$

with $2D=\sum_{j=1}^K j_k$. The claims follow now from embeddings and estimates for the operator $\partial-2\tau$.

We recall the structure of the Nth equation good variables hierarchy. It can be written as

$$\psi_t-(-1)^N\psi^{(2N+1)} = \partial F_N^{GV}(\psi),$$

where the structure of F_N is the following:

Lemma 5.9.

$$F_N^{\mathrm{GV}}(w) = \sum_{K,(j_k)_k,l,M} c_{K,(j_k)_k,l,M}\partial^{j_0}(1+v)^{-M}\tau^{2l}\prod_{k=1}^K v^{(j_k)},$$

$$0 \le M \le 2N+1, \qquad 2 \le K \le 3N+1 \qquad \sum_{k=0}^K j_k = 2N-2l, \qquad j_k \le N-l.$$

Proof. $F_N^{GV}(v)$ is a sum over $(1+v)^{-M}\tau^l \prod_{k=1}^K v^{(j_k)}$ with (cf. Theorem 3.12)

$$0 \le M \le 2N-1, \qquad 2 \le K \le 2N+1,$$

$$l + \sum_{k=1}^K j_k = 2N, \qquad \sum_{k=1}^K j_k \ge M+1 \text{ if } M \ge 1.$$

For the good variable equation, we proceed as in Lemma 3.13. □

Let $s > \frac{1}{2}$. Then we can define the notion of a weak solution to the good variable equation under the regularity assumption

$$v \in L^\infty(I, H^s) \cap L^2(I, H^{s+N})_u.$$

Theorem 5.10. *Let $s > \frac{1}{2}$,$\tau \ge 2$, I an open interval. Suppose $v \in L^\infty(H^s) \cap L^2 H_u^{s+N}$ satisfies $v > -1$. Let $w = v - \frac{1}{2}\partial \log v$. Then $w \in L^\infty(H^{s-1}) \cap L^2 H_u^{s-1+N}$. Vice versa: Suppose that $w \in L^\infty H^{s-1} \cap L^2 H_u^{s-1+N}$. Then there exists a unique $v \in L^\infty$ with $v > -1$ and $(v+1)^{-1}$ which satisfies*

$$\tau v - \partial_x(\log(1+v)) = w.$$

Moreover $v \in L^\infty H^s \cap L^2 H_u^{s+N}$. Under these assumptions w is a solution to the Nth τ Gardner equation iff v is a weak solution to the Nth τ Good variable equation.

Proof. We argue in the same fashion for the second part. We recall the equation for v (see Theorem 3.12

$$v_t = (-1)^N v^{(2N+1)} + \partial_x F_N^{gv}$$

with

$$\partial \frac{\delta H_N^{\mathrm{Gardner}}}{\partial w}\Big|_{w=\tau v - \frac{1}{2}\partial \log(1+v)} = \Big(\tau - \partial \frac{1}{2(1+v)}\Big)\partial\left(F_N^{GV}(v) + \partial^{2N} v\right).$$

Suppose that v is a weak solution the Nth equation and set $v_\varepsilon = J_\varepsilon v$, $w_\varepsilon = \tau v_\varepsilon - \frac{1}{2}\partial\log(1+v_\varepsilon)$. Then

$$\partial_t w_\varepsilon - \partial \frac{\delta H_N^{\mathrm{Gardner}}}{\delta w}(w_\varepsilon) = \Big(\tau - \partial\frac{1}{2(1+v_\varepsilon)}\Big)\partial(J_\varepsilon F_N^{GV}(v) - F_N^{GV}(v_\varepsilon)).$$

The left hand side converges to

$$\partial_t w - \partial \frac{\delta H^{\mathrm{Gardner}}}{\delta w}$$

in a distributional sense. We claim that the right hand side converges to zero. To prove that is suffices to consider convergence for summands as in Lemma 5.9:

$$\tau^{2l}\Big(\tau - \partial\frac{1}{2(1+v_\varepsilon)}\Big)\partial^{1+j_0}\Big((1+v)^{-M}\prod_{k=1}^K v^{(j_k)}\Big).$$

Again we pull the derivatives in front. We obtain a sum of terms

$$\tau^L \partial^{j_0}(1+v)^{-M} \prod_{k=1}^K v^{(j_k)}$$

where

$$0 \le M \le 2N+3, \qquad 2 \le K \le 3N+2 \qquad \sum_{k=0}^K j_k = 2N+1-L \qquad j_k \le N - L/2.$$

We may ignore the powers of τ and of $(1+v)$ for the question of distributional convergence, which now follows from (5.28)

Suppose that w is a weak solution to the N Gardner equation, $w_\varepsilon = J_\varepsilon w$ and v_ϵ satisfies $\tau v - \frac{1}{2}\partial \log(1+v)$. Then

$$\partial_t v_\varepsilon - (-1)^N v_\varepsilon^{(n+1)} - \partial F_N^{GV}(v_\varepsilon)) = \Big(\tau - \partial \frac{1}{2(1+v_\varepsilon)}\Big)^{-1} \partial \Big\{ J_\varepsilon F_N^{\text{Gardner}}(w) - F_N^{\text{Gardner}}(w_\varepsilon) \Big\}.$$

The convergence of the left hand side to the good variable equation follows from the bounds which ensure that the notion of a weak solutions is well defined. We again have to provide bounds for

$$\Big(\tau - \partial \frac{1}{2(1+v_\varepsilon)}\Big)^{-1} \partial F_N^{\text{Gardner}}(w),$$

more precisely for its summands

$$\Big(\tau - \partial \frac{1}{2(1+v_\varepsilon)}\Big)^{-1} \partial^{1+m} \prod_{k=1}^K w^{(j_k)}.$$

We write $w = \tau\tilde{v} - \tilde{v}_x$ and ignore the difference between $\tilde{v}$, v and v_ε in the sequel. We pull derivatives to the left:

$$\begin{aligned}\Big(\tau - \partial \frac{1}{2(1+v)}\Big)^{-1} \partial &= \Big(\tau - \partial \frac{1}{2(1+v)}\Big)^{-1} \partial (2(1+v))^{-1}(2(1+v)) \\ &= -2(1+v) + \tau \Big(\tau - \partial \frac{1}{2(1+v)}\Big)^{-1} (2(1+\tilde{v})).\end{aligned}$$

Consider

$$\tau\phi + \partial \frac{\phi}{(1+v)} = f.$$

Let $\psi = \frac{\phi}{1+v}$ and rewrite it as

$$\partial\psi + \tau(1+v)\phi = f$$

and deduce

$$\|\phi\|_{L^\infty} \le \|\psi\|_{L^\infty} \|1+v\|_{L^\infty} \le c\|1+v\|_{L^\infty} \|(1+v)^{-1}\|_{L^\infty} \|f\|_{L^1_u}.$$

The final estimate now follows from (5.28). □

6. Regularity of weak solutions and Kato smoothing

We analyze the structure of the conservation laws for smooth solutions to the Nth KdV Gardner equation. Moreover, we use the conservation laws in their weak form to show smoothing estimates and precompactness of orbits for weak solutions.

6.1. **The energy-flux identity.** We obtain the energy flux identity for smooth solutions by multiplying the equation by ηw:

$$0=\int_{\mathbb{R}^2}\eta w(w_t-\frac{\delta}{\delta w}H_N^{\text{Gardner}})dxdt=\frac{1}{2}\int_{\mathbb{R}^2}-w^2\eta_t+\frac{\delta}{\delta w}H_N^{\text{Gardner}}\partial_x(w\eta)dxdt.$$

We rewrite the last term for fixed t

$$\int\frac{\delta}{\delta w}H_N^{\text{Gardner}}\partial(\eta u)dx=\frac{d}{ds}H_N^{\text{Gardner}}(w+s\eta\partial_x w))\Big|_{s=0}+\int\frac{\delta}{\delta w}H_N^{\text{Gardner}}w\partial_x\eta dx.$$

We study the first term on the right hand side for differential monomials $I=\int\prod w^{(j_k)}dx$

$$\begin{aligned}\frac{d}{ds}I(w+s\eta w_x)|_{s=0}&=\int\eta\sum_{k=1}^{k}w_x^{(j_k)}\prod_{i\neq k}w^{(j_i)}dx=\int\eta\partial_x\prod_{k=1}^{K}w^{(j_k)}dx\\&=-\int\eta_x\prod_{k=1}^{K}w^{(j_k)}dx.\end{aligned}$$

We obtain the flux

$$Fl_N(w)=\sum_{K=2}^{2N+2}\sum_{l=0}^{2N+2-K}\partial^{j_0}\sum_{\sum j_k=2N+2-K-l}c_{K,l,j_k}\tau^l\prod_{j=1}^{K}w^{(j_k)}\tag{6.1}$$

which has almost the same structure as the energy density so that the energy flux equation

$$\partial_t\frac{1}{2}w^2=\partial_x Fl_N(w)\tag{6.2}$$

holds. In particular the quadratic part of the flux is

$$\partial Fl_{N,2}=2ww^{(2N+1)}=\partial\big((2N+1)|w^{(N)}|^2+\sum_{k=1}^{n}a_k\partial^{2k}|w^{(N-k)}|^2\big),$$

for some combinatorical constants $a_k\in\mathbb{R}$.

Lemma 6.1. *Suppose that $w\in L^\infty(\mathbb{R},L^2)$ with $w^{(N)}\in L^2_u(\mathbb{R}\times\mathbb{R})$ is a weak solution to the Nth Gardner equation. Then*

$$\partial_t w^2=\partial_x Fl_N(w)$$

in the sense of distributions.

Proof. Let $\rho\in C_c^\infty((-1,1))$ be nonnegative and even with $\int\rho dx=1$. We define $\rho_\varepsilon=\varepsilon\rho(x/\varepsilon)$ and $J_\varepsilon^x f=\rho_\varepsilon *_x f$ and similarly J_ε^t is the convolution with respect to time. We recall that weak solutions w satisfy $J_\varepsilon^x w\in W^{1,1}(I,L^2(J))$ whenever I,J are compact intervals. As a consequence

$$t\to\int w(t)\phi dx$$

is continuous for every test function ϕ, or, equivalently, after modifications on the set of times of measure zero $t\to w(t)\in L^2$ is weakly continuous.

Let w be a weak solution satisfying the regularity assumptions. Then with $w_\varepsilon = J^t_{\varepsilon_1} J^x_{\varepsilon_2} w$, similar to above

$$\begin{aligned} 0 &= \int \Big(w_t - \partial \frac{\delta H_N^{\text{Gardner}}}{\delta w}(w)\Big) J^t_{\varepsilon_1} J^x_{\varepsilon_2}(\phi w_\varepsilon) dxdt \\ &= \int -\frac{1}{2} w_\varepsilon^2 \partial_t \phi + \Big(J^t_{\varepsilon_1} J^x_{\varepsilon_2} \frac{\delta H_N^{\text{Gardner}}}{\delta w}(w)\Big)\Big(w_\varepsilon \partial_x \phi + (\partial_x w_\varepsilon)\phi\Big)\, dxdt. \end{aligned} \tag{6.3}$$

All terms converge as the time regularization ε_1 tends to 0. We set $\varepsilon_1 = 0$ and write $\varepsilon = \varepsilon_2$ in the sequel. By the continuity in ε we have to justify

$$\int_{\mathbb{R}^2} \Big(J^t_{\varepsilon_1} J^x_{\varepsilon_2} \frac{\delta H_N^{\text{Gardner}}}{\delta w}(w)\Big)\Big(w_\varepsilon \partial_x \phi + (\partial_x w_\varepsilon)\phi\Big)\, dxdt \to \int FL_N \partial_x \phi dxdt.$$

We obtain for the linear resprectively quadratic term

$$\int -\frac{1}{2} w_\varepsilon^2 \partial_t \eta + FL_{N,2}(w_\varepsilon)\partial_x \eta \, dx\, dt \to \int -\frac{1}{2} w^2 + Fl_{N,2}(w)\partial_x \eta \, dx\, dt$$

since $w^{(j)}$ is in L^2_{loc} of space time for $j \le N$.

Let $h(w) = \tau^l \prod_{k=1}^K w^{(j_k)}$ be monomial in the energy density up to a coefficient with $K \ge 3$ and let FL be the corresponding part of the flux. Let D be the total number of derivatives. Then

$$K + D + l = 2N + 2$$

Then

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} J_\varepsilon h(w)(w_\varepsilon \partial_x \phi + \partial_x w_\varepsilon \phi)\, dx\, dt = \int_{\mathbb{R}^2} h(w)(w\partial_x \phi + \partial_x w \phi dx)\, dt.$$

Summation and comparison with (6.3) implies the energy flux identity. $\square$

6.2. **Consequences of the energy-flux identity: Kato smoothing for weak solutions.** Weak solutions satisfy the energy flux identity (6.2) by Lemma 3.11 which can be written as

$$\int w^2 \partial_t \eta - \mathrm{Fl}_N(w)\partial_x \eta dxdt = 0 \tag{6.4}$$

for smooth compactly supported functions η. We say that $u \in L^\infty(I; H^{-1})$ with $u^{(N-1)} \in L^2_{loc}(I \times \mathbb{R})$ lies in the N th KdV Kato smoothing space if for all compact subintervals $J \subset I$

$$\limsup_{x_0 \to \pm\infty} \|u^{(N-1)}\|_{L^2(J \times (x_0 - 1, x_0 + 1))} = 0. \tag{6.5}$$

We say that $w \in L^\infty(I; L^2)$ with $u^{(N)} \in L^2_{loc}(I \times \mathbb{R})$ lies in the N the Nth Gardner Kato smoothing space if for all compact subintervals $J \subset I$

$$\limsup_{x_0 \to \pm\infty} \|w^{(N)}\|_{L^2(J \times (x_0 - 1, x_0 + 1))} = 0. \tag{6.6}$$

The condition (6.5) is equivalent to

$$\limsup_{x_0 \to \pm\infty} \| \operatorname{sech}(\sigma(x - x_0)) u\|_{L^2(J; H^{N-1}(\mathbb{R}))} = 0.$$

and (6.6) is equivalent to

$$\limsup_{x_0 \to \pm\infty} \| \operatorname{sech}(\sigma(x - x_0)) w\|_{L^2(J; H^{N}(\mathbb{R}))} = 0.$$

If $u = w_x + 2\tau w + w^2$ (for fixed time) and $\sigma < \tau$ then

$$\|\operatorname{sech}(\sigma(x-x_0))w\|_{H^N} \sim \|\operatorname{sech}(\sigma(x-x_0))u\|_{H^{N-1}}$$

with constants depending on τ and $\|w\|_{L^2}$.

Lemma 6.2. *Let $N \geq 1$, $\tau \geq 1$. Suppose that w in the Nth Gardner Kato smoothing space ((6.5), (6.6)) on $\mathbb{R}$ is a weak solution. Then $w \in C(\mathbb{R}, L^2(\mathbb{R}))$ and there exists $\rho = \rho(\tau^{-1/2}\|w\|_{L^2})$ so that it satisfies with $w_0 = w(0)$ the Kato smoothing estimate for $0 < \kappa \leq \tau$*

$$\begin{aligned}\sup_t \int (1+\tanh(\kappa(x-x_0-\rho\tau^{2n}t))|w(t,x)|^2 dx&\\ +\kappa\int_0^\infty \int \operatorname{sech}^2(\kappa(x-x_0-\rho\tau^{2n}t))(|w^{(n)}|^2+\tau^{2n}|w|^2)dxdt&\\ \lesssim \int \big(1+\tanh(\kappa(x-x_0))\big)w_0^2\,dx.&\end{aligned} \tag{6.7}$$

The implicit constant depends only on N.

Proof. Let $\eta_j \in C_c^\infty$, $j = 0, 1$ and assume that η_0 is nonnegative and satisfies $\eta_0(0) = 1$. We define for $R > 0$

$$\eta(t,x) = \eta_1(t)\eta_0(x/R).$$

Then

$$\lim_{R\to\infty}\int_{\mathbb{R}^2} w^2\partial_t\eta - FL_N\partial_x\eta dx = \int_{\mathbb{R}} \|w(t)\|_{L^2}^2\partial_t\eta_0(t)dt$$

hence $\|w\|_{L^2}$ is independent of time (up to sets of measure 0). We have seen that $t \to w_x^\varepsilon \in L^2$ is weakly continuous which together with norm continuity implies that $w \in C(\mathbb{R}; L^2)$.

We want plug in

$$\eta = \chi_{[0,T]}(1+\tanh(\kappa(x-x_0-\rho\tau^{2n}t)))$$

into the integrated energy-flux identity (6.4). There are two obstacles: The characteristic function is not smooth, and $1 + \tanh$ is not compactly supported. We deal with the second problem by multiplying in addition by $\eta_0(x/R)$ as above, and with the first obstacle by regularization. We obtain

$$\begin{aligned}\frac{1}{2}\int (1+\tanh(\kappa(x-x_0-\rho\tau^{2n}t)))w^2dx\Big|_0^T&\\ = -\kappa\int_0^T\int \operatorname{sech}^2(\kappa(x-x_0-\rho\tau^{2n}t)))\big(\frac{\rho\tau^{2n}}{2}w^2 - Fl_N(w)\big)dx.&\\ =: \int_0^T A(t)dt&\end{aligned}$$

The quadratic term on the RHS is - omitting the argument of sech^2 as well as the time integration -

$$-\kappa\int \operatorname{sech}^2\big(\frac{\rho\tau^{2n}}{2}w^2 + (2N+1)|w^{(N)}|^2\big)dx + a_k\sum_{k=1}^N (\operatorname{sech}^2)^{2k+1})|w^{(N-k)}|^2dx$$

and

$$\left|\int \operatorname{sech}^2 Fl_{N\geq 3} dx\right| \leq c(1+\|w\|_{L^2}^2)^N \|\operatorname{sech} w\|_{H^N_\tau} \|\operatorname{sech} w\|_{H^{N-1}_\tau}$$
$$\leq \|\operatorname{sech}(x) w\|^2_{H^N_\tau} + c^2(1+\|w\|_{L^2})^{2N^2} \|\operatorname{sech}^2 w\|^2_{L^2})$$

Together the integrand can be estimated

$$A(t) \leq -\kappa \int \operatorname{sech}^2(\rho\tau 2nw^2 + 2N|w^{(N)}|^2 dx + c^2(1+\|w\|^{N^2}_{L^2})\|\operatorname{sech} w\|^2 + \|\operatorname{sech} w\|^2_{H^{N-1}_\tau}.$$

We choose

$$\rho \geq (C + c^2(1+\|w\|^2_{L^2})^{N^2+1}\tau^{-2N})$$

and subtract the second and the third term on the RHS from both sides. This gives (6.7). $\square$

6.3. **Precompactness of orbits of weak solutions.** In this section we show that the orbit of weak solutions of the Gardner hierarchy with initial data in a precompact subset of $L^2(\mathbb{R})$ are precompact over compact time intervals.

Theorem 6.3. *Let $Q \subset \mathcal{S}(\mathbb{R})$ be a precompact subset of $L^2(\mathbb{R})$, let A be a set of weak solutions with initial data in Q and let I by a bounded interval. Then*

$$\{w(t) : w \in A, t \in I\} \subset L^2$$

is precompact.

This will be essential in the proof of our main theorem, as we want to upgrade weak to strong convergence using a compactness argument. To check precompactness, one can check boundedness, equicontinuity, and tightness.

Proof. We begin with boundedness. Since Q is precompact it is bounded and there exists $R > 0$ so that

$$\|u_0\|_{L^2} < R$$

for $u_0 \in Q$. Weak solution conserve the L^2 norm by Lemma 6.2.

We turn to tightness on the the right. Let $\varepsilon > 0$. Since Q is precompact and hence tight there exists R so that for $x_0 > R$

$$\sup_{w_0 \in Q} \int (1 + \tanh(\kappa(x - x_0))|w_0|^2 dx < \varepsilon.$$

By Lemma 6.2 again

$$\sup_{w_0\in Q} \sup_t \int (1 + \tanh(\kappa(x - x_0 - \gamma\tau^{2N} t)))|w(t)|^2 dx \leq c\varepsilon.$$

Given $\varepsilon > 0$ and a bounded time interval I we find x_0 so that

$$\sup_{t\in I} \sup_{w_0\in Q} \|w\|_{L^2(x_0,\infty)} < \varepsilon. \tag{6.8}$$

We turn to the proof of equicontinuity. The set

$$Q_u = \{w_x + 2\tau w + w^2 : w \in Q\}$$

is precompact since Q is. If $\tau_1 \geq \tau$ it lies in the range of the τ_1 Miura map. Q_v is equicontinuous hence there there exists τ_1 so that

$$\|u\|_{H^{-1}_{\tau_1}} < \varepsilon$$

Let $Q_{\tau_1} = M_{\tau_1}^{-1}(Q_u)$, Then

$$\|w_0\|_{L^2} \le \varepsilon$$

for all $w_0 \in Q_{\tau_1}$. Under these mapping every weak solution to the τ Gardner equation is mapped to a weak solution to the τ_1 Gardner equation. This equation preserves the L^2 norm, hence

$$\|w_x(t) + 2\tau w^{\tau_1}(t) + (w(t))^2\|_{H^{-1}_{\tau_1}} = \|w^{\tau_1}_x(t) + 2\tau_1 w^{\tau_1}(t) + (w^{\tau_1}(t))^2\|_{H^{-1}_{\tau}} < \varepsilon$$

and the orbits are equicontinuous on compact time intervals.

The Kato smoothing estimate for w^{τ_1} implies

$$\sup_{x_0} \kappa \int_0^\infty \|\operatorname{sech}(\kappa(x - x_0 - \gamma\tau_1^{2N} t))w\|^2_{H^N_{\tau_1}} dt \le \varepsilon,$$

hence

$$\sup_{x_0} \kappa \int_0^\infty \|\operatorname{sech}(\kappa(x - x_0 - \gamma\tau_1^{2N} t))w_x + \tau w\|^2_{H^{N-1}_{\tau_1}} dt \le c\varepsilon$$

for all weak solutions with initial datum in Q. This implies the high frequency bound

(6.9)
$$\sup_{x_1} \int_0^\infty \int \operatorname{sech}^2(\kappa_1(x - x_1 - \rho\tau_1^{2N} t))(|w^{(N)}|^2 + \tau_1^{2(N-1)}(|w_x|^2 + w^2))dxdt < \frac{\varepsilon}{\kappa_1}$$

for all weak solutions w to the N-Gardner equation with $w(0) \in Q$ and κ_1 much smaller than τ.

With the high frequency estimate in place we turn to tightness on the left. We claim in the setting of the theorem: Given $I = [0, T]$ and $\varepsilon > 0$ there exists x_0 so that

$$\sup_{0 \le t \le T} \sup_{w_0 \in Q} \|w(t)\|_{L^2(-\infty, x_0)} \le \varepsilon \tag{6.10}$$

To simplify the notation we set $T = 1$. We use

$$\eta(t, x) = \chi_{[0,T]}\Big(1 - \tanh(\kappa(x - x_0 + \rho\tau^{2n} t))\Big)$$

with $\kappa = \delta\tau^{-2n}$ in the integrated energy-flux identity. Again this function is not admissible since it is not a test function, neither in t nor in x. In the same way as for the L^2 conservation we obtain for $T \le 1$ and $x_0 \in \mathbb{R}$.

$$\int \eta w^2 dx \Big|_0^T \le c\kappa_0 \int_0^1 \|\operatorname{sech}(\tau^{-2N}(x - x_0 + \rho\tau^{2n} t)))w^{(N)}\|^2_{L^2}. \tag{6.11}$$

We choose τ_1 so that in the notation above

$$\|w^{\tau_1}(0)\|_{L^2} \le \varepsilon.$$

By tightness of Q we can choose R so that the initial term satisfies

$$\int (1 + \tanh(\tau_1^{-2N}(x - x_0 + \tau_1^{2N})))|w|^2 dx < \varepsilon$$

We set $x_1 = x_0 - 3\tau_1^{2N}$ and $\kappa_1 = \frac{1}{9}\tau_1^{-2N}$ in (6.9) to estimate the right hand side in (6.11) by $c\varepsilon$. $\square$

7. Smoothing and convergence for the difference flow

We recall the formula (3.47)

$$\mathcal{T}_{-1}^{\text{Gardner}}(z,w,\tau)=\frac{-iz}{4z^2+4\tau^2}\log\det{}_2(1+K).$$

where to shorten the notation we define

$$\begin{aligned}K(w)&=\begin{pmatrix}0 & (-iz-\partial)^{-1}w\\ -(-iz+\partial)^{-1}w & (-iz+\partial)^{-1}2\tau(-iz-\partial)^{-1}\end{pmatrix}\\ &=\begin{pmatrix}0 & (-iz-\partial)^{-1}w\\ -(-iz+\partial)^{-1}w & -\frac{\tau}{iz}\big[(-iz+\partial)^{-1}+(-iz-\partial)^{-1}\big]w\end{pmatrix}.\end{aligned}\tag{7.1}$$

Then

$$-\log\det{}_2\big(1+K(w)\big)=\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\operatorname{tr}\,K(w)^n$$

and

$$\mathcal{T}_N^{\text{Gardner}}(z,w,\tau)=\frac{-iz(2z)^{2N+2}}{4z^2+4\tau^2}\log\det{}_2\big(1+K(w)\big)-\sum_{n=0}^{N}(2z)^{2(N-n)}H_n^{\text{Gardner}}(w,\tau).$$

In this section we prove Proposition 2.13, in particular the bound on $\mathcal{T}_N^{\text{Gardner}}$ (2.44), the characterization of equicontinuity of subsets Q in H^N (2.45) and the weak convergence statement (2.46). As a consequence we also obtain the corresponding bound (2.11) and (2.13). A central step consists in the study of $(2z)^{N+1}\operatorname{tr}\,K(w)^n$.

7.1. **Schatten classes and the case of large $n\geq N+2$.** We first deal with the sum over large $n\geq 2N+4$ and collect simple estimates for the operator K. The central estimates are

Lemma 7.1. *Let $N\geq 3$, $N+2\leq n$ and $\operatorname{Im}z\geq\tau$. Then*

$$\begin{aligned}(\operatorname{Im}z)^{N+1}|\operatorname{tr}\,K^n(w)|&\leq c((\operatorname{Im}z)^{-1/2}\|w\|_{L^2})^{n-N-2}\|w\|_{L^{N+2}}^{N+2}\\ &\leq c((\operatorname{Im}z)^{-1/2}\|w\|_{L^2})^{n-N-2}\|w\|_{L^2}^{N}\|w^{(N/4)}\|_{L^2}^2\\ &\leq c((\operatorname{Im}z)^{-1/2}\|w\|_{L^2})^{n-N-2}\|w\|_{L^2}^{N+1}\|w^{(N/2)}\|_{L^2}\end{aligned}\tag{7.2}$$

where $\|w^{(s)}\|_{L^2}$ is defined using the Fourier transform.

Since for $N\geq 4$

$$\int|w|^{N+2}dx\leq\|w\|_{L^2}^2\|w\|_{L^\infty}^N\leq\|w\|_{L^2}^{2+\frac{N}{2}}\|w_x\|_{L^2}^{\frac{N}{2}}\leq\|w\|_{L^2}^{2+\frac{N}{2}}\left(\|w\|_{L^2}^{1-\frac{4}{N}}\|w^{(N/4)}\|_{L^2}^{\frac{4}{N}}\right)^{\frac{N}{2}},$$

$$\|w^{(N/4)}\|_{L^2}^2\leq\|w\|_{L^2}\|w^{(N/2)}\|_{L^2}$$

and

$$\int|w|^3dx\leq\|w\|\|w^{(1/2)}\|_{L^2}^2$$

it suffices to prove the first inequality.

To this end recall the Schatten classes $\mathfrak{I}_p$ of compact operators A with l^p-summable singular values,

$$\|A\|_{\mathfrak{I}_p}^p=\sum_{n=1}^{\infty}\mu_n(A)^p.\tag{7.3}$$

Here we always consider L^2 and its powers as the underlying Hilbert space. Special cases are the trace class operators $\mathfrak{I}_1$, the Hilbert-Schmidt operators $\mathfrak{I}_2$, and the class of compact operators $\mathfrak{I}_\infty$ which are defined with the obvious adaption in (7.3). We will make use of the fact that the $\mathfrak{I}_p$ are $*$-ideals in the sense that for all $A \in \mathfrak{I}_p, B : L^2 \to L^2$

$$AB, BA \in \mathfrak{I}_p, \qquad \|AB\|_{\mathfrak{I}_p}, \|BA\|_{\mathfrak{I}_p} \le \|A\|_{\mathfrak{I}_p} \|B\|_{L^2 \to L^2},$$

and of the Hölder-like inequality

$$\|AB\|_{\mathfrak{I}_r} \le \|A\|_{\mathfrak{I}_p} \|B\|_{\mathfrak{I}_q}, \qquad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

In particular for all $\sum_i \frac{1}{p_i} = 1$,

$$\big| \operatorname{tr} \big(\prod_i A_i \big) \big| \le \big\| \prod_i A_i \big\|_{\mathfrak{I}_1} \le \prod_i \|A_i\|_{\mathfrak{I}_{p_i}}.$$

We refer to [59] for a thorough introduction to these spaces. Finally we note that the classes $\mathfrak{I}_p$ admit the interpolation property, see [3, Proposition 2.1].

Lemma 7.2. *We have with* $\sigma = \operatorname{Im} z$

$$\begin{aligned} \|K(w)\|_{\mathfrak{I}_2} &= \Big(2 + \frac{\tau^2}{|z|^2} \Big)^{\frac{1}{2}} \sigma^{-\frac{1}{2}} \|w\|_{L^2}, \\ \|K(w)\|_{\mathfrak{I}_p} &\le \Big(2 + \frac{\tau}{|z|} \Big) \sigma^{-1+\frac{1}{p}} \|w\|_{L^p}, \qquad p \ge 2 \end{aligned} \tag{7.4}$$

Proof. For the first estimate we calculate K^*K and obtain

$$\begin{aligned} \|K\|_{\mathfrak{I}_2}^2 &= \operatorname{tr}(K^*K) \\ &= \|(-iz - \partial)^{-1} w\|_{\mathfrak{I}_2}^2 + \|(-iz + \partial)^{-1} w\|_{\mathfrak{I}_2}^2 + 4\tau^2 \|(-z^2 - \partial^2)^{-1} w\|_{\mathfrak{I}_2}^2 \\ &= (1 + \tau^2/z^2)(2iz) \int ((-\partial + 2iz) w)^2 dx. \end{aligned}$$

By using the integral kernel of $(-iz - \partial)^{-1}$ and Fubini's theorem we obtain

$$\|(-iz - \partial)^{-1} w\|_{\mathfrak{I}_2}^2 = \int_{x>y} |e^{iz(x-y)} w(y)|^2 \, dy dx = \sigma^{-1} \|w\|_{L^2}^2.$$

The same holds for the second summand. For the third summand we calculate, using Fubini and then Plancherel,

$$\|(-z^2 - \partial^2)^{-1} w\|_{\mathfrak{I}_2}^2 = (2\pi)^{-1} \|(-z^2 + \xi^2)^{-1}\|_{L^2_\xi}^2 \|w\|_{L^2}^2 = \frac{1}{4\sigma |z|^2} \|w\|_{L^2}^2.$$

This proves first identity of (7.4) for the Hilbert-Schmidt norm. Next

$$\|(-iz \pm \partial)^{-1} w\|_{L^2 \to L^2} \le \|(-iz \pm \partial)^{-1}\|_{L^2 \to L^2} \|w\|_{L^\infty} = \sigma^{-1} \|w\|_{L^\infty}$$

which implies

$$\|K(w)\|_{L^2 \to L^2} \le (1 + \frac{\tau}{\sigma}) \sigma^{-1}$$

and for the second inequality we interpolate by viewing the operator $K(w)$ as a map $L^p \to \mathfrak{I}_p$, where for $p = 2$ we use our first inequality and for $p = \infty$ the operator bound. $\square$

Let $N \in \mathbb{N}$, $N \le n$. Then

$$
\begin{aligned}
|\operatorname{tr} K^n| \le \|K^n\|_{\mathfrak{I}_1} &\le \|K^N\|_{\mathfrak{I}_1} \|K\|_{\mathfrak{I}_2}^{n-N} \\
&\le c_N (\operatorname{Im} z)^{1-N} \|w\|_{L^N}^N \Big(\frac{3}{(\operatorname{Im} z)^{1/2}} \|w\|_{L^2} \Big)^{n-N}.
\end{aligned}
\tag{7.5}
$$

There is one more structural observation: The integral kernel of the lower right entry of K is

$$
2\tau \int_{\max\{x,y\}}^\infty e^{-2iz(2t-x-y)} dt = \frac{\tau}{iz} e^{-iz|x-y|}
$$

hence

$$
2(iz-\partial)^{-1} \tau (iz+\partial)^{-1} = -\frac{\tau}{iz}\big((-iz-\partial)^{-1} + (-iz+\partial)^{-1}\big)
$$

If in an expansion the lower right entry of

$$
\begin{pmatrix} 0 & (-iz-\partial)^{-1} w \\ (-iz+\partial)^{-1} w & -\frac{\tau}{iz}\big((-iz-\partial)^{-1} + (-iz+\partial)^{-1}\big) w \end{pmatrix}
$$

is involved (which is always the case if n is odd) then we gain a factor $\frac{\tau}{\operatorname{Im} z}$ in all the estimates above.

7.2. **The case of $n < N+2$: The structure of the terms.** We will encounter a special class of multidimensional integrals frequently in this section.

Definition 7.3 (Primitive integrals)**.** *We call integrals of the type*

$$
\Big(\frac{2\tau}{2iz} \Big)^{2m-n} \int_{x_1 < \dots < x_n} \prod_{j=1}^n e^{2izx \cdot y} w_j(x_j) dx_j, \quad n/2 \le m \le n
\tag{7.6}
$$

where

$$
\sum_j y_j = 0, \quad y \cdot x \ge x_n - x_1 \qquad \text{for all } x \in A = \{x_1 < x_2 < \dots < x_n\}
$$

primitive integrals.

We decompose

$$
\mathcal{T}_{-1}^{\text{Gardner}}(z,w,\tau) = \sum_{n=2}^\infty \mathcal{T}_n(z,w,\tau)
$$

where

$$
\mathcal{T}_n(z,w,\tau) = \frac{(-1)^n}{n} \frac{iz}{4z^2 + 4\tau^2} \operatorname{tr} K^n(w)
$$

is the n-homogeneous part with respect to w. In the previous section we have seen that if $n \ge M+2$ and $\operatorname{Im} z \ge \tau$

$$
|\mathcal{T}_n(z,w,\tau)| \le (c (\operatorname{Im} z)^{-1/2} \|w\|_{L^2}^2)^{M+2-n} \frac{|z|}{\operatorname{Im} z} (\operatorname{Im} z)^{-M} \|w\|_{L^{M+2}}^{M+2}.
\tag{7.7}
$$

We begin with an algebraic decomposition of $\mathcal{T}_n$.

Lemma 7.4. *Suppose that $n < M+2$. Then we can write*

$$
(2z)^{M+1} \mathcal{T}_n = \sum_{j \le M-2} (2z)^{M-j} \int h_{n,j} dx + (2z)^{-1} \mathcal{T}_{M+1,n}
\tag{7.8}
$$

(so that the leading term is $(2z)^M$ tr $K^n(w)$) where $h_{n,j}$ are differential polynomials independent of M and

$$\mathcal{T}_{M+1,n} = \sum_{j=1}^{J} \Big(\frac{z}{\operatorname{Im} z}\Big)^{D_j-1} \tau^{k_j} B_{M+1,n,j} + \frac{4z^2\tau^{M+2-n}}{4z^2+4\tau^2}\Big(\frac{z}{\operatorname{Im} z}\Big)^{n-1} B_{M+1,n} \tag{7.9}$$

where the $h_{n,j}$ are differential polynomials, the $B_{M+1,n,j}$ are D_j dimensional primitive integrals, $D_j \ge 2$,

$$(2\operatorname{Im} z)^{D-1} \int_{x_1<x_2\dots<x_D} e^{2izy\cdot x} P_1(x_1)w(x_2)\dots w(x_{D-1})P_D(x_D)dx_1\dots dx_D$$

where P_1 and P_D are differential monomials. In total there are n w factors (including those with derivatives) and there are at most

$$d \le M+2-n-k_j$$

derivatives which are evenly distributed over P_1 and P_D (equal if $2m-k$ is even, with a difference 1 otherwise). If n is odd then $k_j \ge 1$.

The $B_{M+1,n}$ are sums of n dimensional primitive integrals

$$(2\operatorname{Im} z)^{n-1} \int_{A_n} \exp(izy_j\cdot x) \prod_{k=1}^{n} w(x_k)dx_k.$$

Proof. To shorten the notation we write $R_\pm = (-iz \pm \partial)^{-1}$ so that K (see (7.1)) can be written as

$$K(w) = \begin{pmatrix} 0 & R_- w \\ -R_+ w & -\frac{\tau}{iz}(R_- + R_+)w \end{pmatrix}.$$

Then, since tr $R_+wR_+w =$ tr $R_-wR_-w = 0$

$$\begin{aligned} \operatorname{tr} K^2 &= \operatorname{tr}\Big(-R_+wR_-w - R_-wR_+w - \frac{\tau^2}{z^2}(R_+wR_+w + R_-wR_-w + R_+wR_-w + R_-wR_+w)\Big) \\ &= -\frac{4z^2+4\tau^2}{4z^2}\int e^{2iz|x-y|}w(x)w(y)dxdy, \end{aligned}$$

At $z = i\tilde\tau$ we obtain

(7.10)

$$\begin{aligned} \operatorname{tr} K^2 &= \frac{4\tau^2-4\tilde\tau^2}{4\tilde\tau^2}\int\int_{\mathbb{R}^2} e^{-2\tilde\tau|x_1-x_2|}w(x_1)w(x_2)dx_1\,dx_2 = \frac{4\tau^2-4\tilde\tau^2}{\tilde\tau}\|w\|^2_{H^{-1}_{2\tilde\tau}} \\ &= \frac{4\tau^2-4\tilde\tau^2}{\tilde\tau}\Big(\sum_{n=0}^{N}\frac{(-1)^n}{(2\tilde\tau)^{2n+2}}\|w^{(n)}\|^2_{L^2} + \frac{(-1)^{N+1}}{(2\tau)^{2N+1}}\|w^{(n+1)}\|^2_{H^{-1}_{2\tau}}\Big) \end{aligned}$$

where we used

$$\frac{1}{(2\tau)^2+|\xi|^2} = \sum_{j=0}^{n}(-1)^j\frac{\xi^{2j}}{(2\tau)^{2j+2}} + (-1)^{n+1}(2\tau)^{-2(n+1)}\frac{|\xi|^{2(n+1)}}{(2\tau)^2+|\xi|^2}.$$

Similarly

$$\begin{aligned} \operatorname{tr} K^3 &= \Big(\frac{3\tau^3}{iz^3}+\frac{6\tau}{iz}\Big)\operatorname{tr}\Big((R_-w)^2R_+w + (R_+w)^2R_-w)\Big) \\ &= \Big(\frac{3\tau^3}{iz^3}+\frac{6\tau}{iz}\Big)\int_{x_1<x_2<x_3} e^{2iz(x_3-x_1)}w(x_1)w(x_2)w(x_3)dx_1dx_2dx_3, \end{aligned}$$

$$\begin{aligned}
\operatorname{tr} K^4 &= \operatorname{tr} \Big(\big(2 + \frac{2\tau^2}{z^2} + \frac{\tau^4}{z^4}\big) R_- w R_+ w R_- w R_+ w + \big(\frac{2\tau^2}{z^2} + \frac{4\tau^4}{z^4}\big) R_- w R_- w R_+ w R_+ w \\
&\quad + \Big(\frac{2\tau^2}{z^2} + \frac{4\tau^4}{z^4}\Big)\big((R_- w)^3 R_+ w + R_- w (R_+ w)^3\big)\Big) \\
&= \int\limits_{x_1<x_2<x_3<x_4} \Big(\big(8 + \frac{8\tau^2}{z^2} + \frac{4\tau^4}{z^4}\big) e^{\gamma_1} + \big(\frac{6\tau^2}{z^2} + \frac{12\tau^4}{z^4}\big) e^{\gamma_2}\Big) w(x_1) w(x_2) w(x_3) w(x_4) dx_1 dx_2 dx_3 dx_4
\end{aligned}$$

with

$$\gamma_1 = 2iz(x_4 + x_3 - x_2 - x_1), \qquad \gamma_2 = 2iz(x_4 - x_1).$$

Expanding the 2×2 matrices in general we see that $\operatorname{tr} K^n$ is a linear combination of products of n factors $R_\pm w$. Let Σ be the permutations of n elements. Since

$$\mathbb{R}^m \backslash \{x_j = x_k \text{ for some } j \neq k\} = \bigcup_{\sigma \in \Sigma} \{(x_{\sigma_j})_{1 \le j \le m} : x_1 < x_2 < \dots < x_m\}$$

$\operatorname{tr} K^n$ can be written as a linear combination of expressions of type

$$\Big(\frac{2\tau}{2iz}\Big)^{2m-n} \int_{x_1 < \dots < x_n} \prod_{j=1}^n e^{2iz x \cdot y} w(x_j) dx_j, \quad n/2 \le m \le n \tag{7.11}$$

where

$$\sum_j y_j = 0, \quad y \cdot x \ge x_n - x_1 \qquad \text{for all } x \in \Omega_n = \{x_1 < x_2 < \dots < x_n\}.$$

In Lemma 7.7 it will be shown that n dimensional primitive integrals are bounded by

$$C (2 \operatorname{Im} z)^{1-n} \prod_j \|w_j\|_{L^n}$$

and $(2 \operatorname{Im} z)^{n-1}$ times a primitive integral with $z = i\tau$ converges to

$$\int_{\Omega_n, x_1 = 0} e^{-y \cdot x} dx_2 \dots dx_n \int \prod w_j(x) dx_j.$$

We illustrate the cancellations with an example before stating a general algorithm. Consider

$$T = (2iz)^3 \int_{\Omega_4} e^{-2iz(x_1 + x_2 - x_3 - x_4)} w(x_1) \dots w(x_4)\, dx.$$

Define $\phi = -2iz(x_1 + x_2 - x_3 - x_4)$. Thus T does not decay as $\operatorname{Im} z \to \infty$. We do a partial integration in x_4 to obtain

$$\begin{aligned}
&-(2iz)^2 \int_{\Omega_4} e^{\phi} w(x_1) w(x_2) w(x_3) w'(x_4)\, dx \\
&\qquad - (2iz)^2 \int_{\Omega_3} e^{-2iz(x_1 + x_2 - 2x_3)} w(x_1) w(x_2) w^2(x_3)\, dx.
\end{aligned}$$

The first term has increased decay compared to T, for the price of one derivative. For the second term, we partially integrate again from the right in x_3. We iterate this procedure and arrive at

$$\begin{aligned}
T = &- (2iz)^2 \int_{\Omega_4} e^{\phi} w(x_1)w(x_2)w(x_3)w'(x_4)\,dx \\
&+ iz \int_{\Omega_3} e^{-2iz(x_1+x_2-2x_3)} w(x_1)w(x_2)(w^2)'(x_3)\,dx \\
&- \frac{1}{2}\int_{\Omega_2} e^{-2iz(x_1-x_2)} w(x_1)(w^3)'(x_2)\,dx + \frac{1}{2}\int_{\mathbb{R}} w^4\,dx.
\end{aligned}$$

The one-dimensional integral has no decay, all the higher-dimensional integrals have decay $(\operatorname{Im} z)^{-1}$ if w is sufficiently regular and integrable. We can further iterate, but now from the left, and apply the same procedure in x_1 to all of the remaining multidimensional integrals. We obtain

$$\begin{aligned}
T = &\, 2iz \int_{\Omega_4} e^{\phi} w'(x_1)w(x_2)w(x_3)w'(x_4)\,dx - \frac{1}{2}\int_{\Omega_3} e^{-2iz(2x_1-x_2-x_3)}(w^2)'(x_1)w(x_2)w'(x_3)\,dx \\
&+ \frac{1}{4iz}\int_{\Omega_2} e^{2iz(x_1-x_2)}(w^3)'(x_1)w'(x_2)\,dx - \frac{1}{2}\int_{\Omega_3} e^{-2iz(x_1+x_2-2x_3)} w'(x_1)w(x_2)(w^2)'(x_3)\,dx \\
&+ \frac{1}{8iz}\int_{\Omega_2} e^{-4iz(x_1-x_2)}(w^2)'(x_1)(w^2)'(x_2)\,dx + \frac{1}{4iz}\int_{\Omega_2} e^{-2iz(x_1-x_2)} w'(x_1)(w^3)'(x_2)\,dx \\
&+ \frac{1}{2}\int_{\mathbb{R}} w^4\,dx - \frac{1}{4iz}\int_{\mathbb{R}}(w^3)'w\,dx - \frac{1}{8iz}\int_{\mathbb{R}} w^2(w^2)'\,dx - \frac{1}{4iz}\int_{\mathbb{R}} w^3 w'\,dx.
\end{aligned}$$

Now all the multidimensional integrals here have decay $(\operatorname{Im} z)^{-2}$. Note that the one-dimensional integrals in this step all vanish, which is expected since we know that $\mathcal{T}_{-1}(i\tau) \sim \sum (2i\tau)^{-2j} H^{\text{Gardner}}_{j-1}$ is only nonzero for even powers of τ. Iterating even further will give the quartilinear term from H_2^{Gardner}, and, of H_N^{Gardner} in general.

The following algorithm for connected integrals describes how to treat the cancellations in general: Let $n < M+2$. The term

$$T := (2iz)^M \int_{\Omega_n} e^{2izy\cdot x} \prod w(x_j)dx_j$$

has homogeneity n in w and is written as a n dimensional integral. We integrate by parts in the x_n integral which gives

(7.12)

$$\begin{aligned}
T = &\,(2iz)^{M-1} y_n^{-1} \int_{\Omega_n} e^{2izy\cdot x} w'(x_n) \prod_{j=1}^{n-1} w(x_j) dx_j dx_1 \\
&+ (2iz)^{M-1}\int_{\Omega_{n-1}} e^{2iz\left(\sum\limits_{j=1}^{n-2} y_j x_j + (y_{n-1}+y_n)x_{n-1}\right)} w^2(x_{n-1}) \prod_{j=1}^{n-2} w(x_j) dx_j.
\end{aligned}$$

Let $\hat{y} = (y_1, \dots, y_{m-1} + y_m)$. Then

$$\hat{y}\cdot x \geq x_{m-1} - x_1 \qquad \text{for } x \in \Omega_{m-1}.$$

At this stage we keep the first term on the right hand side of (7.12) and repeat the argument with the second term iteratively if $n < M+1$, ending up with a sum of integrals of dimension $n, n-1, n-2, \dots 1$. For all multidimensional (all besides the one dimensional) integrals we repeat the iterative procedure from x_1. We obtain two one-dimensional integrals plus multidimensional integrals where we gained a decay of z^{-2},

$$\begin{aligned} T &= c_1(2iz)^{M+1-n}\int w^n dx + c_2(2iz)^{M-n}\int w^{n-1}\partial_x w dx \\ &\quad + (2iz)^{M-2}\sum_{D=2}^{n}\sum_{k_1+k_2=n-D}\sum_l c_{M,l,k_1}(2iz)^{D-1}\int_{\Omega_D} e^{2iz\langle y_{D,l},x\rangle}\partial_{x_1}w^{k_1}(x_1)\partial_{x_D}w^{k_2}(x_D) \\ &\qquad \times \prod_{j=2}^{D-1} w(x_j)dx_j dx_1 dx_D \end{aligned}$$

where the homogeneity in w is always n. The second one dimensional integral clearly vanishes, but we keep it at this point since the vanishing of the corresponding terms in the next steps is true but not immediately obvious.

We repeat this iterative procedure $M+2-n$ times and arrive at

$$T = \sum_{k=0}^{M+1-n}(2iz)^k\int P_{M,k}dx + (2iz)^{-1}A_{M+1,n}$$

where $P_{M,k}$ are differential polynomials of homogeneity n with a total of $M+1-n-k$ derivatives. The terms $A_{M+1,n}$ are sums over D dimensional integrals,

$$A_{M+1,n} = \sum_{D=2}^{n} z^{D-1}\sum_j\int_{\Omega_D} e^{iz\langle y_{D,j}\cdot x\rangle}P_{1,D,j}(x_1)w(x_2)\dots P_{D,D,j}(x_D)dx_1\dots dx_D.$$

Here $P_{D,j}$ are homogeneous differential monomials of homogeneity n_j, $n_1+n_D+D-2=n$, with a total of $M+2-n$ derivatives.

We complete the proof of Lemma 7.4: We expand tr K_n into a sum $(\tau/z)^k$, $0\le k\le n$ times the trace of the product of n operators $R_\pm$, which we expand into a sum of n dimensional primitive integrals.

We multiply by the prefactor which we expand as far as we need

$$\frac{(-1)^n}{n}\frac{iz(2z)^{2N+2}}{4z^2+4\tau^2} = \frac{(-1)^n i}{2n}\sum_{l=0}^{L}(-1)^l(2\tau)^{2l}(2z)^{2N-2l} + \frac{(-1)^n i}{2n}\frac{(2z)^{2N-2L}}{4z^2+4\tau^2}.$$

Finally we do the integrations by parts. □

7.3. **The case $n < M+2$: Estimates.** In the previous section we have seen that we can decompose $(2z)^M$ tr $K^n(w)$ into integrals over differential polynomials and a sum of primitive integrals. In the next lemma we provide estimates for the primitive integrals.

Lemma 7.5. *Let $N\ge 0$, $M+2=d+n$, $1\le D\le n$,*

$$I = (2\operatorname{Im} z)^{D-1}\int_{\Omega_D} e^{2izy\cdot x}P_1(x_1)w(x_2)\dots P_D(x_D)dx_1\dots dx_D$$

be a primitive integral (see Definition 7.3) with n terms w, a total number of derivatives d, evenly distributed among P_1 and P_D as above. Then following estimates *hold:*

$$|I| \le c \begin{cases} \|w\|_{L^{M+2}}^{M+2} + \|w^{(\frac{M}{2})}\|_{L^2}^2 \\ \|w\|_{L^2}^{n+2-\frac{d}{M}} \|w^{(\frac{M}{2})}\|_{L^2}^{1+\frac{d}{M}} \\ \|w\|_{L^2}^{n-2} \|u^{(\frac{M+1}{2}-\frac{n}{4})}\|_{L^2}^2 . \end{cases}$$

We apply these estimates to $\mathcal{T}_{M+1,n}$ of Lemma 7.4.

Corollary 7.6. *If $n < M+2$ then*

$$|\mathcal{T}_{M+1,n}| \le c \Big(\frac{|z|}{\operatorname{Im} z}\Big)^{n-1} \begin{cases} \|w\|_{L^{M+2}}^{M+2} + \|w\|_{H_\tau^{\frac{M}{2}}}^2 \\ \|w\|_{L^2}^{n-2} \|w\|_{H_\tau^{\frac{M+1}{2}-\frac{n}{4}}}^2 . \end{cases} \tag{7.13}$$

and if n is in addition odd

$$|\mathcal{T}_{M+1,n}| \le c\tau \Big(\frac{|z|}{\operatorname{Im} z}\Big)^{n-1} \times \begin{cases} \|w\|_{L^{M+1}}^{M+1} + \|w\|_{H_\tau^{\frac{M-1}{2}}}^2 \\ \|w\|_{L^2}^{n-2} \|w\|_{H_\tau^{\frac{M}{2}-\frac{n}{4}}}^2 . \end{cases} \tag{7.14}$$

Proof. We apply Lemma 7.5 to the decomposition of 7.4. We have to estimate I of homogeneity n with a total number $d \le M+2-n-k$ derivatives where k is the power of τ. The corollary follows by Hölder's inequality

$$\int \tau^j |w|^{M+2-j} dx \le \Big(\int w^{M+2} dx\Big)^{1-\frac{j}{M}} \Big(\int \tau^M w^2 dx\Big)^{\frac{j}{M}}$$

and interpolation

$$\tau^{j/2} \|w^{((M-j)/2)}\|_{L^2} \le \|\tau^{M/2} w\|_{L^2}^{\frac{j}{M}} \|w^{(M/2)}\|^{1-\frac{j}{M}} .$$

The sum starts at $j=1$ if n is odd. Tracing the change in the estimates implies the result in that case. ☐

We will use Hölder type estimates for simpler primitive integrals first. To shorten notation, we recall the notation

$$\Omega_D = \{x \in \mathbb{R}^D : x_1 < \dots < x_D\}. \tag{7.15}$$

Lemma 7.7. *Let $D \ge 2$ and $y \in \mathbb{R}^D$ satisfying*

$$\sum_{j=1}^D y_j = 0, \qquad y \cdot x \ge x_D - x_1 \quad \text{if } x \in \Omega_D. \tag{7.16}$$

Define the primitive integral

$$I(z) = \int_{\Omega_D} e^{2izy\cdot x} \prod_{j=1}^D f_j(x_j) dx_j .$$

Let $1 \le p_j \le \infty$ satisfy $\sum_{j=1}^D \frac{1}{p_j} \ge 1$. Then

$$|I(z)| \le (\operatorname{Im} z)^{\sum_{j=1}^D \frac{1}{p_j} - D} \prod \|f_j\|_{L^{p_j}} . \tag{7.17}$$

If $\sum \frac{1}{p_j} = 1$ *and* $Q_j \subset L^{p_j}$ *are equicontinuous sets then*

$$\lim_{\tau\to\infty} (2\tau)^{D-1} I(i\tau) = \int_{\Omega_D : x_1 = 0} e^{-y\cdot x} dx_2 \dots dx_D \int_{\mathbb{R}} \prod_{j=1}^{D} f_j(x)\, dx$$

uniformly for $f_j \in Q_j$.

Proof. Fubini's theorem shows the trivial estimate

$$|I| \le \prod \|f_j\|_{L^1}.$$

We consider the endpoint case $\sum_{j=1}^{M} \frac{1}{p_j} = 1$. We observe that (7.16) implies $y_M \ge 1$. Let $\frac{1}{\tilde p_{M-1}} = \frac{1}{p_{M-1}} + \frac{1}{p_M} \in [1,\infty]$ and

$$\tilde f_{M-1} = f_{M-1}\Big((e^{2izy_M x}\chi_{x>0}) * f_M \Big)$$

We estimate

$$\|\tilde f_{M-1}\|_{L^{\tilde p_{M-1}}} \le \|f_{M-1}\|_{L^{p_{M-1}}} \|(e^{2izy_M x}\chi_{x>0}) * f_M\|_{L^{p_M}}$$

and

$$\begin{aligned} \|(e^{2izy_M x}\chi_{x>0}) * f_M\|_{L^{p_M}} &\le (2\operatorname{Im} z)^{-1} \|(e^{-y_M x}\chi_{x>0})\|_{L^1} \|f_M\|_{L^{p_M}} \\ &\le (2\operatorname{Im} z)^{-1} \|f_M\|_{L^{p_M}}. \end{aligned}$$

Let $(\tilde y)_j = y_j$ if $j < M-1$ and $\tilde y_{M-1} = y_{M-1} + y_M$. Then (7.16) is satisfied:

$$\tilde y \cdot \tilde x = y \cdot (x_1, \dots x_{M-1}, x_{M-1}) \ge x_{M-1} - x_1 \qquad \text{in } \Omega_{M-1}.$$

We complete this case by induction on M. The general case follows now either by a simple variant or an iterative application of complex interpolation by Riesz-Thorin. To see the uniform convergence on equicontinuous sets we observe

$$(2\tau)^{M-1} \chi_{\Omega_M} e^{-2\tau y\cdot x} \to \int_{\Omega_M : x_1 = 0} e^{-y\cdot x} dx_2 \dots dx_M \prod_{j=2}^{M} \delta_{x_j - x_1} \tag{7.18}$$

as $\tau \to \infty$ in the sense of functionals on $C_c(\mathbb{R}^M)$ which is seen by writing

$$(2\tau)^{M-1} I(i\tau) = \int_{0<s_2<s_3 \dots s_M} e^{-2\tau \sum\limits_{j=2}^{M} y_j s_j} \int f_1(x) \prod_{j=2}^{M} f_j(x+s_j) ds_j\, dx.$$

hence

$$\begin{aligned} &\left| (2\tau)^{M-1} I(i\tau) - \int \prod f_j(x) dx \right| \\ &\qquad = \left| \int_{0<s_2<s_3\dots s_M} e^{-2\tau \sum\limits_{j=2}^{M} y_j s_j} \int f_1(x) \left(\prod_{j=2}^{M} f_j(x+s_j) - \prod_{j=2}^{M} f(x) \right) ds_j\, dx \right| \\ &\qquad \le c(\operatorname{supp} f_j) \tau^{-1} \prod_{j=1}^{M} \|f_j\|_{C^1}. \end{aligned}$$

Let $Q_j \subset L^{p_j}(\mathbb{R})$ be bounded equicontinuous sets. Given $\varepsilon > 0$ there exists R and $C > 0$ so that for $f_j \in Q_j$ there exists $F_j \in C^1((-R,R)$ with $\|F_j\|_{C^1} \le C$ with

$\|F_j - f_j\|_{L^{p_j}} < \varepsilon$. Then

$$I(i\tau, f_j) - I(i\tau, F_j) = \sum_{k=1}^{M} I(i\tau, f_1, \dots, f_{j-1}, f_j - F_j, F_{j+1}, \dots, F_M)$$

hence

$$\left|(2\tau)^{M-1} I(i\tau, f_j) - \int_{\Omega_M : x_1 = 0} e^{-y \cdot x} dx_2 \dots dx_M \int \prod f_j dx\right| \le C\tau^{-1} + cM\varepsilon$$

uniformly for $f_j \in Q_j$. □

Proof of Lemma 7.5. Let n_j be the homogeneity of P_j in w and its derivatives. Then $n_1 + n_D + M - 2 = n$ We estimate using Lemma 7.7 to estimate

$$|I| \le \|w\|^{m-2}_{L^{n+d}} \|P_1\|_{L^{\frac{n+d}{n_1+d_1}}} \|P_D\|_{L^{\frac{n+d}{n_2+d_2}}}$$

and, with $P_1 = \prod w^{(\alpha_j)}$ by Hölder's inequality

$$\|P_1\|_{L^{\frac{n+d}{n_1+d_1}}} \le \prod \|w^{(\alpha_j)}\|_{L^{\frac{n+d}{1+\alpha_j}}}$$

with a similar estimate for p_M. The two estimates of 7.5 follow by Lemma 5.4. □

7.4. **Bounding the difference Hamiltonian $\mathcal{T}_N^{\mathrm{Gardner}}$.** The central estimate in this subsection is (2.44). Here we prove the claims on $\mathcal{T}_N^{\mathrm{Gardner}}$ and $\mathcal{T}_N^{\mathrm{KdV}}$ stated in Section 2 which we repeat here.

Proposition 7.8. *The following estimate holds if $\|w\|_{L^2} < \frac{1}{4}\sqrt{\operatorname{Im} z}$:*

$$|\mathcal{T}_N^{\mathrm{Gardner}}(iz, w, \tau)| \le c\left(\frac{|z|}{\operatorname{Im} z}\right)^{2N+1} (\operatorname{Im} z)^{-2}\left(\|w\|^{2N+4}_{L^{2N+4}} + \|w\|^2_{H_\tau^{N+1}}\right).$$

Proof. We write

$$\begin{aligned}
2z\mathcal{T}_N^{\mathrm{Gardner}} &= (2z)^{2N+3}\mathcal{T}_{-1}^{\mathrm{Gardner}} - \sum_{k=0}^{N} (2z)^{2(N-k)+1} H_k^{\mathrm{Gardner}} \\
&= (2z)^{-1}\sum_{n=2}^{2N+3} \mathcal{T}_{2N+3,n} + \sum_{n>2N+3} \frac{(-1)^n}{n} \frac{iz(2z)^{2N+3}}{4z^2+4\tau^2} \operatorname{tr} K^n \\
&\quad - \sum_{k=0}^{N} (2z)^{2(N-k)+1} H_k^{\mathrm{Gardner}} + \sum_{n=2}^{2N+3}\sum_{j=0}^{2N+2} (2z)^{2N+2-j} \int h_{n,j}
\end{aligned} \tag{7.19}$$

and apply (7.5) to the traces and estimate

$$\left|\frac{iz(2z)^{2N+3}}{4z^2+4\tau^2} \operatorname{tr} K^n\right| \le c_N\Big(|z|/\operatorname{Im} z\Big)^{2N+2} (\operatorname{Im} z)^{-1}\Big(\frac{3}{(\operatorname{Im} z)^{1/2}}\|w\|_{L^2}\Big)^{n-2N-4} \|w\|^{2N+4}_{L^{2N+4}}.$$

Summing over n we obtain the desired estimate for the traces. By Lemma 7.4 and Corollary 7.6, if $n \le 2N+3$

$$|\mathcal{T}_{2N+3,n}| \le c\Big(\frac{|z|}{\operatorname{Im} z}\Big)^{n-1}\Big(\|w\|^{2N+4}_{L^{2N+4}} + \|w\|^2_{H^{N+1}}\Big)$$

Comparison of the expansion of Lemma 7.4 with the expansion (3.25) allows to identify the coefficients: The h_j vanish if j is odd and $\int \sum h_{2j,n} dx = H_j^{\mathrm{Gardner}}$. In particular (7.19) vanishes and

$$\mathcal{T}_{2N+2,n} = 2z\mathcal{T}_{2N+1,n} = (2z)^2\mathcal{T}_{2N,n} + H_{N,n}^{\mathrm{Gardner}}. \tag{7.20}$$

□

Lemma 7.9. *The quadratic part of $\mathcal{T}_{-1}^{\text{Gardner}}$ is*

$$\frac{1}{2}\frac{-iz(2z)^{2N+2}}{4z^2+4\tau_0^2} \operatorname{tr} K^2(w) = \frac{1}{2}\int \Big((-\partial+2iz)^{-1}w\Big)^2 dx.$$

The quadratic part of $\mathcal{T}_N^{\text{Gardner}}(z,w,\tau_0)$ is

$$\frac{1}{2}\int ((\partial-2iz)^{-1}w^{(N+1)})^2 dx.$$

Proof. We give two different arguments. First the quadratic part of

$$\int w^2(z) - w^2 dx = \int [(\partial - 2iz + w(iz))^{-1}(\partial+2\tau+w)w]^2 - w^2 dx$$

is

$$\int w(-\partial^2-4z^2)^{-1}(-\partial^2+4\tau^2)w - w^2 dx = -(4z^2+4\tau^2)\int \big[(-\partial+2iz)w\big]^2.$$

Alternatively we computed $\operatorname{tr} K^2$ explicitly in (7.10). □

We recall that we denote the higher order part for $z = i\tau$ by

$$\mathcal{T}_N^{NL} = \mathcal{T}_N^{\text{Gardner}} - \frac{1}{2}\|w^{(N+1)}\|^2_{H^{-1}_{2\tau}}.$$

Lemma 7.10. *We have*

$$|\mathcal{T}_N^{NL}(i\tau,w,\tau_0)| \le c(\tau_0,\|w\|_{L^2})\tau^{-1}\|w\|^2_{H^N_{\tau_0}}.$$

Proof. By the proof of Proposition 7.8 we have to estimate $\mathcal{T}_{2N+2,n}$. The claim follows by Corollary 7.6. More precisely, with $M = 2N+1$ and maximal a total number of $d = 2N+3-n$ derivatives

$$\mathcal{T}_{2N+2,n} \le c\left(\frac{|z|}{\operatorname{Im} z}\right)^{n-1}\|w\|^{n-2}_{L^2}\|w\|^2_{H_\tau^{N+1-\frac{n}{4}}}$$

Then $N+1-\frac{n}{4} \le N$ if $n \ge 4$. Finally

$$|\mathcal{T}_{2N+2,3}| \le c\tau\left(\frac{|z|}{\operatorname{Im} z}\right)^2\|w\|_{L^2}\|w\|^2_{H_\tau^{N-\frac{1}{4}}}$$

since 3 is odd. The claim follows by summation with respect to n. □

We obtain the claim on equicontinuity (2.45) of Proposition 2.13 as corollary.

Corollary 7.11. *A subset $Q \subset H^N$ is bounded and equicontinuous iff*

$$\lim_{\tau\to\infty}\sup_{w\in Q}|\mathcal{T}_N^{\text{Gardner}}(i\tau,w,\tau_0)| = 0.$$

Proof. By Lemma 7.10 is suffices to consider the quadratic part. Then

$$\|w^{(N+1)}\|^2_{H_\tau^{-1}} = \int \frac{\xi^2}{\tau^2+\xi^2}|\xi^N\hat{w}|^2 dx \to 0$$

uniformly for $w^{(N)} \in \tilde{Q}$ iff $\tilde{Q} \subset L^2$ is bounded and equicontinuous. □

We turn to bounding variational derivatives and prove (2.46), which we formulate as a lemma.

Lemma 7.12. *Let $N \geq 2$, $Q \subset H^{N-1}$ be bounded and equicontinuous. Then*

$$\Big\| \frac{\delta}{\delta w} \mathcal{T}_N^{\text{Gardner}}(i\tau, w, \tau_0) \Big\|_{H^{-N-1}} \to 0 \qquad \text{as } \tau \to \infty$$

uniformly in Q.

Proof. We observe that

$$\|(4\tau^2 - \partial^2)^{-1} \partial^{2N+2} w\|_{H^{-N-1}}$$

converges to 0 uniformly in bounded and equicontinuous sets. From the definitions and the properties of the trace

$$\int \phi \frac{\delta}{\delta w} \frac{1}{n} \operatorname{tr} K^n = \operatorname{tr} K(\phi) K^{n-1}(w)$$

and hence all terms with $n \geq 2N+3$ converge uniformly to 0 on bounded sets in H^1. If $n \geq 2N+3$ we hence obtain $\tau = \operatorname{Im} z$ in this estimate

$$\Big| \frac{1}{n} \int \phi \frac{\delta}{\delta w} \operatorname{tr} K^n dx \Big| \leq (c\tau^{-1/2} \|w\|_{L^2})^{2N+3-n} (\operatorname{Im} z)^{-2N-2} \|\phi\|_{L^\infty} \|w\|_{L^{2N+2}}^{2N+2}.$$

For $3 \leq n \leq 2N+2$ we use the decomposition of Lemma 7.4 which reduces the claim to the corresponding claim for primitive integrals as in Lemma 7.4. Let $M = 2N+1$, I be an D dimensional primitive integrals with a prefactor $(\operatorname{Im} z)^{D-1}$ with homogeneity n in w and a total of $d \leq 2N+3-n$ derivatives which are evenly distributed. The variational derivative plus duality amounts to replacing w's by ϕ, resp. n by $n-1$ and M by $M-1$. Of course the situation improves further if derivatives fall on ϕ. Hence, if $n > 2$ by Lemma 7.7

$$\Big| \int \phi \frac{\delta}{\delta w} I dx \Big| \leq c \sum_{m=0}^{d/2} \|\phi^{(m)}\|_{L^\infty} \|w\|_{L^2}^{n-3} \|w\|^2_{H_\tau^{N+\frac{1}{4}-\frac{n}{4}}}.$$

Thus all terms of homogeneity $n \geq 5$ are bounded by $\tau^{-1}\|w\|^2_{H^{N-1}}$ with a constant depending on $\|w\|_{L^2}$.

It remains to consider the quadratic ($n = 3$) and cubic contributions ($n = 4$) with no derivative falling on ϕ. We consider

$$\mathcal{T}_{N,n}^{\text{Gardner}} := (2z)\mathcal{T}_{2N-1,n} + H_{N,n}^{\text{Gardner}}$$

with $n = 3$ or $n = 4$ with suggestive notation. If at least one derivative falls on ϕ or the power of τ is at least one in the cubic case ($n = 4$) or at least 2 in the quadratic case ($n = 3$) we argue as above and obtain a decay like $(\operatorname{Im} z)^{-1}$. For the remain case we omit the final integration by parts and write $(2\tau)^{-1}\mathcal{T}_{2N+2,3}$ as a linear combination of terms of the type

$$\begin{aligned} &\tau_0 (2\tau)^{-2} \int_{\Omega_3} e^{-2\tau(x_3 - x_1)} w^{(N-1)}(x_1) \phi(x_2) w^{(N-1)}(x_3) dx_1 dx_2 dx_3 \\ &\qquad - \int_{0 < x_2 < x-2} e^{-x_3} dx_2 dx_3 \int (w^{(N-1)}(x))^2 \phi(x) dx \end{aligned}$$

which converge to 0 uniformly on bounded equicontinuous sets in H^{N-1} by Lemma 7.7. For $n=4$ we obtain similarly a linear combination of terms of the type

$$\tau_0(2\tau)^{-2}\int_{\Omega_3} e^{-2\tau(x_3-x_1)}w^{(d_1)}(x_1)w^{(d_2)}(x_1)\phi(x_2)w^{(N-1)}(x_3)dx_1dx_2dx_3 - \int_{0<x_2<x-2} e^{-x_3}dx_2dx_3\int w^{(d_1)}w^{(d_2)}w^{(N-1)}\phi\, dx$$

which converge to 0 uniformly on bounded equicontinuous sets in $H^{N-1}\cap L^\infty$, and hence on bounded equicontinuous sets in H^{N-1} if $N\ge 2$.

□

We translate these bounds to the KdV side by the Miura map and prove the estimates of Propositions 2.3. We recall (3.27)

$$4z^2\mathcal{T}_N^{\mathrm{KdV}}(z,w_x+2\tau w+w^2) = 4\tau^2\mathcal{T}_N^{\mathrm{Gardner}}(z,w,\tau)+4z^2\mathcal{T}_{N+1}^{\mathrm{Gardner}}(z,w,\tau).$$

We estimate

$$\begin{aligned}|\mathcal{T}_N^{\mathrm{KdV}}(z,u)| &\le |4(\tau/z)^2\mathcal{T}_N^{\mathrm{Gardner}}(z,w,\tau)|+|\mathcal{T}_{N+1}^{\mathrm{Gardner}}(z,w,\tau)|\\ &\le c((\operatorname{Im} z)^{-1/2}\|w\|_{L^2})(\operatorname{Im} z)^{-2}\Big(\frac{|z|}{\operatorname{Im} z}\Big)^{2N+3}\Big(\|w\|^2_{H_\tau^{N+2}}+\|w\|^{2N+4}_{L^{2N+4}}\Big)\end{aligned}$$

Together with the fact that $\|w\|_{H^{N+2}}\sim\|u\|_{H^{N+1}}$ (see Proposition 4.8), and

$$\|w\|^{2N+4}_{L^{2N+4}}\le c(\|w\|_{L^2})\|u\|^{N+2}_{L^{N+2}},$$

which shows (2.11). The bound (2.12) follows by similar arguments.

7.5. **Weighted estimates.** In this subsection we prove the central estimates used to prove local smoothing of the difference flow (2.58) and convergence of the difference flow (2.57).

Proposition 7.13. *A) The following estimates hold for $n\ge M+2$.*

(7.21)
$$\Big|\int \operatorname{sech}^2\phi\frac{\delta}{\delta w}\operatorname{tr} K^n(w)dx\Big| \le \big(c(\operatorname{Im} z)^{-1/2}\|w\|_{L^2}\big)^{n-M-2}(\operatorname{Im} z)^{-M-1}\|\phi\|_{L^\infty}\|w\|^{M-1}_{L^2}\|\operatorname{sech} w\|^2_{H_\tau^{\frac{M-1}{4}}}$$

(7.22)
$$\Big|\int \tanh w\partial\frac{\delta}{\delta w}\operatorname{tr} K^n(w)dx\Big| \le \big(c(\operatorname{Im} z)^{-1/2}\|w\|_{L^2}\big)^{n-M-2}(\operatorname{Im} z)^{-M-2}|z|\|w\|^{M}_{L^2}\|\operatorname{sech} w\|^2_{H_\tau^{\frac{M}{4}}}.$$

B) Let $2\le n<M+3$ and let

$$I=(\operatorname{Im} z)^{D-1}\int_{\Omega_D} e^{2izy\cdot x}P_1(x_1)w(x_2)\dots P_D(x_D)dx_1\dots dx_D \tag{7.23}$$

be a $D\ge 2$ dimensional primitive integral with homogeneity n, a total number of derivatives $d=M+3-n$ which are evenly distributed. Then

$$\Big|\int \operatorname{sech}^2\phi\frac{\delta}{\delta w}Idx\Big|\le\|\phi\|_{H^M}\|w\|^{n-3}_{L^2}\|\operatorname{sech} w\|^2_{H_\tau^{\frac{2M+3-n}{4}}}\qquad \text{if } n\ge 3 \tag{7.24}$$

and

$$\Big|\int \tanh w\partial_x\frac{\delta}{\delta w}Idx\Big|\le c\|w\|^{n-2}_{L^2}\frac{|z|}{\operatorname{Im} z}\|\operatorname{sech} w\|^2_{H_\tau^{\frac{2M+4-n}{4}}}. \tag{7.25}$$

We postpone the proof and deduce (2.57) and (2.58).

Proof of (2.57) *and* (2.58). In this section constants are allowed to depend on τ_0 which simplifies the notation. Let $M = 2N+2$ and $n \geq 2N+4$. By duality

$$\Big\| \operatorname{sech}^2 \frac{\delta}{\delta w} \mathcal{T}_{2N+3,n} \Big\|_{L^1} \leq (c(\operatorname{Im} z)^{-1/2} \|w\|_{L^2})^{n-2N-3} \Big(\frac{|z|}{\operatorname{Im} z} \Big)^{2N+3} \|w\|_{L^2}^{2N} \|\operatorname{sech} w\|^2_{H^{\frac{N}{2}+\frac14}}.$$

If $2 < n \leq M+2 = 2N+3$,

$$\Big\| \operatorname{sech}^2 \frac{\delta}{\delta w} \mathcal{T}_{2N+3,n}(i\tau, w, \tau_0) \Big\|_{H^{-N}} \leq c \|w\|_{L^2}^{n-3} \|\operatorname{sech} w\|^2_{H^{N+\frac{5-n}{4}}}$$

and, since $n = 3, 5$ is

$$\Big\| \operatorname{sech}^2 \frac{\delta}{\delta w} \mathcal{T}_{2N+3,5}(i\tau, w, \tau_0) \Big\|_{H^{-N}} \leq c \|w\|_{L^2}^{n-3} \|\operatorname{sech} w\|^2_{H^{N-\frac12}}$$

$$\Big\| \operatorname{sech}^2 \frac{\delta}{\delta w} \mathcal{T}_{2N+3,3}(i\tau, w, \tau_0) \Big\|_{H^{-N}} \leq c \|w\|_{L^2}^{n-3} \|\operatorname{sech} w\|^2_{H^{N}}$$

Let $\mathcal{T}_N^{>5}$ be the contributions of homogeneity $n \geq 5$ to $\mathcal{T}_N$. Then, if $\|w\|_{L^2} \leq \frac14 \tau$

$$\| \operatorname{sech}^2 \frac{\delta}{\delta w} \mathcal{T}_N^{>5}(i\tau, w, \tau_0) \|_{H^{-N}} \leq c\tau^{-2}(1 + \|w\|_{L^2}^{2N+1}) \|\operatorname{sech} w\|^2_{H^{N-\frac14}}. \tag{7.26}$$

Let $\chi \in \mathbb{C}_c^\infty([-2,2])$ be an even function, identically 1 on $[-1,1]$. We decompose

$$w = w_< + w_> \qquad w_< = \mathcal{F}^{-1} \chi(|\xi|/\tau) \hat{w}.$$

If $0 \leq s \leq N$

$$\|\operatorname{sech} w\|_{H^s} \leq \|\operatorname{sech} w_<\|_{H^s} + \|\operatorname{sech} w_>\|_{H^s},$$

$$\begin{aligned} \|\operatorname{sech} w_>\|_{H^s} &\leq \|\operatorname{sech} w_>\|_{L^2}^{1-\frac{s}{N}} \Big(\|\operatorname{sech} w_>\|_{L^2} + \|\operatorname{sech} w_>^{(N)}\|_{L^2} \Big)^{\frac{s}{N}} \\ &\leq c(\|\operatorname{sech} w\|_{L^2} + \|\operatorname{sech} w^{(N+1)}\|_{H^{-1}_{2\tau}}) \end{aligned}$$

and

$$\begin{aligned} \|\operatorname{sech} w_<\|_{H^s_\tau} &\leq \|\operatorname{sech} w\|_{L^2}^{1-\frac{s}{N+1}} \|\operatorname{sech} w_<\|_{H^{N+1}}^{\frac{s}{N+1}} \\ &\leq c\tau^{\frac{s}{N+1}} (\|\operatorname{sech} w\|_{L^2} + \|\operatorname{sech} w_<^{(N+1)}\|_{H^{-1}_{2\tau}}). \end{aligned} \tag{7.27}$$

These weighted interpolation estimates are a consequence of the interpolation estimates without weight by first proving them on bounded intervals by restriction and extension, and summation over the intervals. If $n > 4$ or $n = 3$ we arrive at the desired bound by

$$\begin{aligned} \tau^{-2} \|\operatorname{sech} w\|^2_{H^N} &\leq \tau^{-2} \Big(\|\operatorname{sech} w_<\|_{H^N} + \|\operatorname{sech} w_>\|_{H^N} \Big)^2 \\ &\leq c\tau^{-\frac{2}{N+1}} \Big(\|w\|_{L^2} + \|\operatorname{sech} w^{(N+1)}\|_{H^{-1}_\tau} \Big)^2 \end{aligned}$$

The case $n = 4$ is more delicate. We have the two bounds for the quartic terms in $\mathcal{T}_N^{\text{Gardner}}$, which, using a different notation to above,

$$\Big\| \operatorname{sech}^2 \frac{\delta}{\delta w} \mathcal{T}_{N,4}^{\text{Gardner}}(\tau, w, \tau_0) \Big\|_{H^{-N}} \leq c\tau^{-1} \|w\|_{L^2} \|\operatorname{sech} w\|^2_{H^{N-\frac14}}.$$

$$\Big\| \operatorname{sech}^2 \frac{\delta}{\delta w} \mathcal{T}_{N,4}^{\text{Gardner}}(\tau, w, \tau_0) \Big\|_{H^{-N}} \leq c\tau^{-2} \|w\|_{L^2} \|\operatorname{sech} w\|^2_{H^{N+\frac14}}.$$

We split $w = w_< + w_>$, polarize the cubic term ($n = 4$) and expand. There can be 0, 1, 2 or 3 high frequency factors $w_>$. Then, with a suggestive notation

$$\Big\| \operatorname{sech}^2 \frac{\delta}{\delta w} \mathcal{T}^{\text{Gardner}}_{N,4,>>>}(\tau, w, \tau_0) \Big\|_{H^{-N}} + \Big\| \operatorname{sech}^2 \frac{\delta}{\delta w} \mathcal{T}^{\text{Gardner}}_{N,4,<>>}(\tau, w, \tau_0) \Big\|_{H^{-N}}$$
$$\leq c\tau^{-1} \|w\|_{L^2} \| \operatorname{sech} w_> \|^2_{H^{N-\frac14}}$$

since $\|w_<\|_{H^s} \leq 2\|w_>\|_{H^s}$ (and commuting the weight with the Fourier multiplier and putting more derivatives on the low frequency part),

$$\Big\| \frac{\delta}{\delta w} \mathcal{T}^{\text{Gardner}}_{N,4,<<>}(\tau, w, \tau_0) \Big\|_{H^{-N}} \leq c\tau^{-2} \|w\|_{L^2} \| \operatorname{sech} w_< \|_{H^{N+\frac12}} \| \operatorname{sech} w_> \|_{H^N}$$

and

$$\Big\| \frac{\delta}{\delta w} \mathcal{T}^{\text{Gardner}}_{N,4,<<<}(\tau, w, \tau_0) \Big\|_{H^{-N}} \leq c\tau^{-2} \|w\|_{L^2} \| \operatorname{sech} w_< \|^2_{H^{N+\frac14}}.$$

We observe that

$$\| \operatorname{sech} w_> \|_{H^N} \leq c\Big(\| \operatorname{sech} w\|_{L^2} + \| \operatorname{sech} w^{(N+1)} \|_{H^{-1}_{2\tau}} \Big).$$

Together with (7.27) and the easy case $n = 2$ this implies (2.57).

The analogous estimates for Kato smoothing are

$$\Big| \int \tanh(x) w \partial \frac{\delta}{\delta w} \operatorname{tr} K^n dx \Big| \leq \big(c (\operatorname{Im} z)^{-1/2} \|w\|_{L^2} \big)^{n-2N-4} (\operatorname{Im} z)^{-2N-3} \|w\|^{2N+2}_{L^2} \| \operatorname{sech} w \|^2_{H_\tau^{\frac{N+1}{2}}}$$

for $n \geq 2N + 4$ and

$$\Big| \int \tanh(x) w \partial \frac{\delta}{\delta w} \mathcal{T}_{2N+3,n} dx \Big| \leq c \|w\|^{n-2}_{L^2} \| \operatorname{sech} w \|^2_{H_\tau^{N+\frac{6-n}{4}}}$$

We argue as above if $n \geq 6$. The case $n = 5$ is simple due to the gain if n is odd. The case $n = 3$ follows by the very same arguments as for $n = 4$ above. There are only trivial changes if there are two or more high frequency terms. In total we get a bound with c depending on N,τ_0 and $\|w\|_{L^2}$, assuming $\|w\|_{L^2} \leq c\tau^{1/2}$,

$$\Big| \int \tanh(x) w \partial \frac{\delta}{\delta w} \mathcal{T}^{>2}_N dx \Big| \leq c\tau^{-\frac{1}{2(N+2)}} \big(\| \operatorname{sech} w \|^2_{L^2} + \| \operatorname{sech} w^{(N+1)} \|_{H^{-1}_{2\tau}} \big)$$

which implies (2.58). □

Proof of Proposition 7.13. We have

$$\int \phi \operatorname{sech}^2(x/R) \frac{\delta}{\delta w} \operatorname{tr} K^n(w) dx = n \operatorname{tr} \Big(K(\operatorname{sech}^2(x/R)\phi) K^{n-1}(w) \Big).$$

Again we estimate the traces for $n \geq 2N+2$. Suppose that $2 \leq p \leq \infty$, η be slowly τ varying and $1 \leq \tau \leq \operatorname{Im} z$. We claim

$$\|\eta K(w) \eta^{-1}\|_{\mathfrak{I}_p} \leq \big(1 + \frac{\tau}{\operatorname{Im} z}\big) \|w\|_{L^p}. \tag{7.28}$$

To verify the claim we bound the Hilbert-Schmidt norm

$$\|\eta K(w) \eta^{-1}\|^2_{\mathfrak{I}_2} \leq c (\operatorname{Im} z)^{-1/2} \|w\|_{L^2}$$

and

$$\|\eta K(w) \eta^{-1}\|^2_{L^2 \to L^2} \leq c (\operatorname{Im} z)^{-1} \|w\|_{L^\infty}.$$

Estimate (7.28) follows by interpolation. An algebraic manipulation gives

$$K(\operatorname{sech}^2(x/R)\phi)K^M(w) = K(\phi)\big(\operatorname{sech}^2 K(\operatorname{sech}^{\frac{2}{M}} w)\operatorname{sech}^{-2}\big)$$
$$\big(\operatorname{sech}^{\frac{2(M-1)}{M}} K(\operatorname{sech}^{\frac{2}{M}} w)\operatorname{sech}^{-\frac{2(M-1)}{M}}\big)\dots\big(\operatorname{sech}^{\frac{2}{M}} K(\operatorname{sech}^{\frac{2}{M}} w)\operatorname{sech}^{-\frac{2}{M}}\big)$$

and
(7.29)

$$\begin{aligned}(\operatorname{Im} z)^M\Big|\operatorname{tr}\ K(\operatorname{sech}^2\phi)K^n(w)\Big| \le{}& (2(\operatorname{Im} z)^{-1/2}\|w\|_{L^2})^{n-M}\operatorname{Im} z\|K(\phi)\|_{L^2\to L^2}\\ &\times\prod_{j=1}^{M}(\operatorname{Im} z)^{\frac{M-1}{M}}\|\operatorname{sech}^{\frac{2j}{M}} K^M(\operatorname{sech}^{\frac{2}{M}} w)\operatorname{sech}^{-\frac{2j}{M}}\|_{\mathfrak{I}_M}\\ \le{}&\|\phi\|_{L^\infty}(2(\operatorname{Im} z)^{-1/2}\|w\|_{L^2})^{n-M}\|\operatorname{sech}^{\frac{2}{M}} w\|^M_{L^M}.\end{aligned}$$

The argument is clearly more flexible: We may distribute the weight among the factors as we wish. Primitive integrals can be understood as traces, and we may estimate them in the same fashion. We consider a primitive integral defined in (7.23) Then, by Hölder's inequality

$$\Big|\int\operatorname{sech}^2\phi\frac{\delta}{\delta w}Idx\Big| \le c\sum_{k=0}^{d/2}\|\phi^{(k)}\|_{L^\infty}\sum_{\sum\alpha_j=d-k}\prod_{j=1}^{n-1}\|\operatorname{sech}^{\frac{2(1+\alpha_j)}{n+d-k-1}} w^{(\alpha_j)}\|_{L^{\frac{n+d-k-1}{1+\alpha_j}}}.$$

We apply (5.8) of Lemma 5.4 and arrive at

$$\Big|\int\operatorname{sech}^2(x/R)\phi\frac{\delta}{\delta w}Idx\Big| \le c\|\phi\|_{C^{d/2}}\|w\|_{L^2}^{n-3}\|\operatorname{sech}(x/R)w\|^2_{H^{\frac{n+2d-3}{4}}}.$$

We turn to the proof of (2.58) and begin with the calculation, setting $R=1$ for simplicity,

$$\begin{aligned}\int w\tanh^2\partial_x\frac{\delta}{\delta w}\operatorname{tr}\ K^n(w)dx ={}& -\frac{1}{n}\operatorname{tr}\ \Big(K((\tanh w)_x)K^{n-1}(w)\Big)\\ ={}& -\frac{n-1}{n}\operatorname{tr}\ \Big(K(\operatorname{sech}^2 w)K^{n-1}(w)\Big)\\ &+\frac{n-1}{n}\sum_{j=0}^{n-2}\operatorname{tr}\ \Big(K(\tanh w)K^j(w)K(w_x)K^{N-j-2}(w) - K(\tanh w_x)K^{n-1}(w)\Big).\end{aligned}$$

In the fashion as above with obvious modifications we obtain for $n\ge M+2$

$$\Big|\operatorname{tr}\ K(\operatorname{sech}^2 w)K^{n-1}(w)\Big| \le c((\operatorname{Im} z)^{-1/2}\|w\|_{L^2})^{n-M-2}(\operatorname{Im} z)^{1-M}\|w\|^M_{L^2}\|\operatorname{sech} w\|^2_{H^{\frac{M}{4}}}$$

and for I as above

$$\Big|\int\operatorname{sech}^2 w\frac{\delta}{\delta w}Idx\Big| \le c\|\phi\|_{C^{d/2}}\|w\|^{M-2}_{L^2}\|\operatorname{sech} w\|^2_{H^{\frac{n+2d-2}{4}}}.$$

The last line above requires more effort. To remove the derivative (and pay by a factor z) we observe

$$K(w_x) = [\partial, K(w)]$$

and, with $\tilde K(w)$ of the same structure as $K(w)$, but with some sign changes

$$\partial K(w) = -iz\tilde K(w) + \begin{pmatrix}0 & w\\ w & 0\end{pmatrix}.$$

By a small abuse of notation the first summand corresponds to replacing w_x by izw. We estimate (again η is assumed to be τ slowly varying)

$$\left\|\eta\begin{pmatrix} 0 & w_1 \\ w_1 & 0 \end{pmatrix} K(w_2)\eta^{-1}\right\|_{\mathfrak{I}_2} \le c \begin{cases} \|w_1\|_{L^2}\|w_2\|_{L^2} \\ (\operatorname{Im} z)^{-1/2}\|w_1\|_{L^\infty}\|w_2\|_{L^2} \end{cases}$$

and

$$\left\|\eta\begin{pmatrix} 0 & w_1 \\ w_1 & 0 \end{pmatrix} K(w_2)\eta^{-1}\right\|_{L^2\to L^2} \le c\tau^{-1}\|w_1\|_{L^\infty}\|w_2\|_{L^\infty}.$$

We interpolate the estimates for $p \ge 2$

$$\left\|\eta\begin{pmatrix} 0 & w_1 \\ w_1 & 0 \end{pmatrix} K(w_2)\eta^{-1}\right\|_{\mathfrak{I}_p} \le c\tau^{-1+\frac{2}{p}}\|w_1\|_{L^{2p}}\|w_2\|_{L^{2p}}. \tag{7.30}$$

First $K(\tanh w) = K(w)\tanh$. In the last line above we want to commute tanh and $K(w)$. The integral kernel of the commutator of $\tanh(x/R)$ and $R_\pm$

$$g(x,y) = (\tanh(x) - \tanh(y))\chi_{\pm(x-y)>0}e^{2iz|x-y|}\min\{\operatorname{sech}^2(x), \operatorname{sech}^2(y)\}$$

is bounded by

$$\frac{1}{\operatorname{Im} z}e^{-\operatorname{Im} z|x-y|}(\cosh(x)+\cosh(y))^{-1}$$

hence for $p \ge 2$

$$\|\eta[\tanh, R(w)]\eta^{-1}\|_{\mathfrak{I}_p} \le \frac{c}{\operatorname{Im} z}\|w\|_{L^p}.$$

Together, if $n \ge M+2$,

$$\begin{aligned} &|\operatorname{tr}(K(w)\tanh K(w)\dots K(w_x)\dots) - \operatorname{tr} K(w)\dots K(w_x)\tanh| \\ &\quad \le c((\operatorname{Im} z)^{-1/2}\|w\|_{L^2})^{n-M-2}|z|(\operatorname{Im} z)^{-M-2}\|\operatorname{sech}^{\frac{2}{M+2}} w\|^{M+2}_{L^{M+2}}. \end{aligned}$$

Let I be a primitive integral as in (7.23). We apply the very same arguments and obtain

$$\left|\int \tanh\frac{\delta}{\delta w}I dx\right| \le c\frac{|z|}{\operatorname{Im} z}\|w\|^{M-2}_{L^2}\|\operatorname{sech} w\|^2_{H^{\frac{n+2d-2}{4}}}.$$

This completes the proof. □

Appendix A. Calculations

A.1. Proof of Theorem 3.12. In this section we prove Theorem 3.12. To do so we introduce some notation. Let $\omega(\tau) = 2\tau v$ be a rescaling of the good variable. Then if u solves the Nth KdV equation, ω solves

$$\omega_t = 2\partial\Big[(\omega+2\tau)\sum_{n=-1}^{N-1}(2i\tau)^{2(N-1-n)}\frac{\delta H_n^{\mathrm{KdV}}}{\delta u}\Big]. \tag{A.1}$$

The relation between u and ω now reads

$$u = -\frac{1}{2}\frac{\omega_{xx}}{\omega+2\tau} + \frac{3}{4}\frac{\omega_x^2}{(\omega+2\tau)^2} + \frac{1}{4}\omega^2 + \tau\omega.$$

And in the case of the KdV equation $N=1$ we find e.g.

$$\omega_t = \partial_x\Big[-\omega_{xx} + 3\tau\omega^2 + \frac{1}{2}\omega^3 + \frac{3}{2}\frac{\omega_x^2}{\omega+2\tau}\Big]. \tag{A.2}$$

Thus in order to prove Theorem 3.12 we are lead to prove that the Nth equation for ω can be written in the form $\omega_t = \partial_x \tilde{F}_N$, where

$$\tilde{F}_N = \sum_{n,l,K,d} (\omega + 2\tau)^{-n} \tau^l \tilde{f}_{N,n,k,d}(\omega), \tag{A.3}$$

where $\tilde{f}_{N,n,k,d}$ has homogeneity k in ω and a total number of derivatives d, and the sum is restricted by

$$\begin{gathered} 0 \le n \le 2N-1, \qquad l+k+d = 2N+1+n, \qquad l+k \ge n+1, \\ \#\{\text{factors of } \omega \text{ with at least 1 derivative}\} \ge n+1 \quad \text{if} \quad n \ge 1. \end{gathered}$$

Moreover, the linear part of the equation is $(-1)^N w^{(2N+1)}$, and $\tau^l \tilde{f}_{N,n,k,d}$ contains no term of the form $\tau^l \omega^k \omega^{(2N)}$ with $l+k = n+1$. The number of derivatives d is always even.

Define a generalized differential monomial in ω to be an expression of the form

$$\tau^l (\omega + 2\tau)^{-n} \prod_{i=1}^{k} \omega^{(\alpha_i)}. \tag{A.4}$$

We call k its homogeneity, n its negative homogeneity, and denote by $d = \sum_{i=0}^k \alpha_i$ its total number of derivatives. Moreover, we define the degree

$$D = l + k - n + d.$$

A generalized differential polyonomial of degree D is then a sum of generalized differential monomials of degree D. Note that expanding the fraction by powers of $\omega + 2\tau$ leaves its degree invariant. What the first part of Theorem 3.12 now says is that the Nth equation for ω has the form

$$\omega_t = \partial_x F_N,$$

where F_N is a generalized differential polynomial in ω of degree $2N+1$ with some further properties.

Proof of Theorem 3.12. First we note that

$$\partial_x \frac{\prod_{i=1}^k \omega^{(\alpha_i)}}{(\omega+2\tau)^n} = \sum_{j=1} \frac{\omega^{(\alpha_j+1)} \prod_{i \ne j} \omega^{(\alpha_i)}}{(\omega+2\tau)^n} - n \frac{w_x \prod_{i=1}^k \omega^{(\alpha_i)}}{(\omega+2\tau)^{n+1}}.$$

Each derivative falling onto $(\omega+2\tau)^{-1}$ produces a new factor of ω with at least one derivative. The degree of the corresponding generalized monomial gets increased by one. Moreover it is clear that the degree of the product of generalized monomials is just the sum of the two degrees.

For KdV, we know

$$\frac{\delta H_m^{\mathrm{KdV}}}{\delta u} = (-1)^m u^{(2m)} + \sum_{k=2}^{m+1} \sum_{\alpha_1+\cdots+\alpha_k = 2(m+1-k)} c_{\alpha_i} \partial_x^{\alpha_1} u \ldots \partial_x^{\alpha_k} u.$$

We plug in

$$u = -\frac{1}{2} \frac{\omega_{xx}}{\omega+2\tau} + \frac{3}{4} \frac{\omega_x^2}{(\omega+2\tau)^2} + \frac{1}{4}\omega^2 + \tau\omega,$$

which is a generalized differential polynomial in ω of degree 2. Thus for each k and $\alpha_1 + \cdots + \alpha_k = 2(m+1-k)$, the degree of

$$\partial_x^{\alpha_1} u \ldots \partial_x^{\alpha_k} u$$

in ω is $2k + 2(m + 1 - k) = 2(m + 1)$. The same holds for the linear term $u^{(2m)}$. This shows that

$$\sum_{n=-1}^{N-1} (2i\tau)^{2(N-1-n)} \frac{\delta H_n^{\mathrm{KdV}}}{\delta u}$$

has degree $2N$ in ω, and the factor $(\omega + 2\tau)$ in the equation for ω_t increases the degree again by one. It remains to prove that $0 \leq n \leq 2N - 1$, $l + k \geq n + 1$, and to analyze the term for $n = 0$.

To prove $0 \leq n \leq 2N - 1$ we see that in order to create higher n, there has to be either a high power or a big amount of derivatives falling onto $(\omega + 2\tau)^{-2}\omega_x^2$. The highest possible power comes from u^N (for $m = N - 1$) and leads to $n = 2N$, which after multiplication with $(\omega + 2\tau)$ becomes $2N - 1$. The term with most derivatives is $u^{(2N-2)}$ which again could leads to a term with $n = 2N$, or $2N - 1$ after multiplication. The terms in between the extremes are handled likewise.

The inequality $l + k \geq n + 1$ is equivalent to $d \leq 2N$ by the degree condition. This in fact follows again by looking at the worst terms: if the full powers of u^N hit either $(\omega + 2\tau)^{-1}\omega_{xx}$ or $(\omega + 2\tau)^{-2}\omega_x^2$ we arrive at $d = 2N$, and so do we if all derivatives from $u^{(2N-2)}$ hit $(\omega + 2\tau)^{-1}\omega_{xx}$ or $(\omega + 2\tau)^{-2}\omega_x^2$.

To see that the number of factors with at least one derivative is $\geq n + 1$ for $n \geq 1$, note that the only way to produce powers of $(\omega + 2\tau)^{-1}$ is if u is one of the factors $(\omega + 2\tau)^{-1}\omega_{xx}$ and $(\omega + 2\tau)^{-2}\omega_x^2$. In both cases, the number is $\geq n$. Multiplication with other factors will leave these properties invariant, as does taking derivatives. Multiplication by $(\omega + 2\tau)$ in the end leads to the statement. This also gives $d \geq n + 1$ if $n \geq 1$, and $l + k \leq 2N$ if $n \geq 1$, respectively $l + k \leq 2N + 1$ if $n = 0$, because in this case $d = n = 0$ may happen. It also shows $k \geq n + 1$.

The number of derivatives d is always even, because this statement holds for the KdV monomials (which in turn follows from the Lenard recursion), and because writing u in terms of ω only contains terms with an even number of derivatives.

We analyze the term with $n = 0$. Consider $\delta H_m^{\mathrm{KdV}}/\delta u$ with u as above. We can only reach $n = 0$ if all copies of u are $\frac{1}{4}\omega^2 + \tau\omega$, or if exactly one u is $(\omega + 2\tau)^{-1}\omega_{xx}$ (because in this case the factor of $(\omega + 2\tau)$ decreases n by one to zero). In the latter case, all possible derivatives have to fall onto ω_{xx}. A linear term in ω can only be reached by

$$2(\omega + 2\tau)(\tau\omega)^{(2m)} = 4\tau^2\omega^{(2m)} + \text{bilinear},$$

and by

$$2(\omega + 2\tau)\Big(- \frac{1}{2}\frac{\omega_{xx}}{\omega + 2\tau}\Big)^{(2m)} = -\omega^{(2m+2)} + \text{higher homogeneities}.$$

Using the summation over $-1 \leq m \leq N - 1$ from (A.1) cancels out all linear terms but the one of order $2N$.

We consider the term with $d = 2N$ derivatives and where $k + l = n + 1, n \geq 1$. $2N$ derivatives can only be reached if we are in the case

$$2(\omega + 2\tau)u^{(2N-2)}$$

where u is either $\frac{\omega_{xx}}{\omega+2\tau}$ or $\frac{\omega_x^2}{(\omega+2\tau)^2}$. In both cases, the only way to create more homogeneity is by factors of ω_x, which shows the impossibiliy of a term $(\omega + 2\tau)^{-n}(\omega^n\omega^{(2N)})$.

We turn to the form of the equation for $v = \omega/(2\tau)$. In general if we have a k-linear form A_k and an equation

$$\omega_t = \frac{A_k(\omega)}{(\omega + 2\tau)^n},$$

then the corresponding equation for v will be

$$v_t = (2\tau)^{k-1-n} \frac{A_k(v)}{(v+1)^n}.$$

Thus the new power of τ in the transition from ω to v will be $l + k - n - 1$, and the degree condition becomes

$$l + d = 2N.$$

The upper bound on the homogeneity k for ω, $k \leq 2N + 1$ does not change when going to v, because it does not depend on l. Similarly, the number of derivatives d stays even. Due to the condition $d + l = 2N$, l has to be even as well. □

A.2. **Good Variables.** For $N = 1$ the good variable equation is

$$v_t = \partial_x \Big[-v_{xx} + 6\tau^2 v^2 + 2\tau^2 v^3 + \frac{3}{2}\frac{v_x^2}{v+1} \Big],$$

and for $N = 2$

$$\begin{aligned} v_t =& \partial_x \Big[v_{xxxx} - 7\tau^2 v_{xx} v^2 - 4\tau^2 v v_x^2 - 14\tau^2 v_{xx} v \\ & - 4\tau^2 v_x^2 + 6\tau^4 v^5 + 30\tau^4 v^4 + 40\tau^4 v^3 \\ & + (v+1)^{-1}\big(-\frac{5}{2} v_{xx}^2 - 5 v_{xxx} v_x + 18\tau^2 v_x^2 v + \frac{9}{2}\tau^2 v_x^2 v^2 - 6\tau^2 v_x^2 \big) \\ & + (v+1)^{-2}\Big(\frac{25}{2} v_{xx} v_x^2\Big) + (v+1)^{-3}\Big(-\frac{45}{8} v_x^4 \Big) \Big]. \end{aligned}$$

Appendix B. The AKNS hierarchy

B.1. **The hierarchy.** The nonlinear Schrödinger equation (NLS) is the equation

$$iq_t = -q_{xx} \pm 2|q|^2 q, \tag{B.1}$$

where the sign + corresponds to the defocusing NLS and − to the focusing NLS. The complex modified Korteweg-de Vries equation (we will abbreviate both the complex and the real mKdV just by mKdV) is the equation

$$q_t = -q_{xxx} \pm 6|q|^2 q_x, \tag{B.2}$$

and again we talk about defocusing and focusing mKdV. Both equations are Hamiltonian equations of the form

$$iq_t = \frac{\delta H}{\delta r}\Big|_{r = \pm\bar{q}}$$

with Hamiltonian $H = \int q' r' + q^2 r^2 dx$ for NLS and Hamiltonian $H = -i \int q' r'' + 3q^2 r r'$ for mKdV. They are special cases of the AKNS system of equations

$$\begin{aligned} iq_t &= -q_{xx} + 2q^2 r, \\ ir_t &= r_{xx} - 2r^2 q, \end{aligned}$$

for NLS, and

$$iq_t = -iq_{xxx} + 6iqq'r,$$
$$ir_t = -ir_{xxx} - 6irr'q,$$

for mKdV, with the additional restriction $r = \pm\bar{q}$. The AKNS equations can be written as the Hamiltonian system of equations

$$iq_t = \frac{\delta H}{\delta r}, \quad ir_t = -\frac{\delta H}{\delta q}, \tag{B.3}$$

where one takes the Hamiltonians as above with the Poisson structure

Definition B.1. *The AKNS symplectic form on $L^2(\mathbb{R};\mathbb{C}^2)$ is given by*

$$\omega((q_1, r_1), (q_2, r_2)) = \int q_1 r_2 - q_2 r_1 dx.$$

It defines the AKNS Poisson structure

$$\{F, G\} = -i \int \frac{\delta F}{\delta q}\frac{\delta G}{\delta r} - \frac{\delta F}{\delta r}\frac{\delta G}{\delta q} dx.$$

A Lax operator for the AKNS equations is given by

$$L(q,r)\phi := i\begin{pmatrix} -\partial & q \\ -r & \partial \end{pmatrix}\phi \tag{B.4}$$

with a Lax equation

$$L(q,r)\phi = z\phi.$$

The Wadati Laplace operator is a special case (see Subsection 3.4 and it seems worthwhile to explore the various and strong relations between all the hierarchies.

We assume at first that q and r decay fast and $z = \xi \in \mathbb{R}$. There exist two fundamental systems, ψ_{-+}, ψ_{--} normalized at $-\infty$

$$\lim_{x\to-\infty} e^{i\xi x}\psi_{-+}(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \lim_{x\to-\infty} e^{-i\xi x}\psi_{--}(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and a fundamental system normalized at ∞

$$\lim_{x\to\infty} e^{-i\xi x}\psi_{+-}(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \lim_{x\to\infty} e^{i\xi x}\psi_{++}(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The solution space of the problem (B.4) is a two-dimensional vector space. As a consequence these solutions are connected on the real line by

$$\begin{pmatrix} \psi_{+-} \\ \psi_{-+} \end{pmatrix} = \begin{pmatrix} a_+(\xi) & b_+(\xi) \\ b_-(\xi) & a_-(\xi) \end{pmatrix}\begin{pmatrix} \psi_{--} \\ \psi_{++} \end{pmatrix}.$$

There are simple alternative expression

$$a_+(\xi) = W(\psi_{-+}, \psi_{+-}) = \det(\psi_{-+}, \psi_{+-}), \quad a_-(\xi) = W(\psi_{++}, \psi_{--}).$$

Here W is the Wronskian, which is independent of x. The solutions ψ_{-+} and ψ_{+-} have a holomorphic extensions to the upper half plane called left and right Jost functions and we define the transmission coefficient

$$T(z)^{-1} = W(\psi_{-+}, \psi_{+-}) = \lim_{x\to\infty} e^{izx}\psi^1_{-+}(x) = \lim_{x\to-\infty} e^{-izx}\psi^2_{+-}(x).$$

for z in the upper half plane. The solutions ψ_{--} and ψ_{++} have a holomorphic extension to the lower half plane and we define

$$T(z) = W(\psi_{++}, \psi_{--}) = \lim_{x\to-\infty} e^{izx}\psi^1_{++}(x) = \lim_{x\to\infty} e^{-izx}\psi^2_{--}(x)$$

on the lower half plane. The reason for this choice are the same as for KdV: It is a choice which gives simultaneous asymptotic series in the lower and the upper half plane. We will see later that

(B.5)

$$-\log T^{\mathrm{AKNS}}(z,q,r) \sim \frac{i}{2z}\int qr\,dx + \frac{1}{(2z)^2}\int qr_x\,dx + \frac{i}{(2z)^3}\int q_x r_x + q^2r^2\,dx + \dots.$$

Definition B.2. *The functions $\alpha(x,z,q,r)$, $\beta(x,z,q,r)$ and $\gamma(x,z,q,r)$ are defined by*

$$\begin{pmatrix} \frac{\gamma}{2} & \alpha \\ \beta & \frac{\gamma}{2} \end{pmatrix} = \begin{cases} \dfrac{T}{2}\begin{pmatrix} \psi^1_{-+}\psi^2_{+-} + \psi^2_{-+}\psi^1_{+-} & 2\psi^1_{-+}\psi^1_{+-} \\ 2\psi^2_{-+}\psi^2_{+-} & \psi^1_{-+}\psi^2_{+-} + \psi^2_{-+}\psi^1_{+-} \end{pmatrix} & \text{if } \operatorname{Im} z > 0 \\[2ex] \dfrac{1}{2T}\begin{pmatrix} \psi^1_{--}\psi^2_{++} + \psi^2_{--}\psi^1_{++} & 2\psi^1_{--}\psi^1_{++} \\ 2\psi^2_{--}\psi^2_{++} & \psi^1_{--}\psi^2_{++} + \psi^2_{--}\psi^1_{++} \end{pmatrix} & \text{if } \operatorname{Im} z < 0. \end{cases}$$

Observe that for each z, we have $\gamma \to 1$, and $\alpha \to 0$ and $\beta \to 0$ when $x \to \pm\infty$.

Lemma B.3. *For z in the upper half-plane, the Green's function for the operator $L - z1$ is*

$$G(x,y,z) = -iT(z)\begin{cases} \begin{pmatrix} \psi^1_{-+}(x,z)\psi^2_{+-}(y,z) & \psi^1_{-+}(x,z)\psi^1_{+-}(y,z) \\ \psi^2_{-+}(x,z)\psi^2_{+-}(y,z) & \psi^2_{-+}(x,z)\psi^1_{+-}(y,z) \end{pmatrix} & \text{if } x < y, \\[2ex] \begin{pmatrix} \psi^1_{+-}(x,z)\psi^2_{-+}(y,z) & \psi^1_{+-}(x,z)\psi^1_{-+}(y,z) \\ \psi^2_{+-}(x,z)\psi^2_{-+}(y,z) & \psi^2_{+-}(x,z)\psi^1_{-+}(y,z) \end{pmatrix} & \text{if } y < x. \end{cases}$$

Proof. We observe that the columns considered as functions of x satisfy

$$LG = zG$$

whenever $x \neq y$. It is the Green's function since, for x^+ being the limit from above and x^- being the limit from below,

$$G(x^+,x) - G(x^-,x) = -i\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

which shows that $(L - z1)G(x,y) = \delta(x-y)$. □

The function α, β and γ are closely related to the Green's function.

Lemma B.4. *Let*

$$g(z,x) = \lim_{h\to0+} \frac{i}{2}(G(x+h,x,z) + G(x,x+h,z))$$

be the diagonal Green's function. Then

$$g(z,x) = \begin{cases} \begin{pmatrix} \gamma/2 & \alpha \\ \beta & \gamma/2 \end{pmatrix} & \text{if } \operatorname{Im} z > 0 \\[2ex] -\begin{pmatrix} \gamma/2 & \alpha \\ \beta & \gamma/2 \end{pmatrix} & \text{if } \operatorname{Im} z < 0. \end{cases}$$

The importance of these objects is that they characterize $\log T$ and its functional derivatives. We introduce the resolvents $R_\pm = (-iz \pm \partial)^{-1}$ for $\operatorname{Im} z > 0$,

$$R_+ f(x) = \int_{-\infty}^{x} e^{iz(x-y)} f(y)dy, \quad R_- f(x) = \int_{x}^{\infty} e^{-iz(x-y)} f(y)dy. \tag{B.6}$$

Then

$$L(q,r) - z1 = (L(0,0) - z1)\left(1 + \begin{pmatrix} 0 & R_- q \\ -R_+ r & 0 \end{pmatrix}\right).$$

Unfortunately the operator in the bracket is only Hilbert-Schmidt for $q, r \in L^2$, but not trace class, even for Schwartz functions. For trace class operators K one has the expansion

$$\ln \det(1 - K) = \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} K^n \tag{B.7}$$

where $\operatorname{tr} K^n$ is defined for K in the L^n Schatten class. In particular only the first term is problematic for the bracket above. On the other hand, formally at least this trace should be zero due to the off-diagonal block matrix form of the operator. This motivates the use of the renormalized determinant

$$\det{}_2(1 + K) = \det(I + K) \exp(-\operatorname{tr} K)$$

for trace class functions, which has a unique extension to Hilbert Schmidt operators K, see . We refer to Subsection 3.4 for details.

The first set of statements below are elementary results from the theory of ODEs. For the reader's convenience we collect the proofs in the Appendix B.3.

Lemma B.5. *α, β, γ are connected to the transmission coefficient via*

$$\frac{d}{dz} \log T^{\mathrm{AKNS}} = i \int \gamma - 1 \, dx, \quad \alpha = \frac{\delta}{\delta r} \log T, \quad \beta = -\frac{\delta}{\delta q} \log T. \tag{B.8}$$

Let $\operatorname{Im} z > 0$ *and* $r, q \in L^2$*. Then the Fredholm determinant below is well defined (with the interpretation described above) and, if* $\operatorname{Im} z > 0$*,*

$$\det{}_2 \left(1 + i(L(0,0) - z1)^{-1} \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}\right) = T(z, q, r)^{-1}. \tag{B.9}$$

Proof. We provide short conceptional proof for (B.9) (which is well known, see [57]) by calculating the derivative of the functional determinant with respect to the potentials. This requires a bit of care. We observe that both sides are identically 1 if $q = r = 0$. By an abuse of notation, $G(z, q, r) := -i(L(q, r) - z1)^{-1}$ whenever it is defined. We approximate r and q by Schwartz functions and replace $R_\pm$ by the trace class operators $R_\pm^\tau = [(1 - \frac{1}{\tau}\partial)(-iz \pm \partial)]^{-1}$ which have integral kernels

$$k_\pm^\tau(x, y) = (e^{\pm iz(\cdot)} \chi_{\{\pm(\cdot) > 0\}}) * (\tau e^{\tau(\cdot)} \chi_{\{-(\cdot) > 0\}})(x - y).$$

Then, with the obvious notation

$$\ln \det{}_2 \left(1 + G(z, 0, 0) \begin{pmatrix} 0 & tq \\ -tr & 0 \end{pmatrix}\right) = \lim_{\tau \to \infty} \ln \det \left(1 + G^\tau(z, 0, 0) \begin{pmatrix} 0 & tq \\ -tr & 0 \end{pmatrix}\right)$$

since the trace of the second summand on the right hand side vanishes, and the operator converges in the Hilbert-Schmidt norm. Moreover,

$$\int \beta^\tau q - \alpha^\tau r \, dx \to \int \beta q - \alpha r \, dx$$

since the Green's function converges. We use the operator identity $1+(t+s)A = (1+tA)(1+(1+tA)^{-1}sA)$ below, and calculate

$$
\begin{aligned}
&\frac{d}{dt}\log\det\left(1+G^\tau(z,0,0)\begin{pmatrix}0 & tq\\ -tr & 0\end{pmatrix}\right)\\
&\quad=\frac{d}{ds}\log\det\left(1+G^\tau(z,0,0)\begin{pmatrix}0 & (t+s)q\\ -(t+s)r & 0\end{pmatrix}\right)_{s=0}\\
&\quad=\frac{d}{ds}\log\det\left[1+s\left(1+G^\tau(z,0,0)\begin{pmatrix}0 & tq\\ -tr & 0\end{pmatrix}\right)^{-1}G^\tau(z,0,0)\begin{pmatrix}0 & q\\ -r & 0\end{pmatrix}\right]_{s=0}\\
&\quad=\frac{d}{ds}\log\det\left[1+sG^\tau(z,tq,tr)\begin{pmatrix}0 & q\\ -r & 0\end{pmatrix}\right]_{s=0}=\operatorname{tr}\left[G^\tau(z,tq,tr)\begin{pmatrix}0 & q\\ -r & 0\end{pmatrix}\right]\\
&\quad=\int\beta^\tau(z,tq,tr)q-\alpha^\tau(z,tq,tr)r\,dx \qquad\to\qquad -\frac{d}{dt}\log T(z,tq,tr)\qquad\text{as } \tau\to\infty.
\end{aligned}
$$

□

Lemma B.6. *α,β,γ satisfy the ODE*

$$
\begin{aligned}
\gamma' &= 2(q\beta+r\alpha)\\
\alpha' &= -2iz\alpha+q\gamma\\
\beta' &= 2iz\beta+r\gamma.
\end{aligned}
\tag{B.10}
$$

Proof. The statements follow from differentiating products of components of solutions to the ODE (B.4). We carry this out for one term, the others follow likewise.

$$
\frac{d}{dx}(\psi^1_{-+}\psi^2_{+-}+\psi^2_{-+}\psi^1_{+-}) = 2q\psi^2_{-+}\psi^2_{+-}+2r\psi^1_{-+}\psi^1_{+-}.
$$

□

Combining the equations (B.10) gives

$$
(\alpha(z_1)\beta(z_2))' = (2iz_2-2iz_1)\alpha(z_1)\beta(z_2)+q\gamma(z_1)\beta(z_2)+r\gamma(z_2)\alpha(z_1).
$$

This has two important consequences. The first follows from setting $z_1=z_2$ and yields an alternative equation for γ,

$$
\gamma^2 = 1+4\alpha\beta, \tag{B.11}
$$

The second consequence is that

$$
\begin{aligned}
&\left(\alpha(z_1)\beta(z_2)+\alpha(z_2)\beta(z_1)-\frac{1}{2}\gamma(z_1)\gamma(z_2)\right)'\\
&\qquad\qquad = 2i(z_2-z_1)(\alpha(z_1)\beta(z_2)-\alpha(z_2)\beta(z_1))
\end{aligned}
\tag{B.12}
$$

which can be used to show that the transmission coefficients are Poisson commuting, see Theorem B.7.

B.1.1. *Symplectic forms and Poisson structures.* It is convenient to consider Poisson brackets of operators $A(p,q) : X \to Y$ with functions as follows: Let $\phi \in X$ and $\psi \in Y^*$. Then $(p,q) \to \psi(A(p,q)(\phi))$ is a function, and we define the Poisson product of $A(p,q)$ with a function H as the operator defined by

$$\psi(\{A,H\}(\phi)) = \{\psi(A(\phi)),H\}$$

whenever this is defined. In particular, if A is the multiplication by a differential polynomial then the Poisson product of the multiplication operator is the multiplication by the Poisson products.

We compute some Poisson brackets. The proof of the next theorem is strongly inspired by [25].[3]

Theorem B.7. *The transmission coefficients $T(z_1)$ and $T(z_2)$ and its logarithms Poisson commute:*

$$\{\log T(z_1), \log T(z_2)\} = 0. \tag{B.13}$$

Moreover, we have

$$\{q(x), \log T(z)\} = -i\alpha(x,z), \quad \{r(x), \log T(z)\} = -i\beta(x,z). \tag{B.14}$$

(B.15)

$$\begin{aligned}\{-i(L(q,r)-z_1 1), \log T(z_2)\} &= -i\begin{pmatrix} 0 & \alpha(z_2) \\ -\beta(z_2) & 0\end{pmatrix} \\ &= -\frac{1}{2(z_2-z_1)}\left((L-z_1 1)\begin{pmatrix} 0 & \alpha(z_2) \\ -\beta(z_2) & 0\end{pmatrix} + \begin{pmatrix} 0 & \alpha(z_2) \\ -\beta(z_2) & 0\end{pmatrix}(L-z_1 1)\right) \\ &\quad + \frac{1}{4(z_2-z_1)}\left((L-z_1 1)\begin{pmatrix} \gamma(z_2) & 0 \\ 0 & -\gamma(z_2)\end{pmatrix} - \begin{pmatrix} \gamma(z_2) & 0 \\ 0 & -\gamma(z_2)\end{pmatrix}(L-z_1 1)\right).\end{aligned}$$

Also

(B.16)

$$\begin{aligned}\{\alpha(z_1), \log T(z_2)\} &= \frac{1}{2(z_1-z_2)}(\alpha(z_1)\gamma(z_2) - \alpha(z_2)\gamma(z_1)), \\ \{\beta(z_1), \log T(z_2)\} &= \frac{1}{2(z_1-z_2)}(-\beta(z_1)\gamma(z_2) + \beta(z_2)\gamma(z_1)), \\ \{\gamma(z_1), \log T(z_2)\} &= \frac{1}{2i(z_1-z_2)^2}\partial_x\Big\{\alpha(z_1)\beta(z_2) + \beta(z_1)\alpha(z_2) - \frac{1}{2}\gamma(z_1)\gamma(z_2)\Big\}.\end{aligned}$$

Proof. $\log T(z_1)$ and $\log T(z_2)$ Poisson commute as a consequence of (B.12). Since the functional derivatives of the logarithms are proportional to the functional derivatives of the functions, the same argument works for the transmission coefficients itself.

Equations (B.14) follow from $\frac{\delta q(x)}{\delta q}(y) = \delta(x-y)$, the Dirac measure, and $\frac{\delta}{\delta r}q(x) = 0$. Then

$$\{q(x), \log T(z)\} = -i\int \delta(x-y)\alpha(y)dy = -i\alpha(x),$$

[3] Note that our notation slightly differs from theirs. In particular, their $\gamma + 1$ corresponds to our γ.

and similarly for $\{r(x), \log T(z)\}$. Also the first line of (B.15) follows from the definition of the Poisson structure (we drop the evaluation in the notation). Using (B.10) we see that

$$\begin{aligned} L(z_1)&\begin{pmatrix} 0 & \alpha(z_2) \\ -\beta(z_2) & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha(z_2) \\ -\beta(z_2) & 0 \end{pmatrix} L(z_1) \\ &= \begin{pmatrix} -q\alpha(z_2) - r\beta(z_2) & -2iz_1\alpha(z_2) - \alpha'(z_2) \\ 2iz_1\beta(z_2) - \beta'(z_2) & -q\alpha(z_2) - r\beta(z_2) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}\gamma'(z_2) & -q\gamma(z_2) \\ -r\gamma(z_2) & -\frac{1}{2}\gamma'(z_2) \end{pmatrix} + 2i(z_2 - z_1) \begin{pmatrix} 0 & \alpha(z_2) \\ -\beta(z_2) & 0 \end{pmatrix} \end{aligned}$$

and

$$L(z_1)\begin{pmatrix} \gamma(z_2) & 0 \\ 0 & -\gamma(z_2) \end{pmatrix} - \begin{pmatrix} \gamma(z_2) & 0 \\ 0 & -\gamma(z_2) \end{pmatrix} L(z_1) = \begin{pmatrix} -\gamma'(z_2) & -2q\gamma(z_2) \\ -2r\gamma(z_2) & -\gamma'(z_2) \end{pmatrix}.$$

We sum the terms to arrive at (B.15).

We begin with the differentiation of the resolvent $L^{-1} = (L_0 - iz)^{-1}$ to prove the last Poisson brackets. Then using (B.15) and the resolvent identity $(A-B)^{-1} - A^{-1} = (A-B)^{-1}BA^{-1}$, we see that

$$\begin{aligned} \{L^{-1}(z_1), \log T(z_2)\} &= -iL^{-1}(z_1) \begin{pmatrix} 0 & \alpha(z_2) \\ -\beta(z_2) & 0 \end{pmatrix} L^{-1}(z_1) \\ = \frac{1}{2(z_2 - z_1)} &\left\{ \begin{pmatrix} 0 & \alpha(z_2) \\ -\beta(z_2) & 0 \end{pmatrix} L^{-1}(z_1) + L^{-1}(z_1) \begin{pmatrix} 0 & \alpha(z_2) \\ -\beta(z_2) & 0 \end{pmatrix} \right\} \\ - \frac{1}{4(z_2 - z_1)} &\left\{ \begin{pmatrix} \gamma(z_2) & 0 \\ 0 & -\gamma(z_2) \end{pmatrix} L^{-1}(z_1) - L^{-1}(z_1) \begin{pmatrix} \gamma(z_2) & 0 \\ 0 & -\gamma(z_2) \end{pmatrix} \right\} \end{aligned}$$

We evaluate the integral kernel at $x = y$, which is possible since the kernel of RHS is continuous.

$$\begin{aligned} \{g(z_1), \log T(z_2)\} = &\frac{1}{2(z_2 - z_1)} \begin{pmatrix} \alpha(z_2)\beta(z_1) - \alpha(z_1)\beta(z_2) & \alpha(z_2)\gamma(z_1) \\ -\beta(z_2)\gamma(z_1) & \alpha(z_2)\beta(z_1) - \alpha(z_1)\beta(z_2) \end{pmatrix} \\ &- \frac{\gamma(z_2)}{2(z_2 - z_1)} \begin{pmatrix} 0 & \alpha(z_1) \\ -\beta(z_1) & 0 \end{pmatrix}. \end{aligned}$$

The claimed equality follows by (B.12). □

From now on we assume that at least one of the two functions q, r is decaying. Otherwise the coefficients in the asymptotic expansion below might be undefined. In the situation where both functions are nondecaying, one may still be able to do an asymptotic expansion in a different spectral parameter, see [42]. The AKNS Hamiltonians are defined as the coefficients in the asymptotic series

$$\log T(z) \sim -i \sum_{n=1}^{\infty} (2z)^{-n} H_n^{\mathrm{AKNS}}. \tag{B.17}$$

As above we define

$$\mathcal{T}_N^{\mathrm{AKNS}} = (2z)^N \log T(z) + i \sum_{n=1}^{N} (2z)^{N-n} H_n^{\mathrm{AKNS}}.$$

Similarly, we use α, β and γ as generating functions,

$$\gamma \sim 1 + \sum_{n=1}^{\infty} (2z)^{-n}\gamma_n, \quad \alpha \sim \sum_{n=1}^{\infty} (2z)^{-n}\alpha_n, \quad \beta_n \sim \sum_{n=1}^{\infty} (2z)^{-n}\beta_n.$$

Note that these expansions are not absolutely convergent but only asymptotic, and they hold for $z \in \mathbb{C}\backslash\mathbb{R}$. Their precise meaning is contained in the following theorem.

Theorem B.8. *The transmission coefficient is a meromorphic function in z for $z \in \mathbb{C}\backslash\mathbb{R}$. The poles coincide with the eigenvalues in the upper half plane. The following estimates hold for $N \in \mathbb{N}$:*

$$|H_N(q,r)| \le c_N(\|q^{(\frac{N-1}{2})}\|_{L^2}^2 + \|r^{(\frac{N-1}{2})}\|_{L^2}^2 + \|q\|_{L^2}^{2N} + \|r\|_{L^2}^{2N}) \tag{B.18}$$

Let $z \in \mathbb{C}$ and $q, r \in L^2$ be such that

$$|\operatorname{Im} z|^{-\frac{1}{2}}(\|q\|_{L^2} + \|r\|_{L^2}) \le \frac{1}{100}.$$

$$\left|\mathcal{T}_N^{\mathrm{AKNS}}\right| \le c_N \left(\frac{|z|}{\operatorname{Im} z}\right)^{2(N-1)} \left(\|q^{(\frac{N-1}{2})}\|_{L^2}^2 + \|r^{(\frac{N-1}{2})}\|_{L^2}^2 + \|q\|_{L^2}^{2N} + \|r\|_{L^2}^{2N)}\right) \tag{B.19}$$

(B.20)
$$\left\|\frac{\delta}{\delta(q,r)}\mathcal{T}_N^{\mathrm{AKNS}}\right\|_{H^{-\frac{N}{2}}} \le c_N \left(\frac{|z|}{\operatorname{Im} z}\right)^{2(N-1)} \left(1 + \|q^{(\frac{N-1}{2})}\|_{L^2} + \|r^{(\frac{N-1}{2})}\|_{L^2}^2 + \|q\|_{L^2}^{2N-1} + \|r\|_{L^2}^{2N-1}\right)$$
$$\times \left(\|q\|_{H^{\frac{N-1}{2}}} + \|r\|_{H^{\frac{N-1}{2}}}\right).$$

All expression here are holomorphic in z, r and q. Derivatives can be controlled by the Cauchy integral formula.

Proof. The Lax equation has the following symmetries. Suppose that $\left(z, q, r, \begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix}\right)$ satisfy (B.4). Then

(1) (Translation symmetry) $\big(z, q(.+h), r(.+h), \phi(.+h)\big)$, $h \in \mathbb{R}$
(2) (scaling symmetry) $(\lambda z, \lambda q(\lambda .), \lambda r(\lambda .), \phi(\lambda .))$, $\lambda > 0$ and
(3) (Galilean symmetry) $\left(z + \xi, e^{-2i\xi x}q, e^{2i\xi x}r, \begin{pmatrix}e^{-i\xi x}\phi^1\\ e^{i\xi x}\phi^2\end{pmatrix}\right)$, $\xi \in \mathbb{R}$

all satisfy (B.4).

As a consequence

$$T^{\mathrm{AKNS}}(\lambda z, \lambda q(\lambda x), \lambda r(\lambda x)) = T(z,q,r) \tag{B.21}$$

$$T^{\mathrm{AKNS}}(z + \xi, e^{-2i\xi x}q, e^{2i\xi x}r) = T(z,q,r). \tag{B.22}$$

We compare the asymptotic series:

$$\log T^{\mathrm{AKNS}}(\lambda z, \lambda q(\lambda x), \lambda r(\lambda x)) \sim -i\sum_{n=1}^{\infty}(2\lambda z)^{-n} H_n^{\mathrm{AKNS}}(\lambda q(\lambda x), \lambda r(\lambda x)) \sim -i\sum_{n=1}^{\infty}(2z)^{-n}H_n^{\mathrm{AKNS}}(q,r)$$

and

$$H_N^{\mathrm{AKNS}}(\lambda q(\lambda .), \lambda r(\lambda .)) = \lambda^N H_N^{\mathrm{AKNS}}(q,r).$$

Similarly

$$\begin{aligned}\log T^{\mathrm{AKNS}}(i\tau+\xi, e^{-2i\xi x}q, e^{2ixix}r) &\sim -i\sum_{n=1}^{\infty}(2(i\tau+\xi))^{-n}H_n^{\mathrm{AKNS}}(\lambda q(\lambda x),\lambda r(\lambda x))\\ &\sim -i\sum_{n=1}^{\infty}(2i\tau)^{-n}H_n^{\mathrm{AKNS}}(q,r)\end{aligned}$$

and,

$$\begin{aligned}\sum_{n=1}^{\infty}(2i\tau)^{-n}H_n^{\mathrm{AKNS}}(q,r) &\sim \sum_{n=1}^{\infty}(2i\tau)^{-n}(1+\frac{2\xi}{2i\tau})^{-n}H_n^{\mathrm{AKNS}}(e^{-2i\xi x}q, e^{2i\xi x}r)\\ &\sim \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-2\xi)^m(2i\tau)^{-n-m}H_n^{\mathrm{AKNS}}(e^{-2i\xi x}q, e^{2i\xi x}r)\end{aligned}$$

which implies

$$H_N^{\mathrm{AKNS}}(e^{2i\xi x}q, e^{-2i\xi x}r) = \sum_{n=0}^{N}(-2\xi)^{N-n}H_n^{\mathrm{AKNS}}(q,r)$$

Together

$$\mathcal{T}_N^{\mathrm{AKNS}}(i\tau+\xi,q,r) = (\tau-i\xi)^N\mathcal{T}_N^{\mathrm{AKNS}}(i, e^{2\xi x/\tau}q(x/\tau)/\tau, e^{-2\xi x/\tau}r(x/\tau)/\tau)$$

which reduces estimate (B.19) to the case $z=i$ under the assumptions $\|q\|_{L^2}+\|r\|_{L^2}\ll 1$,

$$|\mathcal{T}_N^{\mathrm{AKNS}}(i)| \le c_N(\|q^{(\frac{N-1}{2})}\|_{L^2}^2 + \|r^{(\frac{N-1}{2})}\|_{L^2}^2 + \|q\|_{L^2}^{2N} + \|r\|_{L^2}^{2N}).$$

The proof if this estimate is analogous to Section 7 but simpler and we omit it. It also implies estimates (B.18) and (B.20).

□

With these definitions and Theorem B.8, (B.10) becomes the equivalent of the Lenard recursion

$$\begin{aligned}\gamma_n' &= 2(q\beta_n + r\alpha_n),\\ \alpha_{n+1} &= i\alpha_n' - iq\gamma_n,\\ \beta_{n+1} &= -i\beta_n' + ir\gamma_n\end{aligned}\tag{B.23}$$

with $\gamma_0=1, \alpha_0=\beta_0=0$. Using the alternative equation for γ (B.11), we also find

$$2\gamma_n = \sum_{k=1}^{n-1}4\alpha_k\beta_{n-k} - \gamma_k\gamma_{n-k}.\tag{B.24}$$

Note that since there are no anti-derivatives involved in the recursion, we can directly conclude that H_n^{AKNS} is an integral over a differential polynomial in q and r. The first Hamiltonians and iterates of α,β and γ can be found in Appendix C.

The next theorem shows that the Nth equation in the AKNS hierarchy $q_t=\{q,H_N\}, r_t=\{r,H_N\}$ takes the simple form

$$q_t = \alpha_N, \qquad r_t = \beta_N.\tag{B.25}$$

Theorem B.9. *The Hamiltonians H_N^{AKNS} Poisson commute with $\log T(z)$. Any two Hamiltonians Poisson commute. They are given by*

$$H_N^{\mathrm{AKNS}} = \frac{1}{2N}\int \gamma_{N+1}dx, \quad \frac{\delta}{\delta q}H_N = -i\beta_N, \quad \frac{\delta}{\delta r}H_N = i\alpha_N. \tag{B.26}$$

Moreover,

$$\{q, H_N^{\mathrm{AKNS}}\} = \alpha_N, \quad \{r, H_N^{\mathrm{AKNS}}\} = \beta_N, \tag{B.27}$$

$$\begin{aligned}
\{\alpha(z), H_N^{\mathrm{AKNS}}\} &= -i\sum_{l+j=N-1}(2z)^l\Big[\alpha(z)\gamma_j - \gamma(z)\alpha_j\Big],\\
\{\beta(z), H_N^{\mathrm{AKNS}}\} &= -i\sum_{l+j=N-1}(2z)^l\Big[-\beta(z)\gamma_j + \gamma(z)\beta_j\Big],\\
\{\gamma(z), H_N^{\mathrm{AKNS}}\} &= 2\sum_{l+j=N-2}(l+1)(2z)^l\partial_x\left[\alpha(z)\beta_j + \beta(z)\alpha_j - \frac{1}{2}\gamma(z)\gamma_j\right],\\
\{\gamma_k, H_N^{\mathrm{AKNS}}\} &= 2\sum_{m+j=N+k-2}(m-k+1)\partial_x(\alpha_m\beta_j + \alpha_j\beta_m - \frac{1}{2}\gamma_m\gamma_j).
\end{aligned}\tag{B.28}$$

Proof. The statements follow from Theorem B.7 and by making the asymptotic expansions using Theorem (B.8). □

B.2. **Nonvanishing limits.** In the previous part we assume that $q, r \in L^2$. Surprisingly many results carry over for $L(z, a+q, b+r)$ with $a, b \in \mathbb{C}$ and $q, r \in L^2$ instead of (q, r) as arguments. The reference operator is now

$$L_0 = \begin{pmatrix} -iz - \partial & a \\ -b & -iz + \partial \end{pmatrix}. \tag{B.29}$$

We write the equation $L\psi = 0$ as

$$\psi' = \begin{pmatrix} -iz & q+a \\ r+b & iz \end{pmatrix}\psi \tag{B.30}$$

The characteristic exponents are the roots of $\lambda^2 + z^2 - ab = 0$, $\lambda = \pm iz\sqrt{1 - ab/z^2}$ which motivates the definition $\zeta = -z\sqrt{1 - ab/z^2}$. It cannot be purely imaginary if $|\operatorname{Im} z| \geq c(a, b)$. We choose a basis of the eigenspaces as columns of

$$U = \begin{pmatrix} z+\zeta & -ia \\ ib & z+\zeta \end{pmatrix}, \quad \det U = 2z(z+\zeta), \quad U^{-1} = \frac{1}{2z(z+\zeta)}\begin{pmatrix} z+\zeta & ia \\ -ib & z+\zeta \end{pmatrix}.$$

The Ansatz $\psi = U\phi$ gives

$$\phi' = \begin{pmatrix} -i\zeta & 0 \\ 0 & i\zeta \end{pmatrix}\phi + \frac{1}{2z}\begin{pmatrix} i(bq+ar) & (z+\zeta)q + \frac{a^2}{z+\zeta}r \\ (z+\zeta)r + \frac{b^2}{z+\zeta}q & -i(bq+ar) \end{pmatrix}\phi. \tag{B.31}$$

Similar to the construction in Section 4, there exist unique Jost solutions to (B.30) provided $|\operatorname{Im} z|$ is sufficiently large and $r, q \in L^1$, normalized by

$$\lim_{x\to-\infty} e^{i\zeta x}\psi_l(x) = \begin{pmatrix} z+\zeta \\ ib \end{pmatrix}, \quad \lim_{x\to\infty} e^{-i\zeta x}\psi_r(x) = \begin{pmatrix} -ia \\ z+\zeta \end{pmatrix}$$

or, equivalently, if ϕ_l satisfies (B.31) with the 'initial' condition $\lim_{x\to-\infty} e^{izx}\phi_l = \begin{pmatrix}1\\0\end{pmatrix}$ resp. $\lim_{x\to\infty} e^{-izx}\phi_r = \begin{pmatrix}0\\1\end{pmatrix}$ this allows to define the transmission coefficient as in the fast decaying case,

$$T^{-1}(\zeta) := \lim_{x\to\infty} e^{i\zeta x}\phi_l^1(x) = \lim_{x\to-\infty} e^{-i\zeta x}\phi_r^2(x) = W(\phi_l,\phi_r) = \frac{1}{2z(z+\zeta)}W(\psi_l,\psi_r).$$

We renormalize the transmission coefficient in order to be able to define it for $r, q \in L^2$, without the integrability condition and observe that

$$\rho_l = \exp i\Big(\zeta x - \frac{1}{2z}\int_0^x bq + ardy\Big)\phi_l$$

satisfies

$$\text{(B.32)} \qquad \rho' = \begin{pmatrix}0 & 0\\ 0 & 2i\zeta\end{pmatrix}\rho + \frac{1}{2z}\begin{pmatrix}0 & (z+\zeta)q + \frac{a^2}{z+\zeta}r\\ (z+\zeta)r + \frac{b^2}{z+\zeta}q & -2i(bq+ar)\end{pmatrix}\rho.$$

We again normalize

$$\lim_{x\to-\infty}\rho_l = \begin{pmatrix}1\\0\end{pmatrix}$$

and define the renormalized transmission coefficient $T_r(z)$

$$(T_r)^{-1} = T^{-1}\exp\Big(-\frac{1}{2z}\int bq + ardy\Big) = \lim_{x\to\infty}\rho_l^1.$$

The quantity on the right hand side is now defined for $r, q \in L^2$.

The resolvent is defined for $|\operatorname{Im} z|$ large and we obtain the same relation between resolvent and the logarithm of the transmission coefficient. The effect of the renormalization is transparent:

$$\tilde\alpha = \alpha - \frac{b}{2z}, \quad \tilde\beta = \beta - \frac{a}{2z}.$$

Exactly as in the decaying case (we diagonalize L_0)

$$\det\nolimits_2\left(1 + L_0^{-1}\begin{pmatrix}0 & q\\ -r & 0\end{pmatrix}\right) = T_r^{-1},$$

where L_0 is given by (B.29). Indeed, a close inspection of the argument used in Lemma B.5 shows that it relies on the decomposition $L_0^{-1}L = 1 + L_0^{-1}(L - L_0)$ rather than on the form of L_0.

In particular we obtain the recursion formulas and the calculation of Poisson brackets carries over to this situation.

B.3. **Functional Derivatives of the Transmission Coefficient.** This section contains the proof of Lemma B.5.

Proof of Lemma B.5. The equation

$$L(z)\psi = f$$

has a forward fundamental solution $G(x,y;z)$ given by 0 if $x < y$ and otherwise (observe that $W(\psi_{++},\psi_{+-}) = 1$)

$$-\begin{pmatrix}\psi_{++}^1(x)\psi_{+-}^2(y) - \psi_{+-}^1(x)\psi_{++}^2(y) & \psi_{++}^1(x)\psi_{+-}^1(y) - \psi_{+-}^1(x)\psi_{++}^1(y)\\ \psi_{++}^2(x)\psi_{+-}^2(y) - \psi_{+-}^2(x)\psi_{++}^2(y) & \psi_{++}^2(x)\psi_{+-}^1(y) - \psi_{+-}^2(x)\psi_{++}^1(y)\end{pmatrix}.$$

This can be seen by checking that $L_x G(x,y;z) = 0$ away from the diagonal and by the jump condition

$$G(x^+,x) - G(x^-,x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

as this implies $L_x G(x,y) = \delta(x-y)$. To determine $\frac{\delta T}{\delta q}$ and $\frac{\delta T}{\delta r}$ resp. for $\operatorname{Im} z > 0$, recall

$$\frac{d}{dt} T(z; q + t\dot{q}, r + t\dot{r})|_{t=0} =: \int \frac{\delta T}{\delta q} \dot{q} + \frac{\delta T}{\delta r} \dot{r} dy$$

We differentiate the equation with respect to t (dots are t-derivatives) and consider $\dot{\psi} = \dot{\psi}_{-+}$

$$L(z)\dot{\psi} = \begin{pmatrix} -\dot{q}\psi^2_{-+} \\ \dot{r}\psi^1_{-+} \end{pmatrix}.$$

Hence

$$\begin{aligned} \dot{\psi}(x) &= \psi_{++}(x) \int_{-\infty}^x \psi^2_{+-}(y)\psi^2_{-+}(y)\dot{q}(y) - \psi^1_{+-}(y)\psi^1_{-+}(y)\dot{r}(y) dy \\ &\quad - \psi_{+-}(x) \int_{-\infty}^x \psi^2_{++}(y)\psi^2_{-+}(y)\dot{q}(y) - \psi^1_{+-}(y)\psi^1_{-+}\dot{r} dy \end{aligned}$$

and

$$\frac{d}{dt} T^{-1}(q + t\dot{q}, r + \dot{r}) = \lim_{x\to\infty} e^{izx}\dot{\psi}^1(x) = \int \psi^2_{+-}\psi^2_{-+}\dot{q} dy - \int \psi^1_{+-}\psi^1_{-+}\dot{r} dy$$

Here, the second summand vanishes due to the assumption $\operatorname{Im} z > 0$. Thus

$$\frac{\delta T^{-1}}{\delta q} = \psi^2_{+-}\psi^2_{-+}, \qquad \frac{\delta T^{-1}}{\delta r} = -\psi^1_{+-}\psi^1_{-+}.$$

We turn to the derivative in the spectral parameter. Define $\tilde{\psi} = e^{izx}\psi$. We are interested in $T^{-1} = \lim_{x\to\infty} \tilde{\psi}_1(x)$. We calculate

$$\begin{pmatrix} -\partial & q \\ -r & -2iz + \partial \end{pmatrix} \tilde{\psi} = 0.$$

Thus

$$\begin{pmatrix} -\partial & q \\ -r & -2iz + \partial \end{pmatrix} \dot{\tilde{\psi}} = \begin{pmatrix} 0 \\ 2i\tilde{\psi}_2 \end{pmatrix}.$$

Then $\Psi = e^{izx}\dot{\tilde{\psi}}$ solves

$$\begin{pmatrix} -iz - \partial & q \\ -r & -iz + \partial \end{pmatrix} \Psi = \begin{pmatrix} 0 \\ 2i\psi^2_{-+} \end{pmatrix}.$$

Hence

$$\begin{aligned} \frac{dT^{-1}}{dz} &= \lim_{x\to\infty} e^{izx}\Psi^1(x) = -2i \int \psi^1_{+-}\psi^2_{-+} dy \\ &= -i \int \psi^2_{+-}\psi^1_{-+} + \psi^1_{+-}\psi^2_{-+} - T^{-1} dy \end{aligned} \tag{B.33}$$

If $\operatorname{Im} z < 0$ we use the backward fundamental solution $G(x,y;z)$ which is 0 for $x > y$ and otherwise

$$\begin{pmatrix} \psi^1_{-+}(x)\psi^2_{--}(y) - \psi^1_{--}(x)\psi^2_{-+}(y) & \psi^1_{-+}(x)\psi^1_{--}(y) - \psi^1_{--}(x)\psi^1_{-+}(y) \\ \psi^2_{-+}(x)\psi^2_{--}(y) - \psi^2_{--}(x)\psi^2_{-+}(y) & \psi^2_{-+}(x)\psi^1_{--}(y) - \psi^2_{--}(x)\psi^1_{-+}(y) \end{pmatrix}$$

We differentiate again the equation and consider

$$L(z)\dot\psi = \begin{pmatrix} -\dot q \psi^2_{++} \\ \dot r \psi^1_{++} \end{pmatrix}.$$

Hence

$$\begin{aligned} \dot\psi(x) &= -\psi_{-+}(x)\int_x^\infty \psi^2_{--}(y)\psi^2_{++}(y)\dot q(y) - \psi^1_{--}(y)\psi^1_{++}(y)\dot r(y)dy \\ &\quad + \psi_{--}(x)\int_x^\infty \psi^2_{-+}(y)\psi^2_{++}(y)\dot q(y) + \psi^1_{-+}(y)\psi^1_{++}(y)\dot r(y)dy \end{aligned}$$

We obtain

$$\frac{d}{dt}T(z,q+t\dot q, r+t\dot r)|_{t=0} = \lim_{x\to-\infty} e^{izx}\dot\psi(x) = \int -\psi^2_{--}\psi^2_{++}\dot q + \psi^1_{--}\psi^1_{++}\dot r dy$$

and

$$\frac{\delta}{\delta q}T(z,q,r) = -\psi^2_{--}\psi^2_{++}, \qquad \frac{\delta}{\delta r}T(z,q,r) = \psi^1_{--}\psi^1_{++}.$$

We observe that

$$G(z,q,r)L(z,\tilde q,\tilde r) = 1 + G(z,q,r)\begin{pmatrix} 0 & \tilde q - q \\ -(\tilde r - r) & 0 \end{pmatrix}$$

where the second summand on the right hand side is a trace class operator if $p,q\in L^1$. We calculate for $\operatorname{Im} z > 0$

$$\begin{aligned} \frac{d}{dt}\det\Big(G(z,q,r)L(z,q+t\dot q, r+t\dot r)\Big)\Big|_{t=0} &= \operatorname{tr}\left(G\begin{pmatrix} 0 & \dot q \\ -\dot r & 0\end{pmatrix}\right) \\ &= \int \operatorname{tr}_{\mathbb{R}^2}\left[\begin{pmatrix} \gamma/2 & \alpha \\ \beta & \gamma/2 \end{pmatrix}\begin{pmatrix} 0 & \dot q \\ -\dot r & 0 \end{pmatrix}\right]dx \\ &= \int \beta\dot q - \alpha\dot r dx. \end{aligned}$$

The left hand side can be rewritten as

$$\det(G(z,q,r)L(z,0,0))\frac{d}{dt}\det G(z,0,0)L(z,q+t\dot q,r+t\dot r)\Big|_{t=0} = \int \beta\dot q - \alpha\dot r$$

and we arrive at

$$\frac{d}{dt}\ln\det(G(z,0,0)L(z,q+t\dot q, r+t\dot r) = \int \beta\dot q - \alpha\dot r dx.$$

Since $\det(G(z,0,0)L(z,0,0)) = 1$ we see that

$$\det\left(1 + G(z,0,0)\begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}\right) = T(z,q,r) \tag{B.34}$$

on the upper half plane.

In the same fashion as above

$$\begin{aligned} \frac{dT}{dz} &= 2i\int \psi^1_{--}\psi^2_{++}dy \\ &= i\int \psi^2_{++}\psi^1_{--} + \psi^1_{++}\psi^2_{--} - Tdy. \end{aligned} \tag{B.35}$$

We arrive at

$$\frac{\delta \ln T}{\delta q} = -\begin{cases} T\psi^2_{-+}\psi^2_{+-} & \text{if } \operatorname{Im} z > 0 \\ T^{-1}\psi^2_{--}\psi^2_{++} & \text{if } \operatorname{Im} z < 0 \end{cases}$$

$$\frac{\delta \ln T}{\delta r} = \begin{cases} T\psi^1_{-+}\psi^1_{+-} & \text{if } \operatorname{Im} z > 0 \\ T^{-1}\psi^1_{--}\psi^1_{++} & \text{if } \operatorname{Im} z < 0 \end{cases}$$

$$\frac{d}{dz}\ln T = \begin{cases} i\displaystyle\int T(\psi^1_{-+}\psi^2_{+-} + \psi^2_{-+}\psi^1_{+-}) - 1dx & \text{if } \operatorname{Im} z > 0 \\ i\displaystyle\int T^{-1}(\psi^1_{--}\psi^2_{++} + \psi^2_{--}\psi^1_{++}) - 1dx & \text{if } \operatorname{Im} z < 0, \end{cases}$$

which finishes the proof. □

B.4. **Embedding other hierarchies into the AKNS hierarchy.** In this additional section we will show that the AKNS hierarchy contains the NLS ($r = \pm\bar{q}$), and as a part of it, the real mKdV hierarchy. There is also the complex KdV hierarchy which is obtained by setting $r = 1$, and the Gardner hierarchy related to the Wadati Lax operator (see Subsection 3.4. In all cases we specialize the transmission coefficient and their variational derivatives, study structural properties like symmetry and relations between transmission coefficients for different Lax operators - the most important being the connection between the Wadati operator and KdV via the modified Miura map. Then we deduce real symplectic and Poisson structures from the complex Poisson structure for AKNS and relate that in many cases to Gardner Poisson structures. In particular we find three Hamiltonian structures for KdV: The Gardner structure, the Magri structure, and the interpretation as AKNS Hamiltonian vector field restricted to a subset of functions.

In fact, we could define the Gardner hierarchy resp. the generating function $\mathcal{T}^{\text{Gardner}}_{-1}$ of the Gardner Hamiltonians and a study of their structure merely from our knowledge on AKNS. In the end we took a shortcut to find the Gardner generating functions by studying the good variables in more detail (see Lemma 3.6), but we decided to keep the AKNS approach since it helps in understanding the various connections between classical integrable hierarchies.

B.5. **Complex KdV hierarchy.** In Sections 2 and 3, we constructed the KdV hierarchy by means of its transmission coefficient T^{KdV} in the upper half-plane. Now we consider the AKNS Lax operator

$$L(z,u,1) = \begin{pmatrix} -iz - \partial & u \\ -1 & -iz + \partial \end{pmatrix},$$

the Hamiltonians and the diagonal Green's function evaluated at $q = u$ and $r = 1$. Note that the Lax operator is holomorphic in u. Taking a fresh start we now define

$$T^{\text{KdV}}(z,u) = T^{\text{AKNS}}(z,u,1).$$

An intriguing consequence of the proposition below is that equations of the KdV hierarchy can be understood as three different Hamiltonian evolutions

(1) As Hamiltonian equations with the Hamiltonian H^{KdV}_n with respect to the Gardner Poisson bracket.
(2) As Hamiltonian equations with the Hamiltonian H^{KdV}_n with respect to the Magri Poisson bracket.
(3) As restriction of the Hamiltonian equations with the Hamiltonian H^{AKNS}_{2n+2} with respect to the AKNS symplectic structure, restricted to the set $(u, 1)$.

Proposition B.10. A) Recursion relations. *The functions* $\log T^{\mathrm{KdV}}$ *and* $\beta(z,u,1)$ *are odd in* z. *We define* $H_n^{\mathrm{KdV}}(u) = \frac{1}{2} H_{2n+3}^{\mathrm{AKNS}}(u,1)$. *The recursion relations can be written as*

$$
\begin{aligned}
\gamma &= \beta' - 2iz\beta, \qquad \alpha = \frac{1}{2}(\beta'' - 2iz\beta' - 2u\beta),\\
0 &= \beta''' + 4(z^2-u)\beta' - 2u'\beta, \quad \beta_{2n-1}''' - 4u\beta_{2n-1}' - 2u'\beta_{2n-1} = -\beta_{2n+1}.
\end{aligned}
\tag{B.36}
$$

where β,β_n *and their derivatives are evaluated at* $(z,u,1)$.

B) Poisson brackets. *Let* $z_1, z_2 \in \mathbb{C}\backslash\mathbb{R}$ *and* $n,m\in\mathbb{N}$. *Then* $\log T^{\mathrm{KdV}}(z_1)$ *and* $\log T^{\mathrm{KdV}}(z_2)$ *all Poisson commute with respect to the Gardner structure, whenever* T^{KdV} *is defined (i.e. when* $z_{1,2}$ *is outside the spectrum). As a consequence* H_n^{KdV}, H_m^{KdV} *Poisson commute on sufficiently regular functions. The Gardner Poisson brackets of* β *with* $\log T^{\mathrm{KdV}}$ *and* H_n^{KdV} *satisfy*

$$
\begin{aligned}
\Big\{\frac{1}{\beta(z_1)}, \log T^{\mathrm{KdV}}(z_2)\Big\}_{\mathrm{Gardner}} &= \frac{\partial\Big[\frac{1}{\beta(z_1)}\frac{\delta}{\delta u}\log T(z_2)\Big]}{(2z_1)^2-(2z_2)^2}\ ,\\
\Big\{\frac{1}{\beta(z)}, H_n^{\mathrm{KdV}}\Big\}_{\mathrm{Gardner}} &= -\partial\Big[\frac{1}{\beta(z)}\frac{\delta}{\delta u}\sum_{m=-1}^{n-1}(2z)^{2(n-1-m)}H_m^{\mathrm{KdV}}\Big]
\end{aligned}
\tag{B.37}
$$

and with

$$
\mathcal{T}_N^{\mathrm{KdV}} = \sum_{n=-1}^{N} H_n^{\mathrm{KdV}}(2z)^{2N-2n} + \frac{(2z)^{2N+3}}{2i}\log T^{\mathrm{KdV}},
$$

we have (again with the Gardner bracket) for $N\geq 0$,

$$
\Big\{\frac{1}{\beta(z_1)}, \mathcal{T}_N^{\mathrm{KdV}}(z_2)\Big\} = \frac{\partial\Big[\frac{1}{\beta}\frac{\delta}{\delta u}\Big((2z_2)^2\mathcal{T}_{N-1}^{\mathrm{KdV}}(z_2) + \sum_{m=-1}^{N-1}(2z_1)^{2(N-m)}H_m^{\mathrm{KdV}}\Big)\Big]}{(2z_1)^2-(2z_2)^2}.
\tag{B.38}
$$

C) Hamiltonian structures. *The three Hamiltonian structures (KdV with Gardner Poisson structure, KdV with Magri Poisson structure and the restriction of the even AKNS Hamiltonian vector field) are expressed by*

$$
\begin{aligned}
2iz\partial_x\frac{\delta}{\delta u}\log T^{\mathrm{KdV}} &= 2iz\partial_x\frac{\delta}{\delta q}\log T^{\mathrm{AKNS}}(z)\Big|_{(q,r)=(u,1)}\\
&= \frac{\delta}{\delta r}\Big(\log T^{\mathrm{AKNS}}(z) + \log T^{\mathrm{AKNS}}(-z)\Big)\Big|_{(q,r)=(u,1)}\\
&= (2iz)^{-1}\Big(\partial^{(3)} - 2(u\partial + \partial u)\Big)\frac{\delta}{\delta u}\log T^{\mathrm{KdV}}
\end{aligned}
\tag{B.39}
$$

which implies the identification of the Gardner, Magri and AKNS Hamiltonian vector fields

$$
\partial\frac{\delta}{\delta u}H_n^{\mathrm{KdV}} = -i\frac{\delta}{\delta r}H_{2(n+1)}^{\mathrm{AKNS}}\Big|_{(q,r)=(u,1)} = -\Big(\partial^{(3)} - 2(u\partial+\partial u)\Big)\frac{\delta}{\delta u}H_{n-1}^{\mathrm{KdV}}.
\tag{B.40}
$$

Finally

$$
\frac{\delta}{\delta q}(\ln T^{\mathrm{AKNS}}(z) + \ln T^{\mathrm{AKNS}}(-z))\Big|_{(u,1)} = 0, \qquad \frac{\delta}{\delta q}H_{2n}^{\mathrm{AKNS}}\Big|_{(u,1)} = 0.
\tag{B.41}
$$

Proof. A) Let ϕ_l be a left Jost function for $L(z,u,1)$. Then $\begin{pmatrix} \phi_{l,1}+2iz\phi_{l,2} \\ \phi_{l,2}\end{pmatrix}$ is a left Jost function for $L(-z,u,1)$. This implies that $\log T^{\mathrm{KdV}}$ is an odd function of z, and the same is true for $\beta = -\frac{\delta}{\delta u}\log T^{\mathrm{KdV}}$. Similarly ϕ_2 satisfies $-\phi_2''+u\phi = z^2\phi$ which shows that the definition of T^{KdV} is consistent with the first subsection. We solve the third equation in (B.10) for γ, substitute γ in the first equation and solve for α to obtain the first two identities of (B.36). We substitute γ and α in the second line in (B.10) to obtain the recursion equation for β. The asymptotic expansion $\beta \sim \sum \beta_n (2z)^{-n}$ implies the last identity in (B.36).

B) From AKNS Poisson commutativity and (B.36),

$$\begin{aligned}
0 &= \int \frac{\delta \log T^{\mathrm{AKNS}}(z_1)}{\delta r}\frac{\delta \log T^{\mathrm{AKNS}}(z_2)}{\delta q} - \frac{\delta \log T^{\mathrm{AKNS}}(z_1)}{\delta q}\frac{\delta \log T^{\mathrm{AKNS}}(z_2)}{\delta r}\,dx\Big|_{(u,1)}\\
&= -\int \alpha(z_1)\beta(z_2)-\alpha(z_2)\beta(z_1)\,dx\\
&= \frac{1}{2}\int 2iz_1\beta'(z_1)\beta(z_2)-2iz_2\beta'(z_2)\beta(z_1)\,dx\\
&= i\int (z_1+z_2)\beta'(z_1)\beta(z_2)\,dx\\
&= -i(z_1+z_2)\int \frac{\delta}{\delta u}\log T^{\mathrm{AKNS}}(z_1,u,1)\partial_x \frac{\delta}{\delta u}\log T^{\mathrm{AKNS}}(z_2,u,1)\,dx\Big|_{(u,1)}\\
&= -i(z_1+z_2)\{\log T^{\mathrm{KdV}}(z_1),\log T^{\mathrm{KdV}}(z_2)\}_{\mathrm{Gardner}}.
\end{aligned}$$

Poisson commutation of $\log T^{\mathrm{KdV}}(z_1)$ and $\log T^{\mathrm{KdV}}(z_2)$ implies Poisson commutation of $\log T^{\mathrm{KdV}}(z)$, H_n^{KdV} and H_m^{KdV}.

We turn to (B.37). We specialize (B.16) to deduce for the AKNS Poisson brackets,

$$\Big\{\beta(z_1),\log T^{\mathrm{AKNS}}(z_2)\Big\}\Big|_{(u,1)} = \frac{\beta'(z_1)\beta(z_2)-\beta'(z_2)\beta(z_1)}{2(z_1-z_2)} - i\beta(z_1)\beta(z_2),$$

and a similar formula for $\log T^{\mathrm{AKNS}}(-z_2)$. Summing these two, the term $i\beta(z_1)\beta(z_2)$ drops out since β is odd in z_2. Now we claim that for $G(u)=F(u,1)$ where $F=F(q,r)$, we have

$$\{F,\log T(z)+\log T(-z)\}|_{(u,1)} = 2z\{G,\log T^{\mathrm{KdV}}(z)\}_{\mathrm{Gardner}}$$

Indeed, let $A(z)=\log T(z)+\log T(-z)$. We will later see that by (B.39),

$$2iz\partial\frac{\delta}{\delta u}\log T^{\mathrm{KdV}} = \frac{\delta}{\delta r}A(z)|_{(u,1)},$$

and by (B.41) $\frac{\delta}{\delta q}A(z)|_{(u,1)}=0$. Thus

$$\begin{aligned}
\{F,\log T(z)+\log T(-z)\}|_{(u,1)} &= \frac{1}{i}\int \frac{\delta F}{\delta q}\frac{\delta A}{\delta r}-\frac{\delta F}{\delta r}\frac{\delta A}{\delta q}\,dx|_{(u,1)}\\
&= \frac{1}{i}\int \frac{\delta G}{\delta u}2iz\partial\frac{\delta}{\delta u}\log T^{\mathrm{KdV}}(z)\,dx\\
&= 2z\{G,\log T^{\mathrm{KdV}}\}_{\mathrm{Gardner}}.
\end{aligned}$$

Applying this to $\beta(z_1)=F$ shows

$$\{\beta(z_1),\log T^{\mathrm{KdV}}(z_2)\}_{\mathrm{Gardner}} = \frac{2}{4z_1^2-4z_2^2}(\beta'(z_1)\beta(z_2)-\beta'(z_2)\beta(z_1)),$$

and we arrive at the first part of (B.37) by using

$$\{\frac{1}{\beta}, \log T\}_{\text{Gardner}} = \frac{-1}{\beta^2}\{\beta, \log T\}_{\text{Gardner}}.$$

The second part of (B.37) follows by using the asymptotic expansion. We turn to the proof of (B.38) which we prove inductively. For $N = 0$, the claim holds as can be seen using (B.37) directly. Now since

$$\mathcal{T}_{N+1}(z_2) = -(2z_2)^2\mathcal{T}_N(z_2) + H_{N+1},$$

we have

$$\begin{aligned}
\{\beta^{-1}, \mathcal{T}_{N+1}(z_2)\} &= \frac{-(2z_2)^2\partial\Big[\frac{1}{\beta}\frac{\delta}{\delta u}\Big((2z_2)^2\mathcal{T}^{\text{KdV}}_{N-1} + \sum_{m=-1}^{N-1}(2z_1)^{2(N-m)}H^{\text{KdV}}_m\Big)\Big]}{(2z_1)^2-(2z_2)^2} \\
&\quad - \partial\big(\beta^{-1}\sum_{m=-1}^{N}\frac{\delta}{\delta u}(2z_1)^{2(N-m)}H_m\big) \\
&= \frac{\partial\Big[\frac{1}{\beta}\frac{\delta}{\delta u}\Big((2z_2)^2\mathcal{T}^{\text{KdV}}_{N} - (2z_2)^2\sum_{m=-1}^{N}(2z_1)^{2(N-m)}H^{\text{KdV}}_m\Big)\Big]}{(2z_1)^2-(2z_2)^2} \\
&\quad + \frac{\partial\Big[\beta^{-1}(-(2z_1)^2+(2z_2)^2)\sum_{m=-1}^{N}\frac{\delta}{\delta u}(2z_1)^{2(N-m)}H_m\Big]}{(2z_1)^2-(2z_2)^2} \\
&= \frac{\partial\Big[\beta^{-1}\big((2z_2)^2\mathcal{T}^{\text{KdV}}_{N} - (2z_1)^2\sum_{m=-1}^{N}\frac{\delta}{\delta u}(2z_1)^{2(N-m)}H_m\big)\Big]}{(2z_1)^2-(2z_2)^2},
\end{aligned}$$

which is the right-hand side of (B.38) for $N+1$.

C) The first identity of (B.39) is the definition. The second identity can be equivalently written as

$$-2iz\beta' = \alpha(z) + \alpha(-z)$$

which follows from the second equation of (B.36) and the observation that the first and the last term on the right hand side are odd. The identity of the left hand side and the right hand side in (B.39) is equivalent to the third line in (B.36). The asymptotic series give (B.40). The last identity (B.41) is a direct consequence of the fact that β is odd. $\square$

B.6. **Defocusing NLS hierarchy.** The defocusing NLS hierarchy contains the (complex) mKdV hierarchy as the even part, and hence also the real mKdV hierarchy. The relations between the hierarchies are interesting in themselves, but we will not use them outside of this subsection.

The defocusing NLS hierarchy is the case $r = \bar{q}$. We choose the standard real symplectic form

$$\omega(f, g) = \operatorname{Im}\int f\bar{g} - g\bar{f}dx$$

and define

$$T^{\text{NLS}}(z, q) = T^{\text{AKNS}}(z, q, \bar{q}).$$

Proposition B.11. A) The transmission coefficient. *The transmission coefficient has the following properties*

$$
\begin{aligned}
1 &= T^{\mathrm{NLS}}(z,q)\overline{T^{\mathrm{NLS}}(\bar z,q)},\\
\log T^{\mathrm{NLS}}(z,q) &= -\log \overline{T^{\mathrm{NLS}}(\bar z,q)},\\
\gamma(z,q,\bar q) &= \overline{\gamma(\bar z,q,\bar q)}\\
\beta(z,q,\bar q) &= \overline{\alpha(\bar z,q,\bar q))}.
\end{aligned}
\tag{B.42}
$$

B) The recursion formula. *The following identities hold*

$$\gamma'(z) = 2(q\overline{\alpha(\bar z)} + \bar q\alpha(z)), \quad \alpha' = -2iz\alpha + q\gamma,$$

hence $H_n \in \mathbb{R}$, $\gamma_n(x) \in \mathbb{R}$, $\alpha_n = \bar\beta_n$ *and* $\gamma_n' = 4\,\mathrm{Re}(q\bar\alpha_n)$ *and*

$$\alpha_{n+1} = i\alpha_n' - iq\gamma_n.$$

As a consequence the Hamiltonian flow of H_n preserves the structure $r = \bar q$.

Proof. Let $\phi_l(z,q,\bar q)$ be a Jost solution. Then also $\begin{pmatrix}\bar\phi_l^2\\ \bar\phi_l^1\end{pmatrix}$ is a Jost solution to the spectral value $\bar z$. This implies the first line and the second line in (B.42). Since

$$(L(z,q,\bar q))^* = -L(\bar z,q,\bar q)$$

we obtain $\alpha(z) = \overline{\beta(\bar z)}$ which implies the remaining assertions. □

It is remarkable that the fourth equation is the complex mKdV equation

$$q_t + q_{xxx} - 6|q|^2 q_x = 0.$$

We note that unlike in the case of real potentials, we do not expect to be able to write the even flows with respect to the Gardner Poisson structure. This can already be seen from the complex defocusing mKdV flow, since $|q|^2 q_x$ can in general not be written as a total derivative, and hence not as Hamiltonian equations with respect to the Gardner structure.

B.7. **Defocusing real mKdV.** The real defocusing mKdV hierarchy is a special degenerate version of the Gardner hierarchy. This is the case $r = q = v$ with real valued functions v. It is a special case of the NLS hierarchy, note however that real functions are contained in a Lagrangian subspace of the symplectic form, i.e. it vanishes identically on it. The relevant Poisson structure is the Gardner Poisson structure. The recursion relations allow to relate the symplectic structure of the AKNS hierarchy to the Gardner Poisson structure of mKdV (in particular to deduce Poisson commutation of $\log T$ with respect to the Gardner Poisson structure from the Poisson commutation of $\log T$ for the AKNS structure. The mKdV hierarchy is connected to the KdV hierarchy via the Miura map. This is useful, however it is difficult to work with it, since it is not even a local diffeomorphism from L^2 to H^{-1}, see [31]. We define

$$T^{\mathrm{mKdV}}(z,v) = T^{\mathrm{AKNS}}(z,v,v) \qquad (= T^{\mathrm{NLS}}(z,v)).$$

Proposition B.12. A) Properties of the transmission coefficient. *The generating function* $\ln T^{\mathrm{mKdV}}$ *is odd and real on the imaginary axis. Moreover*

$$T^{\mathrm{mKdV}}(z,v) = T^{\mathrm{KdV}}(z, v_x + v^2). \tag{B.43}$$

We define $H_n^{\mathrm{mKdV}}(v) = H_{n-1}^{\mathrm{KdV}}(v_x + v^2)$ *so that*

$$\ln T^{\mathrm{mKdV}} \sim -2i \sum_{n=0}^{\infty} H_n^{\mathrm{mKdV}} (2z)^{-1-2n}.$$

B) Recursion relations. *The functions α, β and γ are real on the imaginary axis. $\alpha - \beta$ is odd and γ and $\alpha + \beta$ are even, and $\alpha(z) = \beta(-z)$. The recursion formula can be written as*

$$\begin{aligned} (\alpha+\beta)'' &= -4z^2(\alpha+\beta) + 2(v\gamma)', \\ (\alpha-\beta)' &= -2iz(\alpha+\beta), \\ \gamma' &= 2v(\alpha+\beta), \\ \alpha_{2n+2} + \beta_{2n+2} &= (\alpha_{2n}+\beta_{2n})'' - 2(v\gamma_{2n})', \\ \gamma_{2n}' &= 2v(\alpha_{2n}+\beta_{2n}). \end{aligned} \tag{B.44}$$

C) Poisson brackets. *Let $z_1, z_2 \in \mathbb{C}\backslash\mathbb{R}$, $n, m \in \mathbb{N}$. The transmission coefficient $T^{\mathrm{mKdV}}(z_1)$, $T^{\mathrm{mKdV}}(z_2)$, H_n^{mKdV} and H_m^{mKdV} all Poisson commute with respect to the Gardner Poisson structure. The Magri structure and Gardner structure are related by*

$$\big\{f(v_x+v^2), g(v_x+v^2)\big\}_{\mathrm{Gardner}} = \big\{f,g\big\}_{\mathrm{Magri}}\big|_{v_x+v^2}. \tag{B.45}$$

D) Hamiltonian structures. *The relation between the mKdV flows, KdV flows and the even AKNS flow is expressed by*

(B.46)

$$-2iz\Big(\frac{\delta}{\delta r} - \frac{\delta}{\delta q}\Big) \log T^{\mathrm{AKNS}}\Big|_{(z,v,v)} = \partial \frac{\delta}{\delta v} \log T^{\mathrm{mKdV}} = \partial(-\partial+2v) \frac{\delta}{\delta u} \log T^{\mathrm{KdV}}\Big|_{v_x+v^2},$$

$$\frac{\delta}{\delta q} H_{2(n+1)}^{\mathrm{AKNS}}|_{(z,v,v)} = \frac{\delta}{\delta r} H_{2(n+1)}^{\mathrm{AKNS}}|_{(z,v,v)} = \partial \frac{\delta}{\delta v} H_n^{\mathrm{mKdV}}(v) = \partial(-\partial+2v)\frac{\delta}{\delta u} H_{n-1}^{\mathrm{KdV}}\Big|_{v_x+v^2}.$$

Moreover, we have the following chain rule for the time derivative

$$(\partial+2v)\partial \frac{\delta}{\delta v} \log T^{\mathrm{mKdV}} = 4z^2 \partial \frac{\delta}{\delta u} \log T^{\mathrm{KdV}}\big|_{v_x+v^2} \tag{B.47}$$

In particular, $u = v_x + v^2$ solves the nth KdV equation whenever v solves the nth mKdV equation, and v solves the nth mKdV equation if and only if (v, v) satisfies the $2(n+1)$th AKNS Hamiltonian equation.

Proof. A) Let ϕ_l be a left Jost function for $L(z, v, v)$. Then $\psi = \phi_l^1 + \phi_l^2$ satisfies

$$(-z^2 - \partial^2 + v^2 + v_x)\psi = 0$$

and hence by inverting the reasoning from the KdV case,

$$\begin{pmatrix} (\phi_l^1+\phi_l^2)' - iz(\phi_l^1+\phi_l^2) \\ \phi_l^1 + \phi_l^2 \end{pmatrix}$$

is a left Jost function for $L(z, v_x + v^2, 1)$. Thus

$$T^{\mathrm{mKdV}}(z, v) = T^{\mathrm{KdV}}(z, v_x + v^2).$$

In particular $\ln T^{\mathrm{mKdV}}$ and $\alpha - \beta = \frac{\delta}{\delta w} T^{\mathrm{mKdV}}$ are odd in z. Specializing the symmetries of defocusing NLS (B.42) to $q = \bar{q} = v$, we see that $\alpha(i\tau, v, v) = \beta(-i\tau, v, v)$ and $\gamma(i\tau, v, v) = \gamma(-i\tau, v, v)$. Combining these facts we find that $\alpha(i\tau) + \beta(i\tau)$ and $\gamma(i\tau)$ are even. Since α and β are holomorphic also $\alpha(z) + \beta(z)$ and $\gamma(z)$ are even.

B) The recursion relations are an immediate consequence of A) and the recursion relations for the NLS hierarchy.

C) The Poisson commutativity holds, because

$$\begin{aligned}
0 &= \int \frac{\delta \log T^{\mathrm{AKNS}}(z_1)}{\delta q}\frac{\delta \log T^{\mathrm{AKNS}}(z_2)}{\delta r} - \frac{\delta \log T^{\mathrm{AKNS}}(z_1)}{\delta r}\frac{\delta \log T^{\mathrm{AKNS}}(z_2)}{\delta q} \\
&= -\int \beta(z_1)\alpha(z_2) - \alpha(z_1)\beta(z_2) \\
&= \frac{1}{2}\int -(\alpha+\beta)(z_1)(\alpha-\beta)(z_2) + (\alpha-\beta)(z_1)(\alpha+\beta)(z_2) \\
&= \Big(\frac{1}{2iz_1} + \frac{1}{2iz_2}\Big)\int (\alpha-\beta)'(z_1)(\alpha-\beta)(z_2) \\
&= \frac{i(z_1+z_2)}{2z_1z_2}\int \frac{\delta}{\delta v}T^{\mathrm{mKdV}}(z_1,v)\partial_x\frac{\delta}{\delta v}T^{\mathrm{mKdV}}(z_2,v,v) \\
&= \frac{i(z_1+z_2)}{2z_1z_2}\Big\{\log T^{\mathrm{mKdV}}(z_1), \log T^{\mathrm{mKdV}}(z_2)\Big\}_{\mathrm{Gardner}}.
\end{aligned}$$

Moreover, from the operator identity

$$(-\partial^3 + 4(v_x+v^2)\partial + 2(v_x+v^2)_x) = (\partial+2v)\partial(-\partial+2v), \tag{B.48}$$

we see

$$\{f,g\}_{\mathrm{Magri}} = \int (-\partial+2v)\frac{\delta}{\delta u}f\partial(-\partial+2v)\frac{\delta}{\delta u}g\,dx = \{f(v_x+v^2), g(v_x+v^2)\}_{\mathrm{Gardner}}.$$

D) The first part of the first identity in (B.46) is just a consequence of the second recursion relation (B.44). The identity

$$(\partial+2v)(\alpha(z,v,v)+\alpha(-z,v,v)) = 2iz\partial\beta(z,u,1),$$

is an equivalent formulation of the second part of the first identity in (B.46). It is a consequence of the second recursion relation (B.44) to which we apply the operator $(\partial+2v)$, use the chain rule and (B.48). The second identity of the first line follows by the chain rule, and the second line spells the conseqences out for the aymptotic series. Using the second equality in (B.46), the operator identity (B.48) and the Lenard recursion, the last equality follows. □

We specialise Theorem B.9 to $q = r \in \mathbb{R}$ and obtain the following

Lemma B.13. *The mKdV Hamiltonian $H_n^{\mathrm{mKdV}}(v) = \frac{1}{2}H_{2n+1}^{\mathrm{AKNS}}(v,v)$ can be written as an integral over a sum of homogeneous differential polynomials*

$$H_n^{\mathrm{mKdV}}(v) = \frac{1}{2}\int |v^{(n)}|^2 dx + \sum_{k=2}^{2n+2}\int e_{n,k}^{\mathrm{mKdV}}dx$$

where $e_{n,j}^{\mathrm{mKdV}}$ is a sum of monomials of degree $2n+2$, homogeneity $2k$ and weight $2(n+1-k)$, each factor in the monomials having order at most $[(n+1-k)/2]+1$. Moreover

$$e_{n,2n+2}^{\mathrm{mKdV}} = \frac{1}{2(2n+1)}\binom{2n+2}{n+1}v^{2n+2}.$$

Appendix C. Hamiltonians, Equations and Recursions

For convenience, we list here some of the equations and Hamiltonians of the hierarchies contained in the AKNS hierarchy. Recall, (B.23),

$$\begin{aligned}\gamma_n' &= 2(q\beta_n + r\alpha_n),\\ \alpha_{n+1} &= i\alpha_n' - iq\gamma_n,\\ \beta_{n+1} &= -i\beta_n' + ir\gamma_n\end{aligned}$$

with $\gamma_0 = 1, \alpha_0 = \beta_0 = 0$, and the alternative equation for γ

$$2\gamma_n = \sum_{k=1}^{n-1} 4\alpha_k\beta_{n-k} - \gamma_k\gamma_{n-k}.$$

The first few iterates are

$$\begin{aligned}
&\alpha_0 = 0, \quad \beta_0 = 0, \quad \gamma_0 = 1,\\
&\alpha_1 = -iq, \quad \beta_1 = ir, \quad \gamma_1 = 0,\\
&\alpha_2 = q', \quad \beta_2 = r', \quad \gamma_2 = 2qr,\\
&\alpha_3 = iq'' - 2iq^2r, \quad \beta_3 = -ir'' + 2ir^2q, \quad \gamma_3 = -2i(qr' - q'r),\\
&\alpha_4 = -q''' + 6qq'r, \quad \beta_4 = -r''' + 6rr'q, \quad \gamma_4 = -2(qr'' + rq'' - q'r') + 6q^2r^2,\\
&\alpha_5 = -i(q^{(4)} - 8qq''r - 6(q')^2r - 4qq'r' - 2q^2r'' + 6q^3r^2),\\
&\beta_5 = i(r^{(4)} - 8rr''q - 6(r')^2q - 4rr'q' - 2r^2q'' + 6r^3q^2),\\
&\gamma_5 = -2i(q'''r - r'''q - q''r' + r''q' + 6(-qq'r^2 + rr'q^2))\\
&\gamma_6 = 2[qr^{(4)} + rq^{(4)} - (q'r''' + q'''r') + q''r'']\\
&\quad - 10((q')^2r^2 + q^2(r')^2) - 20(q^2rr'' + r^2qq'') + 20q^3r^3
\end{aligned}$$

For the first Hamiltonians we construct the functional antiderivatives by hand and find

$$\begin{aligned}
H_1^{\mathrm{AKNS}} &= \int qr\,dx,\\
H_2^{\mathrm{AKNS}} &= -\frac{i}{2}\int qr' - q'r\,dx = -i\int qr'\,dx,\\
H_3^{\mathrm{AKNS}} &= \int q'r' + q^2r^2\,dx,\\
H_4^{\mathrm{AKNS}} &= -\frac{i}{2}\int q'r'' - q''r' + 3(q^2rr' - r^2qq')\,dx = -i\int q'r'' + 3q^2rr'\,dx,\\
H_5^{\mathrm{AKNS}} &= \int q''r'' + \frac{3}{2}(q^2)'(r^2)' + ((qr)')^2 + 2q^3r^3.
\end{aligned}$$

C.0.1. *Complex KdV.* We set $r = 1$. Then,

$$\begin{aligned}
&\alpha_0 = 0, \quad \beta_0 = 0, \quad \gamma_0 = 1,\\
&\alpha_1 = -iq, \quad \beta_1 = i, \quad \gamma_1 = 0,\\
&\alpha_2 = q', \quad \beta_2 = 0, \quad \gamma_2 = 2q,\\
&\alpha_3 = iq'' - 2iq^2, \quad \beta_3 = 2iq, \quad \gamma_3 = 2iq',\\
&\alpha_4 = -q''' + 6qq', \quad \beta_4 = 0, \quad \gamma_4 = -2q'' + 6q^2,\\
&\alpha_5 = -i(q^{(4)} - 6(q')^2 - 8qq'' + 6q^3), \quad \beta_5 = -i(2q'' + 6q^2), \quad \gamma_5 = 2i(-q''' + 6qq')
\end{aligned}$$

C.0.2. *Defocusing NLS, complex mKdV.* We set $r = \bar{q}$. Then,

$$\begin{aligned}
&\alpha_0 = 0, \quad \beta_0 = 0, \quad \gamma_0 = 1,\\
&\alpha_1 = -iq, \quad \beta_1 = i\bar{q}, \quad \gamma_1 = 0,\\
&\alpha_2 = q', \quad \beta_2 = \bar{q}', \quad \gamma_2 = 2|q|^2,\\
&\alpha_3 = iq'' - 2i|q|^2 q, \quad \beta_3 = -i\bar{q}'' + 2i|q|^2\bar{q}, \quad \gamma_3 = 4\operatorname{Im}(q\bar{q}'),\\
&\alpha_4 = -q''' + 6|q|^2 q', \quad \beta_4 = -\bar{q}''' + 6|q|^2\bar{q}', \quad \gamma_4 = -2(2\operatorname{Re}(q\bar{q}'') - |q'|^2) + 6|q|^4,\\
&\alpha_5 = -i(q^{(4)} - 8|q|^2 q'' - 6(q')^2\bar{q} - 4q|q'|^2 - 2q^2\bar{q}'' + 6|q|^4 q),\\
&\beta_5 = i(\bar{q}^{(4)} - 8|q|^2\bar{q}'' - 6(\bar{q}')^2 q - 4\bar{q}|q'|^2 - 2\bar{q}^2 q'' + 6|q|^4\bar{q}),\\
&\gamma_5 = 4\operatorname{Im}(q'''\bar{q} - q''\bar{q}') + 12\operatorname{Im}(|q|^2 q\bar{q}')
\end{aligned}$$

and

$$\begin{aligned}
H_1 &= \int |q|^2\,dx,\\
H_2 &= \operatorname{Im}\int q\bar{q}'\,dx,\\
H_3 &= \int |q'|^2 + |q|^4\,dx,\\
H_4 &= \operatorname{Im}\int q'\bar{q}'' + 3|q|^2 q\bar{q}'\,dx,\\
H_5 &= \int |q''|^2 + \frac{3}{2}|(q^2)'|^2 + ((|q|^2)')^2 + 2|q|^6.
\end{aligned}$$

C.0.3. *Defocusing real mKdV.* We set $r = q \in \mathbb{R}$. Then,

$$\begin{aligned}
&\alpha_0 = 0, \quad \beta_0 = 0, \quad \gamma_0 = 1,\\
&\alpha_1 = -iq, \quad \beta_1 = iq, \quad \gamma_1 = 0,\\
&\alpha_2 = q', \quad \beta_2 = q', \quad \gamma_2 = 2q^2,\\
&\alpha_3 = iq'' - 2iq^3, \quad \beta_3 = -iq'' + 2iq^3, \quad \gamma_3 = 0,\\
&\alpha_4 = -q''' + 6q^2 q', \quad \beta_4 = -q''' + 6q^2 q', \quad \gamma_4 = -2(2qq'' - (q')^2) + 6q^4,\\
&\alpha_5 = -i(q^{(4)} - 10q^2 q'' - 10(q')^2 q + 6q^5),\\
&\beta_5 = i(q^{(4)} - 10q^2 q'' - 10(q')^2 q + 6q^5), \quad \gamma_5 = 0
\end{aligned}$$

and

$$H_1 = \int q^2\,dx, \quad H_2 = 0,$$
$$H_3 = \int (q')^2 + q^4\,dx, \quad H_4 = 0,$$
$$H_5 = \int (q'')^2 + 10q^2(q')^2 + 2q^6.$$

C.0.4. *Gardner.* We set $q = w, r = w + 2\tau_0 \in \mathbb{R}$. Then,

$$\alpha_0 = 0, \quad \beta_0 = 0, \quad \gamma_0 = 1,$$
$$\alpha_1 = -iw, \quad \beta_1 = i(w + 2\tau_0), \quad \gamma_1 = 0,$$
$$\alpha_2 = w', \quad \beta_2 = w', \quad \gamma_2 = 2w(w + 2\tau_0),$$
$$\alpha_3 = iw'' - 2iw^2(w + 2\tau_0), \quad \beta_3 = -iw'' + 2i(w + 2\tau_0)^2 w, \quad \gamma_3 = 4i\tau_0 w',$$
$$\alpha_4 = -w''' + 6w^2w' + 12\tau_0 ww', \quad \beta_4 = -w''' + 6w^2w' + 12\tau_0 ww',$$
$$\gamma_4 = -2(2\tau_0 w'' - (w')^2) + 6w^2(w + 2\tau_0)^2,$$
$$\begin{aligned}\alpha_5 = -i(w^{(4)} &- 8ww''(w + 2\tau_0) - 6(w')^2(w + 2\tau_0)\\ &- 4w(w')^2 - 2w^2w'' + 6w^3(w + 2\tau_0)^2),\end{aligned}$$
$$\begin{aligned}\beta_5 = i(w^{(4)} &- 8(w + 2\tau_0)w''w - 6(w')^2 w\\ &- 4(w + 2\tau_0)(w')^2 - 2(w + 2\tau_0)^2 w'' + 6(w + 2\tau_0)^3 w^2),\end{aligned}$$
$$\gamma_5 = -2i(2\tau_0 w''' + 6(-ww'(w + 2\tau_0)^2 + (w + 2\tau_0)w'w^2))$$

and the Hamiltonians $H_n^{\mathrm{Wadati}}(w, \tau_0) = H_n^{\mathrm{AKNS}}(w, w + 2\tau_0)$ become

$$H_1^{\mathrm{Wadati}} = \int w^2 + 2\tau_0 w\,dx,$$
$$H_2^{\mathrm{Wadati}} = 0,$$
$$H_3^{\mathrm{Wadati}} = \int (w')^2 + w^4 + w^2(w + 2\tau_0)^2\,dx,$$
$$H_4^{\mathrm{Wadati}} = 0,$$
$$H_5^{\mathrm{Wadati}} = \int (w'')^2 + \frac{3}{2}(w^2)'((w + 2\tau_0)^2)' + ((w(w + 2\tau_0))')^2 + 2w^3(w + 2\tau_0)^3.$$

The Gardner Hamiltonians $H_{n-1}^{\mathrm{Gardner}}(w, \tau_0) = \frac{1}{2}H_{2n+1}^{\mathrm{Wadati}}(w, \tau_0) - 4\tau_0^2 H_{n-1}^{\mathrm{Gardner}}(w, \tau_0)$ if $n \geq 1$ are

$$H_0^{\mathrm{Gardner}} = \int w^2\,dx,$$
$$H_1^{\mathrm{Gardner}} = \int w_x^2 + w^4 + w^4 + 4\tau_0 w^3,$$
$$H_2^{\mathrm{Gardner}} = \int w_{xx}^2 + 10w^2w_x^2 + 2w^6 + 4\tau_0(5ww_x^2 + 3w^5) + 24\tau_0^2 w^4\,dx.$$

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