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CRC Preprint 2023/16, June 2023

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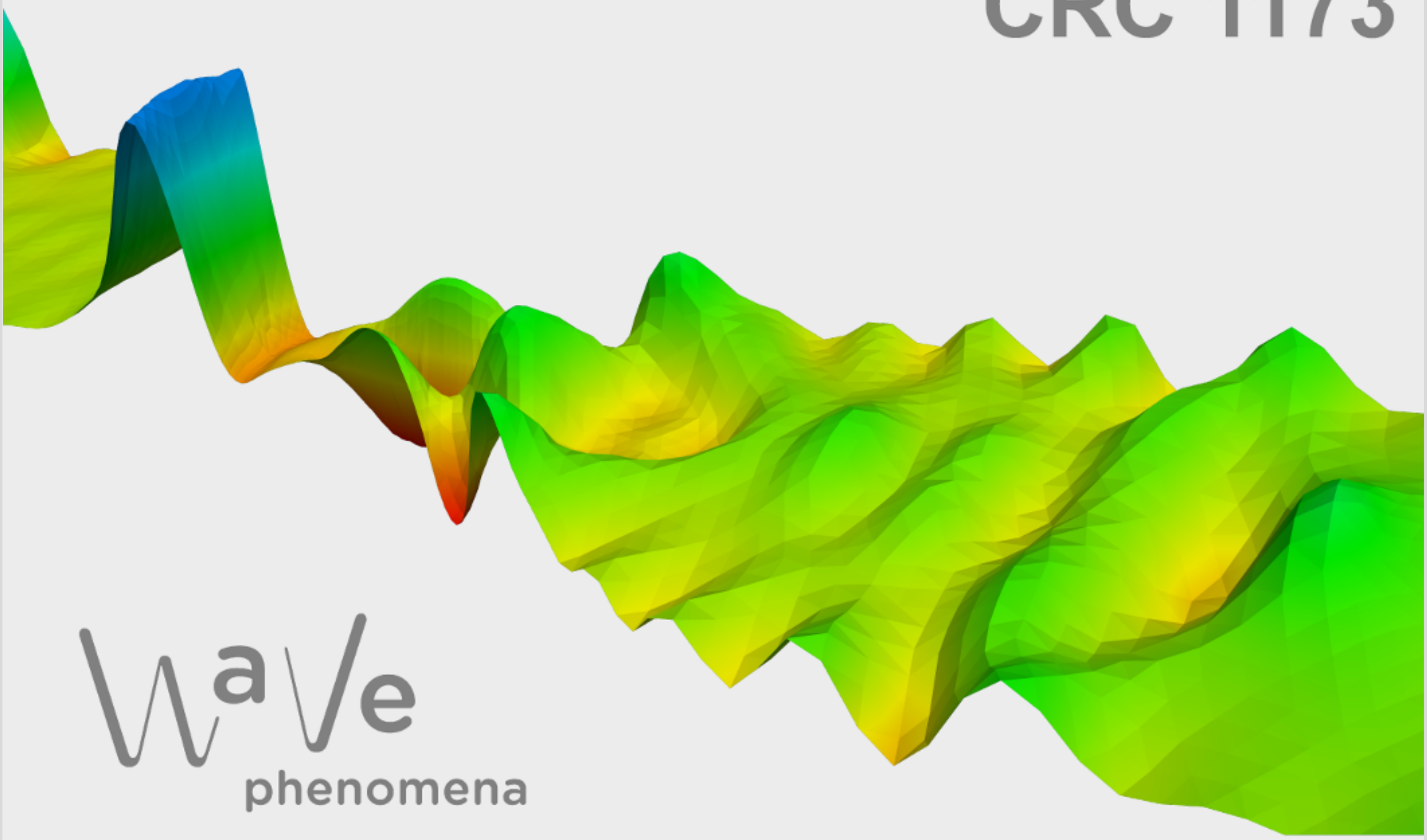
CRC 1173



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ISSN 2365-662X

The domain derivative in inverse obstacle scattering with nonlinear impedance boundary condition

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Abstract. In this paper an inverse obstacle scattering problem for the Helmholtz equation with nonlinear impedance boundary condition is considered. For a certain class of nonlinearities, well-posedness of the direct scattering problem is proven. Furthermore, differentiability of solutions with respect to the boundary is shown by the variational method. A characterization of the derivative allows for iterative regularization schemes in solving the inverse problem, which consists in reconstructing the scattering obstacle from the far field pattern of a scattered wave. An all-at-once Newton-type regularization method is developed to illustrate the use of the domain derivative by some numerical examples.

Keywords: inverse obstacle problem, nonlinear impedance boundary condition, domain derivative, iterative regularization method

1. Introduction

Nonlinear phenomena in acoustic and electromagnetic obstacle scattering have gained importance in the last decade, for example in the case of thinly coated perfectly conducting objects filled with a nonlinear medium [27, 29]. Through an asymptotic analysis, nonlinear material properties in thin layers lead to approximate nonlinear boundary conditions, [26]. Effective boundary conditions for thin films filled with linear media have been derived and investigated in [2, 5, 7, 9]. Scattering problems with such nonlinear boundary conditions have been studied in [25] for electromagnetic waves and in [3, 4] for the acoustic wave equation, based on a boundary element method in space and convolution quadrature in time.

In this work we focus on the reconstruction of an obstacle with material properties near the boundary approximated by an impedance boundary condition with an additional nonlinear term. Generally, in inverse obstacle scattering, we are interested in determining the shape of an obstacle by illuminating the object with plane incident time-harmonic waves and considering measurements of the far-field pattern of corresponding scattered waves. In case of linear scattering models such inverse problems are extensively studied (see [6]) but much less is known in case of nonlinear media. First approaches in recovering the support of nonlinear media can be found in [23] and [8] based on the factorization method and the monotonicity method. These require, at least theoretically, measurements of the far field pattern for all incident directions. But, for the following investigations we assume access to the far field pattern for a single or just a few incident fields.

We consider time-harmonic scattering modeled by the Helmholtz equation in the exterior of an object with a nonlinear impedance boundary condition. Thus, the model may cover acoustic scattering (in $\mathbb{R}^3$) as well as a specific polarization in electromagnetic scattering (in $\mathbb{R}^2$). For the scattering problem with classical Robin boundary condition, i.e., without a nonlinear term, it is observed that iterative regularization schemes based on the domain derivative of the scattering problem lead to feasible reconstruction algorithms (see [22] and references cited therein). Therefore, in extending this approach to nonlinear impedance boundary conditions our task is twofold. First, we have to establish a shape derivative of the scattered field with respect to the scattering obstacle, and second, we are going to develop an iterative regularization scheme in solving the severely ill-posed reconstruction problem.

In view of the variational method for showing existence of the domain derivative we begin with a weak scattering theory for the nonlinear boundary condition. Thus, in the second section we present quite general assumptions on the nonlinear term which ensure a well-posed direct problem. It is shown that Kacurovskii's extension of the Fredholm Theory to nonlinear compact operators can be applied to the scattering problem. Furthermore, we will prove uniqueness and stability of the nonlinear scattering problem by using Rellich's Lemma.

With a well-posed scattering problem the existence of the so called material derivative is proven in the following section. The presented approach is closely related to the ideas established in [13] for semilinear boundary value problems. Similar to the case of linear boundary value problems such a derivative can be represented by its domain derivative (see [11, 14, 18]), which is shown to satisfy a corresponding linearized impedance boundary value problem for the Helmholtz equation assuming suitable regularity of the boundary.

The subsequent sections are devoted to the inverse problem. With the domain derivative at hand it becomes natural to consider iterative regularization schemes in solving the inverse problem. Since any iteration step requires solving a nonlinear boundary value problem we suggest an all-at-once method based on linearizing the scattering problem and applying a regularised Newton step for the reconstruction. In general, it is shown by B. Kaltenbacher in [15] that such an approach leads to regularization schemes by assuming certain mapping properties of the nonlinear operator.

Although, a complete convergence proof is not known even for the inverse object reconstruction for linear scattering problems, we elaborate such an all-at-once approach in the fourth section. We introduce an integral equation method solving for the linearized scattering problems, and develop a Newton type regularization scheme for the reconstruction of a scattering object from a far field pattern. Some examples in the last section illustrate the performance of the scheme. It turns out that the suggested method leads to reconstructions in case of nonlinear boundary conditions comparable to the known results with Robin boundary conditions.

2. The direct problem

Let D be a bounded domain in $\mathbb{R}^N$ $(N = 2, 3)$ with a Lipschitz continuous boundary ∂D. An incident plane wave $u^i(x) = e^{ikd\cdot x}$ with propagation direction $d \in S^{N-1}$ and real positive wave number k is scattered by the object D, generating a scattered wave

u^s. The scattering problem is to find the total wave $u = u^i + u^s$ which satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^N \backslash \overline{D} \tag{1}$$

together with a nonlinear impedance boundary condition

$$\frac{\partial u}{\partial \nu} + ik\lambda u = g(\cdot, u) \quad \text{on} \quad \partial D, \tag{2}$$

where ν denotes the unit normal vector to ∂D oriented towards $\mathbb{R}^N \backslash \overline{D}$, $\lambda \in L^\infty(\partial D)$ is a complex-valued impedance function, and $g : \partial D \times \mathbb{C} \to \mathbb{C}$ gives an additional nonlinear term with respect to u. Furthermore, the scattered field u^s is assumed to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{N-1}{2}} \left(\frac{\partial u^s(x)}{\partial r} - iku^s(x) \right) = 0, \quad r = |x|, \tag{3}$$

uniformly with respect to $\hat{x} = \frac{x}{|x|}$.

In the given scattering problem the propagation region of the wave is unbounded. For a weak formulation we consider an equivalent representation of the problem on a bounded subdomain $\Omega := B_R \backslash \overline{D}$, where the radius $R > 0$ of the ball $B_R \subseteq \mathbb{R}^N$ is chosen large enough such that $\overline{D} \subseteq B_R$. We consider weak solutions in the Sobolev space $H^1(\Omega)$. To include the Sommerfeld radiation condition, we specify a nonlocal boundary condition on the boundary ∂B_R. Therefore, we introduce the Dirichlet-to-Neumann map

$$\Lambda : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R), \quad \Lambda f = \frac{\partial w}{\partial \nu},$$

where w is the radiating solution of the Helmholtz equation in $\mathbb{R}^N \backslash \overline{B_R}$ with Dirichlet trace $w = f$ on ∂B_R (see [6]). Since the scattered wave u^s is a solution of the Helmholtz equation satisfying the Sommerfeld radiation condition we obtain the boundary condition

$$\frac{\partial u^s}{\partial \nu} = \Lambda u^s \quad \text{on} \quad \partial B_R.$$

Thus, a weak formulation of the scattering problem is given by

$$\mathcal{R}(u, v) = (f, v)_{H^1} \quad \text{for all } v \in H^1(\Omega) \tag{4}$$

with

$$\mathcal{R}(u, v) = \int_\Omega \nabla u \cdot \nabla \overline{v} - k^2 u \overline{v} \, \mathrm{d}x - ik \int_{\partial D} \lambda u \overline{v} \, \mathrm{d}s + \int_{\partial D} g(\cdot, u) \overline{v} \, \mathrm{d}s - \int_{\partial B_R} \Lambda u \overline{v} \, \mathrm{d}s,$$

and $f \in H^1(\Omega)$ is defined from the representation theorem by

$$(f, v)_{H^1} = \int_{\partial B_R} \left(\frac{\partial u^i}{\partial \nu} - \Lambda u^i \right) \overline{v} \, \mathrm{d}s.$$

We assume throughout this work the function $g(x, z)$ to be measurable in x, continuous in z, and satisfying a Caratheodory condition

$$|g(x, z)| \leq |\psi(x)| + c|z|^p, \quad \text{a.e. } x \in \partial D, \ z \in \mathbb{C} \tag{5}$$

with $c > 0$, $\psi \in L^2(\partial D)$ and $1 \leq p < \infty$ in case of $N = 2$, and $1 \leq p \leq 2$ in case of $N = 3$. Then, with $u \in H^1(\Omega)$ we have $G(u) \in L^2(\partial D)$ for the Nemytskii operator defined by $G(u)(x) = g(x, u(x))$, see [1, Theorem 5.36]. Further conditions on g will be specified below.

It is well known for the linear scattering problem, i.e. $g = 0$, that there exists a unique solution $u \in H^1(\Omega)$ of (4) for any $f \in H^1(\Omega)$, if the impedance function satisfies $\mathrm{Re}(\lambda) \geq 0$ (see [11], [19]). Thus, throughout we assume $\mathrm{Re}(\lambda) \geq 0$ on ∂D.

To ensure also a well-posed direct boundary value problem in case of the nonlinear impedance condition, the function $g : \partial D \times \mathbb{C} \to \mathbb{C}$ has to fulfil some additional assumptions. The following Theorem, based on an extension of the Fredholm theory, gives a sufficient condition for existence of weak solutions of the scattering problem.

Theorem 2.1 *Let the nonlinearity g be sublinear in z at infinity, i.e.,*

$$|g(x, z)| = o(|z|) \ \text{as} \ |z| \to \infty \quad \text{for a.e. } x \in \partial D. \tag{6}$$

Then, for any $f \in H^1(\Omega)$ there exists a weak solution $u \in H^1(\Omega)$ of the boundary value problem (4).

Proof. The variational formulation is equivalent to the operator equation

$$R_l u + R_n(u) = f \quad \text{in} \quad H^1(\Omega),$$

where the operators $R_l, R_n : H^1(\Omega) \to H^1(\Omega)$ are defined by

$$\begin{aligned}(R_l u, v)_{H^1} \quad &= \int_\Omega \nabla u \cdot \nabla \overline{v} - k^2 u\overline{v}\,\mathrm{d}x - ik \int_{\partial D} \lambda u \overline{v}\,\mathrm{d}s \\ &\quad - \int_{\partial B_R} \Lambda u \overline{v}\,\mathrm{d}s, \\ (R_n(u), v)_{H^1} &= \int_{\partial D} g(\cdot, u)\overline{v}\,\mathrm{d}s.\end{aligned}$$

The linear scattering problem is given by the operator R_l. Thus, as mentioned above, it is an injective Fredholm operator with index zero and has a bounded inverse. From

$$\|(R_n(w) - R_n(u)\|_{H^1(\Omega)} \leq \|G(w) - G(u)\|_{L^2(\partial D)}$$

we see, the operator R_n is continuous, since by the sublinear assumption the Nemytskii operator G is continuous as a mapping from $L^2(\partial D)$ into $L^2(\partial D)$ (see [28]). Furthermore, the trace operator from $H^1(\Omega)$ into $L^2(\partial D)$ is compact and we observe $R_n : H^1(\Omega) \to H^1(\Omega)$ to be continuous and compact.

Using the assumption on g we obtain

$$\begin{aligned}|(R_n(u), v)| &= \left| \int_{\partial D} g(\cdot, u)\,\overline{v}\ \mathrm{d}s \right| \\ &\leq \|g(\cdot, u(\cdot))\|_{L^2(\partial D)} \|v\|_{L^2(\partial D)} = o\left(\|u\|_{L^2(\partial D)}\right)\, \|v\|_{L^2(\partial D)}.\end{aligned}$$

Thus, R_n is sublinear, i.e. $\|R_n(u)\|/\|u\| \to 0$, if $\|u\| \to \infty$. Therefore, the Theorem of Kacurovskii, an extension of the Fredholm alternative, see [31, Theorem 29.A], can be applied and shows existence of a solution to (4) for any $f \in H^1(\Omega)$. □

Let g be differentiable in its second argument, in the sense of existence of $g_z(.,z;w) \in L^\infty(\partial D)$ such that

$$g(x, z+w) - g(x,z) = g_z(x,z;w) + o(|w|)$$

for all $z, w \in \mathbb{C}$, a.e. $x \in \partial D$ and g_z to be $\mathbb{R}$-linear w.r.t. w, then a lower bound on $\mathrm{Re}(\lambda)$ leads to a well-posed direct scattering problem.

Corollary 2.2 *Let $g : \partial D \times \mathbb{C} \to \mathbb{C}$ be a sublinear function as in Theorem 2.1, differentiable in its second argument as described above, and let*

$$k\mathrm{Re}(\lambda) \geq \sup_{z,w\in\mathbb{C}} \frac{\mathrm{Im}(g_z(x,z;w)\overline{w})}{|w|^2} \quad \textit{for any } z,w \in \mathbb{C} \textit{ and a.e. } x \in \partial D.$$

Then, the problem (4) has a unique solution $u \in H^1(\Omega)$ and it exists a constant $c > 0$ with

$$\|u\|_{H^1(\Omega)} \leq c\|f\|_{H^1(\Omega)} \quad \textit{for any } f \in H^1(\Omega). \tag{7}$$

Proof. Rellich's Lemma ensures that a radiating solution of the Helmholtz equation vanishes if

$$\int_{\partial D} \mathrm{Im}\left(\overline{u}\frac{\partial u}{\partial \nu}\right) \mathrm{d}s \leq 0,$$

see [6]. Let u_1 and u_2 be two solutions of (4). We obtain the inequality for the scattered solution $u = u_1 - u_2$ of the Helmholtz equation,

$$\int_{\partial D} \mathrm{Im}\left(\overline{u}\frac{\partial u}{\partial \nu}\right) \mathrm{d}s = -k\int_{\partial D} \mathrm{Re}(\lambda)|u|^2\,\mathrm{d}s + \mathrm{Im}\int_{\partial D} (g(\cdot,u_1) - g(\cdot,u_2))\overline{u})\,\mathrm{d}s \leq 0$$

from

$$\mathrm{Im}\left((g(x,z+w) - g(x,z))\overline{w}\right) = \mathrm{Im}\int_0^1 g_z(x, z+tw; w)\overline{w}\,\mathrm{d}t \leq k\mathrm{Re}(\lambda)|w|^2\,.$$

This implies uniqueness of the solution $u \in H^1(\Omega)$ of problem (4). The uniform bound by a constant $c > 0$ follows analogously as it is known from the Riesz-Fredholm theory for an injective linear Fredholm operator. □

Remark (a) As an example we may consider

$$g(\cdot, z) = \frac{z}{1+|z|^2}$$

in case of $k\mathrm{Re}(\lambda) > \frac{1}{2}$ on ∂D. Obviously, the function is sublinear in z, and with

$$\mathrm{Im}(g_z(\cdot,z;w)\overline{w}) = -\frac{2\mathrm{Re}(z\overline{w})\,\mathrm{Im}(z\overline{w})}{(1+|z|^2)^2}$$

and $\frac{|z|}{1+|z|^2} \leq \frac{1}{2}$ for all $z \in \mathbb{C}$ we observe the assumption of Corollary 2.2.

(b) Note, the restriction of a sublinear function g is quite natural, since in the general well-posed case g can be replaced by a sublinear one without changing the solution. This is seen as follows: Let us assume that the boundary value problem (4) is well-posed, f is defined from an incident field u^i as above, and $u = u^s + u^i$

denotes the solution of the scattering problem. From Green's representation theorem in $\mathbb{R}^n \backslash \overline{D}$ for u^s and in D for u^i, and the boundary condition we obtain

$$\begin{aligned}\frac{1}{2}(u^s - u^i) &= \int_{\partial D} u \frac{\partial \Phi}{\partial \nu} + (ik\lambda u - g(.,u))\Phi \, \mathrm{d}s \\ &= \mathcal{D}u + \mathcal{S}(iku - g(.,u))\end{aligned}$$

on ∂D, where Φ is the fundamental solution of the Helmholtz equation and $\mathcal{D}, \mathcal{S}$ denote the boundary integral operators corresponding to the double and the single layer potential of the Helmholtz equation, see [20]. Let $D \subseteq \mathbb{R}^2$ with $\partial D \in C^{2,\alpha}$, $\alpha \in (0,1)$ we have $\mathcal{D}, \mathcal{S} : L^2(\partial D) \to H^1(\partial D) \subseteq C(\partial D)$, see [17]. Thus, the above representation holds pointwise and the Caratheodory condition implies

$$\begin{aligned}\|u\|_{C(\partial D)} &\leq C\big(\|u^i\|_{C(\partial D)} + \|u\|_{H^1(\Omega_R)} + \|\psi\|_{L^2(\partial D)} + \|u\|^p_{H^1(\Omega_R)}\big) \\ &\leq c\big(\|u^i\|_{C(\partial D)} + \|f\|_{H^1(\Omega_R)} + \|\psi\|_{L^2(\partial D)} + \|f\|^p_{H^1(\Omega_R)}\big) \\ &\leq b_0\end{aligned}$$

for some $b_0 \in \mathbb{R}$. Analogously, this is true in case of $D \subseteq \mathbb{R}^3$, if $\partial D \in C^{3,\alpha}$ and $g(.,u) \in H^1(\partial D)$ for $u \in H^1(\partial D)$, since first we conclude $u \in H^1(\partial D)$ and then we can apply the Sobolev embedding $H^2(\partial D) \subseteq C(\partial D)$ from $\mathcal{D}, \mathcal{S} : H^1(\partial D) \to H^2(\partial D)$.

Now consider g_b defined by $g_b(x,z) = \varphi_b(|z|)g(x,z)$ with $\varphi_b \in C^\infty(\mathbb{R})$ given by $\varphi_b(r) \in [0,1]$ for $r \in \mathbb{R}$ and

$$\varphi_b(r) = \begin{cases} 1\,, & \text{for } |z| \leq b \\ 0\,, & \text{for } |z| > b+1\,. \end{cases}$$

We observe g_b to be sublinear in its second argument and u solves the problem (4) for all $b > b_0$, if g is replaced by g_b. Thus, if g_b satisfies the conditions of Corollary 2.2 and u_b denotes the corresponding unique solution, we conclude $u = u_b$ for all $b > b_0$ by uniqueness.

3. The domain derivative

In view of iterative regularization schemes for the inverse obstacle scattering problem, it is necessary to study differentiability of the domain to far field operator. Therefore, we investigate the derivative of a solution of the scattering problem with respect to the domain D. In this chapter we assume that D is of class C^1. A variation of a domain $D \subseteq B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ is described by a sufficiently small vector field $h \in C^1_c(B_R)$. We denote a perturbed domain by

$$D_h = \{x + h(x) \in \mathbb{R}^n : x \in D\}$$

and $\Omega_h = B_R \backslash \overline{D_h}$. The corresponding scattering problem is given by

$$\mathcal{R}_h(u_h, v_h) = (f, v_h)_{H^1} \tag{8}$$

for all $v_h \in H^1(\Omega_h)$ with

$$\begin{aligned}\mathcal{R}_h(u_h, v_h) &= \int_{\Omega_h} \nabla u_h \cdot \nabla \overline{v_h} - k^2 u_h \overline{v_h} \, \mathrm{d}x - ik \int_{\partial D_h} \lambda u_h \overline{v_h} \, \mathrm{d}s \\ &\quad + \int_{\partial D_h} g(\cdot, u_h) \overline{v_h} \, \mathrm{d}s - \int_{\partial B_R} \Lambda u_h \overline{v_h} \, \mathrm{d}s, \\ (f, v_h)_{H^1} &= \int_{\partial B_R} \left(\frac{\partial u^i}{\partial \nu} - \Lambda u^i \right) \overline{v_h} \, \mathrm{d}s.\end{aligned}$$

Throughout, we consider λ and g on ∂D and on perturbed boundaries to be traces of functions $\lambda : \mathbb{R}^N \to \mathbb{C}$ and $g : \mathbb{R}^N \times \mathbb{C} \to \mathbb{C}$ and assume there exist unique solutions $u \in H^1(\Omega)$, $u_h \in H^1(\Omega_h)$ of (4) and (8), respectively. For instance Corollary 2.2 gives sufficient conditions.

A transformation of variables with $\varphi(x) = x + h(x)$ leads to

$$\begin{aligned} \widetilde{\mathcal{R}}_h(\widetilde{u}_h, v) = &\int_\Omega \Big[\nabla \widetilde{u}_h \cdot J_\varphi^{-1} J_\varphi^{-\top} \nabla \overline{v} - k^2 \widetilde{u}_h \overline{v}\Big] \det(J_\varphi)\,\mathrm{d}x \\ &- ik \int_{\partial D} \widetilde{\lambda} \widetilde{u}_h \overline{v}\ \mathrm{Det}(\varphi)\,\mathrm{d}s - \int_{\partial B_R} \Lambda \widetilde{u}_h \overline{v}\,\mathrm{d}s \\ &+ \int_{\partial D} g(\varphi(\cdot), \widetilde{u}_h) \overline{v}\ \mathrm{Det}(\varphi)\,\mathrm{d}s \\ = &\int_{\partial B_R} \left(\frac{\partial u^i}{\partial \nu} - \Lambda u^i\right) \overline{v}\,\mathrm{d}s = (f, v)_{H^1} \end{aligned} \tag{9}$$

for all $v \in H^1(\Omega)$. Here, we denote by J_φ the Jacobian matrix of the transformation φ and $\widetilde{u}_h(x) = (u_h \circ \varphi)(x)$. Furthermore, the Jacobian with respect to the surface ∂D is given by

$$\mathrm{Det}(\varphi) = \sqrt{\det\left(J_{\widehat{\phi}}^\top J_{\widehat{\phi}}\right)} \Big/ \sqrt{\det\left(J_\phi^\top J_\phi\right)},$$

for local parametrizations ϕ and $\widehat{\phi} = \phi + h \circ \phi$ of ∂D and ∂D_h, respectively. Additionally, note that for any perturbation h we have the same f in (9) as in (4) given by the incident field on ∂B_R.

For the tangential and the normal component of a vector on the boundary ∂D we introduce the notations

$$h_\tau = \nu \times (h \times \nu) \quad \text{and} \quad h_\nu = h \cdot \nu$$

and thus $h = h_\tau + h_\nu \nu$ holds on ∂D. Furthermore, we introduce the tangential gradient, given by $\nabla_\tau v = \nabla v - \frac{\partial v}{\partial \nu}\nu$, and the tangential divergence, $\mathrm{Div} V = \mathrm{div} V - \nu^\top J_V \nu$, for smooth functions v or V in a neighborhood of ∂D, which both can be extended to linear bounded operators

$$\nabla_\tau, \mathrm{Div} : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D).$$

Elementary calculations show the following linearizations of the Jacobians.

Lemma 3.1 *Let $D \subseteq \mathbb{R}^N$ $(N = 2, 3)$ be a bounded domain, $\lambda \in C^1(\mathbb{R}^N, \mathbb{C})$ and κ be the mean curvature of ∂D. Then*

$$\begin{aligned} &\|\det(J_\varphi) - 1 - \mathrm{div}(h)\|_\infty &&= o\left(\|h\|_{C^1}\right), \\ &\|J_\varphi^{-1} J_\varphi^{-\top} \det(J_\varphi) - I + J_h + J_h^\top - \mathrm{div}(h) I\|_\infty &&= o\left(\|h\|_{C^1}\right), \\ &\|\widetilde{\lambda}\mathrm{Det}(\varphi) - \lambda\left(1 + \mathrm{Div}(h_\tau) + 2\kappa h_\nu\right) - \nabla\lambda^\top h\|_\infty &&= o\left(\|h\|_{C^1}\right). \end{aligned}$$

Proof. See [12]. □

Therefore, from now on we assume $\lambda \in C^1(\mathbb{R}^N, \mathbb{C})$. As in [13], we also have to specify assumptions on the nonlinear function g which ensure existence of the domain derivative. We collect sufficient conditions which will be applied in the following considerations.

Assumption (A) A continuous function $g : \mathbb{R}^N \times \mathbb{C} \to \mathbb{C}$ satisfies Assumption (A), if

(i) g is continuously differentiable in $x \in \mathbb{R}^N$, where the partial derivatives g_x satisfy a growth condition as in (5), i.e.

$$|g_x(x,z)| \le |\psi_1(x)| + c_1|z|^{p_1} \tag{10}$$

for a.e. $x \in \partial D$ and all $z \in \mathbb{C}$.

(ii) the derivatives g_z, g_{xz} exist in the sense as above, i.e., $g_z(.,z;w), g_{xz}(.,z;w) \in L^\infty(\partial D_h)$ for any admissible variation h, are $\mathbb{R}$-linear functions with respect to $w \in \mathbb{C}$ with

$$\begin{aligned} g(x,z+w) - g(x,z) \quad &= g_z(x,z;w) + o(|w|), \\ g_x(x,z+w) - g_x(x,z) &= g_{xz}(x,z;w) + o(|w|) \end{aligned} \tag{11}$$

for all $z, w \in \mathbb{C}$ and a.e. $x \in \partial D_h$. Additionally, these derivatives also satisfy growth conditions

$$|g_z(x,z;w)|, |g_{xz}(x,z;w)| \le (\psi(x) + c|z|^p)|w|$$

with corresponding functions $\psi \in L^2(\partial D)$, $c > 0$ and $1 \le p < \infty$ in case of $N = 2$, and $1 \le p \le 2$ in case of $N = 3$.

With Lemma 3.1 and Assumption (A), we conclude continuous dependence on variations of the domain D.

Theorem 3.2 *Let g satisfies (A) and let $u \in H^1(\Omega)$ be the solution of (4) and $u_h \in H^1(\Omega_h)$ the solutions of the scattering problem with respect to perturbed domains. Furthermore, let $|g_z(x,z;w)| \le \eta|w|$ with a sufficiently small constant $\eta > 0$ locally for a.e. $x \in U$ and for all $z \in V$ in open sets $\partial D \subseteq U \subseteq \mathbb{R}^3$ and $\{u(x) : x \in \partial D\} \subseteq V \subseteq \mathbb{C}$. Then, the solution of the boundary value problem depends continuously on the domain D, i.e.*

$$\lim_{\|h\|_{C^1}\to 0} \|\widetilde{u}_h - u\|_{H^1} = 0\,, \tag{12}$$

where the notation $\widetilde{u}_h = u_h \circ \varphi$ with $\varphi = I + h$ is used.

Proof. According to well-posed scattering problems, the operators

$$\begin{aligned} T &: H^1(\Omega) \to H^1(\Omega),\ (T(w), v)_{H^1(\Omega)} = \mathcal{R}(w,v) \\ T_h &: H^1(\Omega) \to H^1(\Omega),\ (T_h(w), v)_{H^1(\Omega)} = \widetilde{\mathcal{R}}_h(w,v) \end{aligned}$$

exist. If we consider the difference between these two operators, we obtain

$$\begin{aligned} \|T(w) - T_h(w)\|_{H^1}^2 &= |\mathcal{R}(w, T(w) - T_h(w)) - \widetilde{\mathcal{R}}_h(w, T(w) - T_h(w))| \\ &= \Bigg| \int_\Omega \Big(\nabla w \cdot \left[I - J_\varphi^{-1} J_\varphi^{-\top} \det(J_\varphi) \right] \nabla \overline{(T(w) - T_h(w))} \\ &\qquad - k^2 (1 - \det(J_\varphi)) w \overline{(T(w) - T_h(w))} \Big) \, \mathrm{d}x \\ &\quad - ik \int_{\partial D} (\lambda - \widetilde{\lambda} \mathrm{Det}(\varphi)) w \overline{(T(w) - T_h(w))} \, \mathrm{d}s \\ &\quad + \int_{\partial D} \big(g(\cdot, w) - g(\varphi(\cdot), w) \mathrm{Det}(\varphi) \big) \overline{(T(w) - T_h(w))} \, \mathrm{d}s \Bigg| . \end{aligned}$$

With the growth conditions (5) and (10), respectively, the expression in the last integral can be estimated by

$$\begin{aligned}
&\|g(\varphi(\cdot),w)\mathrm{Det}(\varphi)-g(\cdot,w)\|_{L^2}\\
&\le \|g(\varphi(\cdot),w)-g(\cdot,w)\|_{L^2}+\|g(\varphi(\cdot),w)(\mathrm{Div}(h_\tau)+2\kappa h_\nu)\|_{L^2}+\mathcal{O}(\|h\|_{C^1})\\
&\le c\|g_x(\cdot,w)\|_{L^2}\|h\|_{C^1}+\|g(\varphi(\cdot),w)(\mathrm{Div}(h_\tau)+2\kappa h_\nu\|_{L^2}+\mathcal{O}(\|h\|_{C^1})\\
&\le c\left(\|\psi_1\|_{L^2}+\|\psi\|_{L^2}+\|w\|_{H^1}\right)\mathcal{O}(\|h\|_{C^1}).
\end{aligned}$$

Thus, for $w\in H^1(\Omega)$ according to Lemma 3.1 we get

$$\|T(w)-T_h(w)\|_{H^1}\le C\|w\|_{H^1}\|h\|_{C^1}+c(\|\psi_1\|_{L^2}+\|\psi\|_{L^2})\|h\|_{C^1}.$$

Furthermore, for solutions of $T(u)=f$ and $T_h(\widetilde{u}_h)=f$ we obtain the identity

$$\begin{aligned}
&T_h(\widetilde{u}_h)-T_h(u)=T(u)-T_h(u)\\
\Leftrightarrow\quad &T_{h,l}(\widetilde{u}_h-u)+(T_{h,n}(\widetilde{u}_h)-T_{h,n}(u))=T(u)-T_h(u)\\
\Leftrightarrow\quad &\widetilde{u}_h-u+T_{h,l}^{-1}(T_{h,n}(\widetilde{u}_h)-T_{h,n}(u))=T_{h,l}^{-1}(T(u)-T_h(u))
\end{aligned}$$

where

$$(T_l(w),v)=\int_\Omega \nabla w\cdot\nabla\overline{v}-k^2w\overline{v}\,\mathrm{d}x-ik\int_{\partial D}\lambda w\overline{v}\,\mathrm{d}s-\int_{\partial B_R}\Lambda w\overline{v}\,\mathrm{d}s$$

denotes the linear part and

$$(T_n(w),v)=\int_{\partial D}g(\cdot,w)\overline{v}\,\mathrm{d}s$$

the nonlinear part of the operator T. Analogous operator splitting holds for T_h.

As mentioned in chapter 2 we know that the operator T_l has a bounded inverse T_l^{-1}. If $\|h\|_{C^1}$ is sufficiently small, then $\|T_l^{-1}(T_{h,l}-T_l)\|\le\frac{1}{2}$. Using Neumann's series, we obtain the existence of the inverse operator $T_{h,l}^{-1}$ and the inequality

$$\|T_{h,l}^{-1}\|\le\frac{\|T_l^{-1}\|}{1-\|T_l^{-1}(T_{h,l}-T_l)\|}\le 2\widetilde{C}$$

for a constant $\widetilde{C}>0$, see [21, Theorem 10.1]. With these considerations, the error estimates

$$\begin{aligned}
&\|\widetilde{u}_h-u+T_{h,l}^{-1}(T_{h,n}(\widetilde{u}_h)-T_{h,n}(u))\|_{H^1}\\
&=\|T_{h,l}^{-1}(T(u)-T_h(u))\|_{H^1}\le\|T_{h,l}^{-1}\|\|T(u)-T_h(u)\|_{H^1}\\
&\le 2\widetilde{C}\left(C\|u\|_{H^1}+c(\|\psi_1\|_{L^2}+\|\psi\|_{L^2})\right)\|h\|\\
&\le 2\widetilde{C}\left(C\|f\|_{H^1}+c(\|\psi_1\|_{L^2}+\|\psi\|_{L^2})\right)\|h\|
\end{aligned}$$

and

$$\begin{aligned}
&\|\widetilde{u}_h-u+T_{h,l}^{-1}(T_{h,n}(\widetilde{u}_h)-T_{h,n}(u))\|_{H^1}\\
&\ge\|\widetilde{u}_h-u\|_{H^1}-\|T_{h,l}^{-1}(T_{h,n}(\widetilde{u}_h)-T_{h,n}(u))\|_{H^1}\\
&\ge\|\widetilde{u}_h-u\|_{H^1}-2\widetilde{C}\|\left(g(\varphi(\cdot),\widetilde{u}_h)-g(\varphi(\cdot),u)\right)\mathrm{Det}(\varphi)\|_{L^2}\\
&\ge\|\widetilde{u}_h-u\|_{H^1}-2\widetilde{C}\|g(\varphi(\cdot),\widetilde{u}_h)-g(\varphi(\cdot),u)\|_{L^2}\\
&\quad-2\widetilde{C}\|g(\varphi(\cdot),\widetilde{u}_h)-g(\varphi(\cdot),u)\|_{L^2}\mathcal{O}(\|h\|)
\end{aligned}$$

are valid. Furthermore,

$$\begin{aligned}
&\|g(\varphi(x),\widetilde{u}_h(x)) - g(\varphi(x),u(x))\|_{L^2}^2 \\
&= \int_{\partial D} \left| \int_0^1 g_z(\varphi(x),u(x)+t(\widetilde{u}_h(x)-u(x));\widetilde{u}_h(x)-u(x))\,\mathrm{d}t \right|^2 \mathrm{d}s \\
&\le \int_0^1 \int_{\partial D} |g_z(\varphi(x),u(x)+t(\widetilde{u}_h(x)-u(x));\widetilde{u}_h(x)-u(x))|^2\,\mathrm{d}s\,\mathrm{d}t\,.
\end{aligned}$$

Overall, with the assumption on g_z we obtain

$$\Big(1-2\eta\widetilde{C}\Big)\|\widetilde{u}_h-u\|_{H^1} = \mathcal{O}(\|h\|).$$

Thus, the solution of the boundary value problem depends continuously on the domain D if $2\eta\widetilde{C}<1$ is satisfied. □

Note from the presented proof that assumptions on the second derivatives of g are not required for the above continuity result.

In the following we also have to consider the $\mathbb{R}$-linear boundary value problem with g replaced by g_z. Thus, we introduce the problem

$$\widehat{R}(u,v) = (f,v)_{H^1} \quad \text{for all } v\in H^1(\Omega) \tag{13}$$

with

$$\begin{aligned}
\widehat{R}(q,v) = &\int_\Omega \nabla q\cdot\nabla\overline{v} - k^2 q\overline{v}\,\mathrm{d}x - ik\int_{\partial D}\lambda q\overline{v}\,\mathrm{d}s \\
&+ \int_{\partial D} g_z(\cdot,u;q)\overline{v}\,\mathrm{d}s - \int_{\partial B_R}\Lambda q\overline{v}\,\mathrm{d}s.
\end{aligned} \tag{14}$$

With the above considerations we now can formulate the main result on existence of a derivative of the solution of the scattering problem with respect to the shape of D.

Theorem 3.3 *We assume the boundary value problems (4) and (13) are well-posed and the solution u of (4) is continuous with respect to the domain D in the sense of (12). With g satisfying assumption (A) the solution u is differentiable with respect to variations $h\in C_c^1(B_R)$ of the domain, i.e., it exists $w\in H^1(\Omega)$, linearly depending on h with*

$$\lim_{\|h\|_{C^1}\to 0} \frac{1}{\|h\|_{C^1}}\|\widetilde{u}_h-u-w\|_{H^1} = 0$$

The material derivative w is given by the unique weak solution of the $\mathbb{R}-$linear boundary value problem

$$\begin{aligned}
\widehat{R}(w,v) = &\int_\Omega \Big[\nabla u\cdot\Big(J_h+J_h^\top-\mathrm{div}(h)I\Big)\nabla\overline{v} + k^2\mathrm{div}(h)u\overline{v}\Big]\mathrm{d}x \\
&+ ik\int_{\partial D}\Big[\lambda\big(\mathrm{Div}(h_\tau)+2\kappa h_\nu\big)+\nabla\lambda^\top h\Big]u\overline{v}\,\mathrm{d}s \\
&- \int_{\partial D}\big(\mathrm{Div}_x(g(\cdot,u)h_\tau)+2\kappa h_\nu g(\cdot,u)\big)\overline{v}\,\mathrm{d}s
\end{aligned} \tag{15}$$

for all $v\in H^1(\Omega)$.

Proof. Since the problem (13) is well posed, $w \in H^1(\Omega)$ exists and according to a support of h close to the boundary ∂D the function w can be extended to a radiating solution of the Helmholtz equation in the exterior of B_R.

Inserting the difference $\widetilde{u}_h - u - w$ and the identity (9) in (14) yields

$$\begin{aligned}
&\widehat{R}(\widetilde{u}_h - u - w, v) \\
&= \int_\Omega \nabla(\widetilde{u}_h - u - w)\cdot\nabla\overline{v} - k^2(\widetilde{u}_h - u - w)\overline{v}\,\mathrm{d}x - ik\int_{\partial D}\lambda(\widetilde{u}_h - u - w)\overline{v}\,\mathrm{d}s \\
&\quad + \int_{\partial D} g_z(\cdot, u; \widetilde{u}_h - u - w)\overline{v}\,\mathrm{d}s - \int_{\partial B_R}\Lambda(\widetilde{u}_h - u - w)\overline{v}\,\mathrm{d}s \\
&= \int_\Omega \nabla\widetilde{u}_h\cdot\Big(I - J_\varphi^{-1}J_\varphi^{-\top}\det(J_\varphi)\Big)\nabla\overline{v} - k^2\big(1 - \det(J_\varphi)\big)\widetilde{u}_h\overline{v}\,\mathrm{d}x \\
&\quad - \int_\Omega \nabla w\cdot\nabla\overline{v} - k^2 w\overline{v}\,\mathrm{d}x - ik\int_{\partial D}\Big(\lambda - \widetilde{\lambda}\mathrm{Det}(\varphi)\Big)\,\widetilde{u}_h\overline{v}\,\mathrm{d}s + ik\int_{\partial D}\lambda w\overline{v}\,\mathrm{d}s \\
&\quad + \int_{\partial D}\big[g(\cdot,u) - g(\varphi(\cdot),\widetilde{u}_h)\mathrm{Det}(\varphi) + g_z(\cdot,u;\widetilde{u}_h - u)\big]\,\overline{v}\,\mathrm{d}s \\
&\quad - \int_{\partial D} g_z(\cdot,u;w)\overline{v}\,\mathrm{d}s + \int_{\partial B_R}\Lambda w\overline{v}\,\mathrm{d}s.
\end{aligned}$$

We insert equation (15) and obtain

$$\begin{aligned}
&\widehat{R}(\widetilde{u}_h - u - w, v) \\
&= \int_\Omega (\nabla\widetilde{u}_h - \nabla u)\cdot\Big(I - J_\varphi^{-1}J_\varphi^{-\top}\det(J_\varphi)\Big)\nabla\overline{v} - k^2\big(1 - \det(J_\varphi)\big)(\widetilde{u}_h - u)\overline{v}\,\mathrm{d}x \\
&\quad + \int_\Omega\Big[\nabla u\cdot\Big(I - J_\varphi^{-1}J_\varphi^{-\top}\det(J_\varphi) - J_h - J_h^\top + \mathrm{div}(h)I\Big)\nabla\overline{v} \\
&\qquad\qquad - k^2\big(1 - \det(J_\varphi) + \mathrm{div}(h)\big)u\overline{v}\Big]\mathrm{d}x \\
&\quad - ik\int_{\partial D}\Big(\lambda - \widetilde{\lambda}\mathrm{Det}(\varphi)\Big)\,(\widetilde{u}_h - u)\overline{v}\,\mathrm{d}s \\
&\quad - ik\int_{\partial D}\Big(\lambda\,(1 + \mathrm{Div}(h_\tau) + 2\kappa h_\nu) + \nabla\lambda^\top h - \widetilde{\lambda}\mathrm{Det}(\varphi)\Big)\,u\overline{v}\,\mathrm{d}s \\
&\quad + \int_{\partial D}\Big[g(\cdot,u) - g(\varphi(\cdot),\widetilde{u}_h)\mathrm{Det}(\varphi) + g_z(\cdot,u;\widetilde{u}_h - u) + \mathrm{Div}_x(g(\cdot,u)h_\tau) \\
&\qquad\qquad + 2\kappa h_\nu g(\cdot,u)\Big]\,\overline{v}\,\mathrm{d}s.
\end{aligned}$$

Here and in the following considerations, terms on the boundary ∂D have to be read in the duality sense of $H^{-\frac{1}{2}}$ and $H^{\frac{1}{2}}$. By Lemma 3.1 and the continuity (12) all integrals except the last are of order $o\left(\|h\|_{C^1}\right)$. To verify an estimate for the last integrand, we sort it to

$$\begin{aligned}
&|g(\varphi(.),\widetilde{u}_h)\mathrm{Det}(\varphi) - g(.,u) - g_z(x,u;\widetilde{u}_h - u) - \mathrm{Div}_x(g(.,u)h_\tau) - 2\kappa h_\nu g(.,u)| \\
&= \big|g(\varphi(.),\widetilde{u}_h)\mathrm{Det}(\varphi) - g(.,u) - g_z(.,u;\widetilde{u}_h - u) - \mathrm{Div}_x(g(.,u)h_\tau) - 2\kappa h_\nu g(.,u) \\
&\qquad - \big(g(\varphi(.),\widetilde{u}_h) - g(\varphi(.),\widetilde{u}_h)\big)(1 + \mathrm{Div}(h_\tau) + 2\kappa h_\nu)\big| \\
&\le \big|g(\varphi(.),\widetilde{u}_h)\big|\,\big|\mathrm{Det}(\varphi) - 1 - \mathrm{Div}(h_\tau) - 2\kappa h_\nu\big| \\
&\quad + \big|g(\varphi(.),\widetilde{u}_h) - g(.,u) - g_z(.,u;\widetilde{u}_h - u) + g(\varphi(.),\widetilde{u}_h)\mathrm{Div}(h_\tau)
\end{aligned}$$

$$- \operatorname{Div}_x(g(.,u)h_\tau) + 2\kappa h_\nu g(\varphi(.),\widetilde{u}_h) - 2\kappa h_\nu g(.,u)\big|.$$

In the first line we can apply (5) and Lemma 3.1. By adding and subtracting further terms it follows

$$\begin{aligned}
&|g(\varphi(.),\widetilde{u}_h)\mathrm{Det}(\varphi) - g(.,u) - g_z(.,u;\widetilde{u}_h - u) - \operatorname{Div}_x(g(.,u)h_\tau) - 2\kappa h_\nu g(.,u)| \\
&\leq (|\psi_0(\varphi(.))| + c_0|\widetilde{u}|^p)\, o(\|h\|_{C^1}) \\
&\quad + \big|g(\varphi(.),\widetilde{u}_h) - g(.,u) - g_z(.,u;\widetilde{u}_h - u) + g(\varphi(.),\widetilde{u}_h)\mathrm{Div}(h_\tau) \\
&\qquad - \operatorname{Div}_x(g(.,u)h_\tau) + 2\kappa h_\nu g(\varphi(.),\widetilde{u}_h) - 2\kappa h_\nu g(.,u) \\
&\qquad - \big(g(.,\widetilde{u}_h) - g(.,\widetilde{u}_h)\big)(1 + \mathrm{Div}(h_\tau) + 2\kappa h_\nu) - \big(g_x(.,\widetilde{u}_h) - g_x(.,\widetilde{u})\big)h_\tau\big| \\
&\leq (|\psi_0(\varphi(.))| + c_0|\widetilde{u}_h|^p) o(\|h\|_{C^1}) + |g(\varphi(.),\widetilde{u}_h) - g(.,\widetilde{u}_h) - g_x(.,\widetilde{u}_h)\, h_\tau| \\
&\quad + |g(.,\widetilde{u}_h) - g(.,u) - g_z(.,u;\widetilde{u}_h - u)| + |\,(g(\varphi(.),\widetilde{u}_h) - g(.,\widetilde{u}_h))\,\mathrm{Div}(h_\tau)| \\
&\quad + |\,(g(.,\widetilde{u}_h) - g(.,u))\,\mathrm{Div}(h_\tau)| + |\,(g_x(.,\widetilde{u}_h) - g_x(.,u)))\, h_\tau| \\
&\quad + |\,(g(\varphi(.),\widetilde{u}_h) - g(.,\widetilde{u}_h))\, 2\kappa h_\nu| + |\,(g(.,\widetilde{u}_h) - g(.,u))\, 2\kappa h_\nu|.
\end{aligned}$$

Using the growth conditions from (A), continuity of the Nemytskii operator, and the continuity of solutions w.r.t. h we obtain

$$\begin{aligned}
&\|g(\varphi(\cdot),\widetilde{u}_h)\mathrm{Det}(\varphi) - g(\cdot,u) - g_z(\cdot,u;\widetilde{u}_h - u) - \operatorname{Div}_x(g(\cdot,u)h_\tau) - 2\kappa h_\nu g(\cdot,u)\|_{L^2} \\
&\leq \big(\|\psi\|_{L^2} + c\|f\|_{L^2}\big) o(\|h\|_{C^1}).
\end{aligned}$$

Finally, the Riesz representation theorem implies that $\widetilde{u}_h - u - w \in H^1(\Omega)$ is the unique solution of

$$\widehat{\mathcal{R}}(\widetilde{u}_h - u - w, v) = (f_h, v)_{H^1}$$

for all $v \in H^1(\Omega)$ with a functional given by $f_h \in H^1(\Omega)$ satisfying $\|f_h\|_{H^1} = o(\|h\|_{C^1})$. By the assumed well-posedness of (13) we conclude

$$\frac{1}{\|h\|_{C^1}}\|\widetilde{u}_h - u - w\|_{H^1} \leq \frac{c\,\|f_h\|_{H^1}}{\|h\|_{C^1}} \to 0 \quad \text{for} \quad \|h\|_{C^1} \to 0.$$

□

As known from shape derivatives for linear boundary value problems the material derivative can be read in the sense of the chain rule, which leads to the notation of the so called domain derivative of u.

Theorem 3.4 *The material derivative of the previous theorem is of the form $w = u' + \nabla u \cdot h$, where u denotes the solution of (4) and the domain derivative $u' \in H^1(\Omega)$ is the unique weak solution of the $\mathbb{R}-$linear boundary value problem*

$$\Delta u' + k^2 u' = 0 \quad \textit{in } \Omega \tag{16}$$

with

$$\begin{aligned}
&\frac{\partial u'}{\partial \nu} + ik\lambda u' + g_z(\cdot,u;u') \\
&= \mathrm{Div}\big(h_\nu \nabla_\tau u\big) + k^2 u h_\nu - ik\lambda\left(2\kappa u h_\nu + \frac{\partial u}{\partial \nu}h_\nu\right) \\
&\quad - ik\frac{\partial \lambda}{\partial \nu}u h_\nu + g_z\left(\cdot,u;\frac{\partial u}{\partial \nu}h_\nu\right) + 2\kappa g(\cdot,u)h_\nu \qquad \textit{on } \partial D
\end{aligned} \tag{17}$$

and

$$\Lambda u' = \nabla u' \cdot \nu \quad on\ \partial B_R\,. \tag{18}$$

Here, the last boundary condition means that u' can be uniquely extended to a weak radiating solution of the Helmholtz equation in $\mathbb{R}^N \backslash \overline{D}$.

Proof. From Green's representation theorem and the radiation condition we obtain

$$u^s = \mathrm{DL}u + \mathrm{SL}(ik\lambda u - g(.,u)) \quad \text{in } \Omega.$$

Thus, the mapping properties of the potentials $\mathrm{DL} : H^{\frac{1}{2}}(\partial D) \to H^2(\Omega)$ and $\mathrm{SL} : H^{-\frac{1}{2}}(\partial D) \to H^2(\Omega)$ if ∂D is C^2, see [24, p.210 ff.], leads to $u \in H^2(\Omega)$. Therefore, the traces of $h \cdot \nabla u$ can be read in $H^{\frac{1}{2}}(\partial D)$ and $H^{-\frac{1}{2}}(\partial D)$, respectively.

We insert $h \cdot \nabla u - w$ in (14), where $\langle \Lambda(h \cdot \nabla u), v\rangle_{\partial B_R} = 0$ vanishes since h has compact support in B_R. The product rule leads to the relation

$$\begin{aligned}\nabla u \cdot \Big(J_h + J_h^\top - \mathrm{div}(h) I\Big)\nabla v = \mathrm{div}\Big((h\cdot\nabla u)\nabla v + (h\cdot\nabla v)\nabla u - (\nabla u\cdot\nabla v)h\Big)\\ - (h\cdot\nabla u)\mathrm{div}(\nabla v) - (h\cdot\nabla v)\mathrm{div}(\nabla u),\end{aligned}$$

and we get

$$\begin{aligned}\int_\Omega \nabla(h\cdot\nabla u)\cdot\nabla\overline{v}\,\mathrm{d}x = \int_\Omega \Big[\nabla u\cdot\Big(J_h + J_h^\top - \mathrm{div}(h)I\Big)\nabla\overline{v} + (h\cdot\nabla\overline{v})\mathrm{div}(\nabla u)\\ - \mathrm{div}\Big((h\cdot\nabla\overline{v})\nabla u - (\nabla u\cdot\nabla\overline{v})h\Big)\Big]\mathrm{d}x.\end{aligned}$$

For the material derivative w we use equation (15) and thus it holds

$$\begin{aligned}\widehat{\mathcal{R}}(h\cdot\nabla u - w, v) =& -\int_\Omega k^2\mathrm{div}(h)u\overline{v} + k^2(h\cdot\nabla u)\overline{v} - (h\cdot\nabla\overline{v})\mathrm{div}(\nabla u)\,\mathrm{d}x\\ &- \int_\Omega \mathrm{div}\Big((h\cdot\nabla\overline{v})\nabla u - (\nabla u\cdot\nabla\overline{v})h\Big)\mathrm{d}x + \mathcal{I}_1 + \mathcal{I}_2\\ =& -\int_\Omega \mathrm{div}\Big((h\cdot\nabla\overline{v})\nabla u - (\nabla u\cdot\nabla\overline{v})h + k^2u\overline{v}h\Big)\mathrm{d}x\\ &+ \mathcal{I}_1 + \mathcal{I}_2\end{aligned}$$

with

$$\begin{aligned}\mathcal{I}_1 =& -ik\int_{\partial D}\lambda(h\cdot\nabla u)\overline{v}\,\mathrm{d}s - ik\int_{\partial D}\Big(\lambda\big(\mathrm{Div}(h_\tau) + 2\kappa h_\nu\big) + \nabla\lambda^\top h\Big)u\overline{v}\,\mathrm{d}s,\\ \mathcal{I}_2 =& \int_{\partial D} g_z(\cdot, u; h\cdot\nabla u)\overline{v}\,\mathrm{d}s + \int_{\partial D}\Big(\mathrm{Div}_x\,(g(\cdot,u)h_\tau) + 2\kappa h_\nu g(\cdot,u)\Big)\overline{v}\,\mathrm{d}s.\end{aligned}$$

Using Gauss' divergence theorem and the compactness of the support of h in B_R we obtain

$$\begin{aligned}\widehat{\mathcal{R}}(h\cdot\nabla u - w, v) =& -\int_{\partial\Omega}\nu\cdot\Big((h\cdot\nabla\overline{v})\nabla u - (\nabla u\cdot\nabla\overline{v})h + k^2u\overline{v}h\Big)\mathrm{d}s + \mathcal{I}_1 + \mathcal{I}_2\\ =& \int_{\partial D}\nu\cdot\Big((h\cdot\nabla\overline{v})\nabla u - (\nabla u\cdot\nabla\overline{v})h + k^2u\overline{v}h\Big)\mathrm{d}s + \mathcal{I}_1 + \mathcal{I}_2\\ =& \int_{\partial D}\Big[(h\cdot\nabla\overline{v})(\nabla u\cdot\nu) - (\nabla u\cdot\nabla\overline{v})h_\nu + k^2u\overline{v}h_\nu\Big]\mathrm{d}s + \mathcal{I}_1 + \mathcal{I}_2.\end{aligned}$$

Applying the product rule to the surface divergence operator leads to

$$\begin{aligned}
&\mathrm{Div}(\lambda uv h_\tau)\\
&= \lambda \mathrm{Div}(h_\tau)uv + \nabla_\tau(\lambda uv)\cdot h_\tau = \lambda\mathrm{Div}(h_\tau)uv + \nabla(\lambda uv)\cdot h - \nabla(\lambda uv)_\nu h_\nu\\
&= \lambda\mathrm{Div}(h_\tau)uv + \nabla\lambda^\top huv + \lambda(h\cdot\nabla u)v + \lambda u(h\cdot\nabla v)\\
&\quad - \frac{\partial\lambda}{\partial\nu}uvh_\nu - \lambda\frac{\partial u}{\partial\nu}vh_\nu - \lambda u(\nabla v\cdot\nu)h_\nu\,,
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Div}(g(\cdot,u)h_\tau v)\\
&= \mathrm{Div}_x(g(\cdot,u)h_\tau)v + g_z(\cdot,u;h_\tau\cdot\nabla u)v + g(\cdot,u)(h_\tau\cdot\nabla_\tau v)\\
&= \mathrm{Div}_x(g(\cdot,u)h_\tau)v + g_z(\cdot,u;h_\tau\cdot\nabla u)v + g(\cdot,u)(h\cdot\nabla v) - g(\cdot,u)(\nabla v\cdot\nu)h_\nu.
\end{aligned}$$

Now we can insert these identities in $\widehat{\mathcal{R}}(h\cdot\nabla u - w, v)$ and get

$$\begin{aligned}
&\widehat{\mathcal{R}}(h\cdot\nabla u - w,v)\\
&= \int_{\partial D}\Big[(h\cdot\nabla\overline{v})(\nabla u\cdot\nu) - (\nabla u\cdot\nabla\overline{v})h_\nu + k^2 u h_\nu\overline{v}\Big]\mathrm{d}s\\
&\quad - ik\int_{\partial D}\left[2\lambda\kappa h_\nu u\overline{v} + \frac{\partial\lambda}{\partial\nu}h_\nu u\overline{v} + \lambda\frac{\partial u}{\partial\nu}h_\nu\overline{v} - \lambda u(h\cdot\overline{v}) + \lambda u(\nabla\overline{v}\cdot\nu)h_\nu\right]\mathrm{d}s\\
&\quad + \int_{\partial D}\left[g_z\left(\cdot,u;\frac{\partial u}{\partial\nu}h_\nu\right)\overline{v} + 2\kappa g(\cdot,u)h_\nu\overline{v} + g(\cdot,u)(h\cdot\overline{v}) - g(\cdot,u)(\nabla\overline{v}\cdot\nu)h_\nu\right]\mathrm{d}s
\end{aligned}$$

for all $v\in H^1(\Omega)$. Using the boundary condition on ∂D finally leads to

$$\begin{aligned}
-\widehat{\mathcal{R}}(u',v) &= \widehat{\mathcal{R}}(h\cdot\nabla u - w,v)\\
&= \int_{\partial D}\Big[(\nabla u\cdot\nu)(\nabla\overline{v}\cdot\nu) - (\nabla u\cdot\nabla\overline{v})\Big]h_\nu + k^2 u\overline{v}h_\nu\,\mathrm{d}s\\
&\quad - ik\int_{\partial D} 2\lambda\kappa h_\nu u\overline{v} + \frac{\partial\lambda}{\partial\nu}u\overline{v}h_\nu + \lambda\frac{\partial u}{\partial\nu}\overline{v}h_\nu\,\mathrm{d}s\\
&\quad - \int_{\partial D} g_z\left(\cdot,u;\frac{\partial u}{\partial\nu}h_\nu\right)\overline{v} + 2\kappa g(\cdot,u)h_\nu\overline{v}\,\mathrm{d}s\\
&= -\int_{\partial D} h_\nu\nabla_\tau u\cdot\nabla_\tau\overline{v} - k^2 u\overline{v}h_\nu ds\\
&\quad - ik\int_{\partial D} 2\lambda\kappa h_\nu u\overline{v} + \frac{\partial\lambda}{\partial\nu}u\overline{v}h_\nu + \lambda\frac{\partial u}{\partial\nu}\overline{v}h_\nu\,\mathrm{d}s\\
&\quad + \int_{\partial D} g_z\left(\cdot,u;\frac{\partial u}{\partial\nu}h_\nu\right)\overline{v} + 2\kappa g(\cdot,u)h_\nu\overline{v}\,\mathrm{d}s\,.
\end{aligned}$$

Thus u' can be extended in $\mathbb{R}^N\backslash\overline{D}$ to the weak scattered solution of (16)-(17). □

4. Shape Reconstruction

The Sommerfeld radiation condition guarantees the asymptotic behaviour of the scattered field

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{\frac{N-1}{2}}}\left(u_\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right)\right) \quad \text{for} \quad |x|\to\infty$$

uniformly in all directions, where the function u_∞ defined on the unit sphere S^{N-1} is called the far field pattern of u^s. Thus, the inverse problem under consideration consists in determining the shape of D from measurements of the far field pattern u_∞. We consider a fixed incident field u^i and define the domain-to-far-field operator F, which maps an admissible boundary onto the far field pattern of the scattered wave. Then, the inverse problem can be read as solving the nonlinear equation

$$F(\partial D) = u_\infty. \tag{19}$$

Since the far field pattern is analytic we expect this nonlinear operator equation to be severely ill-posed.

Throughout the last two chapters, we consider the scattering problem only in $\mathbb{R}^2$ and specify the set of admissible domains to be starlike with respect to the origin. A parametrization of the boundary is given by

$$\partial D = \{z(t) = r(t)\,(\cos(t), \sin(t))^\top \in \mathbb{R}^2 : t \in [0, 2\pi]\}$$

with positive, 2π-periodic radial function $r \in C^2(0, 2\pi)$. For the sake of simplicity, we write $F(r)$ instead of $F(\partial D)$ without changing notation. From Section 3, we know that F is differentiable, with the derivative given by the far field pattern of the domain derivative, i.e., $F'[\partial D]h = u'_\infty$.

We suggest an all-at-once Newton-type method based on linearization of the forward problem and of the domain-to-far-field operator in the sense of [15]. A linearization of the nonlinear operator equation (19) leads to

$$F(r) + F'[r]h \approx u^\delta_\infty \tag{20}$$

with noise level δ, i.e. $\|u_\infty - u^\delta_\infty\|_{L^2} \le \delta$. Since the evaluation of the operators $F(r)$ and $F'[r]$ require the far field u_∞ and the Cauchy-data $u|_{\partial D}$, $\frac{\partial u}{\partial \nu}|_{\partial D}$ of the solution u of the nonlinear scattering problem for the domain represented by r, we replace in any iteration step u by $u_n = u^i + u^s_n$ and r by $r_n = r_{n-1} + h_{n-1}$. Choosing an initial guess ∂D_0, we iteratively compute the solution u_n of the scattering problem

$$\Delta u_n + k^2 u_n = 0 \quad \text{in } \Omega_n \tag{21}$$

with boundary condition

$$\frac{\partial u_n}{\partial \nu} + ik\lambda u_n - g_z(\cdot, u_{n-1}; u_n) = g(\cdot, u_{n-1}) - g_z(\cdot, u_{n-1}; u_{n-1}) \tag{22}$$

on ∂D_n and $u^s_n = u_n - u^i$ satisfying the radiation condition. Note that the boundary condition is obtained by linearizing the nonlinear term $g(\cdot, u_n)$ in the sense of (11), i.e.

$$g(\cdot, u_n) - g(\cdot, u_{n-1}) \approx \; g_z(\cdot, u_{n-1}; u_n - u_{n-1}).$$

The domain-to-far-field operator is then defined by $F_n(r_n) = u_{n,\infty}$ and the derivative of F_n is given by $F'_n[r_n]h_n = u'_{n,\infty}$, where u is replaced by u_n on the right hand side of the boundary condition (17).

Since the ill-posed linear operator equation (20) requires a regularization, we apply as for the regularized Levenberg-Marquardt method a Tikhonov regularization in any iteration step, which finally leads to the update h_n given as the solution of

$$\big((F'_n[r_n])^* F'_n[r_n] + \alpha I\big) h_n = (F'_n[r_n])^* \big(F_n(r_n) - u^\delta_\infty\big), \tag{23}$$

where $\alpha > 0$ denotes a positive regularization parameter, see [10].

For such an iterative regularisation method a stopping rule is required because the approximations will deteriorate for noisy data after a certain number of iterations. The most commonly used stopping rule is the discrepancy principle, where the iterations are terminated at an index n for which

$$\|F_n(r_n) - u_\infty^\delta\| \leq \tau\delta, \quad \tau > 1. \tag{24}$$

is valid for the first time. If the operator F satisfies the tangent cone condition

$$\|F(r) - F(\hat{r}) - F'[\hat{r}](r - \hat{r})\| \leq c\|r - \hat{r}\|\|F(r) - F(\hat{r})\|$$

locally in a neighbourhood of the exact solution in suitable Hilbert spaces, then the approach (23) with stopping condition (24) yields a regularisation method (see [16]). So far it has not been confirmed that the cone condition is fulfilled in case of inverse obstacle scattering problems.

It is known that boundary integral equations solved by Nyström's method are suitable for solving the scattering problem (21)-(22). Therefore we consider a single layer potential ansatz for the scattered wave

$$u_n^s(x) = \mathrm{SL}\varphi_n(x) = \int_{\partial D} \Phi(x,y)\varphi_n(y)\,\mathrm{d}s_y \quad \text{for } x \in \Omega, \tag{25}$$

with a density $\varphi_n \in C(\partial D)$ and the fundamental solution of the Helmholtz equation

$$\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|), \quad x \neq y,$$

in $\mathbb{R}^2$. Using the jump relations for the single-layer potential ([6]), the scattered wave $u_n^s = u_n - u^i$ defined by (25) is a radiating solution of the Helmholtz equation satisfying the impedance boundary condition (22) on ∂D_n if and only if the density φ_n satisfies the integral equation

$$-\frac{1}{2}\varphi_n + \mathcal{K}'\varphi_n + ik\lambda\mathcal{S}\varphi_n - g_z(\cdot, u_{n-1}; \mathcal{S}\varphi_n) = f \tag{26}$$

with

$$f = -\frac{\partial u^i}{\partial \nu} - ik\lambda u^i + g(\cdot, u_{n-1}) + g_z(\cdot, u_{n-1}; u^i) - g_z(\cdot, u_{n-1}; u_{n-1}).$$

Here, $\mathcal{S}$ and $\mathcal{K}'$ denote the boundary integral operators corresponding to the single and the adjoint double layer potential of the Helmholtz equation given by

$$\begin{aligned} \mathcal{S}\varphi(x) &= \int_{\partial D} \Phi(x,y)\varphi(y)\,\mathrm{d}s_y, \quad x \in \partial D, \\ \mathcal{K}'\varphi(x) &= \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu_x}\varphi(y)\,\mathrm{d}s_y, \quad x \in \partial D. \end{aligned}$$

For $\partial D \in C^2$ the operators $\mathcal{S}, \mathcal{K}' : C(\partial D) \to C(\partial D)$ are compact, see [6]. For the implementation of the linearized operator equation (23), we need the evaluation of F and F' as well as its adjoint operator F'^*. To avoid computing the adjoint of the derivative, we have considered the adjoint of the discretised operator F' in our implementation. Overall, the shape reconstruction algorithm for the scattering problem (21)-(22) is given as follows.

(i) First, an initial guess is made for the shape ∂D_0 of the scattering obstacle. In order to solve the direct problem we start with the incident wave $u_0 = u^i$.

(ii) By solving the integral equation (26) using the initial guess u_0, the Cauchy data $u_n|_{\partial D_n}$, $\frac{\partial u_n}{\partial \nu}\big|_{\partial D_n}$ for the approximate boundary curve ∂D_n can be determined. The far field pattern $F_n(r_n) = u_{n,\infty}$ is computed from the density φ_n by

$$u_{n,\infty}(\hat{x}) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_{\partial D_n} e^{-ik\hat{x}\cdot y}\varphi_n(y)\,\mathrm{d}s_y, \quad \hat{x} \in \mathbb{S}^1. \tag{27}$$

(iii) The radial component of the starlike boundary and the variations are approximated by trigonometric polynomials

$$r(t) = \sum_{j=0}^{m} r_j^c \cos(jt) + r_j^s \sin(jt), \quad r_j^c, r_j^s \in \mathbb{R},$$
$$h(t) = \sum_{j=0}^{m} h_j^c \cos(jt) + h_j^s \sin(jt), \quad h_j^c, h_j^s \in \mathbb{R}.$$

For each basis function $h(t) = \cos(jt)$ and $h(t) = \sin(jt)$ the domain derivative from Theorem 3.4 is determined by solving the same integral equation (26) with

$$\begin{aligned} f = & \frac{\mathrm{d}}{\mathrm{d}s}\left(h_\nu \frac{\mathrm{d}}{\mathrm{d}s} u_n\right) + k^2 u_n h_\nu - ik\lambda\left(2\kappa u_n h_\nu + \frac{\partial u_n}{\partial \nu} h_\nu\right) \\ & - ik\frac{\partial \lambda}{\partial \nu} u_n h_\nu + g_z\left(\cdot, u_n; \frac{\partial u_n}{\partial \nu}\right) + 2\kappa g(\cdot, u_n) h_\nu \,. \end{aligned}$$

(iv) Again with (27) the far field pattern of the domain derivatives $F'[r_n]h = u'_{n,\infty}$ is calculated for all basic functions related to the current boundary curve. The collection of these far field patterns leads to a Jacobian matrix J at the points chosen for the discretisation. The application of Tikhonov regularisation yields

$$\big(J^* J + \alpha I\big) h = J^*\big(F_n(r_n) - u_\infty^\delta\big)$$

where h denotes the vector of $2m+1$ Fourier coefficients of h.

(v) The iteration is stopped at index n for which

$$\|F_n(r_n) - u_\infty^\delta\| \le \tau\delta, \quad \tau \in (1, 2),$$

is valid for the first time.

To avoid an inverse crime, we compute synthetic data by a different integral equation approach than in (25). We consider the direct approach based on the Green's representation theorem, which is given by

$$u^s(x) = \mathrm{DL}u(x) - \mathrm{SL}\frac{\partial u(x)}{\partial \nu} = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu_y} u(y) - \Phi(x,y)\frac{\partial u(y)}{\partial \nu_y}\,\mathrm{d}s_y \quad \text{for } x \in \Omega.$$

The jump relations of the single layer and double layer potential at the boundary ∂D and the linearized boundary condition (22) lead to the integral equation

$$\begin{aligned} \frac{1}{2} u_n - \mathcal{K} u_n - \mathcal{S}(ik\lambda u_n) + \mathcal{S} g_z(\cdot, u_{n-1}; u_n) & \\ = u^i + \mathcal{S}\big(g_z(\cdot, u_{n-1}; u_{n-1}) - g(\cdot, u_{n-1})\big) & \end{aligned} \tag{28}$$

where

$$\mathcal{K}\varphi(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu_y} \varphi(y)\,\mathrm{d}s_y, \quad x \in \partial D,$$

denotes the boundary integral operator corresponding to the double layer potential. Note that for the direct approach, the far field pattern is given by

$$F_n(r_n) = u_{n,\infty}(\hat{x}) = -\frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_{\partial D_n} \left(ik(\nu_y \cdot \hat{x})\varphi_n(y) + \frac{\partial \varphi_n(y)}{\partial \nu}\right) e^{-ik\hat{x}\cdot y}\,\mathrm{d}s_y,$$

with $\hat{x} \in \mathbb{S}^1$.

5. Numerical Results

For numerical examples, we choose an apple-shaped and a peanut-shaped obstacle as in [22]. The parameterizations are given by

$$z(t) = \frac{0.5 + 0.4\cos(t) + 0.1\sin(2t)}{1 + 0.7\cos(t)}(\cos(t), \sin(t)), \quad 0 \le t \le 2\pi,$$

and

$$z(t) = \sqrt{\cos^2(t) + 0.25\sin^2(t)}(\cos(t), \sin(t)), \quad 0 \le t \le 2\pi,$$

respectively. The impedance function λ is defined by

$$\lambda(z(t)) = \frac{1}{1 - 0.2\sin(2t)} \quad \text{for} \quad 0 \le t \le 2\pi.$$

Numerically, we solve the integral equations (26) and (28) both by Nyström's method, see [6, chap. 3.5]. For the computation of the synthetic data we use 128 discretisation points on the boundary curve, while for the inverse scheme we evaluate 64 discretisation points on the boundary with equidistant angles. We consider the wave number $k = 2$ and choose trigonometric polynomials of degree $m = 4$ for the approximation of the boundary curve. Furthermore, we use the plane wave $u^i(x) = e^{ikx\cdot d}$ with direction $d = \left(\cos(\frac{\pi}{6}), \sin(\frac{\pi}{6})\right)$ as an incident field.

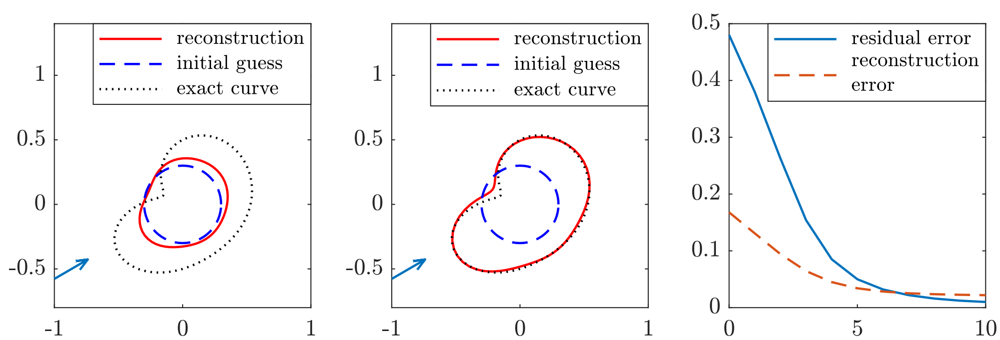

Figure 1. First and 10th iteration in the case of noise-free data as well as the relative discrete L^2 errors.

First we consider the reconstruction from the synthetic data without noise using the regularised Newton scheme, see figure 1. The angle of the incident plane wave is indicated by an arrow, the original boundary curve by a dotted line and the initial guess, a circle with radius $r = 0.3$, by the dashed line. One can observe that choosing the radius of the circle between 0.1 and 1.1 gives similarly results, possibly with slightly more iteration steps. On the right hand side of the figure the relative discrete L^2 errors are plotted. The residual error $\|F_n(r_n) - u_\infty\|/\|u_\infty\|$ by the full line and the reconstruction error $\|r_n - r_{\text{exact}}\|/\|r_{\text{exact}}\|$ by the dashed line. For this example we have chosen

$$g(x, u) = u + \sin(u)$$

as nonlinear function on the boundary. A nonlinearity of this form was already examined for the Laplace problem in [30] by Wendland and Ruotsalainen. This function does not satisfy condition (6), but g is Lipschitz continuous and the assumptions in (A) are guaranteed. In the case of noise-free synthetic data, we observe a stable performance for the apple-shaped object, as well as for other test objects. We have set the regularisation parameter here to $\alpha = 0.05$, whereas a wide range of regularisation parameters leads to comparable results. For a different incident direction d of u^i we obtain similar results, while adding more incident fields gives us a small improvement in our reconstructions, as known from linear scattering problems.

Now we consider reconstructions of the peanut-shape obstacle from running 100 experiments with 5% random noise added. The best and the worst result are shown in figure 2. Comparable results are obtained using the apple-shaped object. In the

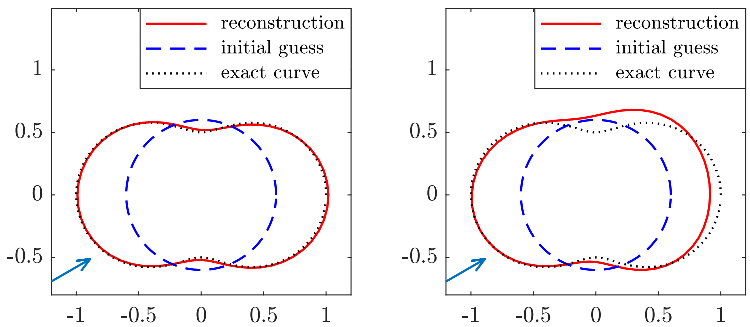

Figure 2. The best and the worst result after 12 iteration and 5% equally distributed random noise.

case of noisy data the regularization parameter has to be increased. In the example considered, we fixed the regularization parameter with $\alpha = 2$, which turned out to be sufficient in all tests. The discrepancy principle (24) with $\tau = 1.3$ was used as stopping condition.

Very similar results are achieved using the nonlinearity $g(x, u) = u/(1 + |u|^2)$ from chapter 2, which satisfies the required assumptions to ensure a well-posed direct problem. If we choose the Kerr-type nonlinearity $g(x, u) = |u|^2 u$ a small deterioration in the results is visible, but the reconstructions are still promising. In conclusion, we can state that the performance of the presented all-at-once regularized Newton method for acoustic scattering problems with nonlinear impedance boundary condition is satisfactory in all tested examples and comparable to the well-known linear cases.

Acknowledgments

This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

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