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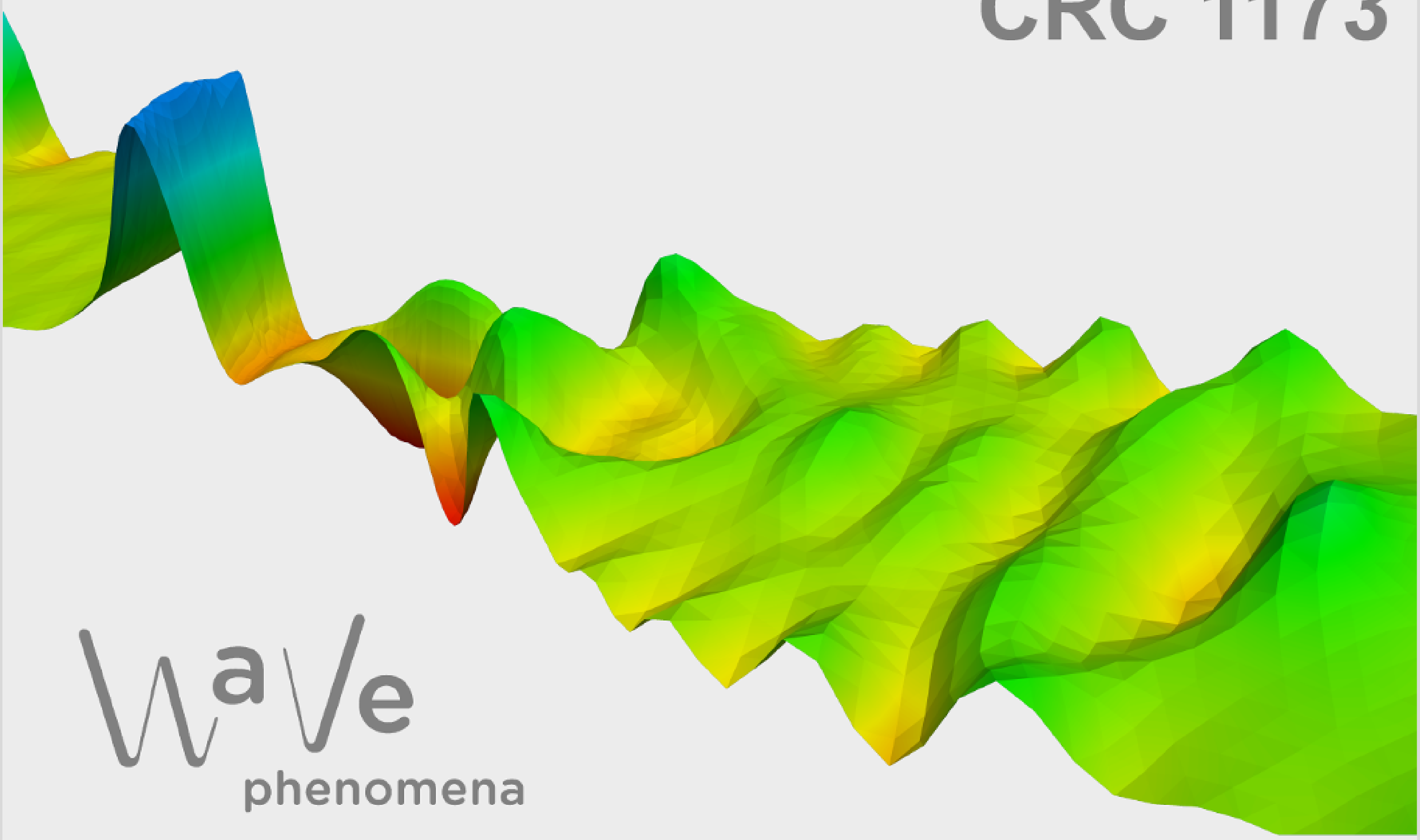

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A scalar product for computing fundamental quantities in matter and its application to the helicity and angular momentum stored in a Hopfion

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We introduce a systematic way to obtain expressions for computing the amount of fundamental quantities such as angular momentum contained in static matter, given its charge and magnetization densities. The expressions are obtained from a scalar product whose form results from imposing invariance under the scale-Euclidian group of transformations. Such group is obtained by restricting the conformal group of invariance of the dynamic Maxwell fields to the static case. In an exemplary application, we compute the helicity and angular momentum squared stored in a Hopfion, and show that the Hopfion is an eigenstate of angular momentum along one axis with eigenvalue zero.

Research in light-matter interactions benefits from theoretical tools where light and matter are treated similarly. Physical concepts that apply to both light and matter are, for example, linear and angular momentum. Both light and matter can have linear and angular momentum, and their physical meaning as the generators of translations and rotations, respectively, is the same for both light and matter. What these quantities have in common is that they are the generators of transformations of symmetry groups that are relevant in physics, such as the Poincaré group of special relativity[1]. The concepts of symmetry transformations and invariance are general enough to be applied to both light and matter [2]. Such concepts are best formulated using the language of operators acting on Hilbert spaces.

In classical electrodynamics, light and matter are treated in a rather similar way: as continuous fields such as the electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ representing radiation, and the densities of charge $\rho(\mathbf{r},t)$, current $\mathbf{J}(\mathbf{r},t)$, and magnetization $\mathbf{M}(\mathbf{r},t)$, representing matter. The light side can be treated using the tools of Hilbert spaces [3–7], which makes the consideration of material symmetries and their consequences for light straightforward [7]. The conformal invariance of the free Maxwell equations [8] and the corresponding conformally invariant scalar product between electromagnetic fields [3, 5, 6] play an important role in such algebraic approach to electrodynamics. In particular, the scalar product allows one to obtain expressions for computing the amount of quantities such as angular momentum, energy, and helicity contained in a given radiation field [5, 6, 9].

In this article, we extend the algebraic approach towards the matter side and introduce a way to obtain expressions for computing the amount of fundamental quantities contained in static material objects from their charge and magnetization densities, $\rho(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$, which are assumed to be spatially confined. The expressions are obtained as a scalar product $\langle\Phi|\Gamma|\Phi\rangle$, where $|\Phi\rangle$ represents $\{\rho(\mathbf{r}),\mathbf{M}(\mathbf{r})\}$, and Γ is the Hermitian operator representing the particular quantity of interest. The expression of the scalar product for static matter is derived for the static fields that are bijectively connected to $\rho(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$ by first considering the largest group of transformations that leave the Maxwell equations invariant, the conformal group, then excluding the transformations that would not preserve the $\omega = 0$ condition of static fields, and finally obtaining a scalar product invariant under the remaining group of transformations, which is the scale-Euclidean group composed by translations, rotations and spatial scalings [10, 11]. The form of the scalar product that we obtain for static fields with $\omega = 0$ is very similar to the form of the scalar product for dynamic fields with $\omega > 0$. Such similarity allows one to reuse for the static case expressions that are known for dynamic fields.

After setting $\rho(\mathbf{r}) = 0$ to focus on magnetization textures, we find that the net angular momentum and linear momentum of $\mathbf{M}(\mathbf{r})$ along any given axis vanishes because of the time-reversal properties of magnetization, and the fact that $\mathbf{M}(\mathbf{r})$ is real-valued, respectively. In contrast, the angular momentum squared and linear momentum squared do not vanish. The helicity does not necessarily vanish either.

In an exemplary application, we use the formalism to compute the helicity and total angular momentum squared stored in a Hopfion [12–16] in FeGe under zero external field. The helicity of the Hopfion, equal to -129.1ħ, results in a lower bound for the number of circularly polarized photons that would be needed in a helicity-dependent all optical switching of the Hopfion onto its mirror image of opposite helicity: $\lceil 129.1 \times 2\rceil = 259$. The number -129.1ħ also bounds the helicity that can be radiated from the Hopfion as it loses its chirality, for example by the action of a large magnetic bias aligning its magnetization density vector along the same direction at all points. The Hopfion is shown to be an eigenstate of angular momentum along one axis with eigenvalue zero, and to contain a total angular momentum squared of $\approx 1.3\times 10^3\hbar^2$. All expected outcomes are numerically verified for the Hopfion, including the vanishing of the momenta and angular momenta, and a relation that time-reversal imposes on the angular momen-

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tum content of static magnetizatoin densitities [Eq. (31)].

The approach makes the computation of fundamental quantities in matter very similar to corresponding computations for the electromagnetic field, and it can be readily applied to analytically derived, numerically obtained, or experimentally measured three-dimensional charge [17] and magnetization density distributions[18].

The rest of the article is organized as follows. Section I introduces the setting and context of the work. The new approach for computing properties of static material objects of finite volume from their charge and magnetization densities is presented in Sec. II. In Sec. III we provide explicit formulas for computing the helicity, angular momentum and angular momentum squared of a static magnetization texture $\mathbf{M}(\mathbf{r})$, and show that the net angular momentum along any given axis vanishes due to the time-reversal properties of magnetization. In Sec IV, we use the formalism for quantitatively studying a Hopfion in FeGe. Section V concludes the article. We foresee that the methodology will in particular be useful for the design and analysis of experiments involving the switching between stable states of a material system, for example as in the use of circularly polarized light for switching the magnetization direction in magnetic films [19], which indicates a path towards much faster and energy efficient computer memories.

I. MOTIVATION AND PROBLEM SETTING

Figure 1 depicts a light-matter interaction sequence. A beam of electromagnetic radiation approaches a material object of finite size. Initially, the object is in equilibrium with the radiation field. Then, the beam and the object interact for a finite period of time. When the interaction stops and equilibrium is reached again, both the beam and the object may have changed. For example, the energy, momentum, and angular momentum contained in the radiation field before and after the interaction may be different. The same can be said about the material system.

There are well-known expressions for computing the amount of a fundamental quantity such as energy or momentum contained in a given electromagnetic field. For example, in SI units which will be used throughout this article, and with ε_0 and μ_0 denoting the permittivity and permeability of vacuum, respectively

$$\int_{\mathbb{R}^3} \mathrm{d}\mathbf{r}\ \varepsilon_0 \mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B},\ \text{and}\ \varepsilon_0 \int_{\mathbb{R}^3} \mathrm{d}\mathbf{r}\ \mathbf{E}\times\mathbf{B} \quad (1)$$

are the energy and momentum of the field, respectively.

Expressions such as those in Eq. (1) can be derived in several different ways. For example, using conservation laws as in [20, Chap. 3], or integrating the electromagnetic stress tensor as in [21, Chap. 12.10]. An alternative approach uses the tools of Hilbert spaces. In such framework, the fields are vectors in the Hilbert space of free space solutions of Maxwell equations, that is, electromagnetic fields that are not interacting with matter. Each particular solution is represented by a ket $\{\mathbf{E}(\mathbf{r},t),\mathbf{B}(\mathbf{r},t)\} \equiv |\Phi\rangle$. The fundamental quantities are represented by Hermitian operators that act on the kets. Then, the total amount of a given fundamental quantity Γ contained in a given electromagnetic field $|\Phi\rangle$ can be written as the scalar product of $\Gamma|\Phi\rangle$ and $|\Phi\rangle$:

$$\langle\Phi|\Gamma|\Phi\rangle. \quad (2)$$

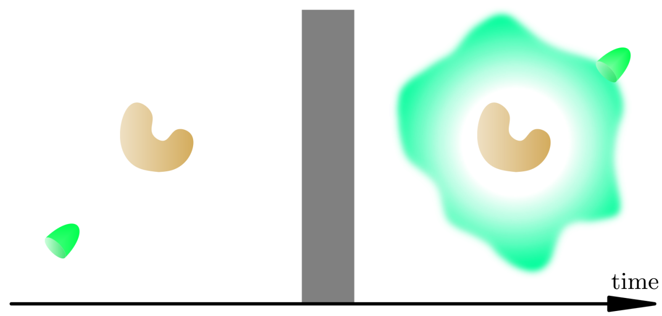

FIG. 1. A beam of electromagnetic radiation interacts with a material object of finite size during the grayed-out period. Before and some time after the interaction the object is in static equilibrium where the time derivatives of all macroscopic quantities vanish. The interaction typically changes fundamental quantities of the field such as energy or momentum. Well-known formulas exist for computing such quantities for the electromagnetic field. We develop a method for computing them in the material object in static equilibrium.

The expression of the scalar product for the free radiation fields, which, in particular, is used for obtaining explicit expressions from Eq. (2), reads [3, 6]:

$$\langle \mathrm{F}|\mathrm{G}\rangle = \int_{\mathbb{R}^3} \frac{d\mathbf{k}}{\hbar c_0|\mathbf{k}|} \begin{bmatrix}\mathbf{F}_+(\mathbf{k})\\ \mathbf{F}_-(\mathbf{k})\end{bmatrix}^\dagger \begin{bmatrix}\mathbf{G}_+(\mathbf{k})\\ \mathbf{G}_-(\mathbf{k})\end{bmatrix}, \quad (3)$$

where $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$, and the two kets are represented by their plane wave components of well-defined helicity $\lambda = 1$ and $\lambda = -1$, corresponding to left- and right-handed circular polarizations, respectively:

$$|\mathrm{F}\rangle \equiv \begin{bmatrix}\mathbf{F}_+(\mathbf{r},t)\\ \mathbf{F}_-(\mathbf{r},t)\end{bmatrix} \equiv \begin{bmatrix}\mathbf{F}_+(\mathbf{k})\\ \mathbf{F}_-(\mathbf{k})\end{bmatrix}, \text{where}$$

$$\mathbf{F}_\pm(\mathbf{r},t) = \sqrt{\frac{\varepsilon_0}{2}}\left[\mathbf{E}(\mathbf{r},t) \pm \mathrm{i}c_0\mathbf{B}(\mathbf{r},t)\right] \quad (4)$$

$$= \int_{\mathbb{R}^3} \frac{d\mathbf{k}}{\sqrt{(2\pi)^3}}\ \mathbf{F}_\pm(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r} - \mathrm{i}c_0|\mathbf{k}|t),$$

with $\mathbf{k}\cdot\mathbf{F}_\pm(\mathbf{k}) = 0$, and, importantly, the angular frequency is restricted to positive values $\omega = c_0|\mathbf{k}| > 0$. The exclusion of $\omega < 0$ is possible in electromagnetism because both sides of the spectrum contain the same information [6, §3.1][22]. The $\omega = 0$ point is also excluded from the domain of the dynamic fields.

The $\mathbf{F}_\pm(\mathbf{r},t)(\mathbf{F}_\pm(\mathbf{k}))$ are eigenstates of the helicity operator Λ. Helicity is the projection of the angular momentum $\mathbf{J}$ on the direction of the linear momentum $\mathbf{P}$. The $\mathbf{k}$-space representation of Λ is particularly simple:

$$\begin{aligned}\Lambda = \frac{\mathbf{J}\cdot\mathbf{P}}{|\mathbf{P}|} \equiv \hbar\mathrm{i}\hat{\mathbf{k}}\times,\\ \hbar\mathrm{i}\hat{\mathbf{k}}\times\mathbf{F}_\pm(\mathbf{k}) = \pm\hbar\mathbf{F}_\pm(\mathbf{k}).\end{aligned} \tag{5}$$

The defining property of Eq. (3) is that it is conformally invariant. That is, the value of $\langle F|G\rangle$ is identical to the scalar product between $\mathrm{X}|\mathrm{F}\rangle$ and $\mathrm{X}|\mathrm{G}\rangle$, for any transformation X in the conformal group. The conformal group is the largest group of invariance of free Maxwell fields [3, 8, 23]. It consists of space-time dilations, four special conformal transformations, and the Poincaré group, which consists of space-time translations, Lorentz boosts, and spatial rotations [24, 25]. It is interesting to note that the result of Eq. (3) is unitless. A conformally invariant scalar product must be unitless, because the space-time scalings in the conformal group can be interpreted as changes in measurement units [26, Sec. 3].

As shown in Ref. 6, the result of $\langle\Phi|\mathrm{H}|\Phi\rangle$ for the energy operator H, and of $\langle\Phi|\mathbf{P}|\Phi\rangle$ for the momentum operator vector $\mathbf{P}$ are equivalent to the corresponding integrals in Eq. (1). The content of the generators of Lorentz boosts in a given field [6, Eq. (4.16)], and an expression for the optical helicity alternative to previous ones [9], can also be derived with this algebraic approach. The result of $\langle\mathrm{F}|\Gamma|\mathrm{F}\rangle$ is neither unitless nor equal in general to $\langle\mathrm{F}|X^\dagger\Gamma X|\mathrm{F}\rangle$, but $\Gamma=\Lambda$ is an exception because, for dynamic Maxwell fields, helicity commutes with all the transformations of the conformal group.

Together, Eq. (2) and Eq. (3) are a general and convenient way of computing the amount of any fundamental quantity in the field. This begs the following question: *Can this algebraic approach be used for the material object?* For an affirmative answer to this question we need an appropriate mathematical representation of matter, and an appropriate scalar product for such representation.

II. SCALE INVARIANT SCALAR PRODUCT FOR STATIC MATTER

With respect to interaction with the free field, matter in static equilibrium can be represented by its electric charge density $\rho(\mathbf{r})$ and its magnetization density $\mathbf{M}(\mathbf{r})$. We assume that the time derivatives of macroscopic quantities vanish, and that there are no static currents $[\mathbf{J}(\mathbf{r})=\mathbf{0}]$. We also assume that $\rho(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$ are contained in a finite volume. The densities $\rho(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$ are equivalent to the static fields that they generate, as per the equations of electrostatics and magnetostatics. The scalar charge density generates the Coulomb field, which is a longitudinal (zero curl) electric field:

$$\varepsilon_0\nabla\cdot\mathbf{E}(\mathbf{r}) = \rho(\mathbf{r}),\ \nabla\times\mathbf{E}(\mathbf{r}) = \mathbf{0}. \tag{6}$$

The vectorial magnetization density generates the $\mathbf{B}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ fields. Outside the material object, both fields are proportional to each other and transverse (zero divergence). Inside the object, where $\mathbf{M}(\mathbf{r})\neq\mathbf{0}$, the $\mathbf{B}(\mathbf{r})$ field is transverse, and the $\mathbf{H}(\mathbf{r})$ field is longitudinal:

$$\begin{aligned}\mathbf{B}(\mathbf{r})/\mu_0 - \mathbf{H}(\mathbf{r}) = \mathbf{M}(\mathbf{r}),\ \nabla\cdot\mathbf{B}(\mathbf{r}) = 0,\\ \nabla\times\mathbf{B}(\mathbf{r})/\mu_0 = \nabla\times\mathbf{M}(\mathbf{r}),\ \nabla\cdot\mathbf{H}(\mathbf{r}) = -\nabla\cdot\mathbf{M}(\mathbf{r}).\end{aligned} \tag{7}$$

Effectively, the equations in (7) are identifications of the transverse and longitudinal parts of $\mathbf{M}(\mathbf{r})$ with the other fields, and we may as well use only $\mathbf{M}(\mathbf{r})$ instead of both $\mathbf{B}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ together. We can therefore represent matter in static equilibrium by means of $\mathbf{E}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$, or, alternatively, by means of their Fourier transforms:

$$|\Phi_{\omega=0}\rangle \equiv \begin{bmatrix}\sqrt{\varepsilon_0}\ \mathbf{E}(\mathbf{r})\\ \sqrt{\mu_0}\ \mathbf{M}(\mathbf{r})\end{bmatrix} \equiv \begin{bmatrix}\sqrt{\varepsilon_0}\ \mathbf{E}(\mathbf{k})\\ \sqrt{\mu_0}\ \mathbf{M}(\mathbf{k})\end{bmatrix}. \tag{8}$$

Both $\mathbf{E}(\mathbf{k})$ and $\mathbf{M}(\mathbf{k})$ can be obtained by integrals in the finite volume V occupied by the object:

$$\begin{aligned}\mathbf{M}(\mathbf{k}) &= \int_{\mathrm{V}} \frac{\mathrm{d}\mathbf{k}}{\sqrt{(2\pi)^3}}\ \mathbf{M}(\mathbf{r})\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r}),\\ \mathbf{E}(\mathbf{k}) &= \frac{-\mathrm{i}\hat{\mathbf{k}}}{\varepsilon_0|\mathbf{k}|}\rho(\mathbf{k}) = \frac{-\mathrm{i}\hat{\mathbf{k}}}{\varepsilon_0|\mathbf{k}|}\int_{\mathrm{V}} \frac{\mathrm{d}\mathbf{k}}{\sqrt{(2\pi)^3}}\ \rho(\mathbf{r})\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r}),\end{aligned} \tag{9}$$

where the first equality in the second line of Eq. (9) follows from Eq. (6).

We now set out to define an appropriate scalar product for $|\Phi_{\omega=0}\rangle$ objects, with the aim of using Eq. (2) for computing the amount of fundamental quantities stored in matter. We pursue the idea of a group invariance for the scalar product starting by the full conformal invariance that Eq. (3) exhibits for $\omega>0$. If we consider matter in static equilibrium, where $\omega=0$, then several transformations in the conformal group are not suitable because the $\omega=0$ condition must be preserved. Lorentz boosts and special conformal transformations change ω and mix it with the components of $\mathbf{k}$ [24, Sec. 3]. We remove such transformations. We also remove time-translations because, while preserving the $\omega=0$ condition, any time-translations will just degenerate into the identity operator for the static case. We are then left with a seven parameter group consisting of spatial translations, spatial rotations and the spatial scaling ($\mathbf{r}\to\alpha\mathbf{r}$ with $\alpha\in\mathbb{R},\ \alpha>0$): The scale-Euclidean group [10, 11]. Appendix A shows that the scalar product

$$\langle\Phi^1_{\omega=0}|\Phi^2_{\omega=0}\rangle = \int_{\mathbb{R}^3}\frac{d\mathbf{k}}{\hbar c_0|\mathbf{k}|}\ \begin{bmatrix}\sqrt{\varepsilon_0}\ \mathbf{E}^1(\mathbf{k})\\ \sqrt{\mu_0}\ \mathbf{M}^1(\mathbf{k})\end{bmatrix}^\dagger \begin{bmatrix}\sqrt{\varepsilon_0}\ \mathbf{E}^2(\mathbf{k})\\ \sqrt{\mu_0}\ \mathbf{M}^2(\mathbf{k})\end{bmatrix}, \tag{10}$$

is invariant upon the action of all the transformation in the scale-Euclidean group. Such invariance can also be argued from the similarity of the expressions for the scalar products in the dynamic and static cases in Eq. (3) and Eq. (10), and the fact that we have reached the scale-Euclidean group by restricting the conformal group. Moreover, when written in the form of Eq. (19), Eq. (10) coincides with the scalar product for the relevant representations of the scale-Euclidean group in [11, Eq. (29)] [1]. The expression in Eq. (10) can also be heuristically reached from Eq. (3) by changing from the $\mathbf{F}_\pm(\mathbf{k})$ to the $\{\mathbf{E}(\mathbf{k}),\mathbf{B}(\mathbf{k})\}$ basis and replacing $\mathbf{B}(\mathbf{k})$ by $\mathbf{M}(\mathbf{k})/\mu_0$. Such replacement is necessary for including the longitudinal degree of freedom in $\mathbf{M}(\mathbf{k})$. This way of obtaining Eq. (10) can be seen as the inclusion of two branches for each $\mathbf{k}$ in the scalar product of Eq. (3), one branch with $(\mathbf{k},\omega=c_0|\mathbf{k}|)$ and another branch with $(\mathbf{k},\omega=0)$. The two branches together cover the domain of dynamic and static electromagnetic fields.

We adopt Eq. (10) as the scalar product for $\omega=0$. We note that demanding invariance under spatial scalings is consistent with and can also be motivated by the fact that such transformations preserve both the electric charge inside and the magnetic flux through the boundary of any given volume (see App. A).

The essentially identical functional form of the expressions for the scalar products in the dynamic and static cases has an important consequence: Known $\mathbf{k}$-space expressions of fundamental operators for the $\omega>0$ case [5, 6, 27] can be also used for the $\omega=0$ case. The same is true for the corresponding $\mathbf{r}$-space expressions. And this provides a new way of obtaining expressions for computing the amount of fundamental quantities stored in matter.

III. HELICITY AND ANGULAR MOMENTUM IN MAGNETIZATION

In the rest of the article, we will focus on fundamental quantities stored in the magnetization density, for which we set $\mathbf{E}(\mathbf{r})=\mathbf{0}$ in Eq. (10):

$$\langle \mathrm{M}_1|\mathrm{M}_2\rangle=\int_{\mathbb{R}^3}\frac{\mu_0 d^3\mathbf{k}}{c_0\hbar|\mathbf{k}|}\mathbf{M}^1(\mathbf{k})^\dagger\mathbf{M}^2(\mathbf{k}) \qquad (11)$$

We start with the helicity stored in $\mathbf{M}(\mathbf{r})$. At each $\mathbf{k}$ point, $\mathbf{M}(\mathbf{k})$ can be decomposed into the three eigenstates of the helicity operator for vectorial fields:

$$\mathbf{M}(\mathbf{k})=\sum_{\lambda=-1,0,1}\mathbf{M}_\lambda(\mathbf{k}),\ \hbar i\hat{\mathbf{k}}\times\mathbf{M}_\lambda(\mathbf{k})=\hbar\lambda\mathbf{M}_\lambda(\mathbf{k}). \qquad (12)$$

[1] The extra $1/|\mathbf{k}|^2$ in [11, Eq. (29)] with respect to Eq. (19) is compensated by the extra $1/|\mathbf{k}|$ factor in the plane wave decomposition in [11, Eq. (57)] as compared to Eq. (9). That $N=-2$ in [11, Eq. (57)] follows from the definition of N in [11, Eq. (25a)], and the transformation properties of $\mathbf{E}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$ under scalings [Eq. (A4)].

The longitudinal $\lambda=0$ component corresponds to non-vanishing $\nabla\cdot\mathbf{M}(\mathbf{r})$. While a net magnetic charge in an isolated object has not been observed, complicated magnetization textures are expected to contain pairs of monopoles of opposite charge [13], and pairs of singularities have indeed been experimentally imaged [18]. The longitudinal term in the magnetization is crucial for their description. In contrast, the free dynamic electromagnetic field is divergenceless, and the $\lambda=0$ component vanishes.

For the explicit decomposition into plane waves we use [28, Eq. (6)]

$$\mathbf{M}_\lambda(\mathbf{k})=f_\lambda(\mathbf{k})\mathbf{Q}_\lambda(\hat{\mathbf{k}}), \qquad (13)$$

where $f_\lambda(\mathbf{k})$ are complex-valued functions, and

$$\mathbf{Q}_0(\hat{\mathbf{k}})=-\frac{\mathbf{k}}{|\mathbf{k}|}=-\hat{\mathbf{k}},\ \mathbf{Q}_\pm(\hat{\mathbf{k}})=\frac{\mp 1}{\sqrt{2}}\begin{bmatrix}\frac{k_x(k_x\pm ik_y)}{|\mathbf{k}|(|\mathbf{k}|+k_z)}-1\\ \frac{k_x(k_x\pm ik_y)}{|\mathbf{k}|(|\mathbf{k}|+k_z)}\mp i\\ \frac{k_x\pm ik_y}{|\mathbf{k}|}\end{bmatrix}, \qquad (14)$$

which meet

$$i\hat{\mathbf{k}}\times\mathbf{Q}_\lambda(\hat{\mathbf{k}})=\lambda\mathbf{Q}_\lambda(\hat{\mathbf{k}}),\ \text{and}\ \mathbf{Q}_\lambda(\hat{\mathbf{k}})^\dagger\mathbf{Q}_{\bar\lambda}(\hat{\mathbf{k}})=\delta_{\lambda\bar\lambda}. \qquad (15)$$

The last equation in (15) implies that

$$f_\lambda(\mathbf{k})=\mathbf{Q}_\lambda(\hat{\mathbf{k}})^\dagger\mathbf{M}(\mathbf{k}). \qquad (16)$$

Other choices for the helicity vectors have been also used in the literature. In particular, the $\mathbf{e}_\lambda(\hat{\mathbf{k}})$ used by Bialynicki-Birula [5, 6] are related to the $\mathbf{Q}_\lambda(\hat{\mathbf{k}})$ vectors used by Moses [11, 28] in this way: $\mathbf{Q}_\lambda(\hat{\mathbf{k}})=-\mathbf{e}_\lambda(\hat{\mathbf{k}})\exp(i\lambda\phi)$.

Using the expression of the helicity operator in Eq. (5), the helicity stored in $\mathbf{M}(\mathbf{r})$ can then be written as:

$$\begin{aligned}\langle \mathrm{M}|\Lambda|\mathrm{M}\rangle &= \int_{\mathbb{R}^3}\frac{\mu_0 d^3\mathbf{k}}{c_0\hbar|\mathbf{k}|}\mathbf{M}(\mathbf{k})^\dagger\hbar i\hat{\mathbf{k}}\times\mathbf{M}(\mathbf{k})\\ &\overset{\text{Eq. (12)}}{=}\int_{\mathbb{R}^3}\frac{\mu_0 d^3\mathbf{k}}{c_0|\mathbf{k}|}\left[\sum_{\bar\lambda=-1,0,1}\mathbf{M}_{\bar\lambda}(\mathbf{k})\right]^\dagger\sum_{\lambda=-1,0,1}\lambda\mathbf{M}_\lambda(\mathbf{k})\\ &\overset{\text{Eqs. (13,15)}}{=}\int_{\mathbb{R}^3}\frac{\mu_0 d^3\mathbf{k}}{c_0|\mathbf{k}|}\left[|f_+(\mathbf{k})|^2-|f_-(\mathbf{k})|^2\right].\end{aligned} \qquad (17)$$

As shown in Ref. 29, the result of Eq. (17) is proportional to the definition of the static magnetic helicity [30–32]:

$$\int_{\mathbb{R}^3} d\mathbf{r}\ \mathbf{B}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})=\int_{\mathbb{R}^3} d\mathbf{k}\mathbf{B}^\dagger(\mathbf{k})\mathbf{A}(\mathbf{k}), \qquad (18)$$

where $\nabla\times\mathbf{A}(\mathbf{r})=\mathbf{B}(\mathbf{r})$, and hence $i\mathbf{k}\times\mathbf{A}(\mathbf{k})=\mathbf{B}(\mathbf{k})$.

More generally, Eqs. (12), (15), and (16) can be used to write the expression of the scalar product in Eq. (11) as a function of the $f_\lambda(\mathbf{k})$

$$\langle \mathrm{M}_1|\mathrm{M}_2\rangle=\sum_{\lambda=-1,0,+1}\int_{\mathbb{R}^3}\frac{\mu_0 d^3\mathbf{k}}{c_0\hbar|\mathbf{k}|}\left[{}^1f_\lambda(\mathbf{k})\right]^*{}^2f_\lambda(\mathbf{k}), \qquad (19)$$

which, in particular, provides an alternative route to the last line of Eq. (17), and in general, to the computation of $\langle \mathrm{M}|\Gamma|\mathrm{M}\rangle$ from the $\mathrm{f}_\lambda(\mathbf{k})$, as long as the action of Γ on $\mathrm{f}_\lambda(\mathbf{k})$ is known. And this is where the reuse of the **k**-space action of Γ from the dynamic case becomes very useful.

Let us now turn our attention to angular momentum. Rather than using $\mathrm{f}_\lambda(\mathbf{k})$, the computation of $\langle \mathrm{M}|\mathrm{J}_i|\mathrm{M}\rangle$ is more conveniently performed in the angular momentum basis:

$$\mathbf{M}(\mathbf{r}) \equiv \sum_{\lambda=-1,0,1} \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \int_0^\infty \mathrm{d}|\mathbf{k}| \mathrm{F}_\lambda^{jm}(|\mathbf{k}|)|\mathbf{k}||k\ j\ m\ \lambda\rangle, \tag{20}$$

where $|k\ j\ m\ \lambda\rangle$ denotes a simultaneous eigenstate of $\mathbf{P}\cdot\mathbf{P}$, J^2, J_z, and Λ, with eigenvalues $\hbar k^2$, $\hbar^2 j(j+1)$, $\hbar m$, and $\hbar\lambda$, respectively, and

$$\mathrm{F}_\lambda^{jm}(|\mathbf{k}|) = |\mathbf{k}|\sqrt{\frac{2j+1}{4\pi}} \int \mathrm{d}\hat{\mathbf{k}}\ \mathrm{D}_j^{m\lambda}(\hat{\mathbf{k}})\mathrm{f}_\lambda(\mathbf{k}), \tag{21}$$

where the Wigner D-matrices for spatial rotations [33, Chap. 4] enter as follows

$$\mathrm{D}_j^{m\lambda}(\hat{\mathbf{k}}) = \mathrm{D}_j^{m\lambda}(\phi,\theta,-\phi) = \exp(-\mathrm{i}m\phi)\mathrm{d}_j^{m\lambda}(\theta)\exp(\mathrm{i}\lambda\phi), \tag{22}$$

$\mathrm{d}_j^{m\lambda}(\theta)$ are the Wigner small D-matrices, and $\int \mathrm{d}\hat{\mathbf{k}} \equiv \int_0^\pi \mathrm{d}\theta \sin\theta\ \int_{-\pi}^{\pi}\mathrm{d}\phi$, with $\phi = \mathrm{atan2}(k_y,k_x)$, and $\theta = \arccos(k_z/|\mathbf{k}|)$. The publicly available EasySpin computer code contains a convenient implementation of the Wigner matrices [34].

As we shown in App. B, the scalar product $\langle \mathrm{M}_1|\mathrm{M}_2\rangle$ can also be written as:

$$\langle \mathrm{M}_1|\mathrm{M}_2\rangle = \sum_{\lambda=-1,0,+1} \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \int_0^\infty \frac{\mu_0 \mathrm{d}|\mathbf{k}|}{\mathrm{c}_0\hbar|\mathbf{k}|} \left[{}^1\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)\right]^* {}^2\mathrm{F}_\lambda^{jm}(|\mathbf{k}|). \tag{23}$$

The action of angular momenta on $\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)$ is [35, Eqs. (2.1)-(2.3)]:

$$\begin{aligned}
&\frac{\mathrm{J}_z}{\hbar}\mathrm{F}_\lambda^{jm}(|\mathbf{k}|) = m\mathrm{F}_\lambda^{jm}(|\mathbf{k}|),\\
&\frac{(\mathrm{J}_y+\mathrm{i}\mathrm{J}_x)}{\hbar}\mathrm{F}_\lambda^{jm}(|\mathbf{k}|) = \sqrt{(j-m)(j+m+1)}\mathrm{F}_\lambda^{j(m+1)}(|\mathbf{k}|),\\
&\frac{(\mathrm{J}_y-\mathrm{i}\mathrm{J}_x)}{\hbar}\mathrm{F}_\lambda^{jm}(|\mathbf{k}|) = \sqrt{(j+m)(j-m+1)}\mathrm{F}_\lambda^{j(m-1)}(|\mathbf{k}|),\\
&\frac{\mathrm{J}^2}{\hbar^2}\mathrm{F}_\lambda^{jm}(|\mathbf{k}|) = \left(\mathrm{J}_x^2+\mathrm{J}_y^2+\mathrm{J}_z^2\right)\mathrm{F}_\lambda^{jm}(|\mathbf{k}|) = j(j+1)\mathrm{F}_\lambda^{jm}(|\mathbf{k}|),
\end{aligned} \tag{24}$$

with which we can write:

$$\begin{aligned}
&\langle \mathrm{M}|\mathrm{J}_z|\mathrm{M}\rangle = \sum_{\lambda=-1,0,+1} \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \int_0^\infty \frac{\mu_0\mathrm{d}|\mathbf{k}|}{\mathrm{c}_0|\mathbf{k}|} m|\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)|^2,\\
&\langle \mathrm{M}|\mathrm{J}_y+\mathrm{i}\mathrm{J}_x|\mathrm{M}\rangle =\\
&\sum_{\lambda jm}\int_0^\infty \frac{\mu_0\mathrm{d}|\mathbf{k}|}{\mathrm{c}_0|\mathbf{k}|}\sqrt{(j-m)(j+m+1)}\left[\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)\right]^*\mathrm{F}_\lambda^{j(m+1)}(|\mathbf{k}|),\\
&\langle \mathrm{M}|\mathrm{J}_y-\mathrm{i}\mathrm{J}_x|\mathrm{M}\rangle =\\
&\sum_{\lambda jm}\int_0^\infty \frac{\mu_0\mathrm{d}|\mathbf{k}|}{\mathrm{c}_0|\mathbf{k}|}\sqrt{(j+m)(j-m+1)}\left[\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)\right]^*\mathrm{F}_\lambda^{j(m-1)}(|\mathbf{k}|),\\
&\langle \mathrm{M}|\mathrm{J}^2|\mathrm{M}\rangle = \sum_{\lambda jm}\int_0^\infty \frac{\mu_0\mathrm{d}|\mathbf{k}|}{\mathrm{c}_0|\mathbf{k}|}\hbar j(j+1)|\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)|^2.
\end{aligned} \tag{25}$$

We will now show that the angular momenta $\langle \mathrm{M}|\mathrm{J}_{\tau\in\{x,y,z\}}|\mathrm{M}\rangle$ of a static magnetization are zero, which is consistent with the static equilibrium condition under zero external fields. The proof that the result of the first three lines in Eq. (25) is zero is reached through the consideration of time reversal.

Let us momentarily consider a time-varying magnetization density $\mathbf{M}(\mathbf{r},t)$. The time-reversal transformation of $\mathbf{M}(\mathbf{r},t)$ produces

$$\mathrm{T}_{\mathrm{rev}}\{\mathbf{M}(\mathbf{r},t)\} = -\mathbf{M}(\mathbf{r},-t), \tag{26}$$

where the overall sign change is the signature behavior of magnetic quantities, as opposed to electric quantities, which do not acquire a minus sign upon time reversal [21, Table 6.1].

For a static magnetization density, Eq. (26) reduces to

$$\mathrm{T}_{\mathrm{rev}}\{\mathbf{M}(\mathbf{r})\} = -\mathbf{M}(\mathbf{r}). \tag{27}$$

The implications of Eq. (27) for the decomposition of $\mathbf{M}(\mathbf{r})$ in Eq. (20) can be elucidated using the transformation properties of angular momentum eigenstates upon time reversal [36, Eq. 12.7-2]

$$\mathrm{T}_{\mathrm{rev}}|k\ j\ m\ \lambda\rangle = (-1)^{j+m}|k\ j\ -m\ \lambda\rangle, \tag{28}$$

and the complex conjugation of the coefficients due to the anti-linear character of $\mathrm{T}_{\mathrm{rev}}$. We obtain:

$$\mathrm{T}_{\mathrm{rev}}\{\mathbf{M}(\mathbf{r})\} \equiv \sum_{\lambda=-1,0,1} \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \int \mathrm{d}|\mathbf{k}| \left(F_\lambda^{jm}(|\mathbf{k}|)\right)^* |\mathbf{k}|(-1)^{j+m}|k\ j\ -m\ \lambda\rangle, \tag{29}$$

which, according to Eq. (27) must also be equal to

$$-\mathbf{M}(\mathbf{r}) \equiv -\sum_{\lambda=-1,0,1} \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \int \mathrm{d}|\mathbf{k}|\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)|\mathbf{k}||k\, j\, m\, \lambda\rangle. \tag{30}$$

Since $|k\ j\ m\ \lambda\rangle$ constitute an orthonormal basis, Eq. (29) and Eq. (30) together force the following relationship:

$$\left[\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)\right]^*(-1)^{j+m} = -\mathrm{F}_\lambda^{j-m}(|\mathbf{k}|). \tag{31}$$

Equation (31) holds for any static magnetization density $\mathbf{M}(\mathbf{r})$, and, in particular, allows to show that the first three lines of Eq. (25) are equal to zero. For $\langle\mathrm{M}|\mathrm{J}_z|\mathrm{M}\rangle$, the proof is straighforward because the sum over m vanishes since Eq. (31) implies that $|\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)|^2 = |\mathrm{F}_\lambda^{j-m}(|\mathbf{k}|)|^2$, and hence $m|\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)|^2 - m|\mathrm{F}_\lambda^{j-m}(|\mathbf{k}|)|^2 = 0$ for all $|m| \in [0, j]$. For $\langle\mathrm{M}|\mathrm{J}_y \pm \mathrm{i}\mathrm{J}_x|\mathrm{M}\rangle$ the proof is slightly more involved (see App. C). Additionally, App. C contains the proof that the net linear momentum $\langle\mathrm{M}|\mathrm{P}_{\tau\in\{x,y,z\}}|\mathrm{M}\rangle$ also vanishes, which is again consistent with the static equilibrium assumption under zero external fields. In contrast, the squared linear and angular momenta $\langle\mathrm{M}|\mathrm{P}^2|\mathrm{M}\rangle$, and $\langle\mathrm{M}|\mathrm{J}^2|\mathrm{M}\rangle$, do not vanish since the corresponding integrands are non-negative (see e.g. the last line of Eq. (25)).

The $\mathrm{f}_\lambda(\mathbf{k})$ and $\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)$ for a given $\mathbf{M}(\mathbf{r})$ can be readily obtained with the following sequence of computations,

$$\mathbf{M}(\mathbf{r}) \overset{\text{Eq. (9)}}{\rightarrow} \mathbf{M}(\mathbf{k}) \overset{\text{Eq. (16)}}{\rightarrow} \mathrm{f}_\lambda(\mathbf{k}) \overset{\text{Eq. (21)}}{\rightarrow} \mathrm{F}_\lambda^{jm}(|\mathbf{k}|). \quad (32)$$

which can be applied to analytically derived, numerically obtained, or experimentally measured three-dimensional magnetization textures confined to finite volumes.

We note that, given $\rho(\mathbf{r})$, $\mathrm{f}_\lambda(\mathbf{k})$ and $\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)$ corresponding to $\mathbf{E}(\mathbf{k})$ can be similarly obtained.

IV. HELICITY AND ANGULAR MOMENTUM OF A HOPFION

We will now compute the helicity and angular momentum squared stored in a Hopfion.

A convenient analytical approximation of the magnetization density of a Hopfion has been provided by P. Sutcliffe in Ref. 12. The Hopfion is hosted in a disk of FeGe of height L=70 nm and diameter 3L. The Cartesian components of its unit magnetization density vector $\hat{\mathbf{m}}(\mathbf{r})$ are given in [12, Eq. (3.3)] as a function of the cylindrical coordinates of the position vector $\mathbf{r} \equiv [\rho, \theta, z] = [\sqrt{x^2+y^2}, \mathrm{atan2}(y,x), z]$. The length of $\mathbf{M}(\mathbf{r})$ is assumed to be constant: $\mathbf{M}(\mathbf{r}) = \mathrm{M}_s\hat{\mathbf{m}}(\mathbf{r})$. A value of M_s=384 kA/m is assumed here.

For the numerical calculations, the cylindrical $\mathbf{r}$ domain of the Hopfion was discretized with [107,65,71] points in cylindrical coordinates, and the $\mathbf{k}$ space with 950 points for $|\mathbf{k}|$ between 0 and $20\mathrm{L}^{-1}$, and with 2592 points for $\hat{\mathbf{k}}$ at each $|\mathbf{k}|$. All the multipolar orders from $j=1$ to $j=9$ were considered.

All expected outcomes such as the time-reversal condition in Eq. (31), or the vanishing of the momenta and angular momenta, are numerically verified for the Hopfion up to numerical inaccuracies at the level of the fifth significant digit.

Table I contains the values for the helicity and angular momentum squared stored in the Hopfion. The helicity stored is equivalent to ≈ 129 right-handed circularly polarized photons, which is about 10 orders of magnitude smaller than the number of photons in a circularly polarized femtosecond laser pulse of 10 mJ cm^{-2} fluence at a central wavelength of 800 nm.

$\langle\mathrm{M}|\Lambda|\mathrm{M}\rangle$	$\langle\mathrm{M}|\mathrm{J}^2|\mathrm{M}\rangle$
-129.1 ħ	1.30×10^3 ħ^2

TABLE I. Helicity Λ, and angular momentum squared J^2 stored in a Hopfion. An analytical approximation of the Hopfion in a chiral FeGe magnet of cylindrical shape [12] was used for the calculations. The height of the cylinder is equal to the magnetic helical period L, and the diameter is equal to 3L. A magnetization density saturation value of M_s=384 kA/m is assumed.

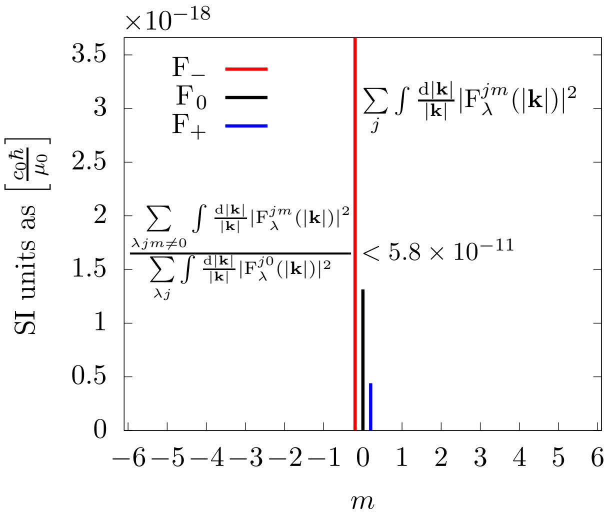

FIG. 2. Angular momentum content of the Hopfion for each helicity. While the numerical errors in the calculations produce $m \neq 0$ components, they are not visible in this scale. Two of the color bars are horizontally offset from the $m=0$ point for clarity.

We will finish this section with a more detailed analysis of the angular momentum content of the Hopfion. Some insight can be gained even without having the $\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)$ at hand. For example, we now show that the Hopfion magnetization density is an eigenstate of J_z with eigenvalue zero.

We start by slightly re-writing the expression in [12, Eq. (3.3)] with the help of a rotation matrix:

$$\hat{\mathbf{m}}(\mathbf{r}) = \begin{bmatrix}\hat{m}_x(\rho,\theta,z)\\ \hat{m}_y(\rho,\theta,z)\\ \hat{m}_z(\rho,\theta,z)\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}4\Xi\Omega\rho\\ 4\Xi\,(\Upsilon-1)\,\rho\\ (1+\Upsilon)^2-8\Xi^2\rho^2\end{bmatrix}\frac{1}{(1+\Upsilon)^2} = R_z(\theta)\hat{\mathbf{m}}_{\theta=0}, \quad (33)$$

where the last equality contains the definition of $\hat{\mathbf{m}}_{\theta=0}$, and

$$\Xi = \left(1 + (2z/\mathrm{L})^2\right) \sec\left(\pi\rho/\left(2\mathrm{L}\right)\right)/\mathrm{L}, \\ \Omega = \tan\left(\pi z/\mathrm{L}\right), \ \Upsilon = \Xi^2\rho^2 + \Omega^2/4. \tag{34}$$

We now apply apply the J_z of [37, Eq. (5.43)] to the Hopfion. In cylindrical coordinates for $\mathbf{r}$ but Cartesian components for the vector, J_z has the following expression:

$$\mathrm{J}_z \equiv -\hbar \mathrm{i}\partial_\theta + \hbar \begin{bmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{35}$$

Since M_s does not depend on θ, we can just apply J_z to $R_z(\theta)\hat{\mathbf{m}}_{\theta=0}$:

$$\frac{1}{\hbar}\mathrm{J}_z R_z(\theta)\hat{\mathbf{m}}_{\theta=0} = -\mathrm{i}\partial_\theta R_z(\theta)\hat{\mathbf{m}}_{\theta=0} + \begin{bmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_z(\theta)\hat{\mathbf{m}}_{\theta=0} \\ = \begin{bmatrix} \mathrm{i}\sin\theta & \mathrm{i}\cos\theta & 0 \\ -\mathrm{i}\cos\theta & \mathrm{i}\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{\mathbf{m}}_{\theta=0} + \begin{bmatrix} -\mathrm{i}\sin\theta & -\mathrm{i}\cos\theta & 0 \\ \mathrm{i}\cos\theta & -\mathrm{i}\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{\mathbf{m}}_{\theta=0} \\ = \mathbf{0}. \tag{36}$$

The result of Eq. (36) is faithfully reproduced by the $\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)$ amplitudes obtained from the sequence of computations written in Eq. (32). Figure 2 shows the distribution of the different helicities across the values of m. We highlight that the values for $m \neq 0$ are not suppressed from the plot, but they are not visible in this scale. The sign of the helicity of the Hopfion in Tab. I can be deduced from the larger value of the negative helicity component in Fig. 2.

V. CONCLUSION

We have introduced a new way to obtain expressions for the computation of the amount of fundamental quantities contained in static matter from its charge and magnetization densities. The method is based on invariance under symmetry transformations and the expressions are obtained as scalar products. The similarity with the case of dynamic fields allows one to reuse in the static case some of the expressions known for dynamic fields.

We have used the formalism to compute the angular momentum squared and helicity stored in a Hopfion in FeGe. The value of helicity, equal to -129.1ħ, implies a lower bound for the number of circularly polarized photons that would be needed in a helicity-dependent all optical switching of the Hopfion onto its mirror image of opposite helicity: $\lceil 129.1 \times 2 \rceil = 259$. Additionally, -129.1ħ also bounds the helicity that can be extracted from the Hopfion as it loses its chirality, for example by the action of a large magnetic bias aligning its magnetization density vector along the same direction at all points.

We foresee that the methodology will in particular be useful for the design and analysis of experiments involving the switching between stable states of a material system.

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Appendix A: Invariance of Eq. (10) under the scale-Euclidean group

It is straightforward to show that the expression

$$\langle \Phi^1_{\omega=0} | \Phi^2_{\omega=0} \rangle = \int_{\mathbb{R}^3} \frac{d\mathbf{k}}{\hbar c_0 |\mathbf{k}|} \begin{bmatrix} \sqrt{\varepsilon_0}\ \mathbf{E}^1(\mathbf{k}) \\ \sqrt{\mu_0}\ \mathbf{M}^1(\mathbf{k}) \end{bmatrix}^\dagger \begin{bmatrix} \sqrt{\varepsilon_0}\ \mathbf{E}^2(\mathbf{k}) \\ \sqrt{\mu_0}\ \mathbf{M}^2(\mathbf{k}) \end{bmatrix}, \tag{A1}$$

is invariant upon the action of all the transformation in the scale-Euclidean group, which is composed by translations, rotations and scalings.

With $\mathbf{X}(\mathbf{k})$ standing for either $\mathbf{E}(\mathbf{k})$ or $\mathbf{M}(\mathbf{k})$, the action of translations and rotations reads

$$\mathbf{X}(\mathbf{k}) \overset{\text{translation by } \mathbf{d}}{\longrightarrow} \overline{\mathbf{X}}(\bar{\mathbf{k}}) = \mathbf{X}(\mathbf{k}) \exp(\mathrm{i}\mathbf{d}\cdot\mathbf{k}), \\ \mathbf{X}(\mathbf{k}) \overset{\text{rotation by R}}{\longrightarrow} \overline{\mathbf{X}}(\bar{\mathbf{k}}) = \mathrm{R}\mathbf{X}(\mathrm{R}^{-1}\mathbf{k}), \tag{A2}$$

where $\mathbf{d}$ is a displacement and R is 3×3 unitary rotation matrix [37, Eq. (5.40)]. Then, the integrands in the expression of the scalar product between the transformed fields will be

$$\left[\mathbf{X}(\mathbf{k})\exp(\mathrm{i}\mathbf{d}\cdot\mathbf{k})\right]^\dagger \mathbf{X}(\mathbf{k})\exp(\mathrm{i}\mathbf{d}\cdot\mathbf{k}) = \mathbf{X}(\mathbf{k})^\dagger\mathbf{X}(\mathbf{k}), \\ \text{and } \left[\mathrm{R}\mathbf{X}(\mathrm{R}^{-1}\mathbf{k})\right]^\dagger \mathrm{R}\mathbf{X}(\mathrm{R}^{-1}\mathbf{k}) = \\ \mathbf{X}(\mathrm{R}^{-1}\mathbf{k})^\dagger \mathrm{R}^\dagger \mathrm{R}\mathbf{X}(\mathrm{R}^{-1}\mathbf{k}) = \mathbf{X}(\mathrm{R}^{-1}\mathbf{k})^\dagger \mathbf{X}(\mathrm{R}^{-1}\mathbf{k}), \tag{A3}$$

where the invariance under translations is obvious, and the invariance under rotations becomes clear after considering the transformed integration measure $\frac{d\mathrm{R}^{-1}\mathbf{k}}{|\mathrm{R}^{-1}\mathbf{k}|}$.

We now show that the result of Eq. (A1) is also invariant under spatial scalings $\mathbf{r} \rightarrow \alpha\mathbf{r}$ with $\alpha \in \mathbb{R}$, $\alpha > 0$, which imply $\mathbf{k} \rightarrow \mathbf{k}/\alpha$.

The transformations of dynamic electromagnetic fields under *spacetime* scalings are given in e.g. the last lines of Eqs. (3.28) and (3.40a) in Ref. 25. Their restriction to *spatial* scalings of time-independent fields reads:

$$\{\mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r})\} \rightarrow \left\{ \overline{\mathbf{E}}(\mathbf{r}) = \frac{\mathbf{E}(\mathbf{r}/\alpha)}{\alpha^2}, \overline{\mathbf{H}}(\mathbf{r}) = \frac{\mathbf{H}(\mathbf{r}/\alpha)}{\alpha^2} \right\}. \tag{A4}$$

That Eq. (A4) is the correct transformation law in the static case is verified by the fact that it preserves the fluxes of $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ across the boundary of any given volume. That is, it preserves the electric charge inside the volume and the magnetic flux through its boundary. The connections between $\mathbf{B}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$ in Eq. (7), together with the requirement of invariant $\mathbf{M}(\mathbf{r})$ flux, imply that $\mathbf{M}(\mathbf{r})$ transforms as $\mathbf{H}(\mathbf{r})$. Then, the transformation properties of the 3D Fourier transforms of $\mathbf{E}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$ can be readily deduced as a special case of the general rule $\bar{f}(\mathbf{r}) = \frac{f(\mathbf{r}/\alpha)}{\alpha^l} \iff \bar{f}(\mathbf{k}) = \alpha^{3-l} f(\alpha\mathbf{k})$:

$$\{\mathbf{E}(\mathbf{k}), \mathbf{M}(\mathbf{k})\} \to \left\{\overline{\mathbf{E}}(\mathbf{k}) = \alpha\mathbf{E}(\alpha\mathbf{k}), \overline{\mathbf{M}}(\mathbf{k}) = \alpha\mathbf{M}(\alpha\mathbf{k})\right\}. \tag{A5}$$

Let us finish the proof by considering the integral of the magnetization density part in the scalar product

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{\mu_0 \mathrm{d}^3\mathbf{k}}{c_0\hbar|\mathbf{k}|} \overline{\mathbf{M}}(\mathbf{k})^\dagger \overline{\mathbf{M}}(\mathbf{k}) &= \int_{\mathbb{R}^3} \frac{\mu_0 \mathrm{d}^3\mathbf{k}}{c_0\hbar|\mathbf{k}|} \alpha^2 \mathbf{M}(\alpha\mathbf{k})^\dagger \mathbf{M}(\alpha\mathbf{k}) \\ &= \int_{\mathbb{R}^3} \frac{\mu_0 \mathrm{d}^3\mathbf{q}}{c_0\hbar|\mathbf{q}|} \mathbf{M}(\mathbf{q})^\dagger \mathbf{M}(\mathbf{q}),\end{aligned} \tag{A6}$$

where the first equality follows from Eq. (A5), and the second from the change of integration variables $\mathbf{q} = \alpha\mathbf{k}$. The invariance of the $\mathbf{E}(\mathbf{k})$ part in the scalar product can be verified by identical steps.

Appendix B: Scalar product in the angular momentum basis

The properties of the Wigner D-matrices imply that the relation inverse to Eq. (21) is

$$\mathrm{f}_\lambda(\mathbf{k}) = \frac{1}{|\mathbf{k}|}\sqrt{\frac{2j+1}{4\pi}} \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \mathrm{F}_\lambda^{jm}(|\mathbf{k}|) \left[\mathrm{D}_j^{m\lambda}(\hat{\mathbf{k}})\right]^*, \tag{B1}$$

which we substitute in Eq. (19)

$$\begin{aligned}\langle \mathrm{M}_1|\mathrm{M}_2\rangle = &\sum_{\lambda=-1,0,+1} \int_{\mathbb{R}^3} \frac{\mu_0 \mathrm{d}^3\mathbf{k}}{c_0\hbar|\mathbf{k}|} \sum_{\bar{j}\bar{m}} \sum_{jm} \sqrt{\frac{2\bar{j}+1}{4\pi}} \sqrt{\frac{2j+1}{4\pi}} \\ &\frac{1}{|\mathbf{k}|^2} \left[{}^1\mathrm{F}_\lambda^{\bar{j}\bar{m}}(|\mathbf{k}|)\right]^* {}^2\mathrm{F}_\lambda^{jm}(|\mathbf{k}|) \mathrm{D}_{\bar{j}}^{\bar{m}\lambda}(\hat{\mathbf{k}}) \left[\mathrm{D}_j^{m\lambda}(\hat{\mathbf{k}})\right]^*.\end{aligned} \tag{B2}$$

After splitting the $\mathrm{d}^3\mathbf{k}$ integral into its radial and angular parts

$$\begin{aligned}\langle \mathrm{M}_1|\mathrm{M}_2\rangle = &\sum_{\lambda=-1,0,+1} \int_{>0}^{\infty} \frac{\mu_0 \mathrm{d}|\mathbf{k}|}{c_0\hbar|\mathbf{k}|} \sum_{\bar{j}\bar{m}} \sum_{jm} \sqrt{\frac{2\bar{j}+1}{4\pi}} \sqrt{\frac{2j+1}{4\pi}} \\ &\left[{}^1\mathrm{F}_\lambda^{\bar{j}\bar{m}}(|\mathbf{k}|)\right]^* {}^2\mathrm{F}_\lambda^{jm}(|\mathbf{k}|) \boxed{\int \mathrm{d}\hat{\mathbf{k}}\ \mathrm{D}_{\bar{j}}^{\bar{m}\lambda}(\hat{\mathbf{k}}) \left[\mathrm{D}_j^{m\lambda}(\hat{\mathbf{k}})\right]^*},\end{aligned} \tag{B3}$$

we solve the angular integral in the box by substituting $\mathrm{D}_j^{m\lambda}(\hat{\mathbf{k}}) = \exp(-\mathrm{i}m\phi)\mathrm{d}_j^{m\lambda}(\theta)\exp(\mathrm{i}\lambda\phi)$ [Eq. (22)], solving the integral in ϕ, and using the orthogonality properties of the small Wigner D-matrices [36, Eq. 8.3-2], whose elements are real-valued:

$$\begin{aligned}&\int_{-\pi}^{\pi} \mathrm{d}\phi \exp\left(\mathrm{i}\left(m-\bar{m}\right)\phi\right) \int_0^{\pi} \mathrm{d}\theta \sin\theta\ \mathrm{d}_{\bar{j}}^{\bar{m}\lambda}(\theta)\mathrm{d}_j^{m\lambda}(\theta) \\ &= 2\pi\delta_{\bar{m}m} \int_0^{\pi} \mathrm{d}\theta \sin\theta\ \mathrm{d}_{\bar{j}}^{\bar{m}\lambda}(\theta)\mathrm{d}_j^{m\lambda}(\theta) \\ &= \frac{4\pi}{2j+1}\delta_{\bar{m}m}\delta_{\bar{j}j}.\end{aligned} \tag{B4}$$

Substituting this result in the box of Eq. (B3) results in Eq. (23).

Appendix C: Linear and angular momentum vanish

1. Angular momentum

Let us consider $\langle \mathrm{M}|\mathrm{J}_y - \mathrm{i}\mathrm{J}_x|\mathrm{M}\rangle$ in Eq. (25), and show that

$$\sum_{m=-j+1}^{m=j} \sqrt{(j+m)(j-m+1)} \left[\mathrm{F}_\lambda^{jm}(|\mathbf{k}|)\right]^* \mathrm{F}_\lambda^{j(m-1)}(|\mathbf{k}|) = 0, \tag{C1}$$

which implies that $\langle \mathrm{M}|\mathrm{J}_y - \mathrm{i}\mathrm{J}_x|\mathrm{M}\rangle = 0$.

The summation in m contains an even number of terms since $\mathrm{F}_\lambda^{j(m-1)}(|\mathbf{k}|)$ is not defined for $m = -j$. The sum can be done pairwise, where one of the terms has $m > 0$ and the other is the $\bar{m} = -m+1$ term. Dropping unnecessary elements from the notation, their sum reads:

$$\begin{aligned}&\sqrt{(j+m)(j-m+1)}\mathrm{F}^{jm^*}\mathrm{F}^{j(m-1)} + \\ &\sqrt{(j+\bar{m})(j-\bar{m}+1)}\mathrm{F}^{j\bar{m}^*}\mathrm{F}^{j(\bar{m}-1)}.\end{aligned} \tag{C2}$$

We now substitute $\bar{m} = -m+1$

$$\begin{aligned}&\sqrt{(j+m)(j-m+1)}\mathrm{F}^{jm^*}\mathrm{F}^{j(m-1)} + \\ &\sqrt{(j-m+1)(j+m)}\mathrm{F}^{j(-m+1)^*}\mathrm{F}^{j-m},\end{aligned} \tag{C3}$$

and find that the sum vanishes after applying Eq. (31) to the two F factors of the second line of Eq. (C3):

$$\begin{aligned}&\sqrt{(j+m)(j-m+1)}[\mathrm{F}^{jm^*}\mathrm{F}^{j(m-1)} + \\ &\mathrm{F}^{j(m-1)}(-1)(-1)^{j-m+1}\mathrm{F}^{jm^*}(-1)(-1)^{j+m}] = 0.\end{aligned} \tag{C4}$$

The steps for the corresponding proof that $\langle \mathrm{M}|\mathrm{J}_y + \mathrm{i}\mathrm{J}_x|\mathrm{M}\rangle = 0$ in the second line of Eq. (25) are very similar.

2. Linear momentum

In $\mathbf{k}$-space, the action of the momentum operator $\mathrm{P}_{\tau\in\{x,y,z\}}$ is $\mathrm{P}_\tau\mathbf{M}(\mathbf{k}) = \hbar k_\tau \mathbf{M}(\mathbf{k})$. Then, the momentum of $\mathbf{M}(\mathbf{r})$ in direction τ can be written:

$$\begin{aligned}\langle \mathrm{M}|\mathrm{P}_\tau|\mathrm{M}\rangle &= \int_{\mathbb{R}^3} \frac{\mu_0 \mathrm{d}^3\mathbf{k}}{\mathrm{c}_0\hbar|\mathbf{k}|}\mathbf{M}(\mathbf{k})^\dagger \hbar k_\tau \mathbf{M}(\mathbf{k}) \\ &= \int_{\mathbb{R}^3} \frac{\mu_0 \mathrm{d}^3\mathbf{k}}{\mathrm{c}_0|\mathbf{k}|} k_\tau |\mathbf{M}(\mathbf{k})|^2.\end{aligned} \quad \text{(C5)}$$

Since $\mathbf{M}(\mathbf{r})$ is a real-valued field, we have that $\mathbf{M}(\mathbf{k}) = \mathbf{M}^*(-\mathbf{k})$, which readily leads to the conclusion that $\langle \mathrm{M}|\mathrm{P}_\tau|\mathrm{M}\rangle = 0$ from Eq. (C5).

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