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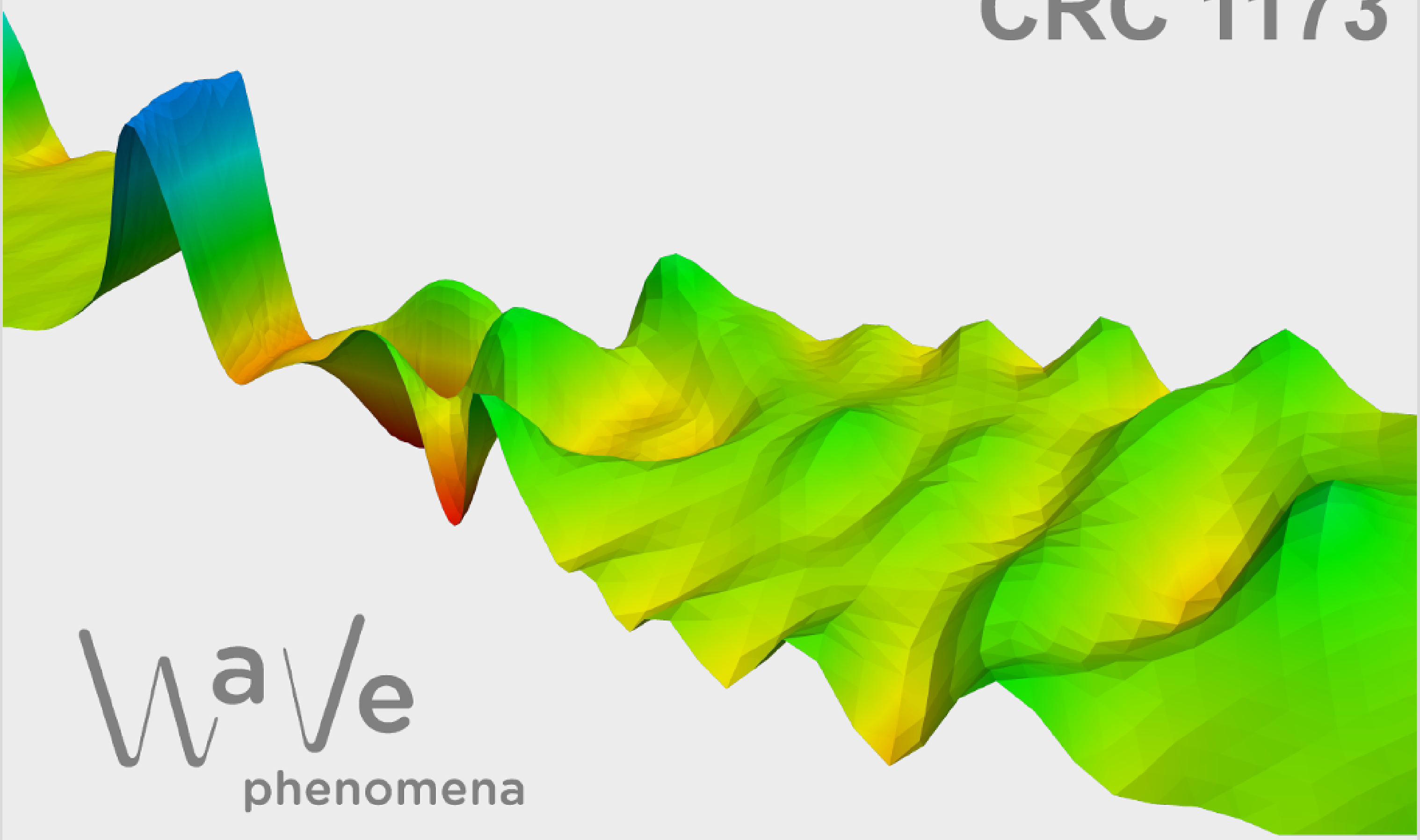

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TANGENTIAL CONE CONDITION FOR THE FULL WAVEFORM FORWARD OPERATOR IN THE ELASTIC REGIME: THE NON-LOCAL CASE

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Abstract. We generalize results of [M. Eller and A. Rieder, *Inverse Problems* 37 (2021) 085011] from the acoustic to the elastic wave equation. That means we show injectivity of the Fréchet derivative of the parameter-to-state map for a semi-discrete seismic inverse problem in the elastic regime. Here, the parameter space is spanned by functions which have a global support in the propagation medium (the non-local case) and are locally linearly independent. As a consequence we derive local conditional wellposedness of this nonlinear inverse problem. Furthermore, the tangential cone condition holds, which is an essential prerequisite in the convergence analysis of a variety of inversion algorithms for nonlinear illposed problems.

1. Introduction

We are interested in a theoretical aspect of seismic imaging in the elastic regime. Mathematically speaking, in this imaging modality one aims to identify parameter functions of the elastic wave equation (mass density, shear and pressure wave moduli) from partial measurements of elastic waves. These waves are initiated by controlled explosions. The resulting nonlinear inverse problem is called *full waveform inversion* (FWI), see, e.g., [4, 14], and it is typically solved using Newton-like iterative regularization schemes. The mathematical analysis of these schemes relies crucially on a structural assumption on the nonlinear forward map known as the tangential cone condition (TCC, sometimes also referred to as the η-condition), which was introduced in [11]. A nonlinear operator $F\colon \mathsf{D}(F) \subset V \to W$ between normed spaces V and W satisfies the TCC at $x^+ \in \operatorname{int}(\mathsf{D}(F))$ if there are an $\eta \in (0,1)$ and an open ball $B_r(x^+) \subset \mathsf{D}(F)$ such that

$$\|F(v) - F(w) - F'(w)[v-w]\|_W \le \eta \, \|F(v) - F(w)\|_W \quad \text{for all } v, w \in B_r(x^+).$$

Here, $F'\colon \mathsf{D}(F) \subset V \to \mathcal{L}(V,W)$ denotes the Fréchet derivative. We only refer to the monographs [7, 13] and the recent publications [6, 10] as evidence for the importance of TCC in the regularization theory of nonlinear illposed problems.

In our previous work [2] we established that TCC holds at x^+ if V is finite-dimensional (the semi-discrete situation) and the Fréchet derivative $F'(x^+)$ has a trivial null space. Using this result we validate the TCC for the FWI forward operator in the elastic regime provided the parameters (density and relaxed S- and P-wave moduli) are restricted to a suitable finite-dimensional space, which is spanned by locally linearly independent $\mathcal{C}^2$-functions having global support in the propagation medium (the non-local case).

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We begin our presentation in the next section introducing first the elastic wave equation as a first order hyperbolic system along with some statements on its wellposedness. Then, we formulate the forward operator Φ of the semi-discrete version of FWI in the elastic regime which we consider in this work. An important property is the Lipschitz continuity of the Fréchet derivative of this forward operator which we state in Theorem 2.1. The rather technical proof is moved to an appendix. In preparation for our main result in Theorem 4.3 we provide a control result for the elastic wave equation in Section 3: given two open subsets Σ and Ω of the propagation medium, we can find a source in Σ such that the resulting velocity field at a sufficiently large time has non-trivial divergence and non-trivial deviator in Ω (see Theorem 3.4). The proof relies on a global Holmgren-John theorem for the homogeneous elastic wave equation across non-characteristic surfaces. As a consequence of this controllability, the Fréchet derivative of Φ must be one-to-one at each inner point of the propagation medium (Theorem 4.2). An application of Lemma C.1 of [2] finally yields the TCC for Φ and the Lipschitz-stability of the inverse problem. We conclude our work with a discussion of possible future research.

2. The setting

In two subsections we introduce the mathematical background of the considered forward and related inverse problem.

2.1. The forward model. We formulate the elastic wave equation as a first order system for the stress tensor $\boldsymbol{\sigma}\colon [0,\infty)\times D \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and the velocity field $\boldsymbol{v}\colon [0,\infty)\times D\to\mathbb{R}^3$. Let $D\subset\mathbb{R}^3$ be a connected bounded domain with boundary ∂D that is piecewise $\mathcal{C}^1$. Then,

$$\partial_t\boldsymbol{\sigma}(t,x) = C\big(\mu(x),\pi(x)\big)\varepsilon(\boldsymbol{v}(t,x)), \qquad (t,x)\in[0,\infty)\times D, \tag{2.1a}$$

$$\varrho(x)\partial_t\boldsymbol{v}(t,x) = \operatorname{div}\boldsymbol{\sigma}(t,x) + \boldsymbol{f}(t,x), \qquad (t,x)\in[0,\infty)\times D, \tag{2.1b}$$

with initial values $\boldsymbol{\sigma}(0,\cdot)=\boldsymbol{\sigma}_0$ and $\boldsymbol{v}(0,\cdot)=\boldsymbol{v}_0$. The boundary conditions on ∂D will be incorporated into the solution spaces in (2.7) below. Here $\varrho\colon D\to(0,\infty)$ denotes the bulk density, $\boldsymbol{f}\colon[0,\infty)\times D\to\mathbb{R}^3$ is a volume force, and $\mu,\pi\colon D\to(0,\infty)$ are the relaxed S- and P-wave moduli. Accordingly, the velocities of shear and pressure waves are

$$v_{\mathrm{sh}} := \sqrt{\mu/\varrho} \qquad \text{and} \qquad v_{\mathrm{pr}} := \sqrt{\pi/\varrho},$$

respectively. The linearized strain rate is given by

$$\varepsilon(\boldsymbol{v}) := \frac{1}{2}\big((\nabla_x\boldsymbol{v})^\top + \nabla_x\boldsymbol{v}\big)$$

and

$$C(m,p)\boldsymbol{\epsilon} := 2m\,\boldsymbol{\epsilon} + (p-2m)\operatorname{trace}(\boldsymbol{\epsilon})\boldsymbol{I}_3, \qquad \boldsymbol{\epsilon}\in\mathbb{R}^{3\times3}_{\mathrm{sym}},\ m,p\in\mathbb{R}, \tag{2.2}$$

specifies Hooke's law. Throughout, we assume that

$$(\varrho,\mu,\pi)\in\mathcal{P} := \big\{\lambda\in L^\infty(D)^3 \ :\ \varrho_{\min}<\lambda_1(\cdot)<\varrho_{\max},\ \mu_{\min}<\lambda_2(\cdot)<\mu_{\max},\ \pi_{\min}<\lambda_3(\cdot)<\pi_{\max}\big\}, \tag{2.3}$$

where $0<\varrho_{\min}<\varrho_{\max}<\infty$, $0<\mu_{\min}<\mu_{\max}<\infty$, and $0<\pi_{\min}<\pi_{\max}<\infty$ are suitable constants with $3\pi_{\min}>4\mu_{\max}$. The latter condition guarantees that $C(\mu,\pi)$ is invertible with

$$C(m,p)^{-1} = C\left(\frac{1}{4m}, \frac{p-m}{m(3p-4m)}\right), \qquad m,p \in \mathbb{R}, \tag{2.4}$$

and as a consequence pressure waves travel faster than shear waves. More precisely, we have that

$$\frac{v_{\mathrm{pr}}}{v_{\mathrm{sh}}} = \sqrt{\frac{\pi}{\varrho}}\sqrt{\frac{\varrho}{\mu}} > \sqrt{\frac{\pi_{\min}}{\mu_{\max}}} > \sqrt{\frac{4}{3}} \approx 1.15.$$

Next, we write (2.1) as an abstract initial value problem

$$B\partial_t u = -Au + f(t), \quad u(0) = \begin{bmatrix} \boldsymbol{\sigma}_0 \\ \boldsymbol{v}_0 \end{bmatrix} =: u_0, \tag{2.5}$$

in the time interval $[0,\infty)$, where $u(t) := (\boldsymbol{\sigma}(t,\cdot), \boldsymbol{v}(t,\cdot))^\top$, $f(t) := (\mathbf{0}, \boldsymbol{f}(t,\cdot))^\top$,

$$B := \begin{bmatrix} C(\mu,\pi)^{-1} & \mathbf{0} \\ \mathbf{0} & \varrho\,\boldsymbol{I}_3 \end{bmatrix}, \qquad \text{and} \qquad A := -\begin{bmatrix} 0 & \varepsilon \\ \operatorname{div} & 0 \end{bmatrix}. \tag{2.6}$$

We define the Hilbert space

$$X := L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \times L^2(D, \mathbb{R}^3)$$

with inner product

$$\big\langle (\boldsymbol{\sigma}, \boldsymbol{v}), (\boldsymbol{\psi}, \boldsymbol{w}) \big\rangle_X := \int_D (\boldsymbol{\sigma} : \boldsymbol{\psi} + \boldsymbol{v}\cdot\boldsymbol{w})\,\mathrm{d}x,$$

where the colon denotes the Frobenius inner product on $\mathbb{R}^{3\times3}$. We split the boundary of D into two disjoint parts, $\partial D = \partial D_D \,\dot\cup\, \partial D_N$, and define

$$\mathsf{D}(A) := \big\{(\boldsymbol{\psi}, \boldsymbol{w}) \in H(\operatorname{div}) \times H^1_D \;:\; \boldsymbol{\psi}\boldsymbol{n} = \mathbf{0} \text{ on } \partial D_N\big\} \tag{2.7}$$

with

$$H^1_D := \big\{\boldsymbol{v} \in H^1(D, \mathbb{R}^3) \;:\; \boldsymbol{v} = \mathbf{0} \text{ on } \partial D_D\big\} \tag{2.8}$$

and

$$H(\operatorname{div}) := \big\{\boldsymbol{\sigma} \in L^2\big(D, \mathbb{R}^{3\times3}_{\mathrm{sym}}\big) \;:\; \operatorname{div}\boldsymbol{\sigma} \in L^2(D, \mathbb{R}^3)\big\}.$$

The operator $A\colon \mathsf{D}(A) \subset X \to X$ is maximal monotone (see, e.g., [8, Lmm. 6.1]). If $(\boldsymbol{\sigma}_0, \boldsymbol{v}_0) \in \mathsf{D}(A)$ and $\boldsymbol{f} \in W^{1,1}\big([0,\infty), L^2(D,\mathbb{R}^3)\big)$, then (2.5) (or equivalently (2.1)) admits a unique classical solution $u = (\boldsymbol{\sigma}, \boldsymbol{v})^\top \in \mathcal{C}([0,\infty), \mathsf{D}(A)) \cap \mathcal{C}^1([0,\infty), X)$ (see, e.g., [8]). On the other hand, if $(\boldsymbol{\sigma}_0, \boldsymbol{v}_0)^\top \in X$ and $\boldsymbol{f} \in L^1_{\mathrm{loc}}([0,\infty), L^2(D,\mathbb{R}^3))$, then (2.5) (or equivalently (2.1)) admits a unique mild solution $u = (\boldsymbol{\sigma}, \boldsymbol{v})^\top \in \mathcal{C}([0,\infty), X)$, which satisfies

$$Bu(t) = Bu_0 - A\int_0^t u(s)\mathrm{d}s + \int_0^t f(s)\mathrm{d}s, \qquad t \in [0,\infty), \tag{2.9}$$

(see, e.g., [12, Pro. 2.15]).

2.2. The (semi-discrete) inverse problem. Let

$$V := \operatorname{span}\{\varphi_j \in \mathcal{C}^2(\overline{D}) \;:\; j = 1, \dots, M\} \subset L^\infty(\overline{D}),$$

where the functions $\{\varphi_j : j = 1, \dots, M\}$ are *locally linearly independent* over D. This means that any linear combination must be trivial that vanishes on a subset O of D with positive Lebesgue-measure:

$$\text{meas}(O) > 0 \;\text{ and }\; \sum_{j=1}^{M} a_j \varphi_j|_O = 0 \quad \Longrightarrow \quad a_j = 0, \; j = 1, \dots, M. \tag{2.10}$$

We write $\|\cdot\|_V := \|\cdot\|_{L^\infty(\overline{D})}$. Specific examples for spaces V with the required properties are polynomial spaces and spaces spanned by certain classes of radial basis functions (see [2, Sec. 3]).

In a seismic experiment, sources are fired at time zero in a non-empty open subset $\Sigma \subset D$ and the resulting wave fields are measured in a different non-empty open subset $\Omega \subset D$ until the observation time $T > 0$ has been reached. Accordingly, the measurements are in $\mathcal{C}([0,T], X_\Omega)$ where $X_\Omega := L^2(\Omega, \mathbb{R}^{3\times 3}_{\text{sym}}) \times L^2(\Omega, \mathbb{R}^3)$. For technical reasons, which will become clear in the proof of Theorem 2.1 below, we confine the prescribed sources to

$$W_0^{2,1}(\Sigma) := \big\{\boldsymbol{f} \in W^{2,1}\big([0,T], L^2(\Sigma, \mathbb{R}^3)\big) \;:\; \boldsymbol{f}(0) = \boldsymbol{f}'(0) = \boldsymbol{0}\big\}. \tag{2.11}$$

To formulate the corresponding semi-discrete inverse problem we set $V_+^3 := V^3 \cap \mathcal{P}$ and define the FWI forward operator (parameter-to-source-to-state map) by

$$\Phi\colon V_+^3 \subset V^3 \to \underbrace{\mathcal{L}\big(W_0^{2,1}(\Sigma), \mathcal{C}([0,T], X_\Omega)\big)}_{=:\,\mathcal{W}}, \quad (\varrho, \mu, \pi) \mapsto \big(\boldsymbol{f} \mapsto \Psi(\boldsymbol{\sigma}, \boldsymbol{v})\big),$$

where $(\boldsymbol{\sigma}, \boldsymbol{v})$ is the unique classical solution of (2.1) with initial values $\boldsymbol{\sigma}_0 = \boldsymbol{0}$, $\boldsymbol{v}_0 = \boldsymbol{0}$, and where $\Psi\colon \mathcal{C}\big([0,T], X\big) \to \mathcal{C}\big([0,T], X_\Omega\big)$, $\Psi(\boldsymbol{\sigma}, \boldsymbol{v}) := (\boldsymbol{\sigma}|_\Omega, \boldsymbol{v}|_\Omega)^\top$, models the measurement process.

Now the semi-discrete seismic inverse problem (time-domain full waveform inversion) in the elastic regime reads:

$$\text{Reconstruct the triple } (\varrho, \mu, \pi) \in V_+^3 \text{ from a measured version of } \Phi(\varrho, \mu, \pi). \tag{2.12}$$

We will verify in Theorem 4.3 below that in contrast to its infinite-dimensional version, which is locally illposed (see [8, Thm. 6.7]), the semidiscrete inverse problem is in fact locally wellposed and Lipschitz stable.

It has been established in [8] that Φ is Fréchet-differentiable at any $(\varrho, \mu, \pi) \in V_+^3$ with derivative $\Phi'\colon V_+^3 \subset V^3 \to \mathcal{L}(V^3, \mathcal{W})$ given by

$$\Phi'(\varrho, \mu, \pi)[\boldsymbol{h}]\boldsymbol{f} = (\overline{\boldsymbol{\sigma}}|_\Omega, \overline{\boldsymbol{v}}|_\Omega)^\top, \qquad \boldsymbol{h} = (h_1, h_2, h_3) \in V^3, \;\; \boldsymbol{f} \in W_0^{2,1}(\Sigma),$$

where $(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{v}}) \in \mathcal{C}\big([0,T], X\big)$ denotes the mild solution of

$$\begin{aligned} \partial_t \overline{\boldsymbol{\sigma}}(t,x) = C(\mu(x), \pi(x))\varepsilon(\overline{\boldsymbol{v}}(t,x)) \qquad\qquad & \\ + C(h_2(x), h_3(x))\varepsilon(\boldsymbol{v}(t,x)) \qquad & (t,x) \in [0,T] \times D, \end{aligned} \tag{2.13a}$$

$$\varrho(x)\partial_t \overline{\boldsymbol{v}}(t,x) = \operatorname{div} \overline{\boldsymbol{\sigma}}(t,x) - h_1(x)\, \partial_t \boldsymbol{v}(t,x) \qquad (t,x) \in [0,T] \times D, \tag{2.13b}$$

with initial values $\overline{\boldsymbol{\sigma}}(0,\cdot) = \boldsymbol{0}$, $\overline{\boldsymbol{v}}(0,\cdot) = \boldsymbol{0}$ and where $\boldsymbol{v}$ is the second component of the classical solution to (2.1) with initial values $\boldsymbol{\sigma}_0 = \boldsymbol{0}$, $\boldsymbol{v}_0 = \boldsymbol{0}$.

The following property of Φ' will become important later. Its proof is given in Appendix A.

Theorem 2.1. *The map $\Phi' \colon V_+^3 \subset V^3 \to \mathcal{L}(V^3, \mathcal{W})$ is Lipschitz continuous, i.e.,*

$$\|\Phi'(\varrho_1, \mu_1, \pi_1) - \Phi'(\varrho_2, \mu_2, \pi_2)\|_{\mathcal{L}(V^3,\mathcal{W})} \lesssim \|(\varrho_1, \mu_1, \pi_1) - (\varrho_2, \mu_2, \pi_2)\|_{V^3}$$ [1]

for all $(\varrho_i, \mu_i, \pi_i) \in V_+^3$, $i = 1, 2$. The Lipschitz constant depends only on the observation time T and the bounds $\varrho_{\min}$, $\varrho_{\max}$, $\mu_{\min}$, $\mu_{\max}$, $\pi_{\min}$, and $\pi_{\max}$ that determine the parameter range $\mathcal{P}$ in (2.3).

3. A control result for the elastic wave equation

In Theorem 3.4 below we will establish the existence of a source $\boldsymbol{f} \in W_0^{2,1}(\Sigma)$, which plugged into (2.1b) initiates a velocity $\boldsymbol{v}(T, \cdot)$ with non-trivial divergence and non-trivial deviator in Ω provided T is large enough.

We define the bounded linear operator

$$\mathcal{T} \colon L^2([0,T] \times \Sigma, \mathbb{R}^3) \to L^2_\varrho(\Omega, \mathbb{R}^3), \quad \boldsymbol{f} \mapsto \boldsymbol{v}(T, \cdot)|_\Omega, \tag{3.1}$$

where $u = (\boldsymbol{\sigma}, \boldsymbol{v})^\top$ is the classical solution of (2.5) with initial values $\boldsymbol{\sigma}_0 = \boldsymbol{0}$, $\boldsymbol{v}_0 = \boldsymbol{0}$. Note that $\mathcal{T}$ is well defined as $\boldsymbol{v}$ is continuous in time. The space $L^2_\varrho(\Omega, \mathbb{R}^3)$ is the same as $L^2(\Omega, \mathbb{R}^3)$ but with the ϱ-weighted inner product

$$\langle \boldsymbol{\psi}, \boldsymbol{w} \rangle_{L^2_\varrho(\Omega,\mathbb{R}^3)} := \langle \varrho\, \boldsymbol{\psi}, \boldsymbol{w} \rangle_{L^2(\Omega,\mathbb{R}^3)}.$$

Both spaces share the same topology.

Lemma 3.1. *The adjoint operator $\mathcal{T}^*$ of $\mathcal{T}$ from (3.1) is given by*

$$\mathcal{T}^* \colon L^2_\varrho(\Omega, \mathbb{R}^3) \to L^2([0,T] \times \Sigma, \mathbb{R}^3), \quad \boldsymbol{r} \mapsto \boldsymbol{w}|_{[0,T]\times\Sigma},$$

where $g = (\boldsymbol{\psi}, \boldsymbol{w})^\top \in \mathcal{C}([0,T], X)$ is the mild solution of the adjoint wave equation

$$B\partial_t g = A^* g, \qquad g(T) = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{r} \end{bmatrix}. \tag{3.2}$$

Proof. Let $\boldsymbol{f} \in L^2([0,T] \times \Sigma, \mathbb{R}^3)$ and $\boldsymbol{r} \in L^2(\Omega, \mathbb{R}^3)$. To work with classical solutions we choose sequences $\{\boldsymbol{f}_k\}_k \subset W^{1,1}([0,T], L^2(D, \mathbb{R}^3))$ and $\{\boldsymbol{r}_k\}_k \subset H^1_D$ from (2.8) with $\boldsymbol{f}_k \to \chi_\Sigma \boldsymbol{f}$ in $L^2([0,T] \times D, \mathbb{R}^3)$ and $\boldsymbol{r}_k \to \chi_\Omega \boldsymbol{r}$ in $L^2_\varrho(D, \mathbb{R}^3)$. Furthermore, let $u_k = (\boldsymbol{\sigma}_k, \boldsymbol{v}_k)^\top$ and $g_k = (\boldsymbol{\psi}_k, \boldsymbol{w}_k)^\top$ be the classical solutions of (2.5) with initial values $u_k(0) = (\boldsymbol{0}, \boldsymbol{0})^\top$ and of (3.2), respectively, when replacing f by $f_k := (\boldsymbol{0}, \boldsymbol{f}_k)^\top$ and $\boldsymbol{r}$ by $\boldsymbol{r}_k$. We note that $g_k \to g$ in $L^2([0,T], X)$ (see [8, Thm. 2.4]).

[1]The notation $A \lesssim B$ indicates the existence of a generic constant $c > 0$ such that $A \le c\,B$.

Integration by parts yields

$$\begin{aligned}
\langle \boldsymbol{v}_k(T,\cdot), \boldsymbol{r}_k\rangle_{L^2_\varrho(\Omega,\mathbb{R}^3)} &= \langle \varrho\, \boldsymbol{v}_k(T,\cdot), \boldsymbol{w}_k(T,\cdot)\rangle_{L^2(\Omega,\mathbb{R}^3)} - \langle \varrho\, \boldsymbol{v}_k(0,\cdot), \boldsymbol{w}_k(0,\cdot)\rangle_{L^2(\Omega,\mathbb{R}^3)}\\
&= \langle Bu_k(T), g_k(T)\rangle_X - \langle u_k(0), Bg_k(0)\rangle_X\\
&= \int_0^T \langle B\partial_t u_k(t), g_k(t)\rangle_X \mathrm{d}t + \int_0^T \langle u_k(t), B\partial_t g_k(t)\rangle_X \mathrm{d}t\\
&= \int_0^T \langle -Au_k(t) + f_k(t), g_k(t)\rangle_X \mathrm{d}t + \int_0^T \langle u_k(t), A^* g_k(t)\rangle_X \mathrm{d}t\\
&= \int_0^T \langle f_k(t), g_k(t)\rangle_X \mathrm{d}t = \langle \boldsymbol{f}_k, \boldsymbol{w}_k\rangle_{L^2([0,T]\times D,\mathbb{R}^3)}.
\end{aligned}$$

Passing to the limit as $k \to \infty$ verifies the assertion. □

Lemma 3.2. *Let $g = (\boldsymbol{\psi}, \boldsymbol{w})^\top \in \mathcal{C}([0,T], X)$ be the mild solution of (3.2) in $[0,T]$. Then,*

$$\widetilde{g}(t) := \begin{bmatrix} -\boldsymbol{I}_3 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_3 \end{bmatrix} g(2T-t) = \begin{bmatrix} -\boldsymbol{\psi}(2T-t) \\ \boldsymbol{w}(2T-t) \end{bmatrix}, \qquad t \in [T, 2T],$$

is the mild, forward in time propagating solution of (3.2) in $[T, 2T]$. Accordingly, $\mathcal{T}^*$ can be extended from $L^2_\varrho(\Omega,\mathbb{R}^3)$ to $L^2([0,2T]\times\Sigma,\mathbb{R}^3)$ by

$$\mathcal{T}^* \boldsymbol{r}(t,\cdot) = \begin{cases} \boldsymbol{w}(t,\cdot)|_\Sigma, & t \in [0,T],\\ \boldsymbol{w}(2T-t,\cdot)|_\Sigma, & t \in (T, 2T]. \end{cases} \tag{3.3}$$

Proof. Obviously, $\widetilde{g} \in \mathcal{C}([T,2T], X)$. Note that g as a mild solution of (3.2) satisfies

$$Bg(t) = Bg(T) + A\int_t^T g(s)\mathrm{d}s, \qquad t \in [0,T],$$

which is (2.9) adapted to (3.2) using $A^* = -A$. We define $D := \left(\begin{smallmatrix} -\boldsymbol{I}_3 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_3 \end{smallmatrix}\right)$. Observing that $DB = BD$, $DA = -AD$, and $Dg(T) = g(T)$ we find that, for $t \in [0,T]$,

$$\begin{aligned}
B\widetilde{g}(2T-t) = BDg(t) = DBg(t) &= DBg(T) + DA\int_t^T g(s)\mathrm{d}s\\
&= BDg(T) - A\int_t^T Dg(s)\mathrm{d}s = Bg(T) - A\int_t^T \widetilde{g}(2T-s)\mathrm{d}s\\
&= Bg(T) + A^*\int_T^{2T-t} \widetilde{g}(s)\mathrm{d}s,
\end{aligned}$$

which, for $t \in [T,2T]$, reduces to

$$B\widetilde{g}(t) = Bg(T) + A^*\int_T^t \widetilde{g}(s)\mathrm{d}s.$$

Thus, $\widetilde{g}$ is the forward in time propagating mild solution of (3.2). □

Theorem 3.3. *Suppose that $\varrho \in \mathcal{C}^1(\overline{D})$ and $\mu, \pi \in \mathcal{C}^2(\overline{D})$. Then the operator $\mathcal{T}$ defined in (3.1) has a dense range, provided that*

$$T > \mathrm{dist}(x,\Sigma) := \inf_{y\in\Sigma} \mathrm{dist}(x,y) \qquad \textit{for all } x \in \Omega. \tag{3.4}$$

Here, dist *denotes the Riemannian distance function in D, which is defined by*

$$\operatorname{dist}(x,y) := \inf_{\gamma} \int_a^b \sqrt{\frac{\varrho\circ\gamma(t)}{\mu\circ\gamma(t)}}\,|\dot\gamma(t)|\,\mathrm{d}t, \qquad x,y\in D\,, \tag{3.5}$$

where the infimum ist taken over all $\mathcal{C}^1$-curves $\gamma\colon [a,b]\to D$ connecting x and y.

Before we will prove the theorem we discuss the physical meaning of (3.4). Since $\sqrt{\varrho/\mu} = 1/v_{\mathrm{sh}}$, the condition says that the allotted observation time T has to be large enough such that the slower shear waves initiated in Σ reach all of Ω within the measurement period.

Proof of Theorem 3.3. From functional analysis we know that $\mathcal{T}$ has a dense range if and only if $\mathcal{T}^*$ is injective. Here, we consider $\mathcal{T}^*$ in its extended version (3.3).

Now, assume that $\mathcal{T}^*\boldsymbol{r} = 0$. Let $g = (\boldsymbol{\psi},\boldsymbol{w}) \in \mathcal{C}([0,2T],X)$ be the corresponding extended mild solution of the adjoint wave equation (3.2), that is

$$\begin{aligned} \partial_t\boldsymbol{\psi}(t,x) &= C(\mu(x),\pi(x))\varepsilon(\boldsymbol{w}(t,x)), && (t,x)\in[0,2T]\times D,\\ \varrho(x)\partial_t\boldsymbol{w}(t,x) &= \operatorname{div}\boldsymbol{\psi}(t,x), && (t,x)\in[0,2T]\times D, \end{aligned}$$

and $\boldsymbol{\psi}(T,\cdot) = \mathbf{0}$, $\boldsymbol{w}(T,\cdot) = \boldsymbol{r}$. This mild solution is the weak solution as well (see, e.g., [8, Cor. 2.5]), and by our assumption on $\boldsymbol{r}$ we immediately obtain that $\boldsymbol{w} = 0$ everywhere in $[0,2T]\times\Sigma$. We will apply a global Holmgren-John theorem [3, Thm. 1.1] to infer that $\boldsymbol{r} = 0$.

By taking the divergence of the first equation and the time derivative of the second equation we eliminate $\boldsymbol{\psi}$ and obtain the second-order system

$$\varrho(x)\partial_t^2\boldsymbol{w}(t,x) = \operatorname{div}\big(C(\mu(x),\pi(x))\varepsilon(\boldsymbol{w}(t,x))\big), \qquad (t,x)\in[0,2T]\times D. \tag{3.6}$$

Recalling (2.2) this equation translates into

$$\begin{aligned} &\varrho(x)\partial_t^2\boldsymbol{w}(t,x) - \mu(x)\big(\Delta\boldsymbol{w}(t,x) + \nabla\operatorname{div}\boldsymbol{w}(t,x)\big)\\ &\quad - \nabla\big((\pi(x)-2\mu(x))\operatorname{div}\boldsymbol{w}(t,x)\big) - 2\varepsilon(\boldsymbol{w}(t,x))\nabla\mu(x) = \mathbf{0}, \qquad (t,x)\in[0,2T]\times D. \end{aligned}$$

We infer that the principal symbol of the associated second-order differential operator on the left hand side is the 3×3 matrix

$$p(x,\tau,\xi) = (\varrho(x)\tau^2 - \mu(x)|\xi|^2)\boldsymbol{I}_3 - (\pi(x)-\mu(x))\xi\xi^\top, \qquad x\in D,\ (\tau,\xi)\in\mathbb{R}\times\mathbb{R}^3,$$

where (τ,ξ) are the Fourier variables corresponding to (t,x). A short calculation using the multilinearity of the determinant shows that

$$\det p(x,\tau,\xi) = (\varrho(x)\tau^2 - \mu(x)|\xi|^2)^2(\varrho(x)\tau^2 - \pi(x)|\xi|^2), \qquad x\in D,\ (\tau,\xi)\in\mathbb{R}\times\mathbb{R}^3. \tag{3.7}$$

Due to our smoothness assumptions on the coefficients ϱ, μ, and π, solutions $\boldsymbol{w}\in H^2_{\mathrm{loc}}((0,T)\times D)$ to the homogeneous elastic wave equation (3.6) satisfy the unique continuation property across non-characteristic $\mathcal{C}^1$-surfaces [1, Thm. 3.6, Lmm. 5.1]. More precisely, let $S\subset(0,2T)\times D$ be a non-characteristic surface of class $\mathcal{C}^1$, that is, $S = \{(t,x)\in(0,2T)\times D : \phi(t,x) = 0\}$ for some $\phi\in\mathcal{C}^1((0,2T)\times D)$ with $\det p(x,\nabla_{t,x}\phi(t,x)) \neq 0$ for all $(t,x)\in S$. If $\boldsymbol{w}\equiv\mathbf{0}$ in $S_+ = \{(t,x)\in(0,2T)\times D : \phi(t,x) > 0\}$, then $\boldsymbol{w}\equiv\mathbf{0}$ in a neighborhood of S. At this point we have to discuss the regularity of $\boldsymbol{w}$. We will demonstrate that the local uniqueness property holds also for mild/weak solutions

$\boldsymbol{w} \in L^2((0,2T)\times D)$. Using a Friedrichs's mollifier $e \in \mathcal{C}_0^\infty(\mathbb{R})$ with $\operatorname{supp} e \subset [-1,1]$ and $\int_{\mathbb{R}} e(t)\,\mathrm{d}t = 1$, we consider the regularizations in time

$$\boldsymbol{w}_n(t) = n\int_{\mathbb{R}} e(n\tau)\boldsymbol{w}(t-\tau)\,\mathrm{d}\tau, \qquad t\in[0,2T],\ n\in\mathbb{N},$$

which satisfy $\boldsymbol{w}_n \to \boldsymbol{w}$ in $L^2((0,2T)\times D)$. For convenience we extend $\boldsymbol{w}$ as a solution to the elastic wave equation to the larger time interval $(-1,2T+1)$ and denote this extension by $\boldsymbol{w}$ as well. Then the regularizations $\boldsymbol{w}_n$ and all their time derivatives are also solutions to the homogeneous elastic wave equation (3.6) in $[0,2T]\times D$ because the coefficients are time-independent. We have $\boldsymbol{w}_n \in \mathcal{C}^\infty([0,2T],L^2(D))$ and interpreting (3.6) at a fixed time $t\in[0,2T]$ as a second order elliptic system with forcing term $\varrho\partial_t^2\boldsymbol{w}(t)$ we obtain $\boldsymbol{w}_n(t)\in H^2_{\mathrm{loc}}(D)$ by elliptic regularity theory. Differentiating (3.6) with respect to t we infer $\partial_t^k\boldsymbol{w}_n(t)\in H^2_{\mathrm{loc}}(D)$ for all $k\in\mathbb{N}$ and any $t\in[0,2T]$. This gives $\boldsymbol{w}_n\in\mathcal{C}^\infty([0,2T],H^2_{\mathrm{loc}}(D))\subset H^2_{\mathrm{loc}}((0,T)\times D)$. Hence, each $\boldsymbol{w}_n$ satisfies the local uniqueness property and consequently it must hold for the limit function $\boldsymbol{w}$ as well.

This local result can be turned into a global statement using the approach developed in [3]. Even though the focus of [3] is the uniqueness of the lateral Cauchy problem, we can use this result to show that $\boldsymbol{w}$ must vanish in a larger set when it is zero in the cylinder $(0,2T)\times\Sigma$. The analysis of the lateral Cauchy problem in [3] is based on the fact that zero Cauchy data on an open subset of the spatial boundary allow us to extend a solution by zero across the boundary (see [3, p. 71]) so that the extended function is a solution to the homogeneous PDE and identically zero in an open subset of the enlarged domain for all $t\in(0,2T)$.

With this understanding we infer that $\boldsymbol{w}(T,x)=\boldsymbol{r}(x)=\boldsymbol{0}$ for all $x\in D$ satisfying $d(x,\Sigma)<T$, where d is the Riemannian distance function from (3.5), which is determined by the linear factor containing μ of the characteristic polynomial $\det p$ in (3.7) considered as a polynomial in τ^2 (see [3, Thm. 1.1]). We have replaced $T/2$ in the reference by T because in our case the equation is valid in $(0,2T)\times D$. Since $\pi>\mu$ everywhere in D we have

$$\mu(x)|\xi|^2/\varrho(x) < \pi(x)|\xi|^2/\varrho(x), \qquad x\in D,\ \xi\in\mathbb{R}^3,$$

and according to [3, Thm. 1.1] d is the metric determined by $\mu(x)|\xi|^2/\varrho(x)$. □

Below we will need the deviator of a vector field. An element $\boldsymbol{\delta}\in L^2(\Omega,\mathbb{R}^{3\times3}_{\mathrm{sym}})$ is called the (weak) deviator of $\boldsymbol{w}\in L^2(\Omega,\mathbb{R}^3)$ if

$$\int_\Omega \boldsymbol{\delta}:\boldsymbol{\phi}\,\mathrm{d}x = -\int_\Omega \boldsymbol{w}\cdot\Big(\operatorname{div}\boldsymbol{\phi}-\frac{1}{3}\nabla\operatorname{trace}(\boldsymbol{\phi})\Big)\,\mathrm{d}x \qquad \text{for all } \boldsymbol{\phi}\in\mathcal{C}_0^\infty(\Omega,\mathbb{R}^{3\times3}_{\mathrm{sym}}).$$

We write $\operatorname{dev}\boldsymbol{w} := \boldsymbol{\delta}$ and note that $\operatorname{dev}\boldsymbol{w} = \varepsilon(\boldsymbol{w}) - \frac{1}{3}(\operatorname{div}\boldsymbol{w})\boldsymbol{I}_3$ for $\boldsymbol{w}\in H^1(\Omega,\mathbb{R}^3)$, i.e., $\operatorname{dev}\boldsymbol{w}$ is the trace-free part of $\varepsilon(\boldsymbol{w})$.

Theorem 3.4. *Suppose that $\varrho\in\mathcal{C}^1(\overline{D})$ and $\mu,\pi\in\mathcal{C}^2(\overline{D})$. Let $\Omega\subset D$ be open and assume that the observation time T satisfies (3.4). Then, there exists a source $\boldsymbol{f}\in W_0^{2,1}(\Sigma)$ such that the second component of the classical solution $u=(\boldsymbol{\sigma},\boldsymbol{v})^\top$ of (2.5) with initial values $u_0=(\boldsymbol{\sigma}_0,\boldsymbol{v}_0)^\top=(\boldsymbol{0},\boldsymbol{0})^\top$ has a non-trivial divergence and a non-trivial deviator in Ω at time T, i.e., $\operatorname{div}\boldsymbol{v}(T,\cdot)|_\Omega\not\equiv 0$ and $\operatorname{dev}\boldsymbol{v}(T,\cdot)|_\Omega\not\equiv\boldsymbol{0}$.*

Proof. Let $\boldsymbol{u}\in L^2(\Omega,\mathbb{R}^3)$ with non-trivial divergence and non-trivial deviator. Such a $\boldsymbol{u}$ can be constructed by choosing $\boldsymbol{u}=\nabla\varphi$ for some $\varphi\in\mathcal{C}^\infty(\overline{\Omega})$ with $\Delta\varphi\neq 0$ and

$\varepsilon_{ij}(\nabla\varphi) = \partial_{x_i}\partial_{x_j}\varphi \not\equiv 0$ for one pair (i,j) with $1 \le i \ne j \le 3$. For instance we can choose $\varphi(x) = \exp(-|x|^2)$, $x \in \Omega$.

Since the spaces of (weak) divergence free and (weak) deviator free vector fields in Ω are proper closed subspaces of $L^2(\Omega,\mathbb{R}^3)$ there is a neighborhood $\mathcal{U} \subset L^2(\Omega,\mathbb{R}^3)$ of $\boldsymbol{u}$ containing only vector fields with non-trivial divergence and non-trivial deviator. By Theorem 3.3, there is an $\boldsymbol{f} \in L^2([0,T]\times\Sigma,\mathbb{R}^3)$ such that $\mathcal{T}\boldsymbol{f} \in \mathcal{U}$, i.e., $\operatorname{div}(\mathcal{T}\boldsymbol{f}) \not\equiv 0$ and $\operatorname{dev}(\mathcal{T}\boldsymbol{f}) \not\equiv \mathbf{0}$. We can even choose $\boldsymbol{f} \in W_0^{2,1}(\Sigma)$ because this space is dense in $L^2([0,T]\times\Sigma,\mathbb{R}^3)$. □

4. Local injectivity of the FWI forward operator yields TCC

Proposition 4.1. *Suppose that $\varrho \in \mathcal{C}^1(\overline{D})$ and $\mu,\pi \in \mathcal{C}^2(\overline{D})$. Let $\boldsymbol{h} \in V^3\backslash\{\mathbf{0}\}$, and let $\Sigma,\Omega \subset D$ be open and disjoint. If the observation time T satisfies (3.4), then there exists an $\boldsymbol{f} \in W_0^{2,1}(\Sigma)$ such that the mild solution $(\overline{\boldsymbol{\sigma}}|_\Omega, \overline{\boldsymbol{v}}|_\Omega)^\top$ of (2.13) with initial values $\overline{\boldsymbol{\sigma}}(0,\cdot) = \mathbf{0}$, $\overline{\boldsymbol{v}}(0,\cdot) = \mathbf{0}$, where $\boldsymbol{v}$ is the second component of the classical solution $u = (\boldsymbol{\sigma},\boldsymbol{v})^\top$ of (2.5) with initial values $u_0 = (\boldsymbol{\sigma}_0,\boldsymbol{v}_0)^\top = (\mathbf{0},\mathbf{0})^\top$, is not identically zero in $(0,T)\times\Omega$. This $\boldsymbol{f}$ does not depend on $\boldsymbol{h}$.*

Proof. We argue by contradiction and assume that there exists an $\boldsymbol{h} = (h_1,h_2,h_3) \in V^3\backslash\{\mathbf{0}\}$ such that for any $\boldsymbol{f} \in W_0^{2,1}(\Sigma)$ the corresponding mild solution of (2.13) with initial values $\overline{\boldsymbol{\sigma}}(0,\cdot) = \mathbf{0}$, $\overline{\boldsymbol{v}}(0,\cdot) = \mathbf{0}$, where $\boldsymbol{v}$ is the second component of the classical solution $u = (\boldsymbol{\sigma},\boldsymbol{v})^\top \in \mathcal{C}([0,\infty),\mathsf{D}(A)) \cap \mathcal{C}^1([0,\infty),X)$ of (2.5) with initial values $u_0 = (\boldsymbol{\sigma}_0,\boldsymbol{v}_0)^\top = (\mathbf{0},\mathbf{0})^\top$, satisfies $(\overline{\boldsymbol{\sigma}}|_{[0,T]\times\Omega}, \overline{\boldsymbol{v}}|_{[0,T]\times\Omega}) = (\mathbf{0},\mathbf{0})$. According to Theorem 3.4 there is an $\boldsymbol{f}$ such that $\boldsymbol{v}$ has a non-trivial divergence and a non-trivial deviator in Ω at time T, i.e., $\operatorname{div}\boldsymbol{v}(T,\cdot)|_\Omega \not\equiv 0$ and $\operatorname{dev}\boldsymbol{v}(T,\cdot)|_\Omega \not\equiv \mathbf{0}$. In particular $\boldsymbol{v}(T,\cdot)|_\Omega \not\equiv \mathbf{0}$.

Now, the integrated version of (2.13), see (2.9), with zero initial values yields that

$$C(h_2(x),h_3(x))\varepsilon(\boldsymbol{v}(t,x)) = \mathbf{0},\; h_1(x)\,\partial_t\boldsymbol{v}(t,x) = \mathbf{0} \quad \text{for a.e. } (t,x) \in [0,T]\times\Omega. \tag{4.1}$$

If $h_1 \not\equiv 0$ then $h_1(\cdot) \ne 0$ almost everywhere in Ω by (2.10). Hence, the equation on the right hand side of (4.1) implies that $\partial_t\boldsymbol{v}|_{[0,T]\times\Omega} = \mathbf{0}$ almost everywhere. In view of the initial values $\boldsymbol{\sigma}_0 = \mathbf{0}$, $\boldsymbol{v}_0 = \mathbf{0}$ we obtain from (2.5) (or equivalently (2.1)) that $\boldsymbol{v}|_{[0,T]\times\Omega} = \mathbf{0}$ almost everywhere. However, this contradicts $\boldsymbol{v}(T,\cdot)|_\Omega \not\equiv \mathbf{0}$. Therefore, $h_1 = 0$.

Next we assume that $h_2 \not\equiv 0$. Then $h_2(\cdot) \ne 0$ almost everywhere in Ω by (2.10). When $C(h_2(\cdot),h_3(\cdot))$ is injective almost everywhere in Ω, the equation on the left hand side of (4.1) shows that $\varepsilon(\boldsymbol{v})|_{[0,T]\times\Omega} = \mathbf{0}$ almost everywhere. Integrating (2.5) with respect to time, using the initial values $\boldsymbol{\sigma}_0 = \mathbf{0}$, $\boldsymbol{v}_0 = \mathbf{0}$, and observing that $\boldsymbol{f}|_{[0,T]\times\Omega} = \mathbf{0}$ because $\Sigma\cap\Omega = \emptyset$, we find that $(\boldsymbol{\sigma}|_{[0,T]\times\Omega}, \boldsymbol{v}|_{[0,T]\times\Omega}) = (\mathbf{0},\mathbf{0})$ almost everywhere. This contradicts $\boldsymbol{v}(T,\cdot)|_\Omega \not\equiv \mathbf{0}$ and thus $h_2 = 0$.

It still remains to discuss the case when $h_2 \not\equiv 0$ and $C(h_2(\cdot),h_3(\cdot))$ fails to be injective on a subset $O \subset \Omega$ with positive Lebesgue measure. Recalling (2.4), this can only happen when $3h_3(\cdot) - 4h_2(\cdot) = 0$ almost everywhere in O, and thus $3h_3(\cdot) - 4h_2(\cdot) = 0$ almost everywhere in Ω by (2.10). Then, the equation on the left of (4.1) reduces to

$$\operatorname{dev}\boldsymbol{v}|_{[0,T]\times\Omega} = \mathbf{0} \qquad \text{almost everywhere,}$$

contradicting $\operatorname{dev}\boldsymbol{v}(T,\cdot)|_\Omega \not\equiv \mathbf{0}$. Thus, $h_2 = 0$.

Finally, we assume that $h_2 = 0$ and $h_3 \not\equiv 0$. Then $h_3(\cdot) \ne 0$ almost everywhere in Ω by (2.10). The equation on the left of (4.1) reduces to

$$\operatorname{trace}(\varepsilon(\boldsymbol{v}|_{[0,T]\times\Omega})) = \operatorname{div}\boldsymbol{v}|_{[0,T]\times\Omega} = 0 \qquad \text{almost everywhere,}$$

contradicting $\operatorname{div}\boldsymbol{v}(T,\cdot)|_\Omega \not\equiv 0$. Thus, $h_2 = h_3 = 0$. □

Local uniqueness of the seismic inverse problem (2.12) follows immediately.

Theorem 4.2. *Suppose that* $(\varrho,\mu,\pi)\in V_+^3$. *The derivative* $\Phi'(\varrho,\mu,\pi)\in\mathcal{L}(V^3,\mathcal{W})$ *is an injective mapping, and we have that*

$$\min\left\{\|\Phi'(\varrho,\mu,\pi)[\boldsymbol{h}]\|_{\mathcal{W}}\ :\ \boldsymbol{h}\in V^3,\ \|\boldsymbol{h}\|_{V^3}=1\right\}>0.$$

Proof. Assume the minimum to be zero. As V^3 is finite dimensional and $\Phi'(\varrho,\mu,\pi)$ is continuous, there exists an $\boldsymbol{h}\in V^3$ with $\|\boldsymbol{h}\|_{V^3}=1$ such that $\Phi'(\varrho,\mu,\pi)[\boldsymbol{h}]\boldsymbol{f}=0$ for all $\boldsymbol{f}\in W_0^{2,1}(\Sigma)$. But then $\boldsymbol{h}=\boldsymbol{0}$ by Proposition 4.1, which contradicts $\|\boldsymbol{h}\|_{V^3}=1$. □

In Theorem 2.1 we have seen that the derivative of the FWI forward operator is Lipschitz continuous. Now an application of Lemma C.1 from [2] yields our main result, that is, Lipschitz stability (4.2) of the semi-discrete seismic inverse problem and the TCC (4.3) for the semi-discrete FWI forward operator.

Theorem 4.3. *For any* $(\varrho^+,\mu^+,\pi^+)\in V_+^3$ *there exists an open ball* $B_r(\varrho^+,\mu^+,\pi^+)\subset V_+^3$ *such that*

$$\|(\varrho_1,\mu_1,\pi_1)-(\varrho_2,\mu_2,\pi_2)\|_{V^3}\lesssim\big\|\Phi(\varrho_1,\mu_1,\pi_1)-\Phi(\varrho_2,\mu_2,\pi_2)\big\|_{\mathcal{W}}\tag{4.2}$$

and

$$\begin{aligned}\big\|\Phi(\varrho_1,\mu_1,\pi_1)-\Phi(\varrho_2,\mu_2,\pi_2)-\Phi'(\varrho_2,\mu_2,\pi_2)[(\varrho_1,\mu_1,\pi_1)-(\varrho_2,\mu_2,\pi_2)]\big\|_{\mathcal{W}}&\\ \lesssim\|(\varrho_1,\mu_1,\pi_1)-(\varrho_2,\mu_2,\pi_2)\|_{V^3}\,\|\Phi(\varrho_1,\mu_1,\pi_1)-\Phi(\varrho_2,\mu_2,\pi_2)\|_{\mathcal{W}}&\end{aligned}\tag{4.3}$$

for all $(\varrho_i,\mu_i,\pi_i)\in B_r(\varrho^+,\mu^+,\pi^+)$, $i=1,2$.

5. Conclusion and discussion

In this work we have extended our results from [2] for FWI in the acoustic regime to the elastic regime: TCC holds for the corresponding forward operator and hence a variety of Newton-like solvers for the seismic inverse problem (2.12) are locally convergent regularization schemes. Moreover, (2.12) is locally wellposed.

There are two directions for future research. 1. Wave propagation in realistic media exhibits attenuation and dispersion which can be accurately modeled by the visco-elastic wave equation. We are confident that a version of Theorem 4.3 holds for this model as well and a proof is in reach as soon as a unique continuation principle for the visco-elastic system has been established. 2. From a practical point of view one would like to span the parameter space V by basis functions having a compact support (the local case). The resulting difficulties have already been discussed in Remarks 3.3 and A.3 of [2] which apply to the present situation with straightforward modifications. To master the local case we plan to adapt the concept of localized potentials [5].

Appendix A. Proof of Theorem 2.1

We decompose

$$\Phi=\Psi\circ F\circ B,\tag{A.1}$$

where $B\colon\mathcal{P}\subset L^\infty(D)^3\to\mathcal{L}^*(X):=\{J\in\mathcal{L}(X):J^*=J\}$ is given by

$$(\varrho,\mu,\pi)\mapsto\left(\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{v}\end{bmatrix}\mapsto\begin{bmatrix}C(\mu,\pi)^{-1}&\boldsymbol{0}\\ \boldsymbol{0}&\varrho\,\boldsymbol{I}_3\end{bmatrix}\begin{bmatrix}\boldsymbol{\sigma}\\ \boldsymbol{v}\end{bmatrix}\right).$$

Furthermore, we define $F\colon \mathsf{D}(F) \subset \mathcal{L}^*(X) \to \mathcal{S} := \mathcal{L}\big(W_0^{2,1}(D), \mathcal{C}([0,T],X)\big)$ by

$$P \mapsto \big(\boldsymbol{f} \mapsto u = \big(\boldsymbol{\sigma}, \boldsymbol{v}\big)^\top\big),$$

where u is the classical solution of (2.5) with $f = (\mathbf{0}, \boldsymbol{f})^\top$, $u_0 = (\mathbf{0}, \mathbf{0})^\top$, A from (2.6), and B is replaced by

$$P \in \mathsf{D}(F) := \big\{\Lambda \in \mathcal{L}^*(X) \ : \ \lambda_- \|x\|_X^2 < \langle \Lambda x, x\rangle_X < \lambda_+ \|x\|_X^2\big\}$$

for some $0 < \lambda_- < \lambda_+ < \infty$. Since, for any $(\varrho, \mu, \pi) \in \mathcal{P}$ and $\boldsymbol{M} \in L^2(D, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$,

$$\begin{aligned}\min\{2\mu_{\min}, 3\pi_{\min} - 4\mu_{\max}\}\, \boldsymbol{M} : \boldsymbol{M} &\le \big(C(\mu,\pi)\boldsymbol{M}\big) : \boldsymbol{M} \\ &\le \max\{2\mu_{\max}, 3\pi_{\max} - 4\mu_{\min}\}\, \boldsymbol{M} : \boldsymbol{M} \quad \text{a.e.,}\end{aligned}$$

(see, e.g., [15, Lmm. 50]), we have $B(\mathcal{P}) \subset \mathsf{D}(F)$ when we set

$$\lambda_+ := \max\{2\mu_{\max}, 3\pi_{\max} - 4\mu_{\min}, \varrho_{\max}\} \quad \text{and} \quad \lambda_- := \min\{2\mu_{\min}, 3\pi_{\min} - 4\mu_{\max}, \varrho_{\min}\}.$$

Then, the factorization of Φ in (A.1) is well defined and we obtain that

$$\Phi'(\boldsymbol{p}) = \Psi F'(B(\boldsymbol{p}))B'(\boldsymbol{p}), \qquad \boldsymbol{p} = (\varrho, \mu, \pi) \in \mathcal{P}.$$

Now, Theorem B.2 in [2] applies to F, and

$$\|F'(P_1) - F'(P_2)\|_{\mathcal{L}(\mathcal{L}^*(X),\mathcal{S})} \lesssim \|P_1 - P_2\|_{\mathcal{L}(X)}, \qquad P_1, P_2 \in \mathsf{D}(F),$$

where the Lipschitz constant only depends on T, λ_-, and λ_+. For any $\boldsymbol{p_i} = (\varrho_i, \mu_i, \pi_i) \in \mathcal{P}$, $i = 1, 2$, we proceed with

$$\begin{aligned}\|\Phi'(\boldsymbol{p}_1) - \Phi'(\boldsymbol{p}_2)\|_{\mathcal{L}(V^3,\mathcal{W})} &\le \|F'(B(\boldsymbol{p}_1))B'(\boldsymbol{p}_1) - F'(B(\boldsymbol{p}_2))B'(\boldsymbol{p}_2)\|_{\mathcal{L}(V^3,\mathcal{S})} \\ &\le \big\|\big(F'(B(\boldsymbol{p}_1)) - F'(B(\boldsymbol{p}_2))\big)B'(\boldsymbol{p}_1)\big\|_{\mathcal{L}(V^3,\mathcal{S})} \\ &\quad + \big\|F'(B(\boldsymbol{p}_2))\big(B'(\boldsymbol{p}_1) - B'(\boldsymbol{p}_2)\big)\big\|_{\mathcal{L}(V^3,\mathcal{S})} \\ &\lesssim \|B(\boldsymbol{p}_1) - B(\boldsymbol{p}_2)\|_{\mathcal{L}(X)}\|B'(\boldsymbol{p}_1)\|_{\mathcal{L}(V^3,\mathcal{L}^*(X))} \\ &\quad + \|F'(B(\boldsymbol{p}_2))\|_{\mathcal{L}(\mathcal{L}^*(X),\mathcal{S})}\|B'(\boldsymbol{p}_1) - B'(\boldsymbol{p}_2)\|_{\mathcal{L}(V^3,\mathcal{L}^*(X))}.\end{aligned} \tag{A.2}$$

In the next step we show uniform boundedness of $\|B'(\boldsymbol{p})\|_{\mathcal{L}(V^3,\mathcal{L}^*(X))}$ and $\|F'(B(\boldsymbol{p}))\|_{\mathcal{L}(\mathcal{L}^*(X),\mathcal{S})}$ for $\boldsymbol{p} = (\varrho, \mu, \pi) \in \mathcal{P}$. Using [8, Lmm. 6.3] we find that

$$B'(\boldsymbol{p})\boldsymbol{h} = \begin{bmatrix} -C(\mu,\pi)^{-1}C(h_1,h_2)C(\mu,\pi)^{-1} & \mathbf{0} \\ \mathbf{0} & h_3\,\boldsymbol{I}_3 \end{bmatrix}, \qquad \boldsymbol{h} = (h_1, h_2, h_3) \in V^3\,. \tag{A.3}$$

This implies $\|B'(\boldsymbol{p})\|_{\mathcal{L}(V^3,\mathcal{L}^*(X))} \lesssim 1$. The estimate for $\|F'(B(\boldsymbol{p}))\|_{\mathcal{L}(\mathcal{L}^*(X),\mathcal{S})}$ is more involved. We will be brief, but more details can be found in [2, App. B] or [9], were the same arguments have been used. Let $\overline{u} := F'(B(\boldsymbol{p}))[H]f$ for some $H \in \mathcal{L}^*(X)$ and $f = (\mathbf{0}, \boldsymbol{f})^\top$ with $\boldsymbol{f} \in W_0^{2,1}(D)$. Then, $\overline{u}$ is the mild solution of the abstract initial value problem

$$B(\boldsymbol{p})\overline{u}'(t) + A\overline{u}(t) = -HF(B(\boldsymbol{p}))f(t), \qquad \overline{u}(0) = 0,$$

in the time interval $[0,T]$. Regularity estimates (see, e.g., [8, Thm. 2.6]) show that

$$\|\overline{u}\|_{\mathcal{C}([0,T],X)} \lesssim \|H\|_{\mathcal{L}(X)}\|F(B(\boldsymbol{p}))f\|_{\mathcal{C}^1([0,T],X)} \lesssim \|H\|_{\mathcal{L}(X)}\|\boldsymbol{f}\|_{W^{2,1}(D)}.$$

Hence, $\|F'(B(\boldsymbol{p}))\|_{\mathcal{L}(\mathcal{L}^*(X),\mathcal{S})} \lesssim 1$ uniformly in $\boldsymbol{p} \in \mathcal{P}$. Substituting these estimates into (A.2) we obtain that

$$\begin{aligned}\|\Phi'(\boldsymbol{p}_1) - \Phi'(\boldsymbol{p}_2)\|_{\mathcal{L}(V^3,\mathcal{W})} &\lesssim \|B(\boldsymbol{p}_1) - B(\boldsymbol{p}_2)\|_{\mathcal{L}(X)} + \|B'(\boldsymbol{p}_1) - B'(\boldsymbol{p}_2)\|_{\mathcal{L}(V^3,\mathcal{L}^*(X))} \\ &\lesssim \|\boldsymbol{p}_1 - \boldsymbol{p}_2\|_{V^3} + \|B'(\boldsymbol{p}_1) - B'(\boldsymbol{p}_2)\|_{\mathcal{L}(V^3,\mathcal{L}^*(X))},\end{aligned}$$

where we used the mean value theorem and the estimate $\|B'(\boldsymbol{p})\|_{\mathcal{L}(V^3,\mathcal{L}^*(X))} \lesssim 1$ for any $\boldsymbol{p} = (\varrho, \mu, \pi) \in \mathcal{P}$ in the last step. The final estimate

$$\|B'(\boldsymbol{p}_1) - B'(\boldsymbol{p}_2)\|_{\mathcal{L}(V^3,\mathcal{L}^*(X))} \lesssim \|\boldsymbol{p}_1 - \boldsymbol{p}_2\|_{V^3}$$

can be validated either directly using (A.3) or by applying again the mean value theorem together with the estimate $\|B''(\boldsymbol{p})[\boldsymbol{h}_1, \boldsymbol{h}_2]\|_{\mathcal{L}(X)} \lesssim \|\boldsymbol{h}_1\|_{V^3}\|\boldsymbol{h}_2\|_{V^3}$ for $\boldsymbol{p} = (\varrho, \mu, \pi) \in \mathcal{P}$ and $\boldsymbol{h}_1, \boldsymbol{h}_2 \in V^3$, where the second derivative of C^{-1} in $B''(\boldsymbol{p})$ is given by eqn. (35) in [9].

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