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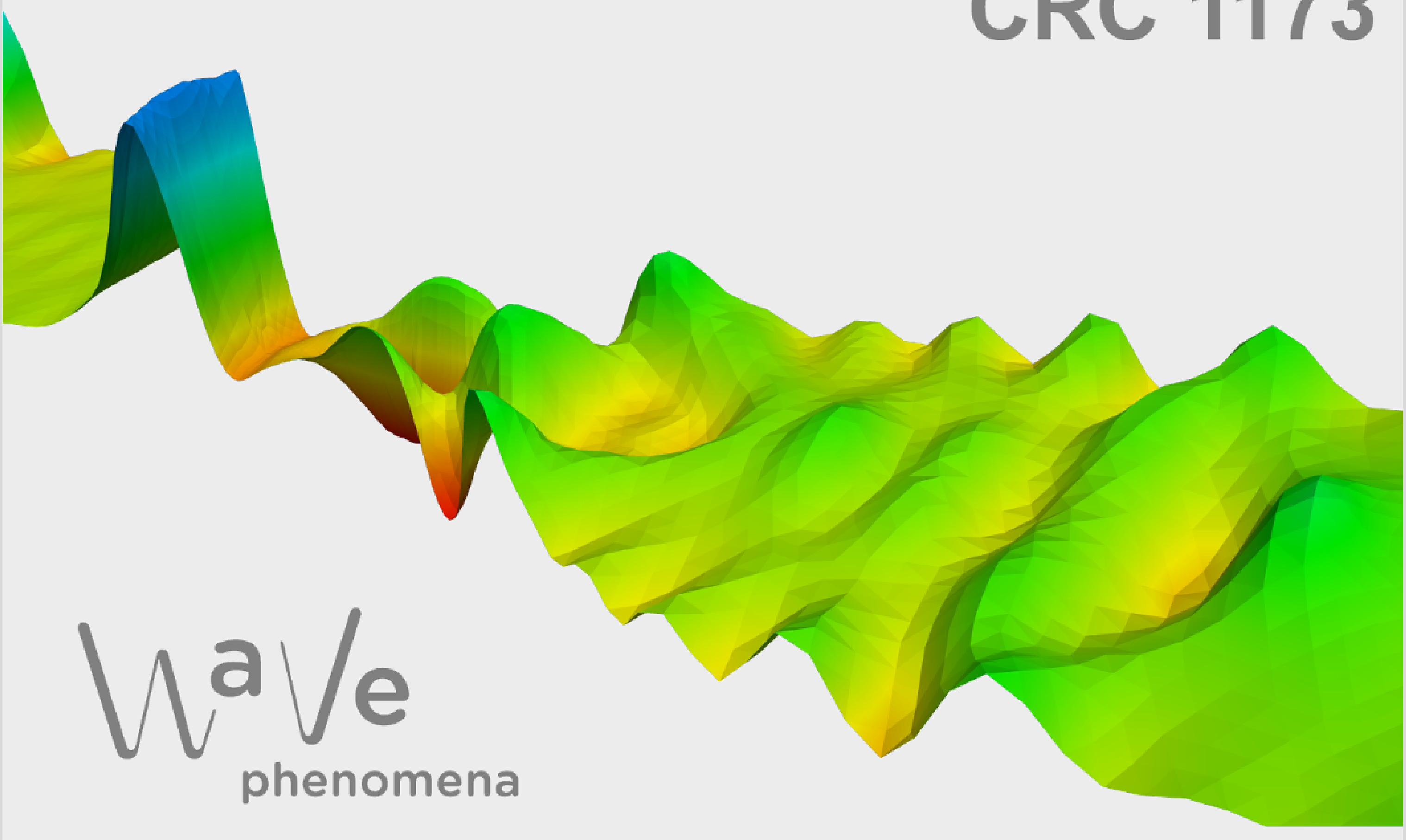

Wave phenomena

Local wellposedness of Maxwell systems with scalar-type retarded material laws

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Abstract. We study the local wellposedness in H^2 for the system of Maxwell equations in a bounded domain equipped with perfectly conducting boundary conditions. The medium is described by a retarded and nonlinear expression for the polarisation having the property that the polarisation is always parallel to the electric field. An example is the Kerr effect in a homogeneous, isotropic and inversion-symmetric medium. The results are obtained using semigroup theory.

Keywords. Nonlinear Maxwell equations, retarded material laws, perfectly conducting boundary conditions, local wellposedness, retarded evolution equations.

Mathematics Subject Classification (2020). 35Q61, 78A25, 47D06, 34G20, 35B30.

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Contents

1 Introduction

The Maxwell equations form the foundation of the classical theory of electromagnetism and play an important role in physics and technical applications. In vacuum, the system of equations (also called microscopic Maxwell equations) is given by

$$\operatorname{div}\mathbf{e} = \rho_{\mathrm{m}}\,, \tag{1.1a}$$
$$\operatorname{div}\mathbf{b} = 0\,, \tag{1.1b}$$
$$\operatorname{curl}\mathbf{e} = -\partial_t\mathbf{b}\,, \tag{1.1c}$$
$$\operatorname{curl}\mathbf{b} = \mathbf{j}_{\mathrm{m}} + \partial_t\mathbf{e}\,, \tag{1.1d}$$

where $\mathbf{e}$ is the electric field and $\mathbf{b}$ the magnetic induction, while ρ_{m} is the charge density and $\mathbf{j}_{\mathrm{m}}$ the corresponding current density. The index m indicates that these are microscopic quantities, i. e. they include all charges in the system. For mathematical convenience, we use a unit system where the physical constants ε_0 and μ_0 both take the value 1. The Gauß laws (1.1a) and (1.1b) describe the facts that charges are the sources of the electric field and that the magnetic induction is source-free (there are no magnetic monopoles). According to Faraday's law (1.1c), vortexes of the electric field are caused by time-variations of the magnetic induction. On the other hand, Ampère's law (1.2d) states that vortexes of the magnetic field are caused by currents as well as time-variations of the electric field. The term $\partial_t\mathbf{e}$ is also called Maxwell's displacement current.

In matter, it is usually hopeless to determine these quantities, since the amount of charged particles is of the order of 10^{23} per cubic centimeter. An exact treatment of the microscopic equations is fortunately not necessary in practice since every measurement involves an averaging process over some macroscopically small but microscopically large volume. Therefore, the rapid spatial and temporal fluctuations of the fields are not accessible in experiments and it thus makes sense to introduce macroscopic fields $\mathbf{E} = \langle \mathbf{e} \rangle$ and $\mathbf{B} = \langle \mathbf{b} \rangle$, where $\langle \cdot \rangle$ represents a suitable averaging process. As described e. g. in [21] or [16], this leads to the macroscopic Maxwell equations

$$\operatorname{div} \mathbf{D} = \rho\,, \tag{1.2a}$$
$$\operatorname{div} \mathbf{B} = 0\,, \tag{1.2b}$$
$$\operatorname{curl} \mathbf{E} = -\partial_t \mathbf{B}\,, \tag{1.2c}$$
$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \partial_t \mathbf{D}\,. \tag{1.2d}$$

Here, ρ is the macroscopic charge density. The bound charges in the material usually compensate each other in the averaging process and therefore ρ is caused by free excess charges. Analogously, the macroscopic current density $\mathbf{J}$ is due to the flow of free charges and linked to ρ by the continuity equation

$$\partial_t \rho + \operatorname{div} \mathbf{J} = 0\,. \tag{1.3}$$

This identity follows from (1.2a) and (1.2d) and implies charge conservation. We divide the current density $\mathbf{J} = \mathbf{J}_0 + \sigma \mathbf{E}$ into an externally applied current density $\mathbf{J}_0$ and induced currents $\sigma \mathbf{E}$ where σ is the conductivity of the material which is assumed to be nonnegative and scalar-valued.

In contrast to the quantities $\mathbf{E}$ and $\mathbf{B}$, which are measurable due to the Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.4}$$

acting on a point charge q with velocity $\mathbf{v}$, the electric displacement $\mathbf{D}$ and the magnetic field $\mathbf{H}$ are auxiliary quantities. So far, the system (1.2) is under-determined and has to be complemented by laws relating $\mathbf{D}$ and $\mathbf{H}$ to $\mathbf{E}$ and $\mathbf{B}$. A result of the averaging process are the expressions

$$\mathbf{D} = \mathbf{E} + \mathbf{P} + \dots\,, \quad \mathbf{H} = \mathbf{B} - \mathbf{M} + \dots\,, \tag{1.5}$$

where $\mathbf{P}$ and $\mathbf{M}$ are the density of electric respectively magnetic dipoles in the medium. The dots indicate the densities of higher order moments which are neglected in the following. Material laws describe the dependencies $\mathbf{P} = \mathbf{P}(\mathbf{E}, \mathbf{B})$ and $\mathbf{M} = \mathbf{M}(\mathbf{E}, \mathbf{B})$. These are obtained from physical models for the material in question and can be very complicated. In this work, we consider a model that is often used in nonlinear optics, see e. g. [6]: We mostly neglect magnetic effects and assume that $\mathbf{M}$ is always proportional to $\mathbf{B}$. The polarisation is assumed to depend only on $\mathbf{E}$. In addition to a linear and instantaneous term analogous to the relationship between $\mathbf{M}$ and $\mathbf{B}$, $\mathbf{P}$ also has nonlinear contributions which depend on past values of the electric field. An example of the type of material laws considered in this work is the Kerr effect described by

$$\mathbf{P}^{(3)}(\mathbf{E})(t) = \int_0^\infty \int_0^\infty \int_0^\infty \widetilde{\mathcal{R}}^{(3)}(\tau_1, \tau_2, \tau_3)\big(\mathbf{E}(t-\tau_2)\cdot \mathbf{E}(t-\tau_3)\big)\mathbf{E}(t-\tau_1)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2\, \mathrm{d}\tau_3\,,$$

where $\widetilde{\mathcal{R}}^{(3)}$ is called a response function. Subsection 3.4 contains a detailed description of the model for the polarisation.

We consider the Maxwell equations in a bounded domain $G \subseteq \mathbb{R}^3$ and need to prescribe boundary conditions on ∂G. A common choice, which we also use here, is to assume that G is surrounded by a perfect conductor. Then the boundary conditions are

$$\nu \times \mathbf{E} = 0\,, \quad \nu \cdot \mathbf{B} = 0 \quad \text{on } \partial G\,, \tag{1.6}$$

where ν is the outer unit normal vector on ∂G.

The goal of this work is to study the local wellposedness of the above model. To this end, we use a well known result, [26], which under certain regularity conditions allows us to absorb the two divergence equations in (1.2) and the boundary condition for the magnetic induction in (1.6) into the initial conditions, see Lemma 3.7. The resulting reduced system of equations can then be written in the form of an abstract retarded evolution equation which is treated using semigroup theory and perturbative methods.

We show local wellposedness in $H^2(G)$ of the Maxwell system subject to the boundary conditions (1.6) and equipped with a retarded and nonlinear material law for P, which is of scalar type, see Assumption 3.19. The latter class includes the Kerr effect. H^2-solutions are needed to deal with the nonlinearity. To obtain such solutions, one has to add further compatibility conditions for the initial fields besides (1.6), see Theorem 3.23. Our assumptions on the material law ensure that P respects these conditions.

This work is split into two parts. In Section 2, the notion of retarded evolution equations is introduced in an abstract setting and the concepts of classical and mild solutions are defined. Using methods from nonlinear evolution equations as described e. g. in [22], local wellposedness for mild solutions is established and conditions are formulated which ensure that a mild solution is a classical one. Section 3 then deals with the application of the abstract results to the above system of Maxwell equations. It begins with a short overview over the trace operators needed in the formulation of the boundary conditions. Then the mentioned reduction of the system of equations is formulated. In order to apply the results from Section 2, two major steps are necessary: It has to be shown that the differential operator appearing in the reduced set of equations generates a strongly continuous semigroup on an appropriate space and that the used model for the polarisation leads to a nonlinearity with properties as required by the local wellposedness results from Section 2. To be able to use the semigroup-based approach of the abstract setting, a restriction to certain scalar-type material laws is necessary. These steps are performed in Subsections 3.3 and 3.4. Finally, the local wellposedness results for the Maxwell system are formulated in Subsection 3.5.

In the following, we use for $a, b \in \mathbb{R}$ the notation $a \lesssim_\lambda b$ if $a \leq cb$ for some $c = c(\lambda)$ independent of a and b, as well as $a \simeq_\lambda b$ if $a \lesssim_\lambda b$ and $b \lesssim_\lambda a$. For Banach spaces X

and Y, we write $X \hookrightarrow Y$ if X is continuously embedded into Y and $X \simeq Y$ if they are isomorphic. If $A : D(A) \subseteq X \to X$ is a closed linear operator, the symbol $[D(A)]$ stands for the Banach space $\big(D(A), \|\cdot\|_A\big)$, with the graph norm $\|\cdot\|_A = \|\cdot\| + \|A\cdot\|$. We write $\mathcal{B}(X,Y)$ for the space of linear bounded operators from $X \to Y$ and often just write $\|\cdot\|$ instead of $\|\cdot\|_{\mathcal{B}(X,Y)}$ for the operator norm. The duality pairing between X and its dual X^* is denoted by $\langle x, x^* \rangle_{X \times X^*}$ for $x \in X$ and $x^* \in X^*$. Furthermore, the space of bounded and uniformly continuous functions from $J \subseteq \mathbb{R}$ to X is denoted by $\mathrm{BUC}(J,X)$ and equipped with the supremum norm. We additionally require the space

$$\mathrm{BUC}^1(J,X) := \left\{ f \in C^1(J,X) \;\middle|\; f, f' \in \mathrm{BUC}(J,X) \right\} .$$

The function spaces used in Section 3 consist of real-valued functions.

We are not aware of works that show wellposedness for the Kerr effect, say, on domains with boundary conditions. Local wellposedness in H^3 was proven for general instantaneous nonlinear material laws, [26], [27], [23], even for interface problems. These papers are based on completely different methods which in future work we want to adapt to the retarded case in order to tackle more general material laws.

Recent results for Maxwell equations with linear retarded material laws on bounded Lipschitz domains can be found e. g. in [20] and [15]. There, also the magnetisation includes retardation effects and in [15], **P** and **M** depend on both **E** and **H** (so called bianisotropic material laws). Treatments of nonlinear material laws with retardation include [19] and [3]. In [3], a more general expression for the polarisation than used here is considered on the full space. The work [19] considers Maxwell equations on a bounded domain and uses a different form of bianisotropic material laws which have to be globally Lipschitz.

2 Abstract retarded evolution equations

Here we introduce the abstract setting used to treat the Maxwell equations in Section 3. The formulation of the problem as an evolution equation with retardation effects is based on Section VI.6 in [9], where the linear case is treated.

2.1 Assumptions and solution concepts

The following assumptions are made throughout this section.

Assumption 2.1. *Let* $(X, \|\cdot\|)$ *be a Banach space and* A *be a linear operator on* X *with domain* $\mathrm{D}(A)$ *that generates a strongly continuous semigroup* $T(\cdot)$ *on* X*. Let* $F : \mathrm{BUC}\big((-\infty, 0], X\big) \to X$ *be a map with* $F(0) = 0$ *that is Lipschitz on bounded sets, i. e., there exists an increasing function* $L : (0, \infty) \to [0, \infty)$ *such that for every* $r > 0$ *it holds*

$$\|F(u) - F(v)\| \leq L(r)\, \|u - v\|_\infty$$

for all $u, v \in \mathrm{BUC}\big((-\infty, 0], X\big)$ *with* $\|u\|_\infty = \sup_{t\leq 0} \|u(t)\| \leq r$ *and* $\|v\|_\infty \leq r$*. Additionally let* $f \in \mathrm{BUC}\big((-\infty, 0], X\big)$ *and* $g \in C(I, X)$ *where* $I = [0, \infty)$ *or* $I = [0, t_{\mathrm{end}})$ *for a* $t_{\mathrm{end}} > 0$*.*

If J is an interval containing $(-\infty, 0]$ and u is a continuous function from J to X we define for $t \in J$ the shifted function $u_t \in C\big((-\infty, 0], X\big)$ by $u_t(s) := u(t + s)$.

The abstract retarded evolution equation treated in this work has the form

$$\begin{aligned} u'(t) &= Au(t) + F(u_t) + g(t)\,, \quad t \in I\,,\\ u(t) &= f(t)\,, \quad t \leq 0\,. \end{aligned} \tag{2.1}$$

The function F describes a nonlinear response of the system which is not instantaneous: The response at time t depends not only on the state $u(t)$ of the system at time t but also on all previous states, i. e., on the whole function u_t. It is therefore not sufficient to prescribe $u(0)$ as an initial condition. Instead, we have to specify the whole history f of the system.

The term $F(u_t)$ is well defined and continuous in t by the lemma below if u is a continuous function fulfilling the second condition of (2.1).

Lemma 2.2. *Let* J *be an interval with* $(-\infty, 0] \subseteq J$ *and* $0 < \sup J \leq \sup I$ *and let* $u \in C(J, X)$ *satisfy* $u(t) = f(t)$ *for all* $t \leq 0$*. Furthermore, let* $b > 0$ *be such that* $(-\infty, b] \subseteq J$*. Then* $u \in \mathrm{BUC}\big((-\infty, b], X\big)$*, the map* $t \mapsto u_t$ *is contained in* $\mathrm{BUC}\big((-\infty, b], \mathrm{BUC}\big((-\infty, 0], X\big)\big)$ *and* $t \mapsto F(u_t) \in C(J, X)$*.*

Proof. 1) The assumption on f implies that u is bounded and uniformly continuous on $(-\infty, 0]$. Since $[0, b]$ is compact and u continuous, this is also the case on $[0, b]$. The continuity of u on $(-\infty, b]$ then yields $u \in \mathrm{BUC}\big((-\infty, b], X\big)$ which implies that for any $t \leq b$, the shifted function u_t is bounded and uniformly continuous on $(-\infty, 0]$.

2) The map $\varphi : (-\infty, b] \to \mathrm{BUC}\big((-\infty,0],X\big)$ given by $\varphi(t) = u_t$ is well defined according to the above observations. Let $\varepsilon > 0$. By 1), there exists a $\delta > 0$ such that $\|u(\widetilde{r}) - u(\widetilde{s})\| \le \varepsilon$ for all $\widetilde{r}, \widetilde{s} \le b$ with $|\widetilde{r} - \widetilde{s}| \le \delta$. Let $r, s \le b$ with $|r - s| \le \delta$. Then

$$\|\varphi(r) - \varphi(s)\|_{\mathrm{BUC}((-\infty,0],X)} = \sup_{\tau \le 0} \|u_r(\tau) - u_s(\tau)\| = \sup_{\tau \le 0} \|u(r+\tau) - u(s+\tau)\| \le \varepsilon$$

which shows that φ is uniformly continuous. The boundedness follows from

$$\sup_{t \le b} \|\varphi(t)\|_{\mathrm{BUC}((-\infty,0],X)} = \sup_{t \le b} \sup_{\tau \le 0} \|u(t+\tau)\| = \sup_{s \le b} \|u(s)\| < \infty.$$

3) The previous steps imply that $F \circ \varphi$ is continuous on $(-\infty, b]$. Since $b > 0$ is arbitrary as long as $(-\infty, b] \subseteq J$, we can conclude $F \circ \varphi \in C(J, X)$. □

Remark 2.3. *Except for Lemma 2.17 and Theorem 2.18, in this section we use the condition $f \in \mathrm{BUC}\big((-\infty,0],X\big)$ only to ensure the continuity of $t \mapsto F(u_t)$. We can thus replace the space $\mathrm{BUC}\big((-\infty,0],X\big)$ by $C_b\big((-\infty,0],X\big)$ if this continuity assumption is true for other reasons. This is the case in the specific model for F used in Section 3. Hence we can work in $C_b\big((-\infty,0],X\big)$ there with the exception of Proposition 3.25 which is based on Theorem 2.18.*

There are different concepts of solutions for (2.1). We consider "classical" and "mild" solutions in the sense of Definitions 2.4 and 2.6.

Definition 2.4. *Let J be an interval with $(-\infty, 0] \subseteq J$ and $0 < \sup J \le \sup I$. A function $u \in C(J,X) \cap C^1(J \cap [0,\infty), X) \cap C(J \cap [0,\infty), [D(A)])$ is called a (classical) solution of (2.1) on J if it satisfies (2.1).*

Classical solutions satisfy Duhamel's formula (2.2) stated below. The lemma easily follows from Corollary 4.2.2 in [22] using the inhomogenity $h(s) = F(u_s) + g(s)$.

Lemma 2.5. *Let J be an interval with $(-\infty, 0] \subseteq J$ and $0 < \sup J \le \sup I$ and let u be a classical solution of (2.1) on J. Then u is given by*

$$u(t) = \begin{cases} T(t)\,[f(0)] + \int_0^t T(t-s)\big(F(u_s) + g(s)\big)\,\mathrm{d}s\,,\ t \in J \cap (0,\infty)\,, \\ f(t)\,,\ t \le 0\,. \end{cases} \tag{2.2}$$

The expression (2.2) is well defined even if u is only continuous with values in X and not necessarily differentiable which motivates the following weaker solution concept.

Definition 2.6. *Let J be an interval with $(-\infty, 0] \subseteq J$ and $0 < \sup J \le \sup I$. A function $u \in C(J, X)$ satisfying (2.2) is called a* mild solution *of (2.1) on J.*

According to Lemma 2.2, a mild solution on an interval J is bounded and uniformly continuous on all closed subsets of J.

2.2 Local wellposedness for mild solutions

The goal of this section is to investigate the local wellposedness of (2.1) for mild solutions. The approach is analogous to the treatment of semilinear evolution equations in Chapter 6 of [22]. The main idea is to consider (2.2) as a fixed point problem which can be treated with Banach's fixed point theorem. We set $M_0 := \sup_{0\le t\le 1}\|T(t)\| \in [1,\infty)$.

Lemma 2.7. *Let Assumption 2.1 be true, $\rho > 0$ and $\tau \in (0, \sup I) \cap (0,1]$. Then there exists a time $b_0 = b_0(\rho, L, M_0, \tau) \in (0,\tau]$ such that for each $f \in \mathrm{BUC}\big((-\infty,0],X\big)$ with $\|f\|_\infty \le \rho$, and each $g \in C(I,X)$ with $\sup_{0\le t\le\tau}\|g(t)\| \le \rho$, there is a mild solution u of (2.1) on $(-\infty,b_0]$. It is the only mild solution of (2.1) on $(-\infty,b_0]$ satisfying $\sup_{t\le b_0}\|u(t)\| \le 1+2M_0\rho$. For each $b \in (0,b_0]$, the restriction $u|_{(-\infty,b]}$ is also the unique mild solution on $(-\infty,b]$ satisfying $\sup_{t\le b}\|u(t)\| \le 1+2M_0\rho$.*

Proof. Let $\rho > 0, \tau \in (0,\sup I)\cap(0,1), b \in (0,\tau], f \in \mathrm{BUC}\big((-\infty,0],X\big)$ and $g \in C(I,X)$ with $\|f\|_\infty \le \rho$ and $\sup_{0\le t\le\tau}\|g(t)\| \le \rho$. We set $r := 1+2M_0\rho$ and define

$$E(b,r) := \left\{ v \in C\big((-\infty,b],X\big) \;\middle|\; v|_{(-\infty,0]} = f,\ \sup_{t\le b}\|v(t)\| \le r \right\}.$$

This is a closed subspace of the Banach space $\mathrm{BUC}\big((-\infty,b],X\big)$ and therefore complete. We define the map Φ on $E(b,r)$ by

$$\Phi(v)(t) = \begin{cases} T(t)[f(0)] + \int_0^t T(t-s)\big(F(v_s)+g(s)\big)\,\mathrm{d}s\,,\ t \in (0,b]\,, \\ f(t)\,,\ t \le 0\,. \end{cases}$$

Clearly, a function $v \in E(b,r)$ is a mild solution of (2.1) on $(-\infty,b]$ if and only if it is a fixed point of Φ. Observe that Φ maps into $\mathrm{BUC}\big((-\infty,b],X\big)$, since $f \in \mathrm{BUC}\big((-\infty,0],X\big)$, Φv is continuous on $(0,b]$ and

$$\|\Phi(v)(t) - f(0)\| \le \|T(t)\,[f(0)] - f(0)\| + tM_0\left(L(r)r + \sup_{s\in[0,b]}\|g(s)\|\right) \to 0$$

as $t \to 0$. For $v, w \in E(b,r)$ and $t \in [0,b]$, using $b < 1$, we have the estimates

$$\begin{aligned}
\|\Phi(v)(t)\| &\le M_0\left(\|f(0)\| + \int_0^t \big(\,\|F(v_s)\| + \|g(s)\|\,\big)\,\mathrm{d}s\right) \\
&\le M_0\left(\rho + bL(r)r + b\sup_{s\in[0,\tau]}\|g(s)\|\right) \le M_0(2\rho + bL(r)r)\,, \\
\|\Phi(v)(t) - \Phi(w)(t)\| &\le M_0\int_0^t \|F(v_s) - F(w_s)\|\,\mathrm{d}s \le M_0 bL(r)\sup_{s\in[0,b]}\|v_s - w_s\|_\infty \\
&\le M_0 bL(r)\sup_{s\in[0,b]}\sup_{\tau\le 0}\|v(s+\tau) - w(s+\tau)\| \\
&= M_0 bL(r)\sup_{\tau\le b}\|v(\tau) - w(\tau)\|\,.
\end{aligned}$$

So with

$$b_0(\rho, L, M_0, \tau) := \min \left\{ \tau, \frac{1}{M_0 L(r) r}, \frac{1}{2 M_0 L(r)} \right\}, \tag{2.3}$$

Φ is a strict contraction (with constant $1/2$) on $E(b,r)$ for all $b \in (0, b_0]$, and the claim follows from Banach's fixed point theorem. □

Next we show that mild solutions can be concatenated and shifted.

Lemma 2.8. *Let Assumption 2.1 be true, $b_1 > 0$ and u be a mild solution of (2.1) on the interval $(-\infty, b_1]$.*

1) *Let $b_2 > 0$ with $b_1 + b_2 < \sup I$. Define $\widetilde{f} : (-\infty, 0] \to X$ and $\widetilde{g} : [0, \sup I - b_1) \to X$ by $\widetilde{f}(t) = u(t + b_1)$ and $\widetilde{g}(t) = g(t + b_1)$. Let $v \in C\big((-\infty, b_2], X\big)$ be a mild solution of*

$$\begin{aligned} v'(t) &= Av(t) + F(v_t) + \widetilde{g}(t), \quad t \in [0, \sup I - b_1), \\ v(t) &= \widetilde{f}(t), \quad t \le 0, \end{aligned}$$

on $(-\infty, b_2]$. Then $w : (-\infty, b_1 + b_2] \to X$ defined by

$$w(t) = \begin{cases} u(t), \ t \le b_1, \\ v(t - b_1), \ b_1 < t \le b_1 + b_2, \end{cases}$$

is a mild solution of (2.1) on $(-\infty, b_1 + b_2]$.

2) *Let $\beta \in (0, b_1)$. Define $\widetilde{f} : (-\infty, 0] \to X$ and $\widetilde{g} : [0, \sup I - \beta) \to X$ by $\widetilde{f}(t) = u(t + \beta)$ and $\widetilde{g}(t) = g(t + \beta)$. Then $v : (-\infty, b_1 - \beta] \to X$ given by $v(t) = u(t + \beta)$ is a mild solution of*

$$\begin{aligned} v'(t) &= Av(t) + F(v_t) + \widetilde{g}(t), \quad t \in [0, \sup I - \beta), \\ v(t) &= \widetilde{f}(t), \quad t \le 0, \end{aligned}$$

on $(-\infty, b_1 - \beta]$.

Proof. 1) The function w is continuous and a mild solution of (2.1) on $(-\infty, b_1]$. For $t \in (b_1, b_1 + b_2]$ we have

$$w(t) = v(t - b_1) = T(t - b_1)[u(b_1)] + \int_0^{t - b_1} T(t - b_1 - s)\big(F(v_s) + \widetilde{g}(s)\big)\, \mathrm{d}s.$$

We substitute $\sigma = s + b_1$ in the integral and insert

$$u(b_1) = T(b_1)[f(0)] + \int_0^{b_1} T(b_1 - s)\big(F(u_s) + g(s)\big)\, \mathrm{d}s,$$

deriving

$$w(t) = T(t - b_1)\left(T(b_1)[f(0)] + \int_0^{b_1} T(b_1 - s)\big(F(u_s) + g(s)\big)\, \mathrm{d}s\right)$$

$$+\int_{b_1}^{t} T(t-\sigma)\big(F(v_{\sigma-b_1})+\widetilde{g}(\sigma-b_1)\big)\,\mathrm{d}\sigma\,.$$

In order to combine the two integrals, we note that for $s \in [0, b_1]$ and $\tau \le 0$ it holds

$$u_s(\tau) = u(s+\tau) = w(s+\tau) = w_s(\tau)$$

and therefore $u_s = w_s$. Similarly, in the second integral for $\sigma \in [b_1, t]$ and $\tau \le 0$ we get

$$w_\sigma(\tau) = w(\sigma+\tau) = v(\sigma - b_1 + \tau) = v_{\sigma-b_1}(\tau)$$

in the case $\sigma + \tau > b_1$, while for $\sigma + \tau \le b_1$ we obtain

$$w_\sigma(\tau) = w(\sigma+\tau) = u(\sigma+\tau) = \widetilde{f}(\sigma - b_1 + \tau) = v(\sigma - b_1 + \tau) = v_{\sigma-b_1}(\tau)\,.$$

It follows $v_{\sigma-b_1} = w_\sigma$, while $\widetilde{g}(\sigma - b_1) = g(\sigma)$ is clear by definition. Thus, we arrive at

$$w(t) = T(t)[f(0)] + \int_0^t T(t-s)\,(F(w_s)+g(s))\,\mathrm{d}s\,,$$

which shows that w is also a mild solution of (2.1) on $(-\infty, b_1 + b_2]$.

2) We have $v \in C\big((-\infty, b_1 - \beta], X\big)$ and for $t \in (0, b_1 - \beta]$ the equation

$$\begin{aligned}
v(t) = u(t+\beta) &= T(t+\beta)[f(0)] + \int_0^{t+\beta} T(t+\beta-s)\big(F(u_s)+g(s)\big)\,\mathrm{d}s \\
&= T(t)\left(T(\beta)[f(0)] + \int_0^{\beta} T(\beta-s)\big(F(u_s)+g(s)\big)\,\mathrm{d}s\right) \\
&\quad + \int_0^t T(t-s)\big(F(u_{s+\beta})+g(s+\beta)\big)\,\mathrm{d}s \\
&= T(t)[u(\beta)] + \int_0^t T(t-s)\big(F(v_s)+g(s+\beta)\big)\,\mathrm{d}s\,.
\end{aligned}$$

□

The mild solution obtained in Lemma 2.7 is only unique under a condition on its size. The next result states that mild solutions are in fact unique unconditionally.

Lemma 2.9. *Let Assumption 2.1 be true and u, v be mild solutions of (2.1) on $(-\infty, T_1]$ respectively $(-\infty, T_2]$. Then $u = v$ on $(-\infty, T_3]$ with $T_3 = \min\{T_1, T_2\}$.*

Proof. Without loss of generality, we can assume $T_1 < T_2$. We define

$$\widehat{t} := \sup\left\{\overline{t} \le T_1 \,\middle|\, u(t) = v(t) \text{ for all } t \le \overline{t}\right\}\,.$$

Then we have $\widehat{t} \ge 0$ and by continuity $u(\widehat{t}) = v(\widehat{t})$. We assume $\widehat{t} < T_1$. According to Lemma 2.8, the functions $\widehat{u} = u(\cdot + \widehat{t})$ and $\widehat{v} = v(\cdot + \widehat{t})$ are mild solutions on $(-\infty, T_1 - \widehat{t}]$ if f and g are replaced by $\widehat{f} := f(\cdot + \widehat{t})$ and $\widehat{g} := g(\cdot + \widehat{t})$. We choose a time $\tau \in (0, \sup I - \widehat{t}) \cap (0, 1]$ and set

$$\rho = \max\left\{\sup_{t\le 0}\left\|\widehat{f}(t)\right\|,\ \sup_{0\le t\le\tau}\|\hat{g}(t)\|\right\}\,.$$

Lemma 2.7 yields a time $b_0 = b_0(\rho, L, M_0, \tau) > 0$ such that there is for each $b \in (0, b_0]$ a unique mild solution $w = w^b$ on $(-\infty, b]$ corresponding to $\widehat{f}$ and $\widehat{g}$ which satisfies $\|w(t)\| \le 1+2M_0\rho$ for all $t \in (-\infty, b]$. Since $\widehat{u}$ and $\widehat{v}$ are continuous and $\sup_{t\le 0}\|\widehat{u}(t)\| = \sup_{t\le 0}\|\widehat{v}(t)\| \le \rho$, there exists a time $b_1 \in (0, b_0]$ with $\widehat{t} + b_1 \le T_1$ and $\|\widehat{u}(t)\|, \|\widehat{v}(t)\| \le 1 + 2M_0\rho$ for all $t \le b_1$. It then follows $\widehat{u} = w = \widehat{v}$ on $(-\infty, b_1]$. Shifting back yields $u(t) = \widehat{u}(t - \widehat{t}) = \widehat{v}(t - \widehat{t}) = v(t)$ for all $t \le \widehat{t} + b_1$ which contradicts the definition of $\widehat{t}$. □

The preceding lemmas lead to the notion of a maximal mild solution.

Definition 2.10. *Let Assumption 2.1 be true. The* maximal existence time *is defined by*

$$t^+(f, g, F, A) := \sup\left\{b > 0 \;\middle|\; \text{there exists a mild solution of (2.1) on } (-\infty, b]\right\}.$$

The interval $J^+(f, g, F, A) := \big(-\infty, t^+(f, g, F, A)\big)$ is called the maximal existence interval *and a mild solution of (2.1) on $J^+(f, g, F, A)$ is called* maximal mild solution.

The quantity b_0 in Lemma 2.7 is a function of M_0 and therefore the maximal existence time depends on the semigroup's generator A. In the application to Maxwell equations, A contains material parameters, which is why we explicitly write this dependence in the form $t^+ = t^+(f, g, F, A)$.

The following theorem states existence and uniqueness of a mild solution on the maximal existence interval and gives a blow-up condition.

Theorem 2.11. *Let Assumption 2.1 be true. Then the following assertions hold.*

1) *There exists a unique mild solution u of (2.1) on $J^+(f, g, F, A)$.*
2) *If $t^+(f, g, F, A) < \sup I$, then there exists a sequence (t_k) in $\big(0, t^+(f, g, F, A)\big)$ with $t_k \to t^+(f, g, F, A)$ and $\|u(t_k)\| \to \infty$ as $k \to \infty$.*

Proof. 1) This follows from Lemma 2.7, Lemma 2.9 and the definition of $J^+(f, g, F, A)$.

2) Let $t^+ := t^+(f, g, F, A) < \sup I$. Assume $C := \sup_{t<t^+}\|u(t)\| < \infty$. We choose $\tau \in (0, \sup I - t^+) \cap (0, 1]$ and $\widetilde{\tau} \in [\tau + t^+, \sup I)$. Let (t_k) be a sequence in $(0, t^+)$ with $t_k \to t^+$ as $k \to \infty$. Then $\tau + t_k \le \widetilde{\tau}$ for all $k \in \mathbb{N}$. We define $g_k : I - t_k \to X$ by $g_k = g(\cdot + t_k)$ for $k \in \mathbb{N}$ and set $\rho := \max\left\{C, \sup_{0\le t\le\widetilde{\tau}}\|g(t)\|\right\}$. Then we have

$$\sup_{0\le t\le\tau}\|g_k(t)\| \le \sup_{0\le t\le\widetilde{\tau}}\|g(t)\| \le \rho$$

for all $k \in \mathbb{N}$. By Lemma 2.7, there exists a time $b_0 = b_0(\rho, L, M_0, \tau) > 0$, independent of k, such that the problem

$$\begin{cases} v'(t) = Av(t) + F(v_t) + g_k(t)\,, & t \in (I - t_k) \cap [0, \infty)\,, \\ v(t) = u(t + t_k)\,, & t \le 0\,, \end{cases}$$

has a mild solution v_k on $(-\infty, b_0]$ for all $k \in \mathbb{N}$. We now pick k large enough that $t_k + b_0 > t^+$. Using u, v_k and Lemma 2.8 we can to construct a mild solution of (2.1) on the interval $(-\infty, t_k + b_0]$ which contradicts the definition of t^+. □

We now look at the continuous dependence of the mild solution on the data f, the external forcing term g, the nonlinearity F and the generator A. To this end we need to specify what it means for two maps F and $\widetilde{F}$ describing the system's response to be close. This can be done with the help of the following definition.

Definition 2.12. *Let $\psi : [0,\infty) \to [0,\infty)$ be a continuous and increasing function with $\psi(x) > 0$ for all $x > 0$. We define*

$$V := \left\{ F : \mathrm{BUC}\big((-\infty,0],X\big) \to X \;\middle|\; F(0) = 0,\ F \text{ is Lipschitz on bounded sets} \right\}$$

and on V the map $|||\cdot|||_\psi$ by

$$|||F|||_\psi = \sup_{\substack{u\in \mathrm{BUC}((-\infty,0],X)\\ u\neq 0}} \frac{\|F(u)\|}{\psi\left(\sup_{t\le 0}\|u(t)\|\right)}$$

as well as the set

$$V_\psi := \left\{ F \in V \;\middle|\; |||F|||_\psi < \infty \right\}.$$

In the linear case we can choose ψ to be the identity and obtain the usual operator norm. In the application to the Maxwell equations we consider nonlinear material laws with nonlinearities up to a fixed order N and will choose $\psi(x) = \sum_{n=1}^N x^n$. The next lemma states that the definition above can be used to estimate how close F and $\widetilde{F}$ are provided they belong to the same space V_ψ.

Lemma 2.13. *Let ψ be as in Definition 2.12. Then $\left(V_\psi, |||\cdot|||_\psi\right)$ is a normed vector space.*

Proof. Let $F, G \in V$ with corresponding functions L_F, L_G describing the Lipschitz property and $\alpha \in \mathbb{C}$. We have $\alpha F(0) + G(0) = 0$ and for every $r > 0$ the estimate

$$\begin{aligned}\|(\alpha F + G)(u) - (\alpha F + G)(v)\| &\le |\alpha|\,\|F(u) - F(v)\| + \|G(u) - G(v)\| \\ &\le \big(\alpha L_F(r) + L_G(r)\big)\,\|u - v\|_\infty\end{aligned}$$

holds for all $u, v \in \mathrm{BUC}\big((-\infty,0],X\big)$ with $\|u\|_\infty \le r$, $\|v\|_\infty \le r$. So V is a vector space. Now we assume $F, G \in V_\psi$. The properties of $\|\cdot\|$ immediately yield $|||\alpha F|||_\psi = |\alpha|\,|||F|||_\psi$ and $|||F+G|||_\psi \le |||F|||_\psi + |||G|||_\psi$. If $|||F|||_\psi = 0$ we get $F(u) = 0$ for all $u \neq 0$ which together with the assumption $F(0) = 0$ leads to $F = 0$. So $|||\cdot|||_\psi$ is a norm. □

We can now formulate a first result on continuous dependence of the mild solution on f, g and F. Here A is kept fixed. Continuity with respect to A is treated in Theorem 2.16.

Theorem 2.14. *Let Assumption 2.1 be true. Assume there exists a function ψ as in Definition 2.12 such that F is contained in V_ψ. We choose some $b \in \big(0, t^+(f,g,F,A)\big)$. Then there exist constants $\delta = \delta(f,g,F,A,b,\psi) > 0$ and $c = c(f,g,F,A,b,\psi) \ge 0$ such that for all $\widetilde{f}$, $\widehat{f} \in \mathrm{BUC}\big((-\infty,0],X\big)$, $\widetilde{g}$, $\widehat{g} \in C(I,X)$ and $\widetilde{F} \in V_\psi$ satisfying*

$$\max\Big\{\big\|f-\widetilde{f}\big\|_\infty,\ \big\|f-\widehat{f}\big\|_\infty,\ \sup_{0\le t\le b}\big\|g(t)-\widetilde{g}(t)\big\|,\ \sup_{0\le t\le b}\big\|g(t)-\widehat{g}(t)\big\|,\ \big|\big|\big|F-\widetilde{F}\big|\big|\big|_\psi\Big\} \le \delta \quad (2.4)$$

the following statements hold.

1) $t^+\big(\widetilde f, \widetilde g, \widetilde F, A\big) > b$.

2) Let $\widetilde u$ *and* $\widehat u$ *be the maximal mild solutions of* (2.1) *with* f, g, F *replaced by* $\widetilde f, \widetilde g, \widetilde F$ *respectively* $\widehat f, \widehat g, F$. *Then we have the estimate*

$$\|\widehat u(t) - \widetilde u(t)\| \le c\Big(\big\|\widehat f - \widetilde f\big\|_\infty + \sup_{0\le\tau\le b}\big\|\widehat g(\tau) - \widetilde g(\tau)\big\| + \big\|\big|F - \widetilde F\big|\big\|_\psi\Big) \tag{2.5}$$

for all $t \in (-\infty, b]$.

Proof. Let u be the maximal mild solution of (2.1) on $\big(-\infty, t^+(f,g,F,A)\big)$ and let $b \in \big(0, t^+(f,g,F,A)\big)$. Since u is bounded on $(-\infty, b]$ we have $R = R(f,g,F,A,b) := 1 + \sup_{t\le b}\|u(t)\| < \infty$. By assumption, F is Lipschitz continuous on the set

$$\overline B_{\mathrm{BUC}}(0,R) = \big\{v \in \mathrm{BUC}\big((-\infty,0],X\big) \;\big|\; \|v\|_\infty \le R\big\}$$

with constant $L = L(f,g,F,A,b)$. Let $\delta_0 \in (0,1)$ and $\widetilde f, \widehat f, \widetilde g, \widehat g$ and $\widetilde F$ as in the claim, but satisfying (2.4) with δ_0 instead of δ. We denote the maximal mild solution corresponding to $\widetilde f, \widetilde g, \widetilde F$ by $\widetilde u$ and define

$$\widetilde b := \sup\Big\{\beta \in (0,b] \;\Big|\; \beta < t^+\big(\widetilde f, \widetilde g, \widetilde F, A\big),\ \sup_{t\le\beta}\|u(t) - \widetilde u(t)\| \le 1\Big\}. \tag{2.6}$$

This number is positive because of $t^+\big(\widetilde f, \widetilde g, \widetilde F, A\big) > 0$, $\delta_0 < 1$ and the continuity of $u - \widetilde u$. Note that $\widetilde b \le b < t^+(f,g,F,A) \le \sup I$. We also have $\widetilde b < t^+\big(\widetilde f, \widetilde g, \widetilde F, A\big)$ as otherwise the blow-up criterion of Theorem 2.11 would yield a sequence (t_k) converging to $\widetilde b$ from below with $\|\widetilde u(t_k)\| \to \infty$ for $k \to \infty$, contradicting (2.6).

We set $M_b := \sup_{0\le t\le b}\|T(t)\| \in [1,\infty)$ and write the difference of the solutions as

$$\begin{aligned} u(t) - \widetilde u(t) &= T(t)\big(f(0) - \widetilde f(0)\big) \\ &\quad + \int_0^t T(t-s)\big(F(u_s) - F(\widetilde u_s) + F(\widetilde u_s) - \widetilde F(\widetilde u_s) + g(s) - \widetilde g(s)\big)\,\mathrm{d}s \end{aligned}$$

for all $t \in [0, \widetilde b)$. For all $r \in [0, \widetilde b)$ we have $\sup_{\tau\le r}\|\widetilde u(\tau)\| \le R$ and thus

$$\big\|F(\widetilde u_s) - \widetilde F(\widetilde u_s)\big\| \le \big\|\big|F - \widetilde F\big|\big\|_\psi\, \psi\Big(\sup_{\tau\le s}\big\|\widetilde u(\tau)\big\|\Big) \le \big\|\big|F - \widetilde F\big|\big\|_\psi\, \psi(R)$$

for all $s \in [0,r]$. Since $r \in \big[0, \widetilde b\big)$ is arbitrary, the inequality holds on $\big[0, \widetilde b\big)$. This leads to the estimate

$$\begin{aligned} \|u(t) - \widetilde u(t)\| &\le M_b\big\|f(0) - \widetilde f(0)\big\| + M_b\bigg(\int_0^t \|F(u_s) - F(\widetilde u_s)\|\,\mathrm{d}s \\ &\qquad + \int_0^t \big\|F(\widetilde u_s) - \widetilde F(\widetilde u_s)\big\|\,\mathrm{d}s + \int_0^t \|g(s) - \widetilde g(s)\|\,\mathrm{d}s\bigg) \end{aligned}$$

$$\leq M_b\Bigg(\big\|f-\widetilde{f}\big\|_\infty + L\int_0^t \sup_{\tau\leq s}\|u(\tau)-\widetilde{u}(\tau)\|\,\mathrm{d}s \qquad (2.7)$$
$$+\, b\big\|\big|F-\widetilde{F}\big\|\big|_\psi \psi(R) + b\sup_{0\leq\tau\leq b}\|g(\tau)-\widetilde{g}(\tau)\|\Bigg)$$

for all $t\in\big[0,\widetilde{b}\big)$. For $t<0$ we have

$$\big\|u(t)-\widetilde{u}(t)\big\| = \big\|f(t)-\widetilde{f}(t)\big\| \leq \sup_{\tau\leq 0}\big\|f(\tau)-\widetilde{f}(\tau)\big\|\,. \qquad (2.8)$$

We define the function $\varphi:\big[0,\widetilde{b}\big]\to[0,\infty)$ by $\varphi(t)=\sup_{\tau\leq t}\|u(\tau)-\widetilde{u}(\tau)\|$. To show that it is continuous, we set $w:=u-\widetilde{u}$ and choose $t_0\in\big[0,\widetilde{b}\big]$ and $\varepsilon>0$. Since w is continuous, there exists a $\rho>0$ such that for all $\tau\in\mathbb{R}$ with $|\tau|\leq\rho$ and $t_0+\tau\in\big[0,\widetilde{b}\big]$ it holds $\big|\|w(t_0+\tau)\|-\|w(t_0)\|\big|\leq\varepsilon$. This leads to

$$|\varphi(t_0+h)-\varphi(t_0)| = \begin{cases}\sup_{\tau\leq t_0+h}\|w(\tau)\| - \sup_{\tau\leq t_0}\|w(\tau)\|\leq\varepsilon\,, & h\in[0,\rho]\,,\\ \sup_{\tau\leq t_0}\|w(\tau)\| - \sup_{\tau\leq t_0+h}\|w(\tau)\|\leq\varepsilon\,, & h\in[-\rho,0)\,.\end{cases}$$

Estimate (2.7) and (2.8) and Gronwall's inequality yield

$$\varphi(t)\leq M_b\Big(\big\|f-\widetilde{f}\big\|_\infty + b\psi(R)\big\|\big|F-\widetilde{F}\big\|\big|_\psi + b\sup_{0\leq\tau\leq b}\|g(\tau)-\widetilde{g}(\tau)\|\Big)\mathrm{e}^{M_bLt} \qquad (2.9)$$

for all $t\in\big[0,\widetilde{b}\big)$. By continuity, (2.9) is true on $\big[0,\widetilde{b}\big]$. We now choose a sufficiently small radius $\delta_0=\delta_0(f,g,F,A,b,\psi)$ such that $\varphi(t)\leq 1/2$ for all $t\in\big[0,\widetilde{b}\big]$.

Assume $\widetilde{b}<b$. Then, using the continuity of u and $\widetilde{u}$, we can find $\widetilde{\varepsilon}>0$ with $\widetilde{b}+\widetilde{\varepsilon}<b$, $b+\widetilde{\varepsilon}<t^+(f,g,F,A)$, $\widetilde{b}+\widetilde{\varepsilon}<\widetilde{t}^+(\widetilde{f},\widetilde{g},\widetilde{F},A)$ and $\sup_{\tau\leq\widetilde{b}+\widetilde{\varepsilon}}\|u(\tau)-\widetilde{u}(\tau)\|\leq 1$. This leads to the contradiction $\widetilde{b}\geq\widetilde{b}+\widetilde{\varepsilon}$ by (2.6). In conclusion we have $b=\widetilde{b}<t^+\big(\widetilde{f},\widetilde{g},\widetilde{F},A\big)$.

Let $\widehat{u}$ be the maximal mild solution corresponding to $\widehat{f},\widehat{g},F$. By the above we have $t^+\big(\widehat{f},\widehat{g},F,A\big)>b$ as well as $\sup_{\tau\leq b}\|\widehat{u}(\tau)\|\leq R$. The same calculation leading to (2.9) with u,f,g replaced by $\widehat{u},\widehat{f},\widehat{g}$ yields

$$\|\widehat{u}(t)-\widetilde{u}(t)\|\leq c\Big(\big\|\widehat{f}-\widetilde{f}\big\| + \sup_{0\leq t\leq b}\|\widehat{g}(t)-\widetilde{g}(t)\| + \big\|\big|F-\widetilde{F}\big\|\big|_\psi\Big)$$

for all $t\in(-\infty,b]$ with

$$c:=M_b\max\{1,b,b\psi(R)\}\,\mathrm{e}^{M_bLb}=c(f,g,F,A,b,\psi)\,. \qquad \square$$

Since in the application to the Maxwell equations the generator A contains material parameters, we also want to study the continuous dependence on A. For this we modify the above proof using the Trotter-Kato theorem. Allowing for variations in A leads to a weaker result for the continuous dependence of the mild solution on the data: Instead of a local Lipschitz continuity as in (2.5), we only get continuity, see (2.10). We formulate an argument used in the proof of Theorem 2.16 as a separate lemma. It is a direct consequence of the Trotter-Kato theorem.

Lemma 2.15. *Let A, A_k be linear operators on X with domain $D(A)$ which generate strongly continuous semigroups $T(\cdot)$ and $T_k(\cdot)$ satisfying $\|T(\cdot)\|, \|T_k(\cdot)\| \leq M\mathrm{e}^{\omega t}$ for all $k \in \mathbb{N}$ and $t \geq 0$ with some constants $M \geq 1$ and $\omega \in \mathbb{R}$. We further assume that $A_k y \to Ay$ as $k \to \infty$ for all $y \in D(A)$. Then for any $b > 0$, the following statements are true.*

1) Let (x_k) be a sequence in X converging to some $x \in X$ as $k \to \infty$. Then

$$\lim_{k\to\infty} \sup_{0\leq t\leq b} \|T_k(t)x_k - T(t)x\| = 0\,.$$

2) Let $K \subseteq X$ be compact. Then

$$\lim_{k\to\infty} \sup_{\substack{0\leq t\leq b,\\ x\in K}} \|T_k(t)x - T(t)x\| = 0\,.$$

Proof. Let $b > 0$. We set $M_b := \sup_{0\leq t\leq b} M\mathrm{e}^{\omega t} < \infty$. By the Trotter-Kato theorem (see Theorem III.4.8 in [9]), we have $T_k(t)x \to T(t)x$ as $k \to \infty$ for all $x \in X$, uniformly on $[0, b]$. Assertion 1) now follows from

$$\begin{aligned}\sup_{0\leq t\leq b} \|T_k(t)x_k - T(t)x\| &\leq \sup_{0\leq t\leq b} \|T_k(t)x_k - T_k(t)x\| + \sup_{0\leq t\leq b} \|T_k(t)x - T(t)x\| \\ &\leq M_b \|x_k - x\| + \sup_{0\leq t\leq b} \|T_k(t)x - T(t)x\| \to 0\end{aligned}$$

as $k \to \infty$. We prove 2) by contradiction: Let the claim be false. Then there exists a subsequence $\big(T_{k_j}(\cdot)\big)$ of $\big(T_k(\cdot)\big)$, a sequence (t_j) in $[0, b]$ and $\varepsilon > 0$ such that

$$\sup_{x\in K} \left\|T_{k_j}(t_j)x - T(t_j)x\right\| \geq \varepsilon$$

for all $j \in \mathbb{N}$. Let $j \in \mathbb{N}$. Since $x \mapsto \left\|T_{k_j}(t_j)x - T(t_j)x\right\|$ is continuous on the compact set K, there exists a vector $x_j \in K$ satisfying

$$\left\|T_{k_j}(t_j)x_j - T(t_j)x_j\right\| = \sup_{x\in K} \left\|T_{k_j}(t_j)x - T(t_j)x\right\|\,.$$

Further, (x_j) has a subsequence (x_{j_l}) converging to some $x_0 \in K$ as $l \to \infty$. This leads to the contradiction

$$\begin{aligned}\varepsilon &\leq \left\|T_{k_{j_l}}(t_{j_l})x_{j_l} - T(t_{j_l})x_{j_l}\right\| \leq \sup_{0\leq t\leq b} \left\|T_{k_{j_l}}(t)x_{j_l} - T(t)x\right\| + \sup_{0\leq t\leq b} \|T(t)x - T(t)x_{j_l}\| \\ &\leq \sup_{0\leq t\leq b} \left\|T_{k_{j_l}}(t)x_{j_l} - T(t)x\right\| + M_b \|x - x_{j_l}\| \to 0\end{aligned}$$

as $l \to \infty$, where we use 1) for the first term on the last line. □

We now prove the main result on continuous dependence in this setting.

Theorem 2.16. *Let Assumption 2.1 be true and u be the maximal mild solution of (2.1) on $\big(-\infty, t^+(f,g,F,A)\big)$. Assume there exists a function ψ as in Definition 2.12 such that $F \in V_\psi$. Let $b \in \big(0, t^+(f,g,F,A)\big)$ and $(f_k), (g_k)$ and (F_k) be sequences in the spaces $\mathrm{BUC}\big((-\infty,0],X\big), C(I,X)$ respectively V_ψ, satisfying*

$$\|f_k - f\|_\infty \to 0\,, \quad \sup_{0\le t\le b}\|g_k(t)-g(t)\| \to 0\,, \quad \|\!|F_k - F|\!\|_\psi \to 0$$

as $k\to\infty$. Let A_k be linear operators on X with domain $D(A)$ which generate strongly continuous semigroups $T_k(\cdot)$ satisfying $\|T(\cdot)\|, \|T_k(\cdot)\| \le M\mathrm{e}^{\omega t}$ for all $k\in\mathbb{N}$ and $t\ge 0$ with some constants $M\ge 1$ and $\omega\in\mathbb{R}$, as well as $A_k y \to Ay$ as $k\to\infty$ for all $y\in D(A)$. For all $k\in\mathbb{N}$, let u_k be the maximal mild solution of (2.1) with f,g,F,A replaced by f_k,g_k,F_k,A_k. Let $\varepsilon>0$. Then there exists an index $K\in\mathbb{N}$ such that

$$t^+(f_k,g_k,F_k,A_k) > b \quad \text{and} \quad \sup_{t\le b}\|u_k(t)-u(t)\| \le \varepsilon \tag{2.10}$$

for all $k\ge K$.

Proof. We use the abbreviations $t^+ := t^+(f,g,F,A)$ and $t_k^+ := t^+(f_k,g_k,F_k,A_k)$. Let $b\in(0,t^+)$. As in the proof of Theorem 2.14, we set $R := \sup_{0\le t\le b}\|u(t)\|+1$, $M_b := \sup_{0\le t\le b} M\mathrm{e}^{\omega t}$, denote the Lipschitz constant of F on $\overline{B}_{\mathrm{BUC}}(0,R)$ by L and set

$$b_k := \sup\left\{\beta\in(0,b] \;\middle|\; \beta < t_k^+\,,\ \sup_{t\le\beta}\|u(t)-u_k(t)\|\le 1\right\}. \tag{2.11}$$

Again we have $0<b_k<t_k^+$ for all $k\in\mathbb{N}$. For $k\in\mathbb{N}$ and $t\in[0,b_k)$, we estimate

$$\|u(t)-u_k(t)\| \le \sum_{i=1}^{5} d_{i,k}(t)$$

with

$$\begin{aligned}
d_{1,k}(t) &:= \|T_k(t)[f_k(0)] - T(t)[f(0)]\|\,,\\
d_{2,k}(t) &:= \int_0^t \big\|\big(T(t-s)-T_k(t-s)\big)\big(F(u_s)+g(s)\big)\big\|\,\mathrm{d}s\,,\\
d_{3,k}(t) &:= \int_0^t \Big\|T_k(t-s)\Big(F(u_s)-F\big((u_k)_s\big)\Big)\Big\|\,\mathrm{d}s\,,\\
d_{4,k}(t) &:= \int_0^t \Big\|T_k(t-s)\Big(F\big((u_k)_s\big)-F_k\big((u_k)_s\big)\Big)\Big\|\,\mathrm{d}s\,,\\
d_{5,k}(t) &:= \int_0^t \big\|T_k(t-s)\big(g(s)-g_k(s)\big)\big\|\,\mathrm{d}s
\end{aligned}$$

and treat the terms separately. Since $s\mapsto F(u_s)+g(s)$ is continuous, the set

$$\big\{F(u_s)+g(s) \;\big|\; s\in[0,b]\big\}$$

is compact in X and Lemma 2.15 implies that $d_{1,k}$ and $d_{2,k}$ converge to 0, uniformly on $[0, b]$. We can estimate the next term by

$$d_{3,k}(t) \le M_b L \int_0^t \sup_{\tau \le s} \|u(\tau) - u_k(\tau)\| \, \mathrm{d}s$$

for all $t \in [0, b_k)$ and $k \in \mathbb{N}$. As in the proof of Theorem 2.14, the last two terms are estimated by

$$d_{4,k}(t) \le M_b b \psi(R) \|F - F_k\|_\psi \,, \quad d_{5,k}(t) \le M_b b \sup_{0 \le s \le b} \|g(s) - g_k(s)\|$$

for all $t \in [0, b_k)$ and $k \in \mathbb{N}$. For negative times, we use

$$\|u(t) - u_k(t)\| \le \|f - f_k\|_\infty \,.$$

Let $\varepsilon \in (0, 1/2]$. Combining the above results, there exists an index $K \in \mathbb{N}$ such that

$$\|u(t) - u_k(t)\| \le \begin{cases} \varepsilon \mathrm{e}^{-M_b L b} + M_b L \int_0^t \sup_{\tau \le s} \|u(\tau) - u_k(\tau)\| \, \mathrm{d}s \,, & t \in [0, b_k) \,, \\ \varepsilon \mathrm{e}^{-M_b L b} \,, & t < 0 \end{cases}$$

for all $k \ge K$. For $k \in \mathbb{N}$ as in Theorem 2.14 we define the continuous function $\varphi_k : [0, b_k] \to [0, \infty)$ by $\varphi_k(t) = \sup_{\tau \le t} \|u(\tau) - u_k(\tau)\|$. It follows

$$\varphi_k(t) \le \varepsilon \mathrm{e}^{-M_b L b} + M_b L \int_0^t \varphi_k(s) \, \mathrm{d}s$$

for all $t \in [0, b_k)$ and $k \ge K$. Gronwall's inequality now yields $\varphi_k(t) \le \varepsilon$ for all $t \in [0, b_k)$ and $k \ge K$. Assume there exists an index $k_0 \ge K$ such that $b_{k_0} < b$. Then we can find as in the proof of Theorem 2.14 a positive number $\widetilde{\varepsilon}$ such that $b_{k_0} + \widetilde{\varepsilon} < b, b + \widetilde{\varepsilon} < t^+$, $b_{k_0} + \widetilde{\varepsilon} < t^+_{k_0}$ and $\sup_{\tau \le b_{k_0} + \widetilde{\varepsilon}} \|u(\tau) - u_{k_0}(\tau)\| \le 1$ which leads to the contradiction $b_{k_0} \ge b_{k_0} + \widetilde{\varepsilon}$ by (2.11). Therefore we have $b = b_k < t^+_k$ for all $k \ge K$.

Finally, the continuity of φ_k yields $\varphi_k(t) \le \varepsilon$ for all $t \in [0, b]$ and $k \ge K$. □

2.3 From mild to classical solutions

The aim of this section is to find conditions for f and g which ensure that a mild solution is also a classical one. As a first step, we show that if the system's history f and the inhomogenity g have additional regularity, then the mild solution is locally Lipschitz continuous in the space $\mathrm{BUC}\big((-\infty, 0], X\big)$.

Lemma 2.17. *Let Assumption 2.1 be true and let $f \in \mathrm{BUC}^1\big((-\infty, 0], X\big)$ with $f(0) \in D(A)$ and $g \in C^1(I, X)$ or $g \in C\big(I, [D(A)]\big)$. Let u be the maximal mild solution of (2.1) on $\big(-\infty, t^+(f, g, F)\big)$. Then the map $\varphi : \big[0, t^+(f, g, F, A)\big) \to \mathrm{BUC}\big((-\infty, 0], X\big)$ defined by $\varphi(t) = u_t$ is locally Lipschitz continuous.*

Proof. We write $t^+ = t^+(f, g, F, A)$ and split $u = v + w$ with

$$v(t) = \begin{cases} \int_0^t T(t-s)g(s)\,\mathrm{d}s\,, & t \in (0, t^+)\,, \\ 0\,, & t \le 0\,, \end{cases}$$

$$w(t) = \begin{cases} T(t)[f(0)] + \int_0^t T(t-s)F(u_s)\,\mathrm{d}s\,, & t \in (0, t^+)\,, \\ f(t)\,, & t \le 0\,. \end{cases}$$

Let $b \in [0, t^+), 0 \le t_0 \le t_0 + h \le b$ and $\tau \le 0$.

1) By Corollaries 4.2.5 and 4.2.6 in [22], v is continuously differentiable on $[0, t^+)$. If $t_0 + \tau \ge 0$, we estimate

$$\|v(t_0 + h + \tau) - v(t_0 + \tau)\| \le \sup_{t_0+\tau \le s \le t_0+h+\tau} \|v'(s)\|\, h$$

In the case $t_0 + \tau \le 0 \le t_0 + h + \tau$, we obtain

$$\|v(t_0 + h + \tau) - v(t_0 + \tau)\| = \|v(t_0 + h + \tau)\| \le \sup_{0 \le s \le t_0+t+\tau} \|v'(s)\|\, h\,.$$

If $t_0 + h + \tau < 0$, it follows $t_0 + \tau < 0$ and therefore $\|v(t_0 + h + \tau) - v(t_0 + \tau)\| = 0$. So with $L_v = L_v(b) := \sup_{\xi \in [0,b]} \|v'(\xi)\|$, we arrive at

$$\sup_{\tau \le 0} \|v(t_0 + h + \tau) - v(t_0 + \tau)\| \le L_v h\,. \tag{2.12}$$

2) We now turn to w and again consider three cases. First, let $t_0+\tau \ge 0$. The formula for w yields

$$\begin{aligned} &w(t_0 + h + \tau) - w(t_0 + \tau) \\ &\quad = T(t_0 + \tau)\big(T(h)[f(0)] - f(0)\big) + \int_0^h T(t_0 + \tau + h - s)F(u_s)\,\mathrm{d}s \\ &\qquad + \int_h^{t_0+\tau+h} T(t_0 + \tau + h - s)F(u_s)\,\mathrm{d}s - \int_0^{t_0+\tau} T(t_0 + \tau - s)F(u_s)\,\mathrm{d}s\,. \end{aligned}$$

Using $f(0) \in D(A)$, we can write

$$T(h)[f(0)] - f(0) = \int_0^h T(s)Af(0)\,\mathrm{d}s\,,$$

which leads together with a substitution to

$$\begin{aligned} w(t_0 + h + \tau) - w(t_0 + \tau) &= \int_0^h T(t_0 + \tau + s)Af(0)\,\mathrm{d}s + \int_0^h T(t_0 + \tau + h - s)F(u_s)\,\mathrm{d}s \\ &\quad + \int_0^{t_0+\tau} T(t_0 + \tau - s)\big(F(u_{s+h}) - F(u_s)\big)\,\mathrm{d}s\,. \end{aligned} \tag{2.13}$$

The quantities $K = K(b) := \sup_{0\le s\le b}\|T(s)\|$, $C = C(b) := \sup_{0\le s\le b}\|F(u_s)\|$ and $r = r(b) := \sup_{0\le s\le b}\sup_{t\le 0}\|u_s(t)\| = \sup_{t\le b}\|u(t)\|$ are all finite. Using the Lipschitz property of F and $t_0+\tau\le t_0$ in the last integral, we obtain from (2.13) the estimate

$$\begin{aligned}&\|w(t_0+h+\tau)-w(t_0+\tau)\|\\ &\quad\le K\|Af(0)\|\,h+KCh+KL(r)\int_0^{t_0}\sup_{\sigma\le 0}\|u_{s+h}(\sigma)-u_s(\sigma)\|\,\mathrm{d}s\,.\end{aligned}\tag{2.14}$$

Now, let $t_0+\tau<0\le t_0+h+\tau$. We similarly arrive at

$$\begin{aligned}w(t_0+h+\tau)-w(t_0+\tau)&=w(t_0+h+\tau)-f(t_0+\tau)\\ &=T(t_0+h+\tau)[f(0)]-f(0)+f(0)-f(t_0+\tau)+\int_0^{t_0+h+\tau}T(t_0+\tau+h-s)F(u_s)\,\mathrm{d}s\\ &=\int_0^{t_0+h+\tau}T(s)A[f(0)]\,\mathrm{d}s+\int_{t_0+\tau}^0 f'(s)\,\mathrm{d}s+\int_0^{t_0+h+\tau}T(t_0+\tau+h-s)F(u_s)\,\mathrm{d}s\,.\end{aligned}$$

Using $t_0+h+\tau\le h$ and $|t_0+\tau|\le h$, we estimate

$$\|w(t_0+h+\tau)-w(t_0+\tau)\|\le K\|Af(0)\|\,h+\sup_{s\le 0}\|f'(s)\|\,h+KCh\,.\tag{2.15}$$

It remains to treat the case $t_0+h+\tau<0$. Here we have

$$\|w(t_0+h+\tau)-w(t_0+\tau)\|=\|f(t_0+h+\tau)-f(t_0+\tau)\|\le\sup_{s\le 0}\|f'(s)\|\,h$$

The inequalities (2.14),(2.15) and (2.3) lead to the bound

$$\sup_{\tau\le 0}\|w_{t_0+h}(\tau)-w_{t_0}(\tau)\|\le\alpha h+\beta\int_0^{t_0}\sup_{\tau\le 0}\|u_{s+h}(\tau)-u_s(\tau)\|\,\mathrm{d}s\tag{2.16}$$

with constants

$$\alpha=\alpha(b):=K\|Af(0)\|+KC+\sup_{s\le 0}\|f'(s)\|\,,\quad \beta=\beta(b):=KL(r)\,.$$

Since

$$\sup_{\tau\le 0}\|u_{t_0+h}(\tau)-u_{t_0}(\tau)\|\le\sup_{\tau\le 0}\|w_{t_0+h}(\tau)-w_{t_0}(\tau)\|+\sup_{\tau\le 0}\|v_{t_0+h}(\tau)-v_{t_0}(\tau)\|\,,$$

we conclude from (2.12) and (2.16)

$$\sup_{\tau\le 0}\|u_{t_0+h}(\tau)-u_{t_0}(\tau)\|\le(\alpha+L_v)\,h+\beta\int_0^{t_0}\sup_{\tau\le 0}\|u_{s+h}(\tau)-u_s(\tau)\|\,\mathrm{d}s\,.\tag{2.17}$$

For $t_0=0$, this yields $\|\varphi(h)-\varphi(0)\|_\infty\le(\alpha+L_v)h$. Now let $t_0>0$, which implies $b>h$. We define $\rho:[0,b-h]\to[0,\infty)$ by

$$\rho(t)=\sup_{\tau\le 0}\|u_{t+h}(\tau)-u_t(\tau)\|\,.$$

Since ρ is continuous (see below), we can apply Gronwall's inequality to (2.17) obtaining

$$\rho(t) \le (\alpha + L_v)\, \mathrm{e}^{\beta b} h$$

for all $t \in (0, b-h]$. These results imply the estimate

$$\|\varphi(t_0 + h) - \varphi(t_0)\|_\infty = \rho(t_0) \le (\alpha + L_v)\, \mathrm{e}^{\beta b} h$$

for all $t_0 \in [0, b-h]$.

It remains to show the continuity of ρ. We define $k : (-\infty, b-h] \to [0, \infty)$ by $k(t) = \|u(t+h) - u(t)\|$. Let $\varepsilon > 0$. Since k is uniformly continuous on $[0, b-h]$, there exists $\delta > 0$ such that

$$\left|k(\tilde{t}) - k(t)\right| \le \frac{\varepsilon}{2} \tag{2.18}$$

for all $\tilde{t}, t \in [0, b-h]$ with $\left|\tilde{t} - t\right| \le \delta$. Let $t \in [0, b-h)$ and $s \in [0, \min\{\delta, b-h-t\}]$. We want to show that

$$|\rho(t+s) - \rho(t)| = \sup_{\tau \le 0} k(t+s+\tau) - \sup_{\tau \le 0} k(t+\tau) = \sup_{\sigma \le t+s} k(\sigma) - \sup_{\sigma \le t} k(\sigma)$$

is bounded by ε. This is of course true if $\sup_{\sigma \le t+s} k(\sigma) = \sup_{\sigma \le t} k(\sigma)$. In the other case we have

$$\sup_{\sigma \le t} k(\sigma) < \sup_{\sigma \le t+s} k(\sigma) = \sup_{t < \sigma \le t+s} k(\sigma)\,.$$

Then there is a time $\sigma_0 \in (t, t+s]$ such that $k(\sigma_0) \ge \sup_{\sigma \le t+s} k(\sigma) - \varepsilon/2$. Using (2.18), we arrive at

$$\sup_{\sigma \le t+s} k(\sigma) \le k(\sigma_0) + \frac{\varepsilon}{2} \le k(t) + \varepsilon \le \sup_{\sigma \le t} k(\sigma) + \varepsilon$$

which proves that ρ is continuous from the right. Continuity from the left is shown in the same way. □

Analogously to Theorem 6.1.5 in [22], using the preceding lemma, we can state a criterion which ensures that the mild solution is a classical one. It essentially requires that the system's history is differentiable and satisfies the evolution equation at the initial time and that the external force is differentiable. Also a technical assumption on the nonlinearity is necessary, which is fulfilled by the specific model used for F in Section 3, see Lemma 3.22.

Theorem 2.18. *Let Assumption 2.1 be true, $g \in C^1(I, X)$ and $f \in \mathrm{BUC}^1\big((-\infty, 0], X\big)$ with $f(0) \in D(A), f'(0) = Af(0) + F(f) + g(0)$. Let $F \in C^1\big(\mathrm{BUC}((-\infty, 0], X), X\big)$ have the property that for all $b > 0$ and $u \in \mathrm{BUC}^1\big((-\infty, b), X\big)$ the map $t \mapsto F(u_t)$ is contained in $C^1\big([0, b), X\big)$ with derivative $F'(u_t)(u')_t$.*

Then the maximal mild solution of (2.1) is a classical solution of (2.1) on the maximal existence interval.

Proof. Let $b \in \big(0, t^+(f,g,F,A)\big)$ and u be the maximal mild solution of (2.1). The main step of the proof consists of showing $u \in \mathrm{BUC}^1\big((-\infty,b),X\big)$.

1) If we assume u to be differentiable, we can set $v = u'$ and formally obtain

$$\begin{aligned} v'(t) &= Av(t) + F'(u_t)v_t + g'(t)\,, \quad t \in [0,b]\,, \\ v(t) &= f'(t)\,, \quad t \le 0\,. \end{aligned} \tag{2.19}$$

This problem serves to define a candidate for the derivative of u.

2) Similarly as in Lemma 2.7, we obtain a solution of (2.19): We set

$$K := \sup_{0\le s\le b} \|T(s)\| < \infty\,, \quad L := \sup_{0\le s\le b} \|F'(u_s)\| < \infty\,,$$

where we use that $s \mapsto F'(u_s)$ is continuous since F is continuously differentiable and $s \mapsto u_s$ is continuous by Lemma 2.2. We equip the space $E := \mathrm{BUC}\big((-\infty,b],X\big)$ with the equivalent norm $\|v\|_\alpha := \sup_{s\le b} \rho_\alpha(s)\,\|v(s)\|$ where $\alpha := 2KL$ and

$$\rho_\alpha(s) = \begin{cases} \mathrm{e}^{-\alpha s}\,, & s \in (0,b]\,, \\ 1\,, & s \le 0\,. \end{cases}$$

On E we define the map Φ by

$$\Phi(v)(t) = \begin{cases} T(t)[f'(0)] + \int_0^t T(t-s)\big(F'(u_s)v_s + g'(s)\big)\,\mathrm{d}s\,, & t \in (0,b]\,, \\ f'(t)\,, & t \le 0\,. \end{cases}$$

Note that $\Phi(v) \in E$ for all $v \in E$. Let $v, w \in E$. We compute

$$\begin{aligned} \|\Phi(v) - \Phi(w)\|_\alpha &= \sup_{0\le t\le b} \left\| \int_0^t \mathrm{e}^{-\alpha(t-s)} T(t-s) F'(u_s) \mathrm{e}^{-\alpha s}(v_s - w_s)\,\mathrm{d}s \right\| \\ &\le KL \sup_{0\le t\le b} \int_0^t \mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}s \sup_{0\le s\le t} \Big(\mathrm{e}^{-\alpha s}\,\|v_s - w_s\|_{\mathrm{BUC}((-\infty,0],X)} \Big) \\ &\le KL \sup_{0\le t\le b} \int_0^t \mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}s \sup_{0\le s\le b} \mathrm{e}^{-\alpha s} \sup_{r\le s} \|v(r) - w(r)\| \\ &\le \frac{KL}{\alpha} \sup_{r\le b} \rho_\alpha(r)\,\|v(r) - w(r)\| = \frac{1}{2}\,\|v - w\|_\alpha\,. \end{aligned}$$

Hence, Φ is a strict contraction on the complete space E, and Banach's fixed point theorem yields a unique $v \in E$ with $\Phi(v) = v$.

3) In this step we verify that the map v is the derivative of u. Let $\widetilde{b} \in (0,b)$ and $h \in (0, b - \widetilde{b})$. We define $w_h : (-\infty, \widetilde{b}] \to X$ by

$$w_h(t) = \frac{1}{h}\big(u(t+h) - u(t)\big) - v(t)\,.$$

Since w_h is continuous, the map $\varphi_h : [0,\widetilde{b}] \to [0,\infty)$ given by $\varphi_h(t) = \sup_{\tau\le t} \|w_h(\tau)\|$ is also continuous (see the proof of Theorem 2.14).

Let $t \in [0, \widetilde{b}]$. As in (2.13) we have

$$u(t+h) - u(t) = T(t)\big(T(h)[f(0)] - f(0)\big) + \int_0^h T(t+h-s)\big(F(u_s) + g(s)\big)\,\mathrm{d}s$$
$$+ \int_0^t T(t-s)\big(F(u_{s+h}) - F(u_s) + g(s+h) - g(s)\big)\,\mathrm{d}s\,.$$

Using $v = \Phi(v)$ and $f'(0) = Af(0) + F(f) + g(0)$, we obtain

$$w_h(t) = S_1(h,t) + S_2(h,t) + S_3(h,t) + S_4(h,t) + S_5(h,t)$$

with

$$S_1(h,t) = T(t)\frac{1}{h}\big(T(h)[f(0)] - f(0)\big) - T(t)Af(0)\,,$$
$$S_2(h,t) = \frac{1}{h}T(t)\int_0^h T(h-s)F(u_s)\,\mathrm{d}s - T(t)F(f)\,,$$
$$S_3(h,t) = \frac{1}{h}T(t)\int_0^h T(h-s)g(s)\,\mathrm{d}s - T(t)g(0)\,,$$
$$S_4(h,t) = \int_0^t T(t-s)\Big[\frac{1}{h}\big(F(u_{s+h}) - F(u_s)\big) - F'(u_s)v_s\Big]\,\mathrm{d}s\,,$$
$$S_5(h,t) = \int_0^t T(t-s)\Big[\frac{1}{h}\big(g(s+h) - g(s)\big) - g'(s)\Big]\,\mathrm{d}s\,.$$

Since $f(0) \in D(A)$, the first term can be estimated by

$$\|S_1(h,t)\| \le K\left\|\frac{1}{h}\big(T(h)[f(0)] - f(0)\big) - Af(0)\right\| =: \alpha_1(h) \to 0$$

as $h \to 0^+$. For the second and third term we use that $T(\cdot)$ is strongly continuous and $s \mapsto F(u_s)$ as well as g are continuous. It follows

$$\|S_2(h,t)\| \le K \sup_{0\le s\le h}\|T(h-s)F(u_s) - F(f)\| =: \alpha_2(h) \to 0\,,$$
$$\|S_3(h,t)\| \le K \sup_{0\le s\le h}\|T(h-s)g(s) - g(0)\| =: \alpha_3(h) \to 0$$

as $h \to 0^+$. The fourth term is split as $S_4(h,t) = S_{4,1}(h,t) + S_{4,2}(h,t)$ with

$$S_{4,1}(h,t) = \int_0^t T(t-s)\frac{1}{h}\big[F(u_{s+h}) - F(u_s) - F'(u_s)(u_{s+h} - u_s)\big]\,\mathrm{d}s\,,$$
$$S_{4,2}(h,t) = \int_0^t T(t-s)F'(u_s)(w_h)_s\,\mathrm{d}s\,.$$

Since $t \mapsto u_t$ is locally Lipschitz continuous by Lemma 2.17, there exists a constant $l \ge 0$ such that $\sup_{\tau\le 0}\|u_{s+h}(\tau) - u_s(\tau)\| \le lh$ for all $s \in [0,t]$. Together with

$$F(u_{s+h}) - F(u_s) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}F\big(u_s + \tau(u_{s+h} - u_s)\big)\,\mathrm{d}\tau$$

$$= \int_0^1 F'\big(u_s + \tau(u_{s+h} - u_s)\big)(u_{s+h} - u_s)\,\mathrm{d}\tau$$

for all $s \in [0,t]$ we find

$$\|S_{4,1}(h,t)\| \le Kbl \sup_{\substack{0\le s\le b-h\\ 0\le\tau\le 1}} \|F'(u_s + \tau(u_{s+h} - u_s)) - F'(u_s)\| =: \alpha_4(h) \to 0$$

as $h \to 0^+$. Here we use that F' is uniformly continuous on the set

$$\{u_s + \tau(u_r - u_s) \,|\, 0 \le \tau \le 1, 0 \le r, s \le b\} \subseteq \mathrm{BUC}\big((-\infty,0],X\big)\,,$$

which is compact as the image of the compact set $[0,1]\times[0,b]^2$ under the continuous map $(\tau,r,s) \mapsto u_s + \tau(u_r - u_s)$. The other part of $S_4(h,t)$ is estimated by

$$\|S_{4,2}(h,t)\| \le KL \int_0^t \sup_{\tau\le 0} \|(w_h)_s(\tau)\|\,\mathrm{d}s = KL \int_0^t \sup_{\tau\le s} \|w_h(\tau)\|\,\mathrm{d}s\,.$$

For the remaining term, as above the uniform continuity of g' on $[0,b]$ yields

$$\|S_5(h,t)\| \le Kb \sup_{\substack{0\le s\le b-h,\\ s\le\tau\le s+h}} \|g'(\tau) - g'(s)\| =: \alpha_5(h) \to 0$$

as $h \to 0^+$. These estimates lead to

$$\|w_h(t)\| \le \alpha_1(h) + \alpha_2(h) + \alpha_3(h) + \alpha_4(h) + \alpha_5(h) + KL \int_0^t \sup_{\sigma\le s} \|w_h(\sigma)\|\,\mathrm{d}s \qquad (2.20)$$

for all $t \in [0,\widetilde{b}]$. In order to apply Gronwall's inequality to φ_h, we also need to estimate $\|w_h(t)\|$ for negative times.

Let $\varepsilon > 0$ and $t \in (-h,0)$. We use $f(0) \in D(A)$ and $Af(0) = f'(0) - F(f) - g(0)$ in the expression

$$w_h(t) = \frac{1}{h}\left(T(t+h)[f(0)] + \int_0^{t+h} T(t+h-s)\big(F(u_s) + g(s)\big)\,\mathrm{d}s - f(t)\right) - f'(t)\,.$$

to obtain

$$\begin{aligned} w_h(t) &= \frac{1}{h}\int_0^{t+h} \big(T(s)f'(0) - f'(0)\big)\,\mathrm{d}s + \frac{1}{h}\int_t^0 \big(f'(s) - f'(0)\big)\,\mathrm{d}s + f'(0) - f'(t) \\ &\quad + \frac{1}{h}\int_0^{t+h} T(t+h-s)\big(F(u_s) + g(s)\big)\,\mathrm{d}s - \int_0^{t+h} T(s)\big(F(f) + g(0)\big)\,\mathrm{d}s\,. \end{aligned}$$

Substituting $\sigma = t + h - s$ in the third integral and using $|t| \le h$ yields

$$\|w_h(t)\| \le \sup_{0\le s\le h} \|T(s)f'(0) - f'(0)\| + 2 \sup_{-h\le s\le 0} \|f'(s) - f'(0)\|$$

$$+ \sup_{0\le s\le h}\|T(s)\| \sup_{0\le\sigma\le h}\|F(u_\sigma)+g(\sigma)-F(f)-g(0)\| \qquad (2.21)$$
$$=: \alpha_6(h)\to 0$$

as $h\to 0^+$.

Now let $t\le -h$. Then we have

$$w_h(t)=\frac{1}{h}\big(f(t+h)-f(t)\big)-f'(t)=\frac{1}{h}\int_t^{t+h}\big(f'(s)-f'(t)\big)\,\mathrm{d}s\,.$$

Let $\varepsilon>0$. Since f' is uniformly continuous by assumption, there exists $\delta>0$ such that

$$\|w_h(t)\|\le \sup_{t\le s\le t+h}\|f'(s)-f'(t)\|\le\varepsilon \qquad (2.22)$$

for all $h\in(0,\delta)$, uniformly in $t\in(-\infty,-h)$.

Combining the estimates(2.20), (2.21) and (2.22) leads to

$$\varphi_h(t)\le\varepsilon+\sum_{i=1}^{6}\alpha_i(h)+KL\int_0^t\varphi_h(s)\,\mathrm{d}s$$

for all $t\in[0,\widetilde{b}]$. Gronwall's inequality now yields

$$\varphi_h(t)\le\big(\varepsilon+\sum_{i=1}^{6}\alpha_i(h)\big)\mathrm{e}^{KLb}$$

for all $t\in[0,\widetilde{b}]$. This implies $w_h(t)\to 0$ as $h\to 0^+$ for all $t\le\widetilde{b}$. Therefore u is differentiable from the right with the continuous right-hand side derivative v. An application of the Hahn-Banach theorem and Corollary 2.1.2 of [22] yield $u\in C^1\big([0,b),X\big)$.

4) To conclude that u is a classical solution of (2.1) on $(-\infty,b)$, we note that $\widetilde{u}:[0,b)\to X$ given by

$$\widetilde{u}(t)=T(t)[f(0)]+\int_0^t T(t-s)\big(F(u_s)+g(s)\big)\,\mathrm{d}s\,,$$

where u is regarded as a given function, is a classical solution of the inhomogeneous evolution equation (without retardation)

$$\widetilde{u}'(t)=A\widetilde{u}(t)+F(u_t)+g(t)\,,\quad t\in[0,b)\,,$$
$$\widetilde{u}(0)=f(0)$$

by Corollary 4.2.5 of [22]. Since $\widetilde{u}=u$ on $[0,b)$, we have $u\in C\big([0,b),[D(A)]\big)$ and

$$u'(t)=Au(t)+F(u_t)+g(t)\,,\quad t\in[0,b)\,,$$

which finishes the proof. □

3 Maxwell equations

Now we aim to apply to results for the abstract retarded evolution equation to the system of Maxwell equations in a domain filled with a material whose polarisation depends on the electric field locally in space but nonlinearly and noninstantaneously. The boundary conditions of a perfect conductor are used.

This section is structured as follows: We first state some necessary properties of trace operators before we formulate the Maxwell system (1.2) with (1.6) in a way suitable for the use of semigroup theory. Subsection 3.3 is concerned with the strongly continuous semigroup generated by the linear differential operator in the Maxwell equations while in Subsection 3.4 the chosen model for the polarisation is studied. A restriction to certain scalar-type material laws is necessary for our semigroup approach. In the last subsection we show the local wellposedness of the Maxwell system.

3.1 Trace operators

We work in Sobolev spaces of real-valued functions on a domain $G \subseteq \mathbb{R}^3$. Since we supplement the system of Maxwell equations with boundary conditions, we need trace operators to give meaning to the restriction of functions (or their normal respectively tangential components) to the boundary. The following two definitions as well as Lemma 3.2 are from Chapter 4 of [17].

Definition 3.1. *Let $G \subseteq \mathbb{R}^3$ be open. We define the spaces*

$$\begin{aligned}
H(\operatorname{curl}, G) := \Big\{ u \in L^2(G)^3 \,\Big|\, & \exists\, v \in L^2(G)^3 : \\
& \int_G u \cdot \operatorname{curl} \psi \,\mathrm{d}x = \int_G v \cdot \psi \,\mathrm{d}x \;\big(\forall\, \psi \in C_c^\infty(G)^3\big) \Big\}, \\
H(\operatorname{div}, G) := \Big\{ u \in L^2(G)^3 \,\Big|\, & \exists\, w \in L^2(G) : \\
& \int_G u \cdot \nabla\varphi \,\mathrm{d}x = -\int_G w\varphi \,\mathrm{d}x \;\big(\forall\, \varphi \in C_c^\infty(G)\big) \Big\}.
\end{aligned}$$

The functions v and w in the above definition are unique if they exist and are denoted by $\operatorname{curl} u$ respectively $\operatorname{div} u$. We write

$$(u|v)_{L^2(G)^m} := \int_G u \cdot v \,\mathrm{d}x$$

for the inner product on $L^2(G)^m$ where $m \in \mathbb{N}$. Recall that we use real function spaces.

Lemma 3.2. *Let $G \subseteq \mathbb{R}^3$ be open. The spaces $H(\operatorname{curl}, G)$ and $H(\operatorname{div}, G)$ are Hilbert spaces when equipped with the inner products*

$$\begin{aligned}
(u|v)_{H(\operatorname{curl},G)} &:= (u|v)_{L^2(G)^3} + (\operatorname{curl} u|\operatorname{curl} v)_{L^2(G)^3} \,, \\
(u|v)_{H(\operatorname{div},G)} &:= (u|v)_{L^2(G)^3} + (\operatorname{div} u|\operatorname{div} v)_{L^2(G)} \,.
\end{aligned}$$

We also need the subspaces obtained by the completion of test functions with respect to the induced norms.

Definition 3.3. *Let $G \subseteq \mathbb{R}^3$ be open. We define the spaces*

$$H_0(\mathrm{curl}, G) := \overline{C_c^\infty(G)^3}^{\|\cdot\|_{H(\mathrm{curl},G)}}, \quad H_0(\mathrm{div}, G) := \overline{C_c^\infty(G)^3}^{\|\cdot\|_{H(\mathrm{div},G)}}.$$

Certain regularity properties of the boundary ∂G are required in the following. We specify these using Definition 1.2.1.1 of [13].

Definition 3.4. *Let $G \subseteq \mathbb{R}^3$ be open. We say that G has a Lipschitz boundary if for every $x \in \partial G$ there exists a neighbourhood V_x of x in $\mathbb{R}^3$, a new system of orthogonal coordinates (y_1, y_2, y_3) and a Lipschitz function ϕ_x with the following properties.*

1) The set V_x is a cube in the new coordinates,

$$V_x = \left\{ y \in \mathbb{R}^3 \,\middle|\, -a_i < y_i < a_i \text{ for all } i \in \{1,2,3\} \right\}.$$

2) The function ϕ_x is defined on $(-a_1, a_1) \times (-a_2, a_2)$ and maps into $[-a_3/2, a_3/2]$ such that

$$G \cap V_x = \left\{ y \in V_x \,\middle|\, y_3 < \phi_x(y_1, y_2) \right\},$$
$$\partial G \cap V_x = \left\{ y \in V_x \,\middle|\, y_3 = \phi_x(y_1, y_2) \right\}.$$

If ϕ_x above can be chosen in C^m (respectively $C^{m,1}$) for a positive integer m, then ∂G is said to be of class C^m (respectively $C^{m,1}$).

If G is an open subset of $\mathbb{R}^3$ with a Lipschitz boundary, the outer unit normal vector ν can be defined almost everywhere on ∂G due to Rademacher's theorem. The next theorem yields the existence of the required trace operators and states some useful properties, taken from Chapter I of [12] and Chapter IX of [7]. We use the notation

$$C_c^\infty(\overline{G}) := \left\{ \varphi|_{\overline{G}} \,\middle|\, \varphi \in C_c^\infty(\mathbb{R}^3) \right\}.$$

Theorem 3.5. *Let $G \subseteq \mathbb{R}^3$ be open with a compact Lipschitz boundary. Then $C_c^\infty(\overline{G})$ is dense in $H^1(G)$ and $C_c^\infty(\overline{G})^3$ is dense in $H(\mathrm{curl}, G)$ as well as in $H(\mathrm{div}, G)$. The traces*

$$\mathrm{tr}\, u = u|_{\partial G}, \quad u \in C_c^\infty(\overline{G}),$$
$$\mathrm{tr}_\tau\, u = u|_{\partial G} \times \nu, \quad u \in C_c^\infty(\overline{G})^3,$$
$$\mathrm{tr}_\nu\, u = u|_{\partial G} \cdot \nu, \quad u \in C_c^\infty(\overline{G})^3$$

can be extended to linear bounded operators (denoted by the same symbols)

$$\mathrm{tr} : H^1(G) \to H^{1/2}(\partial G),$$
$$\mathrm{tr}_\tau : H(\mathrm{curl}, G) \to H^{-1/2}(\partial G)^3,$$

$$\operatorname{tr}_\nu : H(\operatorname{div}, G) \to H^{-1/2}(\partial G)\,.$$

The operators tr *and* tr_ν *are surjective and the kernels of the traces are given by*

$$\mathrm{N}(\operatorname{tr}) = H_0^1(G)\,, \quad \mathrm{N}(\operatorname{tr}_\tau) = H_0(\operatorname{curl}, G)\,, \quad \mathrm{N}(\operatorname{tr}_\nu) = H_0(\operatorname{div}, G)\,.$$

Let $u \in H(\operatorname{div}, G), v \in H(\operatorname{curl}, G), \varphi \in H^1(G)$ *and* $\psi \in H^1(G)^3$*. The following Green's formulas hold*

$$\int_G u \cdot \nabla\varphi \,\mathrm{d}x = -\int_G \operatorname{div} u\, \varphi \,\mathrm{d}x + \langle \operatorname{tr}\varphi, \operatorname{tr}_\nu u\rangle_{H^{1/2}(\partial G)\times H^{-1/2}(\partial G)}\,, \tag{3.1}$$

$$\int_G v \cdot \operatorname{curl}\psi \,\mathrm{d}x = \int_G \operatorname{curl} v \cdot \psi \,\mathrm{d}x + \langle \operatorname{tr}\psi, \operatorname{tr}_\tau v\rangle_{H^{1/2}(\partial G)^3\times H^{-1/2}(\partial G)^3}\,. \tag{3.2}$$

In the last line, $\operatorname{tr}\psi$ *is understood componentwise.*

The next lemma collects some results which are helpful for the formulation of constraints imposed on the fields. It is mostly based on Remark 3.3 in [14] and the comments after Lemma 2.1 in [8].

Lemma 3.6. *Let* $G \subseteq \mathbb{R}^3$ *be open with a compact Lipschitz boundary and let* $\kappa \in W^{1,\infty}(G)$ *satisfy* $\kappa \geq \eta$ *for some constant* $\eta > 0$*.*

1) Let $m \in \{0,1\}$ *and* $\mathbf{F} \in H^m(G)^3$*. In the case* $m = 1$*, let* $\partial_i\kappa$ *additionally be contained in* $W^{1,3}(G)$ *for all* $i \in \{1,2,3\}$*. Then* $\operatorname{div}\mathbf{F} \in H^m(G)$ *is equivalent to* $\operatorname{div}(\kappa\mathbf{F}) \in H^m(G)$ *and we have the estimates*

$$\|\operatorname{div}\mathbf{F}\|_{H^m(G)} \lesssim_{\kappa,\eta,G} \|\mathbf{F}\|_{H^m(G)^3} + \|\operatorname{div}(\kappa\mathbf{F})\|_{H^m(G)}\,, \tag{3.3}$$

$$\|\operatorname{div}(\kappa\mathbf{F})\|_{H^m(G)} \lesssim_{\kappa,G} \|\mathbf{F}\|_{H^m(G)^3} + \|\operatorname{div}\mathbf{F}\|_{H^m(G)}\,. \tag{3.4}$$

2) Let $m \in \{0,1\}$ *and* $\mathbf{F} \in H^m(G)^3$*. In the case* $m = 1$*, let* $\partial_i\kappa$ *additionally be contained in* $W^{1,3}(G)$ *for all* $i \in \{1,2,3\}$*. Then* $\operatorname{curl}\mathbf{F} \in H^m(G)$ *is equivalent to* $\operatorname{curl}(\kappa\mathbf{F}) \in H^m(G)$ *and we have the estimates*

$$\|\operatorname{curl}\mathbf{F}\|_{H^m(G)^3} \lesssim_{\kappa,\eta,G} \|\mathbf{F}\|_{H^m(G)^3} + \|\operatorname{curl}(\kappa\mathbf{F})\|_{H^m(G)^3}\,, \tag{3.5}$$

$$\|\operatorname{curl}(\kappa\mathbf{F})\|_{H^m(G)^3} \lesssim_{\kappa,G} \|\mathbf{F}\|_{H^m(G)^3} + \|\operatorname{curl}\mathbf{F}\|_{H^m(G)^3}\,. \tag{3.6}$$

3) Let $\mathbf{F} \in H(\operatorname{curl}, G)$*. Then* $\operatorname{tr}_\tau(\kappa\mathbf{F}) = 0$ *is equivalent to* $\operatorname{tr}_\tau(\mathbf{F}) = 0$*.*

4) Let $\mathbf{F} \in H(\operatorname{div}, G)$*. Then* $\operatorname{tr}_\nu(\kappa\mathbf{F}) = 0$ *is equivalent to* $\operatorname{tr}_\nu(\mathbf{F}) = 0$*.*

Proof. The assumptions on κ imply that κ^{-1} possesses analogous properties, namely $\kappa^{-1} \in W^{1,\infty}(G)$ and $\kappa^{-1} \geq \|\kappa\|_\infty^{-1} > 0$, see Lemma 3.9.

1) We show (3.3). Let $\mathbf{F} \in L^2(G)^3$ with $\operatorname{div}(\kappa\mathbf{F}) \in L^2(G)$. We compute

$$\operatorname{div}\mathbf{F} = \operatorname{div}\left(\kappa^{-1}\kappa\mathbf{F}\right) = \nabla\left(\kappa^{-1}\right)\cdot(\kappa\mathbf{F}) + \kappa^{-1}\operatorname{div}(\kappa\mathbf{F}) = -\kappa^{-1}\nabla\kappa\cdot\mathbf{F} + \kappa^{-1}\operatorname{div}(\kappa\mathbf{F}) \tag{3.7}$$

in $H^{-1}(G)$. Hence, $\operatorname{div} \mathbf{F}$ belongs to $\in L^2(G)$ and satisfies

$$\begin{aligned}\|\operatorname{div} \mathbf{F}\|_{L^2(G)} &\leq \left\|\kappa^{-1}\right\|_{L^\infty(G)} \|\kappa\|_{W^{1,\infty}(G)} \|\mathbf{F}\|_{L^2(G)^3} + \left\|\kappa^{-1}\right\|_{L^\infty(G)} \|\operatorname{div}(\kappa\mathbf{F})\|_{L^2(G)} \\ &\lesssim_{\kappa,\eta} \|\mathbf{F}\|_{L^2(G)^3} + \|\operatorname{div}(\kappa\mathbf{F})\|_{L^2(G)} \,.\end{aligned}$$

Now we consider $m = 1$. Let $\mathbf{F} \in H^1(G)^3$ with $\operatorname{div}(\kappa\mathbf{F}) \in H^1(G)$. Then we have (3.7) in $L^2(G)$. Since $\kappa^{-1} \in W^{1,\infty}(G)$, the second term is contained in $H^1(G)$. Let $j, m \in \{1, 2, 3\}$. We note that $\xi_j := \kappa^{-1}\partial_j\kappa$ is contained in $W^{1,3}(G) \cap L^\infty(G)$ and therefore $\xi_j F_j \in L^2(G)$. By Theorem 3.5, there exists a sequence (u_k) in $C_c^\infty(\overline{G})$ converging to F_j in $H^1(G)$. Let $\varphi \in C_c^\infty(G)$. Using the embedding $H^1(G) \hookrightarrow L^6(G)$, we obtain

$$\begin{aligned}\int_G \xi_j F_j \partial_m \varphi \,\mathrm{d}x &= \lim_{k\to\infty} \int_G \xi_j u_k \partial_m \varphi \,\mathrm{d}x = -\lim_{k\to\infty} \int_G \left(\xi_j \partial_m u_k + \partial_m \xi_j u_k\right) \varphi \,\mathrm{d}x \\ &= -\int_G \left(\xi_j \partial_m F_j + \partial_m \xi_j F_j\right) \varphi \,\mathrm{d}x \,.\end{aligned}$$

Since $\xi_j \partial_m F_j + \partial_m \xi_j F_j \in L^2(G)$, we conclude $\kappa^{-1}\nabla\kappa \cdot \mathbf{F} \in H^1(G)$ with

$$\partial_m \left(\kappa^{-1}\nabla\kappa \cdot \mathbf{F}\right) = \sum_{j=1}^{3} \left(\xi_j \partial_m F_j + \partial_m \xi_j F_j\right) \,.$$

The estimate (3.3) now follows from (3.7), where the constant depends on G since the constant in the embedding $H^1(G) \hookrightarrow L^6(G)$ depends on the domain. The other half of claim 1) can be shown by an analogous computation using

$$\operatorname{div}(\kappa\mathbf{F}) = \nabla\kappa \cdot \mathbf{F} + \kappa \operatorname{div} \mathbf{F} \,.$$

In this case, κ^{-1} is not needed and the estimate (3.4) does not depend on η.

2) The proof is analogous to that of 1), using

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \operatorname{curl}\left(\kappa^{-1}\kappa\mathbf{F}\right) = \nabla\left(\kappa^{-1}\right) \times (\kappa\mathbf{F}) + \kappa^{-1} \operatorname{curl}(\kappa\mathbf{F}) \\ &= -\kappa^{-1}\nabla\kappa \times \mathbf{F} + \kappa^{-1} \operatorname{curl}(\kappa\mathbf{F}) \,, \\ \operatorname{curl}(\kappa\mathbf{F}) &= \nabla\kappa \times \mathbf{F} + \kappa \operatorname{curl} \mathbf{F} \,.\end{aligned}$$

3) By 2), both $\kappa\mathbf{F}$ and $\mathbf{F}$ have a tangential trace on ∂G. Let $\varphi \in H^1(G)^3$. The claim follows from $\operatorname{tr}\big(H^1(G)^3\big) = H^{1/2}(\partial G)^3$, the fact that multiplication by κ is an isomorphism on $H^1(G)^3$ and the calculation

$$\begin{aligned}\langle \operatorname{tr}\varphi, \operatorname{tr}_\tau(\kappa\mathbf{F})\rangle_{H^{1/2}(\partial G)^3 \times H^{-1/2}(\partial G)^3} &= \int_G \big(\kappa\mathbf{F} \cdot \operatorname{curl}\varphi - \operatorname{curl}(\kappa\mathbf{F}) \cdot \varphi\big) \,\mathrm{d}x \\ &= \int_G \big(\kappa\mathbf{F} \cdot \operatorname{curl}\varphi - (\nabla\kappa \times \mathbf{F}) \cdot \varphi - \kappa\varphi \cdot \operatorname{curl}\mathbf{F}\big) \,\mathrm{d}x \\ &= \int_G \big(\mathbf{F} \cdot \operatorname{curl}(\kappa\varphi) - \kappa\varphi \cdot \operatorname{curl}\mathbf{F}\big) \,\mathrm{d}x = \langle \operatorname{tr}(\kappa\varphi), \operatorname{tr}_\tau \mathbf{F}\rangle_{H^{1/2}(\partial G)^3 \times H^{-1/2}(\partial G)^3} \,,\end{aligned} \tag{3.8}$$

using (3.2).

4) This is proven analogously to 3), using (3.1) instead of (3.2), see Remark 3.3 of [14]. □

3.2 Maxwell equations

Formulated using trace operators, the perfectly conduction boundary conditions (1.6) take the form

$$\operatorname{tr}_\tau \mathbf{E} = 0\,, \quad \operatorname{tr}_\nu \mathbf{B} = 0\,.$$

The next lemma is a slight modification of a result found in [26] (see also Lemma 7.25 in [25]). It states that under certain regularity conditions the two divergence equations in (1.2) and the boundary condition for the magnetic induction in (1.6) are redundant in the sense that if they are fulfilled at the initial time, then they also hold for later times.

Lemma 3.7. *Let G be open with a compact Lipschitz boundary and $b > 0$. Let $\mathbf{E} \in C\big([0,b], H(\operatorname{curl}, G)\big), \mathbf{B} \in C^1\big([0,b], H(\operatorname{div}, G)\big)$, $\mathbf{H} \in C\big([0,b], H(\operatorname{curl}, G)\big)$, $\mathbf{D} \in C\big([0,b], H(\operatorname{div}, G)\big) \cap C^1\big([0,b], L^2(G)^3\big), \rho \in C^1\big(([0,b], H^{-1}(G)\big)$ and $\mathbf{J} \in C\big(([0,b], L^2(G)^3\big)$ be solutions of*

$$\partial_t \mathbf{D} = \operatorname{curl} \mathbf{H} - \mathbf{J}\,, \quad \partial_t \mathbf{B} = -\operatorname{curl} \mathbf{E}\,, \quad \partial_t \rho + \operatorname{div} \mathbf{J} = 0\,, \quad \text{in } G\,,\ t \in [0,b]\,. \tag{3.9}$$

Then the following statements hold.

1) If $\operatorname{div} \mathbf{D}(0) = \rho(0)$, then $\operatorname{div} \mathbf{D}(t) = \rho(t)$ for all $t \in [0,b]$.

2) If $\operatorname{div} \mathbf{B}(0) = 0$, then $\operatorname{div} \mathbf{B}(t) = 0$ for all $t \in [0,b]$.

3) If $\operatorname{tr}_\tau \mathbf{E}(t) = 0$ for all $t \in [0,b]$ and $\operatorname{tr}_\nu \mathbf{B}(0) = 0$, then $\operatorname{tr}_\nu \mathbf{B}(t) = 0$ for all $t \in [0,b]$.

Proof. 1) Let $t \in [0,b], h \in \mathbb{R} \setminus \{0\}$ such that $t+h \in [0,b]$ and $\varphi \in H_0^1(G)$. The calculation

$$\begin{aligned}
&\langle \varphi, \partial_t \operatorname{div} \mathbf{D}(t)\rangle_{H_0^1(G)\times H^{-1}(G)} = \lim_{h\to 0} \frac{1}{h} \langle \varphi, \operatorname{div} \mathbf{D}(t+h) - \operatorname{div} \mathbf{D}(t)\rangle_{H_0^1(G)\times H^{-1}(G)} \\
&\quad = -\lim_{h\to 0} \frac{1}{h} \int_G \nabla\varphi \cdot \big(\mathbf{D}(t+h) - \mathbf{D}(t)\big)\,\mathrm{d}x = -\int_G \nabla\varphi \cdot \partial_t \mathbf{D}(t)\,\mathrm{d}x \\
&\quad = \langle \varphi, \operatorname{div} \partial_t \mathbf{D}(t)\rangle_{H_0^1(G)\times H^{-1}(G)}
\end{aligned}$$

shows $\partial_t \operatorname{div} \mathbf{D}(t) = \operatorname{div} \partial_t \mathbf{D}(t)$ in $H^{-1}(G)$ for all $t \in [0,b]$. Inserting (3.9) yields

$$\partial_t \operatorname{div} \mathbf{D}(t) = \operatorname{div}\big(\operatorname{curl} \mathbf{H}(t) - \mathbf{J}(t)\big) = -\operatorname{div} \mathbf{J}(t) = \partial_t \rho(t)$$

for all $t \in [0,b]$, where we have used $\operatorname{div}\operatorname{curl} \mathbf{H}(t) = 0$ in $H^{-1}(G)$. Therefore we conclude

$$\operatorname{div} \mathbf{D}(t) = \rho(0) + \int_0^t \partial_t \rho(s)\,\mathrm{d}s = \rho(t)$$

for all $t \in [0,b]$.

2) We obtain analogously

$$\partial_t \operatorname{div} \mathbf{B}(t) = \operatorname{div} \partial_t \mathbf{B}(t) = -\operatorname{div}\operatorname{curl} \mathbf{E}(t) = 0$$

which implies $\operatorname{div}\mathbf{B}(t) = \operatorname{div}\mathbf{B}(0) = 0$ for all $t \in [0,b]$.

3) Let $t \in [0,b], h \in \mathbb{R}\setminus\{0\}$ such that $t+h \in [0,b]$ and $\psi \in C_c^\infty(\overline{G})$. We use (3.1) in the calculation

$$\begin{aligned}
&\langle \operatorname{tr}\psi, \partial_t \operatorname{tr}_\nu \mathbf{B}(t)\rangle_{H^{1/2}(\partial G)\times H^{-1/2}(\partial G)} \\
&= \lim_{h\to 0}\frac{1}{h}\big\langle \operatorname{tr}\psi, \operatorname{tr}_\nu\big(\mathbf{B}(t+h)-\mathbf{B}(t)\big)\big\rangle_{H^{1/2}(\partial G)\times H^{-1/2}(\partial G)} \\
&\quad = \lim_{h\to 0}\frac{1}{h}\left(\int_G \nabla\psi\cdot\big(\mathbf{B}(t+h)-\mathbf{B}(t)\big)\,\mathrm{d}x + \int_G \psi\cdot\operatorname{div}\big(\mathbf{B}(t+h)-\mathbf{B}(t)\big)\,\mathrm{d}x\right) \\
&\quad = \int_G \nabla\psi\cdot\partial_t\mathbf{B}(t)\,\mathrm{d}x\,,
\end{aligned}$$

where we have used that the divergence of $\mathbf{B}$ is constant by step 2). Equations (3.9) and using (3.2) yield

$$\begin{aligned}
\langle \operatorname{tr}\psi, \partial_t \operatorname{tr}_\nu \mathbf{B}(t)\rangle_{H^{1/2}(\partial G)\times H^{-1/2}(\partial G)} &= -\int_G \nabla\psi\cdot\operatorname{curl}\mathbf{E}(t)\,\mathrm{d}x \\
&= -\int_G \operatorname{curl}\nabla\psi\cdot\mathbf{E}(t)\,\mathrm{d}x + \langle \operatorname{tr}\nabla\psi, \operatorname{tr}_\tau \mathbf{E}(t)\rangle_{H^{1/2}(\partial G)^3\times H^{-1/2}(\partial G)^3}\,.
\end{aligned}$$

Since $\operatorname{curl}\nabla\psi = 0$ and the tangential trace of $\mathbf{E}(t)$ vanishes by assumption, we get

$$\langle \operatorname{tr}\psi, \partial_t \operatorname{tr}_\nu \mathbf{B}(t)\rangle_{H^{1/2}(\partial G)\times H^{-1/2}(\partial G)} = 0\,.$$

The density of $C_c^\infty(\overline{G})$ in $H^1(G)$ and the surjectivity of tr from $H^1(G)$ to $H^{1/2}(\partial G)$ now imply $\partial_t \operatorname{tr}_\nu \mathbf{B}(t) = 0$ for all $t \in [0,b]$ which together with the initial condition for $\operatorname{tr}_\nu \mathbf{B}$ finishes the proof. □

The above lemma suggests to study the reduced system

$$\begin{aligned}
\partial_t\mathbf{D} = \operatorname{curl}\mathbf{H} - \mathbf{J}\,,\quad \partial_t\mathbf{B} = -\operatorname{curl}\mathbf{E}\,,\quad &\text{in } G\,,\, t\geq 0 \\
\operatorname{tr}_\tau\mathbf{E} = 0\,,\quad t\geq 0\,.&
\end{aligned} \tag{3.10}$$

We do not take the continuity equation (1.3) explicitly into account. The free charge density (which does not appear in the above system) can be obtained from an initial value and the free current density by

$$\rho(t) = \rho(0) - \int_0^t \operatorname{div}\mathbf{J}(s)\,\mathrm{d}s\,,\quad t\geq 0\,.$$

It is still needed to specify to the relationship between $\mathbf{P},\mathbf{M}$ and $\mathbf{E},\mathbf{H}$. We mostly neglect magnetic effects and assume a linear, isotropic and instantaneous relationship of the form

$$\mathbf{M} = \chi_{\mathrm{m}}\mathbf{H}$$

between the magnetisation and the magnetic field characterised by a scalar-valued magnetic susceptibility χ_{m} which can depend on the spatial variable x. The polarisation in our model consists of an analogous term with an additional contribution $\widetilde{\mathbf{P}}$ that is nonlinear and noninstantaneous, i. e.

$$\mathbf{P} = \chi_{\mathrm{e}}\mathbf{E} + \widetilde{\mathbf{P}}(\mathbf{E}) \,,$$

where χ_{e} is scalar-valued, can depend on x and is called the electric susceptibility. The model used for $\widetilde{\mathbf{P}}$ is specified and studied in Subsection 3.4. It requires the field $\mathbf{E}$ to have at least H^2-regularity. Defining the permeability $\mu := 1 + \chi_{\mathrm{m}}$ and the permittivity $\varepsilon := 1 + \chi_{\mathrm{e}}$, we can write

$$\mathbf{B} = \mu\mathbf{H} \,, \quad \mathbf{D} = \varepsilon\mathbf{E} + \widetilde{\mathbf{P}}(\mathbf{E}) \,.$$

We choose $u = (\mathbf{E}, \mathbf{H})$ to describe the state of the fields. Using $\mathbf{J} = \mathbf{J}_0 + \sigma\mathbf{E}$ and assuming that ε and μ have no zeros in G, the system (3.10) takes the form

$$\begin{aligned} \partial_t u &= \begin{pmatrix} -\sigma\varepsilon^{-1} I & \varepsilon^{-1}\operatorname{curl} \\ -\mu^{-1}\operatorname{curl} & 0 \end{pmatrix} u - \varepsilon^{-1}\begin{pmatrix} \partial_t\widetilde{\mathbf{P}} + \mathbf{J}_0 \\ 0 \end{pmatrix} , \quad t \geq 0 \,, \\ \operatorname{tr}_\tau \mathbf{E} &= 0 \,, \quad t \geq 0 \,. \end{aligned} \tag{3.11}$$

As a first step towards interpreting this equation in the abstract setting of (2.1), we want to identify the linear operator appearing in (3.11) as the generator of a strongly continuous semigroup on an appropriate subspace of $H^2(G)^6$. This is the subject of the next subsection.

3.3 Maxwell semigroup

In order to specify the required properties of the material parameters ε, μ and σ, we introduce for an open set $G \subseteq \mathbb{R}^3$ the space

$$Z(G) := \left\{ \alpha \in W^{1,\infty}(G) \;\middle|\; \partial_i\alpha \in W^{1,3}(G) \text{ for all } i \in \{1,2,3\} \right\}$$

and equip it with the norm

$$\|\alpha\|_{Z(G)} := \|\alpha\|_{W^{1,\infty}(G)} + \sum_{i=1}^{3} \|\partial_i\alpha\|_{W^{1,3}(G)} \,.$$

Lemma 3.8. *Let $G \subseteq \mathbb{R}^3$ be open. The space $Z(G)$ is a Banach space.*

Proof. Let (α_k) be a Cauchy sequence in $Z(G)$. Then it is a Cauchy sequence in $W^{1,\infty}(G)$ and therefore converges to some α in $W^{1,\infty}(G)$. Further, for any $i \in \{1,2,3\}$, $(\partial_i\alpha_k)$ is a Cauchy sequence in $W^{1,3}(G)$ and thus converges to a β_i in the space $W^{1,3}(G)$. Let $\varphi \in C_c^\infty(G)$. The calculation

$$\int_G \partial_i\alpha\varphi \,\mathrm{d}x = \lim_{k\to\infty} \int_G \partial_i\alpha_k\varphi \,\mathrm{d}x = \int_G \beta_i\varphi \,\mathrm{d}x$$

shows $\partial_i\alpha = \beta_i \in W^{1,3}(G)$ which implies $\alpha \in Z(G)$. ☐

The next lemma states that under an additional assumption on the boundary ∂G, the space $Z(G)$ is a Banach algebra if $\|\cdot\|_{Z(G)}$ is replaced by an equivalent norm $c\|\cdot\|_{Z(G)}$ with a suitable $c > 0$. Furthermore, if $\alpha \in Z(G)$ is bounded from below by a positive constant, then α^{-1} is also contained in $Z(G)$.

Lemma 3.9. *Let $G \subseteq \mathbb{R}^3$ be open with a compact boundary of class C^1 and let $\alpha, \beta \in Z(G)$. Then $\alpha\beta \in Z(G)$ and we have $\|\alpha\beta\|_{Z(G)} \lesssim \|\alpha\|_{Z(G)} \|\beta\|_{Z(G)}$. If $\alpha \geq \eta$ for some positive constant η, then $\alpha^{-1} \in Z(G)$ with*

$$\left\|\alpha^{-1}\right\|_{Z(G)} \lesssim \eta^{-1} + \eta^{-2} \|\alpha\|_{Z(G)} + \eta^{-3} \|\alpha\|_{Z(G)}^2 . \tag{3.12}$$

Let $m \in \{0, 1, 2\}$ and $u \in H^m(G)$. Then αu is contained in $H^m(G)$ and we have $\|\alpha u\|_{H^m(G)} \lesssim_G \|\alpha\|_{Z(G)} \|u\|_{H^m(G)}$, where for $m \in \{0, 1\}$, the constant is independent of G.

Proof. The assumptions on G imply that we can identify $W^{1,\infty}(G)$ with the space of bounded Lipschitz continuous functions on G.

1) Let $\alpha, \beta \in Z(G)$. Then $\alpha\beta \in W^{1,\infty}(G)$ since the product of two bounded Lipschitz continuous functions on G is again a bounded Lipschitz continuous function. By Rademacher's theorem, $\alpha\beta$ is differentiable almost everywhere on G and the product rule yields $\partial_i(\alpha\beta) = \partial_i\alpha\beta + \alpha\partial_i\beta \in W^{1,3}(G)$ with

$$\partial_j\partial_i(\alpha\beta) = \partial_j\partial_i\alpha\beta + \partial_i\alpha\partial_j\beta + \partial_j\alpha\partial_i\beta + \alpha\partial_j\partial_i\beta$$

for all $i, j \in \{1, 2, 3\}$. Therefore $\alpha\beta$ is contained in $Z(G)$ and we have the estimates

$$\begin{aligned}
\|\alpha\beta\|_{W^{1,\infty}(G)} &= \max\left\{\|\alpha\beta\|_{L^\infty(G)}, \|\partial_1(\alpha\beta)\|_{L^\infty(G)}, \|\partial_2(\alpha\beta)\|_{L^\infty(G)}, \|\partial_3(\alpha\beta)\|_{L^\infty(G)}\right\} \\
&\lesssim \|\alpha\|_{Z(G)} \|\beta\|_{Z(G)} , \\
\|\partial_i(\alpha\beta)\|_{W^{1,3}(G)}^3 &= \|\partial_i(\alpha\beta)\|_{L^3(G)}^3 + \sum_{j=1}^{3} \|\partial_j\partial_i(\alpha\beta)\|_{L^3(G)}^3 \lesssim \|\alpha\|_{Z(G)}^3 \|\beta\|_{Z(G)}^3 ,
\end{aligned}$$

which imply $\|\alpha\beta\|_{Z(G)} \lesssim \|\alpha\|_{Z(G)} \|\beta\|_{Z(G)}$.

2) Now let $\alpha \in Z(G)$ with $\alpha \geq \eta > 0$. We choose functions $f, g \in C^1(\mathbb{R})$ with bounded derivatives and $f(t) = t^{-1}, g(t) = t^{-2}$ for all $t \geq \eta$. Then $\alpha^{-1} = f(\alpha), \alpha^{-2} = g(\alpha)$ and the chain rule yields $\alpha^{-1}, \alpha^{-2} \in W^{1,\infty}(G)$ with

$$\partial_i\left(\alpha^{-1}\right) = f'(\alpha)\partial_i\alpha = -\alpha^{-2}\partial_i\alpha , \quad \partial_i\left(\alpha^{-2}\right) = g'(\alpha)\partial_i\alpha = -2\alpha^{-3}\partial_i\alpha$$

for all $i \in \{1, 2, 3\}$. By the product rule, $\partial_i\left(\alpha^{-1}\right) \in W^{1,3}(G)$ with

$$\partial_j\partial_i\left(\alpha^{-1}\right) = 2\alpha^{-3}\partial_j\alpha\partial_i\alpha - \alpha^{-2}\partial_j\partial_i\alpha .$$

So we obtain $\alpha^{-1} \in Z(G)$ and the estimate (3.12).

3) The last assertion follows from Hölder's inequality in the case $m = 0$, while for $m \in \{1, 2\}$ we use the product rule and in the case $m = 2$ also the embedding $H^1(G) \hookrightarrow L^6(G)$ as in the proof of Lemma 3.6. □

The following assumptions are used throughout the rest of this work.

Assumption 3.10. *The set $G \subseteq \mathbb{R}^3$ is a bounded domain with ∂G of class $C^{2,1}$. There exists a constant $\eta > 0$ such that the conductivity, permittivity and permeability fulfill*

$$\sigma, \varepsilon, \mu \in Z(G)\,, \quad \sigma \geq 0\,, \quad \varepsilon, \mu \geq \eta\,.$$

The assumptions imply that the domain G fulfills the cone condition and therefore the estimate $\|fg\|_{H^2(G)} \lesssim_G \|f\|_{H^2(G)} \|g\|_{H^2(G)}$ holds for all $f, g \in H^2(G)$ by Theorem 4.39 in [1]. (When equipped with an equivalent norm, $H^2(G)$ is a Banach algebra.)

We now define the space $X := L^2(G)^6$, equipped with the inner product

$$\Big((\mathbf{E},\mathbf{H})\Big|(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}})\Big)_X := \Big(\varepsilon\mathbf{E}\Big|\widetilde{\mathbf{E}}\Big)_{L^2(G)^3} + \Big(\mu\mathbf{H}\Big|\widetilde{\mathbf{H}}\Big)_{L^2(G)^3}$$

and the associated norm

$$\|(\mathbf{E},\mathbf{H})\|_X := \Big(\big\|\varepsilon^{1/2}\mathbf{E}\big\|^2_{L^2(G)^3} + \big\|\mu^{1/2}\mathbf{H}\big\|^2_{L^2(G)^3}\Big)^{1/2}\,.$$

Due to the properties of ε and μ, X is a Hilbert space with a norm that is equivalent to the usual L^2-norm, since

$$\|u\|_{L^2(G)^6} \leq \eta^{-1/2}\,\|u\|_X \leq \eta^{-1/2}\Big(\|\varepsilon\|_{L^\infty(G)} + \|\mu\|_{L^\infty(G)}\Big)^{1/2}\,\|u\|_{L^2(G)^6} \tag{3.13}$$

for all $u \in X$. On X, we define the linear operator

$$A := \begin{pmatrix} -\sigma\varepsilon^{-1}I & \varepsilon^{-1}\operatorname{curl} \\ -\mu^{-1}\operatorname{curl} & 0 \end{pmatrix} \tag{3.14}$$

with domain $D(A) := H_0(\operatorname{curl}, G) \times H(\operatorname{curl}, G)$. The boundary condition for the electric field is thus incorporated into $D(A)$. As in [8], the subspace

$$\begin{aligned} X_{\operatorname{div}} :&= \big\{(\mathbf{E},\mathbf{H}) \in X \;\big|\; \operatorname{div}(\mu\mathbf{H}) = 0,\, \operatorname{tr}_\nu(\mu\mathbf{H}) = 0,\, \operatorname{div}(\varepsilon\mathbf{E}) \in L^2(G)\big\} \\ &= \big\{(\mathbf{E},\mathbf{H}) \in X \;\big|\; \operatorname{div}(\mu\mathbf{H}) = 0,\, \operatorname{tr}_\nu(\mathbf{H}) = 0,\, \operatorname{div}(\mathbf{E}) \in L^2(G)\big\} \end{aligned} \tag{3.15}$$

is defined, which also takes into account the divergence condition and the boundary condition for the magnetic induction, as well as a regularity condition for $\varepsilon\mathbf{E}$. Here, $\operatorname{div}(\mu\mathbf{H}) = 0$ and $\operatorname{tr}_\nu(\mu\mathbf{H}) = 0$ are understood in $H^{-1}(G)$, respectively $H^{-1/2}(\partial G)$. The last line in (3.15) follows from Lemma 3.6. On X_{div}, we define the inner product

$$\Big((\mathbf{E},\mathbf{H})\Big|(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}})\Big)_{X_{\operatorname{div}}} := \Big((\mathbf{E},\mathbf{H})\Big|(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}})\Big)_{L^2(G)^6} + \Big(\operatorname{div}(\varepsilon\mathbf{E})\Big|\operatorname{div}(\varepsilon\widetilde{\mathbf{E}})\Big)_{L^2(G)}$$

and the corresponding norm

$$\|(\mathbf{E},\mathbf{H})\|_{X_{\operatorname{div}}} := \Big(\|(\mathbf{E},\mathbf{H})\|^2_{L^2(G)^6} + \|\operatorname{div}(\varepsilon\mathbf{E})\|^2_{L^2(G)}\Big)^{1/2}\,.$$

In [8], the X-norm is used instead of the $L^2(G)^6$-norm in the first term on the right hand side, which leads to an equivalent norm (with constants depending on ε, μ). As stated in [8], X_{div} is a Hilbert space (in [8], G is a cuboid, but the proof also works for domains satisfying Assumption 3.10).

We further define the operator $A_{\text{div}} : D(A_{\text{div}}) \subseteq X_{\text{div}} \to X_{\text{div}}$ as the part of A in X_{div},

$$D(A_{\text{div}}) = \left\{ u \in D(A) \cap X_{\text{div}} \;\middle|\; Au \in X_{\text{div}} \right\}, \quad A_{\text{div}} u = Au$$

for all $u \in D(A_{\text{div}})$. As shown in [8], the identity

$$D(A_{\text{div}}^k) = D(A^k) \cap X_{\text{div}}$$

holds for all $k \in \mathbb{N}$ (again, the proof in [8] for cuboids transfers to our case).

It is known that the Maxwell operator A generates a strongly continuous semigroup $T(\cdot)$ on X: By Lemma 2.2 in [4], $\mathrm{i}A$ is self-adjoint in the case $\sigma = 0$. An application of Stone's theorem and a perturbation argument for the case $\sigma \neq 0$ then show that A generates a contraction semigroup on X, see Proposition 3.5 of [14]. Here we temporarily work in the space $L^2(G, \mathbb{C})^6$ of complex-valued functions. Since the coefficients of A are real-valued, restricting the semigroup to real-valued functions yields a semigroup on $L^2(G)^6$.

It turns out a restriction $T(\cdot)$ to X_{div} is a strongly continuous semigroup on X_{div}, generated by A_{div}. The following result is proved in Proposition 2.3 of [8].

Proposition 3.11. *Let Assumption 3.10 be true. Then the operators A and A_{div} generate C_0–semigroups $T(\cdot)$ on X, respectively $T_{\text{div}}(\cdot)$ on X_{div}. Moreover, $T_{\text{div}}(\cdot)$ is the restriction of $T(\cdot)$ to X_{div}, and for all $t \geq 0$ we have the estimates*

$$\|T(t)\|_{\mathcal{B}(X)} \leq 1, \quad \|T(t)\|_{\mathcal{B}(L^2(G))} \leq c_{\varepsilon,\mu,\eta},$$
$$\|T_{\text{div}}(t)\|_{\mathcal{B}(X_{\text{div}})} \lesssim 1 + c_{\varepsilon,\mu,\eta} + c_{\varepsilon,\mu,\eta} \|\varepsilon\|_{L^\infty(G)} \left\|\sigma\varepsilon^{-1}\right\|_{W^{1,\infty}(G)} t \lesssim_{\varepsilon,\mu,\sigma,\eta} 1 + t$$

with $c_{\varepsilon,\mu,\eta} := \eta^{-1/2} \left(\|\varepsilon\|_{L^\infty(G)} + \|\mu\|_{L^\infty(G)} \right)^{1/2}$.

Proof. We only supplement the proof in [8] by explicitly writing out the dependence of the constants on ε, μ and σ. The estimates for $T(\cdot)$ follow from (3.13) and the fact that $T(\cdot)$ is a contraction on X. For the remaining estimate, let $u_0 = (\mathbf{E}_0, \mathbf{H}_0) \in X_{\text{div}}$ and set $u(t) = T_{\text{div}}(t) u_0 = \big(\mathbf{E}(t), \mathbf{H}(t)\big)$. According to equation (2.7) in [8], we have the identity

$$\operatorname{div}\big(\varepsilon \mathbf{E}(t)\big) = \mathrm{e}^{-\kappa t} \operatorname{div}(\varepsilon \mathbf{E}_0) - \int_0^t \mathrm{e}^{-\kappa t} \varepsilon \nabla\kappa \cdot \mathbf{E}(s) \, \mathrm{d}s,$$

with $\kappa = \sigma\varepsilon^{-1}$, which yields

$$\begin{aligned} \left\|\operatorname{div}\big(\varepsilon \mathbf{E}(t)\big)\right\|_{L^2(G)} &\leq \|\operatorname{div}(\varepsilon \mathbf{E}_0)\|_{L^2(G)} + t \|\varepsilon \nabla\kappa\|_{L^\infty(G)^3} \sup_{0 \leq s \leq t} \|\mathbf{E}(s)\|_{L^2(G)^3} \\ &\lesssim \|\operatorname{div}(\varepsilon \mathbf{E}_0)\|_{L^2(G)} + t \|\varepsilon\|_{L^\infty(G)} \|\kappa\|_{W^{1,\infty}(G)} c_{\varepsilon,\mu,\eta} \|u_0\|_{L^2(G)^6}. \end{aligned} \tag{3.16}$$

Since

$$\|u(t)\|_{X_{\rm div}} \leq \|u(t)\|_{L^2(G)^6} + \left\|\operatorname{div}\big(\varepsilon \mathbf{E}(t)\big)\right\|_{L^2(G)} \leq c_{\varepsilon,\mu,\eta} \|u_0\|_{L^2(G)^6} + \left\|\operatorname{div}\big(\varepsilon \mathbf{E}(t)\big)\right\|_{L^2(G)} ,$$

the estimate for $T_{\rm div}(\cdot)$ is true. □

We now show a regularity result needed below. For $k \in \mathbb{N}$, we denote the space $D(A_{\rm div}^k)$ endowed with the graph norm $\|u\|_{A_{\rm div}^k} := \|u\|_{X_{\rm div}} + \left\|A_{\rm div}^k u\right\|_{X_{\rm div}}$ by $[D(A_{\rm div}^k)]$.

Lemma 3.12. *Let Assumption 3.10 be true. The space $[D(A_{\rm div})]$ is continuously embedded into $H^1(G)^6$ with $\|u\|_{H^1(G)^6} \lesssim_G d_{\varepsilon,\mu,\sigma,\eta} \|u\|_{A_{\rm div}}$ for all $u \in D(A_{\rm div})$, where*

$$d_{\varepsilon,\mu,\sigma,\eta} := \left(1+\eta^{-1}\right)\left(1 + \|\varepsilon\|_{W^{1,\infty}(G)} + \|\mu\|_{W^{1,\infty}(G)} + \|\sigma\|_{L^\infty(G)}\right) .$$

Proof. Let $u \equiv (\mathbf{E},\mathbf{H}) \in D(A_{\rm div})$. Since $\mathbf{E} \in H_0(\operatorname{curl}, G)$, we have $\operatorname{tr}_\tau \mathbf{E} = 0$ by Theorem 3.5. From (3.15) and Lemma 3.6 follows $\operatorname{tr}_\nu \mathbf{H} = 0$ as well as $\operatorname{div} \mathbf{E} \in L^2(G)$ and $\operatorname{div} \mathbf{H} \in L^2(G)$. We further have $(\mathbf{K},\mathbf{L}) := Au \in L^2(G)^6$ which implies $\operatorname{curl} \mathbf{E} = -\mu \mathbf{L} \in L^2(G)^3$ and $\operatorname{curl} \mathbf{H} = \varepsilon \mathbf{K} + \sigma \mathbf{E} \in L^2(G)^3$. Theorem IX.1.3 in [7] now yields $u \in H^1(G)^6$ and

$$\|u\|_{H^1(G)^6}^2 \simeq_G \|u\|_{L^2(G)^6}^2 + \|\operatorname{curl} \mathbf{E}\|_{L^2(G)^3}^2 + \|\operatorname{curl} \mathbf{H}\|_{L^2(G)^3}^2 + \|\operatorname{div} \mathbf{E}\|_{L^2(G)^3}^2 + \|\operatorname{div} \mathbf{H}\|_{L^2(G)^3}^2 .$$

Due to $\operatorname{div}(\mu \mathbf{H}) = 0$, the estimate

$$\|\operatorname{div} \mathbf{H}\|_{L^2(G)} = \left\|\nabla\left(\mu^{-1}\right)\cdot(\mu\mathbf{H})\right\|_{L^2(G)} = \left\|\mu^{-1}\nabla\mu\cdot\mathbf{H}\right\|_{L^2(G)} \leq \eta^{-1}\|\mu\|_{W^{1,\infty}(G)}\|\mathbf{H}\|_{L^2(G)^3}$$

holds. We further have

$$\begin{aligned}
\|\operatorname{curl} \mathbf{E}\|_{L^2(G)^3} &\leq \|\mu\|_{L^\infty(G)} \left\|\mu^{-1}\operatorname{curl} \mathbf{E}\right\|_{L^2(G)^3} , \\
\|\operatorname{curl} \mathbf{H}\|_{L^2(G)^3} &\leq \|\varepsilon\|_{L^\infty(G)} \left\|\varepsilon^{-1}\operatorname{curl} \mathbf{H} - \sigma\varepsilon^{-1}\mathbf{E}\right\|_{L^2(G)^3} + \|\sigma\|_{L^\infty(G)}\|\mathbf{E}\|_{L^2(G)^3} , \\
\|\operatorname{div} \mathbf{E}\|_{L^2(G)} &= \left\|-\varepsilon^{-1}\nabla\varepsilon\cdot\mathbf{E} + \varepsilon^{-1}\operatorname{div}(\varepsilon\mathbf{E})\right\|_{L^2(G)} \\
&\leq \eta^{-1}\left(\|\varepsilon\|_{W^{1,\infty}(G)}\|\mathbf{E}\|_{L^2(G)^3} + \|\operatorname{div}(\varepsilon\mathbf{E})\|_{L^2(G)}\right) .
\end{aligned}$$

Setting $d_{\varepsilon,\mu,\sigma,\eta}$ as in the claim and using $\operatorname{div}\operatorname{curl} = 0$, we obtain the estimate

$$\begin{aligned}
\|u\|_{H^1(G)^6}^2 \lesssim_G d_{\varepsilon,\mu,\sigma,\eta}^2 \Big(&\|u\|_{L^2(G)^6}^2 + \|\operatorname{div}(\varepsilon\mathbf{E})\|_{L^2(G)}^2 + \left\|\varepsilon^{-1}\operatorname{curl}\mathbf{H} - \sigma\varepsilon^{-1}\mathbf{E}\right\|_{L^2(G)^3}^2 \\
&+ \left\|\mu^{-1}\operatorname{curl}\mathbf{E}\right\|_{L^2(G)^3}^2 + \|\operatorname{div}(\sigma\mathbf{E})\|_{L^2(G)}^2 \Big) \\
= d_{\varepsilon,\mu,\sigma,\eta}^2 &\left(\|u\|_{X_{\rm div}}^2 + \|Au\|_{X_{\rm div}}^2\right) \simeq d_{\varepsilon,\mu,\sigma,\eta}^2 \|u\|_{A_{\rm div}}^2
\end{aligned}$$

which finishes the proof. □

Since the expression (3.28) below for the polarisation requires us to work in $H^2(G)^6$, the space X_{div} is not sufficient and we have to further restrict the semigroup. To this end, we define the subspace

$$X_2 := \left\{(\mathbf{E},\mathbf{H}) \in D(A_{\text{div}}^2) \;\middle|\; \operatorname{div}(\varepsilon\mathbf{E}) \in H^1(G)\right\}$$

of X_{div}, equipped with the inner product

$$\begin{aligned}\Big((\mathbf{E},\mathbf{H})\Big|(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}})\Big)_{X_2} := \big((\mathbf{E},\mathbf{H}) \mid (\mathbf{E},\mathbf{H})\big)_{L^2(G)^6} + \Big(A^2(\mathbf{E},\mathbf{H})\Big|A^2(\widetilde{\mathbf{E}},\widetilde{\mathbf{H}})\Big)_{L^2(G)^6} \\ + \Big(\operatorname{div}(\varepsilon\mathbf{E})\Big|\operatorname{div}(\varepsilon\widetilde{\mathbf{E}})\Big)_{H^1(G)} .\end{aligned}$$

The corresponding norm

$$\|(\mathbf{E},\mathbf{H})\|_{X_2} := \left(\|(\mathbf{E},\mathbf{H})\|_{L^2(G)^6}^2 + \left\|A^2(\mathbf{E},\mathbf{H})\right\|_{L^2(G)^6}^2 + \|\operatorname{div}(\varepsilon\mathbf{E})\|_{H^1(G)}^2\right)^{1/2} .$$

is stronger than the norm in X_{div}, so $X_2 \hookrightarrow X_{\text{div}}$.

Lemma 3.13. *Let Assumption 3.10 be true. The space X_2 is a Hilbert space and the continuous embeddings $X_2 \hookrightarrow [D(A_{\text{div}}^2)] \hookrightarrow [D(A_{\text{div}})] \hookrightarrow H^1(G)^6$ hold with constants depending on the norm of ε, μ, σ in $Z(G)$ and on η. In the case $[D(A_{\text{div}})] \hookrightarrow H^1(G)^6$, the constant also depends on G.*

Proof. 1) Let $u = (\mathbf{E},\mathbf{H}) \in D(A_{\text{div}}^2)$. By Proposition 3.11 and Theorem II.3.8 in [9], 1 is in the resolvent set of A_{div} and $\|(I-A_{\text{div}})^{-1}\| \lesssim_{\varepsilon,\mu,\sigma,\eta} 1$. Therefore we have

$$\|Au\|_{X_{\text{div}}} = \left\|(I-A_{\text{div}})^{-1}(I-A_{\text{div}})Au\right\|_{X_{\text{div}}} \lesssim_{\varepsilon,\mu,\sigma,\eta} \left(\|Au\|_{X_{\text{div}}} + \left\|A^2u\right\|_{X_{\text{div}}}\right) . \tag{3.17}$$

The formula

$$Au = T(1)u - u - \int_0^1 (1-s)T(s)A^2u \,\mathrm{d}s$$

(see Section II.1 in [9]) yields

$$\|Au\|_{X_{\text{div}}} \lesssim_{\varepsilon,\mu,\sigma,\eta} \|u\|_{X_{\text{div}}} + \left\|A^2u\right\|_{X_{\text{div}}} . \tag{3.18}$$

Inserting this estimate into (3.17) shows $[D(A_{\text{div}}^2)] \hookrightarrow [D(A_{\text{div}})]$.

2) For $u \in X_2$, we calculate

$$\begin{aligned}\|u\|_{A_{\text{div}}^2}^2 \simeq \|u\|_{X_{\text{div}}}^2 + \left\|A^2u\right\|_{X_{\text{div}}}^2 = \|u\|_{L^2(G)^6}^2 + \|\operatorname{div}(\varepsilon\mathbf{E})\|_{L^2(G)}^2 + \left\|A^2u\right\|_{L^2(G)^6}^2 \\ + \left\|\operatorname{div}\big(\sigma^2\varepsilon^{-1}\mathbf{E} - \sigma\varepsilon^{-1}\operatorname{curl}\mathbf{H}\big)\right\|_{L^2(G)^3}^2\end{aligned}$$

and use

$$\begin{aligned}\operatorname{div}\left(\sigma^2\varepsilon^{-1}\mathbf{E} - \sigma\varepsilon^{-1}\operatorname{curl}\mathbf{H}\right) =& \nabla\left(\sigma^2\varepsilon^{-2}\right)\cdot(\varepsilon\mathbf{E}) + \sigma^2\varepsilon^{-2}\operatorname{div}(\varepsilon\mathbf{E}) \\ & - \varepsilon\nabla\left(\sigma\varepsilon^{-1}\right)\cdot\left(\varepsilon^{-1}\operatorname{curl}\mathbf{H} - \sigma\varepsilon^{-1}\mathbf{E}\right) - \sigma\nabla\left(\sigma\varepsilon^{-1}\right)\cdot\mathbf{E}\end{aligned}$$

to estimate

$$\|u\|_{A_{\mathrm{div}}^2}^2 \lesssim_{\varepsilon,\sigma,\eta} \|u\|_{L^2(G)^6}^2 + \left\|A^2 u\right\|_{L^2(G)^6}^2 + \|\mathrm{div}(\varepsilon \mathbf{E})\|_{H^1(G)}^2 + \|Au\|_{L^2(G)^6}^2 \,.$$

For the last term we use the same calculation as for (3.18) to obtain

$$\|Au\|_{L^2(G)^6} \lesssim_\eta \|Au\|_X \lesssim \|u\|_X + \left\|A^2 u\right\|_X \lesssim_{\varepsilon,\mu,\eta} \|u\|_{L^2(G)^6} + \left\|A^2 u\right\|_{L^2(G)^6} \,. \tag{3.19}$$

This finally leads to

$$\|u\|_{A_{\mathrm{div}}^2}^2 \lesssim_{\varepsilon,\mu,\sigma,\eta} \|u\|_{L^2(G)^6}^2 + \left\|A^2 u\right\|_{L^2(G)^6}^2 + \|\mathrm{div}(\varepsilon \mathbf{E})\|_{H^1(G)}^2 \,.$$

Together with Lemma 3.12, we have the continuous embeddings $X_2 \hookrightarrow [D(A_{\mathrm{div}}^2)] \hookrightarrow [D(A_{\mathrm{div}})] \hookrightarrow H^1(G)^6$.

3) Now let $(u_k) = \big((\mathbf{E}_k, \mathbf{H}_k)\big)$ be a Cauchy sequence in X_2. By the above, (u_k) is a Cauchy sequence in $[D(A_{\mathrm{div}}^2)]$, implying that there exists an element $u = (\mathbf{E}, \mathbf{H})$ of $[D(A_{\mathrm{div}}^2)]$ such that $u_k \to u$ in $[D(A_{\mathrm{div}}^2)]$ as $k \to \infty$. In particular, $(\mathbf{E}_k, \mathbf{H}_k) \to (\mathbf{E}, \mathbf{H})$ and $A^2(\mathbf{E}_k, \mathbf{H}_k) \to A^2(\mathbf{E}, \mathbf{H})$ in $L^2(G)^6$ as $k \to \infty$. Furthermore, $\big(\mathrm{div}(\varepsilon \mathbf{E}_k)\big)$ is a Cauchy sequence in $H^1(G)$, so $\mathrm{div}(\varepsilon \mathbf{E}_k) \to \varphi$ in $H^1(G)$ as $k \to \infty$ for some $\varphi \in H^1(G)$. Since $\mathbf{E}_k \to \mathbf{E}$ in $H^1(G)$, we have $\mathrm{div}(\varepsilon \mathbf{E}_k) \to \mathrm{div}(\varepsilon \mathbf{E})$ in $L^2(G)$ as $k \to \infty$, which implies $\mathrm{div}(\varepsilon \mathbf{E}) = \varphi \in H^1(G)$. In conclusion, $u \in X_2$ and $u_k \to u$ in X_2 as $k \to \infty$. □

We can now prove that further restricting $T_{\mathrm{div}}(\cdot)$ to X_2 yields a strongly continuous semigroup.

Proposition 3.14. *Let Assumption 3.10 be true. The restriction $T_2(\cdot) := T_{\mathrm{div}}(\cdot)|_{X_2}$ is a C_0-semigroup on X_2. The generator is given by $A_2 : D(A_2) \subseteq X_2 \to X_2$, $A_2 u = Au$, with domain $D(A_2) := \{u \in X_2 \mid Au \in X_2\}$. For all $t \geq 0$, we have the estimate*

$$\begin{aligned}\|T_2(t)\|_{\mathcal{B}(X_2)} \lesssim_G\ & 1 + c_{\varepsilon,\mu,\eta} + \Big(\|\kappa\|_{W^{1,\infty}(G)} + a_{\varepsilon,\mu,\sigma,\eta}(1 + \|\varepsilon\|_{L^\infty(G)} \|\kappa\|_{W^{1,\infty}(G)}\Big)\, t \\ & + a_{\varepsilon,\mu,\sigma,\eta} \Big(\Big(1 + \|\varepsilon\|_{L^\infty(G)}\Big) \|\kappa\|_{W^{1,\infty}(G)} + \|\varepsilon\|_{L^\infty(G)} \|\kappa\|_{W^{1,\infty}(G)}^2\Big)\, t^2 \\ & + a_{\varepsilon,\mu,\sigma,\eta} \|\varepsilon\|_{L^\infty(G)} \|\kappa\|_{W^{1,\infty}(G)}^2\, t^3 \\ \lesssim_{\varepsilon,\mu,\sigma,\eta,G}\ & 1 + t^3 \,,\end{aligned}$$

where $\kappa = \sigma \varepsilon^{-1}$ and

$$a_{\varepsilon,\mu,\sigma,\eta} := d_{\varepsilon,\mu,\sigma,\eta} \|\varepsilon\|_{W^{1,\infty}(G)} (1 + c_{\varepsilon,\mu,\eta}) \Big(1 + \|\kappa\|_{L^\infty(G)}\Big) \|\kappa\|_{Z(G)} \,.$$

Proof. Let $u_0 = (\mathbf{E}_0, \mathbf{H}_0) \in X_2$ and $u(t) = \big(\mathbf{E}(t), \mathbf{H}(t)\big) := T_{\mathrm{div}}(t) u_0$ for $t \geq 0$.

1) We first prove that X_2 is invariant under $T_{\mathrm{div}}(\cdot)$. By Lemma II.1.3 in [9], $T_{\mathrm{div}}(\cdot)$ leaves $D(A_{\mathrm{div}}^2)$ invariant. It remains to show that the divergence condition in X_2 is preserved by $T_{\mathrm{div}}(\cdot)$. The map $u \in C^1(\mathbb{R}_{\geq 0}, X_{\mathrm{div}}) \cap C(\mathbb{R}_{\geq 0}, [D(A_{\mathrm{div}})])$ satisfies

$$u'(t) = Au(t)\,,\ t \geq 0\,, \qquad u(0) = u_0$$

and therefore

$$\partial_t \, \varepsilon\mathbf{E}(t) = -\sigma\mathbf{E}(t) + \operatorname{curl}\mathbf{H}(t)\,, \quad t \geq 0$$

in $L^2(G)^3$. Integration yields

$$\varepsilon\mathbf{E}(t) = \varepsilon\mathbf{E}_0 - \int_0^t \sigma\mathbf{E}(s)\,\mathrm{d}s + \int_0^t \operatorname{curl}\mathbf{H}(s)\,\mathrm{d}s\,, \quad t \geq 0\,. \tag{3.20}$$

Since div is a bounded operator from $L^2(G)^3$ to $H^{-1}(G)^3$ and $\operatorname{div}\operatorname{curl} = 0$, equation (3.20) yields

$$\begin{aligned}\operatorname{div}\big(\varepsilon\mathbf{E}(t)\big) &= \operatorname{div}(\varepsilon\mathbf{E}_0) - \int_0^t \operatorname{div}\big(\sigma\mathbf{E}(s)\big)\,\mathrm{d}s \\ &= \operatorname{div}(\varepsilon\mathbf{E}_0) - \int_0^t \sigma\varepsilon^{-1}\operatorname{div}\big(\varepsilon\mathbf{E}(s)\big)\,\mathrm{d}s - \int_0^t \varepsilon\nabla\left(\sigma\varepsilon^{-1}\right)\cdot\mathbf{E}(s)\,\mathrm{d}s\,, \quad t \geq 0\end{aligned} \tag{3.21}$$

in $H^{-1}(G)^3$. Due to the embedding $[D(A_{\mathrm{div}})] \hookrightarrow H^1(G)^6$, we have $u \in C\big(\mathbb{R}_{\geq 0}\, H^1(G)^6\big)$ and the above identity also holds in $L^2(G)$. We consider (3.21) as an integral equation in $L^2(G)$ of the form

$$w(t) = w(0) - \int_0^t \kappa w(s)\,\mathrm{d}s - \int_0^t \varepsilon\nabla\kappa\cdot\mathbf{E}(s)\,\mathrm{d}s \tag{3.22}$$

for $w := \operatorname{div}(\varepsilon\mathbf{E})$, where the last term is a given inhomogenity and we set $\kappa := \sigma\varepsilon^{-1}$. Let w and $\widetilde{w}$ be solutions of (3.22). Then the estimate

$$\|w(t) - \widetilde{w}(t)\|_{L^2(G)} \leq \|\kappa\|_{L^\infty(G)} \int_0^t \|w(s) - \widetilde{w}(s)\|_{L^2(G)}\,\mathrm{d}s$$

holds for all $t \geq 0$ and Gronwall's inequality yields $w = \widetilde{w}$, so a solution of (3.22) is unique. We now set

$$w(t) := \mathrm{e}^{-\kappa t}w(0) - \int_0^t \mathrm{e}^{-\kappa(t-s)}\varepsilon\nabla\kappa\cdot\mathbf{E}(s)\,\mathrm{d}s$$

for $t \geq 0$ and show that this is a solution of (3.22). Since multiplication with an element of $L^\infty(G)$ is a bounded operation on $L^2(G)$ and $t \mapsto \mathrm{e}^{-\kappa t} \in C^1\big(\mathbb{R}_{\geq 0}, W^{1,\infty}(G)\big)$, we have $t \mapsto \mathrm{e}^{-\kappa t}\operatorname{div}(\varepsilon\mathbf{E}_0) \in C^1\big(\mathbb{R}_{\geq 0}, L^2(G)\big)$ with

$$\partial_t\,\mathrm{e}^{-\kappa t}\operatorname{div}(\varepsilon\mathbf{E}_0) = -\kappa\mathrm{e}^{-\kappa t}\operatorname{div}(\varepsilon\mathbf{E}_0)$$

for all $t \geq 0$. Furthermore, $\varepsilon\nabla\kappa\cdot\mathbf{E} \in C\big(\mathbb{R}_{\geq 0}, L^2(G)\big)$ which implies

$$\left[t \mapsto \mathrm{e}^{-\kappa t}\int_0^t \mathrm{e}^{\kappa s}\varepsilon\nabla\kappa\cdot\mathbf{E}(s)\,\mathrm{d}s\right] \in C^1\left(\mathbb{R}_{\geq 0}, L^2(G)\right)$$

with derivative

$$-\kappa \mathrm{e}^{-\kappa t} \int_0^t \mathrm{e}^{\kappa s} \varepsilon \nabla\kappa \cdot \mathbf{E}(s) \, \mathrm{d}s + \varepsilon \nabla\kappa \cdot \mathbf{E}(t) \, .$$

This leads to

$$w'(t) = -\kappa w(t) - \varepsilon \nabla\kappa \cdot \mathbf{E}(t)$$

for all $t \geq 0$ and integration shows that w is a solution of (3.22). Therefore,

$$\operatorname{div}\big(\varepsilon \mathbf{E}(t)\big) = \mathrm{e}^{-\kappa t} \operatorname{div}(\varepsilon \mathbf{E}_0) - \int_0^t \mathrm{e}^{-\kappa(t-s)} \varepsilon \nabla\kappa \cdot \mathbf{E}(s) \, \mathrm{d}s \, . \tag{3.23}$$

Since $\nabla\kappa \in \big(L^\infty(G) \cap W^{1,3}(G)\big)^3$, it follows $\nabla\kappa \cdot \mathbf{E} \in C\big(\mathbb{R}_{\geq 0}, H^1(G)\big)$ as in the first part of the proof of Lemma 3.6. Therefore the integrand is contained in $C\big(\mathbb{R}_{\geq 0}, H^1(G)\big)$. Together with $\operatorname{div}(\varepsilon \mathbf{E}_0) \in H^1(G)$ we infer $\operatorname{div}\big(\varepsilon \mathbf{E}(t)\big) \in H^1(G)$ and so $u(t) \in X_2$ for all $t \geq 0$.

2) In this step, we show that $T_2(\cdot)$ is a C_0-semigroup on X_2. The properties $T_2(0) = I$ and $T_2(t+s) = T_2(t)T_2(s)$ for all $t, s \geq 0$ are directly inherited from $T_{\mathrm{div}}(\cdot)$. Using the strong continuity of $T_{\mathrm{div}}(\cdot)$ and $X_2 \subseteq D(A_{\mathrm{div}}^2)$ yields $u(t) \to u_0$ and $A^2 u(t) = T_{\mathrm{div}}(t) A^2 u_0 \to A^2 u_0$ in $X_{\mathrm{div}} \hookrightarrow L^2(G)$ for $t \to 0^+$. From (3.23), it additionally follows $\operatorname{div}\big(\varepsilon \mathbf{E}(t)\big) \to \operatorname{div}(\varepsilon \mathbf{E}_0)$ in $H^1(G)$ as $t \to 0^+$. We thus have $u(t) \to u_0$ in X_2 as $t \to 0^+$ and thus $T_2(\cdot)$ is a C_0-semigroup on X_2.

3) The generator of $T_2(\cdot)$ is given by the part of A_{div} in X_2, i. e., the restriction of A_{div} to $D(A_2) := \big\{u \in D(A_{\mathrm{div}}) \cap X_2 \;\big|\; Au \in X_2\big\} = \big\{u \in X_2 \;\big|\; Au \in X_2\big\}$, see Section II.2.3 in [9].

4) It remains to estimate the norm of the semigroup. By Proposition 3.11, we have

$$\begin{aligned} \|u(t)\|_{L^2(G)^6} &\leq c_{\varepsilon,\mu,\eta} \|u_0\|_{L^2(G)^6} \\ \big\|A^2 u(t)\big\|_{L^2(G)^6} = \big\|T(t) A^2 u_0\big\|_{L^2(G)^6} &\leq c_{\varepsilon,\mu,\eta} \big\|A^2 u_0\big\|_{L^2(G)^6} \, . \end{aligned} \tag{3.24}$$

We estimate $\operatorname{div}\big(\varepsilon \mathbf{E}(t)\big)$ in $H^1(G)$ using (3.23). The first term can be treated with

$$\begin{aligned} \big\|\mathrm{e}^{-\kappa t} \operatorname{div}(\varepsilon \mathbf{E}_0)\big\|_{H^1(G)} &\leq \big\|\mathrm{e}^{-\kappa t}\big\|_{W^{1,\infty}(G)} \|\operatorname{div}(\varepsilon \mathbf{E}_0)\|_{H^1(G)} \\ &\leq \Big(1 + t \|\kappa\|_{W^{1,\infty}(G)}\Big) \|\operatorname{div}(\varepsilon \mathbf{E}_0)\|_{H^1(G)} \, . \end{aligned}$$

For the integrand, we argue as in part 1) of the proof of Lemma 3.6 to obtain

$$\|\varepsilon \nabla\kappa \cdot \mathbf{E}(s)\|_{H^1(G)} \lesssim_G \|\varepsilon\|_{W^{1,\infty}(G)} \|\kappa\|_{Z(G)} \|\mathbf{E}(s)\|_{H^1(G)^3} \, .$$

Lemma 3.12 yields $\|\mathbf{E}(s)\|_{H^1(G)^3} \lesssim_G d_{\varepsilon,\mu,\sigma,\eta} \|u(s)\|_{A_{\mathrm{div}}}$. We now treat the terms in

$$\begin{aligned} \|u(s)\|_{A_{\mathrm{div}}} &= \|u(s)\|_{X_{\mathrm{div}}} + \|Au(s)\|_{X_{\mathrm{div}}} \\ &\simeq \|u(s)\|_{L^2(G)^6} + \big\|\operatorname{div}\big(\varepsilon \mathbf{E}(s)\big)\big\|_{L^2(G)} + \|Au(s)\|_{L^2(G)^6} + \big\|\operatorname{div}\big(\sigma \mathbf{E}(s)\big)\big\|_{L^2(G)} \, . \end{aligned}$$

The first term has already been considered in (3.24). The third one can be estimated by

$$\|Au(s)\|_{L^2(G)^6} = \|T(s) A u_0\|_{L^2(G)^6} \leq \eta^{-1/2} \|Au_0\|_X \lesssim \eta^{-1/2} \left(\|u_0\|_X + \big\|A^2 u_0\big\|_X\right)$$

$$\leq c_{\varepsilon,\mu,\eta} \left(\|u_0\|_{L^2(G)^6} + \left\|A^2 u_0\right\|_{L^2(G)^6} \right) ,$$

where we have used (3.19). For the last term, we calculate

$$\begin{aligned}\left\|\operatorname{div}\big(\sigma \mathbf{E}(s)\big)\right\|_{L^2(G)} &= \left\|\varepsilon \nabla\kappa \cdot \mathbf{E}(s) + \kappa \operatorname{div}\big(\varepsilon \mathbf{E}(s)\big)\right\|_{L^2(G)} \\ &\leq c_{\varepsilon,\mu,\eta} \|\varepsilon\|_{L^\infty(G)} \|\kappa\|_{W^{1,\infty}(G)} \|u_0\|_{L^2(G)} + \|\kappa\|_{L^\infty(G)} \left\|\operatorname{div}\big(\varepsilon \mathbf{E}(s)\big)\right\|_{L^2(G)} .\end{aligned}$$

We combine these results with the estimate (3.16) for $\left\|\operatorname{div}\big(\varepsilon \mathbf{E}(s)\big)\right\|_{L^2(G)}$ and obtain

$$\begin{aligned}\|u(s)\|_{A_{\mathrm{div}}} &\lesssim c_{\varepsilon,\mu,\eta} \left(1 + \|\varepsilon\|_{L^\infty(G)} \|\kappa\|_{W^{1,\infty}(G)} \left(1 + \left(1 + \|\kappa\|_{L^\infty(G)}\right) s\right)\right) \|u_0\|_{L^2(G)^6} \\ &\quad + c_{\varepsilon,\mu,\eta} \left\|A^2 u_0\right\|_{L^2(G)^6} + \left(1 + \|\kappa\|_{L^\infty(G)}\right) \|\operatorname{div}(\varepsilon \mathbf{E}_0)\|_{L^2(G)^6} \\ &\lesssim (1 + c_{\varepsilon,\mu,\eta}) \left(1 + \|\kappa\|_{L^\infty(G)}\right) \left(1 + \|\varepsilon\|_{L^\infty(G)} \|\kappa\|_{W^{1,\infty}(G)} (1+s)\right) \|u_0\|_{X_2} .\end{aligned}$$

Using $\|\mathrm{e}^{-\kappa t}\|_{W^{1,\infty}(G)} \leq 1 + \|\kappa\|_{W^{1,\infty}(G)}\, t$ and $\|\mathbf{E}(s)\|_{H^1(G)^3} \lesssim_G d_{\varepsilon,\mu,\sigma,\eta} \|u(s)\|_{A_{\mathrm{div}}}$, equation (3.23) next yields

$$\begin{aligned}\left\|\operatorname{div}\big(\varepsilon \mathbf{E}(t)\big)\right\|_{H^1(G)} &\lesssim_G \left(1 + \|\kappa\|_{W^{1,\infty}(G)}\, t\right) \left\|\operatorname{div}\big(\varepsilon \mathbf{E}_0\big)\right\|_{H^1(G)} \\ &+ a_{\varepsilon,\mu,\sigma,\eta} \int_0^t \left(1 + \|\kappa\|_{W^{1,\infty}(G)} (t-s)\right) \left(1 + \|\varepsilon\|_{L^\infty(G)} \|\kappa\|_{W^{1,\infty}(G)} (1+s)\right) \mathrm{d}s\, \|u_0\|_{X_2}\end{aligned}$$

with $a_{\varepsilon,\mu,\sigma,\eta}$ defined as in the claim. Evaluating the integral and combining with (3.24) finally leads to the estimate for $\|T_2(t)\|_{\mathcal{B}(X_2)}$. □

The next lemma allows us to identify X_2 with a subspace of $H^2(G)^6$.

Lemma 3.15. *Let Assumption 3.10 be true. It holds*

$$X_2 = \left\{(\mathbf{E}, \mathbf{H}) \in H^2(G)^6 \;\middle|\; \operatorname{tr}_\tau \mathbf{E} = 0, \operatorname{tr}_\nu \mathbf{H} = 0, \operatorname{tr}_\tau(\operatorname{curl} \mathbf{H}) = 0, \operatorname{div}(\mu \mathbf{H}) = 0\right\}$$

and X_2 is closed in $H^2(G)^6$. We further have $\|u\|_{X_2} \simeq_{\varepsilon,\mu,\sigma,\eta,G} \|u\|_{H^2(G)^6}$ for all $u \in X_2$. The equivalence depends on η, G and the norm of ε, μ, σ in $Z(G)$.

Proof. 1) Let $u = (\mathbf{E}, \mathbf{H}) \in X_2$. In particular, $u \in D(A_{\mathrm{div}}) = D(A) \cap X_{\mathrm{div}}$ which implies $\operatorname{tr}_\tau \mathbf{E} = 0$ and $\operatorname{tr}_\nu \mathbf{H} = 0$. Lemma 3.6 and Lemma 3.13 as well as $\operatorname{div}(\varepsilon \mathbf{E}) \in H^1(G)$ and $\operatorname{div}(\mu \mathbf{H}) = 0$ yield $\operatorname{div} \mathbf{E} \in H^1(G)$ and $\operatorname{div} \mathbf{H} \in H^1(G)$. Since $(\mathbf{K}, \mathbf{L}) := Au \in [D(A_{\mathrm{div}})] \hookrightarrow H^1(G)^6$, both $\operatorname{curl} \mathbf{E} = -\mu \mathbf{L}$ and $\operatorname{curl} \mathbf{H} = \varepsilon \mathbf{K} + \sigma \mathbf{E}$ are contained in $H^1(G)^3$. We can now apply Corollary 2.15 in [2] to conclude $u \in H^2(G)^6$.

2) To prove the other inclusion, let $u = (\mathbf{E}, \mathbf{H})$ be an element in the set on the right-hand side of the claim. By Lemma 3.6, we have $\operatorname{div}(\varepsilon \mathbf{E}) \in H^1(G)$. Also, u is contained in $H_0(\operatorname{curl}, G) \times H(\operatorname{curl}, G) = D(A)$ and in X_{div}, therefore $u \in D(A_{\mathrm{div}})$. We set $(\mathbf{K}, \mathbf{L}) := Au$. Using Lemma 3.6, we see $\operatorname{tr}_\tau \mathbf{K} = 0$ which implies $\mathbf{K} \in H_0(\operatorname{curl}, G)$ by Theorem 3.5. Further, $\mathbf{L} = -\mu^{-1} \operatorname{curl} \mathbf{E} \in H^1(G)^3 \subseteq H(\operatorname{curl}, G)$. Combining, we have $(\mathbf{K}, \mathbf{L}) \in D(A)$ and therefore $u \in D(A^2_{\mathrm{div}})$.

3) Let $(u_k) = \big((\mathbf{E}_k, \mathbf{H}_k)\big)$ be a sequence in X_2, $u = (\mathbf{E}, \mathbf{H}) \in H^2(G)^6$ and $u_k \to u$ in $H^2(G)^6$ as $k \to \infty$. The embeddings $H^2(G)^3 \hookrightarrow H(\operatorname{curl}, G)$ and $H^2(G)^3 \hookrightarrow H(\operatorname{div}, G)$ and the continuity of the trace operators $\operatorname{tr}_\tau, \operatorname{tr}_\nu$ imply

$$\operatorname{tr}_\tau \mathbf{E} = \lim_{k\to\infty} \operatorname{tr}_\tau \mathbf{E}_k = 0 \qquad \text{and} \qquad \operatorname{tr}_\nu \mathbf{H} = \lim_{k\to\infty} \operatorname{tr}_\nu \mathbf{H}_k = 0\,.$$

Similarly, $\operatorname{tr}_\tau(\operatorname{curl}\mathbf{H}) = \lim_{k\to\infty} \operatorname{tr}_\tau(\operatorname{curl}\mathbf{H}_k) = 0$. The continuity of $\operatorname{div} : H^1(G)^3 \to L^2(G)$ yields together with $\mu\mathbf{H}_k \to \mu\mathbf{H}$ in $H^1(G)^3$ as $k \to \infty$ the limit $\operatorname{div}(\mu\mathbf{H}) = \lim_{k\to\infty} \operatorname{div}(\mu\mathbf{H}_k) = 0$. Therefore, $u \in X_2$, showing that X_2 is closed in $H^2(G)^6$.

4) We now prove the norm equivalence. Let $u = (\mathbf{E}, \mathbf{H}) \in X_2$. The definition of the norm in X_2 yields

$$\begin{aligned} \|u\|_{X_2} &\lesssim_{\varepsilon,\mu,\sigma,\eta} \|u\|_{L^2(G)^6} + \|\operatorname{curl}\mathbf{H}\|_{L^2(G)^3} + \left\|\operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathbf{E}\right)\right\|_{L^2(G)^3} \\ &\quad + \left\|\operatorname{curl}\left(\sigma\varepsilon^{-1}\mathbf{E}\right)\right\|_{L^2(G)^3} + \left\|\operatorname{curl}\left(\varepsilon^{-1}\operatorname{curl}\mathbf{H}\right)\right\|_{L^2(G)^3} + \|\operatorname{div}(\varepsilon\mathbf{E})\|_{H^1(G)} \\ &\lesssim_{\varepsilon,\mu,\sigma,\eta,G} \|u\|_{H^2(G)^6}\ , \end{aligned} \tag{3.25}$$

where we have used $\|\operatorname{div}(\varepsilon\mathbf{E})\|_{H^1(G)}^2 \lesssim_{\varepsilon,G} \|\mathbf{E}\|_{H^2(G)^3}^2$, see Lemma 3.6. Corollary 2.15 in [2] (see also Proposition 1.4 in [11]) implies

$$\|u\|_{H^2(G)^6} \lesssim_G \|u\|_{L^2(G)^6} + \|\operatorname{curl}\mathbf{E}\|_{H^1(G)^3} + \|\operatorname{curl}\mathbf{H}\|_{H^1(G)^3} + \|\operatorname{div}\mathbf{E}\|_{H^1(G)} + \|\operatorname{div}\mathbf{H}\|_{H^1(G)}\ .$$

We use Lemmas 3.6 and 3.13 to treat these terms. For the last two, we obtain

$$\begin{aligned} \|\operatorname{div}\mathbf{E}\|_{H^1(G)} &\lesssim_{\varepsilon,\eta,G} \|\mathbf{E}\|_{H^1(G)^3} + \|\operatorname{div}(\varepsilon\mathbf{E})\|_{H^1(G)} \lesssim_{\varepsilon,\mu,\sigma,\eta,G} \|u\|_{X_2}\ , \\ \|\operatorname{div}\mathbf{H}\|_{H^1(G)} &\lesssim_{\mu,\eta,G} \|\mathbf{H}\|_{H^1(G)^3} \lesssim_{\varepsilon,\mu,\sigma,\eta,G} \|u\|_{X_2}\ . \end{aligned}$$

The terms involving the curl operator can be estimated by

$$\begin{aligned} &\|\operatorname{curl}\mathbf{E}\|_{H^1(G)^3} + \|\operatorname{curl}\mathbf{H}\|_{H^1(G)^3} \lesssim_{\varepsilon,\mu,\sigma,\eta,G} \|(\operatorname{curl}\mathbf{E}, \operatorname{curl}\mathbf{H})\|_{A_{\operatorname{div}}} \\ &\quad\lesssim \|\operatorname{curl}\mathbf{E}\|_{L^2(G)^3} + \|\operatorname{curl}\mathbf{H}\|_{L^2(G)^3} + \left\|\sigma\varepsilon^{-1}\operatorname{curl}\mathbf{E}\right\|_{L^2(G)^3} + \left\|\varepsilon^{-1}\operatorname{curl}\operatorname{curl}\mathbf{H}\right\|_{L^2(G)^3} \\ &\qquad + \left\|\mu^{-1}\operatorname{curl}\operatorname{curl}\mathbf{E}\right\|_{L^2(G)^3} + \|\operatorname{div}(\varepsilon\operatorname{curl}\mathbf{E})\|_{L^2(G)} + \|\operatorname{div}(\sigma\operatorname{curl}\mathbf{E})\|_{L^2(G)}\ . \end{aligned} \tag{3.26}$$

Since we have

$$\|\operatorname{curl}\mathbf{E}\|_{L^2(G)^3} + \|\operatorname{curl}\mathbf{H}\|_{L^2(G)^3} + \left\|\sigma\varepsilon^{-1}\operatorname{curl}\mathbf{E}\right\|_{L^2(G)^3} \lesssim_{\sigma,\eta} \|u\|_{H^1(G)^6} \lesssim_{\varepsilon,\mu,\sigma,\eta,G} \|u\|_{X_2}$$

and

$$\begin{aligned} \|\operatorname{div}(\varepsilon\operatorname{curl}\mathbf{E})\|_{L^2(G)} + \|\operatorname{div}(\sigma\operatorname{curl}\mathbf{E})\|_{L^2(G)} &= \|\nabla\varepsilon\cdot\operatorname{curl}\mathbf{E}\|_{L^2(G)} + \|\nabla\sigma\cdot\operatorname{curl}\mathbf{E}\|_{L^2(G)} \\ &\lesssim_{\varepsilon,\sigma} \|u\|_{H^1(G)^6} \lesssim_{\varepsilon,\mu,\sigma,\eta,G} \|u\|_{X_2}\ , \end{aligned}$$

only two terms in (3.26) remain. These can also be bounded by $\|u\|_{X_2}$ via

$$\left\|\varepsilon^{-1}\operatorname{curl}\operatorname{curl}\mathbf{H}\right\|_{L^2(G)^3} \lesssim_\eta \left\|\operatorname{curl}\left(\varepsilon\varepsilon^{-1}\operatorname{curl}\mathbf{H}\right)\right\|_{L^2(G)^3}$$

$$
\begin{aligned}
&= \left\|\nabla\varepsilon \times \left(\varepsilon^{-1}\operatorname{curl}\mathbf{H}\right)\right\|_{L^2(G)^3} + \left\|\varepsilon\operatorname{curl}\left(\varepsilon^{-1}\operatorname{curl}\mathbf{H}\right)\right\|_{L^2(G)^3} \\
&\lesssim_{\varepsilon,\mu,\eta} \|\mathbf{H}\|_{H^1(G)^3} + \left\|\mu^{-1}\operatorname{curl}\left(\varepsilon^{-1}\operatorname{curl}\mathbf{H}\right) - \mu^{-1}\operatorname{curl}\left(\sigma\varepsilon^{-1}\mathbf{E}\right)\right\|_{L^2(G)^3} \\
&\qquad + \left\|\mu^{-1}\operatorname{curl}\left(\sigma\varepsilon^{-1}\mathbf{E}\right)\right\|_{L^2(G)^3} \\
&\lesssim_{\varepsilon,\mu,\sigma,\eta,G} \|u\|_{X_2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\mu^{-1}\operatorname{curl}\operatorname{curl}\mathbf{E}\right\|_{L^2(G)^3} \lesssim_\eta \left\|\operatorname{curl}\left(\mu\mu^{-1}\operatorname{curl}\mathbf{E}\right)\right\|_{L^2(G)^3} \\
&\quad = \left\|\nabla\mu \times \left(\mu^{-1}\operatorname{curl}\mathbf{E}\right)\right\|_{L^2(G)^3} + \left\|\mu\operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathbf{E}\right)\right\|_{L^2(G)^3} \\
&\quad \lesssim_{\varepsilon,\mu,\sigma,\eta} \|\mathbf{E}\|_{H^1(G)^3} + \left\|\sigma^2\varepsilon^{-2}\mathbf{E} - \sigma\varepsilon^{-2}\operatorname{curl}\mathbf{H} - \varepsilon^{-1}\operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathbf{E}\right)\right\|_{L^2(G)^3} \\
&\qquad\qquad + \left\|\sigma^2\varepsilon^{-2}\mathbf{E} - \sigma\varepsilon^{-2}\operatorname{curl}\mathbf{H}\right\|_{L^2(G)^3} \\
&\quad \lesssim_{\varepsilon,\mu,\sigma,\eta,G} \|u\|_{X_2} .
\end{aligned}
$$

□

In addition to the perfectly conducting boundary conditions ($\operatorname{tr}_\nu \mathbf{H} = 0$ is equivalent to $\operatorname{tr}_\nu \mathbf{B} = 0$ by Lemma 3.6) and the magnetic divergence condition, the space X_2 includes the requirement $\operatorname{tr}_\tau(\operatorname{curl}\mathbf{H}) = 0$. We assume $\operatorname{tr}_\tau \mathbf{J}_0 = 0$ and material properties such that the polarisation is always parallel to the electric field, which implies $\operatorname{tr}_\tau \mathbf{P} = 0$. Then this requirement is a natural consequence of the boundary condition $\operatorname{tr}_\tau \mathbf{E} = 0$ and equation (1.2d), since

$$
\operatorname{tr}_\tau \operatorname{curl}\mathbf{H} = \operatorname{tr}_\tau \mathbf{J} + \operatorname{tr}_\tau \partial_t\mathbf{D} = \operatorname{tr}_\tau \mathbf{J}_0 + \operatorname{tr}_\tau(\sigma\mathbf{E}) + \partial_t \operatorname{tr}_\tau \mathbf{D} = 0
$$

by a similar calculation as in the proof of Lemma 3.7.

3.4 Model for the polarisation

This subsection describes the noninstantaneous part $\widetilde{\mathbf{P}}$ of the polarisation. We assume that it depends only on $\mathbf{E}$ and not on $\mathbf{H}$ and contains nonlinearities of up to order N. According to a model commonly used in nonlinear optics (see e. g. Chapter 2 of [6] and also [3]), we can write

$$
\widetilde{\mathbf{P}}(\mathbf{E}) = \sum_{n=1}^{N} \mathbf{P}^{(n)}(\mathbf{E}) \tag{3.27}
$$

with components

$$
\begin{aligned}
&P_{j_0}^{(n)}(\mathbf{E})(t,x) \\
&= \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} R_{j_0 j_1 \ldots j_n}^{(n)}(t-s_1,\ldots,t-s_n,x) E_{j_1}(s_1,x)\ldots E_{j_n}(s_n,x)\,\mathrm{d}s_1\ldots\mathrm{d}s_n \\
&= \int_{\mathbb{R}^n_{>0}} R_{j_0 j_1 \ldots j_n}^{(n)}(\tau,x) \prod_{k=1}^{n} E_{j_k}(t-\tau_k,x)\,\mathrm{d}\tau
\end{aligned} \tag{3.28}
$$

for $j_0 \in \{1,2,3\}$ where $R^{(n)} = \big(R^{(n)}_{j_0 j_1 \cdots j_n}\big) : \mathbb{R}^n_{\geq 0} \times G \to \mathbb{R}^{3^{(n+1)}}$ is called the nth order polarisation response function and $R^{(n)}(\tau_1, \dots, \tau_n, x)$ is a tensor of rank $n+1$ for each $(\tau_1, \dots, \tau_n, x) \in \mathbb{R}^n_{\geq 0} \times G$. Here and in the following we use Einstein's convention of summing over repeated indices. The above model is local in space, i. e., the polarisation at point x is only influenced by the electric field at the same point. It is however noninstantaneous, since $\widetilde{\mathbf{P}}(\mathbf{E})(t,x)$ depends on the values of $\mathbf{E}(s,x)$ for all $s \leq t$ (due to causality, there can be no dependence on $\mathbf{E}(s,x)$ for $s > t$). The model incorporates time invariance in the form that in the second line of (3.28), the response functions only depend on the time differences $t - s_i$ for $i = 1, \dots, n$. From now on, we usually omit the spatial variable x.

Since $N \in \mathbb{N}$ can be chosen freely, (3.28) can contain products of arbitrarily many field components. For the expression to be well defined, we therefore require the fields to have at least H^2-regularity, so the products are again in $H^2(G)$ by the Banach algebra property. Equation (3.28) is understood as a Bochner-Lebesgue integral.

The response function $R^{(n)}$ has a property called intrinsic permutation symmetry (see Section 2.1.3 of [6]): Denoting the group of permutations on $\{1, \dots, n\}$ by S_n, we can split $R^{(n)}$ as

$$R^{(n)} = R^{(n,s)} + R^{(n,a)}$$

with

$$R^{(n,s)}_{j_0 \cdots j_n}(\tau_1, \dots, \tau_n) = \frac{1}{n!} \sum_{\sigma \in S_n} R^{(n)}_{j_0 j_{\sigma(1)} \cdots j_{\sigma(n)}}\big(\tau_{\sigma(1)}, \dots, \tau_{\sigma(n)}\big)\,, \quad R^{(n,a)} = R^{(n)} - R^{(n,s)}\,.$$

Since (3.28) contains summations over $j_1, \dots, j_n$ and integrations over $\tau_1, \dots, \tau_n$, $R^{(n,a)}$ yields no contribution and therefore $R^{(n)}$ can be assumed to be symmetric under the exchange $(j_j, \tau_l) \leftrightarrow (j_k, \tau_k)$ for $l, k \in \{1, \dots, n\}$.

We assume the elements $R^{(n)}_{j_0 \cdots j_n}$ to be continuously differentiable in $\mathbb{R}^n_{>0}$ with values in $H^2(G)$ and that they and their derivatives can continuously be extended onto $\mathbb{R}^n_{\geq 0}$ which we write as $R^{(n)}_{j_0 \cdots j_n} \in C^1\big(\mathbb{R}^n_{\geq 0}, H^2(G)\big)$. We also need integrability of the response functions and their time derivatives, as well as a mild decay property.

Assumption 3.16. *For all $n \in \{1, \dots, N\}, j_0, \dots, j_n \in \{1,2,3\}$ and $l \in \{1, \dots, n\}$ it holds*

$$\begin{aligned} R^{(n)}_{j_0 \cdots j_n} &\in C^1\big(\mathbb{R}^n_{\geq 0}, H^2(G)\big) \cap L^1\big(\mathbb{R}^n_{>0}, H^2(G)\big)\,, \\ \partial_{\tau_l} R^{(n)}_{j_0 \cdots j_n} &\in L^1\big(\mathbb{R}^n_{>0}, H^2(G)\big)\,, \\ R^{(n)}_{j_0 \cdots j_n}\big|_{\partial \mathbb{R}^n_{>0}} &\in L^1\big(\partial \mathbb{R}^n_{>0}, H^2(G)\big)\,. \end{aligned}$$

An example for the case of a bounded domain filled with a homogeneous material would be

$$R^{(n)}(\tau_1, \dots, \tau_n, x) = K^{(n)} \mathrm{e}^{-\sum_{i=1}^n \lambda_i \tau_i}$$

with a tensor $K^{(n)}$ of rank $n+1$ and constants $\lambda_1, \dots, \lambda_n > 0$ characterizing the decay of the material's memory. Appendix A.2 contains a description of the Lorentz oscillator model which can be used to describe the reaction of bound electrons to an electric field. It can be seen that the decay of the corresponding response functions is due to damping effects.

Due to the Banach algebra property of $H^2(G)$, the expression for $\mathbf{P}^{(n)}(\mathbf{E})(t)$ is well-defined as a Bochner integral if $\mathbf{E} \in C_b\big((-\infty,t], H^2(G)^3\big)$. In this case the estimate

$$\left\|\mathbf{P}^{(n)}(\mathbf{E})(t)\right\|_{H^2(G)^3} \lesssim \left(\sup_{s\le t}\|\mathbf{E}(s)\|_{H^2(G)^3}\right)^n \sum_{j_0,\dots,j_n=1}^{3}\left\|R^{(n)}_{j_0\dots j_n}\right\|_{L^1(\mathbb{R}^n_{>0},H^2(G))}$$

holds.

The Maxwell equations involve the time derivative of the polarisation which is given by the following lemma.

Lemma 3.17. *Let G be as in Assumption 3.10 and let Assumption 3.16 be true, $T>0$ and $\mathbf{E} \in C_b\big((-\infty,T), H^2(G)^3\big)$. Then $\mathbf{P}^{(n)}(\mathbf{E}) \in C^1\big((-\infty,T), H^2(G)^3\big)$ with derivative*

$$\begin{aligned}
&\partial_t P^{(n)}_{j_0}(\mathbf{E})(t) \\
&\quad= \sum_{l=1}^{n}\Bigg[\int_0^\infty\cdots\int_0^\infty \partial_{\tau_l}R^{(n)}_{j_0\dots j_n}(\tau_1,\dots,\tau_n)E_{j_1}(t-\tau_1)\dots E_{j_n}(t-\tau_n)\,\mathrm{d}\tau_1\dots\mathrm{d}\tau_n \\
&\qquad+\int_0^\infty\cdots\int_0^\infty R^{(n)}_{j_0\dots j_n}(\tau_1,\dots,\tau_{l-1},0,\tau_{l+1},\dots,\tau_n)E_{j_1}(t-\tau_1)\dots E_{j_{l-1}}(t-\tau_{l-1})E_{j_l}(t) \\
&\qquad\qquad\cdot E_{j_{l+1}}(t-\tau_{l+1})\dots E_{j_n}(t-\tau_n)\,\mathrm{d}\tau_1\dots\mathrm{d}\tau_{l-1}\,\mathrm{d}\tau_{l+1}\dots\mathrm{d}\tau_n\Bigg] \qquad (3.29)\\
&\quad= \sum_{l=1}^{n}\int_{\mathbb{R}^n_{>0}}\partial_{\tau_l}R^{(n)}_{j_0\dots j_n}(\tau)\prod_{k=1}^{n}E_{j_k}(t-\tau_k)\,\mathrm{d}\tau + \int_{\partial\mathbb{R}^n_{>0}}R^{(n)}_{j_0\dots j_n}(\tau)\prod_{k=1}^{n}E_{j_k}(t-\tau_k)\,\mathrm{d}\tau
\end{aligned}$$

for all $t<T, n\in\{1,\dots,N\}$ and $j_0\in\{1,2,3\}$.

Proof. Let $n\in\{1,\dots,N\}, j_0\in\{1,2,3\}, T_0,T_1\in\mathbb{R}$ with $T_0<T_1<T$ and $a<T_0$. We define $v^{(n)}_{j_0}: D\to H^2(G)$ by

$$v^{(n)}_{j_0}(t,s) = R^{(n)}_{j_0\dots j_n}(t-s_1,\dots,t-s_n)E_{j_1}(s_1)\dots E_{j_n}(s_n)$$

with $D := \{(t,s)=(t,s_1,\dots,s_n)\in\mathbb{R}^{n+1}\,|\, t\le T_1, s_1\le t,\dots,s_n\le t\}$. The map $v^{(n)}_{j_0}$ is continuous and the chain rule implies that in every $(t,s)\in D^\circ$, the partial derivative of $v^{(n)}_{j_0}$ with respect to t exists and is given by

$$\partial_t v^{(n)}_{j_0}(t,s) = \sum_{l=1}^{n}\partial_{\tau_l}R^{(n)}_{j_0\dots j_n}(t-s_1,\dots,t-s_n)E_{j_1}(s_1)\dots E_{j_n}(s_n)\,. \qquad (3.30)$$

By Assumption 3.16, $\partial_t v^{(n)}_{j_0}$ can be continuously extended to D (we denote the extension by the same symbol). For $a<T_0$, we further define $f^{(n)}_{a,j_0}$ and $f^{(n)}_{-\infty,j_0}:[T_0,T_1]\to H^2(G)$

by

$$f^{(n)}_{a,j_0}(t) = \int_{(a,t)^n} v^{(n)}_{j_0}(t,s)\,\mathrm{d}s\,, \quad f^{(n)}_{-\infty,j_0}(t) = \int_{(-\infty,t)^n} v^{(n)}_{j_0}(t,s)\,\mathrm{d}s = P^{(n)}_{j_0}(\mathbf{E})(t)\,.$$

1) For any $t \in [T_0, T_1]$ we can estimate

$$\begin{aligned}
&\left\| f^{(n)}_{a,j_0}(t) - f^{(n)}_{-\infty,j_0}(t) \right\|_{H^2(G)} \\
&\quad \lesssim \left(\sup_{\rho \le T_1} \|\mathbf{E}(\rho)\|_{H^2(G)^3} \right)^n \sum_{j_1,\dots,j_n=1}^{3} \int_{(-\infty,t)^n \setminus (a,t)^n} \left\| R^{(n)}_{j_0,\dots,j_n}(t-s_1,\dots,t-s_n) \right\|_{H^2(G)} \,\mathrm{d}s\,.
\end{aligned}$$

We now substitute $\tau_i = t - s_i, i = 1,\dots,n$. Then $s \in (-\infty,t)^n \setminus (a,t)^n$ is equivalent to

$$\tau \in (0,\infty)^n \setminus (0, t-a)^n \subseteq (0,\infty)^n \setminus (0, T_0 - a)^n\,.$$

Therefore we have

$$\begin{aligned}
&\left\| f^{(n)}_{a,j_0}(t) - f^{(n)}_{-\infty,j_0}(t) \right\|_{H^2(G)} \\
&\quad \lesssim \left(\sup_{\rho \le T_1} \|\mathbf{E}(\rho)\|_{H^2(G)^3} \right)^n \sum_{j_1,\dots,j_n=1}^{3} \int_{(0,\infty)^n \setminus (0,T_0-a)^n} \left\| R^{(n)}_{j_0,\dots,j_n}(\tau) \right\|_{H^2(G)} \,\mathrm{d}\tau \to 0
\end{aligned}$$

as $a \to -\infty$, so $f^{(n)}_{a,j_0}$ converges to $f^{(n)}_{-\infty,j_0}$, uniformly on $[T_0, T_1]$.

2) Let $t \in [T_0, T_1)$ and $a < T_0$ be fixed. Let $h \in (0, T_1 - t)$. Then

$$\frac{1}{h}\left(f^{(n)}_{a,j_0}(t+h) - f^{(n)}_{a,j_0}(t) \right) = I_1 + I_2$$

with

$$\begin{aligned}
I_1 &:= \int_{(a,t)^n} \frac{1}{h}\left(v^{(n)}_{j_0}(t+h,s) - v^{(n)}_{j_0}(t,s) \right) \mathrm{d}s\,, \\
I_2 &:= \int_{(a,t+h)^n \setminus (a,t)^n} \frac{1}{h} v^{(n)}_{j_0}(t+h,s)\,\mathrm{d}s\,.
\end{aligned}$$

We first treat I_1. The integrand converges to (3.30) for every $s \in (a,t)^n$ as $h \to 0^+$. It is further bounded by

$$\sup_{(\xi,s) \in [t,T_1] \times [a,t]^n} \left\| \partial_t v^{(n)}_{j_0}(\xi,s) \right\|_{H^2(G)} < \infty$$

for all $h \in (0, T_1 - t)$. Therefore, Lebesgue's theorem yields

$$I_1 \to \int_{(a,t)^n} \partial_t v^{(n)}_{j_0}(t,s)\,\mathrm{d}s$$

as $h \to 0^+$. For I_2, we split the region of integration into disjoint sets via

$$C = C_1 \dot\cup C_2 \dot\cup \dots \dot\cup C_n$$

with

$$C_m := \left\{ s \in (a, t+h)^n \;\middle|\; s_k > t \text{ for exactly } m \text{ components } s_k \right\}$$

and set

$$I_{2,m} := \frac{1}{h} \int_{C_m} v_{j_0}^{(n)}(t+h, s)\, \mathrm{d}s$$

for $m \in \{1, \dots, n\}$. The first contribution to I_2 is given by

$$\frac{1}{h} \int_{C_1} v_{j_0}^{(n)}(t+h, s) ds = \frac{1}{h} \sum_{l=1}^{n} \int_a^t \cdots \int_a^t \int_t^{t+h} \int_a^t \cdots \int_a^t$$
$$v_{j_0}^{(n)}(t, s_1, \dots, s_{l-1}, s_l, s_{l+1}, \dots, s_n)\, \mathrm{d}s_1 \dots \mathrm{d}s_{l-1}\, \mathrm{d}s_l\, \mathrm{d}s_{l+1} \dots \mathrm{d}s_n\,.$$

We now set

$$\begin{aligned} \widetilde{I}_{2,1} : &= \sum_{l=1}^{n} \int_a^t \cdots \int_a^t v_{j_0}^{(n)}(t, s_1, \dots, s_{l-1}, t, s_{l+1}, \dots, s_n)\, \mathrm{d}s_1 \dots, \mathrm{d}s_{l-1}\, \mathrm{d}s_{l+1} \dots \mathrm{d}s_n \\ &= \frac{1}{h} \sum_{l=1}^{n} \int_a^t \cdots \int_a^t \int_t^{t+h} \int_a^t \cdots \int_a^t v_{j_0}^{(n)}(t, s_1, \dots, s_{l-1}, t, s_{l+1}, \dots, s_n)\, \mathrm{d}s \end{aligned}$$

Let $\varepsilon > 0$. Since $v_{j_0}^{(n)}$ is uniformly continuous on the compact set

$$D_a := \left\{ (t, s) \in \mathbb{R}^{n+1} \;\middle|\; t \in [T_0, T_1], s \in [a, t]^n \right\}\,,$$

there exists $\delta \in (0, T_1 - t)$ such that for all $l \in \{1, \dots, n\}$ the estimate

$$\left\| v_{j_0}^{(n)}(t+h, s) - v_{j_0}^{(n)}(t, s_1, \dots, s_{l-1}, t, s_{l+1}, \dots, s_n) \right\|_{H^2(G)} \le \varepsilon$$

holds for all $s_1, \dots, s_{l-1}, s_{l+1}, \dots, s_n \in [a, t]$ and $s_l \in [t, t+h]$ with $h < \delta$. Thus,

$$\left\| I_{2,1} - \widetilde{I}_{2,1} \right\|_{H^2(G)} \le n(t-a)^n \varepsilon$$

for sufficiently small h, implying $I_{2,1} \to \widetilde{I}_{2,1}$ in $H^2(G)$ as $h \to 0^+$. The other integrals vanish as $h \to 0^+$, since

$$\|I_{2,m}\|_{H^2(G)} \le h^{m-1}(t-a)^{n-m} \sup_{(t,s) \in D_a} \left\| v_{j_0}^{(n)}(t, s) \right\|_{H^2(G)}$$

for all $m \in \{2, \dots, n\}$. Combining these results, $f_{a,j_0}^{(n)} \in C\big([T_0, T_1), H^2(G)\big)$ is differentiable from the right with right derivative

$$g_{a,j_0}^{(n)}(t) := \int_{(a,t)^n} \partial_t v_{j_0}^{(n)}(t, s)\, \mathrm{d}s \tag{3.31}$$

$$+\sum_{l=1}^{n}\int_a^t \cdots \int_a^t v_{j_0}^{(n)}(t,s_1,\dots,s_{l-1},t,s_{l+1},\dots,s_n)\,\mathrm{d}s_1\dots\mathrm{d}s_{l-1}\,\mathrm{d}s_{l+1}\dots\mathrm{d}s_n\,.$$

3) We now show that $g_{a,j_0}^{(n)} \in C\big([T_0,T_1),H^2(G)\big)$. Then we can conclude as in the proof of Theorem 2.18 that $f_{a,j_0}^{(n)}$ is contained in $C^1\big([T_0,T_1),H^2(G)\big)$ with $\frac{d}{dt}f_{a,j_0}^{(n)}(t) = g_{a,j_0}^{(n)}(t)$. Let $\varepsilon > 0$. The function $g_{a,j_0}^{(n)}$ is a sum of terms of the form

$$L(t) := \int_{(a,t)^k} w(t,s)\,\mathrm{d}s\,,$$

where $k \in \{n, n-1\}$ and $w \in C\big(\widetilde{D}, H^2(G)\big)$ with

$$\widetilde{D} := \big\{(t,s) \in \mathbb{R}^{k+1} \;\big|\; t \le T_1, s_1 \le t, \dots, s_k \le t\big\}\,.$$

Let $t \in [T_0,T_1)$ and $h \in [0, T_1 - t)$. As in step 2), we use that w is uniformly continuous on

$$\widetilde{D}_a := \big\{(t,s) \in \mathbb{R}^{k+1} \;\big|\; t \in [T_0,T_1], s \in [a,t]^k\big\}$$

to obtain a $\delta \in (0, T_1 - t)$ such that $\|w(t+h,s) - w(t,s)\|_{H^2(G)} \le \varepsilon$ for all $s \in [a,t]^k$ if $h \le \delta$. Then, for $h \le \delta$, we have

$$\begin{aligned}
&\|L(t+h) - L(t)\|_{H^2(G)} \\
&\quad\le \int_{(a,t)^k} \|w(t+h,s) - w(t,s)\|_{H^2(G)}\,\mathrm{d}s + \int_{(a,t+h)^k\setminus(a,t)^k} \|w(t+h,s)\|_{H^2(G)}\,\mathrm{d}s \\
&\quad\le \varepsilon(t-a)^k + \sup_{(t,s)\in\widetilde{D}_a} \|w(t,s)\|_{H^2(G)}\big((t+h-a)^k - (t-a)^k\big)\,,
\end{aligned}$$

which shows $L(t+h) \to L(t)$ as $h \to 0^+$. Next let $t \in (T_0,T_1)$ and $h \in (T_0 - t, 0]$. We can similarly estimate

$$\begin{aligned}
&\|L(t+h) - L(t)\|_{H^2(G)} \\
&\quad\le \int_{(a,t+h)^k} \|w(t+h,s) - w(t,s)\|_{H^2(G)}\,\mathrm{d}s + \int_{(a,t)^k\setminus(a,t+h)^k} \|w(t,s)\|_{H^2(G)}\,\mathrm{d}s \\
&\quad\le \varepsilon(t+h-a)^k + \sup_{(t,s)\in\widetilde{D}_a} \|w(t,s)\|_{H^2(G)}\big((t-a)^k - (t+h-a)^k\big)
\end{aligned}$$

for sufficiently small $|h|$, proving $L(t+h) \to L(t)$ as $h \to 0^-$.

4) We define $g_{-\infty,j_0}^{(n)}$ by (3.31) with a replaced by $-\infty$, so $g_{-\infty,j_0}^{(n)}(t)$ is the right hand side in the claim (3.29) (using $\tau_k = t - s_k$). We have

$$\Big\|g_{a,j_0}^{(n)}(t) - g_{-\infty,j_0}^{(n)}(t)\Big\|_{H^2(G)} \lesssim K_1 + K_2\,.$$

The first term

$$K_1 = \int_{(-\infty,t)^n\setminus(a,t)^n} \Big\|\partial_t v_{j_0}^{(n)}(t,s)\Big\|_{H^2(G)}\,\mathrm{d}s$$

converges to 0 as $a \to -\infty$, uniformly in $t \in [T_0, T_1)$ as in step 1), using that by Assumption 3.16, $\partial_{\tau_l} R^{(n)}_{j_0,\dots,j_n} \in L^1\big((0,\infty), H^2(G)\big)$ for all $l \in \{1,\dots,n\}$. The second term

$$
\begin{aligned}
K_2 = \sum_{l=1}^{n} \sum_{j_1,\dots,j_n=1}^{3} & \Big(\sup_{\rho \le T_1} \|E(\rho)\|_{H^2(G)}\Big)^n \\
& \cdot \int_{(-\infty,t)^{n-1} \setminus (a,t)^{n-1}} \Big\| R^{(n)}_{j_0,\dots,j_n}(t-s_1,\dots,t-s_{l-1},0,t-s_{l+1},\dots,t-s_n) \Big\|_{H^2(G)} \\
& \mathrm{d}s_1 \dots \mathrm{d}s_{l-1}\, \mathrm{d}s_{l+1} \dots \mathrm{d}s_n
\end{aligned}
$$

also vanishes as $a \to -\infty$ as in step 1), uniformly in $t \in [T_0, T_1)$. In particular, $g^{(n)}_{-\infty,j_0}$ is contained in $C\big([T_0,T_1), H^2(G)\big)$.

5) Let $\varepsilon > 0$ and $t \in [T_0, T_1)$. Since $g^{(n)}_{-\infty,j_0}$ is continuous, there exists $\delta > 0$ such that

$$
\left\| \frac{1}{h} \int_t^{t+h} g^{(n)}_{-\infty,j_0}(s)\, \mathrm{d}s - g^{(n)}_{-\infty,j_0}(t) \right\|_{H^2(G)} \le \varepsilon
$$

for all $h \in \mathbb{R} \setminus \{0\}$ with $t + h \in [T_0, T_1)$ and $|h| \le \delta$. Furthermore, the uniform convergence of $\frac{d}{dt} f^{(n)}_{a,j_0} = g^{(n)}_{a,j_0}$ to $g^{(n)}_{-\infty,j_0}$ implies the existence of an $A \in \mathbb{R}$ such that $\left\| \frac{d}{dt} f^{(n)}_{a,j_0}(s) - g^{(n)}_{-\infty,j_0}(s) \right\|_{H^2(G)} \le \varepsilon$ for all $a \le A$ and $s \in [T_0, T_1)$. Thus,

$$
\begin{aligned}
& \left\| \frac{f^{(n)}_{-\infty,j_0}(t+h) - f^{(n)}_{-\infty,j_0}(t)}{h} - g^{(n)}_{-\infty,j_0}(t) \right\|_{H^2(G)} \\
&\quad = \lim_{a \to -\infty} \left\| \frac{f^{(n)}_{a,j_0}(t+h) - f^{(n)}_{a,j_0}(t)}{h} - g^{(n)}_{-\infty,j_0}(t) \right\|_{H^2(G)} \\
&\quad \le \varlimsup_{a \to -\infty} \Bigg(\frac{1}{h} \int_t^{t+h} \left\| \frac{d}{ds} f^{(n)}_{a,j_0}(s) - g^{(n)}_{-\infty,j_0}(s) \right\|_{H^2(G)} \mathrm{d}s \\
&\qquad\qquad + \left\| \frac{1}{h} \int_t^{t+h} g^{(n)}_{-\infty,j_0}(s)\, \mathrm{d}s - g^{(n)}_{-\infty,j_0}(t) \right\|_{H^2(G)} \Bigg) \le 2\varepsilon
\end{aligned}
$$

for all h as above, which proves the claim since $\varepsilon > 0$ and the interval $[T_0, T_1) \subseteq (-\infty, T)$ are arbitrary. □

The lemma motivates the definition of a map $F_{j_0} : C_b\big((-\infty,0]), H^2(G)^6\big) \to H^2(G)^6$ by

$$
\begin{aligned}
F_{j_0}(\mathbf{E}, \mathbf{H}) = -\varepsilon^{-1} \sum_{n=1}^{N} \Bigg[& \sum_{l=1}^{n} \int_{\mathbb{R}^n_{\ge 0}} \partial_{\tau_l} R^{(n)}_{j_0 \dots j_n}(\tau) \prod_{k=1}^{n} E_{j_k}(-\tau_k)\, \mathrm{d}\tau \\
& + \int_{\partial \mathbb{R}^n_{\ge 0}} R^{(n)}_{j_0 \dots j_n}(\tau) \prod_{k=1}^{n} E_{j_k}(-\tau_k)\, \mathrm{d}\tau \Bigg]
\end{aligned}
\tag{3.32}
$$

for $j_0 \in \{1,2,3\}$ and $F_{j_0}(\mathbf{E},\mathbf{H}) = 0$ for $j_0 \in \{4,5,6\}$. Here the variable t is missing in contrast to (3.29) and F is actually independent of $\mathbf{H}$. If Assumption 3.16 is valid and $u = (\mathbf{E},\mathbf{H}) \in C_b\big((-\infty,T), H^2(G)^6\big)$ for some $T > 0$, we can thus write

$$-\varepsilon^{-1}\begin{pmatrix}\partial_t \widetilde{\mathbf{P}}(\mathbf{E})(t)\\ 0\end{pmatrix} = F(u_t)$$

for all $t < T$ (multiplication by ε^{-1} is an isomorphism on $H^2(G)^3$ by Lemma 3.9, so Lemma 3.17 remains true for $\varepsilon^{-1}\widetilde{P}(\mathbf{E})$).

The next lemma yields the Lipschitz property of F required by Assumption 2.1.

Lemma 3.18. *The map F is contained in $C^1\big(C_b\big((-\infty,0], H^2(G)^6\big), H^2(G)^6\big)$. The derivative F' is bounded on bounded sets. In particular, F is Lipschitz on bounded sets.*

Proof. Let $j_0 \in \{1,2,3\}$, Assumption 3.16 be true and $u = (\mathbf{E},\mathbf{H}), v = (\widetilde{\mathbf{E}},\widetilde{\mathbf{H}}) \in C_b\big((-\infty,0], H^2(G)^6\big)$ with $v \neq 0$. We have

$$\begin{aligned} &F_{j_0}(u+v) - F_{j_0}(u)\\ &= -\varepsilon^{-1}\sum_{n=1}^{N}\Bigg[\sum_{l,m=1}^{n}\int_{\mathbb{R}^n_{>0}} \partial_{\tau_l} R^{(n)}_{j_0\dots j_n}(\tau)\Bigg(\prod_{\substack{k=1,\\k\neq m}}^{n} E_{j_k}(-\tau_k)\Bigg)\widetilde{E}_{j_m}(-\tau_m)\,\mathrm{d}\tau \qquad (3.33)\\ &\qquad + \sum_{m=1}^{n}\int_{\partial\mathbb{R}^n_{>0}} R^{(n)}_{j_0\dots j_n}(\tau)\Bigg(\prod_{\substack{k=1,\\k\neq m}}^{n} E_{j_k}(-\tau_k)\Bigg)\widetilde{E}_{j_m}(-\tau_m)\,\mathrm{d}\tau\Bigg]\\ &\qquad + \mathcal{O}\Bigg(\Big(\sup_{t\le 0}\|v(t)\|_{H^2(G)^6}\Big)^2\Bigg). \end{aligned}$$

Let $(L(u)v)_{j_0}$ be given by the linear (in $\widetilde{E}$) terms on the right hand side of (3.33). So we obtain

$$\frac{\|F(u+v) - F(u) - L(u)v\|_{H^2(G)^6}}{\|v\|_{C_b((-\infty,0],H^2(G)^6)}} \to 0$$

as $v \to 0$.

For $j_0 \in \{4,5,6\}$ we have $(L(u)v)_{j_0} = 0$. The estimate

$$\begin{aligned}\|L(u)v\|_{H^2(G)^6} \lesssim_{\varepsilon,\eta} \sum_{n=1}^{N}\sum_{j_0\dots j_n=1}^{3}\Bigg[\sum_{l=1}^{n}\Big\|\partial_{\tau_l} R^{(n)}_{j_0\dots j_n}\Big\|_{L^1(\mathbb{R}^n_{>0},H^2(G))} + \Big\|R^{(n)}_{j_0\dots j_n}\Big\|_{L^1(\partial\mathbb{R}^n_{>0},H^2(G))}\Bigg]\\ \cdot\Big(\sup_{t\le 0}\|u(t)\|_{H^2(G)^6}\Big)^{n-1}\Big(\sup_{t\le 0}\|v(t)\|_{H^2(G)^6}\Big)\end{aligned}$$

shows that $L(u)$ is a bounded operator from $C_b\big((-\infty,0], H^2(G)^6\big)$ to $H^2(G)^6$. Hence F is differentiable with $F'(u) = L(u)$. Also $u \mapsto L(u)$ is bounded on bounded sets.

It remains to show the continuity of the derivative. Let $u = (\mathbf{E}, \mathbf{H}), v = (\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}})$ and $w = (\widehat{\mathbf{E}}, \widehat{\mathbf{H}})$ be contained in $C_b\big((-\infty, 0], H^2(G)^6\big)$. Similar to the above calculation, we get

$$\begin{aligned}
&\left\| \big(F'(u+w) - F'(u)\big)v \right\|_{H^2(G)^6} \\
&\lesssim_{\varepsilon,\eta} \sum_{n=1}^{N} \sum_{m=1}^{n} \left[\sum_{l=1}^{n} \left\| \partial_{\tau_l} R^{(n)}_{j_0\dots j_n} \right\|_{L^1(\mathbb{R}^n_{>0}, H^2(G))} + \left\| R^{(n)}_{j_0\dots j_n} \right\|_{L^1(\partial\mathbb{R}^n_{>0}, H^2(G))} \right] \\
&\quad \cdot \left(\sup_{t\le 0} \left\| \prod_{\substack{k=1,\\ k\neq m}}^{n} \left(E_{j_k}(t) - \widehat{E}_{j_k}(t) \right) - \prod_{\substack{k=1,\\ k\neq m}}^{n} E_{j_k}(t) \right\|_{H^2(G)^3} \right) \left(\sup_{t\le 0} \left\| \widetilde{E}_{j_m}(t) \right\|_{H^2(G)^3} \right).
\end{aligned}$$

The estimates $\sup_{t\le 0} \left\| \widetilde{E}_{j_m}(t) \right\|_{H^2(G)^3} \le \sup_{t\le 0} \|v(t)\|_{H^2(G)^6}$ and

$$\begin{aligned}
&\sup_{t\le 0} \left\| \prod_{\substack{k=1\\ k\neq m}}^{n} \left(E_{j_k}(t) - \widehat{E}_{j_k}(t) \right) - \prod_{\substack{k=1\\ k\neq m}}^{n} E_{j_k}(t) \right\|_{H^2(G)^3} \\
&\quad \lesssim \sum_{i=1}^{n-1} \left(\sup_{t\le 0} \|u(t)\|_{H^2(G)^6} \right)^{n-i} \left(\sup_{t\le 0} \|w(t)\|_{H^2(G)^6} \right)^{i}
\end{aligned}$$

now lead to $\|F'(u+w) - F'(u)\| \to 0$ as $w \to 0$ in $C_b\big((-\infty, 0], H^2(G)^6\big)$. □

To be to able to write (3.11) in the form of equation (2.1) (with A_2 and X_2 instead of A and X), the property

$$F \text{ maps } C_b\big((-\infty, 0], X_2\big) \text{ to } X_2 \tag{3.34}$$

is required. Unfortunately this is in general not the case: $\mathbf{P}$ does not need to be parallel to $\mathbf{E}$, so $\nu \times \mathbf{E} = 0$ on ∂G does not imply $\nu \times \mathbf{P} = 0$ on ∂G. Therefore we need to restrict ourselves to scalar-type material laws, defined in the following way.

Assumption 3.19. *We assume the polarisation to be described by a scalar-type material law, i. e., for every $n \in \{1, \dots, N\}$, there exists a map $\mathcal{R}^{(n)} : \mathbb{R}^n_{\ge 0} \times G \times (\mathbb{R}^3)^{(n-1)} \to \mathbb{R}$ which is multilinear in the last $n-1$ arguments and satisfies*

$$R^{(n)}_{j_0\dots j_n}(\tau, x) \prod_{k=1}^{n} E^{(k)}_{j_k} = \mathcal{R}^{(n)}\left(\tau, x, \mathbf{E}^{(1)}, \dots, \mathbf{E}^{(n-1)}\right) E^{(n)}_{j_0} \tag{3.35}$$

for all $j_0 \in \{1,2,3\}, \tau \in \mathbb{R}^n_{\ge 0}, x \in G$ and $\mathbf{E}^{(1)}, \dots, \mathbf{E}^{(n)} \in \mathbb{R}^3$.

An example is the *Kerr effect* which is a third order effect described by

$$\mathcal{R}^{(3)}\left(\tau, \mathbf{E}^{(1)}, \mathbf{E}^{(2)}\right) = \widetilde{\mathcal{R}}^{(3)}(\tau)\big(\mathbf{E}^{(1)} \cdot \mathbf{E}^{(2)}\big)$$

In this case, the nonlinear part of the polarisation is given by

$$\mathbf{P}^{(3)}(\mathbf{E})(t) = \int_0^\infty \int_0^\infty \int_0^\infty \widetilde{\mathcal{R}}^{(3)}(\tau_1,\tau_2,\tau_3)\big(\mathbf{E}(t-\tau_1)\cdot\mathbf{E}(t-\tau_2)\big)\mathbf{E}(t-\tau_3)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3\,.$$

In Appendix A.1, it is shown that such scalar-type laws arise in the special case of homogeneous and isotropic materials with inversion symmetry.

The required properties on the response functions from Assumption 3.16 carry over to scalar-type laws, as stated in the next lemma. We denote the ith unit vector in $\mathbb{R}^3$ by $\mathbf{e}_i$ and again omit the spatial variable x.

Lemma 3.20. *Let Assumptions 3.16 and 3.19 be true. Then*

$$\mathcal{R}^{(n)}\left(\cdot,\mathbf{e}_{m_1},\dots,\mathbf{e}_{m_{n-1}}\right) \in C^1\big(\mathbb{R}^n_{\geq 0},H^2(G)\big)\cap L^1\big(\mathbb{R}^n_{>0},H^2(G)\big)$$

for all $m_1,\dots,m_{n-1}\in\{1,2,3\}$ and $n\in\{1,\dots,N\}$. Let $\mathbf{E}\in C_b\big((-\infty,0],H^2(G)^3\big)$ and $l\in\{1,\dots,n\}$. For any $n\in\{1,\dots,N\}$, the maps $\mathbf{\Phi}^{(n)},\mathbf{\Psi}_l^{(n)}:\mathbb{R}^n_{\geq 0}\to H^2(G)^3$ and $\mathbf{\Xi}^{(n)}:\partial\mathbb{R}^n_{\geq 0}\to H^2(G)^3$ given by

$$\begin{aligned}
\mathbf{\Phi}^{(n)}(\tau) &= \mathcal{R}^{(n)}\big(\tau,\mathbf{E}(-\tau_1),\dots,\mathbf{E}(-\tau_{n-1})\big)\mathbf{E}(-\tau_n)\,,\\
\mathbf{\Psi}_l^{(n)}(\tau) &= \partial_{\tau_l}\mathcal{R}^{(n)}\big(\tau,\mathbf{E}(-\tau_1),\dots,\mathbf{E}(-\tau_{n-1})\big)\mathbf{E}(-\tau_n)\,,\\
\mathbf{\Xi}^{(n)}(\tau) &= \mathcal{R}^{(n)}\big(\tau,\mathbf{E}(-\tau_1),\dots,\mathbf{E}(-\tau_{n-1})\big)\mathbf{E}(-\tau_n)
\end{aligned}$$

satisfy

$$\mathbf{\Phi}^{(n)}\,,\ \mathbf{\Psi}_l^{(n)}\in L^1\big(\mathbb{R}^n_{>0},H^2(G)^3\big)\,,\quad \mathbf{\Xi}^{(n)}\in L^1\big(\partial\mathbb{R}^n_{>0},H^2(G)^3\big)\,.$$

For the map F from (3.32), we obtain $F(\mathbf{E},\mathbf{H})=(\mathbf{F}(\mathbf{E}),\mathbf{0})$ with

$$\mathbf{F}(\mathbf{E}) = -\varepsilon^{-1}\sum_{n=1}^N\left[\sum_{l=1}^n\int_{\mathbb{R}^n_{>0}}\mathbf{\Psi}_l^{(n)}(\tau)\,\mathrm{d}\tau+\int_{\partial\mathbb{R}^n_{>0}}\mathbf{\Xi}^{(n)}(\tau)\,\mathrm{d}\tau\right]. \tag{3.36}$$

Proof. Inserting the unit vectors $\mathbf{E}^{(k)}=\mathbf{e}_{m_k}$ into (3.35) for $m_k\in\{1,2,3\}$ and $k=1,\dots,n$, we obtain

$$R^{(n)}_{j_0m_1\dots m_n}(\tau)=\mathcal{R}^{(n)}\big(\tau,\mathbf{e}_{m_1},\dots,\mathbf{e}_{m_{n-1}}\big)\delta_{j_0m_n} \tag{3.37}$$

for all $\tau\in\mathbb{R}^n_{\geq 0}, j_0\in\{1,2,3\}$. Hence $R^{(n)}_{j_0m_1\dots m_n}$ is uniquely determined by $\mathcal{R}^{(n)}$. Formula (3.37) and Assumption 3.16 yield

$$\mathcal{R}^{(n)}\big(\cdot,\mathbf{e}_{m_1},\dots,\mathbf{e}_{m_{n-1}}\big)=\frac{1}{3}R^{(n)}_{m_nm_1\dots m_n}\in C^1\big(\mathbb{R}^n_{\geq 0},H^2(G)\big)\cap L^1\big(\mathbb{R}^n_{>0},H^2(G)\big) \tag{3.38}$$

as well as

$$\begin{aligned}
\partial_{\tau_l}\mathcal{R}^{(n)}\big(\cdot,\mathbf{e}_{m_1},\dots,\mathbf{e}_{m_{n-1}}\big) &\in L^1\big(\mathbb{R}^n_{>0},H^2(G)\big)\,,\\
\mathcal{R}^{(n)}\big(\cdot,\mathbf{e}_{m_1},\dots,\mathbf{e}_{m_{n-1}}\big)|_{\partial\mathbb{R}^n_{>0}} &\in L^1\big(\partial\mathbb{R}^n_{>0},H^2(G)\big)
\end{aligned}$$

for all $l \in \{1, \dots, n\}$ (in (3.38), the index m_n appearing twice on the right hand side implies a sum over m_n according to the sum convention).

Since $\mathbf{\Phi}^{(n)}$ is continuous in $H^2(G)^3$ and

$$\left\|\mathbf{\Phi}^{(n)}(\cdot)\right\|_{H^2(G)^3} \lesssim \left(\sup_{s\le 0} \|\mathbf{E}(s)\|_{H^2(G)^3}\right)^n \max_{\substack{m_1,\dots,m_{n-1}\\ \in\{1,2,3\}}} \left\|\mathcal{R}^{(n)}(\cdot, \mathbf{e}_{m_1}, \dots, \mathbf{e}_{m_{n-1}})\right\|_{H^2(G)} \\ \in L^1\big(\mathbb{R}^n_{>0}\big),$$

we have $\mathbf{\Phi}^{(n)} \in L^1\big(\mathbb{R}^n_{>0}, H^2(G)^3\big)$. The statements for $\mathbf{\Psi}_l^{(n)}$ and $\mathbf{\Xi}^{(n)}$ are shown analogously. From (3.37) further follows

$$\begin{aligned}\partial_{\tau_l} R^{(n)}_{j_0 j_1 \dots j_n}(\tau) \prod_{k=1}^n E_{j_k}(-\tau_k) &= \partial_{\tau_l} \mathcal{R}^{(n)}(\tau, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}) \delta_{j_0 j_n} \prod_{k=1}^n E_{j_k}(-\tau_k) \\ &= \partial_{\tau_l} \mathcal{R}^{(n)}\big(\tau, \mathbf{E}(-\tau_1), \dots, \mathbf{E}(-\tau_{n-1})\big) E_{j_0}(-\tau_n)\end{aligned} \tag{3.39}$$

for all $j_0 \in \{1,2,3\}$, which implies (3.36). □

We can now prove that (3.34) is satisfied for scalar-type material laws. In the following, the restriction of F to $C_b\big((-\infty,0], X_2\big)$ is again denoted by F.

Lemma 3.21. *Let Assumptions 3.10, 3.16 and 3.19 be true and let $u \in C_b\big((-\infty,0], X_2\big)$. Then $F(u) \in X_2$. Further, F is contained in $C^1\big(C_b\big((-\infty,0]), X_2\big), X_2\big)$ and the derivative F' is bounded on bounded sets. In particular, F is Lipschitz on bounded sets.*

Proof. Let $u, v \in C_b\big((-\infty,0], X_2\big)$. As a consequence of Lemma 3.18, we only need to show $F(u) \in X_2$ and $F'(u)v \in X_2$. Since the magnetic components of $F(u)$ vanish, i. e., $F(u) = (\mathbf{F}(\mathbf{E}), \mathbf{0})$, by Lemma 3.15 it suffices to prove $\operatorname{tr}_\tau \mathbf{F}(\mathbf{E}) = 0$ to conclude $F(u) \in X_2$. We use (3.36). Let $n \in \{1, \dots, N\}$ and $l \in \{1, \dots, n\}$. We have

$$\operatorname{tr}_\tau \int_{\mathbb{R}^n_{>0}} \mathbf{\Psi}_l^{(n)}(\tau)\, \mathrm{d}\tau = \int_{\mathbb{R}^n_{>0}} \operatorname{tr}_\tau \mathbf{\Psi}_l^{(n)}(\tau)\, \mathrm{d}\tau$$

due to the continuity of tr_τ. The same calculation as in (3.8) yields

$$\begin{aligned}&\left\langle \operatorname{tr}\varphi, \operatorname{tr}_\tau \mathbf{\Psi}_l^{(n)}(\tau)\right\rangle_{H^{1/2}(\partial G)^3 \times H^{-1/2}(\partial G)^3} \\ &= \left\langle \operatorname{tr}\left(\partial_{\tau_l} \mathcal{R}^{(n)}\big(\tau, \mathbf{E}(-\tau_1), \dots, \mathbf{E}(-\tau_{n-1})\big)\varphi\right), \operatorname{tr}_\tau \mathbf{E}(-\tau_n)\right\rangle_{H^{1/2}(\partial G)^3 \times H^{-1/2}(\partial G)^3} = 0\end{aligned}$$

for all $\varphi \in H^1(G)^3$ which implies $\operatorname{tr}_\tau \mathbf{\Psi}_l^{(n)}(\tau) = 0$. Analogously $\operatorname{tr}_\tau \mathbf{\Xi}^{(n)}(\tau)$ vanishes. It follows $\operatorname{tr}_\tau\big(\varepsilon \mathbf{F}(\mathbf{E})\big) = 0$ and therefore $\operatorname{tr}_\tau \mathbf{F}(\mathbf{E}) = 0$ by Lemma 3.6. Using the expression for $F'(u)v$ from Lemma 3.18, we see as above that $F'(u)v \in X_2$. □

To use Theorem 2.18, we need a differentiability property of F.

Lemma 3.22. *Let Assumptions 3.10, 3.16 and 3.19 be true and $b > 0$.*

1) *Let* $u \in C_b\big((-\infty,b),X_2\big)$*. Then the map* $t \mapsto F(u_t)$ *is contained in* $C\big((-\infty,b),X_2\big)$*.*

2) *Let* $u \in C_b^1\big((-\infty,b),X_2\big)$*. Then the map* $t \mapsto F(u_t)$ *is contained in* $C^1\big((-\infty,b),X_2\big)$ *with derivative* $F'(u_t)(u')_t$*.*

Proof. Let $b>0, u=(\mathbf{E},\mathbf{H}) \in C_b\big((-\infty,b),X_2\big)$ and $j_0 \in \{1,2,3\}$. We have

$$F_{j_0}(u_t) = -\varepsilon^{-1}\sum_{n=1}^{N}\left[\sum_{l=1}^{n}\int_{\mathbb{R}^n_{>0}} \psi^{(n)}_{l,j_0}(t,\tau)\,\mathrm{d}\tau + \int_{\partial\mathbb{R}^n_{>0}} \xi^{(n)}_{j_0}(t,\tau)\,\mathrm{d}\tau\right]$$

with functions $\psi^{(n)}_{l,j_0} : (-\infty,b)\times\mathbb{R}^n_{>0} \to H^2(G), \xi^{(n)}_{j_0} : (-\infty,b)\times\partial\mathbb{R}^n_{>0} \to H^2(G)$ defined by

$$\begin{aligned}\psi^{(n)}_{l,j_0}(t,\tau) &= \partial_{\tau_l}\mathcal{R}^{(n)}\big(\tau,\mathbf{E}(t-\tau_1),\dots,\mathbf{E}(t-\tau_{n-1})\big)E_{j_0}(t-\tau_n)\,,\\ \xi^{(n)}_{j_0}(t,\tau) &= \mathcal{R}^{(n)}\big(\tau,\mathbf{E}(t-\tau_1),\dots,\mathbf{E}(t-\tau_{n-1})\big)E_{j_0}(t-\tau_n)\,.\end{aligned}$$

Let $n \in \{1,\dots,N\}$. The Banach algebra property of $H^2(G)^3$ implies that the map $(t,\tau)\mapsto E_{j_0}(t-\tau_n)E_{j_1}(t-\tau_1)\dots E_{j_{n-1}}(t-\tau_{n-1})$ is contained in $C_b\big((-\infty,b)\times\mathbb{R}^n_{\geq 0},H^2(G)\big)$ for all $j_1,\dots,j_{n-1}\in\{1,2,3\}$. Writing

$$\begin{aligned}\psi^{(n)}_{l,j_0}(t,\tau) &= \partial_{\tau_l}\mathcal{R}^{(n)}\big(\tau,\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_{n-1}}\big)E_{j_1}(t-\tau_1)\dots E_{j_{n-1}}(t-\tau_{n-1})E_{j_0}(t-\tau_n)\,,\\ \xi^{(n)}_{j_0}(t,\tau) &= \mathcal{R}^{(n)}\big(\tau,\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_{n-1}}\big)E_{j_1}(t-\tau_1)\dots E_{j_{n-1}}(t-\tau_{n-1})E_{j_0}(t-\tau_n)\,,\end{aligned}$$

and using Lemma 3.20 we see that

i) $\psi^{(n)}_{l,j_0}(t,\cdot) \in L^1\big(\mathbb{R}^n_{>0},H^2(G)\big)$ and $\xi^{(n)}_{j_0}(t,\cdot) \in L^1\big(\partial\mathbb{R}^n_{>0},H^2(G)\big)$ for all $t\in(-\infty,b)$,

ii) $\psi^{(n)}_{l,j_0}(\cdot,\tau) \in C\big((-\infty,b),H^2(G)\big)$ for all $\tau\in\mathbb{R}^n_{>0}$ as well as $\xi^{(n)}_{j_0}(\cdot,\tau) \in C\big((-\infty,b),H^2(G)\big)$ for all $\tau\in\partial\mathbb{R}^n_{>0}$,

iii) $\left\|\psi^{(n)}_{l,j_0}(t,\tau)\right\|_{H^2(G)} \lesssim_G g^{(n)}_{l,j_0}(\tau)$ for all $(t,\tau)\in(-\infty,b)\times\mathbb{R}^n_{>0}$ as well as $\left\|\xi^{(n)}_{j_0}(t,\tau)\right\|_{H^2(G)} \lesssim_G h^{(n)}_{j_0}(\tau)$ for all $(t,\tau)\in(-\infty,b)\times\partial\mathbb{R}^n_{>0}$,

where the functions $g^{(n)}_{l,j_0}\in L^1(\mathbb{R}^n_{>0})$ and $h^{(n)}_{j_0}\in L^1(\partial\mathbb{R}^n_{>0})$ are given by

$$\begin{aligned}g^{(n)}_{l,j_0}(\tau) &= \left(\sup_{s<b}\|u(s)\|_{H^2(G)^6}\right)^n \left\|\partial_{\tau_l}\mathcal{R}^{(n)}\big(\tau,\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_{n-1}}\big)\right\|_{H^2(G)}\,,\\ h^{(n)}_{j_0}(\tau) &= \left(\sup_{s<b}\|u(s)\|_{H^2(G)^6}\right)^n \left\|\mathcal{R}^{(n)}\big(\tau,\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_{n-1}}\big)\right\|_{H^2(G)}\,.\end{aligned}$$

By a corollary to Lebesgue's theorem, $t\mapsto F_{j_0}(u_t)$ is contained in $C\big((-\infty,b),H^2(G)\big)$. Since $u_t\in C_b\big((-\infty,0],X_2\big)$ for all $t<b$ and the H^2- and X_2-norms are equivalent by Lemma 3.15, Lemma 3.21 yields $t\mapsto F(u_t)\in C\big((-\infty,b),X_2\big)$.

To show 2), let $u = (\mathbf{E}, \mathbf{H}) \in C_b^1\big((-\infty, b), X_2\big)$. Then the map

$$t \mapsto E_{j_1}(t-\tau_1)\dots E_{j_{n-1}}(t-\tau_{n-1})E_{j_0}(t-\tau_n)$$

is contained in $C_b^1\big((-\infty, b), H^2(G)\big)$ for all $j_1, \dots, j_{n-1} \in \{1,2,3\}$ and $\tau_1, \dots, \tau_n \in \mathbb{R}_{\geq 0}$. The derivative is given by

$$\partial_t E_{j_1}(t-\tau_1)E_{j_2}(t-\tau_2)\dots E_{j_0}(t-\tau_n) + \dots + E_{j_1}(t-\tau_1)\dots E_{j_{n-1}}(t-\tau_{n-1})\partial_t E_{j_0}(t-\tau_n)\,.$$

Then we have

ii') $\psi_{l,j_0}^{(n)}(\cdot, \tau) \in C^1\big((-\infty, b), H^2(G)\big)$ for all $\tau \in \mathbb{R}^n_{>0}$ as well as $\xi_{j_0}^{(n)}(\cdot, \tau) \in C^1\big((-\infty, b), H^2(G)\big)$ for all $\tau \in \partial\mathbb{R}^n_{>0}$,

iii') $\Big\|\partial_t \psi_{l,j_0}^{(n)}(t,\tau)\Big\|_{H^2(G)} \lesssim_G \widetilde{g}_{l,j_0}(\tau)$ for all $(t,\tau) \in (-\infty, b) \times \mathbb{R}^n_{>0}$ as well as $\Big\|\partial_t \xi_{j_0}^{(n)}(t,\tau)\Big\|_{H^2(G)} \lesssim_G \widetilde{h}_{j_0}(\tau)$ for all $(t,\tau) \in (-\infty, b) \times \partial\mathbb{R}^n_{>0}$,

where the functions $\widetilde{g}_{l,j_0}^{(n)} \in L^1(\mathbb{R}^n_{>0})$ and $\widetilde{h}_{j_0}^{(n)} \in L^1(\partial\mathbb{R}^n_{>0})$ are given by

$$\widetilde{g}_{l,j_0}^{(n)}(\tau) = \left(\sup_{s<b} \|u(s)\|_{H^2(G)^6}\right)^{n-1} \left(\sup_{s<b} \|u'(s)\|_{H^2(G)^6}\right) \big\|\partial_{\tau_l} \mathcal{R}^{(n)}\big(\tau, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}\big)\big\|_{H^2(G)}\,,$$

$$\widetilde{h}_{j_0}^{(n)}(\tau) = \left(\sup_{s<b} \|u(s)\|_{H^2(G)^6}\right)^{n-1} \left(\sup_{s<b} \|u'(s)\|_{H^2(G)^6}\right) \big\|\mathcal{R}^{(n)}\big(\tau, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}\big)\big\|_{H^2(G)}\,.$$

As above, it follows that $t \mapsto F_{j_0}(u_t)$ is contained in $C^1\big((-\infty, b), H^2(G)\big)$ with

$$\partial_t F_{j_0}(u_t) = -\varepsilon^{-1} \sum_{n=1}^{N} \left[\sum_{l=1}^{n} \int_{\mathbb{R}^n_{>0}} \partial_t \psi_{l,j_0}^{(n)}(t,\tau)\, \mathrm{d}\tau + \int_{\partial\mathbb{R}^n_{>0}} \partial_t \xi_{j_0}^{(n)}(t,\tau)\, \mathrm{d}\tau\right].$$

We insert the expressions

$$\begin{aligned}
\partial_t \psi_{l,j_0}^{(n)}(t,\tau) &= \partial_{\tau_l} \mathcal{R}^{(n)}\big(\tau, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}\big) \Big[\partial_t E_{j_1}(t-\tau_1)E_{j_2}(t-\tau_2)\dots E_{j_0}(t-\tau_n) \\
&\qquad + \dots + E_{j_1}(t-\tau_1)\dots E_{j_{n-1}}(t-\tau_{n-1})\partial_t E_{j_0}(t-\tau_n)\Big] \\
&= \partial_{\tau_l} \mathcal{R}^{(n)}\big(\tau, \partial_t \mathbf{E}(t-\tau_1), \mathbf{E}(t-\tau_2), \dots, \mathbf{E}(t-\tau_{n-1})\big) E_{j_0}(t-\tau_n) \\
&\qquad + \dots + \mathcal{R}^{(n)}\big(\tau, \mathbf{E}(t-\tau_1), \mathbf{E}(t-\tau_2), \dots, \mathbf{E}(t-\tau_{n-1})\big) \partial_t E_{j_0}(t-\tau_n)
\end{aligned}$$

and

$$\begin{aligned}
\partial_t \xi_{j_0}^{(n)}(t,\tau) &= \mathcal{R}^{(n)}\big(\tau, \partial_t \mathbf{E}(t-\tau_1), \mathbf{E}(t-\tau_2), \dots, \mathbf{E}(t-\tau_{n-1})\big) E_{j_0}(t-\tau_n) \\
&\qquad + \dots + \mathcal{R}^{(n)}\big(\tau, \mathbf{E}(t-\tau_1), \mathbf{E}(t-\tau_2), \dots, \mathbf{E}(t-\tau_{n-1})\big) \partial_t E_{j_0}(t-\tau_n)\,.
\end{aligned}$$

Comparison with (3.33) shows $\partial_t F_{j_0}(u_t) = \big(F'(u_t)(u')_t\big)_{j_0}$. In particular, we have that $t \mapsto F(u_t)$ is contained not only in $C^1\big((-\infty, b), H^2(G)^6\big)$ but in $C^1\big((-\infty, b), X_2\big)$. □

3.5 Local wellposedness results

We are now in a position to formulate the reduced system of Maxwell equations (3.11) as an abstract retarded evolution equation of the form (2.1), provided Assumptions 3.10, 3.16 and 3.19 hold. Under the condition that the external current density $\mathbf{J}_0$ is contained in $C\big(I, H^2(G)^3\big)$ for $I = [0,\infty)$ or $I = [0, t_{\text{end}})$ for some $t_{\text{end}} > 0$ and satisfies $\operatorname{tr}_\tau \mathbf{J}_0(t) = 0$ for all $t \in I$, we can define $g : I \to X_2$ by

$$g(t) = -\varepsilon^{-1}\begin{pmatrix}\mathbf{J}_0(t)\\ 0\end{pmatrix}.$$

In this setting, the problem (3.11) takes the form

$$\begin{aligned} u'(t) &= A_2 u(t) + F(u_t) + g(t)\,, \quad t \in I\,, \\ u(t) &= f(t)\,, \quad t \le 0\,, \end{aligned} \tag{3.40}$$

in the space X_2, where $f = (\mathbf{E}_{\text{h}}, \mathbf{H}_{\text{h}}) \in C_b\big((-\infty,0], X_2\big)$ contains the history of the fields. The boundary condition for $\mathbf{E}$ is included in the definition of X_2. The expression (2.2) for a mild solution $u = (\mathbf{E},\mathbf{H}) \in C(J, X_2)$ of (3.40) on an interval J with $(-\infty,0] \subseteq J$ and $0 < \sup J \le \sup I$ can here be written in the form

$$\begin{pmatrix}\mathbf{E}(t)\\ \mathbf{H}(t)\end{pmatrix} = T_2(t)\begin{pmatrix}\mathbf{E}_{\text{h}}(0)\\ \mathbf{H}_{\text{h}}(0)\end{pmatrix} - \int_0^t T_2(t-s)\varepsilon^{-1}\begin{pmatrix}\partial_t\widetilde{\mathbf{P}}(\mathbf{E})(s) + \mathbf{J}_0(s)\\ 0\end{pmatrix}\mathrm{d}s$$

for $t \in J \cap (0,\infty)$ and $\big(\mathbf{E}(t), \mathbf{H}(t)\big) = \big(\mathbf{E}_{\text{h}}(t), \mathbf{H}_{\text{h}}(t)\big)$ for $t \le 0$.

An application of the Theorems 2.11, 2.14 and 2.16 to the reduced Maxwell system leads to the central result of this work (where A and X in Section 2 are now given by A_2 and X_2). As in the abstract setting, we treat the continuous dependence on A, i. e., on σ, ε, μ as a separate statement, since for fixed A, a stronger result for the continuous dependence on $\mathbf{J}_0, \mathbf{E}_{\text{h}}, \mathbf{H}_{\text{h}}$ and the response functions can be obtained.

Theorem 3.23. *Let Assumption 3.10, 3.16 and 3.19 be true. Let $I = [0,\infty)$ or $I = [0, t_{\text{end}})$ for some $t_{\text{end}} > 0$ and $\mathbf{J}_0 \in C(I, H^2(G)^3)$ satisfy $\operatorname{tr}_\tau \mathbf{J}_0(t) = 0$ for all $t \in I$. Let $\mathbf{E}_{\text{h}}, \mathbf{H}_{\text{h}} \in C_b\big((-\infty,0], H^2(G)^3\big)$ satisfy*

$$\operatorname{tr}_\tau \mathbf{E}_{\text{h}}(t) = \operatorname{tr}_\tau\big(\operatorname{curl}\mathbf{H}_{\text{h}}(t)\big) = 0\,, \quad \operatorname{tr}_\nu\big(\mu\mathbf{H}_{\text{h}}(t)\big) = 0\,, \quad \operatorname{div}\big(\mu\mathbf{H}_{\text{h}}(t)\big) = 0$$

for all $t \le 0$. Then the following statements hold.

1) *Equation (3.40) has a unique mild solution $u = (\mathbf{E},\mathbf{H})$ on a maximal existence interval $(-\infty, t^+)$ with $t^+ = t^+\big(\mathbf{E}_{\text{h}}, \mathbf{H}_{\text{h}}, \mathbf{J}_0, \mathcal{R}^{(1)}, \dots, \mathcal{R}^{(N)}, \sigma, \varepsilon, \mu, \eta, G\big) > 0$.*

2) *It $t^+ < \sup I$, there exists a sequence (t_k) in $(0, t^+)$ with $t_k \to t^+$ and $\|u(t_k)\|_{H^2(G)^6} \to \infty$ as $k \to \infty$.*

3) *Let $b \in (0, t^+)$. There exist constants $\delta > 0$ and $c \ge 0$ depending on $\mathbf{E}_{\text{h}}, \mathbf{H}_{\text{h}}, \mathbf{J}_0$, $\mathcal{R}^{(1)}, \dots, \mathcal{R}^{(N)}, \sigma, \varepsilon, \mu, \eta, G$ and b such that for all*

$$\widetilde{\mathbf{J}}_0, \widehat{\mathbf{J}}_0 \in C\big(I, H^2(G)^3\big)\,, \quad \widetilde{\mathbf{E}}_{\text{h}}, \widetilde{\mathbf{H}}_{\text{h}}, \widehat{\mathbf{E}}_{\text{h}}, \widehat{\mathbf{H}}_{\text{h}} \in C_b\big((-\infty,0], H^2(G)^3\big)$$

and $\widetilde{\mathcal{R}}^{(1)}, \dots, \widetilde{\mathcal{R}}^{(N)}$ *satisfying Assumptions 3.16 and 3.19 and*

$$\operatorname{tr}_\tau \widetilde{\mathbf{J}}_0(t) = \operatorname{tr}_\tau \widehat{\mathbf{J}}_0(t) = 0\,, \quad t \in I\,,$$

$$\operatorname{tr}_\tau \widetilde{\mathbf{E}}_{\mathrm{h}}(t) = \operatorname{tr}_\tau\big(\operatorname{curl} \widetilde{\mathbf{H}}_{\mathrm{h}}(t)\big) = 0\,, \quad \operatorname{tr}_\nu\big(\widetilde{\mathbf{H}}_{\mathrm{h}}(t)\big) = 0\,, \quad \operatorname{div}\big(\mu \widetilde{\mathbf{H}}_{\mathrm{h}}(t)\big) = 0\,, \quad t \le 0\,,$$

$$\operatorname{tr}_\tau \widehat{\mathbf{E}}_{\mathrm{h}}(t) = \operatorname{tr}_\tau\big(\operatorname{curl} \widehat{\mathbf{H}}_{\mathrm{h}}(t)\big) = 0\,, \quad \operatorname{tr}_\nu\big(\widehat{\mathbf{H}}_{\mathrm{h}}(t)\big) = 0\,, \quad \operatorname{div}\big(\mu \widehat{\mathbf{H}}_{\mathrm{h}}(t)\big) = 0\,, \quad t \le 0\,,$$

$$\sup_{0 \le t \le b} \Big\| \mathbf{J}_0(t) - \widetilde{\mathbf{J}}_0(t) \Big\|_{H^2(G)^3} \le \delta\,, \quad \sup_{0 \le t \le b} \Big\| \mathbf{J}_0(t) - \widehat{\mathbf{J}}_0(t) \Big\|_{H^2(G)^3} \le \delta\,,$$

$$\sup_{t \le 0} \Big\| \big(\mathbf{E}_{\mathrm{h}}(t), \mathbf{H}_h(t)\big) - \big(\widetilde{\mathbf{E}}_{\mathrm{h}}(t), \widetilde{\mathbf{H}}_{\mathrm{h}}(t)\big) \Big\|_{H^2(G)^6} \le \delta\,,$$

$$\sup_{t \le 0} \Big\| \big(\mathbf{E}_{\mathrm{h}}(t), \mathbf{H}_{\mathrm{h}}(t)\big) - \big(\widehat{\mathbf{E}}_{\mathrm{h}}(t), \widehat{\mathbf{H}}_{\mathrm{h}}(t)\big) \Big\|_{H^2(G)^6} \le \delta\,,$$

$$\max_{\substack{n \in \{1,\dots,N\},\\ j_1,\dots,j_{n-1} \in \{1,2,3\},\\ l \in \{1,\dots,n\}}} \Big\| \partial_{\tau_l} \mathcal{R}^{(n)}\big(\cdot, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}\big) - \partial_{\tau_l} \widetilde{\mathcal{R}}^{(n)}\big(\cdot, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}\big) \Big\|_{L^1(\mathbb{R}^n_{>0}, H^2(G))} \le \delta\,,$$

$$\max_{\substack{n \in \{1,\dots,N\},\\ j_1,\dots,j_{n-1} \in \{1,2,3\}}} \Big\| \mathcal{R}^{(n)}\big(\cdot, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}\big) - \widetilde{\mathcal{R}}^{(n)}\big(\cdot, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}\big) \Big\|_{L^1(\partial\mathbb{R}^n_{>0}, H^2(G))} \le \delta\,,$$

we have $t^+\Big(\widetilde{\mathbf{E}}_{\mathrm{h}}, \widetilde{\mathbf{H}}_{\mathrm{h}}, \widetilde{\mathbf{J}}_0, \widetilde{\mathcal{R}}^{(1)}, \dots, \widetilde{\mathcal{R}}^{(N)}, \sigma, \varepsilon, \mu, \eta, G\Big) > b$ *as well as the estimate*

$$\begin{aligned}
&\Big\| \big(\widehat{\mathbf{E}}(t), \widehat{\mathbf{H}}(t)\big) - \big(\widetilde{\mathbf{E}}(t), \widetilde{\mathbf{H}}(t)\big) \Big\|_{H^2(G)^6} \\
&\le c \Bigg(\sup_{\tau \le 0} \Big\| \big(\widehat{\mathbf{E}}_{\mathrm{h}}(\tau), \widehat{\mathbf{H}}_{\mathrm{h}}(\tau)\big) - \big(\widetilde{\mathbf{E}}_{\mathrm{h}}(\tau), \widetilde{\mathbf{H}}_{\mathrm{h}}(\tau)\big) \Big\|_{H^2(G)^6} + \sup_{0 \le \tau \le b} \Big\| \widehat{\mathbf{J}}_0(\tau) - \widetilde{\mathbf{J}}_0(\tau) \Big\|_{H^2(G)^3} \\
&+ \max_{\substack{n \in \{1,\dots,N\},\\ j_1,\dots,j_{n-1} \in \{1,2,3\},\\ l \in \{1,\dots,n\}}} \Big\| \partial_{\tau_l} \mathcal{R}^{(n)}\big(\cdot, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}\big) - \partial_{\tau_l} \widetilde{\mathcal{R}}^{(n)}\big(\cdot, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}\big) \Big\|_{L^1(\mathbb{R}^n_{>0}, H^2(G))} \\
&+ \max_{\substack{n \in \{1,\dots,N\},\\ j_1,\dots,j_{n-1} \in \{1,2,3\}}} \Big\| \mathcal{R}^{(n)}\big(\cdot, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}\big) - \widetilde{\mathcal{R}}^{(n)}\big(\cdot, \mathbf{e}_{j_1}, \dots, \mathbf{e}_{j_{n-1}}\big) \Big\|_{L^1(\partial\mathbb{R}^n_{>0}, H^2(G))} \Bigg)
\end{aligned}$$

for all $t \le b$*, where* $\big(\widetilde{\mathbf{E}}, \widetilde{\mathbf{H}}\big)$ *and* $\big(\widehat{\mathbf{E}}, \widehat{\mathbf{H}}\big)$ *are the mild solutions of* (3.40) *with* $\mathbf{J}_0, \mathbf{E}_{\mathrm{h}}, \mathbf{H}_{\mathrm{h}}, \mathcal{R}^{(1)}, \dots, \mathcal{R}^{(N)}$ *replaced by* $\widetilde{\mathbf{J}}_0, \widetilde{\mathbf{E}}_{\mathrm{h}}, \widetilde{\mathbf{H}}_{\mathrm{h}}, \widetilde{\mathcal{R}}^{(1)}, \dots, \widetilde{\mathcal{R}}^{(N)}$ *respectively* $\widehat{\mathbf{J}}_0, \widehat{\mathbf{E}}_{\mathrm{h}}, \widehat{\mathbf{H}}_{\mathrm{h}}, \mathcal{R}^{(1)}, \dots, \mathcal{R}^{(N)}$.

4) Let $b \in (0, t^+)$ *and* $\Big(\mathbf{J}_0^{(k)}\Big)$, $\Big(\mathbf{E}_{\mathrm{h}}^{(k)}\Big)$, $\Big(\mathbf{H}_{\mathrm{h}}^{(k)}\Big)$ *be sequences in* $C\big(I, H^2(G)^3\big)$ *respectively* $C_b\big((-\infty, 0], H^2(G)^3\big)$ *with*

$$\sup_{0 \le t \le b} \Big\| \mathbf{J}_0^{(k)}(t) - \mathbf{J}_0(t) \Big\|_{H^2(G)^3} \to 0\,,$$

$$\sup_{t \le 0} \Big\| \Big(\mathbf{E}_{\mathrm{h}}^{(k)}(t), \mathbf{H}_{\mathrm{h}}^{(k)}(t)\Big) - \big(\mathbf{E}_{\mathrm{h}}(t), \mathbf{H}_{\mathrm{h}}(t)\big) \Big\|_{H^2(G)^6} \to 0$$

as $k \to \infty$. Let $\left(\mathcal{R}^{(1,k)}\right), \ldots, \left(\mathcal{R}^{(N,k)}\right)$ satisfy Assumptions 3.16 and 3.19 and

$$\begin{aligned}\left\|\partial_{\tau_l}\mathcal{R}^{(n,k)}\left(\cdot, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{n-1}}\right) - \partial_{\tau_l}\mathcal{R}^{(n)}\left(\cdot, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{n-1}}\right)\right\|_{L^1(\mathbb{R}^n_{>0}, H^2(G))} \to 0\,,\\ \left\|\mathcal{R}^{(n,k)}\left(\cdot, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{n-1}}\right) - \mathcal{R}^{(n)}\left(\cdot, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{n-1}}\right)\right\|_{L^1(\mathbb{R}^n_{>0}, H^2(G))} \to 0\end{aligned}$$

as $k \to \infty$ and all $n \in \{1, \ldots, N\}, j_1, \ldots, j_{n-1} \in \{1,2,3\}, l \in \{1, \ldots, n\}$. Let $(\sigma_k), (\varepsilon_k), (\mu_k)$ be sequences in $Z(G)$ with $\sigma_k \to \sigma, \varepsilon_k \to \varepsilon, \mu_k \to \mu$ in $Z(G)$ as $k \to \infty$ and $\varepsilon_k \geq \eta, \mu_k \geq \eta$ for all $k \in \mathbb{N}$ with $\eta > 0$ from Assumption 3.10. Additionally let

$$\begin{aligned}\operatorname{tr}_\tau \mathbf{J}_0^{(k)}(t) = 0\,, \quad t \in I\,,\\ \operatorname{tr}_\tau \mathbf{E}_{\mathrm{h}}^{(k)}(t) = \operatorname{tr}_\tau\left(\operatorname{curl}\mathbf{H}_{\mathrm{h}}^{(k)}(t)\right) = 0\,, \quad t \leq 0\,,\\ \operatorname{tr}_\nu\left(\mathbf{H}_{\mathrm{h}}^{(k)}(t)\right) = 0\,,\ \operatorname{div}\left(\mu_k \mathbf{H}_{\mathrm{h}}^{(k)}(t)\right) = 0\,, \quad t \leq 0\,,\end{aligned}$$

for all $k \in \mathbb{N}$. Let $\zeta > 0$ and $\left(\mathbf{E}^{(k)}, \mathbf{H}^{(k)}\right)$ be the maximal mild solution on $(0, t_k^+)$ of (3.40) with $\mathbf{E}_{\mathrm{h}}, \mathbf{H}_{\mathrm{h}}, \mathbf{J}_0, \mathcal{R}^{(1)}, \ldots, \mathcal{R}^{(N)}, \sigma, \varepsilon, \mu$ replaced by $\mathbf{E}_{\mathrm{h}}^{(k)}, \mathbf{H}_{\mathrm{h}}^{(k)}, \mathbf{J}_0^{(k)}$, $\mathcal{R}^{(1,k)}, \ldots, \mathcal{R}^{(N,k)}, \sigma_k, \varepsilon_k, \mu_k$. Then there exists an index $K \in \mathbb{N}$ such that

$$t_k^+ > b \quad \text{and} \quad \sup_{t \leq b}\left\|\left(\mathbf{E}^{(k)}, \mathbf{H}^{(k)}\right) - \left(\mathbf{E}, \mathbf{H}\right)\right\| \leq \zeta$$

for all $k \geq K$.

Proof. The first two assertions follow directly from Theorem 2.11 and Remark 2.3. In order to conclude 3) from Theorem 2.14, we define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(x) = \sum_{n=1}^N x^n$ and for $v \in C_b\left((-\infty, 0], H^2(G)^6\right)$ using Lemma 3.9, (3.37) and a variant for $\partial\mathbb{R}^n_{>0}$, we estimate

$$\begin{aligned}&\|F(v)\|_{H^2(G)^6}\\ &\lesssim \left(\frac{1}{\eta} + \frac{\|\varepsilon\|_{Z(G)}}{\eta^2} + \frac{\|\varepsilon\|^2_{Z(G)}}{\eta^3}\right)\sum_{n=1}^N \sum_{j_1,\ldots,j_{n-1}=1}^3 \left[\sum_{l=1}^n \left\|\partial_{\tau_l}\mathcal{R}^{(n)}\left(\cdot, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{n-1}}\right)\right\|_{L^1(\mathbb{R}^n_{>0}, H^2(G))}\right.\\ &\qquad\left. + \left\|\mathcal{R}^{(n)}\left(\cdot, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{n-1}}\right)\right\|_{L^1(\partial\mathbb{R}^n_{>0}, H^2(G))}\right]\left(\sup_{t\leq 0}\|v(t)\|_{H^2(G)^6}\right)^n\\ &\lesssim_{\varepsilon,\eta}\left(\max_{\substack{n\in\{1,\ldots,N\},\\ j_1,\ldots,j_{n-1}\in\{1,2,3\},\\ l\in\{1,\ldots,n\}}}\left\|\partial_{\tau_l}\mathcal{R}^{(n)}\left(\cdot, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{n-1}}\right)\right\|_{L^1(\mathbb{R}^n_{>0}, H^2(G))}\right.\\ &\qquad\left. + \max_{\substack{n\in\{1,\ldots,N\},\\ j_1,\ldots,j_{n-1}\in\{1,2,3\}}}\left\|\mathcal{R}^{(n)}\left(\cdot, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{n-1}}\right)\right\|_{L^1(\partial\mathbb{R}^n_{>0}, H^2(G))}\right)\psi\left(\sup_{t\leq 0}\|v(t)\|_{H^2(G)^6}\right).\end{aligned}$$

It follows

$$|||F|||_\psi \lesssim_{\varepsilon,\eta} \max_{\substack{n\in\{1,\ldots,N\},\\ j_1,\ldots,j_{n-1}\in\{1,2,3\},\\ l\in\{1,\ldots,n\}}}\left\|\partial_{\tau_l}\mathcal{R}^{(n)}\left(\cdot, \mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{n-1}}\right)\right\|_{L^1(\mathbb{R}^n_{>0}, H^2(G))}$$

$$+ \max_{\substack{n\in\{1,\dots,N\},\\ j_1,\dots,j_{n-1}\in\{1,2,3\}}} \left\|\mathcal{R}^{(n)}(\cdot,\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_{n-1}})\right\|_{L^1(\partial\mathbb{R}^n_{>0},H^2(G))} .$$

Statement 3) now follows from Theorem 2.14 (in which we replace δ by 2δ since we split $|||F|||_\psi$ into two contributions).

4) By assumption,

$$C := \max\left\{\|\sigma\|_{Z(G)}, \|\mu\|_{Z(G)} \|\varepsilon\|_{Z(G)}, \sup_{k\in\mathbb{N}}\|\sigma_k\|_{Z(G)}, \sup_{k\in\mathbb{N}}\|\varepsilon_k\|_{Z(G)}, \sup_{k\in\mathbb{N}}\|\mu_k\|_{Z(G)}\right\} < \infty .$$

Let ψ and v be as above and let $F^{(k)}$ denote the map obtained from F by replacing ε by ε_k as well as $\mathcal{R}^{(m)}$ by $\mathcal{R}^{(k,m)}$ for all $m \in \{1,\dots,N\}$. Using the triangle inequality, Lemma 3.9 and an analogous computation as in step 3), we estimate

$$\begin{aligned}
&\left\|F(v) - F^{(k)}(v)\right\|_{H^2(G)^6} \\
&\lesssim \max_{\substack{n\in\{1,\dots,N\},\\ j_1,\dots,j_{n-1}\in\{1,2,3\},\\ l\in\{1,\dots,n\}}} \Bigg[\left\|\varepsilon^{-1}\right\|_{Z(G)} \\
&\quad\Bigg(\left\|\partial_{\tau_l}\mathcal{R}^{(n,k)}(\cdot,\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_{n-1}}) - \partial_{\tau_l}\mathcal{R}^{(n)}(\cdot,\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_{n-1}})\right\|_{L^1(\mathbb{R}^n_{>0},H^2(G))} \\
&\qquad + \left\|\mathcal{R}^{(n,k)}(\cdot,\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_{n-1}}) - \mathcal{R}^{(n)}(\cdot,\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_{n-1}})\right\|_{L^1(\mathbb{R}^n_{>0},H^2(G))}\Bigg) \\
&\quad + \left\|\frac{\varepsilon_k - \varepsilon}{\varepsilon\varepsilon_k}\right\|_{Z(G)} \Bigg(\left\|\partial_{\tau_l}\mathcal{R}^{(n,k)}(\cdot,\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_{n-1}})\right\|_{L^1(\mathbb{R}^n_{>0},H^2(G))} \\
&\qquad + \left\|\mathcal{R}^{(n,k)}(\cdot,\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_{n-1}})\right\|_{L^1(\mathbb{R}^n_{>0},H^2(G))}\Bigg)\Bigg] \psi\left(\sup_{t\le 0}\|v(t)\|_{H^2(G)^6}\right) .
\end{aligned}$$

Since

$$\left\|\frac{\varepsilon_k - \varepsilon}{\varepsilon\varepsilon_k}\right\|_{Z(G)} \lesssim_{G,C,\eta} \|\varepsilon_k - \varepsilon\|_{Z(G)} ,$$

the above estimate yields $|||F - F^{(k)}|||_\psi \to 0$ for $k \to \infty$. We define for all $k \in \mathbb{N}$ the linear operator $A_{2,k}$ on X_2 with domain $D(A_2)$ by replacing σ, ε, μ in A by $\sigma_k, \varepsilon_k, \mu_k$, see (3.14). Proposition 3.14 shows that $A_{2,k}$ generates a strongly continuous semigroup $T_{2,k}(\cdot)$ on X_2 with

$$\|T_{2k}(t)\|_{\mathcal{B}(X_2)} \lesssim_{G,C,\eta} 1 + t^3$$

for all $k \in \mathbb{N}$ and $t \ge 0$, with a constant independent of k. So for any $\omega > 0$, there is a constant $M \ge 1$ such that

$$\|T_2(t)\|_{\mathcal{B}(X_2)}, \|T_{2,k}(t)\|_{\mathcal{B}(X_2)} \le M\mathrm{e}^{\omega t}$$

for all $t \geq 0$. For the application of Theorem 2.16, it only remains to show $A_{2,k}w \to A_2 w$ in X_2 as $k \to \infty$ for all $w = (\mathbf{K}, \mathbf{L}) \in D(A_2)$. To prove this convergence, we write

$$A_{2,k}w - A_2 w = \frac{\sigma\varepsilon_k - \sigma_k\varepsilon}{\varepsilon\varepsilon_k}\begin{pmatrix}\mathbf{K}\\ \mathbf{0}\end{pmatrix} + (B_k - I)\left(A_2 w + \frac{\sigma}{\varepsilon}\begin{pmatrix}\mathbf{K}\\ \mathbf{0}\end{pmatrix}\right)$$

with

$$B_k := \begin{pmatrix}\varepsilon\varepsilon_k^{-1}I & 0\\ 0 & \mu\mu_k^{-1}I\end{pmatrix}$$

for all $k \in \mathbb{N}$. Using again Lemma 3.9, as well as

$$\|\sigma_k\varepsilon - \sigma\varepsilon_k\|_{Z(G)} \lesssim \|\varepsilon\|_{Z(G)}\|\sigma - \sigma_k\|_{Z(G)} + \|\sigma\|_{Z(G)}\|\varepsilon - \varepsilon_k\|_{Z(G)} ,$$

we obtain

$$\begin{aligned}\left\|\frac{\sigma\varepsilon_k - \sigma_k\varepsilon}{\varepsilon\varepsilon_k}\mathbf{K}\right\|_{H^2(G)^3} &\lesssim_{C,\eta,\varepsilon,\sigma} \|\sigma_k - \sigma\|_{Z(G)}\|\mathbf{K}\|_{H^2(G)^3} ,\\ \|B_k - I\|_{\mathcal{B}(H^2(G)^6)} &\lesssim_{C,\eta,\varepsilon,\mu} \|\varepsilon_k - \varepsilon\|_{Z(G)} + \|\mu_k - \mu\|_{Z(G)} ,\end{aligned}$$

which implies

$$\begin{aligned}&\|A_{2,k}w - A_2 w\|_{H^2(G)^6}\\ &\quad\lesssim_{C,\eta,\varepsilon,\mu,\sigma} \left(\|\sigma_k - \sigma\|_{Z(G)} + \|\varepsilon_k - \varepsilon\|_{Z(G)} + \|\mu_k - \mu\|_{Z(G)}\right)\left(\|A_2 w\|_{H^2(G)^6} + \|\mathbf{K}\|_{H^2(G)^3}\right)\\ &\quad\to 0\end{aligned}$$

as $k \to \infty$. Theorem 2.16 now implies part 4). □

In the next step, we show that a mild solution has additional time regularity in $H^1(G)^6$. In combination with Lemma 3.7, this implies that the mild solution of the reduced system (3.10) of Maxwell equations is a (classical) solution of the full system (1.2) and (1.6) in $H^1(G)^6$, provided the divergence conditions and the magnetic boundary condition are fulfilled at the initial time and that the continuity equation (1.3) holds.

Lemma 3.24. *Let the assumptions of Theorem 3.23 be true and $u \in C\big((-\infty, b), X_2\big)$ be a mild solution of (3.40) for some $b > 0$. Then $u \in C^1\big([0, b), [D(A_{\mathrm{div}})]\big)$ and therefore $u \in C^1\big([0, b), H^1(G)^6\big)$ by Lemma 3.12. We further have $u'(t) = Au(t) + F(u_t) + g(t)$ for all $t \in [0, b)$.*

Proof. 1) We define $\chi : [0, b) \to X_2$ by $\chi(s) = F(u_s) + g(s)$. Then $\chi \in C\big([0, b), [D(A^2_{\mathrm{div}})]\big)$ by Lemma 3.22 and Lemma 3.13. Hence, the map $v : [0, b) \to X_2$ given by

$$v(t) = \int_0^t T(t - s)\chi(s)\,\mathrm{d}s$$

is contained in $C^1\big([0, b), X_{\mathrm{div}}\big) \cap C\big([0, b), [D(A^2_{\mathrm{div}})]\big)$ with

$$v'(t) = Av(t) + \chi(t)$$

for all $t \in [0, b)$. We further have

$$Av(t) = \int_0^t T(t-s)A\chi(s)\,\mathrm{d}s\,.$$

Since $A\chi \in C\big([0,b),[D(A_{\mathrm{div}})]\big)$, we infer $Av \in C^1\big([0,b),X_{\mathrm{div}}\big) \cap C\big([0,b),[D(A_{\mathrm{div}})]\big)$ with

$$(Av)'(t) = A^2 v(t) + A\chi(t)$$

for all $t \in [0, b)$. This implies $v \in C^1\big([0,b),[D(A_{\mathrm{div}})]\big)$.

2) We set $w(t) := T(t)[f(0)]$ for $t \geq 0$. Since $f(0) \in D(A_{\mathrm{div}}^2)$, the map w is contained in $C^1\big(\mathbb{R}_{\geq 0},X_{\mathrm{div}}\big) \cap C\big(\mathbb{R}_{\geq 0},[D(A_{\mathrm{div}})]\big)$ with $w'(t) = Aw(t)$ for all $t \geq 0$. Also, $Af(0) \in D(A_{\mathrm{div}})$ and the identity $Aw(t) = T(t)Af(0)$ implies $Aw \in C^1\big(\mathbb{R}_{\geq 0},X_{\mathrm{div}}\big) \cap C\big(\mathbb{R}_{\geq 0},D[(A_{\mathrm{div}})]\big)$ with $(Aw)'(t) = A^2 w(t)$ for all $t \geq 0$. Therefore w is contained in $C^1\big(\mathbb{R}_{\geq 0},[D(A_{\mathrm{div}})]\big)$.

3) The claim now follows from $u = w + v$. □

We finally apply Theorem 2.18 to the Maxwell system to obtain a classical solution in $H^2(G)^6$. The theorem's conditions are formulated below using Remark 2.3, Proposition 3.14, Lemma 3.15 and Lemma 3.22.

Proposition 3.25. *Let Assumption 3.10, 3.16 and 3.19 be true. Let $I = [0,\infty)$ or $I = [0, t_{\mathrm{end}})$ for some $t_{\mathrm{end}} > 0$ and $\mathbf{J}_0 \in C^1(I, H^2(G)^3)$ satisfy $\operatorname{tr}_\tau \mathbf{J}_0(t) = 0$ for all $t \in I$. Let $\mathbf{E}_{\mathrm{h}}, \mathbf{H}_{\mathrm{h}} \in \mathrm{BUC}^1\big((-\infty,0],H^2(G)^3\big)$ satisfy*

$$\operatorname{tr}_\tau \mathbf{E}_{\mathrm{h}}(t) = \operatorname{tr}_\tau\big(\operatorname{curl}\mathbf{H}_{\mathrm{h}}(t)\big) = 0\,, \quad \operatorname{tr}_\nu\big(\mu\mathbf{H}_{\mathrm{h}}(t)\big) = 0\,, \quad \operatorname{div}\big(\mu\mathbf{H}_{\mathrm{h}}(t)\big) = 0$$

for all $t \leq 0$, as well as

$$\begin{gathered}\operatorname{tr}_\tau\left(-\sigma\varepsilon^{-1}\mathbf{E}_{\mathrm{h}}(0) + \varepsilon^{-1}\operatorname{curl}\mathbf{H}_{\mathrm{h}}(0)\right) = 0\,, \quad \operatorname{tr}_\tau\left(\operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathbf{E}_{\mathrm{h}}(0)\right)\right) = 0\,,\\ \operatorname{tr}_\nu\left(\mu^{-1}\operatorname{curl}\mathbf{E}_{\mathrm{h}}(0)\right) = 0\,, \quad \mathbf{H}_{\mathrm{h}}'(0) = -\mu^{-1}\operatorname{curl}\mathbf{E}_{\mathrm{h}}(0)\,,\\ \mathbf{E}_{\mathrm{h}}'(0) = -\sigma\varepsilon^{-1}\mathbf{E}_{\mathrm{h}}(0) + \varepsilon^{-1}\operatorname{curl}\mathbf{H}_{\mathrm{h}}(0) - \varepsilon^{-1}\partial_t\widetilde{\mathbf{P}}(\mathbf{E}_{\mathrm{h}})(0) - \varepsilon^{-1}\mathbf{J}_0(0)\,.\end{gathered}$$

Then the maximal mild solution of (3.40) is a classical solution of (3.40) on the maximal existence interval.

Appendix

A.1 Homogeneous, isotropic and inversion-symmetric materials

In this section we consider how spatial symmetries of the material lead to relations between the components of the response functions. As a result, Assumption 3.19 is fulfilled at least in the special case of homogeneous and isotropic materials with inversion symmetry which includes gases, liquids and amorphous materials. The discussion here is based on Neumann's principle which states that if the medium has a spatial symmetry then for any tensor describing its physical properties, the components must be invariant under changing between two coordinate systems related by the transformation corresponding to the spatial symmetry, see Chapter 5 of [6].

We consider for $n \in \{1, \dots, N\}$ and $b > 0$ the expression (3.28) for the nth-order contribution to the polarisation for two fields $\mathbf{E}, \widetilde{\mathbf{E}} \in C_b\big((-\infty, b), H^2(G)^3\big)$ related by $\widetilde{\mathbf{E}}(t) = C\mathbf{E}(t)$ for all $t < b$, where $C \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix describing a geometric transformation like a rotation, reflection or inversion. The polarisation caused by the transformed electric field is given by

$$\begin{aligned}
\widetilde{P}^{(n)}_{i_0}(\widetilde{\mathbf{E}})(t,x) &= \int_{\mathbb{R}^n_{>0}} R^{(n)}_{i_0 i_1 \dots i_n}(\tau, x) \prod_{k=1}^{n} \widetilde{E}_{i_k}(t - \tau_k, x)\, \mathrm{d}\tau \\
&= \int_{\mathbb{R}^n_{>0}} R^{(n)}_{i_0 i_1 \dots i_n}(\tau, x) \prod_{k=1}^{n} C_{i_k j_k} E_{j_k}(t - \tau_k, x)\, \mathrm{d}\tau
\end{aligned}$$

for all $t < b, x \in G$ and $i_0 \in \{1, 2, 3\}$. Let C be a symmetry transformation for the material, i. e., it leaves the structure invariant (at least in a microscopically large volume around x). Then $\widetilde{\mathbf{P}}$ has to be identical to $C\mathbf{P}$, the transformed original polarisation. Using the orthogonality of C, we conclude

$$\begin{aligned}
P^{(n)}_{j_0}(\mathbf{E})(t,x) &= \big(C^{-1}\big)_{j_0 i_0} \widetilde{P}^{(n)}_{i_0}(\widetilde{\mathbf{E}})(t,x) \\
&= \int_{\mathbb{R}^n_{>0}} C_{i_0 j_0} R^{(n)}_{i_0 i_1 \dots i_n}(\tau, x) \prod_{k=1}^{n} C_{i_k j_k} E_{j_k}(t - \tau_k, x)\, \mathrm{d}\tau\,.
\end{aligned}$$

This has to hold for all fields $\mathbf{E}$. Comparing with (3.28) and applying Lemma A.1 below, we derive

$$R^{(n)}_{j_0 j_1 \dots j_n} = R^{(n)}_{i_0 i_1 \dots i_n} \prod_{k=0}^{n} C_{i_k j_k} \tag{A.1}$$

for all $j_0, \dots, j_n \in \{1, 2, 3\}$. In particular, the inversion transformation described by $C = -I$ leads to

$$R^{(n)}_{j_0 j_1 \dots j_n} = R^{(n)}_{i_0 i_1 \dots i_n} \prod_{k=0}^{n} (-\delta_{i_k j_k}) = (-1)^{n+1} R^{(n)}_{j_0 j_1 \dots j_n}\,,$$

which implies that all even order response functions have to vanish for inversion symmetric materials.

Lemma A.1. *Let $G \subseteq \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary, $n \in \mathbb{N}$ and $R^{(n)} \in C\big(\mathbb{R}^n_{\geq 0}, H^2(G)^{3^{n+1}}\big) \cap L^1\big(\mathbb{R}^n_{>0}, H^2(G)^{3^{n+1}}\big)$ have the property*

$$R^{(n)}_{j_0 j_1 \dots j_n}(\tau_1, \dots, \tau_n) = R^{(n)}_{j_0 j_{\sigma(1)} \dots j_{\sigma(1)}}\big(\tau_\sigma(1), \dots, \tau_\sigma(1)\big)$$

for all $\tau \in \mathbb{R}^n_{>0}, j_0, \dots, j_n \in \{1,2,3\}$ and permutations $\sigma \in S_n$. If

$$\int_{\mathbb{R}^n_{>0}} R^{(n)}_{j_0 j_1 \dots j_n}(\tau_1, \dots, \tau_n) \prod_{k=1}^n F_{j_k}(\tau_k)\,\mathrm{d}\tau = 0$$

for all $\mathbf{F} \in C_b\big(\mathbb{R}_{\geq 0}, H^2(G)^3\big)$ and $j_0 \in \{1,2,3\}$, then $R^{(n)} = 0$.

Proof. Let $\rho \in C^\infty_c(\mathbb{R})$ with $\rho \geq 0, \operatorname{supp}\rho \subseteq [-1,1]$ and $\int_{-\infty}^\infty \rho(t)\,\mathrm{d}t = 1$. Let $\theta \in \mathbb{R}^n_{>0}$ and $\varepsilon > 0$. By continuity, there exists a $\delta > 0$ with $B(\theta,\delta) \subseteq \mathbb{R}^n_{>0}$ such that $\big\|R^{(n)}(\tau) - R^{(n)}(\theta)\big\|_{H^2(G)^{3^{n+1}}} \leq \varepsilon$. Let $j_0, \dots, j_n \in \{1,2,3\}$. We define for $c \in \mathbb{R}^n$ the map $\mathbf{F} : \mathbb{R}_{\geq 0} \to H^2(G)^3$ by

$$\mathbf{F}(t) = \sum_{l=1}^n c_l\,\mathbf{e}_{j_l} \mathbb{1}_G \delta^{-1} \rho\left(\delta^{-1}(t - \theta_l)\right) .$$

The estimate

$$\begin{aligned}
&\left\| \sum_{l_1=1}^n \cdots \sum_{l_n=1}^n c_{l_1} \dots c_{l_n} R^{(n)}_{j_0 j_{l_1} \dots j_{l_n}}(\theta_{l_1}, \dots, \theta_{l_n}) \right\|_{H^2(G)} \\
&= \left\| \sum_{l_1=1}^n \cdots \sum_{l_n=1}^n c_{l_1} \dots c_{l_n} R^{(n)}_{j_0 j_{l_1} \dots j_{l_n}}(\theta_{l_1}, \dots, \theta_{l_n}) - \int_{\mathbb{R}^n_{>0}} R^{(n)}_{j_0 m_1 \dots m_n}(\tau) \prod_{k=1}^n F_{m_k}(\tau_k)\,\mathrm{d}\tau \right\|_{H^2(G)} \\
&\leq \sum_{l_1=1}^n \cdots \sum_{l_n=1}^n |c_{l_1} \dots c_{l_n}|\,\delta^{-n} \int_{\mathbb{R}^n_{>0}} \left\| R^{(n)}_{j_0 j_{l_1} \dots j_{l_n}}(\theta_{l_1}, \dots, \theta_{l_n}) - R^{(n)}_{j_0 j_{l_1} \dots j_{l_n}}(\tau) \right\|_{H^2(G)} \\
&\qquad \cdot \prod_{k=1}^n \rho\left(\delta^{-1}(\tau_{l_k} - \theta_{l_k})\right)\,\mathrm{d}\tau \\
&\leq n^n\,|c|^n\,\varepsilon
\end{aligned}$$

shows

$$\sum_{l_1=1}^n \cdots \sum_{l_n=1}^n c_{l_1} \dots c_{l_n} R^{(n)}_{j_0 j_{l_1} \dots j_{l_n}}(\theta_{l_1}, \dots, \theta_{l_n}) = 0$$

for all $c \in \mathbb{R}^n$. Using the permutation symmetry of $R^{(n)}$, we see that the polynomial in c given by

$$\underbrace{\sum_{l_1=1}^n \cdots \sum_{l_n=1}^n}_{l_1 \leq \dots \leq l_n} c_{l_1} \dots c_{l_n} \alpha_{l_1 \dots l_n} R^{(n)}_{j_0 j_{l_1} \dots j_{l_n}}(\theta_{l_1}, \dots, \theta_{l_n})$$

vanishes, where the positive factors $\alpha_{l_1 \dots l_n}$ result from combining terms related by a permutation. Therefore the coefficients have to be zero from which we conclude the claim since $\theta \in \mathbb{R}^n_{>0}$ and $j_0, j_1, \dots, j_n \in \{1,2,3\}$ are arbitrary. $\square$

We say that a material is isotropic if (A.1) holds for all rotations C and all $n \in \{1, \dots, N\}$. This is e. g. the case for amorphous materials in optics. Although there is a short-range order in the atomic structure, over microscopically large but macroscopically small ranges the material has a random order and therefore there are no distinguished directions concerning the macroscopic polarisation. For the same reason, amorphous materials are inversion symmetric on the macroscopic scale. If the assumption of isotropy holds in the whole domain G, the material has to be homogeneous, i. e., the material properties do not depend on $x \in G$. In a real system, the physical properties near the material's surface change with respect to the bulk properties due to the different atomic arrangements. This is neglected here. We now consider the idealised case of a homogeneous, isotropic and inversion-symmetric material. Combining the intrinsic permutation symmetry with a general result on isotropic tensors yields that the response functions have a simple form in this case and satisfy Assumption 3.19.

Lemma A.2. *Let the material be homogeneous, isotropic and inversion-symmetric. Then all even order response functions vanish and for all odd $n \in \{1, \dots, N\}$, there exists a function $\widetilde{\mathcal{R}}^{(n)} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ such that*

$$R^{(n)}_{j_0 \dots j_n}(\tau, x) = \widetilde{\mathcal{R}}^{(n)}(\tau)\delta_{j_0 j_n}\delta_{j_1 j_2} \dots \delta_{j_{n-2} j_{n-1}} \tag{A.2}$$

for all $j_0, \dots, j_n \in \{1, 2, 3\}, \tau \in \mathbb{R}^n_{\geq 0}$ and $x \in G$. In particular, Assumption 3.19 is true.

Proof. We fix a $n \in \{1, \dots, N\}$ and use that $R^{(n)}(\tau, x)$ is a tensor of rank $n+1$ for all $\tau \in \mathbb{R}^n_{\geq 0}$ (we drop the x-dependence, since a homogeneous medium is assumed). As seen above, all even order response functions vanish. Therefore let n be odd. To prove (A.2), we define

$$\mathcal{T} := \left\{ T \,\middle|\, T \text{ is a real-valued isotropic tensor of rank } n+1 \right\} ,$$

which forms a finite dimensional vector space which we equip with the scalar product

$$(T|S) = T_{j_0 \dots j_n} S_{j_0 \dots j_n} \,.$$

As a consequence of Theorem 2.9.A in [28], every $T \in \mathcal{T}$ can be expressed as a linear combination of products of $(n+1)/2$ Kronecker deltas. (See also the Supplemental Material of [10] for a more direct proof.) So there exist an integer $q \in \mathbb{N}$ and permutations $\sigma_k \in S_n$ for $k \in \{1, \dots, q\}$ such that $\{T_k \,|\, k \in \{1, \dots, q\}\}$ forms a basis of $\mathcal{T}$ where

$$T_{k, j_0 \dots j_n} := \delta_{j_0 j_{\sigma_k(n)}} \delta_{j_{\sigma_k(1)} j_{\sigma_k(2)}} \dots \delta_{j_{\sigma_k(n-2)} j_{\sigma_k(n-1)}}$$

for $k \in \{1, \dots, q\}$ and $j_0, \dots, j_n \in \{1, 2, 3\}$. Using the Gram-Schmidt procedure, we find coefficients $c_{k,l} \in \mathbb{R}$ such that $\{\widetilde{T}_k \,|\, k \in \{1, \dots, q\}\}$ is an orthonormal basis of $\mathcal{T}$ where $\widetilde{T}_k = \sum_{l=1}^q c_{k,l} T_l$ for $k \in \{1, \dots, q\}$. So for a fixed $\tau \in \mathbb{R}^n_{\geq 0}$, we can write

$$R^{(n)}(\tau) = \sum_{k=1}^q r_k(\tau) \widetilde{T}_k = \sum_{k,l=1}^q c_{k,l} r_k(\tau) T_l \tag{A.3}$$

with

$$r_k(\tau) = \left(R^{(n)}(\tau) \middle| \widetilde{T}_k \right) \tag{A.4}$$

for all $k \in \{1, \dots, q\}$. Letting τ vary, we obtain functions $r_k : \mathbb{R}^n_{\geq 0} \to H^2(G)$. Equation (A.4) implies that r_k inherits the properties of Assumption 3.16 with $R^{(n)}_{j_0 \cdots j_n}$ replaced by r_k for $k \in \{1, \dots, q\}$. The same arguments that led to the intrinsic permutation symmetry of $R^{(n)}$ can be applied to every term in (A.3), so we obtain

$$\begin{aligned} R^{(n)}_{j_0 \cdots j_n}(\tau) &= \sum_{k,l=1}^{q} c_{k,l} r_k(\tau_1, \dots, \tau_n) \delta_{j_0 j_{\sigma_l(n)}} \delta_{j_{\sigma_l(1)} j_{\sigma_k(2)}} \dots \delta_{j_{\sigma_l(n-2)} j_{\sigma_l(n-1)}} \\ &= \left[\sum_{k,l=1}^{q} c_{k,l} r_k \left(\tau_{\sigma_l^{-1}(1)}, \dots, \tau_{\sigma_l^{-1}(n)} \right) \right] \delta_{j_0 j_n} \delta_{j_1 j_2} \dots \delta_{j_{n-2} j_{n-1}} . \end{aligned}$$

Equation (A.2) now follows with

$$\widetilde{\mathcal{R}}^{(n)}(\tau) := \sum_{k,l=1}^{q} c_{k,l} r_k \left(\tau_{\sigma_l^{-1}(1)}, \dots, \tau_{\sigma_l^{-1}(n)} \right) .$$

Let $\tau \in \mathbb{R}^n_{\geq 0}$, $j_0 \in \{1, 2, 3\}$ and $\mathbf{E}^{(1)}, \dots, \mathbf{E}^{(n)} \in \mathbb{R}^3$. Equation (A.2) yields

$$R^{(n)}_{j_0 \cdots j_n}(\tau) \prod_{k=1}^{n} E^{(k)}_{j_k} = \widetilde{\mathcal{R}}^{(n)}(\tau) \left(\mathbf{E}^{(1)} \cdot \mathbf{E}^{(2)} \right) \dots \left(\mathbf{E}^{(n-2)} \cdot \mathbf{E}^{(n-1)} \right) E^{(n)}_{j_0} .$$

Therefore, the identity (3.35) is satisfied with

$$\mathcal{R}^{(n)}\left(\tau, \mathbf{E}^{(1)}, \dots, \mathbf{E}^{(n-1)}\right) = \widetilde{\mathcal{R}}^{(n)}(\tau) \left(\mathbf{E}^{(1)} \cdot \mathbf{E}^{(2)} \right) \dots \left(\mathbf{E}^{(n-2)} \cdot \mathbf{E}^{(n-1)} \right) . \qquad \square$$

A.2 Lorentz oscillator model

The Lorentz oscillator model [18] is often used in physics to describe the optical properties of dielectric materials caused by the motion of bound electrons. We consider an electron with charge q and mass m which is bound to an atom (or ion) by a potential V and make the following assumptions (see Section 7.5 in [16]).

Since the atom has a much higher mass, its position is treated as fixed. We further neglect the difference between the macroscopic electric field and the local electric field at the atom's position in the medium (see Section 4.5 in [16] for a discussion of the local field). Also the magnetic part of the Lorentz force (1.4) is dropped and we assume that the amplitude of the electron's oscillations is small which allows us to approximate the electric field by its value at the electron's rest position. In [16], V is an isotropic harmonic potential and an isotropic damping term is included in the equation of motion. Since we are interested in nonlinear terms for the polarisation, we add anharmonic contributions to V as in Section 1.4 of [5]. Additionally, the potential and the damping term are allowed to be anisotropic as in [24].

The potential is modelled by

$$V(\mathbf{x}) = \frac{1}{2} m K^{(1)}_{ij} x_i x_j + \frac{1}{3} m K^{(2)}_{ijl} x_i x_j x_l + \frac{1}{4} m K^{(3)}_{ijlm} x_i x_j x_l x_m$$

for all $\mathbf{x} \in \mathbb{R}^3$ with symmetric tensors $K^{(1)}, K^{(2)}, K^{(3)}$ of ranks two, three and four (using the sum convention). The equation of motion for the electron's displacement $\mathbf{x}$ then has the form

$$\ddot{x}_i + 2\Gamma_{ij}\dot{x}_j + K^{(1)}_{ij} x_j + K^{(2)}_{ijl} x_j x_l + K^{(3)}_{ijlm} x_j x_l x_m = F_i(t) \tag{A.5}$$

for $i = 1, 2, 3$ with $\mathbf{F}(t) = \frac{q}{m}\mathbf{E}(t)$ and a symmetric tensor Γ of rank two describing damping. We assume $K^{(1)}$ and Γ to be positive definite with eigenvalues $\omega_{0,1}^2, \omega_{0,2}^2, \omega_{0,3}^2$ and $\gamma_1, \gamma_2, \gamma_3$ respectively. We further assume the damping to be weak which we characterise by the condition

$$\max_{i\in\{1,2,3\}} \gamma_i^2 < \frac{1}{2} \min_{i\in\{1,2,3\}} \omega_{0,i}^2 \,.$$

Since (A.5) can not be solved in general, we use perturbation theory to obtain an approximation for $\mathbf{x}$. To this end, we replace $\mathbf{F}$ by $\lambda\mathbf{F}$ with a perturbation parameter λ ranging continuously between zero and one. The solution for $\mathbf{x}$ is then expanded in a power series of the form

$$\mathbf{x} = \lambda\mathbf{x}^{(1)} + \lambda^2\mathbf{x}^{(2)} + \lambda^3\mathbf{x}^{(3)} + \dots \,.$$

The zero order contribution describes just a damped motion independent of $\mathbf{E}$ and can be omitted for our purposes. If N stands for the number density of atoms, the polarisation can be written as $\mathbf{P} = Nq\mathbf{x}$ leading to a corresponding series

$$\mathbf{P} = \lambda\mathbf{P}^{(1)} + \lambda^2\mathbf{P}^{(2)} + \lambda^3\mathbf{P}^{(3)} + \dots$$

with $\mathbf{P}^{(n)} = Nq\mathbf{x}^{(n)}$. Here we consider only the case of one electron per atom. The general case with several electrons per atom having different potentials and damping constants can be treated by summing up the individual contributions, weighted with appropriate oscillator strengths, see [16].

Equation (A.5) is solved for a given order $\mathbf{x}^{(n)}$ by gathering all the coefficients in (A.5) of λ^n and setting them to zero. In the calculations below, we assume that $\mathbf{F}$ and $\mathbf{x}^{(n)}$ are sufficiently regular to justify the use of the Fourier transform and the interchange of the order of integrations.

We start with the first order term and obtain the equation

$$\ddot{\mathbf{x}}^{(1)} + 2\Gamma\dot{\mathbf{x}}^{(1)} + K^{(1)}\mathbf{x}^{(1)} = \mathbf{F}(t) \,. \tag{A.6}$$

Using the Fourier transform given by

$$\mathbf{F}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\mathbf{F}}(\omega) \mathrm{e}^{-\mathrm{i}\omega t} \,\mathrm{d}\omega \,, \quad \widehat{\mathbf{F}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{F}(t) \mathrm{e}^{\mathrm{i}\omega t} \,\mathrm{d}t$$

and analogous for $\mathbf{x}^{(1)}$, equation (A.6) leads to

$$A(\omega)\widehat{\mathbf{x}}^{(1)}(\omega) = \widehat{\mathbf{F}}(\omega) \,, \quad A(\omega) := K^{(1)} - \omega^2 I - 2\mathrm{i}\omega\Gamma \,.$$

We first show that $A(\omega) = \big(A_{ij}(\omega)\big)$ is invertible for all $\omega \in \mathbb{R}$. The case $\omega = 0$ is clear since $K^{(1)}$ is positive definite. Let $\omega \neq 0$ and assume there is a $\mathbf{v} \in \mathbb{C}^3$ with $A(\omega)\mathbf{v} = \mathbf{0}$. We write $\mathbf{v} = \mathbf{a} + \mathrm{i}\mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Using the symmetry of $K^{(1)}$ and Γ, we compute

$$\begin{aligned} 0 &= (\mathbf{a} - \mathrm{i}\mathbf{b})^T A(\omega)\,(\mathbf{a} + \mathrm{i}\mathbf{b}) \\ &= \mathbf{a}^T \left(K^{(1)} - \omega^2 I\right)\mathbf{a} + \mathbf{b}^T \left(K^{(1)} - \omega^2 I\right)\mathbf{b} - 2\mathrm{i}\omega \left(\mathbf{a}^T\Gamma\mathbf{a} + \mathbf{b}^T\Gamma\mathbf{b}\right) . \end{aligned}$$

The imaginary part leads to $\mathbf{a} = \mathbf{b} = \mathbf{0}$ due to the positive definiteness of Γ from which we conclude that $A(\omega)$ is invertible. Therefore we can write

$$\begin{aligned} \mathbf{x}^{(1)}(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\mathbf{x}}^{(1)}(\omega) \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\omega)^{-1} \widehat{\mathbf{F}}(\omega) \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\omega)^{-1}\mathbf{F}(s)\mathrm{e}^{-\mathrm{i}\omega(t-s)}\,\mathrm{d}s\,\mathrm{d}\omega = \frac{m}{Nq^2}\int_{-\infty}^{\infty} R^{(1)}(t-s)\mathbf{F}(s)\,\mathrm{d}s \end{aligned}$$

with the response function $R^{(1)} : \mathbb{R} \to \mathbb{R}^{3\times 3}$ given by

$$R^{(1)}(\tau) = \frac{Nq^2}{2\pi m} \int_{-\infty}^{\infty} A(\omega)^{-1} \mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\omega\,. \tag{A.7}$$

Here we have chosen the prefactor such that

$$\mathbf{P}^{(1)}(t) = \int_{-\infty}^{\infty} R^{(1)}(t-s)\mathbf{E}(s)\,\mathrm{d}s\,,$$

in accordance with (3.28). Causality requires $R^{(1)}(\tau) = 0$ for $\tau < 0$. To see that this is the case, we consider the elements $\alpha_{ij}(\omega) := \big(A(\omega)^{-1}\big)_{ij}$ for $i, j \in \{1, 2, 3\}$. As a result of Cramer's rule,

$$\alpha_{ij}(\omega) = \frac{p_{ij}(\omega)}{\det(A(\omega))}$$

where $p_{ij}(\omega)$ is up to a sign given by the determinant of the matrix obtained by deleting row j and column i in $A(\omega)$. Therefore p_{ij} is a polynomial in ω of degree less than or equal to four. Let $\lambda_k(\omega)$ be an eigenvalue of $A(\omega)$ and $\mathbf{v}_k = \mathbf{a}_k + \mathrm{i}\mathbf{b}_k$ be a corresponding eigenvector with $\mathbf{a}_k, \mathbf{b}_k \in \mathbb{R}^3$ and $|\mathbf{v}_k| = 1$. As above, we obtain from

$$(\mathbf{a}_k - \mathrm{i}\mathbf{b}_k)^T \big(A(\omega) - \lambda_k(\omega)\big)(\mathbf{a}_k + \mathrm{i}\mathbf{b}_k) = 0$$

the identity

$$\lambda_k(\omega) = \kappa_k - 2\mathrm{i}\omega\xi_k - \omega^2$$

with

$$\kappa_k = \mathbf{a}_k^T K^{(1)}\mathbf{a}_k + \mathbf{b}_k^T K^{(1)}\mathbf{b}_k\,, \quad \xi_k = \mathbf{a}_k^T\Gamma\mathbf{a}_k + \mathbf{b}_k^T\Gamma\mathbf{b}_k\,.$$

The estimate

$$\xi_k^2 \le 2\Big(\left(\mathbf{a}_k^T\Gamma\mathbf{a}_k\right)^2 + \left(\mathbf{b}_k^T\Gamma\mathbf{b}_k\right)^2\Big) \le 2 \max_{m\in\{1,2,3\}} \gamma_m^2 \Big(|\mathbf{a}_k|^4 + |\mathbf{b}_k|^4\Big)$$

$$\leq 2 \max_{m\in\{1,2,3\}} \gamma_m^2 < \min_{m\in\{1,2,3\}} \omega_{0,m}^2 \leq \kappa_k$$

shows that $\lambda_k(\omega)$ has the roots $\omega_{k,\pm} = \pm\sqrt{\kappa_k - \xi_k^2} - \mathrm{i}\xi_k$ which both lie in the open lower half plane. Thus we have

$$\alpha_{ij}(\omega) = -\frac{p_{ij}(\omega)}{\prod_{k=1}^3 (\omega - \omega_{k,+})(\omega - \omega_{k,-})} \,.$$

In the case $\tau < 0$, the residue theorem yields

$$R_{ij}^{(1)}(\tau) = \frac{Nq^2}{2\pi m} \int_{-\infty}^{\infty} \alpha_{ij}(\omega) \mathrm{e}^{-\mathrm{i}\omega\tau} \,\mathrm{d}\omega = 0 \,,$$

since we can close the integration contour in the upper half plane.

Now we look at the second order. Here the equation of motion is given by

$$\ddot{x}_i^{(2)} + 2\Gamma_{ij}\dot{x}_j^{(2)} + K_{ij}^{(1)} x_j^{(2)} + K_{ijl}^{(2)} x_j^{(1)} x_l^{(1)} = 0$$

for $i = 1, 2, 3$. Application of the Fourier transform results in

$$\int_{-\infty}^{\infty} A_{ij}(\omega_1) \widehat{x}_j^{(2)}(\omega_1) \mathrm{e}^{-\mathrm{i}\omega_1 t} \,\mathrm{d}\omega_1 = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{ijl}^{(2)} \widehat{x}_j^{(1)}(\omega_1) \widehat{x}_l^{(1)}(\omega_2) \mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t} \,\mathrm{d}\omega_1 \,\mathrm{d}\omega_2 \,.$$

Inserting $1 = \int_{-\infty}^{\infty} \delta(\omega_2) \mathrm{e}^{-\mathrm{i}\omega_2 t} \,\mathrm{d}\omega_2$ on the left side leads to

$$\begin{aligned} \widehat{x}_i^{(2)}(\omega_1)\delta(\omega_2) &= -\frac{1}{\sqrt{2\pi}} \alpha_{ij}(\omega_1) K_{jlm}^{(2)} \widehat{x}_l^{(1)}(\omega_1) \widehat{x}_m^{(1)}(\omega_2) \\ &= -\frac{1}{\sqrt{2\pi}} K_{jlm}^{(2)} \alpha_{ij}(\omega_1) \alpha_{lp}(\omega_1) \alpha_{mq}(\omega_2) \widehat{F}_p(\omega_1) \widehat{F}_q(\omega_2) \,, \end{aligned}$$

where we have used $\widehat{\mathbf{x}}^{(1)}(\omega) = A(\omega)^{-1}\widehat{F}(\omega)$. The Fourier transform now yields

$$\begin{aligned} x_i^{(2)}(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{x}_i^{(2)}(\omega_1)\delta(\omega_2) \mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t} \,\mathrm{d}\omega_1 \,\mathrm{d}\omega_2 \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{jlm}^{(2)} \alpha_{ij}(\omega_1) \alpha_{lp}(\omega_1) \alpha_{mq}(\omega_2) \widehat{F}_p(\omega_1) \widehat{F}_q(\omega_2) \mathrm{e}^{-\mathrm{i}(\omega_1+\omega_2)t} \,\mathrm{d}\omega_1 \,\mathrm{d}\omega_2 \\ &= \frac{m^2}{Nq^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{ipq}^{(2)}(t - s_1, t - s_2) F_p(s_1) F_q(s_2) \,\mathrm{d}s_1 \,\mathrm{d}s_2 \end{aligned}$$

with

$$R_{ipq}^{(2)}(\tau_1, \tau_2) = -\frac{Nq^3}{(2\pi)^2 m^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{jlm}^{(2)} \alpha_{ij}(\omega_1) \alpha_{lp}(\omega_1) \alpha_{mq}(\omega_2) \mathrm{e}^{-\mathrm{i}(\omega_1\tau_1+\omega_2\tau_2)} \,\mathrm{d}\omega_1 \,\mathrm{d}\omega_2 \,. \tag{A.8}$$

Again using the residue theorem, we see $R_{ipq}^{(2)}(\tau_1, \tau_2) = 0$ for all $i, p, q \in \{1, 2, 3\}$ if $\tau_1 < 0$ or $\tau_2 < 0$.

The third order equation takes the form

$$\ddot{x}_i^{(3)} + 2\Gamma_{ij}\dot{x}_j^{(3)} + K_{ij}^{(1)}x_j^{(3)} + 2K_{ijl}^{(2)}x_j^{(1)}x_l^{(2)} + K_{ijlm}^{(3)}x_j^{(1)}x_l^{(1)}x_m^{(1)} = 0\,.$$

An analogous computation leads to the third order response function

$$\begin{aligned} &R_{iabc}^{(3)}(\tau_1,\tau_2,\tau_3)\\ &= \frac{Nq^4}{(2\pi)^{5/2}m^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[2K_{jlm}^{(2)}K_{pqr}^{(2)}\alpha_{ij}(\omega_1)\alpha_{la}(\omega_1)\alpha_{mp}(\omega_2)\alpha_{qb}(\omega_2)\alpha_{rc}(\omega_3) \\ &\qquad - K_{jhlm}^{(3)}\alpha_{ij}(\omega_1)\alpha_{ha}(\omega_1)\alpha_{lb}(\omega_2)\alpha_{mc}(\omega_3)\Big]\mathrm{e}^{-\mathrm{i}(\omega_1\tau_1+\omega_2\tau_2+\omega_3\tau_3)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3\,, \end{aligned} \tag{A.9}$$

which again vanishes if $\tau_k < 0$ for an index $k \in \{1,2,3\}$.

To illustrate the results from Subsection A.1, we consider the case special of an isotropic and inversion-symmetric material. Here we have $\Gamma = \gamma I$ and $K^{(1)} = \omega_0^2 I$ for positive constants γ, ω_0^2 satisfying $w_0^2 > 2\gamma^2$. We further see that $K^{(2)}$ must vanish due to the inversion symmetry since if $\mathbf{x}$ is a solves the equation of motion, $-\mathbf{x}$ must also be a solution if $\mathbf{F}$ is replaced by $-\mathbf{F}$, but the term involving $K^{(2)}$ depends quadratically on $\mathbf{x}$. As discussed in [5], the only possible form for the cubic term in (A.5) for isotropic materials is $\widetilde{K}^{(3)}\,(\mathbf{x}\cdot\mathbf{x})\,x_i$ for some scalar $\widetilde{K}^{(3)}$, i. e., $K_{ijlm}^{(3)} = \widetilde{K}^{(3)}\delta_{im}\delta_{jl}$ for all $i,j,l,m \in \{1,2,3\}$. Therefore we have

$$A(\omega) = \left(\omega_0^2 - \omega^2 - 2\mathrm{i}\gamma\omega\right) I\,, \quad \alpha_{ij}(\omega) = \frac{\delta_{ij}}{\omega_0^2 - \omega^2 - 2\mathrm{i}\gamma\omega}$$

for all $i \in \{1,2,3\}$. Using (A.7) and the residue theorem, we calculate

$$R_{j_0j_1}^{(1)}(\tau) = \delta_{j_0j_1}\frac{Nq^2}{2\pi m}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{\omega_0^2-\omega^2-2\mathrm{i}\gamma\omega}\,\mathrm{d}\omega = \delta_{j_0j_1}\frac{Nq^2}{m\nu}\mathrm{e}^{-\gamma\tau}\sin(\nu\tau)$$

for all $\tau > 0$ and $i,j \in \{1,2,3\}$, where $\nu := \sqrt{\omega_0^2 - \gamma^2}$. From (A.8), we see that $R^{(2)}$ vanishes and (A.9) yields

$$\begin{aligned} &R_{j_0j_1j_2j_3}^{(3)}(\tau_1,\tau_2,\tau_3)\\ &\quad = -\delta_{j_0j_3}\delta_{j_1j_2}\frac{Nq^4\widetilde{K}^{(3)}\sqrt{2\pi}}{2m^3\nu^5}\mathrm{e}^{-\gamma(\tau_1+\tau_2+\tau_3)}\sin(\nu\tau_2)\sin(\nu\tau_3)\big(\sin(\nu\tau_1)-\nu\tau_1\cos(\nu\tau_1)\big) \end{aligned}$$

for all $\tau \in (0,\infty)^3$ and $j_0, j_1, j_2, j_3 \in \{1,2,3\}$. This expression does not show the intrinsic permutation symmetry mentioned in Subsection 3.4. As discussed there, we can replace $R^{(3)}$ by a symmetrised version without changing the polarisation. This leads to

$$\begin{aligned} R_{j_0j_1j_2j_3}^{(3)}(\tau_1,\tau_2,\tau_3) = -&\frac{Nq^4\widetilde{K}^{(3)}\sqrt{2\pi}}{12m^3\nu^5}\mathrm{e}^{-\gamma(\tau_1+\tau_2+\tau_3)}\\ &\Big[2\left(\delta_{j_0j_1}\delta_{j_2j_3}+\delta_{j_0j_2}\delta_{j_1j_3}+\delta_{j_0j_3}\delta_{j_1j_2}\right)\sin(\nu\tau_1)\sin(\nu\tau_2)\sin(\nu\tau_3)\\ &- \delta_{j_0j_1}\delta_{j_2j_3}\big(\nu\tau_2\cos(\nu\tau_2)\sin(\nu\tau_3)+\nu\tau_3\cos(\nu\tau_3)\sin(\tau_2)\big)\sin(\nu\tau_1)\\ &- \delta_{j_0j_2}\delta_{j_1j_3}\big(\nu\tau_1\cos(\nu\tau_1)\sin(\nu\tau_3)+\nu\tau_3\cos(\nu\tau_3)\sin(\tau_1)\big)\sin(\nu\tau_2)\\ &- \delta_{j_0j_3}\delta_{j_1j_2}\big(\nu\tau_1\cos(\nu\tau_1)\sin(\nu\tau_2)+\nu\tau_2\cos(\nu\tau_2)\sin(\tau_1)\big)\sin(\nu\tau_3)\Big]\,. \end{aligned}$$

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