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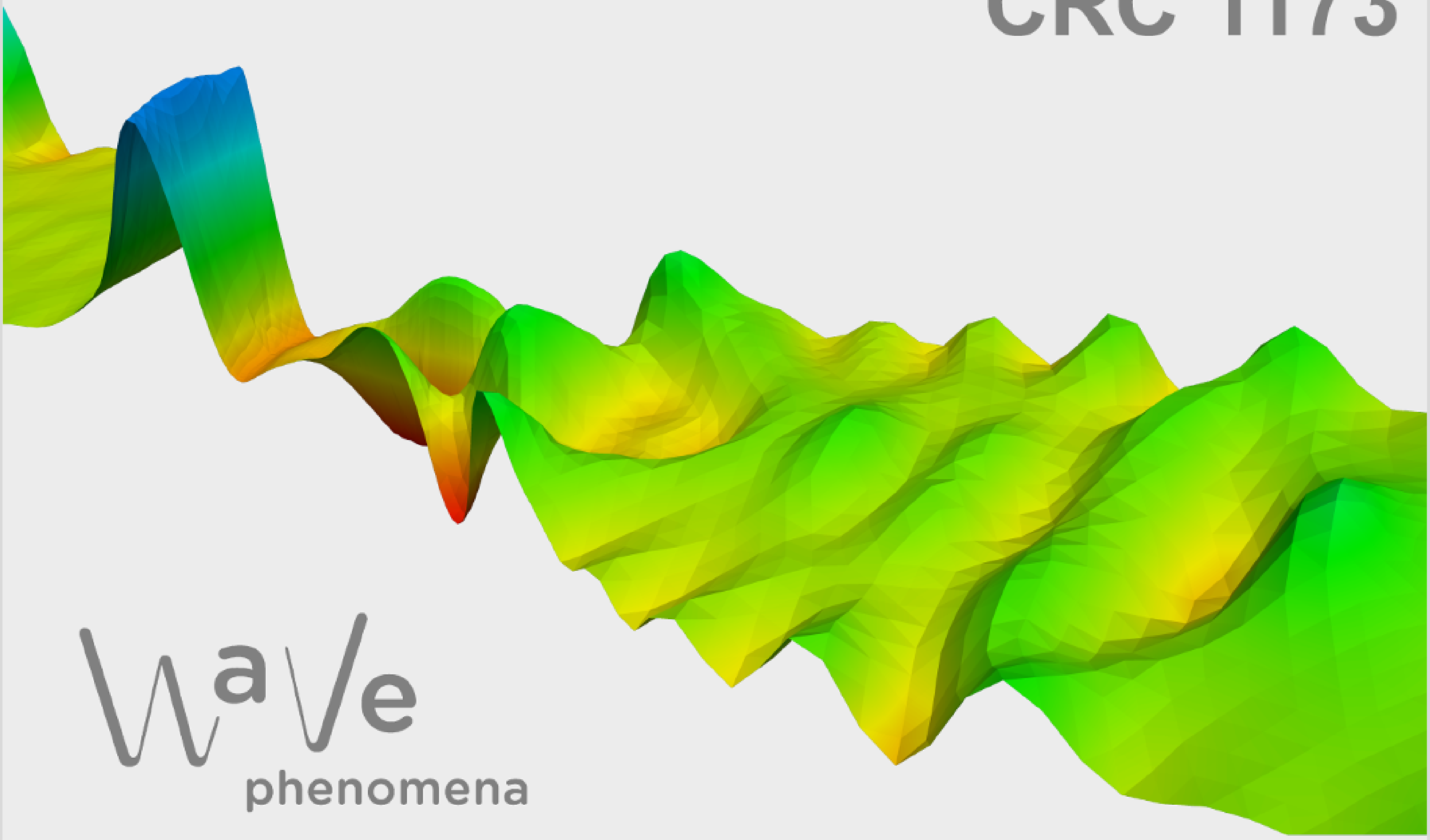

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MULTIPLICATION OPERATORS ON A CLASS OF HARDY SPACES FOR FOURIER INTEGRAL OPERATORS

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Abstract. We show boundedness of multiplication operators M_g on Hardy spaces for Fourier integral operators $H^p_{FIO,a}(\mathbb{R}^d)$, which are adapted to structured Lipschitz coefficients. The boundedness is described in terms of regularity of g in adapted Besov and BMO spaces. This improves a recent result on multiplication operators on $H^p_{FIO,a}(\mathbb{R}^d)$, where more regularity on g was required.

Mathematics Subject Classification (2020): Primary 42B35. Secondary 35L05, 42B30, 42B37, 35S30.

1. Introduction

The function space $H^1_{FIO}(\mathbb{R}^d)$, called Hardy space for Fourier integral operators, was first introduced in [15], with the purpose to have a function space at hand that is preserved by Fourier integral operators of order 0 associated to canonical transformations. In the more recent work [7], the construction of [15] was then generalised to all $p \in [1, \infty]$, and it was shown that the Hardy spaces $H^p_{FIO}(\mathbb{R}^d)$ are again preserved by Fourier integral operators of order 0. The results in particular imply the boundedness of the wave operator $\cos(\sqrt{-\Delta})$ on $H^p_{FIO}(\mathbb{R}^d)$, and, because of sharp Sobolev embeddings between H^p_{FIO} and L^p spaces, one recovers the fixed-time L^p estimates for wave equations of [11,12] as a special case. Much more general results have been obtained in the subsequent work [8], where the well-posedness of linear wave equations with $C^{1,1}$ coefficients on $H^p_{FIO}(\mathbb{R}^d)$ was shown. The spaces $H^p_{FIO}(\mathbb{R}^d)$ play a key role in [8] for the construction of a parametrix.
In our recent work [6] together with P. Portal, on the other hand, we introduced a class of Hardy spaces for Fourier integral operators $H^p_{FIO,a}(\mathbb{R}^d)$ that are adapted to structured Lipschitz coefficients. The precise condition on the coefficients is described below. We use the Hardy spaces $H^p_{FIO,a}(\mathbb{R}^d)$ to show fixed-time L^p estimates for the half-wave operator $(e^{it\sqrt{L}})_{t\in\mathbb{R}}$, where e.g. $L = -\sum_{j=1}^d \partial_j \widetilde{a}_j \partial_j$, see also the definitions of L_1 and L_2 in Section 3 below. This was done by showing boundedness of the half-wave group on $H^p_{FIO,a}(\mathbb{R}^d)$ together with sharp Sobolev embeddings for $H^p_{FIO,a}(\mathbb{R}^d)$.
A natural question occurring in this context is the boundedness of multiplication operators on $H^p_{FIO}(\mathbb{R}^d)$ or $H^p_{FIO,a}(\mathbb{R}^d)$. Results for $H^p_{FIO}(\mathbb{R}^d)$ have been obtained in [14] and rely on Coifman-Meyer type paraproduct arguments with additional decompositions in angular directions in order to respect the anisotropy inherent to $H^p_{FIO}(\mathbb{R}^d)$. Since the construction of $H^p_{FIO,a}(\mathbb{R}^d)$ relies on the Phillips calculus as a generalisation of the Fourier multiplier

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construction of $H^p_{FIO}(\mathbb{R}^d)$, the arguments of [14] cannot be applied directly for an analogous result on $H^p_{FIO,a}(\mathbb{R}^d)$. In [6], we give a first result on multiplication operators on $H^p_{FIO,a}(\mathbb{R}^d)$ through Leibniz type rules for the operator D_a as defined in the next section. In this article, we are now able to significantly improve the result of [6, Theorem 10.1] and obtain boundedness of

$$M_g : \; H^p_{FIO,a}(\mathbb{R}^d) \to H^p_{FIO,a}(\mathbb{R}^d),$$

whenever $p \in (1,\infty)$, $s > \frac{s_p}{2} = \frac{d-1}{2}\left|\frac{1}{p} - \frac{1}{2}\right|$, and $g \in L^\infty(\mathbb{R}^d)$ is such that $g \in \dot{B}^{0,L_k}_{\infty,\infty}$, $\nabla L_m^{-\frac{1}{2}} g \in \dot{B}^{0,L_k}_{\infty,\infty}$ and $L^s_m g \in BMO_{L_m}$ for $m = 1, 2$.
As a corollary of [6, Theorem 10.1], we obtained in [6] the boundedness of the half-wave operator under first order perturbations with g in the above class of functions for all $s > s_p$. We refer to [6, Corollary 10.3] for the precise statement. With our new result in Theorem 3.1, we can now weaken the assumptions on g to $s > \frac{s_p}{2}$ instead of $s > s_p$.
In follow-up work, we will apply Theorem 3.1 to pseudodifferential operators with symbols of limited smoothness and establish their boundedness on $H^p_{FIO,a}(\mathbb{R}^d)$, cf. [14] for corresponding results on $H^p_{FIO}(\mathbb{R}^d)$. We moreover expect that Theorem 3.1 will play a crucial for the investigation of nonlinear wave equations with structured Lipschitz coefficients.

2. Wave packet transform and Hardy spaces

For a detailed description of the setting, we refer to [6]. We recall the most important definitions.

For $j \in \{1, \ldots, 2d\}$, let $a_j \in C^{0,1}(\mathbb{R})$ with $\frac{d}{dx} a_j \in L^\infty$, and assume that there exist $0 < \lambda \le \Lambda$ such that $\lambda \le a_j(x) \le \Lambda$ for all $x \in \mathbb{R}$. We denote by $\widetilde{a_j} \in C^{0,1}(\mathbb{R}^d)$ the map defined by $\widetilde{a_j} : x \mapsto a_j(x_j)$.

Definition 2.1. *For $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$, define*

$$\xi.D_a := \sum_{j=1}^d \xi_j \begin{pmatrix} 0 & -\widetilde{a_{j+d}}\partial_j \\ \widetilde{a_j}\partial_j & 0 \end{pmatrix},$$

$$\xi.\sqrt{D_a^2} := \sum_{j=1}^d \xi_j \begin{pmatrix} \sqrt{-\widetilde{a_{j+d}}\partial_j \widetilde{a_j}\partial_j} & 0 \\ 0 & \sqrt{-\widetilde{a_j}\partial_j\widetilde{a_{j+d}}\partial_j} \end{pmatrix},$$

as an unbounded operator acting on $L^2(\mathbb{R}^d;\mathbb{C}^2)$, with domain $W^{1,2}(\mathbb{R}^d;\mathbb{C}^2)$.

As in [10, Section 4, Case II], $ie_j.D_a$ generates a bounded C_0 group on $L^2(\mathbb{R}^d;\mathbb{C}^2)$ for all $j = 1, .., d$, since $e_j.D_a$ is self-adjoint with respect to an equivalent inner product of the form $(u, v) \mapsto \langle A^{-1}u, v\rangle$, where A is a diagonal multiplication operator with $C^{0,1}$ entries.

[6, Proposition 4.3] states that for $\xi \in \mathbb{R}^d$ and $p \in (1,\infty)$, the group $(\exp(it\xi.\sqrt{D_a^2}))_{t\in\mathbb{R}}$ is bounded on $L^p(\mathbb{R}^d;\mathbb{C}^2)$. Given $\Psi \in \mathcal{S}(\mathbb{R}^d)$, we can therefore define $\Psi(\sqrt{D_a^2})$ using the

Phillips functional calculus associated with the commutative group $(\exp(i\xi.\sqrt{D_a^2}))_{\xi\in\mathbb{R}^d}$:

$$\Psi(\sqrt{D_a^2}) := \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d} \widehat{\Psi}(\xi)\exp(i\xi.\sqrt{D_a^2})d\xi.$$

We restrict our attention to functions Ψ that satisfy $\Psi = \Psi^s$, where

$$\Psi^s(x) := 2^{-d}\sum_{(\delta_j)_{j=1}^d\in\{-1,1\}^d} \Psi(\delta_1 x_1, ..., \delta_d x_d),$$

and write $\Psi(D_a)$ instead of $\Psi(\sqrt{D_a^2})$ when $\Psi = \Psi^s$ is symmetrised.

We recall the definition of wave packets from [6] (see also [7,13]). In the following we only consider functions that are symmetrised, i.e. $\Psi_{\omega,\sigma} = \Psi^s_{\omega,\sigma}$.

Let $\Psi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative radial function with $\Psi(\zeta) = 0$ for $|\zeta| \notin [\frac{1}{2}, 2]$, and

$$\int_0^\infty \Psi(\sigma\zeta)^2\,\frac{d\sigma}{\sigma} = 1 \tag{2.1}$$

for $\zeta \neq 0$. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a radial, non-negative function with $\varphi(\zeta) = 1$ for $|\zeta| \leq \frac{1}{2}$ and $\varphi(\zeta) = 0$ for $|\zeta| > 1$.

For $\omega \in S^{d-1}$, $\sigma > 0$ and $\zeta \in \mathbb{R}^d \setminus \{0\}$, set $\underline{\varphi_{\omega,\sigma}}(\zeta) := c_\sigma \varphi\left(\frac{\hat{\zeta}-\omega}{\sqrt{\sigma}}\right)$, and $\varphi_{\omega,\sigma} = \underline{\varphi_{\omega,\sigma}}^s$, where $c_\sigma := \left(\int_{S^{d-1}} \varphi\left(\frac{e_1-\nu}{\sqrt{\sigma}}\right)^2 d\nu\right)^{-1/2}$. Set $\varphi_{\omega,\sigma}(0) := 0$. Set furthermore $\Psi_\sigma(\zeta) := \Psi(\sigma\zeta)$ and $\psi_{\omega,\sigma}(\zeta) := \Psi_\sigma(\zeta)\varphi_{\omega,\sigma}(\zeta)$ for $\omega \in S^{d-1}$, $\sigma > 0$ and $\zeta \in \mathbb{R}^d$. By construction, we then have

$$\int_0^\infty \int_{S^{d-1}} \psi_{\omega,\sigma}(\zeta)^2\, d\omega \frac{d\sigma}{\sigma} = 1 \tag{2.2}$$

for all $\zeta \in \mathbb{R}^d \setminus \{0\}$, see [7, Lemma 4.1]. For $\omega \in S^{d-1}$ and $\zeta \in \mathbb{R}^d$, we moreover set

$$\varphi_\omega(\zeta) := \int_0^4 \psi_{\omega,\tau}(\zeta)\,\frac{d\tau}{\tau}.$$

We then have, for all $f \in L^2(\mathbb{R}^d)$, the following resolution of identity

$$\frac{1}{|S^{d-1}|}\int_{S^{d-1}}\int_1^\infty \Psi(\sigma D_a)^2 f\,\frac{d\sigma}{\sigma}d\omega \quad + \int_{S^{d-1}}\int_0^1 \varphi_\omega(D_a)^2\Psi(\sigma D_a)^2 f\,\frac{d\sigma}{\sigma}d\omega = f \tag{2.3}$$

We recall the definition of Hardy spaces $H^p_{FIO,a}(\mathbb{R}^d)$, and refer to [6, Section 7] for more details. In order to define $H^p_{FIO,a}(\mathbb{R}^d)$, we first define test function spaces adapted to our setting.

Definition 2.2. *Define*

$$\mathcal{S}_1 = \{f \in H^1_L(\mathbb{R}^d) \,:\, \exists g \in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)\;\; \exists \tau > 0\;\; f = \Psi(\tau D_a)g\},$$

and for $p \in (1,\infty)$

$$\mathcal{S}_p = \{f \in L^p(\mathbb{R}^d) \,:\, \exists g \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \ \exists \tau > 0 \ \ f = \Psi(\tau D_a)g\}.$$

We can then define $H^p_{FIO,a}(\mathbb{R}^d)$ in the following way. For the definition of tent spaces $T^{p,2}(\mathbb{R}^d)$, we refer to [4].

Definition 2.3. *Let* $p \in [1,\infty)$. *We define the space* $H^p_{FIO,a}(\mathbb{R}^d)$ *as the completion of* $\mathcal{S}_p$ *for the norm defined by*

$$\begin{aligned}&\|f\|_{H^p_{FIO,a}(\mathbb{R}^d)}\\ &\quad := \|\omega \mapsto [(\sigma,x) \mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma D_a)f(x) + 1_{[0,1]}(\sigma)\varphi_\omega(D_a)\Psi(\sigma D_a)f(x)]\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}.\end{aligned}$$

3. Multiplication on Hardy spaces

We denote by L_1 and L_2 the operators

$$L_1 := -\sum_{j=1}^d \widetilde{a_{j+d}}\partial_j \widetilde{a_j}\partial j, \qquad L_2 := -\sum_{j=1}^d \widetilde{a_j}\partial_j \widetilde{a_{j+d}}\partial j,$$

and set $L := \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} = D_a^2$.
For a function $g : \mathbb{R}^d \to \mathbb{R}$, we denote by M_g the multiplication operator $(f,F) \mapsto (gf, gF)$. For the definition of Besov spaces $\dot{B}^{0,L_k}_{\infty,\infty}$ associated with the operators L_k, we refer to [2], and for the definition of BMO spaces BMO_{L_k}, we refer to [5].

We obtain the following improvement of [6, Theorem 10.1].

Theorem 3.1. *Let* $p \in (1,\infty)$, $s_p = (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$ *and* $s > \frac{s_p}{2}$. *Let* $g \in L^\infty(\mathbb{R}^d)$ *be such that* $g \in \dot{B}^{0,L_k}_{\infty,\infty}$, $\nabla L_m^{-\frac{1}{2}} g \in \dot{B}^{0,L_k}_{\infty,\infty}$ *and* $L_m^s g \in BMO_{L_m}$ *for* $m = 1,2$. *Then*

$$M_g : H^p_{FIO,a}(\mathbb{R}^d) \to H^p_{FIO,a}(\mathbb{R}^d)$$

is bounded.

The proof will be a consequence of Lemma [6, Lemma 10.4], Lemma 3.3 and Lemma 3.6 below.

We use the following paraproduct decomposition, which is a refinement of the decomposition in [6]. Let $\Phi \in \mathcal{S}(\mathbb{R}^d), \phi \in \mathcal{S}(\mathbb{R}^d)$ with $\phi(0) = 1$ and $\Phi_\sigma(\zeta) = \phi(\sigma^2|\zeta|^2)$ for $\sigma > 0$, $\zeta \in \mathbb{R}^d$. We denote by $M_{\phi(L)g}$ and $M_{\phi(\underline{L})g}$, respectively, the multiplication operators

$$\begin{aligned}M_{\phi(L)g} &: (f,F) \mapsto (\phi(L_1)g.f, \phi(L_2)g.F),\\ M_{\phi(\underline{L})g} &: (f,F) \mapsto (\phi(L_2)g.f, \phi(L_1)g.F).\end{aligned}$$

For $f \in \mathcal{S}_p$ and $g \in \mathcal{S}(\mathbb{R}^d)$, we use (2.3) to decompose the product gf as follows.

$$
\begin{aligned}
M_g f = &\int_1^\infty M_{\phi(\tau L)g} \Psi(\tau D_a)^2 f \, \frac{d\tau}{\tau} + \int_1^\infty (M_g - M_{\phi(\tau L)g}) \Psi(\tau D_a)^2 f \, \frac{d\tau}{\tau} \\
&+ \int_{S^{d-1}} \int_0^1 M_{\phi(\tau L)g} \varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f \, \frac{d\tau}{\tau} d\nu \\
&+ \int_{S^{d-1}} \int_0^1 (M_g - M_{\phi(\tau L)g}) \varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f \, \frac{d\tau}{\tau} d\nu. \qquad (3.1)
\end{aligned}
$$

We omit the proof for the two low-frequency terms in the first line, as they are similar but simpler than the high-frequency terms and can be treated with similar arguments as in e.g. [1] without additional angular decomposition. We now further decompose the term in the third line of (3.1) and write for $\tau \in (0,1)$

$$
(I - \phi(\tau L))g = \sum_{k=1}^{j_0} (\phi(2^{k-1}\tau^2 L) - \phi(2^k \tau^2 L))g + (I - \phi(\tau^2 L))g,
$$

where, for notational simplicity, we assume that $j_0 \in \mathbb{N}$ can be chosen such that $2^{j_0} = \tau^{-1}$ (otherwise one adds another similar term to the decomposition). Now note that for $k = 1, \ldots, j_0$,

$$
\phi(2^{k-1}\tau^2 L) - \phi(2^k \tau^2 L) = \psi(2^k \tau^2 L),
$$

for some function $\psi \in C_c^\infty(\mathbb{R}^d)$, with $\psi(\zeta) = 0$ outside a compact annulus not containing 0.

In order to prove Theorem 3.1, it is therefore enough to estimate the term in the second line of (3.1), the term

$$
\int_{S^{d-1}} \int_0^1 (M_g - M_{\phi(\tau^2 L)g}) \varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f \, \frac{d\tau}{\tau} d\nu, \qquad (3.2)
$$

as well as the term

$$
\int_{S^{d-1}} \int_0^1 \sum_{k=1}^{j_0} (M_{\psi(2^k \tau^2 L)g}) \varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f \, \frac{d\tau}{\tau} d\nu. \qquad (3.3)
$$

For the term in the second line of (3.1), we recall [6, Lemma 10.4].

Lemma 3.2. *Let $p \in (1, \infty)$. Let $g \in L^\infty$ be such that $g \in \dot{B}^{0,L_m}_{\infty,\infty}$ and $\nabla L_m^{-\frac{1}{2}} g \in \dot{B}^{0,L_m}_{\infty,\infty}$ for $m = 1, 2$. For all $f \in H^p_{FIO,a}(\mathbb{R}^d)$, we have that*

$$
\begin{aligned}
&\|(\omega, \sigma, \cdot) \mapsto \psi_{\omega,\sigma}(D_a) \int_{S^{d-1}} \int_0^1 M_{\phi(\tau L)g} \varphi_\nu(D_a)^2 \Psi(\tau D_a)^2 f \, \frac{d\tau}{\tau} \, d\nu \|_{L^p(S^{d-1}; T^{p,2}(\mathbb{R}^d))} \\
&\lesssim (\|g\|_\infty + \max_{m=1,2} \|g\|_{\dot{B}^{0,L_m}_{\infty,\infty}} + \max_{m=1,2} \|\nabla L_m^{-\frac{1}{2}} g\|_{\dot{B}^{0,L_m}_{\infty,\infty}}) \|f\|_{H^p_{FIO,a}(\mathbb{R}^d)}.
\end{aligned}
$$

Since we have

$$
(I - \phi(\tau^2 L))g = -\int_0^\tau \partial_s \phi(s^2 L) g \, ds = \int_0^\tau \psi(s^2 L) g \, \frac{ds}{s},
$$

with some $\psi \in C_c^\infty$ with compact support away from 0, we obtain with integration by parts

$$\begin{aligned}\int_0^1 (I-\phi(\tau^2 L))g\cdot\varphi_\nu(D_a)^2\Psi(\tau D_a)f\,\frac{dt}{t} &= \int_0^1\int_0^\tau \psi(s^2L)g\cdot\varphi_\nu(D_a)^2\Psi(\tau D_a)f\,\frac{ds}{s}\frac{d\tau}{\tau}\\ &\simeq \int_0^1 \psi(s^2L)g\cdot\int_s^1 \varphi_\nu(D_a)^2\Psi(\tau D_a)f\,\frac{\tau}{\tau}\frac{ds}{s}\\ &\simeq \int_0^1 \psi(s^2L)g\cdot\varphi_\nu(D_a)^2\Phi(\tau D_a)f\,\frac{ds}{s},\end{aligned}$$

omitting possible additional low frequency terms. The term in (3.2) can therefore be dealt with as in [6, Lemma 10.6], noting that because the scaling of $M_{\Psi(\tau^2L)g}$ is now in τ^2 instead of τ, we only require that $L^s g\in BMO_{L_m}$ for $m=1,2$ and $s>\frac{s_p}{2}$. Note also that $\Psi(\tau D_a)$ can be changed to $\Psi(\tau D_a)^2$ by renormalisation.

Lemma 3.3. *Let $p\in(1,\infty)$, let $s>\frac{s_p}{2}$. Let $g\in L^\infty$ be such that $L_m^s g\in BMO_{L_m}$ for $m=1,2$, and let $f\in H^p_{FIO,a}(\mathbb{R}^d)$. Then*

$$\begin{aligned}&\|(\omega,\sigma,\cdot)\mapsto \psi_{\omega,\sigma}(D_a)\int_{S^{d-1}}\int_0^1 M_{\Psi(\tau^2L)g}\cdot\varphi_\nu(D_a)^2\Phi(\tau D_a)f\,\frac{d\tau}{\tau}d\nu\|_{L^p(T^{p,2})}\\ &\lesssim \max_{m=1,2}\|L_m^s g\|_{BMO_{L_m}}\|f\|_{H^p_{FIO,a}(\mathbb{R}^d)}.\end{aligned}$$

The main part of the proof is the estimate for (3.3) with g restricted to intermediate frequencies. We recall the following result for the boundedness of wave packets on $L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))$ from [6, Remark 8.2].

Lemma 3.4. *For all $p\in(1,\infty)$, we have*

$$\|(\omega,\sigma,\,.\,)\mapsto \sigma^{\frac{s_p}{2}}\psi_{\omega,\sigma}(D_a)F(\sigma,\,.\,)\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}\lesssim \|F\|_{T^{p,2}(\mathbb{R}^d)}$$

for all $F\in T^{p,2}(\mathbb{R}^d)$.

We also recall the following result on factorizations for tent spaces.

Theorem 3.5 ([3, Theorem 1.1])**.** *Let $p,q\in(1,\infty)$. If $F\in T^{p,\infty}(\mathbb{R}^d)$ and $G\in T^{\infty,q}(\mathbb{R}^d)$, then $FG\in T^{p,q}(\mathbb{R}^d)$ and*

$$\|F\cdot G\|_{T^{p,q}(\mathbb{R}^d)}\le C\|F\|_{T^{p,\infty}(\mathbb{R}^d)}\|G\|_{T^{\infty,q}(\mathbb{R}^d)},$$

with a constant $C>0$ which is independent of F and G.

For the estimate of intermediate frequencies, we refine the angular decomposition used in [6, Lemma 10.4]. We then however proceed with a tent space factorization $T^{p,2}=T^{p,\infty}\cdot T^{\infty,2}$ as in [6, Lemma 10.6], since the cancellative term acts on g and not on f (i.e. we are dealing an operator of type ψ_k instead of φ_k acting on g).

Lemma 3.6. *Let $p\in(1,\infty)$ and $s>\frac{s_p}{2}$. Let $g\in L^\infty(\mathbb{R}^d)$ be such that $L_m^s g\in BMO_{L_m}$ for $m=1,2$, and let $f\in H^p_{FIO,a}(\mathbb{R}^d)$. Let $k\in\mathbb{N}$, and set $\psi(2^k\tau^2L)=0$ for $\tau\in(0,1)$ with*

$2^k > \frac{1}{\tau}$. *Then*

$$\|(\omega,\sigma,\,.\,)\mapsto \psi_{\omega,\sigma}(D_a)\int_{S^{d-1}}\int_0^1 M_{\psi(2^k\tau^2L)g}\,\varphi_\nu(D_a)^2\Phi(\tau D_a)f\,\frac{d\tau}{\tau}d\nu\|_{L^p(T^{p,2})}$$
$$\lesssim \max_{m=1,2}\|L_m^s g\|_{BMO_{L_m}}\|f\|_{H^p_{FIO,a}(\mathbb{R}^d)}.$$

Proof. Since the result is true for $p=2$ with $s=0$, by interpolation and duality it suffices to show the statement for $s=\frac{d-1}{4}=\frac{s_1}{2}$ and $p\in(1,2)$. We decompose the integral in τ into the two regions $0<\tau<\sigma$ and $\sigma<\tau<1$ and the integral in ν into the two regions $|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}$ and its complement.

Part 1: We first consider the case $0<\tau<\sigma$ and $|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}$. By [6, Lemma 6.8], [9, Theorem 5.2] and Hardy's inequality we then have

$$\|\psi_{\omega,\sigma}(D_a)\int_0^{\min(\sigma,1)}\int_{|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}} M_{\psi(2^k\tau^2L)g}\,\varphi_\nu(D_a)^2\Phi(\tau D_a)f\,\frac{d\tau}{\tau}d\nu\|_{L^p(T^{p,2})}$$
$$\lesssim \|\sigma^{-\frac{d-1}{4}}\int_0^{\min(\sigma,1)}\int_{|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}} M_{\psi(2^k\tau^2L)g}\,\varphi_\nu(D_a)^2\Phi(\tau D_a)f\,\frac{d\tau}{\tau}d\nu\|_{L^p(T^{p,2})}$$
$$\lesssim \int_{S^{d-1}}\|\tau^{-\frac{d-1}{4}}\int_{|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}} M_{\psi(2^k\tau^2L)g}\,\varphi_\nu(D_a)^2\Phi(\tau D_a)f\,d\nu\|_{T^{p,2}}\,d\omega. \tag{3.4}$$

Now observe that by [6, Lemma 6.1] we have that $\|\mathcal{F}^{-1}(\varphi_\nu\Phi_\tau)\|_{L^1(\mathbb{R}^d)}\lesssim \tau^{-\frac{d-1}{4}}$ uniformly in τ and ν, thus

$$\tau^{-\frac{d-1}{4}}\int_{|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}}\|\mathcal{F}^{-1}(\varphi_\nu\Phi_\tau)\|_{L^1}\,d\nu \lesssim \tau^{-\frac{d-1}{4}}(2^{(j_0-k)/2}\sqrt{\tau})^{d-1}\tau^{-\frac{d-1}{4}} = 2^{(j_0-k)\frac{d-1}{2}}.$$

We can therefore modify the arguments of [6, Proposition 7.9] and replace the operator $\Phi_\sigma(D_a)\varphi_\omega(D_a)$ (with σ replaced by τ, ω replaced by ν) in the second part of the proof of [6, Proposition 7.9] by

$$\tau^{-\frac{d-1}{4}}\int_{|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}}\varphi_\nu(D_a)\Phi_\tau(D_a)\,d\nu.$$

Since we have the corresponding off-diagonal estimates, we obtain the following estimate for the nontangential maximal function

$$\int_{S^{d-1}}\|\int_{|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}} 2^{k\frac{d-1}{2}}\tau^{\frac{d-1}{4}}\varphi_\nu(D_a)^2\Phi(\tau D_a)f\,d\nu\|_{T^{p,\infty}}\,d\omega \lesssim \|f\|_{H^p_{FIO,a}(\mathbb{R}^d)}.$$

The factorization of tent spaces stated in Theorem 3.5 then yields that the expression in (3.4) is bounded by a constant times

$$\begin{aligned}
&2^{-k\frac{d-1}{2}}\|\tau^{-\frac{d-1}{2}}\psi(2^k\tau^2 L)g\|_{T^{\infty,2}}\\
&\quad\cdot\int_{S^{d-1}}\|\int_{|\nu-\omega|\leq 2^{(j_0-k)/2}\sqrt{\tau}} 2^{k\frac{d-1}{2}}\tau^{\frac{d-1}{4}}\varphi_\nu(D_a)^2\Phi(\tau D_a)f\,d\nu\|_{T^{p,\infty}}\,d\omega\\
&\lesssim 2^{-k\frac{d-1}{4}}\|(2^k\tau^2 L)^{-\frac{d-1}{4}}\psi(2^k\tau^2 L)L^{\frac{d-1}{4}}g\|_{T^{\infty,2}}\|f\|_{H^p_{FIO,a}(\mathbb{R}^d)}\\
&\lesssim 2^{-k\frac{d-1}{4}}\max_{m=1,2}\|L_m^{\frac{d-1}{4}}g\|_{BMO_{L_m}}\|f\|_{H^p_{FIO,a}(\mathbb{R}^d)},
\end{aligned}$$

where in the last line we use [5, Lemma 4.3].

Part 2: Now consider the integral over $0<\tau<\sigma$ and $|\nu-\omega|\geq 2^{(j_0-k)/2}\sqrt{\tau}$. In this case $|\omega.\nu|\lesssim 2^{(j_0-k)/2}\sqrt{\tau}$. In order to estimate

$$\|\psi_{\omega,\sigma}(D_a)\int_0^{\min(\sigma,1)}\int_{|\nu-\omega|\geq 2^{(j_0-k)/2}\sqrt{\tau}} M_{\Psi(2^k\tau^2 L)g}\cdot\varphi_\nu(D_a)^2\Psi(\tau D_a)^2 f\,\frac{d\tau}{\tau}d\nu\|_{L^p(T^{p,2})},$$

we apply the product rule by distributing derivatives from the factor $\varphi_\nu(D_a)^2\Psi(\tau D_a)^2 f$ onto the other parts. More precisely, we use that

$$M_{\Psi(2^k\tau^2 L)g}(e_j.D_a) = (e_j.D_a)M_{\Psi(2^k\tau^2\underline{L})g} - M_{(e_j.D_a)\Psi(2^k\tau^2 L)g},$$

for $j=1,\dots,d$, where

$$M_{(e_j.D_a)\Psi(2^k\tau^2 L)g}:(f,F)\mapsto(-\widetilde{a_{j+d}}\partial_j\Psi(2^k\tau^2 L_1)g\cdot F,\tilde{a_j}\partial_j\Psi(2^k\tau^2 L_2)g\cdot f).$$

In order to handle derivatives onto the operator on the outside, we denote by $(\omega,\omega_1,\dots,\omega_{d-1})$ an orthonormal basis of $\mathbb{R}^d$, and write

$$\begin{aligned}
\tau(\nu.D_a)\psi_{\omega,\sigma}(D_a) &= \frac{\tau}{\sigma}(\nu.\omega)\sigma(\omega.D_a)\psi_{\omega,\sigma}(D_a)+\sqrt{\tau}\sqrt{\frac{\tau}{\sigma}}\sum_{j=1}^{d-1}(\nu.\omega_j)\sqrt{\sigma}(\omega_j.D_a)\psi_{\omega,\sigma}(D_a)\\
&\simeq \sqrt{\tau}(2^{(j_0-k)/2}\frac{\tau}{\sigma}+\sqrt{\frac{\tau}{\sigma}})\tilde{\psi}_{\omega,\sigma}(D_a),
\end{aligned}$$

with $\tilde{\psi}_{\omega,\sigma}$ some function satisfying the same assumptions as $\psi_{\omega,\sigma}$ (up to constants). On the other hand, we can write

$$\tau(\nu.D_a)\Psi(2^k\tau^2 L)g\simeq 2^{-k/2}\tilde{\Psi}(2^k\tau^2 L)g\simeq 2^{(j_0-k)/2}\sqrt{\tau}\tilde{\Psi}(2^k\tau^2 L)g,$$

again with $\tilde{\Psi}$ being similar to Ψ. Applying the product rule $2M$ times, we obtain - suppressing similar terms with L replaced by $\underline{L}$ -

$$\begin{aligned}
&\|\psi_{\omega,\sigma}(D_a)\int_0^{\min(\sigma,1)}\int_{|\nu-\omega|\geq 2^{(j_0-k)/2}\sqrt{\tau}} M_{\Psi(2^k\tau^2 L)g}\cdot\varphi_\nu(D_a)^2\Psi(\tau D_a)^2 f\,\frac{d\tau}{\tau}d\nu\|_{L^p(T^{p,2})}\\
&\lesssim \max_{l=0,\dots,2M}\|\tau^M(2^{(j_0-k)/2}\frac{\tau}{\sigma}+\sqrt{\frac{\tau}{\sigma}})^l(2^{(j_0-k)/2})^{2M-l}\tilde{\psi}_{\omega,\sigma}(D_a)\\
&\qquad\int_0^{\min(\sigma,1)}\int_{|\nu-\omega|\geq 2^{(j_0-k)/2}\sqrt{\tau}} M_{\tilde{\Psi}(2^k\tau^2 L)g}\cdot\tilde{\varphi}_\nu(D_a)^2\tilde{\Psi}(\tau D_a)^2 f\,\frac{d\tau}{\tau}d\nu\|_{L^p(T^{p,2})}.
\end{aligned}$$

Repeating the arguments of Part 1 and applying [6, Lemma 6.8], [9, Theorem 5.2] and Hardy's inequality together with Theorem 3.5, we get that the last expression is bounded by a constant times

$$\begin{aligned}
&\max_{l=0,\dots,2M}\int_{S^{d-1}}\|(2^{(j_0-k)/2}+1)^l(2^{(j_0-k)/2})^{2M-l}\tau^M\tau^{-\frac{d-1}{4}}M_{\tilde\Psi(2^k\tau^2L)g}\cdot\tilde\varphi_\nu(D_a)^2\tilde\Psi(\tau D_a)^2f\|_{T^{p,2}}\,d\nu\\
&\quad\lesssim 2^{-kM}\|\tau^{-\frac{d-1}{4}}\tau^{-\frac{d-1}{4}}\tilde\Psi(2^k\tau^2L)g\|_{T^{\infty,2}}\int_{S^{d-1}}\|\tau^{\frac{d-1}{4}}\tilde\varphi_\nu(D_a)^2\tilde\Psi(\tau D_a)^2f\|_{T^{p,\infty}}\,d\nu\\
&\quad\simeq 2^{-kM}2^{k\frac{d-1}{4}}\|(2^k\tau^2)^{-\frac{d-1}{4}}\tilde\Psi(2^k\tau^2L)g\|_{T^{\infty,2}}\int_{S^{d-1}}\|\tau^{\frac{d-1}{4}}\tilde\varphi_\nu(D_a)^2\tilde\Psi(\tau D_a)^2f\|_{T^{p,\infty}}\,d\nu\\
&\quad\lesssim 2^{-k\tilde M}\max_{m=1,2}\|L_m^{\frac{d-1}{4}}g\|_{BMO_{L_m}}\|f\|_{H^p_{FIO,a}(\mathbb{R}^d)}.
\end{aligned}$$

Part 3: Now consider the integral over $0<\sigma<\tau$ and $|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}$. We choose $M\in\mathbb{N}$ sufficiently large depending on p,d and write $\widetilde{\psi}_{\omega,\sigma}(D_a):=\psi_{\omega,\sigma}(D_a)(\sigma^2L)^{-M}$ with $\widetilde{\psi}_{\omega,\sigma}$ a function with the same properties as $\psi_{\omega,\sigma}$ up to constants. We can then write with similar arguments as in Part 1, using [6, Lemma 6.8] and Hardy's inequality,

$$\begin{aligned}
&\|\psi_{\omega,\sigma}(D_a)\int_{\min(\sigma,1)}^1\int_{|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}}M_{\psi(2^k\tau^2L)g}\,\varphi_\nu(D_a)^2\Phi(\tau D_a)f\,\frac{d\tau}{\tau}d\nu\|_{L^p(T^{p,2})}\\
&\quad\lesssim\|\sigma^{2M-\frac{d-1}{4}}\int_{\min(\sigma,1)}^1\int_{|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}}L^M[M_{\psi(2^k\tau^2L)g}\,\varphi_\nu(D_a)^2\Phi(\tau D_a)f]\,\frac{d\tau}{\tau}d\nu\|_{L^p(T^{p,2})}\\
&\quad\lesssim\|\tau^{2M-\frac{d-1}{4}}\int_{|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}}L^M[M_{\psi(2^k\tau^2L)g}\,\varphi_\nu(D_a)^2\Phi(\tau D_a)f]\,d\nu\|_{L^p(T^{p,2})}.
\end{aligned}$$

Now use for $j=1,\dots,d$ the product rule

$$(e_j.D_a)M_{\Psi(2^k\tau^2L)g}=M_{\Psi(2^k\tau^2\underline{L})g}(e_j.D_a)+M_{(e_j.D_a)\Psi(2^k\tau^2L)g},$$

and note that with $l\in\{0,\dots,2M\}$ even, $m=1,2$ and $\delta\in\{0,1\}$, we can bound the above expression by multiples of terms of the form

$$\|\tau^{-\frac{d-1}{4}}\int_{|\nu-\omega|\le 2^{(j_0-k)/2}\sqrt{\tau}}M_{\tau^\delta(e_j.D_a)^\delta(\tau^2L_m)^{l/2}\psi(2^k\tau^2L)g}\,(\tau D_a)^{2M-l-\delta}\varphi_\nu(D_a)^2\Phi(\tau D_a)f\,d\nu\|_{L^p(T^{p,2})}.$$

Terms of this form can now be estimated similarly as in Part 1. The estimate of the factor involving f does not change significantly, as the operator $(\tau D_a)^{2M-l-\delta}\varphi_\nu(D_a)^2\Phi(\tau D_a)$ has the same (for $2M-l-\delta=0$) or better properties than $\varphi_\nu(D_a)^2\Phi(\tau D_a)$. For the estimate of the factor involving g, we have to deal with the following term, which again behaves the same or better than the corresponding term in Part 1,

$$\begin{aligned}
&2^{-k\frac{d-1}{2}}\|\tau^{-\frac{d-1}{2}}\tau^\delta(e_j.D_a)^\delta(\tau^2L_m)^{l/2}\psi(2^k\tau^2L)g\|_{T^{\infty,2}}\\
&\qquad\lesssim 2^{-k\frac{d-1}{2}}2^{-k\frac{\delta+l}{2}}\|\tau^{-\frac{d-1}{2}}\widetilde{\psi}(2^k\tau^2L)g\|_{T^{\infty,2}}\lesssim 2^{-k\frac{d-1}{4}}\max_{m=1,2}\|L_m^{\frac{d-1}{4}}g\|_{BMO_{L_m}}.
\end{aligned}$$

Part 4: For the integral over $0<\sigma<\tau$ and $|\nu-\omega|\ge 2^{(j_0-k)/2}\sqrt{\tau}$, we combine the arguments of Part 2 with the use of the Leibniz rule in Part 3. This is done just as in the last part of the proof of [6, Lemma 10.4], with first applying the product rule argument of Part 2 in order to gain a factor $\tau^{\frac{M'}{2}}$, but still keeping some operator $\widetilde{\psi}_{\omega,\sigma}(D_a)$ in the expression. One can then apply the product rule argument of Part 3 with $M > \frac{M'}{2}$ sufficiently large in order to get into a position where Hardy's inequality in $\sigma < \tau$ is applicable. The occurring terms change from Part 2 to Part 4 in the same way as they did from Part 1 to Part 3. □

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