

Strong norm error bounds for quasilinear wave equations under weak CFL-type conditions

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CRC Preprint 2022/52, October 2022

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CRC 1173



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ABSTRACT. In the present paper we consider a class of quasilinear wave equations on a smooth, bounded domain. We discretize it in space with isoparametric finite elements, and apply a semi-implicit Euler and midpoint rule as well as the exponential Euler method to obtain three fully discrete schemes. We derive rigorous error bounds of optimal order for the semi-discretization in space and the fully discrete methods in norms which are stronger than the classical $H^1 \times L^2$ energy norm under weak CFL-type conditions.

1. INTRODUCTION

In the present paper we consider the quasilinear wave equation

$$(1.1) \quad \lambda(u(t, x)) \partial_{tt} u(t, x) = \Delta u(t, x) + g(t, x, u(t, x), \partial_t u(t, x)),$$

for $t \in [0, T]$, $x \in \Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$. We assume the domain Ω to be bounded with a sufficiently regular boundary, and impose homogeneous Dirichlet boundary conditions. We discretize (1.1) in space using isoparametric finite elements, and employ for the time discretization a semi-implicit Euler and midpoint rule as well as an exponential Euler method. We derive error bounds in norms stronger than the standard energy $H^1 \times L^2$ -norm.

The first wellposedness results for a large class of quasilinear wave type equation was given by Kato in [23, 24]. This approach was refined in [9] for the problem (1.1) to account for the state-dependent norms necessary in the analysis. A typical example in nonlinear acoustics is the model $\lambda(u) = 1 - u^m$ for some $m \in \mathbb{N}$. Hence, in order to ensure $\lambda(u) > 0$, a key ingredient in the proof is to establish pointwise bounds on u (as well as $\partial_t u$), often via Sobolev's embedding $H^2 \hookrightarrow L^\infty$. To inherit this property in the spatial discretization, we need pointwise bounds on the numerical approximations in the error analysis. However, since the finite element space is not H^2 -conforming, we cannot follow the above approach.

So far in the literature, bounds in $H^1 \times L^2$ are shown by inverse estimates which yield a factor $h^{-\beta}$ for some $\beta \geq 1$ with the spatial mesh width h . This induces unsatisfactory CFL-type conditions and excludes linear finite elements. In contrast to this, we adapt the idea from the wellposedness and perform the error analysis not in the energy space $H^1 \times L^2$, but employ a discrete version of the H^2 -norm. A discrete variant of Sobolev's embedding and a suitably defined solution space for

1991 *Mathematics Subject Classification.* Primary: 65M12, 65M15, 65J15, Secondary: 65M60, 35L05.

Key words and phrases. error analysis, full discretization, quasilinear wave equation, nonconforming space discretization, isoparametric finite elements, a-priori error bounds.

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

the numerical approximation allow us to remove lower bounds on the polynomial degree of the finite element space and significantly improve the CFL-type condition compared to the literature. For the temporal step size τ and the spatial mesh width h , we show convergence in $N = 2$ under the restriction $\tau \lesssim h^\alpha$, for any $\alpha > 0$, and in $N = 3$ we have $\tau \lesssim h^{1/2+\alpha}$ for the first-order methods in time and $\tau \lesssim h^{1/4+\alpha}$ for the second-order method. In addition, we fully remove the CFL-type condition for $N = 1$.

The strategy of the semi discrete proof relies on a bootstrap argument. We set up a suitable solution space for the numerical approximation, and show that the initial value lies in this. Instead of the usual choice of interpolated initial values, we have to use a Ritz map for which we provide a computable alternative of the correct order. Since we are working with a finite dimensional subspace, this directly yields local wellposedness up to some time $t_h^* > 0$. On this possibly short time interval, we prove convergence in the stronger norm, and use this to extend t_h^* beyond T and to close the argument. For the fully discrete error bounds, this approach is generalized using an induction argument.

We give a brief overview of the literature on the numerical treatment of quasilinear wave equations. In the pioneering works [8, 22, 25, 33], existence of solutions to quasilinear and nonlinear evolution equations is established, and one can find approximation rates of the implicit and semi-implicit Euler method. Within an (extended) Kato framework, optimal order for these methods was achieved in [20] and rigorous error bounds for the time discretization by higher-order Runge-Kutta methods are derived in [21, 26].

Concerning the spatial discretization, the results in [19] yields optimal order of convergence for the equation (1.1), however only for polynomials of degree greater than two. For the strongly damped Westervelt equation, continuous and discontinuous Galerkin methods were analyzed in [1, 31]. Very recently, mixed finite elements for the Kuznetsov and Westervelt equations were studied in [30].

In [28], error bounds for two variant of the midpoint rule are derived of optimal order, but only for polynomials of degree greater than two and under a stronger CFL-type condition compared to our results. In the case of one-dimensional wave equation subject to periodic boundary conditions, full discretization error bounds are established in [17]. A sophisticated energy technique combined with the properties of the spectral discretization yields convergence without a CFL-type condition.

For a slightly different quasilinear wave equation, optimal error bounds in L^2 for continuous finite elements were considered in the literature. One-step methods of different order are analyzed in [2, 3, 15], and two-step methods are considered in [4]. For a class of linearly implicit single-step schemes as well as a linearly and a fully implicit two-step scheme, optimal error bounds are derived in [29]. However, all of these results require a CFL-type condition at least as strong as $\tau \lesssim h$, and do not allow for linear finite elements. We expect that our technique can be generalized to these problems, but this will be part of future research.

The paper is organized as follows: We describe in [Section 2](#) the analytical framework and the space discretization by isoparametric Lagrange finite element, present the schemes and state our main results. The proof of the spatial convergence rates is given in [Section 3](#), where we first reduce the main result to error bounds in a stronger energy norm which is established afterwards. In [Section 4](#), we extend this

technique to the fully discrete case for the three presented methods. Certain stability estimates and the bounds on the defects are given in [Section 5](#), and some postponed results are shown in [Appendices A to C](#).

Notation. In the rest of the paper we use the notation

$$a \lesssim b,$$

if there is a constant $C > 0$ independent of the spatial parameter h and the time step size τ such that $a \leq Cb$. For the sake of readability, we introduce the notation $t^n = n\tau$ and

$$x^n := x(t^n)$$

for an arbitrary time-dependent, continuous object $x(t)$. If it is clear from the context, we write L^p instead of $L^p(\Omega)$ or $L^p(\Omega_h)$.

2. GENERAL SETTING

For a bounded domain $\Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$, with boundary $\partial\Omega \in C^{s,1}$, $s \in \mathbb{N}$, we study the quasilinear wave equation [\(1.1\)](#) with homogeneous Dirichlet boundary conditions, and initial values

$$u(0) = u^0, \quad \partial_t u(0) = v^0.$$

We note that the operator $-\Delta$ is positive and self-adjoint on $L^2(\Omega)$, and we define the spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Throughout the paper we impose the following conditions on the function λ and g . Additional requirements are stated in our main results.

Assumption 2.1. (λ_1) The function $\lambda: \mathbb{R} \rightarrow \mathbb{R}$ satisfies $\lambda \in C^2(\mathbb{R}, \mathbb{R})$.

(λ_2) There is some radius $\hat{r}_\infty > 0$ such that there is a constant $c_\lambda = c_\lambda(\hat{r}_\infty) > 0$ such that

$$c_\lambda \leq \lambda(x), \quad |x| \leq \hat{r}_\infty.$$

(g_1) The function $g: [0, T] \times \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies $g \in C^2([0, T] \times \bar{\Omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

(g_2) For $x \in \partial\Omega$ and $y = z = 0$ it holds $g(t, x, y, z) = 0$.

The conditions (λ_1) , (g_1) are structural assumptions which allow us to show crucial stability estimates. The lower bound in (λ_2) prevents the degeneracy of [\(1.1\)](#). The main effort in the discretization and error analysis is to ensure that this condition is inherited. We note that condition (g_2) implies in particular that for $u, v \in V$ one has $g(t, u, v) \in V$, and that all conditions are already required for the wellposedness. We recall an example for the quasilinear problem [\(1.1\)](#) given in [\[9\]](#).

Example 2.2. Let $K \in C^4(\mathbb{R}, \mathbb{R})$ with $1 + K'(0) > 0$ and consider the problem

$$\partial_{tt}(u + K(u)) = \Delta u,$$

for example with the Kerr model $K(z) = \alpha z^3$ for $\alpha \in \mathbb{R}$. If we rewrite it in the form [\(1.1\)](#), we obtain

$$\lambda(z) = 1 + K'(z), \quad g(t, x, u, v) = -K''(u)v^2,$$

which satisfy [Assumption 2.1](#). Denoting the fractional powers of the Laplacian by $\mathcal{H}_k := \mathcal{D}((-\Delta)^{k/2})$, under suitable smallness assumptions on the initial values, the existence of a solution

$$u \in C([0, T], \mathcal{H}_3) \cap C^1([0, T], \mathcal{H}_2) \cap C^2([0, T], \mathcal{H}_1)$$

is shown in [9, Thm. 4.1]. \diamond

Equivalently to (1.1), we consider the quasilinear wave equation in first-order formulation

$$(2.1) \quad \Lambda(y(t))\partial_t y(t) = Ay(t) + G(t, y(t)), \quad t \in [0, T], \quad y = \begin{pmatrix} u \\ \partial_t u \end{pmatrix},$$

with initial value $y(0) = y^0$ in the product space $X = V \times H$, and

$$y^0 = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \quad \Lambda(y) = \begin{pmatrix} \text{Id} & 0 \\ 0 & \lambda(u) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \text{Id} \\ \Delta & 0 \end{pmatrix}, \quad G(t, y) = \begin{pmatrix} 0 \\ g(t, u, \partial_t u) \end{pmatrix}.$$

Remark 2.3. The assumption on the regularity of the boundary is not essential in the error analysis, which also works on a convex, polygonal domain. Hence, one could apply a conforming finite element method. However, since the wellposedness of quasilinear equations requires a regular boundary, we will work in the nonconforming framework in the following.

Space discretization. We study the nonconforming space discretization of (2.1) based on isoparametric finite elements. For further details on this approach, we refer to [13, 14]. In particular, we introduce a shape-regular and quasi-uniform triangulation \mathcal{T}_h , consisting of isoparametric elements of degree $k \in \mathbb{N}$ and let $\partial\Omega \in C^{k+1,1}$. The computational domain Ω_h is given by

$$\Omega_h = \bigcup_{K \in \mathcal{T}_h} K \approx \Omega,$$

where the subscript h denotes the maximal diameter of all elements $K \in \mathcal{T}_h$. Based on the transformations F_K mapping the reference element \hat{K} to $K \in \mathcal{T}_h$, we introduce the finite element space of degree k

$$W_h = \{\varphi \in C_0(\bar{\Omega}_h) \mid \varphi|_K = \hat{\varphi} \circ (F_K)^{-1} \text{ with } \hat{\varphi} \in \mathcal{P}^k(\hat{K}) \text{ for all } K \in \mathcal{T}_h\}.$$

Here, $\mathcal{P}^k(\hat{K})$ consists of all polynomials on \hat{K} of degree at most k . The discrete approximation spaces are given by

$$H_h = (W_h, (\cdot | \cdot)_{L^2(\Omega_h)}), \quad V_h = (W_h, (\cdot | \cdot)_{H_0^1(\Omega_h)}),$$

and we set $X_h = V_h \times H_h$.

Following the detailed construction in [14, Sec. 5], we introduce the lift operator $\mathcal{L}_h: H_h \rightarrow H$. In particular, for $p \in [1, \infty]$ there are constants $c_p, C_p > 0$ with

$$(2.2a) \quad c_p \|\varphi_h\|_{L^p(\Omega_h)} \leq \|\mathcal{L}_h \varphi_h\|_{L^p(\Omega)} \leq C_p \|\varphi_h\|_{L^p(\Omega_h)}, \quad \varphi_h \in L^p(\Omega_h),$$

$$(2.2b) \quad c_p \|\varphi_h\|_{W^{1,p}(\Omega_h)} \leq \|\mathcal{L}_h \varphi_h\|_{W^{1,p}(\Omega)} \leq C_p \|\varphi_h\|_{W^{1,p}(\Omega_h)}, \quad \varphi_h \in W^{1,p}(\Omega_h),$$

cf. [14, Prop. 5.8]. By construction, the boundary nodes of Ω_h lie on $\partial\Omega$ and zero boundary conditions are preserved by \mathcal{L}_h , see [14, Sec. 8.5]. Further by [13, Sec. 4], the lift preserves values at the nodes, i.e., in particular

$$(2.3) \quad I_h \mathcal{L}_h \varphi_h = \varphi_h, \quad \varphi_h \in V_h,$$

where we denote the nodal interpolation operator by $I_h: C_0(\Omega) \rightarrow V_h$ and, enriching the space W_h by basis functions corresponding to the boundary nodes, its extension

$I_h^e: C(\Omega) \rightarrow C(\Omega_h)$. Further, we define the adjoint lift operators $\mathcal{L}_h^{H*}: H \rightarrow H_h$ and $\mathcal{L}_h^{V*}: V \rightarrow V_h$ by

$$(2.4a) \quad (\mathcal{L}_h^{H*} \varphi | \psi_h)_{H_h} = (\varphi | \mathcal{L}_h \psi_h)_H, \quad \varphi \in H, \psi_h \in H_h,$$

$$(2.4b) \quad (\mathcal{L}_h^{V*} \varphi | \psi_h)_{V_h} = (\varphi | \mathcal{L}_h \psi_h)_V, \quad \varphi \in V, \psi_h \in V_h.$$

We note that in the conforming case \mathcal{L}_h^{H*} and \mathcal{L}_h^{V*} coincide with the L^2 - and the Ritz projection, respectively.

For $u_h, v_h \in V_h$ we define the discrete operator $\lambda_h(u_h): H_h \rightarrow H_h$ and the discrete right-hand side g_h by

$$(2.5) \quad \lambda_h(u_h) \varphi_h = I_h(\lambda(\mathcal{L}_h u_h) \mathcal{L}_h \varphi_h), \quad g_h(t, u_h, v_h) = I_h g(t, \mathcal{L}_h u_h, \mathcal{L}_h v_h),$$

respectively, and its first-order counterparts

$$(2.6) \quad y_h = \begin{pmatrix} u_h \\ v_h \end{pmatrix}, \quad \Lambda_h(y_h) = \begin{pmatrix} \text{Id} & 0 \\ 0 & \lambda_h(u_h) \end{pmatrix}, \quad G_h(t, y_h) = \begin{pmatrix} 0 \\ g_h(t, u_h, v_h) \end{pmatrix}.$$

Moreover, as I_h is a nodal interpolation operator, the inverse operators satisfy

$$\lambda_h(u_h)^{-1} \varphi_h = I_h(\lambda(\mathcal{L}_h u_h)^{-1} \mathcal{L}_h \varphi_h), \quad \Lambda_h(y_h)^{-1} = \begin{pmatrix} \text{Id} & 0 \\ 0 & \lambda_h(u_h)^{-1} \end{pmatrix}.$$

Finally, we introduce the operators $\Delta_h: V_h \rightarrow H_h$ and $A_h: X_h \rightarrow X_h$ given by

$$(2.7) \quad -(\Delta_h \varphi_h | \psi_h)_{H_h} = (\varphi_h | \psi_h)_{V_h}, \quad A_h = \begin{pmatrix} 0 & \text{Id} \\ \Delta_h & 0 \end{pmatrix}, \quad \varphi_h, \psi_h \in V_h.$$

Note that Δ_h is symmetric and A_h is skew-symmetric with respect to H_h and X_h , respectively, but they are not uniformly bounded with respect to h . The spatially discrete quasilinear wave equation in first-order formulation then reads

$$(2.8) \quad \Lambda_h(y_h) \partial_t y_h(t) = A_h y_h(t) + G_h(t, y_h), \quad t \in [0, T],$$

with the initial value $y_h(0) = y_h^0$.

2.1. Choice of the initial value. As already mentioned, an appropriately chosen initial value is a key ingredient in the subsequent error analysis. An ideal initial value would include the adjoint lift operator \mathcal{L}_h^{V*} defined in (2.4b). However, in order to compute this operator, integrals over the exact domain Ω have to be evaluated.

We thus propose an alternative that involves to use a finite element space of degree $k' \geq k + 1$ denoted by \tilde{V}_h over the computational domain $\tilde{\Omega}_h$. Further, let $\tilde{\mathcal{L}}_h$ and \tilde{I}_h be the corresponding lift and interpolation operators. Then, for $u \in H^2$, we define the modified Ritz map $\tilde{R}_h u$ via

$$(2.9) \quad \left(\tilde{R}_h u | \varphi_h \right)_{V_h} = \left(\tilde{I}_h u | \tilde{\mathcal{L}}_h^{-1} \mathcal{L}_h \varphi_h \right)_{\tilde{V}_h}, \quad \varphi_h \in V_h.$$

We use this operator together with the interpolation to define the initial value by

$$(2.10) \quad y_h^0 = \begin{pmatrix} u_h^0 \\ v_h^0 \end{pmatrix} = \begin{pmatrix} \tilde{R}_h u^0 \\ \tilde{I}_h v^0 \end{pmatrix}.$$

In [Appendix A](#), we prove the following approximation property and discuss the computation of \tilde{R}_h .

Proposition 2.4. *For $u^0 \in H^{k+2}(\Omega) \cap V$, the difference of the adjoint lift \mathcal{L}_h^{V*} defined in (2.4b) and \tilde{R}_h in (2.9) satisfies the bound*

$$\|\mathcal{L}_h^{V*}u^0 - \tilde{R}_hu^0\|_{H^1(\Omega_h)} + h\|\Delta_h(\mathcal{L}_h^{V*}u^0 - \tilde{R}_hu^0)\|_{L^2(\Omega_h)} \leq Ch^{k+1}\|u^0\|_{H^{k+2}},$$

where the constant C is independent of h .

We emphasize that the precise construction of the initial value is not important in the error analysis, but only the bounds obtained in Proposition 2.4. Hence, if we can compute the adjoint lift exactly, which is the Ritz projection in the conforming case, then one can also choose $u_h^0 = \mathcal{L}_h^{V*}u^0$. However, we cannot make the standard choice $u_h^0 = I_hu^0$, since this would imply the statement of Proposition 2.4 only with k instead of $k+1$.

2.2. Main result for the semi-discretization in space. Before we state our main error bounds we chose some exponent p^* , depending on the dimension $N = 1, 2, 3$, as

$$(2.11) \quad N < p^* \begin{cases} \leq \infty, & N = 1, \\ < \infty, & N = 2, \\ < 6, & N = 3. \end{cases}$$

This choice in particular implies the Sobolev embeddings

$$(2.12) \quad H^1 \hookrightarrow L^{p^*} \quad \text{and} \quad H^2 \hookrightarrow W^{1,p^*} \hookrightarrow L^\infty.$$

Our first main result gives an error bound on the spatially discrete solution defined in (2.8), and the proof is given in Section 3. Recall the fractional powers of the Dirichlet Laplacian $\mathcal{H}_k := \mathcal{D}((-\Delta)^{k/2})$.

Theorem 2.5. *Let $\partial\Omega \in C^{k+1,1}$, and Assumption 2.1 hold. Further, let the solution u satisfy*

$$(2.13) \quad \begin{aligned} u &\in C([0, T], \mathcal{H}_3 \cap H^{k+3}(\Omega)) \cap C^2([0, T], V \cap W^{k+1, \infty}(\Omega)), \\ \lambda(u) &\in C([0, T], W^{k+1, \infty}(\Omega)), \quad g(t, u, \partial_t u) \in C([0, T], H^{k+1}(\Omega)), \end{aligned}$$

and choose the initial value (2.10). Then, there is $h_0 > 0$ such that for all $h \leq h_0$, it holds for $t \in [0, T]$

$$\|u(t) - \mathcal{L}_hu_h(t)\|_{W^{1,p^*}(\Omega)} + \|\partial_t u(t) - \mathcal{L}_hv_h(t)\|_{H^1(\Omega)} \leq Ch^k,$$

with a constant $C > 0$ which is independent of h .

Using (2.12), the theorem implies convergence in the maximum norm for u_h and in L^{p^*} for v_h , and is in particular applicable to linear finite elements. We note that the results from the literature so far had the limitation $k \geq 2$.

2.3. Main results for full discretization. We further discuss the convergence of three different fully discrete schemes. We recall that by $\tau > 0$ we denote the time step size and define for $n = 0, \dots, N$ the times $t^n = n\tau$, with $T = N\tau$. The proofs of the convergence results are given in Section 4.

Semi-implicit Euler method. For a variant of the implicit Euler method, we introduce the discrete derivative

$$(2.14) \quad \partial_\tau a_n := \frac{1}{\tau}(a^n - a^{n-1}), \quad n \geq 1, \quad \partial_\tau a_0 := a_0,$$

and consider as in [8, 20] the semi-implicit Euler method

$$(2.15) \quad \Lambda_h(y_h^n) \partial_\tau y_h^{n+1} = \Lambda_h y_h^{n+1} + G_h(t^n, y_h^n), \quad n \geq 0,$$

by freezing the nonlinear parts at the numerical approximation in the last step. The computation of the next approximation thus only requires the solution of a linear system. For the analysis we impose the following weak CFL-type condition

$$(2.16) \quad \tau \leq ch^{N/p^* + \varepsilon_0}$$

with p^* from (2.11) and some arbitrary $\varepsilon_0 > 0$. This yields the following convergence result.

Theorem 2.6. *Let $\partial\Omega \in C^{k+1,1}$, and Assumption 2.1 hold. Further, let the solution u in addition to (2.13) satisfy*

$$u \in C^3([0, T], L^2(\Omega)),$$

and choose the initial value (2.10). Then, under the condition (2.16) there are $h_0, \tau_0 > 0$ such that for all $h \leq h_0$ and $\tau \leq \tau_0$, it holds for $0 \leq t^n \leq T$

$$\|u(t^n) - \mathcal{L}_h u_h^n\|_{W^{1,p^*}(\Omega)} + \|\partial_t u(t^n) - \mathcal{L}_h v_h^n\|_{H^1(\Omega)} \leq C(\tau + h^k),$$

with a constant $C > 0$ which is independent of h and τ .

We emphasize that the CFL-type condition in (2.16) is essentially no restriction for $N = 2$ since p^* can be chosen arbitrarily large due to (2.11). For $N = 3$, the CFL roughly yields $\tau \lesssim h^{1/2+\varepsilon}$. However, even for $k = 1$, the error behaves as $\tau + h$, and one would choose $\tau \sim h$ anyway.

Semi-implicit midpoint rule. As a second-order in time method, we consider a variant of the midpoint rule proposed in [26]

$$(2.17a) \quad \Lambda_h(\bar{y}_h^{n+1/2}) \partial_\tau y^{n+1} = \Lambda_h y^{n+1/2} + G_h(t^n, \bar{y}_h^{n+1/2}), \quad n \geq 1,$$

with average $y_h^{n+1/2}$ and extrapolation $\bar{y}_h^{n+1/2}$ given by

$$(2.17b) \quad y_h^{n+1/2} = \frac{1}{2}(y_h^{n+1} + y_h^n), \quad \bar{y}_h^{n+1/2} = \frac{3}{2}y_h^n - \frac{1}{2}y_h^{n-1}.$$

The first approximation y^1 is computed with the Euler method (2.15), and as before, in every time step only a linear system has to be solved. For the analysis of the second-order method, we can weaken the CFL-type condition compared to (2.16) and require only

$$(2.18) \quad \tau \leq ch^{N/2p^* + \varepsilon_0}.$$

Theorem 2.7. *Let $\partial\Omega \in C^{k+1,1}$, and Assumption 2.1 hold. Further, let the solution u in addition to (2.13) satisfy*

$$u \in C^2([0, T], \mathcal{H}_3) \cap C^3([0, T], \mathcal{H}_2) \cap C^4([0, T], L^2(\Omega)),$$

and choose the initial value (2.10). Then, under the condition (2.18) there are $h_0, \tau_0 > 0$ such that for all $h \leq h_0$ and $\tau \leq \tau_0$, it holds for $0 \leq t^n \leq T$

$$\|u(t^n) - \mathcal{L}_h u_h^n\|_{W^{1,p^*}(\Omega)} + \|\partial_t u(t^n) - \mathcal{L}_h v_h^n\|_{H^1(\Omega)} \leq C(\tau^2 + h^k),$$

where C is independent of h .

Since, there is again essentially no CFL-type condition for $N = 2$, we only discuss the case $N = 3$. We require $\tau \lesssim h^{1/4+\varepsilon}$, whereas in [28] and [29] not only $k \geq 2$ but also

$$\tau \lesssim h^{3/4+\varepsilon} \quad \text{or} \quad \tau \lesssim h^{3/2+\varepsilon},$$

respectively, has to be imposed.

Exponential Euler method. We turn to the exponential method which employs the variation-of-constants formula and an exact evaluation of the matrix exponential applied to a vector. For the approximation $y_h^n \approx y(t^n)$, we use the shorthand notation $\mathbf{A}_h^n = \Lambda_h(y_h^n)^{-1} \mathbf{A}_h$ and consider the method which was proposed in [10]

$$\begin{aligned} y_h^{n+1} &= e^{\tau \mathbf{A}_h^n} y_h^n + \tau \varphi_1(\tau \mathbf{A}_h^n) \Lambda_h^{-1}(y_h^n) G_h(t^n, y_h^n) \\ &= y_h^n + \tau \varphi_1(\tau \mathbf{A}_h^n) (\mathbf{A}_h^n y_h^n + \Lambda_h^{-1}(y_h^n) G_h(t^n, y_h^n)), \end{aligned}$$

with the analytic function $\varphi_1(z) = \int_0^1 e^{sz} ds$. We obtain the following error bound.

Theorem 2.8. *Let $\partial\Omega \in C^{k+1,1}$, and Assumption 2.1 hold. Further, let the solution u satisfy (2.13), and choose the initial value (2.10). Then, under the condition (2.16) there are $h_0, \tau_0 > 0$ such that for all $h \leq h_0$ and $\tau \leq \tau_0$, it holds for $0 \leq t^n \leq T$*

$$\|u(t^n) - \mathcal{L}_h u_h^n\|_{W^{1,p^*}(\Omega)} + \|\partial_t u(t^n) - \mathcal{L}_h v_h^n\|_{H^1(\Omega)} \leq C(\tau + h^k),$$

where C is independent of h and τ .

We note that the CFL-type condition is the same as in the error bound of the semi-implicit Euler in Theorem 2.6. A natural question is to ask if one can generalize these results to a second-order exponential integrator, for example the exponential midpoint rule proposed in [10]. Using the notation in (2.17b), we define

$$\mathbf{A}_h^{n+1/2} := \Lambda_h^{-1}(\bar{y}_h^{n+1/2}) \mathbf{A}_h$$

and consider the scheme

$$\begin{aligned} y_h^{n+1} &= e^{\tau \mathbf{A}_h^{n+1/2}} y_h^n + \tau \varphi(\tau \mathbf{A}_h^{n+1/2}) \Lambda_h^{-1}(\bar{y}_h^{n+1/2}) G_h(t^{n+1/2}, \bar{y}_h^{n+1/2}) \\ &= y_h^n + \tau \varphi(\tau \mathbf{A}_h^{n+1/2}) (\mathbf{A}_h^{n+1/2} y_h^n + \Lambda_h^{-1}(\bar{y}_h^{n+1/2}) G_h(t^{n+1/2}, \bar{y}_h^{n+1/2})). \end{aligned}$$

We expect that the techniques established for the proofs of Theorem 2.6 and Theorem 2.8, combined with the techniques in [10], allow for a convergence result as in Theorem 2.7 under the weaker CFL-type condition (2.18). However, we will not investigate this in the present paper.

2.4. Additional results for isoparametric finite elements. In this section, we provide further estimates on the spatially discrete objects which are used throughout the paper.

As shown in [14, Thm. 5.9], we have for the nodal interpolation operator for $m \in \{0, 1\}$, $1 \leq p \leq \infty$, and $1 \leq \ell \leq k$ the estimates

$$(2.19) \quad \|(\text{Id} - \mathcal{L}_h I_h^e) \varphi\|_{W^{m,p}(\Omega)} \lesssim h^{\ell+1-m} \|\varphi\|_{W^{\ell+1,p}(\Omega)}, \quad \varphi \in W^{\ell+1,p}(\Omega).$$

Further, by [7, Thm. 3.1.6] $\ell = 0$ is allowed for $N < p \leq \infty$. Another crucial property of the interpolation concerns the stability when applied to the product of functions. We give a proof in Appendix B.

Lemma 2.9. *Let $\psi_h \in V_h$, $\delta > 0$, and $\varphi \in W^{1,N+\delta}(\Omega)$. Then,*

$$\begin{aligned} \|I_h(\varphi \mathcal{L}_h \psi_h)\|_{L^2} &\leq C \|\varphi\|_{W^{1,N+\delta}} \|\psi_h\|_{L^2}, \\ \|I_h(\varphi \mathcal{L}_h \psi_h)\|_{H^1} &\leq C \|\varphi\|_{W^{1,N+\delta}} \|\psi_h\|_{H^1}, \end{aligned}$$

where the constant $C > 0$ is independent of h .

Concerning the adjoint lifts defined in (2.4), we show in [Appendix B](#) the following bounds for $1 \leq \ell \leq k$

$$(2.20a) \quad \|\mathcal{L}_h^{H^*} \varphi\|_{H_h} \lesssim \|\varphi\|_{L^2(\Omega)}, \quad \varphi \in L^2(\Omega),$$

$$(2.20b) \quad \|(I_h - \mathcal{L}_h^{H^*})\varphi\|_{H_h} \lesssim h^{\ell+1} \|\varphi\|_{H^{\ell+1}(\Omega)}, \quad \varphi \in H^{\ell+1}(\Omega) \cap V.$$

An interpolation argument between [14, Lem. 3.8] and [11, Thm. 2.5], yields

$$(2.21) \quad \|(\text{Id} - \mathcal{L}_h \mathcal{L}_h^{V^*})\varphi\|_{W^{1,p}(\Omega)} \lesssim h^\ell \|\varphi\|_{W^{\ell+1,p}(\Omega)}, \quad \varphi \in H^{\ell+1}(\Omega) \cap V,$$

for $2 \leq p \leq \infty$, $0 \leq \ell \leq k$. We will further make use of the inverse estimates, cf. [6, Thm. 4.5.11] or [27, Lem. 5.6],

$$(2.22) \quad \|\varphi_h\|_{V_h} \leq Ch^{-1} \|\varphi_h\|_{L^2(\Omega_h)}, \quad \|\varphi_h\|_{L^q} \leq Ch^{N/q-N/p} \|\varphi_h\|_{L^p},$$

for $1 \leq p \leq q \leq \infty$.

Finally, we introduce the first-order lift operator $\mathcal{L}_h: W^{\ell,p}(\Omega_h)^2 \rightarrow W^{\ell,p}(\Omega)^2$, $\ell = 0, 1$, $2 \leq p \leq \infty$, the adjoint lift $\mathcal{L}_h^*: X \rightarrow X_h$, and the reference operator $J_h: V \times V \rightarrow X_h$ defined by

$$(2.23) \quad \mathcal{L}_h = \begin{pmatrix} \mathcal{L}_h & 0 \\ 0 & \mathcal{L}_h \end{pmatrix}, \quad \mathcal{L}_h^* = \begin{pmatrix} \mathcal{L}_h^{V^*} & 0 \\ 0 & \mathcal{L}_h^{H^*} \end{pmatrix}, \quad J_h = \begin{pmatrix} \mathcal{L}_h^{V^*} & 0 \\ 0 & \mathcal{L}_h^{V^*} \end{pmatrix},$$

which are bounded uniformly in h due to (2.2), (2.20), and (2.21). From the proof of [18, Lem. 4.7], we then obtain the identity

$$(2.24) \quad A_h J_h = \mathcal{L}_h^* A,$$

which is used several times in the proofs.

3. ERROR ANALYSIS FOR THE SPACE DISCRETIZATION

In this section, we give the proof of [Theorem 2.5](#). We decompose the error into

$$\begin{aligned} y(t) - \mathcal{L}_h y_h(t) &= (\text{Id} - \mathcal{L}_h J_h)y(t) + \mathcal{L}_h (J_h y(t) - y_h(t)) \\ &=: e_{J_h}(t) + \mathcal{L}_h e_h(t), \end{aligned}$$

where the projection error e_{J_h} is easily bounded using (2.21). The first part of the proof consists in reducing the bound on $\|e_h\|_{W^{1,p^*} \times H^1}$ to an estimate in the stronger norm induced by $\|A_h \cdot\|_{X_h}$, and not in the standard X_h -norm.

The second part consists in establishing the stronger norm bound on $\|A_h e_h\|_{X_h}$ in [Section 3.2](#). We note that a key idea is to set up an appropriate solution space for the numerical approximation, see (3.3) below, which allows for an appropriate formulation of the error equation. We give a detailed explanation in [Remark 3.6](#).

3.1. Reduction to stronger norm estimates. For p^* defined in (2.11), we chose some fixed $\delta > 0$ such that

$$(3.1) \quad \frac{1}{2} - \frac{1}{N + \delta} \leq \frac{1}{p^*},$$

a radius $r_\infty < \widehat{r}_\infty$ from Assumption 2.1, and another radius $r'_\infty > 0$, such that

$$(3.2) \quad \|u\|_{L^\infty(L^\infty)} \leq r_\infty, \quad \text{and} \quad \max\{\|u\|_{L^\infty(W^{1,N+\delta})}, \|\partial_t u\|_{L^\infty(W^{1,N+\delta})}\} \leq \frac{1}{2}r'_\infty,$$

where $\|x\|_{L^\infty(X)} := \max_{[0,T]} \|x(t)\|_X$. We denote by t_h^* the time with

$$(3.3) \quad t_h^* := \sup\{t \in [0, T] \mid \sup_{s \in [0,t]} \|\mathcal{L}_h u_h(s)\|_{L^\infty} \leq \widehat{r}_\infty \quad \text{and} \\ \sup_{s \in [0,t]} \|\mathcal{L}_h u_h(s)\|_{W^{1,N+\delta}}, \sup_{s \in [0,t]} \|\mathcal{L}_h v_h(s)\|_{W^{1,N+\delta}} \leq r'_\infty\}.$$

We assume for a moment that the set is not empty and hence $t_h^* > 0$, see Proposition 3.5. The following result is a direct consequence of Lemma 2.9 and the key ingredient to ensure wellposedness of the discrete equation. In addition, it enables us to employ energy techniques in the error analysis.

Lemma 3.1. *Let Assumption 2.1 hold. We have for $t \in [0, t_h^*]$, $1 \leq p \leq \infty$, and $j = 0, 1$ the bounds*

$$\|\partial_t^j \lambda_h(u_h(t)) \varphi_h\|_{L^p} \leq C_\lambda \|\varphi_h\|_{L^p}, \quad \|\partial_t^j \lambda_h(u_h(t))^{-1} \varphi_h\|_{L^p} \leq C_\lambda \|\varphi_h\|_{L^p},$$

$$\|\lambda_h(u_h(t)) \varphi_h\|_{H^1} \leq C_\lambda \|\varphi_h\|_{H^1}, \quad \|\lambda_h(u_h(t))^{-1} \varphi_h\|_{H^1} \leq C_\lambda \|\varphi_h\|_{H^1},$$

with a constant $C_\lambda > 0$ depending only on λ , its derivatives and $\widehat{r}_\infty, r'_\infty$, but is independent of h and t_h^* .

Proof. We use the definition of λ_h in (2.5), to conclude the assertion from Assumption 2.1, Lemma 2.9, and (3.3). \square

Making extensive use of Lemma 3.1, we show via energy techniques in Section 3.2 the following error bound on

$$(3.4) \quad \left(\|\Delta_h(\mathcal{L}_h^{V^*} u(t) - u_h(t))\|_{L^2}^2 + \|\mathcal{L}_h^{V^*} \partial_t u(t) - v_h(t)\|_{H^1}^2 \right)^{1/2} = \|A_h e_h(t)\|_{X_h}.$$

Note that initially the result is only valid as long as the bounds in (3.3) hold.

Proposition 3.2. *Under the assumptions of Theorem 2.5, it holds for $0 \leq t \leq t_h^*$*

$$\|A_h e_h(t)\|_{X_h} \leq Ch^k,$$

where C is independent of h and t_h^* .

From this bound, we are able to extract convergence as well as to extend the final time t_h^* beyond T for sufficiently small h . Concerning u_h , we show in the following lemma how to obtain convergence in the maximum norm and first-order Sobolev norms, but postpone the proof to Appendix C. Further, we may directly deduce the bounds on u_h in (3.3). Note that this lemma can be seen as a discrete analogue to (2.12), and is an improved variant of the results in [5, 12]. Similar bounds were already shown in [16, Thm. 1.12] and [32, Thm. 3].

Lemma 3.3. *Let p^* be given by (2.11). Then, there is a constant C independent of h such that*

$$\|\varphi_h\|_{L^\infty(\Omega_h)} + \|\varphi_h\|_{W^{1,p^*}(\Omega_h)} \leq C \|\Delta_h \varphi_h\|_{L^2(\Omega_h)}$$

for all $\varphi_h \in V_h$.

A further ingredient in the proof of the main result is to employ the H^1 -bound on v_h in Proposition 3.2 to derive boundedness in L^∞ and $W^{1,N+\delta}$, and thus extend the final time t_h^* .

Lemma 3.4. *Let $\varphi_h \in V_h$ and $\varphi \in W^{k+1,\infty}(\Omega) \cap V$, and assume that*

$$(3.5) \quad \|\mathcal{L}_h^{V^*} \varphi - \varphi_h\|_{H^1(\Omega_h)} \leq Ch^k.$$

Then, we have for p^* defined in (2.11) and δ chosen in (3.1)

$$\begin{aligned} \|\mathcal{L}_h \varphi_h\|_{L^\infty(\Omega)} &\leq \|\varphi\|_{L^\infty(\Omega)} + Ch^{k-N/p^*}, \\ \|\mathcal{L}_h \varphi_h\|_{W^{1,N+\delta}(\Omega)} &\leq \|\varphi\|_{W^{1,N+\delta}(\Omega)} + Ch^{k-N/p^*}, \end{aligned}$$

with a constant C independent of h .

Since $k \geq 1$, the choice in (2.11) enables us to deduce the desired bounds (3.3) in L^∞ and $W^{1,N+\delta}$ from approximation properties in H^1 , and hence allows us to extend the final time t_h^* .

Proof of Lemma 3.4. For $\psi_h \in V_h$ we combine the inverse estimate (2.22) and the Sobolev embedding $H^1(\Omega_h) \hookrightarrow L^{p^*}(\Omega_h)$, and conclude by (3.1)

$$(3.6a) \quad \|\psi_h\|_{L^\infty(\Omega_h)} \leq Ch^{-N/p^*} \|\psi_h\|_{L^{p^*}(\Omega_h)} \leq Ch^{-N/p^*} \|\psi_h\|_{H^1(\Omega_h)}$$

$$(3.6b) \quad \|\psi_h\|_{W^{1,N+\delta}(\Omega_h)} \leq Ch^{N/(N+\delta)-N/2} \|\psi_h\|_{H^1(\Omega_h)} \leq Ch^{-N/p^*} \|\psi_h\|_{H^1(\Omega_h)}$$

with a constant C independent of h . For the desired bound, we expand with the adjoint lift $\mathcal{L}_h^{V^*}$ and obtain by (2.21)

$$\begin{aligned} \|\mathcal{L}_h \varphi_h\|_{L^\infty(\Omega)} &\leq \|\varphi\|_{L^\infty(\Omega)} + \|\varphi - \mathcal{L}_h \mathcal{L}_h^{V^*} \varphi\|_{L^\infty(\Omega)} + \|\mathcal{L}_h \mathcal{L}_h^{V^*} \varphi - \mathcal{L}_h \varphi_h\|_{L^\infty(\Omega)} \\ &\leq \|\varphi\|_{L^\infty(\Omega)} + Ch^k \|\varphi\|_{W^{k+1,\infty}(\Omega)} + C \|\mathcal{L}_h^{V^*} \varphi - \varphi_h\|_{L^\infty(\Omega_h)}. \end{aligned}$$

Since $\mathcal{L}_h^{V^*} \varphi - \varphi_h \in V_h$, the first assertion then follows from (3.6a) together with (3.5). The second estimate is derived fully analogously. \square

Hence, once we have shown Proposition 3.2, we can give the proof of our first main result.

Proof of Theorem 2.5. Inserting the adjoint lift, we obtain for $t \in [0, t_h^*]$ with (2.21), (3.4), and Lemma 3.3 the bound

$$\|u(t) - \mathcal{L}_h u_h(t)\|_{W^{1,p^*}} \leq \|(\text{Id} - \mathcal{L}_h \mathcal{L}_h^{V^*})u(t)\|_{W^{1,p^*}} + C \|A_h e_h(t)\|_{X_h} \leq Ch^k,$$

and similarly

$$\|\partial_t u(t) - \mathcal{L}_h v_h(t)\|_{H^1} \leq \|(\text{Id} - \mathcal{L}_h \mathcal{L}_h^{V^*})\partial_t u(t)\|_{H^1} + C \|A_h e_h(t)\|_{X_h} \leq Ch^k,$$

with a constant C independent of h and t_h^* . Hence, it remains to show $t_h^* = T$. Combining the bounds in [Proposition 3.2](#) and [Lemma 3.3](#), we show by (3.2) for h sufficiently small that

$$\begin{aligned}\|\mathcal{L}_h u_h(t_h^*)\|_{L^\infty(\Omega)} &\leq \|u(t_h^*)\|_{L^\infty(\Omega)} + Ch^k < \widehat{r}_\infty, \\ \|\mathcal{L}_h u_h(t_h^*)\|_{W^{1,N+\delta}(\Omega)} &\leq \|u(t_h^*)\|_{W^{1,N+\delta}(\Omega)} + Ch^k < r'_\infty,\end{aligned}$$

as well as with [Proposition 3.2](#) and [Lemma 3.4](#)

$$\|\mathcal{L}_h v_h(t_h^*)\|_{W^{1,N+\delta}(\Omega)} \leq \|\partial_t u(t_h^*)\|_{W^{1,N+\delta}(\Omega)} + Ch^{k-N/p^*} < r'_\infty.$$

Thus, the continuity of the discrete solution y_h and the equivalence of all norms in finite dimensional spaces yields $t_h^* \geq T$. In particular, the statement of [Theorem 2.5](#) is true for $t \in [0, T]$. \square

3.2. Proof of [Proposition 3.2](#). The rest of this section is devoted to the proof of [Proposition 3.2](#). The first step is to show that the set defined in (3.3) is not empty.

Proposition 3.5. *The initial error satisfies*

$$\|A_h e_h(0)\|_{X_h} \leq Ch^k,$$

where C is independent of h . In particular, it holds $0 < t_h^* \leq T$.

Proof. The bound directly follows from the choice (2.10), the interpolation properties in (2.19), and the bounds in [Proposition 2.4](#). To show that $t_h^* > 0$, we proceed as in the proof of [Theorem 2.5](#) with $t = 0$ instead of $t = t_h^*$. \square

With the aid of [Lemma 3.1](#), we are able to define with $\Lambda_h(y_h)$ from (2.6) the state-dependent inner products

$$(\varphi_h | \psi_h)_{\Lambda_h, t} := (\Lambda_h(y_h(t))\varphi_h | \psi_h)_{X_h}, \quad t \in [0, t_h^*], \varphi_h, \psi_h \in X_h.$$

The corresponding norm is equivalent to the norm of X_h , i.e., we have

$$(3.7) \quad c_{\Lambda_h} \|\varphi_h\|_{X_h} \leq \|\varphi_h\|_{\Lambda_h, t} \leq C_{\Lambda_h} \|\varphi_h\|_{X_h}, \quad t \in [0, t_h^*], \varphi_h \in X_h,$$

with the constants from [Lemma 3.1](#).

Error equation. We study the bound on the discrete error e_h and derive an evolution equation for it. Inserting the projected solution $J_h y$ of (2.1) in (2.8), we obtain

$$\begin{aligned}\Lambda_h(y_h(t))J_h \partial_t y(t) &= A_h J_h y(t) + G_h(t, J_h y) + (\Lambda_h(y_h(t)) - \Lambda_h(J_h y(t)))J_h \partial_t y(t) \\ &\quad + \delta_h(t)\end{aligned}$$

with defect

$$(3.8) \quad \begin{aligned}\delta_h(t) &= (\Lambda_h(J_h y(t))J_h - J_h \Lambda(y(t)))\partial_t y(t) \\ &\quad + (J_h A - A_h J_h)y(t) + (J_h G(t, y) - G_h(t, J_h y)).\end{aligned}$$

This leads us to the error equation

$$(3.9) \quad \Lambda_h(y_h(t))\partial_t e_h(t) = A_h e_h(t) + \Gamma_h(t) + \delta_h(t),$$

where the stability term is given by

$$(3.10) \quad \Gamma_h(t) := (G_h(t, J_h y(t)) - G_h(t, y_h(t))) + (\Lambda_h(y_h(t)) - \Lambda_h(J_h y(t)))J_h \partial_t y(t).$$

Remark 3.6. Let us explain the main differences to the error analysis presented by Maier and Hochbruck [19, 28] and Makridakis [29]. In [19, 28], instead of $\Lambda_h(y_h)$ they use $\Lambda_h(I_h y)$ which has the properties from [Lemma 3.1](#). However, to bound the stability term, inverse inequalities are used which induce restrictions on the polynomial degree and also the CFL-type condition. Our technique is more related to [29], where bounds on $\|u_h\|_{W^{1,\infty}}$ replace (3.3).

However, in both approaches the error analysis is performed in $H^1 \times L^2$. They thus have to impose stronger CFL-type conditions to close the argument. \diamond

We introduce the state-dependent operator

$$\mathbf{A}_h(t) = \Lambda_h(y_h(t))^{-1} A_h$$

and define the modified error as

$$\tilde{e}_h(t) := \mathbf{A}_h(t) e_h(t).$$

Differentiating the term $\Lambda_h(y_h(t)) \tilde{e}_h(t)$ and using (3.9), leads to the following modified error equation

$$(3.11) \quad \begin{aligned} \Lambda_h(y_h(t)) \partial_t \tilde{e}_h(t) &= A_h \tilde{e}_h(t) - (\partial_t \Lambda_h(y_h(t))) \tilde{e}_h(t) \\ &\quad + A_h \Lambda_h(y_h(t))^{-1} (\Gamma_h(t) + \delta_h(t)). \end{aligned}$$

We state two results on the stability term and the defect, and postpone their proofs to [Section 5](#).

Lemma 3.7. *For $0 \leq t \leq t_h^*$ it holds*

$$\|A_h \Gamma_h(t)\|_{X_h} \leq C \|A_h e_h(t)\|_{X_h}$$

with a constant C independent of h and t_h^* .

Similarly, we show the optimal consistency error of the defect in the stronger norm.

Lemma 3.8. *For $0 \leq t \leq t_h^*$ it holds*

$$\|A_h \delta_h(t)\|_{X_h} \leq C h^k$$

with a constant C independent of h and t_h^* .

In addition, we note that by (2.6) and [Lemma 3.1](#) there is a constant C independent of h and t_h^* such that for all $x_h \in X_h$ it holds

$$(3.12) \quad \|A_h \Lambda_h(y_h(t)) x_h\|_{X_h} \leq C \|A_h x_h\|_{X_h}, \quad 0 \leq t \leq t_h^*.$$

With these two lemmas and the bound on the initial error in [Proposition 3.5](#), we conclude the remaining estimate.

Proof of Proposition 3.2. We first compute

$$\partial_t \|\tilde{e}_h(t)\|_{\Lambda_h, t}^2 = ((\partial_t \Lambda_h(y_h(t))) \tilde{e}_h(t) \mid \tilde{e}_h(t))_{X_h} + 2 (\Lambda_h(y_h(t)) \partial_t \tilde{e}_h(t) \mid \tilde{e}_h(t))_{X_h}.$$

Inserting the error equation (3.11), we use the skew-symmetry of A_h and combine the bounds in (3.12) and [Lemmas 3.1, 3.7, and 3.8](#) to obtain

$$\partial_t \|\tilde{e}_h(t)\|_{\Lambda_h, t}^2 \leq C \|\tilde{e}_h(t)\|_{\Lambda_h, t}^2 + C h^{2k}.$$

The application of a Gronwall lemma together with [Proposition 3.5](#) and (3.7) then yields the assertion. \square

4. ERROR ANALYSIS FOR THE FULL DISCRETIZATION

We carry over the results and techniques established in the last section to the fully discrete schemes. We work with a discrete analogous of (3.3) given by a final time step n^* which allows us to perform the next time step to t^{n^*+1} , that is

$$(4.1) \quad n^* := \max \left\{ 0 \leq n \leq N-1 \mid \max_{k=0, \dots, n} \|\mathcal{L}_h u_h^k\|_{L^\infty} \leq \widehat{r}_\infty, \text{ and} \right. \\ \left. \max_{k=0, \dots, n} \max \left\{ \|\mathcal{L}_h u_h^k\|_{W^{1, N+\delta}}, \|\mathcal{L}_h v_h^k\|_{W^{1, N+\delta}}, \|\mathcal{L}_h \partial_\tau u_h^k\|_{W^{1, N+\delta}} \right\} \leq r'_\infty \right\}.$$

In particular, we will establish $n^* \geq N-1$. Note that by (2.14) formally, we have to show that $n^* \geq 1$ for the last term in (4.1), which can be interpreted as providing both the base cases $n = 0, 1$ in the induction. However, the case $n = 0$ is already covered by Proposition 3.2, such that the set in (4.1) is not empty and it holds $n^* \geq 0$.

Further note that similar to Lemma 3.1 we conclude from the bounds in (4.1) that for $0 \leq n \leq n^*$, $1 \leq p \leq \infty$, and $j = 0, 1$ it holds

$$(4.2) \quad \|\partial_\tau^j \lambda_h(u_h^n) \varphi_h\|_{L^p} \leq C_\lambda \|\varphi_h\|_{L^p}, \quad \|\partial_\tau^j \lambda_h(u_h^n)^{-1} \varphi_h\|_{L^p} \leq C_\lambda \|\varphi_h\|_{L^p},$$

and the bounds in Lemma 3.1 in the H^1 -norm remain valid.

Throughout this section, we employ several times the estimate from Lemma 3.3, and also a straightforward extension of Lemma 3.4 including the temporal convergence rate.

Lemma 4.1. *Let $\varphi_h \in V_h$ and $\varphi \in W^{k+1, \infty}(\Omega) \cap V$, and assume that for some $\ell \in \{1, 2\}$ it holds*

$$\|\mathcal{L}_h^{V*} \varphi - \varphi_h\|_{H^1(\Omega_h)} \leq C(\tau^\ell + h^k).$$

Then, we have

$$\|\mathcal{L}_h \varphi_h\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)} + Ch^{-N/p^*} (\tau^\ell + h^k), \\ \|\mathcal{L}_h \varphi_h\|_{W^{1, N+\delta}(\Omega)} \leq \|\varphi\|_{W^{1, N+\delta}(\Omega)} + Ch^{-N/p^*} (\tau^\ell + h^k),$$

with a constant C independent of h and τ .

4.1. Euler. First note that for the Euler method (2.15), we have by construction $\partial_\tau u_h^k = v_h^k$ such that it is sufficient to check the first three conditions. As above, we define the discrete error by $e_h^n = J_h y(t^n) - y_h^n$ and aim to show as in Proposition 3.2 the following bound.

Proposition 4.2. *Under the assumptions of Theorem 2.6, for $0 \leq n \leq n^* + 1$ it holds the bound*

$$\|\mathbf{A}_h e_h^n\|_{X_h} \leq C(\tau + h^k),$$

where C is independent of h , τ and n^* .

As in the spatially discrete case, this estimate allows us to immediately conclude our main result.

Proof of Theorem 2.6. We proceed along the lines of the proof of Theorem 2.5 to conclude the convergence up to t^{n^*+1} . In addition, Lemma 3.3 and the CFL-type condition (2.16) together with Lemma 4.1 for $\ell = 1$ further allow us to prove $n^* \geq N-1$ for h, τ sufficiently small, and the assertion is shown for all n . \square

The rest of this section is devoted to the proof of [Proposition 4.2](#). In order to derive the error equation, we insert the projected exact solution $J_h y$ of [\(2.1\)](#) in the scheme [\(2.15\)](#) and derive

$$\begin{aligned} \Lambda_h(y_h^n) J_h \partial_\tau y(t^{n+1}) &= \Lambda_h J_h y(t^{n+1}) + G_h(t^n, J_h y(t^n)) \\ &\quad + (\Lambda_h(y_h^n) - \Lambda_h(J_h y(t^n))) J_h \partial_\tau y(t^{n+1}) + \delta_{\text{Eu}}^{n+1}, \end{aligned}$$

with defect $\delta_{\text{Eu}}^{n+1} = \delta_{h,\text{Eu}}^{n+1} + \delta_{\tau,\text{Eu}}^{n+1}$ given by

$$(4.3a) \quad \delta_{h,\text{Eu}}^{n+1} = (J_h \mathbf{A} - \mathbf{A}_h J_h) y(t^{n+1}) + J_h G(t^n, y(t^n)) - G_h(t^n, J_h y(t^n)) \\ + (\Lambda_h(J_h y(t^n)) J_h - J_h \Lambda(y(t^n))) \partial_\tau y(t^{n+1}),$$

$$(4.3b) \quad \delta_{\tau,\text{Eu}}^{n+1} = J_h \Lambda(y(t^n)) \partial_\tau y(t^{n+1}) - J_h \Lambda(y(t^{n+1})) \partial_t y(t^{n+1}) \\ + J_h G(t^{n+1}, y(t^{n+1})) - J_h G(t^n, y(t^n)).$$

This yields the discrete error equation

$$(4.4) \quad \Lambda_h(y_h^n) \partial_\tau e_h^{n+1} = \mathbf{A}_h e_h^{n+1} + \Gamma_h^n + \delta_{\text{Eu}}^{n+1},$$

where the stability term is given by

$$(4.5) \quad \Gamma_h^n := (G_h(t^n, J_h y(t^n)) - G_h(t^n, y_h)) + (\Lambda_h(y_h^n) - \Lambda_h(J_h y(t^n))) J_h \partial_\tau y(t^{n+1}).$$

In order to obtain a recursion for e_h^{n+1} , we recall the state-dependent operator and define the corresponding resolvent

$$(4.6) \quad \mathbf{A}_h^n = \Lambda_h(y_h^n)^{-1} \mathbf{A}_h, \quad \mathbf{R}_{\text{Eu},n} := (I - \tau \mathbf{A}_h^n)^{-1}.$$

A simple calculation shows that for the inner product,

$$(\varphi_h | \psi_h)_n := (\Lambda_h(y_h^n) \varphi_h | \psi_h)_{X_h}, \quad \varphi_h, \psi_h \in X_h,$$

which satisfies by [\(4.1\)](#) the same bounds as in [\(3.7\)](#), we obtain

$$\|\mathbf{R}_{\text{Eu},n} \varphi_h\|_n \leq \|\varphi_h\|_n,$$

and rewrite [\(4.4\)](#) as

$$e_h^{n+1} = \mathbf{R}_{\text{Eu},n} e_h^n + \tau \mathbf{R}_{\text{Eu},n} \Lambda_h(y_h^n)^{-1} (\Gamma_h^n + \delta_{\text{Eu}}^{n+1}).$$

Since \mathbf{A}_h^n commutes with $\mathbf{R}_{\text{Eu},n}$, we obtain

$$\mathbf{A}_h^n e_h^{n+1} = \mathbf{R}_{\text{Eu},n} \mathbf{A}_h^n e_h^n + \tau \mathbf{R}_{\text{Eu},n} \mathbf{A}_h^n \Lambda_h(y_h^n)^{-1} (\Gamma_h^n + \delta_{\text{Eu}}^{n+1})$$

which has to be resolved. Proceeding as in [Lemma 3.7](#), and noting that for any norm it holds

$$(4.7) \quad \|\partial_\tau y(t^n)\| \leq \max_{t \in [t^{n-1}, t^n]} \|\partial_t y(t)\|,$$

we have for $0 \leq n \leq n^*$ the stability bound

$$\|\mathbf{A}_h \Gamma_h^n\|_{X_h} \leq C \|\mathbf{A}_h e_h^n\|_{X_h},$$

with a constant C independent of h , τ and n^* . Similarly, we show the optimal consistency error of the defect in the stronger norm, see [Section 5](#) for the proof.

Lemma 4.3. *For $0 \leq n \leq n^*$ it holds*

$$\|\mathbf{A}_h \delta_{\text{Eu}}^{n+1}\|_{X_h} \leq C(\tau + h^k)$$

with a constant C independent of h , τ and n^* .

Hence, we have already established the estimate

$$(4.8) \quad \|\mathbf{A}_h^n e_h^{n+1}\|_n \leq \|\mathbf{A}_h^n e_h^n\|_n + C\tau \|\mathbf{A}_h e_h^n\|_{X_h} + C\tau(\tau + h^k),$$

and the last step towards the main result is to change the norms.

Lemma 4.4. *For $1 \leq n \leq n^*$ it holds for all $\varphi_h \in V_h$*

$$\|\mathbf{A}_h^n \varphi_h\|_n \leq (1 + C\tau) \|\mathbf{A}_h^{n-1} \varphi_h\|_{n-1}$$

with a constant C independent of h , τ and n^* .

Proof. Expanding the norm as

$$\begin{aligned} \|\mathbf{A}_h^n \varphi_h\|_n^2 &= (\mathbf{A}_h \varphi_h \mid \Lambda_h (y_h^n)^{-1} \mathbf{A}_h \varphi_h)_{X_h} \\ &= \|\mathbf{A}_h^{n-1} \varphi_h\|_{n-1}^2 + \tau (\mathbf{A}_h \varphi_h \mid \partial_\tau \Lambda_h (y_h^n)^{-1} \mathbf{A}_h \varphi_h)_{X_h} \end{aligned}$$

and using (4.2) several times, gives the assertion. \square

With this we are able to proof the estimate on $\mathbf{A}_h e_h^{n+1}$.

Proof of Proposition 4.2. We first consider the case $n^* = 0$. Hence, (4.8) with $n = 0$ directly yields the assertion without the use of Lemma 4.4 and hence without any bound on $\partial_\tau u_h^k$. With this, we established $n^* \geq 1$.

In the case $n^* \geq 1$, we employ Lemma 4.4 in (4.8) and make use of the norm equivalences to obtain

$$\|\mathbf{A}_h^n e_h^{n+1}\|_n \leq (1 + C\tau) \|\mathbf{A}_h^{n-1} e_h^n\|_{n-1} + C\tau(\tau + h^k).$$

Resolving the recursion and using Proposition 3.5 yields the result. \square

4.2. Midpoint. The proof is very similar to the Euler method and hence, we only sketch the relevant details. First note that by construction in (2.17) it holds

$$(4.9) \quad \partial_\tau u_h^k = \frac{1}{2}(v_h^k + v_h^{k-1}),$$

such that the last bound in (4.1) does not have to be shown separately. Again, we aim at the following bound.

Proposition 4.5. *Under the assumptions of Theorem 2.7, for $0 \leq n \leq n^* + 1$ it holds the bound*

$$\|\mathbf{A}_h e_h^n\|_{X_h} \leq C(\tau^2 + h^k),$$

where C is independent of h , τ and n^* .

Combining Lemma 4.1 with the weaker CFL-type condition (2.18) yields the convergence result.

Proof of Theorem 2.7. As in the proof of Theorem 2.6, the convergence follows directly. To show that $n^* \geq N - 1$, we employ Lemma 4.1 with $\ell = 2$ together with the CFL-type condition (2.18). \square

Hence, it remains to show Proposition 4.5. As for the Euler method, we derive the following error equation

$$(4.10) \quad \Lambda_h (\bar{y}_h^{n+1/2}) \partial_\tau e_h^{n+1} = \mathbf{A}_h e_h^{n+1/2} + \Gamma_h^n + \delta_M^{n+1},$$

with a stability term similar to the one in (4.5) satisfying

$$(4.11) \quad \|\mathbf{A}_h \Gamma_h^n\|_{X_h} \leq C(\|\mathbf{A}_h e_h^n\|_{X_h} + \|\mathbf{A}_h e_h^{n-1}\|_{X_h}),$$

and a composed defect $\delta_M^{n+1} = \delta_{h,M}^{n+1} + \delta_{\tau,M}^{n+1}$. The first component is basically the same as $\delta_{h,\text{Eu}}^{n+1}$ in (4.3b), and the second satisfies

$$(4.12) \quad \begin{aligned} \delta_{\tau,M}^{n+1} &= J_h \Lambda(\bar{y}^{n+1/2}) \partial_\tau y(t^{n+1}) - J_h \Lambda(y(t^{n+1/2})) \partial_t y(t^{n+1/2}) \\ &\quad + J_h \mathbf{A}(y(t^{n+1/2}) - \frac{1}{2}(y(t^{n+1}) + y(t^n))), \\ &\quad + J_h G(t^{n+1/2}, y(t^{n+1/2})) - J_h G(t^{n+1/2}, \bar{y}^{n+1/2}), \end{aligned}$$

such that we derive in Section 5 the desired order of convergence.

Lemma 4.6. *For $0 \leq n \leq n^*$ it holds*

$$\|\mathbf{A}_h \delta_M^{n+1}\|_{X_h} \leq C(\tau^2 + h^k)$$

with a constant C independent of h , τ and n^* .

We solve for e_h^{n+1} in the error equation (4.10) and define for the solution-dependent operator $\mathbf{A}_h^{n+1/2} = \Lambda_h(\bar{y}^{n+1/2})^{-1} \mathbf{A}_h$ the maps

$$\mathcal{R}_{\pm, n+1/2} := I \pm \frac{\tau}{2} \mathbf{A}_h^{n+1/2}, \quad \mathbf{R}_{m, n+1/2} := \mathcal{R}_{-, n+1/2}^{-1} \mathcal{R}_{+, n+1/2}.$$

A simple calculation shows that for the inner product

$$(\varphi_n | \psi_h)_{n+1/2} := (\Lambda_h(\bar{y}^{n+1/2}) \varphi_n | \psi_h)_{X_h}, \quad \varphi_n, \psi_h \in X_h,$$

we have

$$\|\mathcal{R}_{-, n+1/2}^{-1} \varphi_h\|_{n+1/2} \leq \|\varphi_h\|_{n+1/2}, \quad \|\mathbf{R}_{m, n+1/2} \varphi_h\|_{n+1/2} = \|\varphi_h\|_{n+1/2}.$$

Rewriting (4.10) and multiplying by $\mathbf{A}_h^{n+1/2}$, we obtain

$$(4.13) \quad \begin{aligned} \mathbf{A}_h^{n+1/2} e_h^{n+1} &= \mathbf{R}_{m, n+1/2} \mathbf{A}_h^{n+1/2} e_h^n \\ &\quad + \tau \mathcal{R}_{-, n+1/2}^{-1} \mathbf{A}_h^{n+1/2} \Lambda_h(\bar{y}^{n+1/2})^{-1} (\Gamma_h^n + \delta_M^{n+1}). \end{aligned}$$

Finally, we have as in Lemma 4.4 the following bound when changing the norm.

Lemma 4.7. *For $1 \leq n \leq n^*$ it holds*

$$\left\| \mathbf{A}_h^{n+1/2} \varphi_h \right\|_{n+1/2} \leq (1 + C\tau) \left\| \mathbf{A}_h^{n-1/2} \varphi_h \right\|_{n-1/2}$$

with a constant C independent of h , τ and n^* .

Proof. First note that

$$\left\| \partial_\tau \left(\frac{3}{2} u_h^k - \frac{1}{2} u_h^{k-1} \right) \right\|_{L^\infty} \leq \frac{3}{2} \left\| \partial_\tau u_h^k \right\|_{L^\infty} + \frac{1}{2} \left\| \partial_\tau u_h^{k-1} \right\|_{L^\infty},$$

as well as by (4.9)

$$\left\| \frac{3}{2} u_h^k - \frac{1}{2} u_h^{k-1} \right\|_{L^\infty} = \left\| u_h^k + \frac{\tau}{4} (v_h^k + v_h^{k-1}) \right\|_{L^\infty}.$$

This allows us to proceed as in Lemma 4.4 and to bound $\partial_\tau^j \Lambda_h(\bar{y}^{n+1/2})$, $j = 0, 1$, and the inverse $\Lambda_h(\bar{y}^{n+1/2})^{-1}$. \square

We are then able to conclude the error bound for the midpoint rule.

Proof of Proposition 4.5. Using error equation (4.13), we employ Lemmas 4.6 and 4.7 and (4.11) to obtain

$$\begin{aligned} \|\mathbf{A}_h^{n+1/2} e_h^{n+1}\|_{n+1/2} &\leq (1 + C\tau) \|\mathbf{A}_h^{n-1/2} e_h^n\|_{n-1/2} \\ &\quad + C\tau (\|\mathbf{A}_h e_h^n\|_{X_h} + \|\mathbf{A}_h e_h^{n-1}\|_{X_h}) + C\tau(\tau^2 + h^k). \end{aligned}$$

With the bound on $\mathbf{A}_h e_h^0$ from Proposition 3.5 and using the fact the first step is given by the Euler method, we obtain with Proposition 4.2

$$\|\mathbf{A}_h e_h^1\|_{X_h} \leq C\tau(\tau + h^k),$$

which yields with a Gronwall lemma the assertion. \square

4.3. Exponential Euler. For the exponential method, we apply a similar approach and derive the necessary bound in the stronger energy norm. However, there is no direct relation to the discrete derivatives of the error. In this case, we have to proof an additional error estimate.

Proposition 4.8. *Under the assumptions of Theorem 2.8, for $0 \leq n \leq n^* + 1$ there hold the bounds*

$$\|\mathbf{A}_h e_h^n\|_{X_h} \leq C(\tau + h^k),$$

and for $1 \leq n \leq n^* + 1$

$$\|\partial_\tau e_h^n\|_{X_h} \leq C(\tau + h^k),$$

where C is independent of h , τ and n^* .

Once this is established, the last main result directly follows.

Proof of Theorem 2.8. In order to conclude the convergence rates, we only employ the first estimate in Proposition 4.8. To show $n^* \geq N - 1$, again the first estimate allows us to guarantee the first three bounds in (4.1). The bound on $\mathcal{L}_h \partial_\tau u_h^k$ follows from the second estimate in Proposition 4.8 combined with Lemma 4.1 and the CFL-type condition (2.16). \square

The rest of this section is devoted to the proof of Proposition 4.8. We introduce the auxiliary approximation $\tilde{y}^n(t^n + s)$ for $s \in [0, \tau]$ as the solution of

$$(4.14) \quad \Lambda_h(y_h^n) \partial_t \tilde{y}^n(t^n + s) = \mathbf{A}_h \tilde{y}^n(t^n + s) + G_h(t^n, y_h^n), \quad \tilde{y}^n(t^n) = y_h^n,$$

and thus satisfies $\tilde{y}^n(t^n + \tau) = y_h^{n+1}$. In order to derive the error equation, we insert the projected exact solution in (4.14)

$$\begin{aligned} \Lambda_h(y_h^n) J_h \partial_t y(t^n + s) &= \mathbf{A}_h J_h y(t^n + s) + G_h(t^n, J_h y(t^n)) \\ &\quad + (\Lambda_h(y_h^n) - \Lambda_h(J_h y(t^n))) J_h \partial_t y(t^n + s) + \delta_{\text{ExEu}}^{n+1}(t^n + s), \end{aligned}$$

with defect $\delta_{\text{ExEu}}^{n+1} = \delta_{h, \text{ExEu}}^{n+1} + \delta_{\tau, \text{ExEu}}^{n+1}$ given by

$$\begin{aligned} \delta_{h, \text{ExEu}}^{n+1}(t^n + s) &= (J_h \Lambda - \mathbf{A}_h J_h) y(t^n + s) + J_h G(t^n, y(t^n)) - G_h(t^n, J_h y(t^n)) \\ &\quad + (\Lambda_h(J_h y(t^n)) J_h - J_h \Lambda(y(t^n))) \partial_t y(t^n + s), \\ \delta_{\tau, \text{ExEu}}^{n+1}(t^n + s) &= J_h \Lambda(y(t^n)) \partial_t y(t^n + s) - J_h \Lambda(y(t^n + s)) \partial_t y(t^n + s) \\ &\quad + J_h G(t^n + s, y(t^n + s)) - J_h G(t^n, y(t^n)). \end{aligned}$$

Similarly, we define the auxiliary error by

$$\tilde{e}_h^n(t^n + s) := J_h y(t^n + s) - \tilde{y}^n(t^n + s), \quad \tilde{e}_h^n(t^n) = e_h^n, \quad \tilde{e}_h^n(t^n + \tau) = e_h^{n+1}.$$

This yields the discrete error equation for $s \in [0, \tau]$

$$(4.15) \quad \Lambda_h(y_h^n) \partial_t \tilde{e}_h^n(t^n + s) = A_h \tilde{e}_h^n(t^n + s) + \Gamma_h^n(t^n + s) + \delta_{\text{ExEu}}^{n+1}(t^n + s),$$

with stability term

$$\Gamma_h^n(t^n + s) := (G_h(t^n, J_h y(t^n)) - G_h(t^n, y_h)) + (\Lambda_h(y_h^n) - \Lambda_h(J_h y(t^n))) J_h \partial_t y(t^n + s).$$

Using the variation-of-constants formula with the state-dependent operator defined in (4.6), we obtain from (4.15)

$$\tilde{e}_h^n(t^n + s) = e^{sA_h^n} e_h^n + \int_0^s e^{(s-\sigma)A_h^n} \Lambda_h(y_h^n)^{-1} (\Gamma_h^n(t^n + \sigma) + \delta_{\text{ExEu}}^{n+1}(t^n + \sigma)) d\sigma.$$

To obtain the error bounds stated in [Proposition 4.8](#), we need the following two estimates which follow along the lines of [Lemmas 3.7](#) and [4.3](#): For $0 \leq n \leq n^*$ it holds

$$(4.16a) \quad \sup_{s \in [0, \tau]} \|A_h \Gamma_h^n(t^n + s)\|_{X_h} \leq C \|A_h e_h^n\|_{X_h},$$

$$(4.16b) \quad \sup_{s \in [0, \tau]} \|A_h \delta_{\text{ExEu}}^{n+1}(t^n + s)\|_{X_h} \leq C(\tau + h^k),$$

with a constant C independent of h , τ and n^* . This allows us to conclude the bounds in the two stronger norms.

Proof of [Proposition 4.8](#). We proceed as in the proof of [Proposition 4.2](#) in order to obtain the bound on $A_h \tilde{e}_h^n$ in the form

$$(4.17) \quad \sup_{s \in [0, \tau]} \|A_h \tilde{e}_h^n(t^n + s)\|_{X_h} \leq C(\tau + h^k),$$

which implies the first statement in the proposition. For the discrete derivative of the error we employ (4.7), (4.15), and (4.16) to conclude

$$\begin{aligned} \|\partial_\tau e_h^n\|_{X_h} &\leq \sup_{s \in [0, \tau]} \|\partial_t \tilde{e}_h^n(t^n + s)\|_{X_h} \\ &\leq C \sup_{s \in [0, \tau]} \|A_h \tilde{e}_h^n(t^n + s)\|_{X_h} + C \|A_h e_h^n\|_{X_h} + C(\tau + h^k) \\ &\leq C(\tau + h^k), \end{aligned}$$

where we used (4.17) in the last step. \square

5. ESTIMATES FOR STABILITY TERMS AND DEFECTS

This section is devoted to the proofs of the postponed stability and consistency estimate from [Sections 3](#) and [4](#).

5.1. Stability. In the following, we give a detailed proof for the stability term given in (3.10). We emphasize that the corresponding bounds used in Section 4, are derived fully analogously, and we thus refrain from giving the details here.

Proof of Lemma 3.7. We consider the two contributions of Γ_h in (3.10) separately.

(a) We first note by (2.5) and (2.6) that

$$A_h(G_h(t, J_h y) - G_h(t, y_h)) = \begin{pmatrix} I_h g(t, \mathcal{L}_h \mathcal{L}_h^{V*} u, \mathcal{L}_h \mathcal{L}_h^{V*} \partial_t u) - I_h g(t, \mathcal{L}_h u_h, \mathcal{L}_h v_h) \\ 0 \end{pmatrix}.$$

Without loss of generality, we show the assertion only for $g(t, u, \partial_t u) = g(\partial_t u)$, and obtain

$$\begin{aligned} & \|A_h(G_h(t, J_h y) - G_h(t, y_h))\|_{X_h} \\ &= \|I_h g(\mathcal{L}_h \mathcal{L}_h^{V*} \partial_t u) - I_h g(\mathcal{L}_h v_h)\|_{V_h} \\ &= \|I_h \left(\int_0^1 g'(\sigma \mathcal{L}_h \mathcal{L}_h^{V*} \partial_t u + (1 - \sigma) \mathcal{L}_h v_h) d\sigma (\mathcal{L}_h \mathcal{L}_h^{V*} \partial_t u - \mathcal{L}_h v_h) \right)\|_{V_h} \\ &\lesssim \left\| \int_0^1 g'(\sigma \mathcal{L}_h \mathcal{L}_h^{V*} \partial_t u + (1 - \sigma) \mathcal{L}_h v_h) d\sigma \right\|_{W^{1, N+\delta}} \|\mathcal{L}_h \mathcal{L}_h^{V*} \partial_t u - \mathcal{L}_h v_h\|_{V_h}, \end{aligned}$$

where we used Lemma 2.9 for the last estimate. The latter term is estimated with (3.4) by $A_h e_h$ in the X_h -norm. For the integral part, we use the stability of $\mathcal{L}_h \mathcal{L}_h^{V*}$ in (2.21) with $\ell = 0$ to bound it by a constant depending on the $W^{1, N+\delta}$ -norms of $\partial_t u$ and v_h . Hence, the bounds in (3.3) yield the stability for G_h .

(b) Next, we consider by (2.5) and (2.6)

$$A_h(\Lambda_h(J_h y(t)) - \Lambda_h(y_h(t)) J_h \partial_t y(t)) = \begin{pmatrix} I_h \left((\lambda(\mathcal{L}_h \mathcal{L}_h^{V*} u) - \lambda(\mathcal{L}_h u_h)) \mathcal{L}_h^{V*} \partial_t^2 u \right) \\ 0 \end{pmatrix},$$

and estimate with the stability of the interpolation (2.19) the algebra property of $W^{1, N+\delta}$, and the stability of \mathcal{L}_h^{V*} in (2.21)

$$\begin{aligned} & \|A_h(\Lambda_h(J_h y(t)) - \Lambda_h(y_h(t)) J_h \partial_t y(t))\|_{X_h} \\ &\leq C \|(\lambda(\mathcal{L}_h \mathcal{L}_h^{V*} u) - \lambda(\mathcal{L}_h u_h))\|_{W^{1, N+\delta}} \|\mathcal{L}_h^{V*} \partial_t^2 u\|_{W^{1, N+\delta}} \\ &\leq C \|\mathcal{L}_h^{V*} u - u_h\|_{W^{1, N+\delta}} \|\partial_t^2 u\|_{W^{1, N+\delta}}, \end{aligned}$$

with a constant depending on the $W^{1, N+\delta}$ -norms of u and u_h . Using Lemma 3.3 and (3.4), the first term is bounded by $A_h e_h$ in the X_h -norm. \square

5.2. Defects. We first estimate the spatial defect (3.8) which will reappear in a modified form in the defects of the full discretization.

Proof of Lemma 3.8. (a) We compute with (2.23) and (2.24)

$$A_h(J_h A - A_h J_h) y(t) = A_h(J_h - \mathcal{L}_h^*) A y(t) = \begin{pmatrix} (\mathcal{L}_h^{V*} - \mathcal{L}_h^{H*}) \Delta u \\ 0 \end{pmatrix}$$

and, inserting the interpolation, estimate with (2.19), (2.20), (2.21), and (2.22),

$$\|A_h(J_h A - A_h J_h) y(t)\|_{X_h} \leq Ch^k \|\Delta u\|_{H^{k+1}}.$$

(b) As above, we only consider the case $g(t, u, \partial_t u) = g(\partial_t u)$ and obtain with (2.5) and (2.6)

$$A_h(J_h G(t, y) - G_h(t, J_h y)) = \begin{pmatrix} \mathcal{L}_h^{V*} g(\partial_t u) - I_h g(\mathcal{L}_h \mathcal{L}_h^{V*} \partial_t u) \\ 0 \end{pmatrix}.$$

From this, we conclude with (2.19) and (2.21)

$$\begin{aligned} & \|A_h(J_h G(t, y) - G_h(t, J_h y))\|_{X_h} \\ & \leq \|(\mathcal{L}_h^{V*} - I_h)g(\partial_t u)\|_{V_h} + \|I_h g(\partial_t u) - I_h g(\mathcal{L}_h \mathcal{L}_h^{V*} \partial_t u)\|_{V_h} \\ & \leq Ch^k \|g(\partial_t u)\|_{H^{k+1}} + C \|(\text{Id} - \mathcal{L}_h \mathcal{L}_h^{V*})\partial_t u\|_{W^{1,\infty}} \\ & \leq C(\|g(\partial_t u)\|_{H^{k+1}} + \|\partial_t u\|_{W^{k+1,\infty}}) h^k, \end{aligned}$$

which gives the desired convergence rate.

(c) We compute with (2.5) and (2.6)

$$A_h(J_h \Lambda(y) - \Lambda_h(J_h y) J_h) \partial_t y = \begin{pmatrix} \mathcal{L}_h^{V*} (\lambda(u) \partial_t^2 u) - I_h (\lambda(\mathcal{L}_h \mathcal{L}_h^{V*} u) \mathcal{L}_h \mathcal{L}_h^{V*} \partial_t^2 u) \\ 0 \end{pmatrix},$$

such that again (2.19) and (2.21) yield the estimate

$$\begin{aligned} & \|A_h(J_h \Lambda(y) - \Lambda_h(J_h y) J_h) \partial_t y\|_{X_h} \\ & \leq \|(\mathcal{L}_h^{V*} - I_h) \lambda(u) \partial_t^2 u\|_{V_h} + C \|\lambda(u) \partial_t^2 u - \lambda(\mathcal{L}_h \mathcal{L}_h^{V*} u) \mathcal{L}_h \mathcal{L}_h^{V*} \partial_t^2 u\|_{W^{1,\infty}} \\ & \leq C(\|\lambda(u)\|_{W^{k+1,\infty}}, \|\partial_t^2 u\|_{W^{k+1,\infty}}) h^k, \end{aligned}$$

and the stability is established. \square

For the fully discrete defects, we rely on further Lipschitz bounds of the nonlinearities Λ and G which we collect in the next lemma. Since, we work in a first-order framework we denote in the following for any function $x \in X$, the projection onto the first and second component by x_1 or x_2 , respectively.

Lemma 5.1. *Let $x, y, z \in X$, and let Assumption 2.1 hold.*

(a) *If $x_1, y_1, z_2 \in W^{1,\infty}(\Omega)$, then*

$$\|A(\Lambda(x) - \Lambda(y))z\|_X \leq C \|x_1 - y_1\|_{H^1(\Omega)}$$

where the constant depends on the $W^{1,\infty}$ -norms of x_1, y_1, z_2 .

(b) *If $x_1, x_2, y_1, y_2 \in W^{1,\infty}(\Omega)$, then*

$$\|A(G(t, x) - G(s, y))\|_X \leq C(|t - s| + \|x_1 - y_1\|_{H^1(\Omega)} + \|x_2 - y_2\|_{H^1(\Omega)})$$

where the constant depends on the $W^{1,\infty}$ -norms of x_1, x_2, y_1, y_2 .

Proof. We expand the difference in part (a) as

$$\begin{aligned} \|A(\Lambda(x) - \Lambda(y))z\|_X &= \|(\lambda(x_1) - \lambda(y_1))z_2\|_{H^1} \\ &= \left\| \int_0^1 \lambda'(\sigma x_1 + (1 - \sigma y_1))(x_1 - y_1)z_2 \right\|_{H^1} \\ &\lesssim \sup_{\sigma \in [0,1]} \|\lambda'(\sigma x_1 + (1 - \sigma y_1))\|_{W^{1,\infty}} \|x_1 - y_1\|_{H^1} \|z_2\|_{W^{1,\infty}}, \end{aligned}$$

and the assumptions in the lemma yield the bound. The very same computation yields the second estimate (b). \square

Thus, the defect of the Euler method can be bounded in a straightforward way.

Proof of Lemma 4.3. We recall the splitting $\delta_{\text{Eu}}^{n+1} = \delta_{h,\text{Eu}}^{n+1} + \delta_{\tau,\text{Eu}}^{n+1}$ of the defect in (4.3), and note that by proof of Lemma 3.8 it holds

$$\|A_h \delta_{h,\text{Eu}}^{n+1}\|_{X_h} \leq Ch^k.$$

We treat the two parts in (4.3b) separately. The second term involving G is bounded using (2.24) and Lemma 5.1. Further, we expand

$$\begin{aligned} & A_h J_h \Lambda(y(t^n)) \partial_\tau y(t^{n+1}) - A_h J_h \Lambda(y(t^{n+1})) \partial_t y(t^{n+1}) \\ &= \mathcal{L}_h^* A (\Lambda(y(t^n)) - \Lambda(y(t^{n+1}))) \partial_t y(t^{n+1}) + \mathcal{L}_h^* A \Lambda(y(t^n)) (\partial_\tau y(t^{n+1}) - \partial_t y(t^{n+1})), \end{aligned}$$

such that Lemma 5.1 is employed on the first part. For the second part, note that by the fundamental theorem of calculus we obtain in any norm

$$\|\partial_\tau z(t^{n+1}) - \partial_t z(t^{n+1})\| \leq \frac{\tau}{2} \sup_{s \in [0, \tau]} \|\partial_t^2 z(t^n + s)\|,$$

and the claim follows. \square

Similarly, we bound the defect of the midpoint rule in (4.12), and as above we do not have to treat the spatial part $\delta_{h,\text{M}}^{n+1}$.

Proof of Lemma 4.6. We first note that from Taylor expansions and the Peano kernel theorem, we conclude in any norm the bounds

$$\begin{aligned} \|\partial_\tau z(t^{n+1}) - \partial_t z(t^{n+1/2})\| &\leq \frac{\tau^2}{24} \sup_{s \in [0, \tau]} \|\partial_t^3 z(t^n + s)\|, \\ \|z(t^{n+1/2}) - \frac{1}{2}(z(t^{n+1}) + z(t^n))\| &\leq \frac{\tau^2}{4} \sup_{s \in [0, \tau]} \|\partial_t^2 z(t^n + s)\|, \\ \|z(t^{n+1/2}) - (\frac{3}{2}z(t^n) - \frac{1}{2}z(t^{n-1}))\| &\leq \frac{3\tau^2}{8} \sup_{s \in [0, \tau]} \|\partial_t^2 z(t^n + s)\|. \end{aligned}$$

Combining this with Lemma 5.1 and the proof of Lemma 4.3 yields the desired bounds on the defect. \square

ACKNOWLEDGMENTS

The author thanks Selina Burkhard and Constantin Carle for their careful reading of this manuscript.

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APPENDIX A. MODIFIED RITZ MAP

In this section, we discuss the approximation property of \tilde{R}_h as well as its computation. Note that the same reasoning is valid in the conforming case.

Proof of Proposition 2.4. We use the definitions in (2.4b) and (2.9) to compute

$$\begin{aligned} \left| \left(\mathcal{L}_h^{V*} u - \tilde{R}_h u \mid \varphi_h \right)_{V_h} \right| &= \left| \left(u \mid \mathcal{L}_h \varphi_h \right)_V - \left(\tilde{I}_h u \mid \tilde{\mathcal{L}}_h^{-1} \mathcal{L}_h \varphi_h \right)_{\tilde{V}_h} \right| \\ &\leq \left| \left(u - \tilde{\mathcal{L}}_h \tilde{I}_h u \mid \mathcal{L}_h \varphi_h \right)_V \right| \\ &\quad + \left| \left(\tilde{\mathcal{L}}_h \tilde{I}_h u \mid \mathcal{L}_h \varphi_h \right)_V - \left(\tilde{I}_h u \mid \tilde{\mathcal{L}}_h^{-1} \mathcal{L}_h \varphi_h \right)_{\tilde{V}_h} \right| \\ &= \Delta_1 + \Delta_2. \end{aligned}$$

We employ the stability of the lift in (2.2) and the interpolation property in (2.19) to obtain

$$\Delta_1 \lesssim h^{k+1} \|u\|_{H^{k+2}(\Omega)} \|\varphi_h\|_{V_h}.$$

The geometric estimate in [14, Lem. 8.24] together with (2.2) allows us to bound

$$\Delta_2 \lesssim h^{k+1} \|\tilde{I}_h u\|_{\tilde{V}_h} \|\tilde{\mathcal{L}}_h^{-1} \mathcal{L}_h \varphi_h\|_{V_h} \lesssim h^{k+1} \|u\|_{H^{k+2}(\Omega)} \|\varphi_h\|_{V_h},$$

and the claim follows. Further, we use the definition of Δ_h in (2.7) and the inverse estimate (2.22)

$$\|\Delta_h u_h\|_{L^2}^2 = - (u_h \mid \Delta_h u_h)_{V_h} \leq \|u_h\|_{V_h} \|\Delta_h u_h\|_{V_h} \lesssim h^{-1} \|u_h\|_{V_h} \|\Delta_h u_h\|_{L^2},$$

and obtain the second bound with one power less in h . \square

In order to compute (2.9), we have to solve a linear system with the stiffness matrix corresponding to the bilinear form $(\cdot \mid \cdot)_{V_h}$ and right-hand side $\tilde{\ell}_u$. For a basis φ_i , $i = 1, \dots, L$, of V_h , the entries are given by

$$(\tilde{\ell}_u)_i = (\tilde{I}_h u \mid \tilde{\mathcal{L}}_h^{-1} \mathcal{L}_h \varphi_i)_{\tilde{V}_h}.$$

Going through the construction explained in [13, Sec. 4.1.2–4.2], one observes that $\tilde{\varphi}_i := \tilde{\mathcal{L}}_h^{-1} \mathcal{L}_h \varphi_i \in \tilde{V}_h$, however $\tilde{\varphi}_i$ is not a nodal basis function. Since also $\tilde{I}_h u \in \tilde{V}_h$, one only needs to modify the routines which are used to assemble the stiffness matrix corresponding to $(\cdot \mid \cdot)_{\tilde{V}_h}$.

APPENDIX B. INTERPOLATION AND ADJOINT LIFT

In this appendix, we provide the proof of Lemma 2.9 and (2.20). The following estimate appears to be standard, but since we could not find a reference in the literature, we provide its proof here.

Lemma B.1. *For $m = 0, 1$, there is a constant $C_m > 0$ independent of h such that*

$$\|I_h(\mathcal{L}_h \varphi_h \cdot \mathcal{L}_h \psi_h)\|_{W^{m,2}(\Omega_h)} \leq C_m \|\mathcal{L}_h \varphi_h \cdot \mathcal{L}_h \psi_h\|_{W^{m,2}(\Omega_h)},$$

for all $\varphi_h, \psi_h \in V_h$.

Proof. Writing $\mathcal{L}_h I_h = \text{Id} + (\mathcal{L}_h I_h - \text{Id})$, by (2.2) it is sufficient to show the assertion for $\mathcal{L}_h I_h - \text{Id}$ instead of I_h . Passing to the reference cell \hat{K} , we only consider the case $m = 1$. We define the map

$$\text{Id} - \hat{I}_h : (\mathcal{P}_{2k}(\hat{K}), |\cdot|_{H^1}) \rightarrow (\mathcal{P}_k(\hat{K}), \|\cdot\|_{H^1}),$$

which is a bounded linear operator with a constant C , such that for any $\varphi \in \mathcal{P}_{2k}(\widehat{K})$

$$\|(\text{Id} - \widehat{I}_h)\varphi\|_{H^1(\widehat{K})} \leq C |\varphi|_{H^1(\widehat{K})}.$$

Then, employing [14, Lem. 4.12] yields the result on an arbitrary cell K . \square

With this, we directly conclude the desired stability estimate.

Proof of Lemma 2.9. By the nodal interpolation property (2.3) and Lemma B.1, we obtain

$$\|I_h(\varphi \cdot \mathcal{L}_h \psi_h)\|_{V_h} = \|I_h(\mathcal{L}_h I_h^e \varphi \cdot \mathcal{L}_h \psi_h)\|_{V_h} \leq C \|\mathcal{L}_h I_h^e \varphi \cdot \mathcal{L}_h \psi_h\|_{V_h}.$$

Using Sobolev's embedding and the stability of the interpolation from (2.19), we further estimate

$$\|\mathcal{L}_h I_h^e \varphi \cdot \mathcal{L}_h \psi_h\|_{V_h} \leq C \|\mathcal{L}_h I_h^e \varphi\|_{W^{1,N+\delta}} \|\varphi_h\|_{V_h} \leq C \|\varphi\|_{W^{1,N+\delta}} \|\varphi_h\|_{V_h}.$$

By the same reasoning, we obtain the bound in the L^2 -norm. \square

Proof of (2.20). The stability of $\mathcal{L}_h^{H^*}$ directly follows from the definition (2.4) and the stability of the lift \mathcal{L}_h in (2.2). For the error bound, we first observe

$$\begin{aligned} \|(I_h - \mathcal{L}_h^{H^*})\varphi\|_{H_h} &= \sup_{\|\psi_h\|_{H_h}=1} ((I_h - \mathcal{L}_h^{H^*})\varphi | \psi_h)_{L^2(\Omega_h)} \\ &= \sup_{\|\psi_h\|_{H_h}=1} \left(((\mathcal{L}_h I_h - \text{Id})\varphi | \mathcal{L}_h \psi_h)_{L^2(\Omega)} \right. \\ &\quad \left. + (I_h \varphi | \psi_h)_{L^2(\Omega_h)} - (\mathcal{L}_h I_h \varphi | \mathcal{L}_h \psi_h)_{L^2(\Omega)} \right). \end{aligned}$$

For the first term we apply (2.19), and for the difference we use [14, Lem. 8.24] to obtain

$$\|(I_h - \mathcal{L}_h^{H^*})\varphi\|_{H_h} \lesssim h^{\ell+1} \|\varphi\|_{H^{\ell+1}(\Omega)} + h^\ell \|\mathcal{L}_h I_h \varphi\|_{L^2(U_h)}$$

with the boundary layer $U_h := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) \leq h\}$. Below, we show

$$(B.1) \quad \|\mathcal{L}_h I_h \varphi\|_{L^\infty(U_h)} \lesssim Ch \|\mathcal{L}_h I_h \varphi\|_{W^{1,\infty}(\Omega_h)}$$

and use this together with $\text{vol}(U_h)^{1/2} \lesssim h^{1/2}$ to estimate

$$\|\mathcal{L}_h I_h \varphi\|_{L^2(U_h)} \lesssim h^{1/2} \|\mathcal{L}_h I_h \varphi\|_{L^\infty(U_h)} \lesssim h^{3/2} \|\mathcal{L}_h I_h \varphi\|_{W^{1,\infty}(\Omega_h)} \lesssim h \|\mathcal{L}_h I_h \varphi\|_{W^{1,6}(\Omega_h)},$$

where we used the inverse inequality (2.22) in the last step. The stability of the interpolation (2.19) and the Sobolev embedding yield

$$\|\mathcal{L}_h I_h \varphi\|_{L^2(U_h)} \lesssim h \|\varphi\|_{H^2(\Omega)},$$

and thus the assertion. To show (B.1), we pick some $x_0 \in U_h$ and $y_0 \in \partial\Omega$ with $|x_0 - y_0| \leq h$ such that

$$(\mathcal{L}_h I_h \varphi)(y_0) = 0, \quad |(\mathcal{L}_h I_h \varphi)(x_0)| = \|\mathcal{L}_h I_h \varphi\|_{L^\infty(U_h)}.$$

Then we use the fundamental theorem of calculus to see

$$\begin{aligned} |(\mathcal{L}_h I_h \varphi)(x_0)| &= |(\mathcal{L}_h I_h \varphi)(x_0) - (\mathcal{L}_h I_h \varphi)(y_0)| \\ &= \left| \int_0^1 (\nabla \mathcal{L}_h I_h \varphi)(sx_0 + (1-s)y_0)(x_0 - y_0) ds \right| \\ &\leq \|\mathcal{L}_h I_h \varphi\|_{W^{1,\infty}(\Omega)} |x_0 - y_0|, \end{aligned}$$

which gives the assertion. \square

APPENDIX C. DISCRETE SOBOLEV EMBEDDING

The proof is adapted from the conforming case presented in [5, Lem. 4.1], but is able to cover a larger range of exponents. Similar results including the discrete differential operator Δ_h are shown in [16, Thm. 1.12] and [32, Thm. 3].

Proof of Lemma 3.3. First, we define the inverse S_h of Δ_h from (2.7) by

$$(S_h \varphi_h | \psi_h)_{V_h} = -(\varphi_h | \psi_h)_{H_h}, \quad \varphi_h, \psi_h \in V_h,$$

and its continuous counterpart $S = \Delta^{-1}$ satisfying

$$(S\varphi | \psi)_V = -(\varphi | \psi)_H, \quad \varphi, \psi \in V.$$

We further define the modified solution operator $\tilde{S}_h = \mathcal{L}_h^{V*} S \mathcal{L}_h$, and write $S_h = \tilde{S}_h + (S_h - \tilde{S}_h)$. For the first term, we use the stability of the Ritz map in W^{1,p^*} from (2.21) with $\ell = 0$ and (2.12) to obtain

$$\|\tilde{S}_h \varphi_h\|_{W^{1,p^*}(\Omega_h)} \lesssim \|S \mathcal{L}_h \varphi_h\|_{W^{1,p^*}(\Omega)} \lesssim \|S \mathcal{L}_h \varphi_h\|_{H^2(\Omega)} \lesssim \|\varphi_h\|_{L^2(\Omega_h)}.$$

It remains to bound the difference, stemming from the nonconformity, by the inverse estimate (2.22)

$$\begin{aligned} \|\tilde{S}_h \varphi_h - S_h \varphi_h\|_{W^{1,p^*}(\Omega_h)} &\leq Ch^{N/p^* - N/2} \|\tilde{S}_h \varphi_h - S_h \varphi_h\|_{V_h} \\ &\leq Ch^{-1} \sup_{\|\psi_h\|_{V_h}=1} \left(\tilde{S}_h \varphi_h - S_h \varphi_h | \psi_h \right)_{V_h} \\ &= Ch^{-1} \sup_{\|\psi_h\|_{V_h}=1} \left((\varphi_h | \psi_h)_{H_h} - (\mathcal{L}_h \varphi_h | \mathcal{L}_h \psi_h)_H \right). \end{aligned}$$

We use [14, Lem. 8.24] to obtain

$$\left| (\varphi_h | \psi_h)_{H_h} - (\mathcal{L}_h \varphi_h | \mathcal{L}_h \psi_h)_H \right| \lesssim h \|\varphi_h\|_{L^2} \|\psi_h\|_{V_h},$$

which yields

$$\|S_h \varphi_h\|_{L^\infty(\Omega_h)} + \|S_h \varphi_h\|_{W^{1,p^*}(\Omega_h)} \leq C \|\varphi_h\|_{L^2(\Omega_h)},$$

and hence the assertion. \square

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