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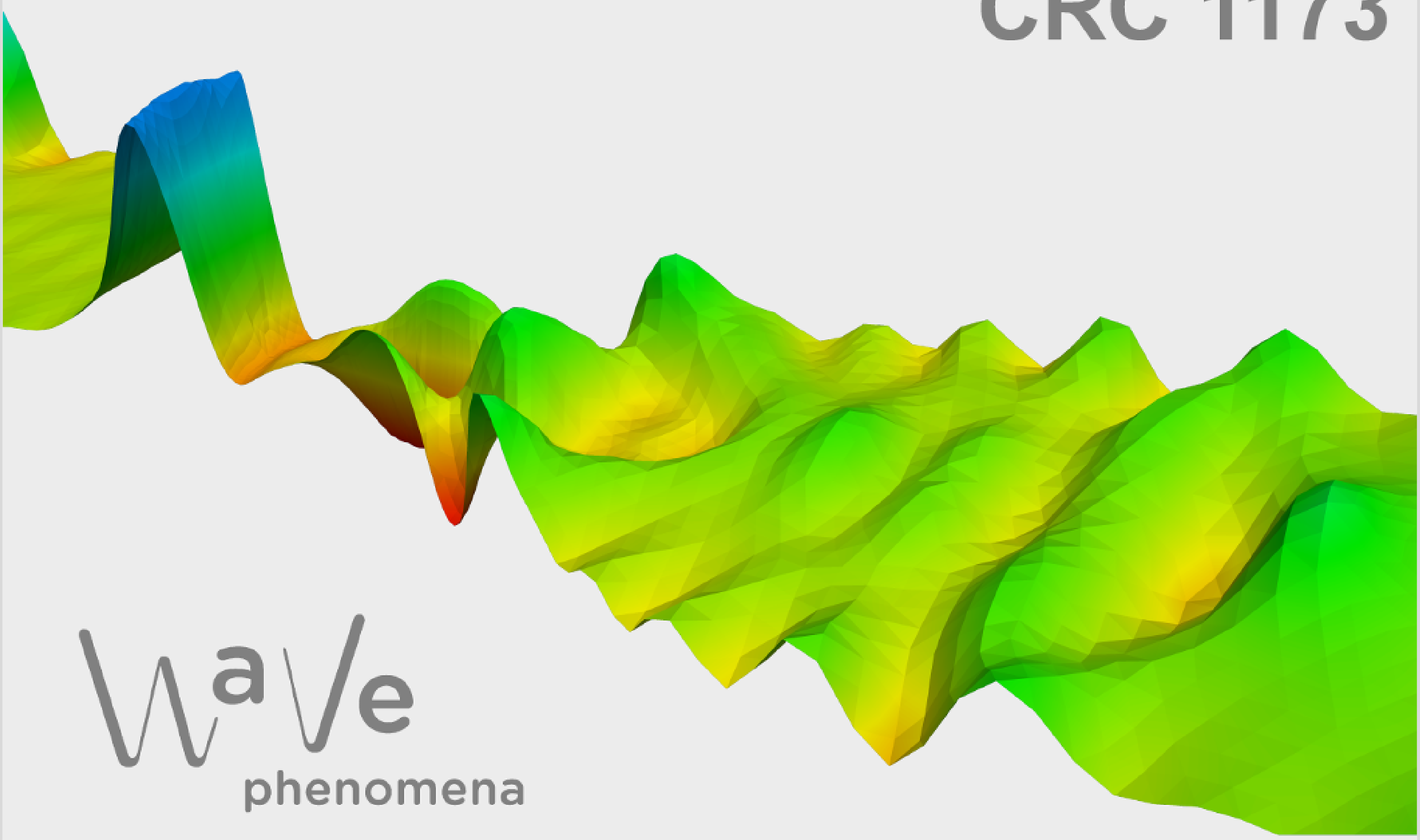

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ANALYSIS OF A DIMENSION SPLITTING SCHEME FOR MAXWELL EQUATIONS WITH LOW REGULARITY IN HETEROGENEOUS MEDIA

KONSTANTIN ZERULLA

Abstract. We analyze a dimension splitting scheme for the time integration of linear Maxwell equations in a heterogeneous cuboid. The domain contains several homogeneous subcuboids, and serves as a model for a rectangular embedded waveguide. Due to discontinuities of the material parameters and irregular initial data, the solution of the Maxwell system has regularity below H^1. The splitting scheme is adapted to the arising singularities, and is shown to converge with order one in L^2. The error result only imposes assumptions on the model parameters and the initial data, but not on the unknown solution. To achieve this result, the regularity of the Maxwell system is analyzed in detail, giving rise to sharp explicit regularity statements. In particular, the regularity parameters are given in explicit terms of the largest jump of the material parameters. The analysis is based on semigroup theory, interpolation theory, and regularity analysis for elliptic transmission problems.

1. Introduction

Maxwell equations belong to the fundamental equations in physics, and are in particular used to describe a large number of phenomena in optics, see [32, 25, 9, 17]. Their solutions are hence of great interest in many applications, like the design of waveguides, see Section 9.3 in [45]. To model waveguides, heterogeneous media are often studied that consist of several homogeneous submedia. This approach leads to material parameters that are discontinuous at the interfaces between different submedia. Maxwell equations with discontinuous material parameters, however, usually have irregular solutions, see [15, 8, 7, 11, 12, 13] for instance. This poses severe difficulties for the analysis of numerical schemes for the considered Maxwell equations. To tackle these difficulties, we employ semigroup theory, interpolation theory, and a detailed regularity analysis of an elliptic transmission problem.

On domains with tensor-structure, alternating direction implicit (ADI) schemes are very attractive methods for the time integration of linear isotropic Maxwell equations. In the ADI splitting from [56, 42], the Maxwell operator is split according to the spatial dimensions in which derivatives arise. The split system is then integrated in time by means of the Peaceman-Rachford scheme, see [44]. The splitting from [56, 42] can also be integrated in an energy conserving way, see [10]. These schemes are implicit and can be shown to be unconditionally stable, see [56, 42, 10, 29, 31, 38] for instance. Despite being implicit, the mentioned ADI schemes are also computationally cheap. In particular, the implicit steps can be shown to decouple into essentially one-dimensional problems amounting to linear complexity, see [56, 42, 10, 29, 30, 38].

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In [47, 48], the Peaceman-Rachford ADI scheme is transformed into an even more efficient formulation, being called fundamental ADI-FDTD scheme. There is also a modified ADI scheme that uniformly preserves the exponential decay behavior of the Maxwell equations with interior damping, see [53].

Despite their practical relevance, it seems to the best of our knowledge that only few rigorous error results are known about ADI schemes. In [29, 20, 21, 19, 18], the material parameters are required to be $W^{1,\infty}$ respectively $W^{1,\infty} \cap W^{2,3}$ regular on the entire cuboidal domain. In presence of appropriate initial data, the ADI schemes from [56, 42, 10] are then shown to be of order two in H^{-1} and L^2, respectively. While the mentioned error statements focus only on the time discretization, a fully discrete error analysis is performed in [38, 31] for the Peaceman-Rachford ADI scheme in combination with a discontinuous Galerkin discretization in space. [31] moreover provides estimates on time- and space-derivative errors. In [54], the Maxwell equations are considered with positive material parameters being piecewise constant on two adjacent cuboids. Assuming appropriate initial data, time discrete approximations of the Maxwell system provided by the Peaceman-Rachford ADI scheme from [56, 42] are here shown to be of order 3/2 in L^2. The error analysis of ADI schemes on the heterogeneous medium from the current paper is not covered by the existing literature, to the best of our knowledge. In particular, the solution of the considered Maxwell system has lower regularity than required in the above mentioned literature.

We study the time dependent linear isotropic Maxwell equations

$$\begin{aligned} \partial_t \mathbf{E} &= \tfrac{1}{\varepsilon} \operatorname{curl} \mathbf{H} - \tfrac{1}{\varepsilon} \mathbf{J}, & \partial_t \mathbf{H} &= -\tfrac{1}{\mu} \operatorname{curl} \mathbf{E}, \\ \mathbf{E}(0) &= \mathbf{E}_0, & \mathbf{H}(0) &= \mathbf{H}_0, \end{aligned} \tag{1.1}$$

for $t \geq 0$ on the cuboid

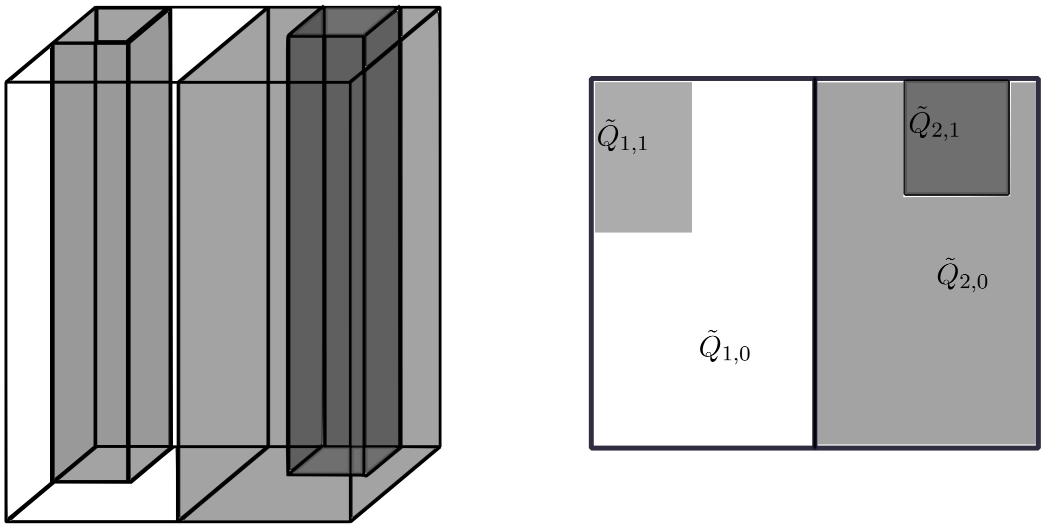
$$Q = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+)$$

with the boundary conditions of a perfect conductor

$$\mathbf{E} \times \nu = 0, \qquad \mu \mathbf{H} \cdot \nu = 0$$

on the boundary ∂Q. Conditions on the divergence of $\mathbf{E}$ and $\mathbf{H}$ are incorporated in an appropriate state space for (1.1), see (2.9) and Remark 2.2. The vector ν denotes the unit exterior normal vector at ∂Q, $\mathbf{E} = \mathbf{E}(x,t) \in \mathbb{R}^3$ stands for the electric field, $\mathbf{H} = \mathbf{H}(x,t) \in \mathbb{R}^3$ for the magnetic field, and $\mathbf{J} = \mathbf{J}(x,t) \in \mathbb{R}^3$ is a given external electric current. The functions $\varepsilon = \varepsilon(x) > 0$ and $\mu = \mu(x) > 0$ are the electric permittivity and magnetic permeability, respectively, and describe the properties of the material Q consists of.

The following assumptions on the parameters ε and μ are essential throughout the paper. The conditions are inspired by a model of a rectangular embedded waveguide, see Section 9.3 in [45] for instance. To formulate the preconditions, we make the following geometric constructions. The cuboid Q is divided into a chain of smaller cuboids $\tilde{Q}_1, \dots, \tilde{Q}_L$, where the interfaces between adjacent cuboids should be parallel to the x_2-x_3-plane. We collect these interfaces in a set $\tilde{\mathscr{F}}_{\text{int}}$. Each cuboid $\tilde{Q}_i$ further contains smaller subcuboids $\tilde{Q}_{i,1}, \dots, \tilde{Q}_{i,K}$, that are separated from each other, and touch the planes $\{x_3 = a_3^-\}$ and $\{x_3 = a_3^+\}$. The smaller subcuboids $\tilde{Q}_{i,1}, \dots, \tilde{Q}_{i,K}$ are, however, not allowed to touch an interface in $\tilde{\mathscr{F}}_{\text{int}}$. The remainder of $\tilde{Q}_i$ is then denoted by $\tilde{Q}_{i,0}$. The resulting partition of Q corresponds to a specific composition of materials. The subcuboids $\tilde{Q}_{i,1}, \dots, \tilde{Q}_{i,K}$ play the role of embedded waveguide structures, while $\tilde{Q}_{i,0}$ serves as the surrounding medium. An example of the considered configuration is given in Figure 1 with $L = 2$ and $K = 1$. Note that our analysis can in a straightforward way also be transferred to the case that each cuboid $\tilde{Q}_{i,j}$, $j \in \{1, \dots, K\}$, contains further embedded subcuboids that again



(A) Three dimensional picture of model domain

(B) Illustration of subdomain notation

FIGURE 1. Example of a heterogeneous model domain Q

touch the planes $\{x_3 = a_3^-\}$ and $\{x_3 = a_3^+\}$, but no other face of $\tilde{Q}_{i,j}$. For the sake of a clear presentation, we however omit this extension.

For the material parameters ε and μ, we throughout impose the assumptions

$$\varepsilon|_{\tilde{Q}_{i,j}}, \mu|_{\tilde{Q}_{i,j}} \in \mathbb{R}_{>0}, \qquad \varepsilon|_{\tilde{Q}_{i,0}} \leq \varepsilon|_{\tilde{Q}_{i,l}}, \qquad \mu|_{\tilde{Q}_{i,0}} = \mu|_{\tilde{Q}_{i,l}}, \tag{1.2}$$

for $i \in \{1, \dots, L\}$, $j \in \{0, \dots, K\}$, and $l \in \{1, \dots, K\}$. These assumptions mean that each subdomain $\tilde{Q}_{i,j}$ should consist of a homogeneous medium. Additionally, μ is assumed to be constant in each cuboid $\tilde{Q}_i$. One main goal of this paper is to express the regularity of the solutions in terms of the largest relative jump of ε inside a cuboid $\tilde{Q}_i$, see Corollary 5.2 and Remark 5.3.

Due to low regularity in the x_1-x_2-plane of the solutions of (1.1), see Remark 5.3, we use a different directional splitting of the Maxwell operator than the standard one from [56, 42]. (The solution of (1.1) is not contained in the domains of the standard splitting operators. Hence the standard Peaceman-Rachford ADI scheme is not applicable to the original solution, see [29].) The idea behind the directional splitting, we consider, is to treat the x_3-direction independently, and to leave the x_1-x_2-directions coupled, see Section 6.1. The split system is then integrated in time by means of the Peaceman-Rachford scheme [44], see (6.3). The resulting scheme is shown to be unconditionally stable, see Lemma 6.3. For the implicit steps in the scheme (6.3), decoupled two-dimensional elliptic problems have to be solved for the third components of the approximations to the electric and magnetic fields, see Remark 6.4. All other components of the electromagnetic field approximations are obtained by solving only one-dimensional elliptic problems.

Note that the present geometry only allows singularities at the interior edges. This is essential for the numerical splitting scheme (6.3). In particular, the scheme relies on the fact that the solution of the Maxwell equations is H^1-regular in the x_3-direction. The scheme is not applicable to the solution of (1.1) in absence of this regularity.

Our main result is given in Theorem 6.5, stating that the directional splitting scheme (6.3) converges with order one in L^2 to the solution of (1.1). The error result is rigorous in the sense that we impose assumptions only on the material parameters and the initial data. Furthermore, we can deal with less regular initial data than comparative literature [29, 21, 20, 19]. For these irregular data, we can, however, only show convergence of order one instead of order two. Indeed, the local error can only be expanded to terms of second order in the time step size, since higher order error terms cannot be controlled properly in our regularity setting. We are also going to provide a rigorous convergence result of order $2 - \kappa$ for scheme (6.3) in a subsequent work in preparation. (The number $\kappa > 0$ depends on the largest jump of the parameter ε at the interior edges.) There we, however, have to impose stronger assumptions on the initial data.

To establish Theorem 6.5, we study the regularity of (1.1) in detail. The regularity of the time-harmonic counterpart of (1.1) on more general heterogeneous polyhedral domains has been analyzed in several papers, see [8, 15, 13, 7, 11, 12] for instance. We provide a regularity analysis here to have sharp regularity statements for our model problem that explicitly link the size of the jumps of the material parameters to the regularity of the problem, see Corollary 5.2 and Remark 5.3. Moreover, we obtain that some components of the electric and magnetic field have differing regularity. This turns out to be crucial for the numerical approximation scheme. The regularity statement and the associated reasoning will additionally be employed in the above mentioned follow-up work to derive higher regularity statements. For the sake of a clear presentation, we hence give a detailed account of the arguments. In particular, we localize at the interior edges in our medium, and study elliptic transmission problems in a neighborhood of these edges, see Section 3 and [14, 15, 12]. To obtain the desired sharp and explicit statement of Corollary 5.2 and Remark 5.3, the first nonzero eigenvalue of a one-dimensional transmission problem has to be determined, see Lemma 3.5. The actual regularity and wellposedness statement in Corollary 5.2 is then deduced by constructing a regular state space X_1 in (2.9) and by applying semigroup theory on the latter space in Proposition 5.1.

Structure of the paper. In Section 2 we recall useful function spaces, and introduce an analytical framework for the Maxwell system (1.1). In particular, we construct a space X_1 in (2.9) that turns out to be a regular state space for (1.1). In the spirit of [15], we then study the regularity of a transmission problem for the Laplacian in Section 3. Using these findings, the space X_1 is shown to embed into a space of fractional Sobolev regularity, see Section 4. In Section 5 we then prove the wellposedness of (1.1) in X_1, and in this way the desired regularity statement. A directional splitting scheme is constructed in Section 6. It is shown to be unconditionally stable, and a rigorous error estimate is established there, see Theorem 6.5.

Notation. For convenience, we use a partition of Q that is different from the above $\overline{Q} = \bigcup_{i=1}^{L} \bigcup_{j=0}^{K} \overline{\widetilde{Q}}_{i,j}$. The new one is subordinate to the one above, and obtained by appropriate refinement. In particular, the material parameters ε and μ are assumed to be constant on each element of the new partition. We arrive at N smaller open cuboids $Q_1, \dots, Q_N$ with $\overline{Q} = \bigcup_{i=1}^{N} \overline{Q}_i$. These cuboids should not overlap and again touch both planes $\{x_3 = a_3^-\}$ and $\{x_3 = a_3^+\}$. It is also assumed that if two subcuboids share an interface, that the edges of the corresponding faces then coincide.

We denote the open faces of Q by

$$\Gamma_j^{\pm} := \{x \in \partial Q \mid x_j = a_j^{\pm},\ x_l \in (a_l^-, a_l^+) \text{ for } l \neq j\}, \qquad \Gamma_j := \Gamma_j^+ \cup \Gamma_j^- \tag{1.3}$$

for $j \in \{1,2,3\}$. The set of interfaces of the fine partition $Q_1, \dots, Q_N$ is called $\mathscr{F}_{\mathrm{int}}$, and the set of exterior faces is $\mathscr{F}_{\mathrm{ext}}$. We also assign a unit normal vector $\nu_F \in \mathbb{R}^3$ to every face $F \in \mathscr{F}_{\mathrm{int}} \cup \mathscr{F}_{\mathrm{ext}}$ in the following way. In case F is an interface being parallel to the x_j-x_3-plane, we choose ν_F as the canonical unit vector e_l, $l \neq j \in \{1,2\}$. Otherwise, F is an exterior face, and ν_F coincides with the outer unit normal vector ν of ∂Q. We also employ a set of effective interfaces $\mathscr{F}_{\mathrm{int}}^{\mathrm{eff}}$ that contains all physical interfaces. It is defined via

$$\mathscr{F}_{\mathrm{int}}^{\mathrm{eff}} := \{\mathscr{F} \subseteq Q \text{ is a face of } \tilde{Q}_{i,j},\ i \in \{1,\dots,L\},\ j \in \{1,\dots,K\}\} \cup \tilde{\mathscr{F}}_{\mathrm{int}}. \tag{1.4}$$

Normal vectors for interfaces in $\mathscr{F}_{\mathrm{int}}^{\mathrm{eff}}$ are defined similarly as for interfaces in $\mathscr{F}_{\mathrm{int}}$.

The restriction of a function $f \in L^2(Q)$ to a subcuboid Q_i is denoted by $f^{(i)}$ for $i \in \{1,\dots,N\}$. We also need a notation for jumps of functions at interfaces in Q. To that end, let F be an interface between two cuboids Q_{i1} and Q_{i2} with face vector ν_F pointing from Q_{i1} to Q_{i2}. Assume additionally that the restrictions $f^{(i1)}$ and $f^{(i2)}$ have well defined traces $\operatorname{tr}_F f^{(i1)}$ and $\operatorname{tr}_F f^{(i2)}$ at F. The jump $[\![f]\!]_F$ of f at F is then defined as $[\![f]\!]_F := \operatorname{tr}_F f^{(i2)} - \operatorname{tr}_F f^{(i1)}$.

For a linear operator A on a normed vector space $(X, \|\cdot\|)$, we denote its domain by $\mathscr{D}(A)$, and its graph norm by $\|x\|_{\mathscr{D}(A)}^2 := \|x\|^2 + \|Ax\|^2$, $x \in \mathscr{D}(A)$.

2. Analytical preliminaries

This section is structured into two parts. The first one collects useful analytical concepts and results about several function spaces that will throughout be employed without further notice. We then proceed in the second part by interpreting the Maxwell system (1.1) as an evolution equation on an appropriate state space.

2.1. **Important function spaces.** For our reasoning, the divergence operator div, and the two- and three-dimensional curl_2 and curl are essential. Formally, they are defined by

$$\operatorname{div}\phi = \sum_{i=1}^{3} \partial_i \phi_i, \qquad \operatorname{curl}_2 v = \partial_1 v_2 - \partial_2 v_1,$$
$$\operatorname{curl}\phi = \big(\partial_2\phi_3 - \partial_3\phi_2, \partial_3\phi_1 - \partial_1\phi_3, \partial_1\phi_2 - \partial_2\phi_1\big),$$

for distributions $\phi = (\phi_1, \phi_2, \phi_3)$ on a Lipschitz domain $\Omega \subseteq \mathbb{R}^3$ and $v = (v_1, v_2)$ on a Lipschitz domain $S \subseteq \mathbb{R}^2$.

For the sake of a clear presentation, we subsequently introduce only spaces and trace operators related to the curl_2, curl, and div operators on the cuboid Q and a rectangle S. The definitions and results, however, can be transferred to the subdomains $Q_1, \dots, Q_N$ by appropriate adaptions. We first recall the Banach spaces

$$\begin{aligned}
H(\operatorname{curl}_2, S) &:= \{v \in L^2(S)^2 \mid \operatorname{curl}_2 v \in L^2(S)\}, & \|v\|_{\operatorname{curl}_2}^2 &:= \|v\|_{L^2}^2 + \|\operatorname{curl}_2 v\|_{L^2}^2,\\
H(\operatorname{curl}, Q) &:= \{\phi \in L^2(Q)^3 \mid \operatorname{curl}\phi \in L^2(Q)^3\}, & \|\phi\|_{\operatorname{curl}}^2 &:= \|\phi\|_{L^2}^2 + \|\operatorname{curl}\phi\|_{L^2}^2\,,\\
H(\operatorname{div}, Q) &:= \{\phi \in L^2(Q)^3 \mid \operatorname{div}\phi \in L^2(Q)\}, & \|\phi\|_{\operatorname{div}}^2 &:= \|\phi\|_{L^2}^2 + \|\operatorname{div}\phi\|_{L^2}^2\,.
\end{aligned}$$

We further use the subspaces $H_0(\operatorname{curl}_2, S)$, $H_0(\operatorname{curl}, Q)$ and $H_0(\operatorname{div}, Q)$, being the completion of the space of test functions on S and Q with respect to the norms $\|\cdot\|_{\operatorname{curl}_2}$, $\|\cdot\|_{\operatorname{curl}}$ and $\|\cdot\|_{\operatorname{div}}$, respectively. For these spaces, Theorems I.2.5–I.2.6 in [24] state the following. The normal trace operator $v \mapsto v \cdot \nu|_{\partial Q}$ extends continuously from $C^\infty(\overline{Q})^3$ to the space $H(\operatorname{div}, Q)$, now mapping into $H^{-1/2}(\partial Q)$ with kernel $H_0(\operatorname{div}, Q)$. Moreover, Green's formula can be extended to $H(\operatorname{div}, Q)$, stating

$$\int_Q v \cdot \nabla \varphi \, \mathrm{d}x + \int_Q (\operatorname{div} v) \varphi \, \mathrm{d}x = \langle v \cdot \nu, \varphi \rangle_{H^{-1/2}(\partial Q) \times H^{1/2}(\partial Q)}$$

for functions $v \in H(\operatorname{div}, Q)$ and $\varphi \in H^1(Q)$.

Concerning the curl operator, Theorems I.2.11–I.2.12 in [24] establish similar results. The tangential trace operator $v \mapsto v \times \nu|_{\partial Q}$ has an extension to the space $H(\operatorname{curl}, Q)$ with kernel $H_0(\operatorname{curl}, Q)$ and range $H^{-1/2}(\partial Q)^3$. Green's formula reads

$$\int_Q (\operatorname{curl} v) \cdot \varphi \, \mathrm{d}x - \int_Q v \cdot \operatorname{curl} \varphi \, \mathrm{d}x = \langle v \times \nu, \varphi \rangle_{H^{-1/2}(\partial Q) \times H^{1/2}(\partial Q)}$$

for vectors $v \in H(\operatorname{curl}, Q)$ and $\varphi \in H^1(Q)^3$.

To cover the two-dimensional case, we additionally introduce the unit tangent ν_t on ∂S. Denoting by $\nu_S = (\nu_1, \nu_2)$ the unit exterior normal vector of ∂S, it is defined by $\nu_t = (-\nu_2, \nu_1)$. For the two-dimensional case, Theorems I.2.10–I.2.12 in [24] then yield that $C^\infty(\overline{S})^2$ is dense in $H(\operatorname{curl}_2, S)$, and the tangential trace $\gamma_t : v \mapsto v \cdot \nu_t|_{\partial S}$ extends continuously to $H(\operatorname{curl}_2, S)$ with kernel $H_0(\operatorname{curl}_2, S)$ and range $H^{-1/2}(\partial S)$. In this setting, the Green's formula is given by

$$\int_S (\operatorname{curl}_2 v) \phi \, \mathrm{d}x - \int_S v \cdot (\partial_2 \phi, -\partial_1 \phi) \, \mathrm{d}x = \langle v \cdot \nu_t, \phi \rangle_{H^{-1/2}(\partial S) \times H^{1/2}(\partial S)}$$

for $v \in H(\operatorname{curl}_2, S)$ and $\phi \in H^1(S)$. We simply call the application of all three Green's formulas integration by parts.

Closely related are intersections of the above spaces, that are useful to derive regularity statements. We define the spaces

$$\begin{aligned} H_T(\operatorname{curl}, \operatorname{div}, Q) &:= H(\operatorname{curl}, Q) \cap H_0(\operatorname{div}, Q), \\ H_N(\operatorname{curl}, \operatorname{div}, Q) &:= H_0(\operatorname{curl}, Q) \cap H(\operatorname{div}, Q). \end{aligned}$$

Both spaces embed continuously into $H^1(Q)^3$, meaning that there is a constant $C_T > 0$ with

$$\|\mathbf{H}\|_{H^1(Q)}^2 \le C_T (\|\operatorname{curl} \mathbf{H}\|_{L^2(Q)}^2 + \|\operatorname{div} \mathbf{H}\|_{L^2(Q)}^2) \tag{2.1}$$

for all $\mathbf{H} \in H_T(\operatorname{curl}, \operatorname{div}, Q) \cup H_N(\operatorname{curl}, \operatorname{div}, Q)$, see for example Lemmas I.3.4, I.3.6 and Theorems I.3.7, I.3.9 in [24].

During the proof of the global error bound in Theorem 6.5, we also use extrapolation theory, see Section V.1.3 in [1] and Section 2.10 in [51]. Let A be a closed and densely defined operator on a Banach space $(X, \|\cdot\|_X)$ with nonempty resolvent set. Let additionally λ be an element of the resolvent set of A. Then the extrapolation space X_{-1}^A with respect to A is defined as the completion of X in the norm $\|\cdot\|_{X_{-1}^A} = \|(\lambda I - A)^{-1} \cdot\|_X$. Note that this definition is independent of the choice of the resolvent value λ. The operator A then has a unique and bounded extension A_{-1} from X to X_{-1}^A. It is called the extrapolation operator of A to X. The resolvent operator $(\lambda I - A)^{-1}$ moreover extends to the bounded operator $(\lambda I - A_{-1})^{-1}$ from X_{-1}^A to X.

Interpolation theory is another important tool for our analysis. Throughout, we only employ real interpolation on Hilbert spaces, which can be defined via the K-method, see Section 1.1 in [41] for instance. By means of interpolation spaces, we in particular define fractional order Sobolev spaces, see [40, 49]. These spaces throughout serve as a measure for regularity statements. Let $s \in [0, 2]$, $k \in \{1, 2\}$, $\theta \in (0, 1) \setminus \{1/2\}$, $d \in \mathbb{N}$, $O \subseteq \mathbb{R}^d$ open with a Lipschitz boundary, and define

$$H^s(O) := (L^2(O), H^2(O))_{s/2,2}, \qquad H_0^\theta(O) := (L^2(O), H_0^1(O))_{\theta,2}. \tag{2.2}$$

We additionally note that the spaces $H^\theta(O)$ and $H_0^\theta(O)$ coincide for $\theta \in (0, 1/2)$ (this can be verified by means of Corollary 1.4.4.5 in [27] for instance).

The spaces of functions with piecewise Sobolev regularity are also important. Let Γ^* be a union of some faces of Q. Define the spaces

$$\begin{aligned} PH^q(Q) &:= \{f \in L^2(Q) \mid f^{(i)} \in H^q(Q_i),\ i \in \{1, \dots, N\}\}, && q \in [0, 2], \\ PH^s_{\Gamma^*}(Q) &:= \{f \in PH^s(Q) \mid f^{(i)} = 0 \text{ on } \partial Q_i \cap \Gamma^*,\ i \in \{1, \dots, N\}\}, && s \in (1/2, 2], \end{aligned}$$

equipped with the norms

$$\|f\|_{PH^q}^2 := \sum_{i=1}^N \left\| f^{(i)} \right\|_{H^q(Q_i)}^2, \qquad \|g\|_{PH^s_{\Gamma^*}} := \|g\|_{PH^s},$$

for $f \in PH^q(Q)$ and $g \in PH^s_{\Gamma^*}(Q)$.

The next lemma serves as a technical tool, establishing a useful density result for function spaces related to the electric and the magnetic field. It uses to approximate with piecewise regular functions, that satisfy prescribed transmission conditions, and that vanish in a neighborhood of all exterior and interior edges of Q. The result is applied in the proof for Lemma 3.2, and it will play a crucial role in a subsequent work that is in preparation. For the statement, let Γ^* be a (possibly empty) union of opposite faces of the cuboid Q, and let $\mathscr{F}_{\mathrm{int},j}$ denote the set of all interfaces whose normal vector is parallel to the j-th canonical unit vector e_j, $j \in \{1, 2\}$.

Lemma 2.1. *Let ε satisfy (1.2). Define the spaces*

$$\begin{aligned} V &:= \{\varphi \in PH^1_{\Gamma^*}(Q) \mid [\![\varepsilon\varphi]\!]_{\mathscr{F}} = 0,\ [\![\varphi]\!]_{\mathscr{F}'} = 0 \textit{ for all } \mathscr{F} \in \mathscr{F}_{\textit{int},j}, \\ &\qquad \mathscr{F}' \in \mathscr{F}_{\mathrm{int}} \setminus \mathscr{F}_{\textit{int},j}\}, \\ W &:= \{\varphi \in PH^2(Q) \cap V \mid \varphi^{(i)} \textit{ is smooth},\ \operatorname{supp}(\varphi) \cap \overline{\Gamma^*} = \emptyset, \\ &\qquad \varphi \textit{ vanishes in a neighborhood of all edges of } Q_1, \dots, Q_N, \\ &\qquad \partial_{\nu_{\mathscr{F}}} \varphi^{(i)} = 0 \textit{ for faces } \mathscr{F} \subseteq \partial Q_i,\ i \in \{1, \dots, N\}\}. \end{aligned}$$

The space W is dense in V with respect to the norm in $PH^1(Q)$.

Proof. We show only the density of W in V in the case $\Gamma^* = \Gamma_2 \cup \Gamma_3$, and assume $j = 2$. All remaining settings can be established with the same techniques, up to appropriate modifications.

1) Let $\varphi \in V$ and $\delta > 0$. Applying Lemma 2.5 in [15] to every interior and exterior edge of Q, there is a function $\mathring{\varphi} \in V$, that vanishes in an open neighborhood of all edges of $Q_1, \dots, Q_N$ and satisfies

$$\|\mathring{\varphi} - \varphi\|_{PH^1(Q)} \leq \delta. \tag{2.3}$$

Hence, there is a union $\mathscr{T}$ of tubes of inner radius $\zeta > 0$ around all edges with $\mathring{\varphi}$ vanishing on $Q \cap \mathscr{T}$.

We next construct a piecewise smooth function fulfilling the required transmission, support, and normal derivative conditions. We only deal with the cuboid

$$Q_1 = (a_1^{-,1}, a_1^{+,1}) \times (a_2^{-,1}, a_2^{+,1}) \times (a_3^{-,1}, a_3^{+,1}),$$

(setting $a_3^{\pm,1} := a_3^{\pm}$) and we assume that Q_1 touches the faces Γ_1^+ and Γ_2^+ of Q, see (1.3). All other cuboids can be treated in the same way with slight modifications. Let $l \in \{1, 2, 3\}$, and

$\chi_{m,l} : \mathbb{R} \to [0,1]$ be a smooth cut-off function with $\operatorname{supp} \chi_{m,l} \subseteq [a_l^{-,1}, a_l^{-,1} + \frac{1}{m}] \cup [a_l^{+,1} - \frac{1}{m}, a_l^{+,1}]$, $\chi_{m,l} = 1$ on $[a_l^{-,1}, a_l^{-,1} + \frac{1}{2m}] \cup [a_l^{+,1} - \frac{1}{2m}, a_l^{+,1}]$, and $\|\chi'_{m,l}\|_\infty \le Cm$ for a uniform constant $C > 0$ for all $m \ge m_l \in \mathbb{N}$. Let

$$\Gamma_l^{\pm,1} = \{x \in \partial Q_1 \mid x_l \in \{a_l^{\pm,1}\},\ x_j \in (a_j^{-,1}, a_j^{+,1}) \text{ for } j \neq l\},$$

and denote the pyramid with basis $\Gamma_l^{\pm,1}$ and peak $(\frac{a_1^{-,1}+a_1^{+,1}}{2}, \frac{a_2^{-,1}+a_2^{+,1}}{2}, \frac{a_3^{-}+a_3^{+}}{2})$ by $P_l^{\pm,1}$. Its reflection at the face $\Gamma_l^{\pm,(1)}$ is called $\check{P}_l^{\pm,1}$. Let further Q_{i_k} be the adjacent cuboid of Q_1 in coordinate direction $k \in \{1,2\}$.

We then define the larger set $\check{Q}_1 := \overline{Q}_1 \cup \bigcup_{l=1}^3 (\check{P}_l^{+,1} \cup \check{P}_l^{-,1})$, and put

$$g_{m,(1)}(x) := \begin{cases} \mathring{\varphi}^{(1)}(x) & \text{for } x \in P_1^{\pm,1} \cup P_2^{-,1}, \\ (1-\chi_{m,3}(x_3))\mathring{\varphi}^{(1)}(x) & \text{for } x = (x_1,x_2,x_3) \in P_3^{\pm,1}, \\ (1-\chi_{m,2}(x_2))\mathring{\varphi}^{(1)}(x) & \text{for } x = (x_1,x_2,x_3) \in P_2^{+,1}, \\ \mathring{\varphi}^{(i_1)}(x) & \text{for } x \in \check{P}_1^{-,1}, \\ \mathring{\varphi}^{(1)}(-x_1 + 2a_1^{+,1}, x_2, x_3) & \text{for } x = (x_1,x_2,x_3) \in \check{P}_1^{+,1}, \\ \frac{\varepsilon^{(i_2)}}{\varepsilon^{(1)}}\mathring{\varphi}^{(i_2)}(x) & \text{for } x \in \check{P}_2^{-,1}, \\ 0 & \text{for } x \in \check{P}_3^{\pm,1} \cup \check{P}_2^{+,1}, \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus \check{Q}_1. \end{cases}$$

Since $\mathring{\varphi}$ is an element of V and vanishes on $\overline{Q} \cap \mathscr{T}$, there is a number $m_4 \in \mathbb{N}$ and an open superset $\mathring{Q}_1$ of $\overline{Q_1}$ with $g_{m,(1)}|_{\mathring{Q}_1} \in H^1(\mathring{Q}_1)$ for $m \ge m_4$. We then repeat the same reasoning for all other subcuboids, by appropriately changing the definition of the function $g_{m,(i)}$ for each subcuboid Q_i. Define then a function g_m on Q by $g_m|_{Q_i} := g_{m,(i)}|_{Q_i}$ for $i \in \{1,\dots,N\}$.

Taking the exterior face conditions for $\mathring{\varphi}$ into account, the arguments from the proof of Lemma 2.1 in [21] show that $g_m^{(i)}$ converges to $\mathring{\varphi}^{(i)}$ in $H^1(P)$ for $P \in \{P_l^{\pm,i} \mid l \in \{1,2,3\}\}$ as $m \to \infty$. There consequently is a number $\check{m} \ge m_4$ with

$$\|g_{\check{m}} - \mathring{\varphi}\|_{PH^1(Q)} \le \delta. \tag{2.4}$$

We next employ the standard mollifier $\rho_{n,l}$ that acts on the l-th coordinate, and that is supported within $[-\frac{1}{n}, \frac{1}{n}]$. Let

$$\tilde{\psi}_{n,i} := \rho_{n,3} * \rho_{n,2} * \rho_{n,1} * g_{\check{m},(i)}, \qquad n \in \mathbb{N}, i \in \{1,\dots,N\}.$$

By construction, the function $\tilde{\psi}_{n,i}$ is smooth, and it vanishes in a union $\tilde{\mathscr{T}}$ of tubes with radius $\frac{3}{4}\zeta$ around all edges as well as in a neighborhood of all exterior faces in $\Gamma_2 \cup \Gamma_3$, provided that $n \ge n_0 \in \mathbb{N}$. We also remark that the function $\tilde{\psi}_n$, being defined by $\tilde{\psi}_n|_{Q_i} := \tilde{\psi}_{n,i}$, satisfies all required transmission conditions in Q for sufficiently large n. As a consequence of standard mollifier theory, the sequence $(\tilde{\psi}_{n,i})_n$ furthermore converges in $H^1(\mathring{Q}_i)$ to $g_{\check{m},(i)}$. There consequently is a number $\check{n} \ge n_0$ with

$$\left\|\tilde{\psi}_{\check{n},i} - g_{\check{m},(i)}\right\|_{H^1(\mathring{Q}_i)} \le \delta, \qquad i \in \{1,\dots,N\}. \tag{2.5}$$

2) It remains to incorporate also the Neumann boundary conditions at the faces of Q_1. This is done by transferring a technique from the proof of Lemma 3.3 in [21] to our setting. Let $\kappa \in (0, \frac{a_1^{+,1}-a_1^{-,1}}{2})$ be a fixed number. Let $\tilde{\alpha} : [a_1^{-,1}, a_1^{+,1}] \to [0,1]$ be a smooth function with

$\operatorname{supp}\tilde\alpha \subseteq [a_1^{-,1}, a_1^{-,1}+\frac{\kappa}{2}]$, and $\tilde\alpha = 1$ on $[a_1^{-,1}, a_1^{-,1}+\frac{\kappa}{4}]$. Define then the function

$$\begin{aligned} h_{k,1}^-(x_1,x_2,x_3) &:= \tilde\psi_{\check n,1}(x_1,x_2,x_3) - \tilde\alpha(x_1)\int_{a_1^{-,1}}^{x_1} \chi_{k,1}(s)\partial_1\tilde\psi_{\check n,1}(s,x_2,x_3)\,\mathrm{d}s \\ &=: \tilde\psi_{\check n,1}(x) - r_k(x) \end{aligned}$$

for $x=(x_1,x_2,x_3)\in P_1^{-,1}$ and $k\in\mathbb{N}$. By construction of $\tilde\psi_{\check n,1}$, the functions $h_{k,1}^-$ and r_k are smooth. We next deduce that r_k tends to zero in $H^1(P_1^{-,1})$ as $k\to\infty$. The integrand of r_k is uniformly bounded in k, and converges pointwise to zero. Thus, $(r_k)_k$ is uniformly bounded. Applying now Lebesgue's theorem of dominated convergence twice, we infer that r_k converges pointwise and in $L^2(P_1^{-,1})$ to zero as $k\to\infty$. A simple computation further gives rise to the formulas

$$\begin{aligned} \partial_1 r_k &= (\partial_1\tilde\alpha)\int_{a_1^{-,1}}^{x_1}\chi_{k,1}(s)\partial_1\tilde\psi_{\check n,1}(s,\cdot)\,\mathrm{d}s + \tilde\alpha\chi_{k,1}\partial_1\tilde\psi_{\check n,1}, \\ \partial_l r_k &= \tilde\alpha\int_{a_1^{-,1}}^{x_1}\chi_{k,1}(s)\partial_1\partial_l\tilde\psi_{\check n,1}(s,\cdot)\,\mathrm{d}s, \qquad l\in\{2,3\}. \end{aligned}$$

Similar arguments to the ones above now imply that $(\partial_1 r_k)_k$ and $(\partial_l r_k)_k$ are null sequences in $L^2(P_1^{-,1})$. As a result, $(h_{k,1}^-)_k$ converges to $\tilde\psi_{\check n,1}$ in $H^1(P_1^{-,1})$, and $\partial_1 h_{k,1}^- = 0$ on $\Gamma_1^{-,1}$. By analogous constructions on all other pyramids $P_1^{+,1}$, $P_2^{\pm,1}$, and $P_3^{\pm,1}$, we further obtain similar functions $h_{k,1}^+$, $h_{k,2}^\pm$ and $h_{k,3}^\pm$ for $k\in\mathbb{N}$. They are in particular smooth and coincide with $\tilde\psi_{\check n,1}$, provided that the distance to the associated face is larger than $\frac{\kappa}{2}$. Define now a new mapping $\psi_{k,1}$ on Q_1 via its restrictions $\psi_{k,1}|_{P_j^{\pm,1}} := h_{k,j}^\pm$. As the function $\tilde\psi_{\check n,1}$ vanishes in $\tilde{\mathscr{T}}$ (union of tubes around all edges with radius $3/4\zeta$), we can choose $\kappa>0$ so small that $\psi_{k,1}$ is smooth on Q_1. We then repeat the analogous construction for all remaining cuboids $Q_2,\dots,Q_N$, obtaining functions $\psi_{k,2},\dots,\psi_{k,N}$ for $k\in\mathbb{N}$. Finally, we define the mapping ψ_k elementwise by $\psi_k^{(i)} := \psi_{k,i}$, for $i\in\{1,\dots,N\}$.

By construction, ψ_k is smooth on every cuboid, and it vanishes in an open neighborhood of $\Gamma_2\cup\Gamma_3$ and of all edges of the subcuboids. It further satisfies the required normal derivative condition at all faces for sufficiently large k. Using finally that the function $\tilde\psi_{\check n}$ satisfies the required transmission conditions, we conclude that ψ_k also fulfills by definition the transmission conditions $[\![\varepsilon\psi_k]\!]_{\mathscr{F}} = 0$, $[\![\psi_k]\!]_{\mathscr{F}'} = 0$ for all $\mathscr{F}\in\mathscr{F}_{\mathrm{int},2}$ and $\mathscr{F}'\in\mathscr{F}_{\mathrm{int},1}$. Taking also (2.3)–(2.5) into account, ψ_k is contained in W, and the estimate $\|\psi_k-\varphi\|_{PH^1(Q)}\le 4\delta$ is valid for sufficiently large k. □

2.2. **Analytical framework for the Maxwell system.** Throughout, we consider the Maxwell equations (1.1) as an evolution equation on the space $X := L^2(Q)^6$. The space is equipped with the weighted inner product

$$\left(\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}, \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix}\right) := \int_Q \varepsilon\mathbf{E}\cdot\tilde{\mathbf{E}} + \mu\mathbf{H}\cdot\tilde{\mathbf{H}}\,\mathrm{d}x, \quad \begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}, \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix}\in X,$$

inducing the norm $\|\cdot\|$ on X. The positivity and boundedness assumption on ε and μ implies that $\|\cdot\|$ is equivalent to the standard L^2-norm.

On X we consider the Maxwell operator

$$M := \begin{pmatrix} 0 & \frac{1}{\varepsilon}\operatorname{curl} \\ -\frac{1}{\mu}\operatorname{curl} & 0\end{pmatrix}, \qquad \mathscr{D}(M) := H_0(\operatorname{curl},Q)\times H(\operatorname{curl},Q). \tag{2.6}$$

Note that fields in $\mathscr{D}(M)$ have continuous tangential components at the interfaces.

We next incorporate the boundary conditions for the magnetic field, as well as divergence and normal transmission conditions. Recall to that end the set of effective interfaces $\mathscr{F}_{\mathrm{int}}^{\mathrm{eff}}$. The latter contains all interfaces between the submedia $\tilde{Q}_{i,l}$, $i \in \{1, \dots, L\}$, $l \in \{0, \dots, K\}$. For each effective interface $\mathscr{F} \in \mathscr{F}_{\mathrm{int}}^{\mathrm{eff}}$, we put

$$V(\mathscr{F}) := (L^2(\mathscr{F}), H^1_{\mathscr{F}})_{\frac{1}{2},2}, \qquad H^1_{\mathscr{F}} := \{u \in H^1(\mathscr{F}) \mid u = 0 \text{ on } \mathscr{F} \cap \partial Q\}. \tag{2.7}$$

We then define the subspace

$$\begin{aligned} X_0 := \{(\mathbf{E}, \mathbf{H}) \in L^2(Q)^6 \mid \operatorname{div}(\varepsilon \mathbf{E}|_{\tilde{Q}_{i,l}}) \in L^2(\tilde{Q}_{i,l}),\ [\![\varepsilon \mathbf{E} \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}} \in V(\mathscr{F}), \\ \operatorname{div}(\mu \mathbf{H}) = 0,\ \mu \mathbf{H} \cdot \nu = 0 \text{ on } \partial Q,\ \mathscr{F} \in \mathscr{F}_{\mathrm{int}}^{\mathrm{eff}},\ i \in \{1, \dots, L\}, l \in \{0, \dots, K\}\}, \end{aligned} \tag{2.8}$$

of X, which is inspired by the spaces X_{div} and X_0 in [29, 21, 20]. The space X_0 is complete with respect to the norm

$$\|(\mathbf{E}, \mathbf{H})\|_{X_0}^2 := \|(\mathbf{E}, \mathbf{H})\|^2 + \sum_{i=1}^{N} \left\| \operatorname{div}(\varepsilon^{(i)} \mathbf{E}^{(i)}) \right\|_{L^2(Q_i)}^2 + \sum_{\mathscr{F} \in \mathscr{F}_{\mathrm{int}}^{\mathrm{eff}}} \|[\![\varepsilon \mathbf{E} \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}}\|_{V(\mathscr{F})}^2 .$$

To equip the Maxwell operator with the magnetic boundary conditions as well as the electric and magnetic divergence conditions, we introduce the restriction M_0 of the Maxwell operator to the space X_0, and consider it on the space

$$X_1 := \mathscr{D}(M_0) := \mathscr{D}(M) \cap X_0, \tag{2.9}$$

which is equipped with the norm

$$\left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_1}^2 := \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_0}^2 + \left\| M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|^2, \qquad \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \in X_1.$$

Remark 2.2. By interpreting the Maxwell equations (1.1) on X_1, we only assume that the divergence of the electric field is an L^2-function on every submedium $\tilde{Q}_{i,l}$, $i \in \{1, \dots, L\}$, $l \in \{0, \dots, K\}$. In particular, we allow for nonzero jumps of the normal component of the field $\varepsilon \mathbf{E}$ across effective interfaces in $\mathscr{F}_{\mathrm{int}}^{\mathrm{eff}}$. These discontinuities represent surface charges on the interfaces, see Section 3.5 in [25]. ♢

Although the space X_1 is mainly defined by means of the domains of the divergence and curl operators, which themselves allow for irregular functions, the space X_1 indeed embeds into a space of functions with piecewise fractional Sobolev regularity above $2/3$, see Proposition 4.6.

The next lemma deals with M_0, and it shows that M_0 is not only the restriction of M to X_0, but also its part in this space. The statement corresponds to relation (2.5) in [21].

Lemma 2.3. *The identity $\mathscr{D}(M_0^k) = \mathscr{D}(M^k) \cap X_0$ is valid for all $k \in \mathbb{N}$, and $M(\mathscr{D}(M))$ is a subset of X_0. In particular, M_0 is the part of M in X_0, and the space X_1 is complete.*

Proof. We show only that the space X_1 is complete. The remaining statements can be established in the same way as identity (2.5) in [21].

To deduce the completeness of X_1, we first note that M_0 is closed in X_0 as the part of a closed operator, and thus its domain X_1 is complete with respect to the graph norm of M_0. It hence suffices to show that the graph norm of M_0 coincides with the standard norm on X_1.

Let $(\mathbf{E},\mathbf{H}) \in X_1 = \mathscr{D}(M) \cap X_0$. Combining the relation $\operatorname{div}(\varepsilon(M(\mathbf{E},\mathbf{H}))_1) = \operatorname{div}(\operatorname{curl}\mathbf{H}) = 0$ with the transmission conditions in $H(\operatorname{curl}, Q)$ and $H(\operatorname{div}, Q)$ and the definition of the norms on X_0 and X_1, the identities

$$\left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_1}^2 = \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_0}^2 + \left\| M \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{X_0}^2 = \left\| \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right\|_{\mathscr{D}(M_0)}^2$$

immediately follow. □

The part of M in X_1 is denoted by M_1, and it is shown to generate a strongly continuous semigroup on X_1. Thus, the space X_1 serves as a state space for the Maxwell equations (1.1), see Proposition 5.1. Using a regularity statement for the space X_1, we can then conclude that the system (1.1) possesses solutions of piecewise $H^{1-\theta}$-regularity, $\theta \in (0, 1)$ appropriate, see Corollary 5.2 and Remark 5.3. As a starting point, the following result states the generator property of the Maxwell operator on X. The statement is part of Proposition 3.5 in [29].

Proposition 2.4. *Let ε and μ satisfy (1.2). The Maxwell operator M generates a unitary C_0-group $(\mathrm{e}^{tM})_{t\in\mathbb{R}}$ on X.*

3. Analysis of an elliptic transmission problem

This section is concerned with investigations of transmission problems for a Laplacian on the cuboid Q, see (3.1). The considered elliptic transmission problem arises several times in literature, see [15, 36, 37, 39, 43, 34, 11, 12, 13] for instance. Note, however, that there are no explicit regularity statements for our particular application of the embedded waveguide at hand, to the best of our knowledge. In other words, we are interested in precise regularity results in terms of the size of jumps of the parameters ε and μ. This is because the below system (3.1) arises naturally when analyzing the regularity of the electric and magnetic field, see the proof of Lemma 4.2 and [15, 11, 12]. Because we are also going to transfer some arguments from the analysis of (3.1) to a different elliptic transmission problem in a subsequent work, we analyze the problem here in detail to have a self-contained presentation.

Let $\eta \in \{\varepsilon, \mu\}$ satisfy the assumptions (1.2). The function η will throughout serve as a placeholder for the material parameters ε and μ. Let further Γ^* be a nonempty union of some of the sets $\Gamma_1, \Gamma_2, \Gamma_3$, consisting of opposite boundary faces of Q, see (1.3). Consider the elliptic transmission problem

$$\begin{aligned} -\Delta\psi^{(i)} &= f^{(i)} && \text{on } Q_i \text{ for } i \in \{1,\dots,N\}, \\ \psi &= 0 && \text{on } \Gamma^*, \\ \nabla\psi\cdot\nu &= 0 && \text{on } \partial Q \setminus \Gamma^*, \\ [\![\psi]\!]_{\mathscr{F}} &= 0 = [\![\eta\nabla\psi\cdot\nu_{\mathscr{F}}]\!]_{\mathscr{F}} && \text{on } \mathscr{F} \in \mathscr{F}_{\mathrm{int}}, \end{aligned} \tag{3.1}$$

involving a given function $f \in L^2(Q)$. System (3.1) can also be expressed equivalently by the formula

$$\Delta_{\Gamma^*} u = f, \tag{3.2}$$

involving the Laplacian

$$\begin{aligned} &(\Delta_{\Gamma^*}u)^{(i)} := \Delta u^{(i)}, \qquad \text{on } Q_i,\ i \in \{1,\dots,N\}, \\ &u \in \mathscr{D}(\Delta_{\Gamma^*}) := \{v \in H^1_{\Gamma^*}(Q) \mid \operatorname{div}(\eta\nabla v) \in L^2(Q),\ \nabla v\cdot\nu = 0 \text{ on } \partial Q \setminus \Gamma^*\}. \end{aligned} \tag{3.3}$$

We next recall the decomposition $\overline{Q} = \bigcup_{i=1}^{L} \bigcup_{j=0}^{K} \overline{\tilde{Q}_{ij}}$ from Section 1. To measure the regularity of the solution of (3.1) in the case $\eta = \varepsilon$, we introduce the number $\overline{\kappa} \in (2/3, 1]$ with

$$\max_{\substack{i \in \{1,\dots,L\},\\ l \in \{1,\dots,K\}}} \frac{(\varepsilon|_{\tilde{Q}_{i,l}} - \varepsilon|_{\tilde{Q}_{i,0}})^2}{\varepsilon|_{\tilde{Q}_{i,l}} \varepsilon|_{\tilde{Q}_{i,0}}} = -\frac{4\sin^2(\overline{\kappa}\pi)}{\sin(\frac{\overline{\kappa}}{2}\pi)\sin(\frac{3\overline{\kappa}}{2}\pi)}. \tag{3.4}$$

Note that $\overline{\kappa}$ decreases if the relative discontinuities of the material parameter ε in the subcuboids $\tilde{Q}_1, \dots, \tilde{Q}_L$ become stronger, meaning the material becomes more heterogeneous. In the limit case of homogeneous subcuboids $\tilde{Q}_1, \dots, \tilde{Q}_L$, on the other hand, the number $\overline{\kappa}$ is one.

The central result of this section is the following regularity statement for (3.1). To state it, we employ the following notation. We define $\tilde{Q} := Q \cap \{x_3 = 1/2\}$, $\tilde{Q}_i := Q_i \cap \{x_3 = 1/2\}$, $i \in \{1, \dots, N\}$, and interpret them as rectangles in $\mathbb{R}^2$. Piecewise Sobolev regularity on $\tilde{Q}$ is then defined with respect to the partition $\tilde{Q}_1, \dots, \tilde{Q}_N$. We put

$$\mathscr{V}_{2-\kappa} := H^2_{x_3}((0,1), L^2(\tilde{Q})) \cap H^1_{x_3}((0,1), H^1(\tilde{Q})) \cap L^2_{x_3}((0,1), PH^{2-\kappa}(\tilde{Q})) \tag{3.5}$$

for $\kappa \in [0, 1)$. This space is canonically equipped with the sum of the norms.

Proposition 3.1. *Let $\eta \in \{\varepsilon, \mu\}$, and let ε and μ satisfy (1.2). Let further $\kappa > 1 - \overline{\kappa}$ if $\eta = \varepsilon$, and $\kappa = 0$ if $\eta = \mu$. Assume also that $f \in L^2(Q)$, and let Γ^* be nonempty. There is a unique solution $\psi \in \mathscr{V}_{2-\kappa}$ of (3.1) with $\|\psi\|_{\mathscr{V}_{2-\kappa}} \leq C \|f\|_{L^2(Q)}$ for a constant $C = C(Q, \eta, \kappa) > 0$.*

The remainder of this section is concerned with the proof of Proposition 3.1. The argument is oriented towards the papers [36, 39, 15]: Using the Lax-Milgram Lemma, the Laplacian Δ_{Γ^*} is bijective. Consequently, the regularity of functions in $\mathscr{D}(\Delta_{\Gamma^*})$ has to be studied. By means of a cut-off argument, we separately study the behavior of functions in $\mathscr{D}(\Delta_{\Gamma^*})$ in cylinders around interior edges and in the remainder of Q.

We first analyze the behavior in a cylinder around an interior edge. By means of cylindrical coordinates, we can decompose the problem into a related elliptic transmission problem on the unit disc, see Section 3.2, and a one-dimensional elliptic problem for the height variable, see the proof of Lemma 3.11. On the remainder of Q, functions in $\mathscr{D}(\Delta_{\Gamma^*})$ are piecewise H^2-regular, see Lemma 3.12. The global regularity statement is then concluded in Section 3.3.

In the next subsection, we first derive an inequality that is useful to establish the energy estimate in Proposition 3.1. It comes into play when we consider functions in $\mathscr{D}(\Delta_{\Gamma^*})$ on the part of Q away from the interior edges.

3.1. **Energy estimate for the Laplacian with transmission conditions.** The next two lemmas provide a useful energy identity and an a priori estimate for the Laplacian on $\mathscr{D}(\Delta_{\Gamma^*}) \cap PH^2(Q)$. This is done in the spirit of Grisvard, see [26].

Lemma 3.2. *Let $\eta \in \{\varepsilon, \mu\}$ satisfy (1.2). The identity*

$$\begin{aligned}\sum_{i=1}^{N} \eta^{(i)} \Big(\left\|\partial_1^2 u^{(i)}\right\|^2_{L^2(Q_i)} + \left\|\partial_2^2 u^{(i)}\right\|^2_{L^2(Q_i)} + \left\|\partial_3^2 u^{(i)}\right\|^2_{L^2(Q_i)} + 2\left\|\partial_1\partial_2 u^{(i)}\right\|^2_{L^2(Q_i)} \\ + 2\left\|\partial_1\partial_3 u^{(i)}\right\|^2_{L^2(Q_i)} + 2\left\|\partial_2\partial_3 u^{(i)}\right\|^2_{L^2(Q_i)} \Big) = \sum_{i=1}^{N} \eta^{(i)} \left\|\Delta u^{(i)}\right\|^2_{L^2(Q_i)}\end{aligned}$$

is valid for $u \in \mathscr{D}(\Delta_{\Gamma^*}) \cap PH^2(Q)$.

Proof. 1) We only treat the case $\Gamma^* = \Gamma_1$. A simple calculation first leads to

$$\begin{aligned}\left\|\Delta u^{(i)}\right\|^2_{L^2(Q_i)} &= \left\|\partial_1^2 u^{(i)}\right\|^2_{L^2(Q_i)} + \left\|\partial_2^2 u^{(i)}\right\|^2_{L^2(Q_i)} + \left\|\partial_3^2 u^{(i)}\right\|^2_{L^2(Q_i)} \\ &\quad + 2\int_{Q_i} (\partial_1^2 u^{(i)})(\partial_2^2 u^{(i)})\,\mathrm{d}x + 2\int_{Q_i} (\partial_1^2 u^{(i)})(\partial_3^2 u^{(i)})\,\mathrm{d}x + 2\int_{Q_i} (\partial_2^2 u^{(i)})(\partial_3^2 u^{(i)})\,\mathrm{d}x\end{aligned} \tag{3.6}$$

for $i \in \{1, \dots, N\}$.

2) By Lemma 2.1, there are two sequences $(\varphi_n)_n$ and $(\psi_n)_n$ in $PH^2(Q)$ satisfying $\varphi_n^{(i)} \to \partial_3 u^{(i)}$, $\psi_n^{(i)} \to \partial_2 u^{(i)}$ in $H^1(Q_i)$ as $n \to \infty$, and fulfilling the boundary and transmission conditions $\varphi_n^{(i)} = 0$ on $\Gamma_3^{(i)}$, $\psi_n^{(i)} = 0$ on $\Gamma_2^{(i)} \cap \partial Q$, and $[\![\varphi_n]\!]_{\mathscr{F}} = 0 = [\![\eta\psi_n]\!]_{\mathscr{F}}$ for $\mathscr{F} \in \mathscr{F}_{\mathrm{int},2}$ for all $i \in \{1, \dots, N\}$ and $n \in \mathbb{N}$. Employing Lemma 2.1 of [21] and Lemma 7.1 of [54], the relations

$$\begin{aligned}\partial_2\varphi_n^{(i)} = 0 \text{ on } \Gamma_3^{(i)}, \qquad \partial_3\psi_n^{(i)} = 0 \text{ on } \Gamma_2^{(i)} \cap \partial Q,\\ [\![\partial_3\varphi_n^{(i)}]\!]_{\mathscr{F}} = 0 = [\![\eta\partial_3\psi_n^{(i)}]\!]_{\mathscr{F}} \text{ for } \mathscr{F} \in \mathscr{F}_{\mathrm{int},2},\end{aligned}$$

are furthermore valid. An integration by parts then leads to

$$\sum_{i=1}^N \int_{Q_i} \eta^{(i)}(\partial_3\varphi_n^{(i)})(\partial_2\psi_n^{(i)})\,\mathrm{d}x = \sum_{i=1}^N \int_{Q_i} \eta^{(i)}(\partial_2\varphi_n^{(i)})(\partial_3\psi_n^{(i)})\,\mathrm{d}x.$$

Taking limits, we infer the formula

$$\sum_{i=1}^N \int_{Q_i} \eta^{(i)}(\partial_3^2 u^{(i)})(\partial_2^2 u^{(i)})\,\mathrm{d}x = \sum_{i=1}^N \int_{Q_i} \eta^{(i)}(\partial_2\partial_3 u^{(i)})^2\,\mathrm{d}x$$

for the last term on the right hand side of (3.6). Treating the fourth and fifth terms on the right hand side of (3.6) analogously, we arrive at the desired statement. □

We next provide an energy estimate for piecewise H^2-regular functions in the domain of Δ_{Γ^*}. It is obtained by combining Lemma 3.2 with the Poincaré inequality, see Theorem 13.6.9 in [51].

Lemma 3.3. *Let $u \in \mathscr{D}(\Delta_{\Gamma^*}) \cap PH^2(Q)$, and $\eta \in \{\varepsilon, \mu\}$ satisfy (1.2). The estimate $\|u\|_{PH^2(Q)} \le C\|\Delta_{\Gamma^*} u\|_{L^2(Q)}$ is valid with a uniform constant $C = C(\eta, Q) > 0$.*

3.2. **Analysis of a Laplacian on the unit disc with transmission conditions.** The goal of this subsection is a precise regularity statement for functions in the domain of a Laplacian on the unit disc with transmission conditions representing the ones in system (3.1) for $\eta = \varepsilon$ near an interior edge, see (3.10). To that end, we derive an explicit spectral decomposition of the Laplacian in terms of Bessel functions for the radial part and piecewise smooth functions for the angular part. The resulting eigenbasis can be decomposed into a set of piecewise H^2-regular functions, and a one containing functions with lower regularity, see (3.14) and Lemma 3.6. Treating both sets separately, the final regularity statement is derived in Corollary 3.9.

The next definition involves the union of all edges of the interfaces, being denoted by $\mathscr{S}$ and called skeleton.

Definition 3.4. *Let $e \subseteq \mathscr{S} \cap Q$ be an interior edge, and let $Q_{\mathrm{in},1}, \dots, Q_{\mathrm{in},4}$ be the four adjacent cuboids to e. The material parameter ε has a **strong discontinuity** at e if $\varepsilon|_{Q_{\mathrm{in},1}\cup\dots\cup Q_{\mathrm{in},4}}$ has a strictly larger value on one cuboid than on the remaining three.*

In the following, we fix an interior edge $e_{\mathrm{in}} \subseteq \mathscr{S} \cap Q$. After translation and scaling we assume the identity

$$e_{\mathrm{in}} = \{(0,0)\} \times [0,1]. \tag{3.7}$$

We moreover assume that ε has a strong discontinuity at e_{in}, and fix four cuboids $Q_{\mathrm{in},1}, \dots, Q_{\mathrm{in},4}$ having e_{in} as a common edge. We denote by $\varepsilon_{\mathrm{in}}$ the restriction of ε to the latter cuboids. The notation $\varepsilon_{\mathrm{in}}^{(i)}$ then refers to $\varepsilon_{\mathrm{in}}|_{Q_{\mathrm{in},i}}$. As ε satisfies (1.2), it then suffices to treat the configuration

$$\varepsilon_{\mathrm{in}}^{(1)} = \varepsilon_{\mathrm{in}}^{(2)} = \varepsilon_{\mathrm{in}}^{(3)} < \varepsilon_{\mathrm{in}}^{(4)}. \tag{3.8}$$

As in [14, 15], we localize in a cylinder around the interior edge e_{in}, see Section 3.3. Let $\mathscr{Z}$ be a cylinder around e_{in} with radius 1, that touches the faces Γ_3^+ and Γ_3^- of Q. After scaling, we can assume that $\mathscr{Z}$ touches no interior edge (except e_{in}, of course). Rotating appropriately, we can assume the representation

$$\begin{aligned} \mathscr{Z} \cap Q_{\mathrm{in},i} &= \{(x,y,z) \mid (x,y) \in D_i,\ z \in [0,1]\}, \\ D_i &= \{(r\cos\varphi, r\sin\varphi) \mid r \in (0,1),\ \varphi \in I_i\} \end{aligned} \tag{3.9}$$

for $i \in \{1, \dots, 4\}$ with the intervals

$$I_1 := (0, \tfrac{\pi}{2}),\quad I_2 := (\tfrac{\pi}{2}, \pi),\quad I_3 := (\pi, \tfrac{3}{2}\pi),\quad I_4 := (\tfrac{3}{2}\pi, 2\pi).$$

By (r, φ) we throughout denote polar coordinates. Note that $D_1, \dots, D_4$ give rise to a partition of the unit disc D. The partition represents the regions, where $\varepsilon_{\mathrm{in}}$ is constant.

In this subsection, we study the two-dimensional Laplacian

$$\begin{aligned} \Delta_D \psi &:= \tfrac{1}{\varepsilon_{\mathrm{in}}} \operatorname{div}(\varepsilon_{\mathrm{in}} \nabla \psi), \\ \psi \in \mathscr{D}(\Delta_D) &:= \{\psi \in H_0^1(D) \mid \operatorname{div}(\varepsilon_{\mathrm{in}} \nabla \psi) \in L^2(D)\}, \end{aligned} \tag{3.10}$$

with transmission conditions on the unit disc D. Note that the transmission conditions fit to the ones of the Laplacian Δ_{Γ^*}, see (3.3).

It is well known that Δ_D is invertible with compact resolvent, and selfadjoint on $L^2(D)$ with respect to the inner product

$$(f, g)_{\varepsilon_{\mathrm{in}}, D} := \int_D \varepsilon_{\mathrm{in}} f g \,\mathrm{d}x, \qquad f, g \in L^2(D), \tag{3.11}$$

see [36] for instance. (Indeed, bijectivity is obtained via a Lax-Milgram-Lemma argument and symmetry is derived with an integration by parts.)

The eigenvalue problem for Δ_D can be handled by transferring the reasoning in [50]. This means that we switch into polar coordinates, and separate between angular and radial variable. As the coefficient $\varepsilon_{\mathrm{in}}$ depends only on the angle φ, it can be interpreted as a piecewise constant function on the union $I_1 \cup \dots \cup I_4$.

The angular part leads to the eigenvalue problem

$$(\psi^{(i)})'' = -\kappa^2 \psi^{(i)} \quad \text{on } I_i,\ i \in \{1, \dots, 4\}, \qquad \psi, \varepsilon_{\mathrm{in}} \psi' \in H^1_{\mathrm{per}}(0, 2\pi), \tag{3.12}$$

where $H^1_{\mathrm{per}}(0, 2\pi)$ refers to the periodic H^1-space on $(0, 2\pi)$. By Lemma 4.2 in [37], (3.12) has countably many eigenvalues $0 = \kappa_0^2 < \kappa_1^2 \le \dots \to \infty$, and associated piecewise smooth eigenfunctions $\psi_0, \psi_1, \dots$. The latter form an orthonormal basis of $L^2(0, 2\pi)$ with respect to the L^2-inner product with weight $\varepsilon_{\mathrm{in}}$.

A lengthy calculation leads to the following relation for the square root of the first nonzero eigenvalue of (3.12), see [55]. It provides a crucial sharp lower bound involving the number $\overline{\kappa}$ from (3.4).

Lemma 3.5. *Let $\varepsilon_{\rm in}$ satisfy (3.8). Then $\overline{\kappa} \leq \kappa_1 < 1 \leq \kappa_2$, and κ_1 satisfies*

$$\frac{(\varepsilon_{\rm in}^{(4)} - \varepsilon_{\rm in}^{(1)})^2}{\varepsilon_{\rm in}^{(4)} \varepsilon_{\rm in}^{(1)}} = -\frac{4\sin^2(\kappa_1\pi)}{\sin(\frac{\kappa_1}{2}\pi)\sin(\frac{3\kappa_1}{2}\pi)}.$$

In the following, we construct an orthonormal basis of eigenfunctions for the Dirichlet Laplacian Δ_D. To this end, we employ the Bessel function J_ν of order $\nu \geq 0$, and the eigenvalues of (3.12). The positive zeros of J_ν are denoted by $0 < \mu_1^{(\nu)} < \mu_2^{(\nu)} < \cdots \to \infty$. We define

$$\Psi_{k,l}(r,\varphi) := J_{\kappa_l}(\mu_k^{(\kappa_l)} r)\psi_l(\varphi), \qquad r \in (0,1),\ \varphi \in (0, 2\pi), \tag{3.13}$$

with $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$. Note that the functions $\Psi_{k,1}$ have second weak derivatives with singularities at $r = 0$. This eventually causes the weaker regularity statement than H^2 in Proposition 3.1. These singular functions are hence incorporated separately by means of the spaces

$$\mathscr{M} := \operatorname{span}\{\Psi_{k,l} \mid k \in \mathbb{N},\ l \in \mathbb{N}_0 \setminus \{1\}\}, \qquad \mathscr{N} := \operatorname{span}\{\Psi_{k,1} \mid k \in \mathbb{N}\}. \tag{3.14}$$

In the next lemma, we derive useful spectral properties of the Laplace operator Δ_D. The proof employs ideas from Theorem 2 in Section 5.5.2, Lemma 1 in Section 6.4.2, and Theorem 1 in Section 6.4.2 of [50].

Lemma 3.6. *Let $\varepsilon_{\rm in}$ satisfy (3.8).*

a) The family $\{\Psi_{k,l} \mid k \in \mathbb{N},\ l \in \mathbb{N}_0\}$ is an orthonormal basis of $L^2(D)$ with respect to the inner product $(\cdot,\cdot)_{\varepsilon_{\rm in},D}$ from (3.11).

b) The spaces $\mathscr{M}$ and $\mathscr{N}$ are contained in the domain $\mathscr{D}(\Delta_D)$. Furthermore, $\mathscr{M}$ is a subspace of $PH^2(D)$. The eigenvector relation $\Delta_D \Psi_{k,l} = -(\mu_k^{(\kappa_l)})^2 \Psi_{k,l}$ is satisfied for $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$.

Proof. a) The asserted orthogonality follows by combining the choice of the functions $\{\psi_l \mid l \in \mathbb{N}_0\}$ with Theorem 2 in Section 5.5.2 of [50]. The completeness of the system $\{\Psi_{k,l} \mid k \in \mathbb{N},\ l \in \mathbb{N}_0\}$ can be concluded in the same manner as in the proof of Lemma 1 in Section 6.4.2 of [50].

b.i) Let $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$. The function $\Psi_{k,l}^{\rm in}$ satisfies the transmission and boundary conditions for Δ_D due to the choice of ψ_l, see (3.12), and the definition of $\mu_k^{(\kappa_l)}$. Note further that every function $\Psi_{k,l}$ is at least H^1-regular. To show that claim, let $i \in \{1, \ldots, 4\}$. Since ψ_l solves (3.12), the function $\Psi_{k,l}^{(i)}$ is as regular as the function $(a_l^{(i)} \cos(\kappa_l\varphi) + b_l^{(i)} \sin(\kappa_l\varphi))r^{\kappa_l}$ with appropriate real numbers $a_l^{(i)}, b_l^{(i)}$. If $l = 0$, this means that $\Psi_{k,0}$ is piecewise smooth. In case $l \in \mathbb{N}$, $\Psi_{k,l}$ then belongs to the space $H^{1+\kappa}(D_i)$ for every $\kappa < \min\{1, \kappa_l\}$, see [4, 5, 3] for instance.

The stated eigenvalue-eigenvector relations are obtained in the same way as in Theorem 2 in Section 5.5.2 of [50] by means of the choice of ψ_l, see (3.12). As a result, $\mathscr{M}$ and $\mathscr{N}$ are contained in $\mathscr{D}(\Delta_D)$.

b.ii) It remains to show that every function in $\mathscr{M}$ is at least piecewise H^2-regular. Let $l \in \mathbb{N}$ with $\kappa_l > 1$. (The case $\kappa_l \in \{0, 1\}$ can be handled by switching into cartesian coordinates.) Since $\Psi_{k,l}$ is as regular as the function $\psi_l(\varphi)r^{\kappa_l}$, it satisfies the estimate

$$\int_0^1 \int_{I_i} \Big(\tfrac{1}{r}|\partial_r(\Psi_{k,l})^{(i)}|^2 + r|\partial_r^2(\Psi_{k,l})^{(i)}|^2 + \tfrac{1}{r}|\partial_r\partial_\varphi(\Psi_{k,l})^{(i)}|^2 + \tfrac{1}{r^3}|\partial_\varphi(\Psi_{k,l})^{(i)}|^2 + \tfrac{1}{r^3}|\partial_\varphi^2(\Psi_{k,l})^{(i)}|^2 \Big)\, \mathrm{d}\varphi\, \mathrm{d}r < \infty,$$

proving that $\Psi_{k,l}$ is an element of $PH^2(D)$. □

In view of the stated inequality in Proposition 3.1, we also need an a-priori energy estimate for functions in $\mathscr{D}(\Delta_D)$ in terms of the Laplacian Δ_D. In Lemma 2.2 and the following Remark in [36], the estimate

$$\|\psi\|_{PH^2(D)} \leq C(\|\psi\|_{L^2(D)} + \|\Delta_D\psi\|_{L^2(D)}), \qquad \psi \in \mathscr{D}(\Delta_D) \cap PH^2(D),$$

is derived with a uniform constant $C = C(\varepsilon_{\text{in}}) > 0$. By Lemma 3.6, $\mathscr{M}$ is a subspace of $PH^2(D)$. Standard reasoning then leads to the inequality

$$\|\psi\|_{PH^2(D)} \leq C\|\Delta_D\psi\|_{L^2(D)}, \qquad \psi \in \mathscr{M}, \tag{3.15}$$

with a uniform constant $C = C(\varepsilon_{\text{in}}) > 0$.

To derive a counterpart of (3.15) for functions in the space $\mathscr{N}$ from (3.14), we transfer ideas by Kellogg in the next two lemmas to our setting, see Theorem 5.2 and Lemma 5.6 in [36].

Let $\nu \in (1/2, 1)$ and $f \in C([0, 1])$. In a first step, a norm estimate is derived for the solution of the one-dimensional problem

$$\begin{aligned} &r^{1/2}\psi''(r) + r^{-1/2}\psi'(r) - \nu^2 r^{-3/2}\psi(r) = f(r), \qquad r \in (0, 1), \\ &\psi(0) = \psi(1) = 0, \qquad \psi \in L^2(0, 1). \end{aligned} \tag{3.16}$$

The solution ψ is given by

$$\psi(r) = \alpha r^\nu + \tfrac{1}{2\nu} r^\nu \int_0^r t^{1/2-\nu} f(t)\,\mathrm{d}t - \tfrac{1}{2\nu} r^{-\nu} \int_0^r t^{1/2+\nu} f(t)\,\mathrm{d}t, \qquad r \in (0, 1),$$

involving the number

$$\alpha := -\tfrac{1}{2\nu} \int_0^1 (t^{1/2-\nu} - t^{1/2+\nu}) f(t)\,\mathrm{d}t.$$

We note that the expression on the left hand side of (3.16) corresponds to the radial part of the Laplacian Δ_D, acting on functions in $\mathscr{N}$. The inequality provided by the next lemma will thus be crucial for an energy estimate in $\mathscr{N}$, see the proof of Lemma 3.8.

Lemma 3.7 is obtained in a straightforward way: Each summand on the right hand side of the solution formula for ψ is estimated separately by means of the Young and Cauchy-Schwarz inequalities. We thus omit the proof.

Lemma 3.7. *Let $\kappa > 2(1 - \nu)$ with parameter $\nu \in (1/2, 1)$ from (3.16). The solution ψ of (3.16) satisfies the inequality*

$$\int_0^1 r^\kappa (r^{-1}(\psi')^2 + r^{-3}\psi^2)\,\mathrm{d}r \leq C \int_0^1 f^2\,\mathrm{d}r$$

with a uniform constant $C = C(\kappa, \nu) > 0$.

We next establish the desired a-priori estimate in fractional order Sobolev spaces for the operator Δ_D on the space $\mathscr{N}$ from (3.14). Recall the number $\overline{\kappa}$ from (3.4).

Lemma 3.8. *Let ε_{in} satisfy (3.8), $\kappa_0 \in (2(1 - \overline{\kappa}), 1)$, and $\phi \in \overline{\mathscr{N}}^{\|\cdot\|_{\mathscr{D}(\Delta_D)}}$. The inequality*

$$\|\phi\|_{PH^{2-\kappa_0/2}(D)} \leq C\|\Delta_D\phi\|_{L^2(D)}$$

is valid with a uniform constant $C = C(\kappa_0) > 0$.

Proof. 1) Let $\phi \in \mathscr{N}$. We use the sets

$$D_{i,\xi} := \{(x, y) \in D_i \mid |(x, y)| \geq \xi\}$$

for $\xi > 0$. Recall the definition of D_i in (3.9). Note moreover that ϕ is smooth on each $D_{i,\xi}$ by definition of $\mathscr{N}$ in (3.14).

Combining the inequality $|x_1|^{\kappa_0/2}|x_2|^{\kappa_0/2} \le \sqrt{x_1^2+x_2^2}^{\kappa_0}$, $(x_1,x_2)\in\mathbb{R}^2$, with Lemma 2.12 from [6], it suffices to prove the estimate

$$\|r^{\frac{\kappa_0}{2}}\phi\|_{L^2(D)} + \sum_{j=1}^2 \|r^{\frac{\kappa_0}{2}}\partial_j\phi\|_{L^2(D)} + \sum_{j,k=1}^2 \|r^{\frac{\kappa_0}{2}}\partial_j\partial_k\phi\|_{L^2(D)} \le C\|\Delta_D\phi\|_{L^2(D)}.$$

Transforming to polar coordinates and integrating by parts with respect to the r- and φ-variables, the formula

$$\begin{aligned}
&\sum_{i=1}^4 \int_{D_{i,\xi}} r^{\kappa_0}\varepsilon_{\mathrm{in}}^{(i)}[(\partial_x^2\phi^{(i)})(\partial_y^2\phi^{(i)}) - (\partial_x\partial_y\phi^{(i)})^2]\,\mathrm{d}(x,y)\\
&\qquad = \sum_{i=1}^4 \int_\xi^1\int_{I_i} \varepsilon_{\mathrm{in}}^{(i)}[r^{\kappa_0}(\partial_r^2\phi^{(i)})(\partial_r\phi^{(i)}) + \tfrac{\kappa_0(1-\kappa_0)}{2} r^{\kappa_0-3}(\partial_\varphi\phi^{(i)})^2]\,\mathrm{d}\varphi\,\mathrm{d}r\\
&\qquad\quad + \sum_{i=1}^4 \int_{I_i} \varepsilon_{\mathrm{in}}^{(i)}[(\tfrac{1+\kappa_0}{2}(\partial_\varphi\phi^{(i)})^2)|_{r=1} - (r^{\kappa_0-1}(\partial_r\phi^{(i)})(\partial_\varphi^2\phi^{(i)})\\
&\qquad\quad\quad + \tfrac{1+\kappa_0}{2} r^{\kappa_0-2}(\partial_\varphi\phi^{(i)})^2)|_{r=\xi}]\,\mathrm{d}\varphi
\end{aligned} \tag{3.17}$$

is obtained.

2) The first term on the right hand side of (3.17) is next treated separately by means of Lemma 3.7. We first note that ϕ has the representation

$$\phi(r\cos\varphi, r\sin\varphi) = \sum_{k=1}^Z \alpha_k J_{\kappa_1}(\mu_k^{(\kappa_1)} r)\psi_1(\varphi), \tag{3.18}$$

for $r\in(0,1)$, $\varphi\in(0,2\pi)$, with numbers $Z\in\mathbb{N}$ and $\alpha_1,\dots,\alpha_Z\in\mathbb{R}$. Since ψ_1 is an eigenfunction of (3.12) to the eigenvalue κ_1^2, we deduce the identity

$$r^{1/2}\Delta\phi^{(i)} = r^{1/2}\partial_r^2\phi^{(i)} + r^{-1/2}\partial_r\phi^{(i)} - \kappa_1^2 r^{-3/2}\phi^{(i)} =: f^{(i)}, \quad r\in(0,1). \tag{3.19}$$

Combining (3.19) with Lemma 3.7, we then infer the estimate

$$\sum_{i=1}^4 \int_0^1\int_{I_i} r^{\kappa_0-1}(\partial_r\phi^{(i)})^2\,\mathrm{d}\varphi\,\mathrm{d}r \le \tilde{C}\sum_{i=1}^4\int_0^1\int_{I_i} r(\Delta\phi^{(i)})^2\,\mathrm{d}\varphi\,\mathrm{d}r$$

with a constant $\tilde{C} = \tilde{C}(\kappa_0) > 0$. For the first term on the right hand side of (3.17), we consequently arrive at the inequalities

$$\begin{aligned}
&\sum_{i=1}^4 \int_\xi^1\int_{I_i} \varepsilon_{\mathrm{in}}^{(i)} r^{\kappa_0}(\partial_r^2\phi^{(i)})(\partial_r\phi^{(i)})\,\mathrm{d}\varphi\,\mathrm{d}r\\
&\ge -\sum_{i=1}^4 \big(\tfrac{1}{16}\int_\xi^1\int_{I_i}\varepsilon_{\mathrm{in}}^{(i)} r^{\kappa_0+1}(\partial_r^2\phi^{(i)})^2\,\mathrm{d}\varphi\,\mathrm{d}r + 4\int_0^1\int_{I_i}\varepsilon_{\mathrm{in}}^{(i)} r^{\kappa_0-1}(\partial_r\phi^{(i)})^2\,\mathrm{d}\varphi\,\mathrm{d}r\big)\\
&\ge -\big(\sum_{i=1}^4 \tfrac{1}{16}\int_\xi^1\int_{I_i}\varepsilon_{\mathrm{in}}^{(i)} r^{\kappa_0+1}(\partial_r^2\phi^{(i)})^2\,\mathrm{d}\varphi\,\mathrm{d}r + 4\tilde{C}\|\sqrt{\varepsilon_{\mathrm{in}}}\Delta_D\phi\|_{L^2(D)}^2\big).
\end{aligned} \tag{3.20}$$

3) We next focus on the face integrals on the right hand side of (3.17). To that end, we analyze the behavior of ϕ near the center of D. In view of (3.18), it suffices to consider the function

$\tilde{\phi}(r,\varphi) := J_{\kappa_1}(r)\psi_1(\varphi)$. Using that ψ_1 is an eigenfunction of (3.12) and that $\kappa_0 > 2(1-\kappa_1)$, one can show that the functions $r^{\kappa_0-1}(\partial_r\tilde{\phi}^{(i)})(\partial_\varphi^2\tilde{\phi}^{(i)})$ and $r^{\kappa_0-2}(\partial_\varphi\tilde{\phi}^{(i)})^2$ possess continuous extensions to $[0,1]\times I_i$, and that they tend to zero as $r\to 0$. The dominated convergence theorem hence yields

$$\lim_{\xi\to 0}\sum_{i=1}^4\int_{I_i}\varepsilon_{\text{in}}^{(i)}[r^{\kappa_0-1}(\partial_r\phi^{(i)})(\partial_\varphi^2\phi^{(i)})+\tfrac{1+\kappa_0}{2}r^{\kappa_0-2}(\partial_\varphi\phi^{(i)})^2]|_{r=\xi}\,\mathrm{d}\varphi = 0. \tag{3.21}$$

4) For the next step, the formula

$$\partial_r^2\phi^{(i)} = \tfrac{x^2}{r^2}(\partial_x^2\phi^{(i)}) + 2\tfrac{xy}{r^2}\partial_x\partial_y\phi^{(i)} + \tfrac{y^2}{r^2}\partial_y^2\phi^{(i)}$$

is useful. Combining (3.17) and (3.20), we derive the estimate

$$\begin{aligned}
&\sum_{i=1}^4\int_{D_{i,\xi}} r^{\kappa_0}\varepsilon_{\text{in}}^{(i)}(\Delta\phi^{(i)})^2\,\mathrm{d}(x,y) + 4\tilde{C}\|\sqrt{\varepsilon_{\text{in}}}\Delta_D\phi\|_{L^2(D)}^2\\
&\quad\geq \tfrac{1}{2}\sum_{i=1}^4\Big(\int_{D_{i,\xi}} r^{\kappa_0}\varepsilon_{\text{in}}^{(i)}[(\partial_x^2\phi^{(i)})^2+(\partial_y^2\phi^{(i)})^2+2(\partial_x\partial_y\phi^{(i)})^2]\,\mathrm{d}(x,y)\\
&\qquad\quad - 2\int_{I_i}\varepsilon_{\text{in}}^{(i)}[r^{\kappa_0-1}(\partial_r\phi^{(i)})(\partial_\varphi^2\phi^{(i)})+\tfrac{1+\kappa_0}{2}r^{\kappa_0-2}(\partial_\varphi\phi^{(i)})^2]|_{r=\xi}\,\mathrm{d}\varphi\Big).
\end{aligned}$$

In the limit $\xi\to 0$, the monotone convergence principle and (3.21) lead to the relation

$$\begin{aligned}
&(1+4\tilde{C})\|\sqrt{\varepsilon_{\text{in}}}\Delta_D\phi\|_{L^2(D)}^2\\
&\qquad\geq \tfrac{1}{2}\sum_{i=1}^4\int_{D_i} r^{\kappa_0}\varepsilon_{\text{in}}^{(i)}[(\partial_x^2\phi^{(i)})^2+(\partial_y^2\phi^{(i)})^2+2(\partial_x\partial_y\phi^{(i)})^2]\,\mathrm{d}(x,y).
\end{aligned}$$

The H^1-norm of ϕ can finally also be bounded by $\|\Delta_D\phi\|_{L^2(D)}$ using the Cauchy-Schwarz and Poincaré inequality. $\square$

The following corollary is an important and direct consequence of Lemmas 3.6 and 3.8, as well as (3.15). It provides the desired regularity statement and energy estimate in the domain $\mathscr{D}(\Delta_D)$.

Corollary 3.9. *Let ε_{in} satisfy (3.8), and $\kappa > 1-\overline{\kappa}$. Then $\mathscr{D}(\Delta_D)\subseteq PH^{2-\kappa}(D)$ with $\|u\|_{PH^{2-\kappa}(D)}\leq C\|\Delta_D u\|_{L^2(D)}$, $u\in\mathscr{D}(\Delta_D)$, for a constant $C=C(\kappa)>0$.*

3.3. **Conclusion of the regularity statement.** We now establish the desired regularity statement for functions in the domain of the operator Δ_{Γ^*} from (3.3), resulting in a regularity result for the solution to the interface problem (3.1). To that end, we first use a cut-off argument to focus on thin cylinders around interior edges. This principle is well known to experts in the field, see [16, 14, 15] for instance. To have a self-contained presentation, we however sketch the arguments.

Let us first fix some notation for the next statements. Recall that $\mathscr{S}$ is the union of all edges of the interfaces. Let $e_{\text{in}}\subseteq\mathscr{S}$ be an interior edge. Without loss of generality, we assume in the following that all cylinders around the interior edges with radius 1 are disjoint from each other. We denote by $\operatorname{dist}(e_{\text{in}},\cdot):\overline{Q}\to[0,\infty]$ the distance function to e_{in}. Let additionally $\chi:[0,\infty)\to[0,1]$ be a smooth cut-off function with $\chi=1$ on $[0,1/4]$ and $\operatorname{supp}\chi\subseteq[0,9/16]$.

Lemma 3.10. *Let ε satisfy (1.2), and let $e_{\text{in}}\subseteq\mathscr{S}$ be an interior edge. Let furthermore $u\in\mathscr{D}(\Delta_{\Gamma^*})$. The function $\chi(\operatorname{dist}(e_{\text{in}},\cdot)^2)u$ belongs to $\mathscr{D}(\Delta_{\Gamma^*})$.*

Proof. We can assume that e_{in} satisfies (3.7), and abbreviate $v := \chi(\operatorname{dist}(e_{\text{in}}, \cdot)^2)u$.

Recall definition (3.3). Using the product rule and construction of $\chi(\operatorname{dist}(e_{\text{in}}, \cdot)^2)$, it suffices to verify the first order transmission condition for v at interfaces touching e_{in}. Let $\mathscr{F}$ be such an interface. We assume the representation $\mathscr{F} = \{0\} \times [0,1] \times [0,1]$, and take $x = (x_1, x_2, x_3) = (0, x_2, x_3) \in \mathscr{F}$. A straightforward computation then shows that $\nabla\chi(\operatorname{dist}(e_{\text{in}}, x)^2) \cdot \nu_{\mathscr{F}} = 0$. This means that v fulfills the same interface conditions as u, whence v is an element of the domain $\mathscr{D}(\Delta_{\Gamma^*})$. □

Recall for the next statement Definition 3.4, (3.5), and (3.4).

Lemma 3.11. *Let ε satisfy (1.2), and let $e_{\text{in}} \subseteq \mathscr{S}$ be an interior edge, where ε has a strong discontinuity. Let furthermore $u \in \mathscr{D}(\Delta_{\Gamma^*})$ and $\kappa > 1 - \overline{\kappa}$. The function $\chi(\operatorname{dist}(e_{\text{in}}, \cdot)^2)u$ belongs to $\mathscr{V}_{2-\kappa} \cap \mathscr{D}(\Delta_{\Gamma^*})$ and*

$$\|\chi(\operatorname{dist}(e_{\text{in}}, \cdot)u\|_{\mathscr{V}_{2-\kappa}} \leq C\|\Delta_{\Gamma^*}(\chi(\operatorname{dist}(e_{\text{in}}, \cdot)u)\|_{L^2(Q)}$$

with a number $C = C(\varepsilon, \kappa)$.

Proof. 1) Without loss of generality, we can assume that e_{in} satisfies (3.7). We moreover assume that $\Gamma_3 \subseteq \Gamma^*$. (The case $\Gamma_3 \not\subseteq \Gamma^*$ can be handled with the usual modifications for homogeneous Neumann boundary conditions.) Throughout, $C = C(\varepsilon, \kappa)$ is a constant that changes from line to line. As in the proof of Lemma 3.10, we set

$$v := \chi(\operatorname{dist}(e_{\text{in}}, \cdot)^2)u \in \mathscr{D}(\Delta_{\Gamma^*}).$$

Recall the cylinder $\mathscr{Z} = D \times (0,1)$. By construction of v, it suffices to prove

$$\|v\|_{L^2((0,1),PH^{2-\kappa}(D))} + \|v\|_{H^1((0,1),H^1(D))} + \|v\|_{H^2((0,1),L^2(D))} \\ \leq C\|\Delta_{\Gamma^*}v\|_{L^2(\mathscr{Z})}. \tag{3.22}$$

2) The function v is odd reflected at Γ_3 to the large cylinder $\tilde{\mathscr{Z}} := D \times (-1, 2)$. The parameter ε_{in} is reflected in an even way. Note that v belongs to $H^1(\tilde{\mathscr{Z}})$, and that $\varepsilon_{\text{in}}\nabla v|_{D_i \times (-1,2)}$ is an element of $H(\operatorname{div}, \tilde{\mathscr{Z}})$.

3) Let $\mathring{\chi}_3 : \mathbb{R} \to [0,1]$ be a smooth cut-off function with $\mathring{\chi}_3 = 1$ on $[0,1]$ and $\operatorname{supp}\mathring{\chi}_3 \subseteq [-1/2, 3/2]$. We analyze the product $\mathring{\chi}_3(x_3)v$ in the following, and thereby use ideas and techniques from [16, 14]. To that end, we extend the function $\mathring{\chi}_3(x_3)v$ trivially by zero in x_3-direction to the infinite cylinder $D \times \mathbb{R}$.

Put now

$$-\Delta(\mathring{\chi}_3 v^{(i)}) =: \tilde{f}^{(i)} \in L^2(D_i \times \mathbb{R}), \tag{3.23}$$

for $i \in \{1, 2, 3, 4\}$. The above extension procedure then implies the fact

$$\|\tilde{f}\|_{L^2(D\times\mathbb{R})} \leq C\|f\|_{L^2(\tilde{\mathscr{Z}})}. \tag{3.24}$$

Next we apply a partial Fourier-Transform with respect to the x_3-variable, and we denote the resulting function by $\hat{w}$ for $w \in L^2(D \times \mathbb{R})$. The inverse transform of a function $v \in L^2(D \times \mathbb{R})$ is denoted by $\check{v}$. We moreover call the new variable in Fourier space ξ. Relation (3.23) then gives rise to the formula

$$(\xi^2 - \partial_{x_1}^2 - \partial_{x_2}^2)\hat{}(\mathring{\chi}_3 v^{(i)})(x_1, x_2, \xi) = \hat{}(\tilde{f}^{(i)}), \qquad (x_1, x_2, \xi) \in D_i \times \mathbb{R}, \tag{3.25}$$

and we note that $\hat{}(\mathring{\chi}_3 v)(\cdot, \xi) \in \mathscr{D}(\Delta_D)$.

4) Corollary 3.9, the triangle inequality and (3.25) provide the relation

$$\|\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\|^2_{PH^{2-\kappa}(D)} \leq C\Big(\|\xi^2\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\|^2_{L^2(D)} + \|\hat{}(\tilde{f})(\cdot,\xi)\|^2_{L^2(D)}\Big). \tag{3.26}$$

We next take also the estimate

$$0 \leq -\Re\big(\varepsilon_{\text{in}}\xi^2\hat{}(\mathring{\chi}_3 v)(\cdot,\xi), \Delta_D\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\big)_{L^2(D)}, \tag{3.27}$$

into account, that is a consequence of the positivity of $-\Delta_D$. Combining (3.25)–(3.27), we arrive at the inequalities

$$\begin{aligned}\|\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\|^2_{PH^{2-\kappa}(D)} \leq C\Big(&\|\sqrt{\varepsilon_{\text{in}}}\xi^2\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\|^2_{L^2(D)}\\ &- 2\Re\big(\varepsilon_{\text{in}}\xi^2\hat{}(\mathring{\chi}_3 v)(\cdot,\xi), \Delta_D\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\big)_{L^2(D)}\\ &+ \|\sqrt{\varepsilon_{\text{in}}}\Delta_D\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\|^2_{L^2(D)} + \|\sqrt{\varepsilon_{\text{in}}}\hat{}(\tilde{f})(\cdot,\xi)\|^2_{L^2(D)}\Big)\\ \leq C&\|\sqrt{\varepsilon_{\text{in}}}\hat{}(\tilde{f})(\cdot,\xi)\|^2_{L^2(D)}.\end{aligned} \tag{3.28}$$

Integrating now with respect to ξ and using Plancherel's Theorem, we conclude

$$\|\mathring{\chi}_3 v\|^2_{L^2(\mathbb{R},PH^{2-\kappa}(D))} \leq C\|\tilde{f}\|^2_{L^2(D\times\mathbb{R})}.$$

In view of (3.24), we consequently derive the relation

$$\|\mathring{\chi}_3 v\|_{L^2(\mathbb{R},PH^{2-\kappa}(D))} \leq C\|f\|_{L^2(\tilde{\mathscr{Z}})}. \tag{3.29}$$

5) Relation (3.27) further yields the inequality

$$\|\xi^2\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\|^2_{L^2(D)} \leq C\|\hat{}(\tilde{f})(\cdot,\xi)\|^2_{L^2(D)}.$$

Integrating with respect to ξ and using (3.24), we infer the estimate

$$\|\mathring{\chi}_3 v\|_{H^2(\mathbb{R},L^2(D))} \leq C\|f\|_{L^2(\tilde{\mathscr{Z}})}. \tag{3.30}$$

With the selfadjointness of the operator $(-\Delta_D)^{1/2}$ and the Cauchy-Schwarz estimate, we next deduce the inequality

$$\begin{aligned}\|\sqrt{\varepsilon_{\text{in}}}|\xi|(-\Delta_D)^{1/2}\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\|^2_{L^2(D)} &= \big(\varepsilon_{\text{in}}|\xi|^2\hat{}(\mathring{\chi}_3 v)(\cdot,\xi), (-\Delta_D)\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\big)_{L^2(D)}\\ &\leq \|\sqrt{\varepsilon_{\text{in}}}|\xi|^2\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\|_{L^2(D)}\|\sqrt{\varepsilon_{\text{in}}}\Delta_D\hat{}(\mathring{\chi}_3 v)(\cdot,\xi)\|_{L^2(D)}.\end{aligned}$$

We now integrate with respect to ξ and use the equivalence of the H^1-norm and the graph norm in $\mathscr{D}(-\Delta_D)^{1/2}$. Relations (3.24) and (3.28) now imply

$$\|\sqrt{\varepsilon_{\text{in}}}\mathring{\chi}_3 v\|_{H^1(\mathbb{R},H^1(D))} \leq C\|f\|_{L^2(\tilde{\mathscr{Z}})}. \tag{3.31}$$

Combining (3.29)–(3.31), inequality (3.22) is valid. □

For the next statement, we collect all interior edges e_{in}, at which ε has a strong discontinuity, into a set $\mathscr{E}(\varepsilon)$, see Definition 3.4. We also set $\mathscr{E}(\mu) := \emptyset$, and recall that χ is introduced at the beginning of this Subsection. The below lemma then states that functions in the domain $\mathscr{D}(\Delta_{\Gamma^*})$ from (3.3) are H^2-regular near every edge of an interface at which ε has no strong discontinuity.

Lemma 3.12. *Let $\eta \in \{\varepsilon,\mu\}$ satisfy (1.2), and let $u \in \mathscr{D}(\Delta_{\Gamma^*})$. The function $w := (1 - \sum_{e\in\mathscr{E}(\eta)}\chi(\operatorname{dist}(e,\cdot)^2))u$ belongs to $PH^2(Q)$ and $\|w\|_{PH^2(Q)} \leq \|\Delta_{\Gamma^*}w\|_{L^2(Q)}$.*

Proof. We only treat the case $\eta = \varepsilon$ and $\Gamma_3 \subseteq \Gamma^*$, as the remaining can be handled with similar arguments. By Lemma 3.10, the function w is an element of $\mathscr{D}(\Delta_{\Gamma^*})$. In view of Lemma 3.3, it suffices to show that w is piecewise H^2-regular. To reach this goal, we analyze w on two adjacent cuboids Q_1 and Q_2 that share an interface $\mathscr{F}$ with two interior edges. (The case of Q_1 and Q_2 touching the exterior faces Γ_1 or Γ_2 can be treated similarly.) Without loss of generality, we can assume the identities

$$Q_1 = (-1,0) \times (-1,1)^2, \quad Q_2 = (0,1) \times (-1,1)^2, \quad \mathscr{F} = \{0\} \times [-1,1]^2.$$

A smooth cut-off function $\tilde{\chi} : [-1,1] \to [0,1]$ is furthermore employed. It satisfies $\operatorname{supp} \tilde{\chi} \subseteq [-7/8, 7/8]$ and $\tilde{\chi} = 1$ on $[-3/4, 3/4]$. Set also $\tilde{Q} := (-1,1)^3$. By construction, the function $f(x_1,x_2,x_3) := \eta\tilde{\chi}(x_1)\tilde{\chi}(x_2)w(x_1,x_2,x_3)$ is then an element of the space

$$\begin{aligned}\{f \in PH^1(\tilde{Q}) \mid \Delta f|_{Q_i} \in L^2(Q_i),\ i \in \{1,2\},\ [\![\tfrac{1}{\eta} f]\!]_{\mathscr{F}} = [\![\partial_1 f]\!]_{\mathscr{F}} = 0,\\ f(\cdot,\pm1,\cdot) = 0,\ f(\pm1,\cdot,\cdot) = 0,\ \partial_3 f(\cdot,\cdot,\pm1) = 0\}.\end{aligned}$$

By Proposition 8.1 in [54], the mapping f is then H^2-regular on Q_1 and Q_2. □

Combining Lemmas 3.11 and 3.12, we derive the desired regularity statement for functions in the domain $\mathscr{D}(\Delta_{\Gamma^*})$. Recall for the statement definitions (3.4) and (3.5).

Lemma 3.13. *Let $u \in \mathscr{D}(\Delta_{\Gamma^*})$, and let $\eta \in \{\varepsilon, \mu\}$ satisfy (1.2). Choose further $\kappa = 0$ if $\eta = \mu$, and $\kappa > 1 - \overline{\kappa}$ if $\eta = \varepsilon$. Then $\|u\|_{\mathscr{V}_{2-\kappa}} \le C\|\Delta_{\Gamma^*} u\|_{L^2(Q)}$ with a uniform constant $C = C(\kappa, \eta, Q)$.*

Proof. 1) In the following, $C = C(\kappa,\eta,Q) > 0$ is a constant that changes from line to line. Integration by parts and the Poincaré inequality imply the well known estimate

$$\|\Delta_{\Gamma^*} u\|_{L^2(Q)} \ge C\|u\|_{H^1(Q)}. \tag{3.32}$$

2) For $e \in \mathscr{E}(\eta)$, we set $v_e := \chi(\operatorname{dist}(e,\cdot)^2)u$ (assuming that the distance between interior edges is greater than 2). Combining the triangle inequality with Lemmas 3.11 and 3.12, we infer the inequality

$$\|u\|_{\mathscr{V}_{2-\kappa}} \le C\Big(\sum_{e\in\mathscr{E}(\eta)} \|\Delta_{\Gamma^*} v_e\|_{L^2(Q)} + \|\Delta_{\Gamma^*}(u - \sum_{e\in\mathscr{E}(\eta)} v_e)\|_{L^2(Q)}\Big).$$

With Young's inequality, we then infer the estimate

$$\|u\|^2_{\mathscr{V}_{2-\kappa}} \le C\Big(\sum_{e\in\mathscr{E}(\eta)} \|\Delta_{\Gamma^*} v_e\|^2_{L^2(Q)} + \|\Delta_{\Gamma^*}(u - \sum_{e\in\mathscr{E}(\eta)} v_e)\|^2_{L^2(Q)}\Big). \tag{3.33}$$

3) Let $e \in \mathscr{E}(\eta)$, and abbreviate $w_e := \chi(\operatorname{dist}(e,\cdot)^2)$. Employing the product rule for the Laplacian as well as Young's inequality, we deduce the relation

$$\begin{aligned}&\|\Delta_{\Gamma^*} v_e\|^2_{L^2(Q)}\\ &\quad\le C\sum_{i=1}^N \big(\|(\Delta w_e)u^{(i)}\|^2_{L^2(Q_i)} + \|(\nabla w_e)\cdot(\nabla u^{(i)})\|^2_{L^2(Q_i)} + \|w_e \Delta u^{(i)}\|^2_{L^2(Q_i)}\big).\end{aligned}$$

We next take into account that all functions w_e have disjoint support. In view of the regularity of w_e on $\overline{Q}$ and inequality (3.32), we arrive at the result

$$\sum_{e\in\mathscr{E}(\eta)} \|\Delta_{\Gamma^*} v_e\|^2_{L^2(Q)} \le C\Big(\|\sum_{e\in\mathscr{E}(\eta)} (\Delta w_e)u\|^2_{L^2(Q)} + \|\sum_{e\in\mathscr{E}(\eta)} (\nabla w_e)\cdot(\nabla u)\|^2_{L^2(Q)}$$

$$+ \| \sum_{e\in\mathscr{E}(\eta)} w_e \Delta_{\Gamma^*} u\|^2_{L^2(Q)}\Big)$$
$$\leq C\|\Delta_{\Gamma^*} u\|^2_{L^2(Q)}. \tag{3.34}$$

Analogous reasoning also establishes the statement

$$\|\Delta_{\Gamma^*}(u - \sum_{e\in\mathscr{E}(\eta)} v_e)\|^2_{L^2(Q)} \leq C\|\Delta_{\Gamma^*} u\|^2_{L^2(Q)}. \tag{3.35}$$

The desired estimate is a consequence of (3.33)–(3.35). □

Proposition 3.1 is now a direct consequence of Lemma 3.13.

Proof of Proposition 3.1. Using the Lax-Milgram Lemma, one can show that the operator Δ_{Γ^*} from (3.2) is bijective. Hence, there is a unique solution $\psi \in \mathscr{D}(\Delta_{\Gamma^*})$ of (3.1). Lemma 3.13 now implies the asserted regularity and energy statements. □

We can also treat the pure Neumann case $\Gamma^* = \emptyset$, as the difference only arises in the energy estimates. For the statement, recall the space $\mathscr{V}_{2-\kappa}$ from (3.5) and the number $\overline{\kappa}$ from (3.4).

Proposition 3.14. *Let $\eta \in \{\varepsilon, \mu\}$ satisfy (1.2), and let $f \in L^2(Q)$. We set $\kappa = 0$ if $\eta = \mu$, and $\kappa > 1 - \overline{\kappa}$ if $\eta = \varepsilon$. There is a unique function $\psi \in \mathscr{V}_{2-\kappa}$ solving*

$$\begin{aligned} (1-\Delta)\psi^{(i)} &= f^{(i)} && \text{on } Q_i \text{ for } i \in \{1,\dots,N\}, \\ \nabla\psi\cdot\nu &= 0 && \text{on } \partial Q, \\ [\![\psi]\!]_{\mathscr{F}} = 0 &= [\![\eta\nabla\psi\cdot\nu_{\mathscr{F}}]\!]_{\mathscr{F}} && \text{for } \mathscr{F} \in \mathscr{F}_{\mathrm{int}}. \end{aligned} \tag{3.36}$$

It satisfies $\|\psi\|_{\mathscr{V}_{2-\kappa}} \leq C\,\|f\|_{L^2(Q)}$ with a constant $C = C(\kappa,\eta,Q) > 0$.

Proof. To unify the arguments, we introduce the appropriate Neumann-Laplacian

$$(\Delta_\emptyset v)^{(i)} := \Delta v^{(i)}, \quad \mathscr{D}(\Delta_\emptyset) := \{v \in H^1(Q) \mid \operatorname{div}(\eta\nabla v) \in L^2(Q),\ \nabla v\cdot\nu = 0 \text{ on } \partial Q\}.$$

As the reasoning in Lemmas 3.11–3.12 focuses only on the local behavior of functions in the domain of Δ_{Γ^*} around the interior edges and also allows homogeneous Neumann boundary conditions, the mentioned statements are also valid for functions in the domain $\mathscr{D}(\Delta_\emptyset)$. (In the proof of Lemma 3.12, one uses Proposition 8.2 from [54] instead of Proposition 8.1.)

Adapting the arguments in the proofs of Lemmas 3.3 and 3.13 to the current setting of Neumann boundary conditions, we furthermore derive the energy estimate $\|u\|_{\mathscr{V}_{2-\kappa}} \leq C\|(I - \Delta_\emptyset)u\|_{L^2(Q)}$ for $u \in \mathscr{D}(\Delta_\emptyset)$ with a uniform constant $C = C(\kappa,\eta,Q) > 0$.

Using the Lax-Milgram Lemma, we moreover obtain that system (3.36) has a unique solution in $\mathscr{D}(\Delta_\emptyset)$. The above regularity statement and energy estimate now imply the asserted result. □

4. Regularity result for the space X_1

This section is devoted to an embedding result for the space X_1 from (2.9). To this end, we extend the well known regularity results for the spaces $H_N(\mathrm{curl},\mathrm{div},Q)$ and $H_T(\mathrm{curl},\mathrm{div},Q)$, see Sections I.3.4 and I.3.5 in [24] for instance. Throughout, we assume that ε and μ satisfy (1.2). The corresponding spaces for our setting of discontinuous coefficients are

$$\begin{aligned} H_{N,00}(\mathrm{curl},\operatorname{div}\varepsilon,Q) &:= \{\mathbf{E} \in H_0(\mathrm{curl},Q) \mid \operatorname{div}(\varepsilon\mathbf{E}) = 0\}, \\ H_{N,0}(\mathrm{curl},\operatorname{div}\varepsilon,Q) &:= \{\mathbf{E} \in H_0(\mathrm{curl},Q) \mid \operatorname{div}(\varepsilon\mathbf{E}) \in L^2(Q)\}, \end{aligned} \tag{4.1}$$

$$H_{T,00}(\mathrm{curl}, \mathrm{div}\,\mu, Q) := \{\mathbf{H} \in H(\mathrm{curl}, Q) \mid \mathrm{div}(\mu\mathbf{H}) = 0,\ \mu\mathbf{H}\cdot\nu = 0 \text{ on } \partial Q\}.$$

The first and last space are already complete with respect to the norm in $H(\mathrm{curl}, Q)$ (making use of the bounded normal trace operator from $H(\mathrm{div}, Q)$ into $H^{-1/2}(\partial Q)$). The second space in (4.1) is complete with respect to the norm

$$\|\mathbf{E}\|^2_{H_{N,0}} := \|\mathbf{E}\|^2_{L^2(Q)} + \|\mathrm{curl}\,\mathbf{E}\|^2_{L^2(Q)} + \|\mathrm{div}(\varepsilon\mathbf{E})\|^2_{L^2(Q)}.$$

Our first goal is to establish embeddings of the spaces from (4.1) into appropriate fractional Sobolev spaces. In a next step, we then derive the desired embedding of X_1, see Proposition 4.6. In the literature, we could detect neither the precise explicit dependence of κ on ε and μ, nor the distinction between the regularity of the single components of the electric and magnetic field. These results, however, turn out to be essential for the error analysis in Section 6.2 and another work that is in preparation. For a clear presentation, we hence deduce the desired embeddings in a sequence of lemmas. Note that [7, 11, 12, 8] contain regularity statements for the above or related spaces in a more general setting, allowing general polyhedral domains for instance. Our plan is to transfer parts of the reasoning in paragraphs I.3.3–I.3.5 in [24] to our setting of a transmission problem.

We start with the study of $H_{N,00}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q)$. Combining Theorem I.3.4 in [24] with an integration by parts, we first obtain the injectivity of the curl-operator on this space.

Lemma 4.1. *Let ε satisfy (1.2). The* curl*-operator is injective on $H_{N,00}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q)$.*

We next introduce the space

$$H_\varepsilon := \{\mathbf{E} \in L^2(Q)^3 \mid \mathrm{div}(\varepsilon\mathbf{E}) = 0,\ \varepsilon\mathbf{E}\cdot\nu = 0 \text{ on } \partial Q\}.$$

The following statement characterizes the preimage of the curl-operator for the space H_ε. The result corresponds to Theorem I.3.6 in [24], and extends Lemma 6.3 in [8] to our setting of multiple submedia in a cuboid. For the statement, we recall the number $\overline{\kappa}$ from (3.4), and introduce the space

$$\mathscr{V}_{1-\kappa} := (PH^{1-\kappa}(Q)^2 \times H^1(Q)) \cap \{v \in L^2(Q)^3 \mid \partial_3 v \in L^2(Q)^3\}, \tag{4.2}$$
$$\|v\|^2_{\mathscr{V}_{1-\kappa}} := \|v\|^2_{PH^{1-\kappa}(Q)^2\times H^1(Q)} + \|\partial_3 v\|^2_{L^2(Q)^3}, \qquad v \in \mathscr{V}_{1-\kappa}.$$

Lemma 4.2. *Let ε satisfy (1.2), and let $\kappa > 1-\overline{\kappa}$. Each function $\boldsymbol{E} \in H_\varepsilon$ has the representation*

$$\boldsymbol{E} = \frac{1}{\varepsilon}\,\mathrm{curl}\,\Phi$$

with a unique function $\Phi \in H_{N,00}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q)$. Moreover, Φ belongs to the space $\mathscr{V}_{1-\kappa}$, and it satisfies the estimate $\|\Phi\|_{\mathscr{V}_{1-\kappa}} \le C\,\|\boldsymbol{E}\|_{L^2(Q)}$ with a constant $C > 0$ depending only on ε, κ, Q.

Proof. 1) Throughout the proof, $C = C(\varepsilon, \kappa, Q) > 0$ is a constant that is allowed to change from line to line. In view of Lemma 4.1, it suffices to show the existence of the desired vector Φ as well as its regularity.

Using Theorem I.3.6 in [24], there is a vector $\tilde{\Phi} \in H^1(Q)^3 \cap H_0(\mathrm{curl}, Q)$ with $\frac{1}{\varepsilon}\,\mathrm{curl}\,\tilde{\Phi} = \mathbf{E}$ and $\mathrm{div}\,\tilde{\Phi} = 0$ on Q. By (1.2), this implies

$$\mathrm{div}(\varepsilon^{(i)}\tilde{\Phi}^{(i)}) = 0. \tag{4.3}$$

In general, $\tilde{\Phi}$ does, however, not satisfy the additional transmission condition $[\![\varepsilon\tilde{\Phi}\cdot\nu_{\mathscr{F}}]\!]_{\mathscr{F}} = 0$ for all interfaces $\mathscr{F}$.

2) We next extend the traces $[\![\varepsilon\tilde{\Phi}\cdot\nu_{\mathscr{F}}]\!]_{\mathscr{F}}$ for the effective interfaces $\mathscr{F}\in\mathscr{F}_{\mathrm{int}}^{\mathrm{eff}}$, see the notation paragraph in Section 1. There is a function $\hat{\psi}\in H^1(Q)\cap PH^2(Q)$ with $\nabla\hat{\psi}\times\nu=0$ on ∂Q, $[\![\varepsilon\nabla\hat{\psi}\cdot\nu_{\mathscr{F}}]\!]_{\mathscr{F}}=[\![\varepsilon\tilde{\Phi}\cdot\nu_{\mathscr{F}}]\!]_{\mathscr{F}}$ for $\mathscr{F}\in\mathscr{F}_{\mathrm{int}}^{\mathrm{eff}}$, and

$$\|\hat{\psi}\|_{PH^2(Q)}\le C\sum_{\mathscr{F}\in\mathscr{F}_{\mathrm{int}}^{\mathrm{eff}}}\|[\![\varepsilon\tilde{\Phi}\cdot\nu_{\mathscr{F}}]\!]_{\mathscr{F}}\|_{V(\mathscr{F})}\le C\|\mathbf{E}\|_{L^2(Q)}. \tag{4.4}$$

Recall that $V(\mathscr{F})$ is defined in (2.7). To show this claim, we consider the model case of four subcuboids

$$\begin{aligned}&Q_1=(-1,0)^2\times(0,1),\quad Q_2=(0,1)\times(-1,0)\times(0,1),\quad Q_3=(0,1)^3,\\&Q_4=(-1,0)\times(0,1)^2,\quad \mathscr{F}_j=\overline{Q_j}\cap\overline{Q_{j+1}},\ j\in\{1,2,3\},\quad \mathscr{F}_4=\overline{Q_1}\cap\overline{Q_4},\\&\varepsilon^{(1)}>\varepsilon^{(2)}=\varepsilon^{(3)}=\varepsilon^{(4)},\end{aligned}$$

and construct a function $\mathring{\psi}$ on $\tilde{Q}:=(-1,1)^2\times(0,1)$ that satisfies the extension property $[\![\varepsilon\nabla\mathring{\psi}\cdot\nu_{\mathscr{F}_1}]\!]_{\mathscr{F}_1}=[\![\varepsilon\tilde{\Phi}\cdot\nu_{\mathscr{F}_1}]\!]_{\mathscr{F}_1}$, homogeneous Neumann boundary conditions on $\partial\tilde{Q}\setminus\Gamma_3$, homogeneous Dirichlet boundary conditions on $\partial\tilde{Q}\cap\Gamma_3$, and the required regularity and energy properties of $\hat{\psi}$. Due to symmetry, the trace $[\![\varepsilon\tilde{\Phi}\cdot\nu_{\mathscr{F}_4}]\!]_{\mathscr{F}_4}$ can be extended in a similar way. The desired function $\hat{\psi}$ is then obtained by combining this reasoning with a cut-off argument around the edges in Q and the extension result from Propositions 2.2 and 2.3 in [2].

In the following, we use techniques from the proof of Lemma 3.1 in [20] and Lemma 8.13 in [54]. Set $g:=\tilde{\Phi}_1|_{\mathscr{F}_1}$, with $\tilde{\Phi}_1$ denoting the first component of $\tilde{\Phi}$. Identify $\mathscr{F}_1$ with $[-1,0]\times[0,1]$, and consider the Laplacian $\Delta_{\mathscr{F}_1}$ on $\mathscr{F}_1$ with domain

$$\mathscr{D}(\Delta_{\mathscr{F}_1}):=\{u\in H^2(\mathscr{F}_1)\mid u(\cdot,0)=u(\cdot,1)=0,\ \partial_2u(0,\cdot)=\partial_2u(1,\cdot)=0\}.$$

The operator $-\Delta_{\mathscr{F}_1}$ is then selfadjoint and positive definite on $L^2(\mathscr{F}_1)$. We can hence define positive definite and selfadjoint fractional powers $(-\Delta_{\mathscr{F}_1})^{\gamma}$, $\gamma>0$, of $-\Delta_{\mathscr{F}_1}$. Hence, $-(-\Delta_{\mathscr{F}_1})^{\gamma}$ generates an analytic semigroup $(\mathrm{e}^{-t(-\Delta_{\mathscr{F}_1})^{\gamma}})_{t\ge0}$. Note further that

$$\mathscr{D}((-\Delta_{\mathscr{F}_1})^{1/2})=\{\varphi\in H^1(\mathscr{F}_1)\mid \varphi(\cdot,0)=\varphi(\cdot,1)=0\}.$$

(This identity can for instance be obtained by means of Theorem VI.2.23 in [35].) Combining furthermore the trace theorem with the boundary conditions for $\tilde{\Phi}$, we conclude $g\in(L^2(\mathscr{F}_1),\mathscr{D}(-\Delta_{\mathscr{F}_1})^{1/2})_{1/2,2}$ with

$$\|g\|_{(L^2(\mathscr{F}_1),\mathscr{D}(-\Delta_{\mathscr{F}_1})^{1/2})_{1/2,2}}\le C\|\tilde{\Phi}_1\|_{H^1(Q)}. \tag{4.5}$$

Let $\chi:[-1,1]\to[0,1]$ be a smooth cut-off function with $\chi=1$ on $[-1/2,1/2]$ and support in $[-3/4,3/4]$. We then set

$$\mathring{\psi}^{(1)}(x_1,x_2,x_3):=\chi(x_1)x_1\big(\mathrm{e}^{-x_1(-\Delta_{\mathscr{F}_1})^{1/2}}g\big)(x_2,x_3),\qquad (x_1,x_2,x_3)\in Q_1.$$

In consideration of the analyticity of $(\mathrm{e}^{-t(-\Delta_{\mathscr{F}_1})^{1/2}})_{t\ge0}$, we conclude the identities

$$\mathring{\psi}^{(1)}|_{\mathscr{F}_1}=0,\quad \partial_1\mathring{\psi}^{(1)}|_{\mathscr{F}_1}=g,\quad \mathring{\psi}^{(1)}|_{\Gamma_3}=0,$$

as well as homogeneous Neumann boundary conditions on all other faces of Q_1. We further calculate

$$\begin{aligned}\partial_1\mathring{\chi}^{(1)}&=\Big(\chi'(x_1)x_1+\chi(x_1)-\chi(x_1)x_1(-\Delta_{\mathscr{F}_1})^{1/2}\Big)\mathrm{e}^{-x_1(-\Delta_{\mathscr{F}_1})^{1/2}}g,\\\partial_1^2\mathring{\psi}^{(1)}&=\Big(\chi''(x_1)x_1+2\chi'(x_1)-2\chi'(x_1)x_1(-\Delta_{\mathscr{F}_1})^{1/2}-2\chi(x_1)(-\Delta_{\mathscr{F}_1})^{1/2}\end{aligned}$$

$$- \chi(x_1)x_1\Delta_{\mathscr{F}_1}\Big)\mathrm{e}^{-x_1(-\Delta_{\mathscr{F}_1})^{1/2}}g.$$

We moreover note that the H^1- and H^2-norm on $\mathscr{F}_1$ are equivalent to the norms $\|(-\Delta_{\mathscr{F}_1})^{1/2}\cdot\|_{L^2(\mathscr{F}_1)}$ and $\|\Delta_{\mathscr{F}_1}\cdot\|_{L^2(\mathscr{F}_1)}$ on $\mathscr{D}(-\Delta_{\mathscr{F}_1})^{1/2}$ and $\mathscr{D}(\Delta_{\mathscr{F}_1})$, respectively. Using Remark 6.3 and Proposition 6.4 in [41], we hence conclude $\mathring{\psi}^{(1)} \in H^2(Q_1)$ with

$$\|\mathring{\psi}^{(1)}\|_{H^2(Q_1)} \le C\|g\|_{(L^2(\mathscr{F}_1),\mathscr{D}(-\Delta_{\mathscr{F}_1})^{1/2})_{1/2,2}} \le C\|\tilde{\Phi}_1\|_{H^1(Q)},$$

see (4.5). Define now

$$\begin{aligned}
\mathring{\psi}^{(2)}(x_1,x_2,x_3) &:= -\mathring{\psi}^{(1)}(-x_1,x_2,x_3), && (x_1,x_2,x_3) \in Q_2,\\
\mathring{\psi}^{(3)}(x_1,x_2,x_3) &:= \mathring{\psi}^{(2)}(x_1,-x_2,x_3), && (x_1,x_2,x_3) \in Q_3,\\
\mathring{\psi}^{(4)}(x_1,x_2,x_3) &:= \mathring{\psi}^{(1)}(x_1,-x_2,x_3), && (x_1,x_2,x_3) \in Q_4.
\end{aligned}$$

By construction, $\mathring{\psi}$ belongs to $PH^2(\tilde{Q}) \cap H^1(\tilde{Q})$, and satisfies the extension property $[\![\varepsilon\nabla\mathring{\psi} \cdot \nu_{\mathscr{F}_1}]\!]_{\mathscr{F}_1} = [\![\varepsilon\tilde{\Phi} \cdot \nu_{\mathscr{F}_1}]\!]_{\mathscr{F}_1}$ as well as the continuity relation $[\![\varepsilon\nabla\mathring{\psi} \cdot \nu_{\mathscr{F}_j}]\!]_{\mathscr{F}_j} = 0$ for $j \in \{2,3,4\}$. Taking also (2.1) into account, we obtain the energy estimate

$$\|\mathring{\psi}\|_{PH^2(\tilde{Q})} \le C\|\tilde{\Phi}_1\|_{H^1(Q)} \le C\|\mathbf{E}\|_{L^2(Q)}.$$

Altogether, $\mathring{\psi}$ is the desired extension on $\tilde{Q}$.

3) Proposition 3.1 provides a unique function $\tilde{\psi} \in \mathscr{D}(\Delta_{\partial Q}) \hookrightarrow \mathscr{V}_{2-\kappa}$ with $\Delta\tilde{\psi}^{(i)} = \Delta\hat{\psi}^{(i)}$ on Q_i and

$$\|\tilde{\psi}\|_{\mathscr{V}_{2-\kappa}} \le C\|\hat{\psi}\|_{PH^2(Q)} \le C\|\mathbf{E}\|_{L^2(Q)}. \tag{4.6}$$

Altogether, $\Phi := \tilde{\Phi} - \nabla\hat{\psi} + \nabla\tilde{\psi}$ is the desired function. The asserted norm estimate is a consequence of (2.1), (4.3)–(4.6) and the definition of $\mathscr{V}_{2-\kappa}$ in (3.5). □

The next proposition summarizes the results of the last two lemmas. The proof is a modification of the one for Theorem I.3.7 in [24]. As an intermediate result of the proof is crucial for the below reasoning, we elaborate the argument.

Lemma 4.3. *Let ε satisfy (1.2), and choose $\kappa > 1 - \overline{\kappa}$. Then $H_{N,00}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q)$ embeds continuously into $\mathscr{V}_{1-\kappa}$.*

Proof. Let $\mathbf{E} \in H_{N,00}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q)$. Lemma 4.2 yields that the operator $\frac{1}{\varepsilon}\,\mathrm{curl}$ is bijective from $H_{N,00}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q)$ into H_ε. Using the open mapping principle, $\frac{1}{\varepsilon}\,\mathrm{curl}$ hence is an isomorphism between these spaces. Lemma 4.2 and Remark I.2.5 in [24] further lead to the identities

$$\begin{aligned}
\frac{1}{\varepsilon}\,\mathrm{curl}\,\big(H_{N,00}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q)\big) &= H_\varepsilon\\
&= \frac{1}{\varepsilon}\,\mathrm{curl}\,\big(H_{N,00}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q) \cap \mathscr{V}_{1-\kappa}\big).
\end{aligned}$$

This implies $H_{N,00}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q) \subseteq \mathscr{V}_{1-\kappa}$. Lemma 4.2 furthermore yields

$$\|\mathbf{E}\|_{H(\mathrm{curl},Q)} + \|\mathbf{E}\|_{\mathscr{V}_{1-\kappa}} \le C\|\tfrac{1}{\varepsilon}\,\mathrm{curl}\,\mathbf{E}\|_{L^2(Q)}, \tag{4.7}$$

with a uniform constant $C = C(\varepsilon, \kappa, Q) > 0$. Altogether, the identity $I = (\frac{1}{\varepsilon}\,\mathrm{curl})^{-1} \circ \frac{1}{\varepsilon}\,\mathrm{curl}$ is bounded from $H_{N,00}(\mathrm{curl}, \mathrm{div}\,\varepsilon, Q)$ into $\mathscr{V}_{1-\kappa}$. □

To show the embedding property of the space X_1 from (2.9) into $\mathscr{V}_{1-\kappa} \times PH^1(Q)^3$, we next prove that one can omit the L^2-norm in the definition of $\|\cdot\|_{H_{N,0}}$.

Lemma 4.4. *Let ε satisfy (1.2). The estimate*

$$\|\boldsymbol{E}\|_{L^2(Q)} \leq C_{N0}(\|\operatorname{curl} \boldsymbol{E}\|_{L^2(Q)} + \|\operatorname{div}(\varepsilon \boldsymbol{E})\|_{L^2(Q)})$$

is valid for all $\boldsymbol{E} \in H_{N,0}(\operatorname{curl}, \operatorname{div} \varepsilon, Q)$ with a constant $C_{N0} = C_{N0}(\varepsilon, Q) > 0$.

Proof. Let $\mathbf{E} \in H_{N,0}(\operatorname{curl}, \operatorname{div} \varepsilon, Q)$. By Proposition 3.1, there is a unique function $\phi \in \mathscr{D}(\Delta_{\partial Q})$ with $\Delta\phi^{(i)} = \operatorname{div} \mathbf{E}^{(i)}$ on Q_i for $i \in \{1, \dots, N\}$. The difference $\psi := \nabla\phi - \mathbf{E}$ then belongs to $H_{N,00}(\operatorname{curl}, \operatorname{div} \varepsilon, Q)$, and we can apply inequality (4.7) to it. In this way, we obtain

$$\|\mathbf{E}\|_{L^2(Q)} \leq \|\psi\|_{L^2(Q)} + \|\nabla\phi\|_{L^2(Q)} \leq \frac{C}{\min \varepsilon}\|\operatorname{curl} \mathbf{E}\|_{L^2(Q)} + \|\nabla\phi\|_{L^2(Q)},$$

where C is the uniform constant from (4.7). In view of the weak formulation of the identity $\Delta_{\partial Q}\phi = \frac{1}{\varepsilon} \operatorname{div}(\varepsilon\mathbf{E})$ and the Poincaré inequality, we infer the estimates

$$\begin{aligned} \|\nabla\phi\|_{L^2(Q)}^2 &\leq -\frac{1}{\min \varepsilon} \int_Q \phi \operatorname{div}(\varepsilon\mathbf{E}) \, \mathrm{d}x \leq \frac{1}{\min \varepsilon} \|\phi\|_{L^2(Q)} \|\operatorname{div}(\varepsilon\mathbf{E})\|_{L^2(Q)} \\ &\leq \frac{C_P}{\min \varepsilon}\|\nabla\phi\|_{L^2(Q)} \|\operatorname{div}(\varepsilon\mathbf{E})\|_{L^2(Q)}, \end{aligned} \tag{4.8}$$

employing the Poincaré constant $C_P > 0$ for Q. □

In view of the assumptions (1.2), the parameter μ is piecewise constant on the chain $\tilde{Q}_1, \dots, \tilde{Q}_L$ of cuboids. As the setting of two cuboids from [54] transfers to the partition $\bigcup_{l=1}^{L} \tilde{Q}_L$ in a straightforward way, the reasoning for Proposition 9.7 in [54] yields the following statement.

Lemma 4.5. *Let μ satisfy (1.2). The space $H_{T,00}(\operatorname{curl}, \operatorname{div} \mu, Q)$ embeds continuously into $PH^1(Q)^3$.*

We now deduce the desired regularity statement for functions in the space X_1. For the statement, recall the number $\overline{\kappa}$ from (3.4) and the space $\mathscr{V}_{1-\kappa}$ from (4.2).

Proposition 4.6. *Let ε, μ satisfy (1.2), and $\kappa > 1 - \overline{\kappa}$. The space X_1 embeds continuously into $\mathscr{V}_{1-\kappa} \times PH^1(Q)^3$.*

Proof. 1) Let $(\mathbf{E}, \mathbf{H}) \in X_1 = \mathscr{D}(M) \cap X_0$. We first show the asserted regularity of $(\mathbf{E}, \mathbf{H})$. In view of Lemma 4.5, it remains to deal with the electric field $\mathbf{E}$.

Consider the elliptic transmission problem

$$\begin{aligned} \Delta\psi^{(i)} &= \operatorname{div} \mathbf{E}^{(i)} && \text{on } Q_i \text{ for } i \in \{1, \dots, N\}, \\ \psi &= 0 && \text{on } \partial Q, \\ [\![\psi]\!]_{\mathscr{F}} &= 0 && \text{for } \mathscr{F} \in \mathscr{F}_{\text{int}}, \\ [\![\varepsilon\nabla\psi \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}} &= [\![\varepsilon\mathbf{E} \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}} && \text{for } \mathscr{F} \in \mathscr{F}_{\text{int}}, \end{aligned} \tag{4.9}$$

which has a unique solution $\psi \in \mathscr{V}_{2-\kappa} \cap H_0^1(Q)$. (The space $\mathscr{V}_{2-\kappa}$ is defined in (3.5).) Indeed, a modification of the reasoning in the proof for Lemma 4.2 and the precondition $[\![\varepsilon\mathbf{E} \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}} \in V(\mathscr{F})$, $\mathscr{F} \in \mathscr{F}_{\text{int}}^{\text{eff}}$, see (2.8), yield a unique mapping $\tilde{\psi} \in \mathscr{V}_{2-\kappa} \cap H_0^1(Q)$ with $\Delta\tilde{\psi}^{(i)} = 0$ on Q_i, satisfying the required boundary and transmission conditions in (4.9). By Proposition 3.1, there also is a function $\check{\psi} \in \mathscr{D}(\Delta_{\partial Q}) \subseteq \mathscr{V}_{2-\kappa}$ with $\Delta_{\partial Q}\check{\psi} = \frac{1}{\varepsilon} \operatorname{div}(\varepsilon\mathbf{E})$. Altogether, $\psi := \tilde{\psi} + \check{\psi} \in \mathscr{V}_{2-\kappa} \cap H_0^1(Q)$ is the solution of (4.9).

Hence, $\mathbf{E} - \nabla\psi$ is an element of $H_{N,00}(\operatorname{curl}, \operatorname{div} \varepsilon, Q) \subseteq \mathscr{V}_{1-\kappa}$, see Lemma 4.3. The vector $\nabla\psi$ being an element of $\mathscr{V}_{1-\kappa}$, we infer the stated regularity result.

2) It remains to show the asserted embedding property. In the following, $C = C(\varepsilon, \kappa, Q) > 0$ is a constant that changes from line to line. Using Lemma 4.5, it suffices to deal with $\mathbf{E}$. Proposition 3.1 yields

$$\|\check{\psi}\|_{\mathscr{V}_{2-\kappa}} \leq C \sum_{i=1}^{N} \|\operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)})\|_{L^2(Q_i)}. \tag{4.10}$$

The reasoning for (4.4) and (4.6) furthermore leads to the bound

$$\|\nabla\tilde{\psi}\|_{\mathscr{V}_{1-\kappa}} \leq C \sum_{\mathscr{F}\in\mathscr{F}_{\mathrm{int}}^{\mathrm{eff}}} \|[\![\varepsilon\mathbf{E}\cdot\nu_{\mathscr{F}}]\!]_{\mathscr{F}}\|_{V(\mathscr{F})}. \tag{4.11}$$

Applying Lemma 4.3 to $\mathbf{E} - \nabla\psi$, the relations

$$\begin{aligned}
\|\mathbf{E}\|_{\mathscr{V}_{1-\kappa}} &\leq \|\mathbf{E} - \nabla\psi\|_{\mathscr{V}_{1-\kappa}} + \|\nabla\psi\|_{\mathscr{V}_{1-\kappa}} \\
&\leq C\big(\|\mathbf{E}\|_{L^2(Q)} + \|\operatorname{curl}\mathbf{E}\|_{L^2(Q)} + \|\nabla\psi\|_{L^2(Q)}\big) + \|\nabla\psi\|_{\mathscr{V}_{1-\kappa}} \\
&\leq C\big(\|M(\mathbf{E},\mathbf{H})\| + \|\mathbf{E}\|_{L^2(Q)} + \|\check{\psi}\|_{\mathscr{V}_{2-\kappa}} + \|\nabla\tilde{\psi}\|_{\mathscr{V}_{1-\kappa}}\big)
\end{aligned}$$

follow. The desired embedding is a consequence of (4.10) and (4.11). □

5. Wellposedness of the Maxwell system in X_1

The main result of Section 4 establishes a regularity statement for the space X_1, see Proposition 4.6. To conclude a corresponding regularity result for the solutions of the Maxwell system (1.1), we show in this section that X_1 is a state space of (1.1). This is done by means of semigroup theory.

We next transfer techniques from the proof of Proposition 2.3 in [21] to the current setting. Recall moreover that M_1 is the part of M in X_1.

Proposition 5.1. *Let ε and μ satisfy (1.2). The operator M_1 generates a contractive C_0-semigroup $(\mathrm{e}^{tM_1})_{t\geq 0}$ on X_1. The latter is the restriction of $(\mathrm{e}^{tM})_{t\geq 0}$ to X_1.*

Proof. 1) Employing the theory of subspace semigroups, see for instance Paragraph II.2.3 in [22], the asserted generator property follows by showing that the family $(\mathrm{e}^{tM})_{t\geq 0}$ leaves the space X_1 invariant, and that it is strongly continuous on it.

We first note that $\mathrm{e}^{tM}(\mathscr{D}(M)) \subseteq \mathscr{D}(M)$ for $t \geq 0$. Regarding the magnetic conditions, the arguments in the proof of Proposition 2.3 in [21] apply also here. This reasoning results in the invariance of the space

$$X_{\mathrm{mag}} := \{(u, v) \in X \mid \operatorname{div}(\mu v) = 0,\ (\mu v)\cdot\nu = 0 \text{ on } \partial Q\}$$

under the resolvent map $R(\lambda, M)$ for $\lambda > 0$, and in the invariance of X_{mag} with respect to the family $(\mathrm{e}^{tM})_{t\geq 0}$.

2) Let $(\tilde{u}, \tilde{v}) \in X_1$, and set $(u(t), v(t)) := \mathrm{e}^{tM}(\tilde{u}, \tilde{v})$ for $t \geq 0$. Semigroup theory then yields that the function (u, v) belongs to $C([0,\infty), \mathscr{D}(M))$. The Maxwell equations (1.1) with $\mathbf{J} = 0$ lead to the formula $\partial_t u = \frac{1}{\varepsilon}\operatorname{curl} v$. Taking the divergence of this equation, the relation $\partial_t \operatorname{div}(\varepsilon u(t)) = 0$ follows in $L^2(\tilde{Q}_{i,l})$, $i \in \{1, \ldots, L\}$, $l \in \{0, \ldots, K\}$, for the subdomains $(\tilde{Q}_{i,l})$ from Section 1. This is equivalent to

$$\operatorname{div}(\varepsilon u(t)) = \operatorname{div}(\varepsilon\tilde{u}) \tag{5.1}$$

on $\tilde{Q}_{i,l}$. As a result, the mapping $[0,\infty) \to H(\operatorname{div}, \tilde{Q}_{l,k})$, $t \mapsto \varepsilon u(t)$ is continuously differentiable. Due to the continuity of the normal trace operator on $H(\operatorname{div}, \tilde{Q}_{i,l})$, the relations

$$\partial_t [\![\varepsilon u(t) \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}} = [\![\operatorname{curl} \tilde{v} \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}} = 0, \qquad t \geq 0,$$

follow in $H^{-1/2}(\mathscr{F})$ for every effective interface $\mathscr{F} \in \mathscr{F}_{\text{int}}^{\text{eff}}$, see the notation paragraph in Section 1. This shows that the function

$$[\![\varepsilon u(t) \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}} = [\![\varepsilon \tilde{u} \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}} \tag{5.2}$$

belongs to the space $V(\mathscr{F})$ from (2.7) for $\mathscr{F} \in \mathscr{F}_{\text{int}}^{\text{eff}}$, and that the mapping $[0,\infty) \to V(\mathscr{F})$, $t \mapsto [\![\varepsilon u(t) \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}}$ is continuously differentiable.

3) We finally note that Proposition 2.4 and (5.1)-(5.2) imply the contractivity of $(\mathrm{e}^{tM}|_{X_1})_{t\geq 0}$ on X_1. □

The next statement is a conclusion of Proposition 5.1. It transfers parts of Proposition 2.3 from [21] to our setting of discontinuous coefficients. Although the proof basically follows the lines of the one for Corollary 9.24 in [54], we present it here for the sake of a self-contained presentation. Note that the formula for $\rho_{\mathscr{F}}$ is also deduced in the Appendix of [46]. For the external current density $\mathbf{J}$, the space

$$W := L^1([0,T], \mathscr{D}(M_1)) + W^{1,1}([0,T], X_1),$$
$$\|f\|_W := \inf_{\substack{f = f_1 + f_2, \\ f_1 \in L^1([0,T],\mathscr{D}(M_1)), \\ f_2 \in W^{1,1}([0,T],X_1)}} \big(\|f_1\|_{L^1([0,T],\mathscr{D}(M_1))} + \|f_2\|_{W^{1,1}([0,T],X_1)}\big), \qquad f \in W,$$

is employed for fixed $T > 0$.

Corollary 5.2. *Let ε and μ satisfy (1.2). Let $T > 0$, $w_0 = (\boldsymbol{E}_0, \boldsymbol{H}_0)$ be initial data from $\mathscr{D}(M_1) = \mathscr{D}(M^2) \cap X_0$, and let $g := (\frac{1}{\varepsilon}\boldsymbol{J}, 0) : [0,T] \to X_1$ be the weighted external current density that is continuous, and an element of W. The following items are valid.*

a) The Maxwell system (1.1) possesses a unique classical solution $w = (\boldsymbol{E}, \boldsymbol{H})$, belonging to $C([0,T], \mathscr{D}(M_1)) \cap C^1([0,T], X_1)$. It satisfies the bounds

$$\|w(t)\|_{X_1} \leq \|w_0\|_{X_1} + \|g\|_{L^1([0,t],X_1)},$$
$$\|Mw(t)\|_{X_1} \leq \|w_0\|_{\mathscr{D}(M_1)} + (\tfrac{2}{T} + 3)\,\|g\|_W,$$

for $t \in [0,T]$.

b) The volume charge density $\rho^{(i)}$ on Q_i and the surface charge $\rho_{\mathscr{F}}$ are given via

$$\rho^{(i)}(t) = \operatorname{div}(\varepsilon^{(i)} \boldsymbol{E}^{(i)}(t)) = \operatorname{div}(\varepsilon^{(i)} \boldsymbol{E}_0^{(i)}) - \int_0^t \operatorname{div}(\boldsymbol{J}^{(i)}(s))\,\mathrm{d}s,$$
$$\rho_{\mathscr{F}}(t) = [\![\varepsilon \boldsymbol{E}(t) \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}} = [\![\varepsilon \boldsymbol{E}_0 \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}} - \int_0^t [\![\boldsymbol{J}(s) \cdot \nu_{\mathscr{F}}]\!]_{\mathscr{F}}\,\mathrm{d}s,$$

for $t \in [0,T]$, $i \in \{1,\ldots,N\}$, and $\mathscr{F} \in \mathscr{F}_{\text{int}}^{\text{eff}}$.

Proof. a) The stated classical wellposedness of (1.1) on X_1 follows from Proposition 5.1 and semigroup theory, see Theorem 8.1.4 in [52] for instance. Duhamel's formula leads to the representation

$$w(t) = \mathrm{e}^{tM_1} w_0 + \int_0^t \mathrm{e}^{(t-s)M_1} g(s)\,\mathrm{d}s = \mathrm{e}^{tM_1} w_0 + \int_0^t \mathrm{e}^{(t-s)M_1} (\tfrac{1}{\varepsilon}\mathbf{J}(s), 0)\,\mathrm{d}s.$$

Taking the contractivity of $(\mathrm{e}^{tM_1})_{t\geq 0}$ into account, the first asserted estimate is obtained.

Let $\zeta > 0$ and $(\frac{1}{\varepsilon}\mathbf{J}, 0) = \mathbf{J}_1 + \mathbf{J}_2 \in L^1([0,T], \mathscr{D}(M_1)) + W^{1,1}([0,T], X_1)$ with

$$\|(\tfrac{1}{\varepsilon}\mathbf{J}, 0)\|_W \geq \|\mathbf{J}_1\|_{L^1([0,T],\mathscr{D}(M_1))} + \|\mathbf{J}_2\|_{W^{1,1}([0,T],X_1)} - \zeta.$$

An integration by parts in the above Duhamel formula leads to the identities

$$\begin{aligned} Mw(t) &= \mathrm{e}^{tM} Mw_0 + \int_0^t \mathrm{e}^{(t-s)M} M\mathbf{J}_1(s)\,\mathrm{d}s - \int_0^t (\frac{d}{ds}\mathrm{e}^{(t-s)M})\mathbf{J}_2(s)\,\mathrm{d}s \\ &= \mathrm{e}^{tM} Mw_0 + \int_0^t \mathrm{e}^{(t-s)M} M\mathbf{J}_1(s)\,\mathrm{d}s - \mathbf{J}_2(t) + \mathrm{e}^{tM}\mathbf{J}_2(0) \\ &\quad + \int_0^t \mathrm{e}^{(t-s)M}\mathbf{J}_2'(s)\,\mathrm{d}s. \end{aligned}$$

Combining Lemma 7.6 in [54] with Proposition 5.1, the relations

$$\begin{aligned} \|Mw(t)\|_{X_1} &\leq \|w_0\|_{\mathscr{D}(M_1)} + \|\mathbf{J}_1\|_{L^1([0,T],\mathscr{D}(M_1))} + (\tfrac{2}{T} + 3)\, \|\mathbf{J}_2\|_{W^{1,1}([0,T],X_1)} \\ &\leq \|w_0\|_{\mathscr{D}(M_1)} + (\tfrac{2}{T} + 3)(\|(\tfrac{1}{\varepsilon}\mathbf{J}, 0)\|_{W_1} + \zeta) \end{aligned}$$

are derived. Letting ζ tend to zero, we infer the second stated estimate.

b) The representation for the current density is obtained by modifying the arguments from Proposition 2.3 in [21] and part 2) from the proof of Proposition 5.1. The linear mapping $X_1 \to L^2(Q_i)$, $(u, v) \mapsto \operatorname{div}(\varepsilon^{(i)}u^{(i)})$ being continuous for $i \in \{1, \dots, N\}$, the regularity of w implies that $\rho^{(i)} : [0,T] \to L^2(Q_i)$ is continuously differentiable. Similar reasoning further shows that $[0,T] \to L^2(Q_i)$, $s \mapsto \operatorname{div}(\mathbf{J}^{(i)}(s))$ is continuous. Taking the divergence in (1.1), leads to

$$\partial_t \operatorname{div}(\varepsilon^{(i)}\mathbf{E}^{(i)}(t)) = -\operatorname{div}(\mathbf{J}^{(i)}(t)), \qquad t \in [0,T],$$

in $L^2(Q_i)$. The first asserted formula is obtained by integration with respect to t. Analogously, the arguments in part 2) of the proof for Proposition 5.1 result in the stated formula for the surface charge $\rho_{\mathscr{F}}$ in $V(\mathscr{F})$ for every effective interface $\mathscr{F} \in \mathscr{F}_{\mathrm{int}}^{\mathrm{eff}}$. □

Remark 5.3. In view of Proposition 4.6, Corollary 5.2 provides a classical solution of the Maxwell system (1.1) in the space $C^1([0,T], \mathscr{V}_{1-\kappa} \times PH^1(Q)^3)$ for $\kappa > 1 - \overline{\kappa}$ with the number $\overline{\kappa}$ from (3.4) and the space $\mathscr{V}_{1-\kappa}$ from (4.2). ♢

6. Analysis of a directional splitting scheme

This section is concerned with the construction and analysis of a directional splitting scheme for (1.1). The scheme can deal with the low regularity of the solution of the Maxwell system, see Remark 5.3. In particular, the regularity requirement for the initial data is weaker than for the ADI schemes from [56, 42, 10], see [29, 21, 19, 20, 23]. In Section 6.1, we introduce the splitting and analyze the splitting operators. We furthermore comment on the efficiency of the scheme. Subsequently, we bound the error of the scheme in Section 6.2. Here the regularity results from Section 5 are essential.

6.1. **Construction of a directional splitting scheme.**

In view of the H^1-regularity in x_3-direction of the solution to (1.1), see Remark 5.3, we split the x_3-coordinate off and leave the

x_1, x_2 coordinates coupled. This strategy leads to the splitting

$$M\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}\frac{1}{\varepsilon}\operatorname{curl}\mathbf{H}\\ -\frac{1}{\mu}\operatorname{curl}\mathbf{E}\end{pmatrix} = \begin{pmatrix}\frac{1}{\varepsilon}\partial_2\mathbf{H}_3\\ -\frac{1}{\varepsilon}\partial_1\mathbf{H}_3\\ \frac{1}{\varepsilon}\operatorname{curl}_2(\mathbf{H}_1,\mathbf{H}_2)\\ -\frac{1}{\mu}\partial_2\mathbf{E}_3\\ \frac{1}{\mu}\partial_1\mathbf{E}_3\\ -\frac{1}{\mu}\operatorname{curl}_2(\mathbf{E}_1,\mathbf{E}_2)\end{pmatrix} + \begin{pmatrix}-\frac{1}{\varepsilon}\partial_3\mathbf{H}_2\\ \frac{1}{\varepsilon}\partial_3\mathbf{H}_1\\ 0\\ \frac{1}{\mu}\partial_3\mathbf{E}_2\\ -\frac{1}{\mu}\partial_3\mathbf{E}_1\\ 0\end{pmatrix}$$

$$=: A\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} + B\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} \tag{6.1}$$

involving the curl_2-operator from Section 2.1. To define appropriate domains for the operators A and B, we denote $S := (a_1^-, a_1^+) \times (a_2^-, a_2^+)$, using the representation $Q = (a_1^-, a_1^+) \times (a_2^-, a_2^+) \times (a_3^-, a_3^+)$. Employing also the space $H_0(\operatorname{curl}_2, S)$ from Section 2.1, we consider the splitting operators A and B on the domains

$$\begin{aligned}\mathscr{D}(A) := \{(\mathbf{E},\mathbf{H}) \in L^2(Q)^6 \mid\ & (\mathbf{E}_1,\mathbf{E}_2) \in L^2((a_3^-, a_3^+), H_0(\operatorname{curl}_2, S)),\\ & (\mathbf{H}_1,\mathbf{H}_2) \in L^2((a_3^-, a_3^+), H(\operatorname{curl}_2, S)),\\ & \partial_1\mathbf{E}_3, \partial_2\mathbf{E}_3, \partial_1\mathbf{H}_3, \partial_2\mathbf{H}_3 \in L^2(Q),\\ & \mathbf{E}_3 = 0 \text{ on } \Gamma_1 \cup \Gamma_2\},\\ \mathscr{D}(B) := \{(\mathbf{E},\mathbf{H}) \in L^2(Q)^6 \mid\ & \partial_3\mathbf{E}_1, \partial_3\mathbf{E}_2, \partial_3\mathbf{H}_1, \partial_3\mathbf{H}_2 \in L^2(Q),\\ & \mathbf{E}_1 = 0 = \mathbf{E}_2 \text{ on } \Gamma_3\}.\end{aligned} \tag{6.2}$$

With these domains, the operators A and B are closed and densely defined on $X = L^2(Q)^6$. Note additionally that Corollary 5.2 provides a classical solution of the Maxwell system (1.1) that is contained in $\mathscr{D}(A) \cap \mathscr{D}(B)$. (This follows from Remark 5.3 and the embedding of X_1 into $\mathscr{D}(M)$.)

Let $\tau \in (0,T)$ be a fixed time step size, $n \in \mathbb{N}$ with $n\tau \le T$, and $(\frac{1}{\varepsilon}\mathbf{J}, 0) \in C([0,T], X_1)$. We then approximate the solution $(\mathbf{E},\mathbf{H})$ of (1.1) with initial datum $(\mathbf{E}_0,\mathbf{H}_0)$ at time $t_n := \tau n \le T$ by means of the Peaceman-Rachford directional splitting

$$\begin{aligned}\begin{pmatrix}\mathbf{E}^n\\ \mathbf{H}^n\end{pmatrix} = \mathscr{T}_{\tau,n}\Big[\begin{pmatrix}\mathbf{E}^{n-1}\\ \mathbf{H}^{n-1}\end{pmatrix}\Big] = (I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)\Big[&(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B)\begin{pmatrix}\mathbf{E}^{n-1}\\ \mathbf{H}^{n-1}\end{pmatrix}\\ &- \tfrac{\tau}{2\varepsilon}\begin{pmatrix}\mathbf{J}(t_{n-1}) + \mathbf{J}(t_n)\\ 0\end{pmatrix}\Big]\end{aligned} \tag{6.3}$$

with exact initial data $(\mathbf{E}^0,\mathbf{H}^0) = (\mathbf{E}_0,\mathbf{H}_0) \in X_1$. For a different operator splitting, this Peaceman-Rachford time integrator is employed in [44, 56, 42, 21, 29, 30, 47, 38, 18, 20, 31, 48, 54], for instance.

In the next two lemmas we derive that both splitting operators are skewadjoint. This implies that the scheme (6.3) is well defined and unconditionally stable, see Lemma 6.3. Recall that the inner product on $X = L^2(Q)^6$ is defined in Section 2.2.

Lemma 6.1. *Let ε and μ satisfy (1.2). Then A and B are skewsymmetric on X.*

Proof. Let $(\mathbf{E},\mathbf{H})$, $(\tilde{\mathbf{E}},\tilde{\mathbf{H}}) \in \mathscr{D}(A)$. We next employ Green's identities from Section 2.1. Taking the boundary conditions in $\mathscr{D}(A)$ into account, we infer the equations

$$\begin{aligned}\Big(A\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix},\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix}\Big) &= \int_Q [(\partial_2\mathbf{H}_3)\tilde{E}_1 - (\partial_1\mathbf{H}_3)\tilde{\mathbf{E}}_2 + (\operatorname{curl}_2(\mathbf{H}_1,\mathbf{H}_2))\tilde{\mathbf{E}}_3 - (\partial_2\mathbf{E}_3)\tilde{\mathbf{H}}_1\\ &\qquad + (\partial_1\mathbf{E}_3)\tilde{\mathbf{H}}_2 - (\operatorname{curl}_2(\mathbf{E}_1,\mathbf{E}_2))\tilde{\mathbf{H}}_3]\,\mathrm{d}x\\ &= \int_Q [\mathbf{H}_3\operatorname{curl}_2(\tilde{\mathbf{E}}_1,\tilde{\mathbf{E}}_2) + \mathbf{H}_1\partial_2\tilde{\mathbf{E}}_3 - \mathbf{H}_2\partial_1\tilde{\mathbf{E}}_3 - \mathbf{E}_3\operatorname{curl}_2(\tilde{\mathbf{H}}_1,\tilde{\mathbf{H}}_2)\\ &\qquad - \mathbf{E}_1\partial_2\tilde{\mathbf{H}}_3 + \mathbf{E}_2\partial_1\tilde{\mathbf{H}}_3]\,\mathrm{d}x\\ &= -\Big(\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}, A\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix}\Big).\end{aligned}$$

This shows that A is skewsymmetric. The operator B can be treated similarly, meaning that the prescribed boundary conditions in $\mathscr{D}(B)$ are used in an integration by parts. □

Using arguments from the proof of Lemma 4.1 in [29], we next conclude that both splitting operators are skewadjoint.

Lemma 6.2. *Let ε,μ satisfy (1.2). Then A and B are skewadjoint on X. In particular, $(I-\tau L)^{-1}$ and $(I+\tau L)(I-\tau L)^{-1}$ are contractive for $L\in\{A,B\}$, $\tau>0$.*

Proof. 1) As A and B are densely defined, closed, and skewsymmetric, see Lemma 6.1, it suffices to show that the operators $I\pm A$ and $I\pm B$ have dense range in X. We only consider the operators $I-A$ and $I-B$, and show that the space of test functions $C_c^\infty(Q)^6$ is contained in their range. (The operators $I+A$ and $I+B$ can be treated with the same arguments.)

2) Let $(\tilde{\mathbf{E}},\tilde{\mathbf{H}}) \in C_c^\infty(Q)^6$. We want to show the existence of a vector $(\mathbf{E},\mathbf{H}) \in \mathscr{D}(A)$ with $(I-A)(\mathbf{E},\mathbf{H}) = (\tilde{\mathbf{E}},\tilde{\mathbf{H}})$. This is equivalent to the system

$$\begin{aligned} \mathbf{E}_1 - \tfrac{1}{\varepsilon}\partial_2\mathbf{H}_3 &= \tilde{\mathbf{E}}_1, & \mathbf{H}_1 + \tfrac{1}{\mu}\partial_2\mathbf{E}_3 &= \tilde{\mathbf{H}}_1,\\ \mathbf{E}_2 + \tfrac{1}{\varepsilon}\partial_1\mathbf{H}_3 &= \tilde{\mathbf{E}}_2, & \mathbf{H}_2 - \tfrac{1}{\mu}\partial_1\mathbf{E}_3 &= \tilde{\mathbf{H}}_2,\\ \mathbf{E}_3 - \tfrac{1}{\varepsilon}\partial_1\mathbf{H}_2 + \tfrac{1}{\varepsilon}\partial_2\mathbf{H}_1 &= \tilde{\mathbf{E}}_3, & \mathbf{H}_3 - \tfrac{1}{\mu}\partial_2\mathbf{E}_1 + \tfrac{1}{\mu}\partial_1\mathbf{E}_2 &= \tilde{\mathbf{H}}_3. \end{aligned} \tag{6.4}$$

By formally inserting the left equations of the first and second line into the right equation of the third line, we derive the formula

$$\mu\mathbf{H}_3 - \partial_1(\tfrac{1}{\varepsilon}\partial_1\mathbf{H}_3) - \partial_2(\tfrac{1}{\varepsilon}\partial_2\mathbf{H}_3) = \mu\tilde{\mathbf{H}}_3 + \partial_2\tilde{\mathbf{E}}_1 - \partial_1\tilde{\mathbf{E}}_2 =: f \in L^2(Q). \tag{6.5}$$

2) Recall the rectangle $S := (a_1^-,a_1^+)\times(a_2^-,a_2^+)$, being the projection of $Q = (a_1^-,a_1^+)\times(a_2^-,a_2^+)\times(a_3^-,a_3^+)$ to the x_1-x_2 plane. We consider the equation

$$\begin{aligned}\int_{a_3^-}^{a_3^+}\int_S [\mu w\varphi + \tfrac{1}{\varepsilon}(\nabla_{x_1,x_2}w)\cdot(\nabla_{x_1,x_2}\varphi)]\,\mathrm{d}(x_1,x_2)\,\mathrm{d}x_3 \\ = \int_{a_3^-}^{a_3^+}\int_S f\varphi\,\mathrm{d}(x_1,x_2)\,\mathrm{d}x_3, \quad \varphi\in L^2((a_3^-,a_3^+),H^1(S)),\end{aligned} \tag{6.6}$$

being the weak formulation of (6.5). The Lax-Milgram Lemma provides a unique solution $w\in L^2((a_3^-,a_3^+),H^1(S))$ of (6.6). Note moreover that $H_3 := w$ satisfies (6.5) and that $\frac{1}{\varepsilon}\nabla_{x_1,x_2}\mathbf{H}_3$ is an element of $L^2((a_3^-,a_3^+),H_0(\operatorname{div},S))$. Put

$$\mathbf{E}_1 := \tilde{\mathbf{E}}_1 + \tfrac{1}{\varepsilon}\partial_2\mathbf{H}_3, \qquad \mathbf{E}_2 := \tilde{\mathbf{E}}_2 - \tfrac{1}{\varepsilon}\partial_1\mathbf{H}_3. \tag{6.7}$$

By construction, the left equations in the first and second line of (6.4) are then fulfilled. Using (6.5), we then derive the relation

$$\partial_2 \mathbf{E}_1 - \partial_1 \mathbf{E}_2 = \partial_2 \tilde{\mathbf{E}}_1 - \partial_1 \tilde{\mathbf{E}}_2 + \operatorname{div}_{x_1,x_2}(\tfrac{1}{\varepsilon}\nabla_{x_1,x_2}\mathbf{H}_3) = \mu(\mathbf{H}_3 - \tilde{\mathbf{H}}_3). \tag{6.8}$$

As a result, $\operatorname{curl}_2(\mathbf{E}_1, \mathbf{E}_2)$ is an element of $L^2(Q)$.

We next deal with the boundary conditions for $(\mathbf{E}_1, \mathbf{E}_2)$. Let ϕ be an element of $L^2((a_3^-, a_3^+), H^1(S))$. With (6.7)–(6.8) and the fact that $\frac{1}{\varepsilon}\nabla_{x_1,x_2}\mathbf{H}_3$ is an element of $L^2((a_3^-, a_3^+), H_0(\operatorname{div}, S))$, we calculate

$$\begin{aligned}
&\int_{a_3^-}^{a_3^+} \int_S (\mathbf{E}_1, \mathbf{E}_2) \cdot (\partial_2\phi, -\partial_1\phi) \,\mathrm{d}(x_1,x_2)\,\mathrm{d}x_3 \\
&\qquad = \int_{a_3^-}^{a_3^+} \int_S \tilde{\mathbf{E}}_1 \partial_2\phi + \tfrac{1}{\varepsilon}(\partial_2 \mathbf{H}_3)\partial_2\phi - \tilde{\mathbf{E}}_2\partial_1\phi + \tfrac{1}{\varepsilon}(\partial_1\mathbf{H}_3)\partial_1\phi \,\mathrm{d}(x_1,x_2)\,\mathrm{d}x_3 \\
&\qquad = \int_{a_3^-}^{a_3^+} \int_S -(\partial_2\tilde{\mathbf{E}}_1)\phi + (\partial_1\tilde{\mathbf{E}}_2)\phi - \operatorname{div}(\tfrac{1}{\varepsilon}\nabla_{x_1,x_2}\mathbf{H}_3)\phi \,\mathrm{d}(x_1,x_2)\,\mathrm{d}x_3 \\
&\qquad = \int_{a_3^-}^{a_3^+} \int_S \operatorname{curl}_2(\mathbf{E}_1, \mathbf{E}_2)\phi \,\mathrm{d}(x_1,x_2)\,\mathrm{d}x_3.
\end{aligned}$$

With Lemma I.2.4 in [24] we conclude $(\mathbf{E}_1, \mathbf{E}_2) \in L^2((a_3^-, a_3^+), H_0(\operatorname{curl}_2, S))$.

3) Treating the remaining equations in (6.4) in a similar fashion, we arrive at a desired vector $(\mathbf{E}, \mathbf{H}) \in \mathscr{D}(A)$ with $(I - A)(\mathbf{E}, \mathbf{H}) = (\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$.

4) We next deal with the splitting operator B, and proceed similar to the above case for A. Solving the formula $(I - B)(\check{\mathbf{E}}, \check{\mathbf{H}}) = (\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$ for $(\check{\mathbf{E}}, \check{\mathbf{H}}) \in \mathscr{D}(B)$ amounts to determining the solution of the system

$$\begin{aligned}
\check{\mathbf{E}}_1 + \tfrac{1}{\varepsilon}\partial_3\check{\mathbf{H}}_2 &= \tilde{\mathbf{E}}_1, & \check{\mathbf{H}}_1 - \tfrac{1}{\mu}\partial_3\check{\mathbf{E}}_2 &= \tilde{\mathbf{H}}_1, \\
\check{\mathbf{E}}_2 - \tfrac{1}{\varepsilon}\partial_3\check{\mathbf{H}}_1 &= \tilde{\mathbf{E}}_2, & \check{\mathbf{H}}_2 + \tfrac{1}{\mu}\partial_3\check{\mathbf{E}}_1 &= \tilde{\mathbf{H}}_2, \\
\check{\mathbf{E}}_3 &= \tilde{\mathbf{E}}_3, & \check{\mathbf{H}}_3 &= \tilde{\mathbf{H}}_3.
\end{aligned} \tag{6.9}$$

Formally inserting the equation on the right hand side of the second line of (6.9) into the one on the left hand side of the first line yields

$$\check{\mathbf{E}}_1 - \tfrac{1}{\varepsilon\mu}\partial_3^2\check{\mathbf{E}}_1 = \tilde{\mathbf{E}}_1 - \tfrac{1}{\varepsilon}\partial_3\tilde{\mathbf{H}}_2 \in L^2(Q). \tag{6.10}$$

As in the proof of Lemma 4.3 in [29], we obtain a unique $\check{\mathbf{E}}_1 \in L^2(S, H^2(a_3^-, a_3^+))$ solving (6.10). (We use here the fact that ε and μ are constant in x_3-direction.) It satisfies the boundary condition $\check{\mathbf{E}}_1 = 0$ on Γ_3. We put $\check{\mathbf{H}}_2 := \tilde{\mathbf{H}}_2 - \frac{1}{\mu}\partial_3\check{\mathbf{E}}_1$. Then, $\partial_3\check{\mathbf{H}}_2$ is an element of $L^2(Q)$, and (6.10) leads to the identity

$$\check{\mathbf{E}}_1 + \tfrac{1}{\varepsilon}\partial_3\check{\mathbf{H}}_2 = \check{\mathbf{E}}_1 + \tfrac{1}{\varepsilon}\partial_3\tilde{\mathbf{H}}_2 - \tfrac{1}{\varepsilon\mu}\partial_3^2\check{\mathbf{E}}_1 = \tilde{\mathbf{E}}_1.$$

The remaining relations of (6.9) can be handled in the same way. Altogether, we obtain a vector $(\check{\mathbf{E}}, \check{\mathbf{H}}) \in \mathscr{D}(B)$ with $(I - B)(\check{\mathbf{E}}, \check{\mathbf{H}}) = (\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$. □

Combining formula (4.5) in [21] with Lemma 6.2, we can now conclude the unconditional stability of scheme (6.3).

Lemma 6.3. *Let ε and μ satisfy (1.2), $\tau > 0$, and $T > n\tau$. Let also $(\boldsymbol{E}^0, \boldsymbol{H}^0) \in \mathscr{D}(B)$, and $(\frac{1}{\varepsilon}\boldsymbol{J}, 0) \in C([0,T], \mathscr{D}(A))$. Then the estimate*

$$\|(\boldsymbol{E}^n, \boldsymbol{H}^n)\| \le \|(\boldsymbol{E}^0, \boldsymbol{H}^0)\|_{\mathscr{D}(B)} + T \max_{t\in[0,T]} \|(\tfrac{1}{\varepsilon}\boldsymbol{J}, 0)\|_{\mathscr{D}(A)}$$

is valid.

Using the reasoning in the proof of Lemma 6.2, we can also draw an important conclusion on the complexity of scheme (6.3).

Remark 6.4. Let ε and μ satisfy (1.2), and let $\tau > 0$. Each application of scheme (6.3) essentially amounts to solving only two-dimensional decoupled elliptic transmission problems for $\mathbf{E}_3$ and $\mathbf{H}_3$, and one-dimensional decoupled elliptic problems for $\mathbf{E}_1$ and $\mathbf{E}_2$. To show this claim, we first note that the main effort for (6.3) consists in evaluating the resolvents of A and B. In the following, we analyze both resolvent operators separately.

1) Let $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) \in X = L^2(Q)^6$, and $(\mathbf{E}, \mathbf{H}) = (I - \frac{\tau}{2}A)^{-1}(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$. We then arrive at system (6.4) (with $\frac{\tau}{2\varepsilon}$ instead of $\frac{1}{\varepsilon}$ and $\frac{\tau}{2\mu}$ instead of $\frac{1}{\mu}$). From the identity on the right hand side of the third line in (6.4), we obtain the relations

$$\begin{aligned}\int_Q \mu\tilde{\mathbf{H}}_3\varphi \,\mathrm{d}x &= \int_{a_3^-}^{a_3^+}\int_S \mu\mathbf{H}_3\varphi + \tfrac{\tau}{2}\operatorname{curl}_2(\mathbf{E}_1, \mathbf{E}_2)\varphi \,\mathrm{d}(x_1, x_2)\\ &= \int_{a_3^-}^{a_3^+}\int_S [\mu\mathbf{H}_3\varphi + \tfrac{\tau}{2}\mathbf{E}_1\partial_2\varphi - \tfrac{\tau}{2}\mathbf{E}_2\partial_1\varphi] \,\mathrm{d}(x_1, x_2)\,\mathrm{d}x_3\end{aligned}$$

for all $\varphi \in L^2((a_3^-, a_3^+), H^1(S))$. Inserting the equations on the left hand side of the first and second line of (6.4), we arrive at the relation

$$\begin{aligned}&\int_Q \mu\tilde{\mathbf{H}}_3\varphi - \tfrac{\tau}{2}\tilde{\mathbf{E}}_1\partial_2\varphi + \tfrac{\tau}{2}\tilde{\mathbf{E}}_2\partial_1\varphi \,\mathrm{d}x\\ &\qquad = \int_{a_3^-}^{a_3^+}\int_S \mu\mathbf{H}_3\varphi + \tfrac{\tau^2}{4\varepsilon}(\nabla_{x_1,x_2}\mathbf{H}_3)\cdot(\nabla_{x_1,x_2}\varphi) \,\mathrm{d}(x_1, x_2)\,\mathrm{d}x_3 \end{aligned} \tag{6.11}$$

for all $\varphi \in L^2((a_3^-, a_3^+), H^1(S))$. Having solved the essentially two-dimensional problem (6.11), $\mathbf{E}_1$ and $\mathbf{E}_2$ are directly obtained via the formulas on the left hand side of the first and second line of (6.4). A similar statement is true for $\mathbf{E}_3$.

2) Let $(\check{\mathbf{E}}, \check{\mathbf{H}}) = (I - \frac{\tau}{2}B)^{-1}(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})$. Then system (6.9) is valid with $\frac{\tau}{2\varepsilon}$ instead of $\frac{1}{\varepsilon}$ and $\frac{\tau}{2\mu}$ instead of $\frac{1}{\mu}$. The identity on the left hand side of the first line in (6.9) leads to the equation

$$\int_Q \varepsilon\tilde{\mathbf{E}}_1\phi \,\mathrm{d}x = \int_{a_3^-}^{a_3^+}\int_S \varepsilon\check{\mathbf{E}}_1\phi - \tfrac{\tau}{2}\check{\mathbf{H}}_2\partial_3\phi \,\mathrm{d}(x_1, x_2)\,\mathrm{d}x_3, \quad \phi \in H_0^1((a_3^-, a_3^+), L^2(S)).$$

Plugging in the formula on the right hand side of the second line of (6.9), we conclude

$$\int_Q \varepsilon\tilde{\mathbf{E}}_1\phi + \tfrac{\tau}{2}\tilde{\mathbf{H}}_2\partial_3\phi \,\mathrm{d}x = \int_{a_3^-}^{a_3^+}\int_S \varepsilon\check{\mathbf{E}}_1\phi + \tfrac{\tau^2}{4\mu}(\partial_3\check{\mathbf{E}}_1)(\partial_3\phi) \,\mathrm{d}(x_1, x_2)\,\mathrm{d}x_3$$

for $\phi \in H_0^1((a_3^-, a_3^+), L^2(S))$, being the weak formulation of (6.10). Having solved this one-dimensional elliptic problem, $\check{\mathbf{H}}_2$ is directly obtained as $\check{\mathbf{H}}_2 = \tilde{\mathbf{H}}_2 - \frac{\tau}{2\mu}\partial_3\check{\mathbf{E}}_1$. Similar statements are true for $\check{\mathbf{E}}_2$ and $\check{\mathbf{H}}_1$. $\Diamond$

6.2. **Error bound for the directional splitting scheme.** This section is devoted to a first order convergence result for scheme (6.3). The statement is proved by combining the regularity results from Section 5 with the statements about the splitting operators from Section 6.1.

In order to expand the semigroup $(\mathrm{e}^{tM})_{t\geq 0}$ for positive times, we additionally employ the functions

$$\Lambda_j(t)w := \frac{1}{t^j(j-1)!}\int_0^t (t-s)^{j-1}\mathrm{e}^{sM} w\,\mathrm{d}s, \qquad \Lambda_0(t) := \mathrm{e}^{tM}, \tag{6.12}$$

for $w \in X, t \geq 0$ and $j \in \mathbb{N}$, see [28, 29] for instance. Note that Proposition 5.1 implies that $\Lambda_j(t)$ leaves the space X_1 invariant for $j \in \mathbb{N}_0$, and $t \geq 0$.

Semigroup theory and Proposition 5.1 moreover lead to the useful relations

$$\|\Lambda_j(t)\|_{\mathcal{L}(X_1)} \leq \frac{1}{j!}, \qquad \|\Lambda_j(t)\|_{\mathcal{L}(\mathscr{D}(M_1))} \leq \frac{1}{j!}, \tag{6.13}$$

$$tM\Lambda_{j+1}(t) = \Lambda_j(t) - \frac{1}{j!}I \qquad \text{on } \mathscr{D}(M),\ j \in \mathbb{N}_0, \tag{6.14}$$

for $t \geq 0$, see Section 4 in [29]. Note also that $\Lambda_j(t)(\mathscr{D}(M_1)) \subseteq \mathscr{D}(M_1)$ for $t \geq 0$.

We next derive an error bound for scheme (6.3). Here arguments from the proofs of Theorem 4.2 in [29], Theorem 5.1 in [21], and Theorem 10.7 in [54] are employed. Throughout the statement and the associated proof, the solution of the Maxwell system (1.1) is denoted by $w = (\mathbf{E}, \mathbf{H})$, while the approximate solution at time $t_n = n\tau$ is w_n. For the current $\mathbf{J}$ in (1.1), we also use the space

$$W_T := W^{1,1}([0,T], X_1) \cap C([0,T], \mathscr{D}(M_1))$$

with corresponding norm

$$\|\cdot\|_{W_T} := \|\cdot\|_{W^{1,1}([0,T],X_1)} + \|\cdot\|_{C([0,T],\mathscr{D}(M_1))},$$

for a fixed final time $T > 0$, see Section 2.2. Note the relation $\mathscr{D}(M_1) = \mathscr{D}(M^2) \cap X_0$.

Theorem 6.5. *Let ε and μ satisfy (1.2), $T \geq 1$, and $w_0 = w(0) \in \mathscr{D}(M^2) \cap X_0$. Let also $(\frac{1}{\varepsilon}\boldsymbol{J}, 0) \in W_T$, and $\tau \in (0, T)$. There is a constant $C = C(\varepsilon, \mu, Q) > 0$ with*

$$\|w(t_n) - w_n\|_{L^2(Q)} \leq C\tau T\big(\|w_0\|_{\mathscr{D}(M_1)} + \|(\tfrac{1}{\varepsilon}\boldsymbol{J}, 0)\|_{W_T}\big)$$

for all $n \in \mathbb{N}_0$ with $n\tau \leq T$.

Proof. 1) We first estimate the local error. Throughout the proof, $C > 0$ denotes a constant that depends only on ε, μ, and Q. It is allowed to change from line to line. Let $k \in \mathbb{N}_0$ with $(k+1)\tau \leq T$, and recall the notation $t_k = k\tau$. Inserting the identity

$$\tfrac{1}{\varepsilon}\mathbf{J}(t_k + s) = \tfrac{1}{\varepsilon}\mathbf{J}(t_k) + \int_0^s \tfrac{1}{\varepsilon}\mathbf{J}'(t_k + r)\,\mathrm{d}r, \qquad s \in [0, \tau], \tag{6.15}$$

into the Duhamel formula for w, we infer the representation

$$\begin{aligned} w(t_{k+1}) &= \mathrm{e}^{\tau M} w(t_k) + \int_0^\tau \mathrm{e}^{(\tau - s)M}(-\tfrac{1}{\varepsilon}\mathbf{J}(t_k + s), 0)\,\mathrm{d}s \\ &= \mathrm{e}^{\tau M} w(t_k) + \int_0^\tau \mathrm{e}^{rM}(-\tfrac{1}{\varepsilon}\mathbf{J}(t_k), 0)\,\mathrm{d}r \\ &\quad + \int_0^\tau \mathrm{e}^{(\tau - s)M}\int_0^s (-\tfrac{1}{\varepsilon}\mathbf{J}'(t_k + r), 0)\,\mathrm{d}r\,\mathrm{d}s \\ &= \mathrm{e}^{\tau M} w(t_k) + \tau\Lambda_1(\tau)(-\tfrac{1}{\varepsilon}\mathbf{J}(t_k), 0) + R_k(\tau), \end{aligned}$$

involving the remainder term

$$R_k(\tau) := \int_0^\tau \mathrm{e}^{(\tau-s)M} \int_0^s (-\tfrac{1}{\varepsilon}\mathbf{J}'(t_k+r), 0)\,\mathrm{d}r\,\mathrm{d}s.$$

Using (6.15) in scheme (6.3), we on the other hand obtain the equations

$$\begin{aligned}
\mathscr{T}_{\tau,k+1}(w(t_k)) &= (I-\tfrac{\tau}{2}B)^{-1}(I+\tfrac{\tau}{2}A)\Big[(I-\tfrac{\tau}{2}A)^{-1}(I+\tfrac{\tau}{2}B)w(t_k) + \tau(-\tfrac{1}{\varepsilon}\mathbf{J}(t_k),0)\\
&\qquad + \tfrac{\tau}{2}\int_0^\tau (-\tfrac{1}{\varepsilon}\mathbf{J}'(t_k+r),0)\,\mathrm{d}r\Big]\\
&= (I-\tfrac{\tau}{2}B)^{-1}\Big[(I-\tfrac{\tau}{2}A_{-1})^{-1}(I+\tfrac{\tau}{2}A_{-1})(I+\tfrac{\tau}{2}B)w(t_k)\\
&\qquad + \tau(I+\tfrac{\tau}{2}A)(-\tfrac{1}{\varepsilon}\mathbf{J}(t_k),0) + \tfrac{\tau}{2}(I+\tfrac{\tau}{2}A)\int_0^\tau (-\tfrac{1}{\varepsilon}\mathbf{J}'(t_k+r),0)\,\mathrm{d}r\Big].
\end{aligned}$$

Note that A is extrapolated in the second identity, as $Bw(t_k)$ is in general not contained in $\mathscr{D}(A)$, see Remark 5.3. Note furthermore that the functions $(-\frac{1}{\varepsilon}\mathbf{J}(t),0)$, $\Lambda_1(\tau)(-\frac{1}{\varepsilon}\mathbf{J}(t),0)$ and $(-\frac{1}{\varepsilon}\mathbf{J}'(t),0)$ belong to $\mathscr{D}(A)\cap\mathscr{D}(B)$ for every $t\in[0,T]$, see Propositions 4.6 and 5.1, as well as (2.9).

Subtracting the representations for $w(t_{k+1})$ and $\mathscr{T}_{\tau,k+1}(w(t_k))$, we conclude

$$\begin{aligned}
&\mathscr{T}_{\tau,k+1}(w(t_k)) - w(t_{k+1})\\
&= (I-\tfrac{\tau}{2}B)^{-1}(I-\tfrac{\tau}{2}A_{-1})^{-1}\Big[(I+\tfrac{\tau}{2}A_{-1})(I+\tfrac{\tau}{2}B) - (I-\tfrac{\tau}{2}A_{-1})(I-\tfrac{\tau}{2}B)\mathrm{e}^{\tau M}\Big]w(t_k)\\
&\quad + \tau(I-\tfrac{\tau}{2}B)^{-1}\Big[(I+\tfrac{\tau}{2}A) - (I-\tfrac{\tau}{2}B)\Lambda_1(\tau)\Big](-\tfrac{1}{\varepsilon}\mathbf{J}(t_k),0)\\
&\quad + \tfrac{\tau}{2}(I-\tfrac{\tau}{2}B)^{-1}(I+\tfrac{\tau}{2}A)\int_0^\tau(-\tfrac{1}{\varepsilon}\mathbf{J}'(t_k+r),0)\,\mathrm{d}r - R_k(\tau)\\
&=: e_{1,k}(\tau) + e_{2,k}(\tau) + e_{3,k}(\tau) - R_k(\tau).
\end{aligned} \tag{6.16}$$

We next estimate the summands on the right hand side of (6.16).

2) We first deal with $e_{1,k}(\tau)$. Recall that $\Lambda_1(\tau)$ and $\Lambda_2(\tau)$ leave the spaces $\mathscr{D}(M_1)$ and X_1 invariant. Thus $M\Lambda_1(\tau)w(t_k)$ is an element of $\mathscr{D}(B)$, and $\Lambda_j(\tau)w(t_k)$ belongs to $\mathscr{D}(M^2)$ for $j\in\{1,2\}$. Algebraic manipulations and (6.14) hence lead to

$$\begin{aligned}
&e_{1,k}(\tau)\\
&= (I-\tfrac{\tau}{2}B)^{-1}(I-\tfrac{\tau}{2}A_{-1})^{-1}\Big[(I+\tfrac{\tau}{2}A_{-1})(I+\tfrac{\tau}{2}B) - (I-\tfrac{\tau}{2}A_{-1})(I-\tfrac{\tau}{2}B)\mathrm{e}^{\tau M}\Big]w(t_k)\\
&= (I-\tfrac{\tau}{2}B)^{-1}(I-\tfrac{\tau}{2}A_{-1})^{-1}\Big[I+\tfrac{\tau}{2}M+\tfrac{\tau^2}{4}A_{-1}B - (I-\tfrac{\tau}{2}M+\tfrac{\tau^2}{4}A_{-1}B)\mathrm{e}^{\tau M}\Big]w(t_k)\\
&= (I-\tfrac{\tau}{2}B)^{-1}(I-\tfrac{\tau}{2}A_{-1})^{-1}\Big[I-\mathrm{e}^{\tau M} + \tfrac{\tau}{2}M(I+\mathrm{e}^{\tau M}) + \tfrac{\tau^2}{4}A_{-1}B(I-\mathrm{e}^{\tau M})\Big]w(t_k)\\
&= (I-\tfrac{\tau}{2}B)^{-1}(I-\tfrac{\tau}{2}A_{-1})^{-1}\Big[-\tau M\Lambda_1(\tau) + \tau M + \tfrac{\tau^2}{2}M^2\Lambda_1(\tau)\\
&\qquad - \tfrac{\tau^3}{4}A_{-1}BM\Lambda_1(\tau)\Big]w(t_k)\\
&= (I-\tfrac{\tau}{2}B)^{-1}(I-\tfrac{\tau}{2}A_{-1})^{-1}\Big[-\tau^2M^2\Lambda_2(\tau) + \tfrac{\tau^2}{2}M^2\Lambda_1(\tau) - \tfrac{\tau^3}{4}A_{-1}BM\Lambda_1(\tau)\Big]w(t_k).
\end{aligned}$$

Combining Corollary 5.2, Remark 5.3 and (6.13), we conclude the estimates

$$\|e_{1,k}(\tau)\| + \|(I+\tfrac{\tau}{2}B)e_{1,k}(\tau)\| \le C\tau^2\|w(t_k)\|_{\mathscr{D}(M^2)\cap X_0}. \tag{6.17}$$

3) We next deal with $e_{2,k}(\tau)$. Note that $M\Lambda_2(\tau)(-\frac{1}{\varepsilon}\mathbf{J}(t),0)$ is contained in $\mathscr{D}(B)$ for every $t \in [0,T]$, as $W_T \hookrightarrow C([0,T],\mathscr{D}(M_1))$, $\Lambda_2(\tau)$ leaves $\mathscr{D}(M_1)$ invariant, and $X_1 \hookrightarrow \mathscr{D}(B)$, see Proposition 4.6. With (6.14), algebraic manipulations then lead to

$$e_{2,k}(\tau) = \tau^2(I - \tfrac{\tau}{2}B)^{-1}\big[\tfrac{1}{2}M - M\Lambda_2(\tau) + \tfrac{\tau}{2}BM\Lambda_2(\tau)\big](-\tfrac{1}{\varepsilon}\mathbf{J}(t_k),0).$$

Proposition 4.6 and (6.13) then imply

$$\|e_{2,k}(\tau)\| + \|(I + \tfrac{\tau}{2}B)e_{2,k}(\tau)\| \le C\tau^2\|(-\tfrac{1}{\varepsilon}\mathbf{J},0)\|_{W_T}. \tag{6.18}$$

4) To bound $e_{3,k}(\tau)$ and $R_k(\tau)$, we employ the embedding of X_1 into $\mathscr{D}(A) \cap \mathscr{D}(B)$, see Proposition 4.6, as well as the contractivity of $(\mathrm{e}^{tM})_{t\ge 0}$ in X_1, see Proposition 5.1. We then infer the inequalities

$$\|e_{3,k}(\tau)\| + \|(I + \tfrac{\tau}{2}B)e_{3,k}(\tau)\| \le C\tau\|(-\tfrac{1}{\varepsilon}\mathbf{J},0)\|_{W^{1,1}([t_k,t_{k+1}],X_1)}, \tag{6.19}$$

$$\|R_k(\tau)\| + \|(I + \tfrac{\tau}{2}B)R_k(\tau)\| \le C\tau\|(-\tfrac{1}{\varepsilon}\mathbf{J},0)\|_{W^{1,1}([t_k,t_{k+1}],X_1)}. \tag{6.20}$$

5) The stated bound on the global error is now obtained in the standard way from the above results for the local error and the stability of scheme (6.3). Using the Lady Winderemere's fan argument, we first derive the global error formula

$$w_n - w(t_n) = \sum_{k=0}^{n-1}\Big[(I - \tfrac{\tau}{2}B)^{-1}(I + \tfrac{\tau}{2}A)(I - \tfrac{\tau}{2}A)^{-1}(I + \tfrac{\tau}{2}B)\Big]^{n-1-k} \cdot \Big(\mathscr{T}_{\tau,k+1}w(t_k) - w(t_{k+1})\Big).$$

We next combine (6.16)–(6.20) with the stability statement in Lemma 6.2. Abbreviating $(I + \frac{\tau}{2}L)(I - \frac{\tau}{2}L)^{-1}$ by $\gamma_\tau(L)$ for $L \in \{A, B\}$, we conclude

$$\begin{aligned}\|w_n - w(t_n)\| &\le \sum_{k=0}^{n-2}\|(I - \tfrac{\tau}{2}B)^{-1}\|\|\big(\gamma_\tau(A)\gamma_\tau(B)\big)^{n-2-k}\gamma_\tau(A)\| \\ &\qquad \cdot \|(I + \tfrac{\tau}{2}B)(\mathscr{T}_{\tau,k+1}w(t_k) - w(t_{k+1}))\| + \|\mathscr{T}_{\tau,n}w(t_{n-1}) - w(t_n)\| \\ &\le C\sum_{k=0}^{n-1}\big(\tau^2\|w(t_k)\|_{\mathscr{D}(M^2)\cap X_0} + \tau^2\|(-\tfrac{1}{\varepsilon}\mathbf{J},0)\|_{W_T} + \tau\|(-\tfrac{1}{\varepsilon}\mathbf{J},0)\|_{W^{1,1}([t_k,t_{k+1}],X_1)}\big) \\ &\le C\tau T\big(\|w_0\|_{\mathscr{D}(M_1)} + \|(-\tfrac{1}{\varepsilon}\mathbf{J},0)\|_{W_T}\big).\end{aligned}$$

For the last estimate we employ Corollary 5.2 and the relation $n\tau \le T$. □

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