

# Infinite-energy solutions to energy-critical nonlinear Schrödinger equations in modulation spaces

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# INFINITE-ENERGY SOLUTIONS TO ENERGY-CRITICAL NONLINEAR SCHRÖDINGER EQUATIONS IN MODULATION SPACES

ROBERT SCHIPPA

ABSTRACT. We prove new well-posedness results for energy-critical nonlinear Schrödinger equations in modulation spaces. This covers initial data with infinite mass and energy. The proof is carried out via bilinear refinements and adapted function spaces.

## 1. INTRODUCTION

In this paper we continue the study of modulation spaces as initial data for nonlinear Schrödinger equations started in [33]. Modulation spaces in the present context are used to model initial data, which are decaying slower than functions in  $L^2$ -based Sobolev spaces. These spaces are natural because of their invariance under the linear Schrödinger evolution in sharp contrast with the  $L^p$ -based Sobolev spaces for  $p \neq 2$ . Modulation spaces were introduced by Feichtinger [16]; see also subsequent joint works with Gröchenig [17, 18, 19]. The body of literature on modulation spaces is already huge, so we refer to [33, 12, 36, 2] and references therein for an overview with an emphasis on the use of modulation spaces in the context of dispersive equations.

In the work [33]  $L^p$ -smoothing estimates in modulation spaces were considered:

$$(1) \quad \|e^{it\Delta} u_0\|_{L^p([0,1], L^p(\mathbb{R}^d))} \lesssim \|u_0\|_{M_{p,2}^s(\mathbb{R}^d)}.$$

These turned out to be useful to prove well-posedness results for the cubic NLS

$$(2) \quad \begin{cases} i\partial_t u + \Delta u &= \pm |u|^2 u, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) &= u_0 \in M_{p,2}^s(\mathbb{R}). \end{cases}$$

The solution was placed in Strichartz spaces, in which the linear part was estimated by (1) and the nonlinear part was iterated with Strichartz estimates.

By frequency localization and rescaling arguments, the estimates (1) followed from  $\ell^2$ -decoupling for the paraboloid due to Bourgain–Demeter [8]. Let  $\mathcal{E}$  denote the Fourier extension operator for the (truncated) paraboloid:

$$\mathcal{E}f(t, x) = \int_{\{\xi \in \mathbb{R}^d; |\xi| < 1\}} e^{i(x \cdot \xi + t|\xi|^2)} f(\xi) d\xi.$$

Bourgain–Demeter proved the following estimates, which are sharp up to the  $\varepsilon$ -loss:

$$(3) \quad \|\mathcal{E}f\|_{L^p(B_{d+1}(0,R))} \lesssim_\varepsilon R^{s_{\text{dec}} + \varepsilon} \left( \sum_\sigma \|\mathcal{E}f_\sigma\|_{L^p(w_{B_{d+1}(0,R)})}^2 \right)^{1/2}$$

with  $s_{\text{dec}} = s_{\text{dec}}(p, d)$  given by

$$s_{\text{dec}} = \begin{cases} 0, & 2 \leq p \leq \frac{2(d+2)}{d}, \\ \frac{d}{4} - \frac{d+2}{2p}, & \frac{2(d+2)}{d} \leq p \leq \infty, \end{cases}$$

and  $f_\sigma$  denotes  $f \cdot 1_{B(x_\sigma, R^{-1/2})}$  such that the family of  $R^{-1/2}$ -balls are finitely overlapping. In [33] was pointed out how the right-hand side is related to the modulation space norm of the initial value by rescaling and a kernel estimate. Thus, (3) indeed gives (1) with  $s > 2s_{\text{dec}}(p, d)$ . It was also shown in [33] that  $s \geq 2s_{\text{dec}}(p, d)$  is necessary for (1) to hold true.

**Theorem 1.1** ( *$L^p$ -smoothing estimates in modulation spaces, [33, Theorem 1.1]*). *Let  $d \geq 1$  and  $p \geq 2$ . Then, (1) holds true for  $s > 2s_{\text{dec}}(p, d)$ .*

In the present work, we show bilinear refinements via Galilean invariance:

**Proposition 1.2.** *Let  $d \geq 3$ ,  $\varepsilon > 0$ , and  $N_1, N_2 \in 2^{\mathbb{N}_0}$  with  $N_2 \lesssim N_1$ . Then, we find the following estimate to hold:*

$$(4) \quad \|P_{N_1} e^{it\Delta} f_1 P_{N_2} e^{it\Delta} f_2\|_{L_{t,x}^2([0,1] \times \mathbb{R}^d)} \lesssim_\varepsilon N_2^{\frac{d-2}{2} + \varepsilon} \|P_{N_1} f_1\|_{M_{4,2}(\mathbb{R}^d)} \|P_{N_2} f_2\|_{M_{4,2}(\mathbb{R}^d)}.$$

Bilinear refinements go back to Bourgain [4, 5].

Next, we apply bilinear Strichartz estimates in modulation spaces to extend the local well-posedness theory of nonlinear Schrödinger equations. We consider the energy-critical nonlinear Schrödinger equation for  $d \in \{3, 4\}$ :

$$(5) \quad \begin{cases} i\partial_t u + \Delta u &= \pm |u|^{\frac{4}{d-2}} u & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) &= u_0 \in M_{4,2}^{1+\varepsilon}(\mathbb{R}^d). \end{cases}$$

The equation (5) is energy critical because the scaling

$$u(t, x) \rightarrow \lambda^{\frac{d-2}{2}} u(\lambda^2 t, \lambda x)$$

leaves the energy invariant:

$$E[u] = \int_{\mathbb{R}^d} \frac{|\nabla u(t, x)|^2}{2} \pm \frac{d-2}{d+2} |u|^{\frac{d+2}{d-2}} dx.$$

The corresponding scaling critical Sobolev space is  $\dot{H}^1(\mathbb{R}^d)$ . For local well-posedness in  $\dot{H}^1(\mathbb{R}^d)$  we refer to the survey by Bourgain [7]. Global well-posedness and scattering for the defocusing case is much harder and was proved for  $d = 3$  by Colliander *et al.* [14] and for  $d = 4$  by Ryckman–Viřan [32]; see also references therein and Bourgain’s seminal contribution [6] in the radially symmetric case. Sharp conditions for global well-posedness and scattering of the focusing equation in the radial case were proved by Kenig–Merle [27]. We refer to the solutions with initial data in  $M_{4,2}^{1+\varepsilon}$  as infinite energy solutions because we can find a sequence of Schwartz functions  $(f_n) \subseteq \mathcal{S}(\mathbb{R}^d)$  with  $\|f_n\|_{M_{4,2}^{1+\varepsilon}} \leq C$ , but  $\|f_n\|_{H^1} \uparrow \infty$ . For this purpose, consider a mollified indicator function  $f_n = \chi_1 * 1_{B(0,n)}$  supported in  $B(0, n+1)$ , which has Fourier support rapidly decaying off  $B(0, 2)$ . We normalize  $1 = \|f_n\|_{L^4(\mathbb{R}^d)} \sim \|f_n\|_{M_{4,2}^{1+\varepsilon}(\mathbb{R}^d)}$ . But then,

$$\|f_n\|_{H^1(\mathbb{R}^d)} \sim \|f_n\|_{L^2(\mathbb{R}^d)} \uparrow \infty.$$

The reason is that we allow for  $L^4$ -admissible decay at infinity in  $M_{4,2}^{1+\varepsilon}$ , which is slower than for  $L^2$ -based Sobolev spaces.

Previous results on infinite energy solutions to nonlinear Schrödinger equations are due to Braz e Silva *et al.* [9] with initial data in weak  $L^p$ -spaces. The results in [9] do not cover the energy critical equations though; see also [10]. Moreover, weak  $L^p$ -spaces are not invariant under the linear propagation in contrast with modulation

spaces. The first results on infinite energy solutions are due to Cazenave–Weissler [11], who consider initial data with finite linear solution in a certain  $L^p$ -norm. The results in [11] do not cover the energy critical case.  $L^2$ -based Besov spaces were considered by Planchon [31]. We also mention the recent contributions of Correia *et al.* [15, 1].

Moreover, we remark how the arguments of [33] extend to  $L^2$ -critical equations for  $d \in \{1, 2\}$ , i.e., the quintic NLS on the real line or the cubic NLS in  $\mathbb{R}^2$ . Note that

$$H^s(\mathbb{R}^d) \sim M_{2,2}^s(\mathbb{R}^d) \hookrightarrow M_{p,2}^s(\mathbb{R}^d)$$

for  $p \geq 2$  and  $s \geq 0$ . In this sense, the following well-posedness results are almost critical:

**Theorem 1.3.** *Let  $s > 0$  and  $T > 0$ .*

(1) *Then, the equation*

$$(6) \quad \begin{cases} i\partial_t u + \Delta u &= \pm |u|^4 u, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0) &= u_0 \in M_{6,2}^s(\mathbb{R}) + L^2(\mathbb{R}) \end{cases}$$

*is locally well-posed in  $X_T = C([0, T], L^2(\mathbb{R}) + M_{6,2}^s(\mathbb{R})) \cap L_t^6([0, T], L^6(\mathbb{R}))$  provided that  $\|u_0\|_{M_{6,2}^s(\mathbb{R}) + L^2(\mathbb{R})} \leq \varepsilon(T)$ .*

(2) *The equation*

$$(7) \quad \begin{cases} i\partial_t u + \Delta u &= \pm |u|^2 u, & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0) &= u_0 \in M_{4,2}^s(\mathbb{R}^2) + L^2(\mathbb{R}^2) \end{cases}$$

*is locally well-posed in  $X_T = C([0, T], L^2(\mathbb{R}^2) + M_{4,2}^s(\mathbb{R}^2)) \cap L_t^4([0, T], L^4(\mathbb{R}^2))$  provided that  $\|u_0\|_{M_{4,2}^s(\mathbb{R}^2) + L^2(\mathbb{R}^2)} \leq \varepsilon(T)$ .*

Note how above we choose the size of the initial data in terms of the existence time. It would be more practical to consider  $T = T(u_0)$ , which is possible, but not detailed for simplicity of presentation (see the proof of Theorem 1.4 below).

For  $d \geq 3$  the derivative loss in the high frequencies of the  $L^4$ -Strichartz estimate has to be ameliorated via bilinear estimates. We show the following:

**Theorem 1.4.** *Let  $d \in \{3, 4\}$ , and  $\varepsilon > 0$ . Then (5) is analytically locally well-posed in  $X_T \hookrightarrow C([0, T], M_{4,2}^{1+\varepsilon}(\mathbb{R}^d))$ . For any  $u_0 \in M_{4,2}^{1+\varepsilon}(\mathbb{R}^d)$  there is  $T = T(u_0)$  such that there is a unique solution  $u \in X_T$  to (5), and the data-to-solution mapping analytically depends on the initial value.*

The first local well-posedness results on energy critical nonlinear Schrödinger equations in the periodic setting are due to Herr–Tataru–Tzvetkov [23, 24]. In these works, improved bilinear or trilinear estimates were proved via orthogonality in time. This proof was simplified by Killip–Vişan [28], which is transferred to modulation spaces presently. Killip–Vişan pointed out how the bilinear refinements can be used to show the well-posedness result for the energy critical equation. A few remarks on global results in the periodic setting are in order: Herr–Tataru–Tzvetkov [23] proved global well-posedness for small initial data by energy conservation. Since the Sobolev embedding  $H^1(\mathbb{T}^d) \hookrightarrow L^{\frac{d+2}{d-2}}(\mathbb{T}^d)$  is sharp, the straight-forward use of energy conservation requires smallness of the  $H^1(\mathbb{T}^d)$  norm. Ionescu–Pausader [25] subsequently proved global well-posedness for large initial data in the defocusing case for  $d = 3$  (see also [35, 34]). But the global results fundamentally build on energy conservation, which is not at disposal for initial data in  $M_{4,2}^s(\mathbb{R}^d)$ , since these possibly have infinite mass and energy. Thus, global results, even in the defocusing

case remain open for initial data in  $M_{4,2}^s(\mathbb{R}^d)$ . On the other hand, the classical blow-up arguments (cf. [27]) in the focusing case show that global solutions need not exist, if the energy is negative. For ill-posedness results for the nonlinear Schrödinger equation with initial data in modulation spaces we refer to Bhimani–Carles [3].

For further reading, we also refer to the very recent contribution [13], in which *unconditional uniqueness* of solutions in  $C([0, T], H^1(X^d))$  for energy critical Schrödinger equations is proved for  $d \in \{3, 4\}$  and  $X \in \{\mathbb{T}, \mathbb{R}\}$ . Chen and Holmer [13] use the Gross–Pitaevskii hierarchy, which was previously considered by Herr and Sohinger [22] to cover the whole subcritical range for  $d = 4$ ; see also references therein.

*Outline of the paper.* In Section 2 we recall basic facts about modulation spaces, and we introduce the function spaces used in the proof of Theorem 1.4. In Section 3 we show Proposition 3.1, by which we prove Theorem 1.4 in Section 4. Theorem 1.3 is proved in Section 4 with linear Strichartz estimates for comparison.

## 2. PRELIMINARIES

**2.1. Modulation spaces.** The modulation spaces  $M_{p,q}^s(\mathbb{R}^d)$  for  $d \geq 1$ ,  $s \in \mathbb{R}$ ,  $p, q \in [1, \infty]$  are defined through an isometric decomposition in Fourier space. Let  $(\sigma_k)_{k \in \mathbb{Z}^d}$  with  $\sigma_k = \sigma(\cdot - k)$  and  $\sigma \in C_c^\infty(B(0, 1))$  denote a smooth partition of unity. We define

$$M_{p,q}^s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M_{p,q}^s(\mathbb{R}^d)} = \|(\langle k \rangle^s \|\sigma_k(D)f\|_{L^p(\mathbb{R}^d)})_{k \in \mathbb{Z}^d}\|_{\ell^q} < \infty\}.$$

We write  $M_{p,q}(\mathbb{R}^d) := M_{p,q}^0(\mathbb{R}^d)$  for brevity. We have the following embeddings in the standard Besov scale (cf. [33, Section 1]): By the embedding  $\ell^{q_1} \hookrightarrow \ell^{q_2}$  for  $q_1 \leq q_2$  and Bernstein’s inequality, we have

$$\begin{aligned} M_{p,q_1}^s(\mathbb{R}^d) &\hookrightarrow M_{p,q_2}^s(\mathbb{R}^d) \quad (q_1 \leq q_2), \\ M_{p_1,q}^s(\mathbb{R}^d) &\hookrightarrow M_{p_2,q}^s(\mathbb{R}^d) \quad (p_1 \leq p_2). \end{aligned}$$

By Plancherel’s theorem, we have

$$(8) \quad M_{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d).$$

Moreover, we have from kernel estimates with  $p = 1$  and  $p = \infty$  and interpolation with (8) the estimates

$$\begin{aligned} M_{p,p'} &\hookrightarrow L^p \hookrightarrow M_{p,p} \quad (2 \leq p \leq \infty), \\ M_{p,p} &\hookrightarrow L^p \hookrightarrow M_{p,p'} \quad (1 \leq p \leq 2). \end{aligned}$$

Lastly, we note that

$$M_{p,q_1}^{s_1}(\mathbb{R}^d) \hookrightarrow M_{p,q_2}^{s_2}(\mathbb{R}^d)$$

provided that  $s_1 - s_2 > d(\frac{1}{q_2} - \frac{1}{q_1}) > 0$  as a consequence of Hölder’s inequality.

**2.2. Adapted function spaces.** We use  $U^p$ -/ $V^p$ -spaces taking values in modulation spaces as iteration spaces.  $U^p$ -/ $V^p$ -spaces based on  $L^2$ -based Sobolev spaces go back to unpublished notes of Tataru in the context of wave maps (cf. [29]). In case the base space is a general Banach space, we refer to [30] (see also the previous work by Hadac–Herr–Koch [20, 21]). We shall be brief in the following.

Let  $\mathcal{Z}$  be the set of finite partitions  $-\infty < t_0 < t_1 < \dots < t_K = \infty$  and let  $\mathcal{Z}_0$  be the set of finite partitions  $-\infty < t_0 < t_1 < \dots < t_K \leq \infty$ . We consider  $U^p$ -/ $V^p$ -spaces taking values in modulation spaces  $M_{p,q}(\mathbb{R}^d)$ . Denote the value space in the following by  $E$ .

**Definition 2.1.** Let  $1 < p < \infty$  and  $\{t_k\}_{k=0}^K \in \mathcal{Z}$  and  $\{\phi\}_{k=0}^{K-1} \subseteq E$  with  $\sum_{k=0}^{K-1} \|\phi_k\|_E^p = 1$  and  $\phi_0 = 0$ . The function  $a : \mathbb{R} \rightarrow E$  defined by  $a = \sum_{k=1}^K 1_{[t_{k-1}, t_k)} \phi_{k-1}$  is said to be a  $U^p$ -atom. We define the atomic space

$$U^p(E) = \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j : a_j : U^p\text{-atom}, (\lambda_j) \in \ell^1 \right\}$$

with norm

$$\|u\|_{U^p} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j, (\lambda_j) \in \ell^1, a_j : U^p\text{-atom} \right\}.$$

Subspaces are considered as in [20, Proposition 2.2]. The spaces of  $p$ -variation were already considered by Wiener [37] (see [30, Definition 4.8]).

**Definition 2.2.** Let  $1 \leq p < \infty$ .  $V^p(E)$  is defined as normed space of all functions  $v : \mathbb{R} \rightarrow E$  such that  $\lim_{t \rightarrow \pm\infty} v(t)$  exists,  $v(\infty) := 0$  (this is purely conventional and does not necessarily coincide with the limit), and  $v(-\infty) = \lim_{t \rightarrow -\infty} v(t)$ . The norm is given by

$$\|v\|_{V^p} = \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left( \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_E^p \right)^{\frac{1}{p}}$$

is finite. Let  $V_-^p$  denote the closed subspace of  $V^p$  with  $\lim_{t \rightarrow -\infty} v(t) = 0$ .

We have the embeddings (cf. [30, Lemma 4.13]):

$$U^p \hookrightarrow V_{rc,-}^p \hookrightarrow U^q$$

for  $1 < p < q < \infty$ . Recall the duality  $(M_{p,q}(\mathbb{R}^d))' \simeq M_{p',q'}(\mathbb{R}^d)$  for  $1 < p, q < \infty$ , which is established via the dual pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : M_{p,q}(\mathbb{R}^d) \times M_{p',q'}(\mathbb{R}^d) &\rightarrow \mathbb{C} \\ (f, g) &\mapsto \int_{\mathbb{R}^d} f \bar{g} dx. \end{aligned}$$

We have the following duality:

**Theorem 2.3** ([30, Theorem 4.14]). *Let  $1 < p < \infty$ . We have*

$$(U^p(E))^* \simeq V^{p'}(E')$$

*in the sense that*

$$T : V^{p'}(E') \rightarrow (U^p(E))^*, \quad T(v) = B(\cdot, v)$$

*is an isometric isomorphism.*

We have an explicit description of  $B$  for sufficiently regular functions:

**Proposition 2.4.** *Let  $1 < p < \infty$ ,  $u \in V_-^1$  be absolutely continuous on compact intervals and  $v \in V^{p'}(E')$ . Then,*

$$B(u, v) = - \int_{-\infty}^{\infty} \langle u'(t), v(t) \rangle dt.$$

Later we rely on computing the  $U^p$ -norm with the aid of duality:

$$(9) \quad \|u\|_{U^p(E)} = \sup_{v \in V^{p'}(E') : \|v\|_{V^{p'}(E')} = 1} |B(u, v)|.$$

We remark that the spaces can as well be localized to an interval, in which case we write  $U^p(I; E)$ ,  $V^p(I; E)$ . We furthermore define the space  $DU^p(I; E)$ :

$$DU^p(I; E) = \{f = u' : u \in U^p(I; E)\}$$

with the derivative considered in the distributional sense and

$$\|f\|_{DU^p(I; E)} = \|u\|_{U^p(I; E)}.$$

By Theorem 2.3, we have  $(DU^p(I; E))^* \simeq V^{p'}(I; E')$  with respect to a bilinear mapping, which for  $f \in L^1(I) \hookrightarrow DU^p(I; E)$  is given by

$$\tilde{B}(f, v) = \int_a^b \langle f(t), v(t) \rangle dt.$$

We adapt  $U^p$ -/ $V^p$ -spaces to the linear Schrödinger propagation  $e^{it\Delta}$  as usual:

$$(10) \quad \|u\|_{X_{\Delta}^p(I; E)} = \|e^{-it\Delta}u\|_{X^p(I; E)}$$

with  $X \in \{U; V; DU\}$ .

### 3. BILINEAR REFINEMENTS

By Galilean invariance, we can show bilinear estimates with derivative loss only in the low frequency. In the context of Strichartz estimates on tori, we refer to [28, 4]. Starting point is the following linear Strichartz estimate:

$$(11) \quad \|e^{it\Delta}u_0\|_{L^4([0,1] \times \mathbb{R}^d)} \lesssim \|u_0\|_{M_{4,2}^s(\mathbb{R}^d)}.$$

**Proposition 3.1.** *Let  $1 \leq K \ll N$  and suppose that (11) holds true. Then, we find the following estimate to hold:*

$$\|P_N e^{it\Delta}u_0 P_K e^{it\Delta}v_0\|_{L^2([0,1] \times \mathbb{R}^d)} \lesssim K^{2s} \|P_N u_0\|_{M_{4,2}} \|P_K v_0\|_{M_{4,2}}.$$

*Proof.* Let  $(Q_{K'})_{K'}$  be a family of frequency projections to balls of size  $K$  in  $\mathbb{R}^d$  whose supports are covering  $B(0, 2N) \setminus B(0, N/2)$  finitely overlapping. By almost orthogonality, we find

$$\begin{aligned} & \|P_N e^{it\Delta}u_0 P_K e^{it\Delta}v_0\|_{L^2([0,1] \times \mathbb{R}^d)}^2 \\ & \lesssim \sum_{K'} \|P_N Q_{K'} e^{it\Delta}u_0 P_K e^{it\Delta}v_0\|_{L^2([0,1] \times \mathbb{R}^d)}^2 \\ & = \sum_{K'} \|P_N Q_{K'} e^{it\Delta}u_0\|_{L^4([0,1] \times \mathbb{R}^d)}^2 \|P_K e^{it\Delta}v_0\|_{L^4([0,1] \times \mathbb{R}^d)}^2. \end{aligned}$$

We apply (11) to the second factor and to the first factor after Galilean transform, which yields

$$\lesssim K^{4s} \sum_{K'} \|Q_{K'} u_0\|_{M_{4,2}}^2 \|P_K v_0\|_{M_{4,2}}^2 \lesssim K^{4s} \|P_N u_0\|_{M_{4,2}}^2 \|P_K v_0\|_{M_{4,2}}^2.$$

The ultimate estimate follows by the finitely overlapping property and the definition of the modulation spaces.  $\square$

This yields Proposition 1.2 by Theorem 1.1. In the next step we use the transfer principle to derive an estimate for  $V_{\Delta}^2 M_{4,2}$ -functions.

**Proposition 3.2.** *Let  $K, N \in 2^{\mathbb{N}_0}$  and  $1 \leq K \ll N$ . Suppose that (11) holds. Then, we find the following estimate to hold:*

$$(12) \quad \|P_N u P_K v\|_{L^2_{t,x}([0,1] \times \mathbb{R}^d)} \lesssim K^{2s} \|P_N u\|_{V^2_{\Delta} M_{4,2}} \|P_K v\|_{V^2_{\Delta} M_{4,2}}.$$

*Proof.* By almost orthogonality, we can write

$$\|P_N u P_K v\|_{L^2}^2 \lesssim \sum_{K'} \|Q_{K'} P_N u P_K v\|_{L^2_{t,x}([0,1] \times \mathbb{R}^d)}^2$$

with  $(Q_{K'})_{K'}$  like above. We apply Hölder's inequality to find

$$\lesssim \sum_{K'} \|Q_{K'} P_N u\|_{L^4_{t,x}([0,1] \times \mathbb{R}^d)}^2 \|P_K v\|_{L^4_{t,x}([0,1] \times \mathbb{R}^d)}^2.$$

We write  $P_K v = \sum_m a_m g_m$  with  $g_m$  a  $U^4_{\Delta} M_{4,2}$ -atom:

$$g_m = \sum_j 1_{I_j^m} e^{it\Delta} f_j^m, \quad \sum_j \|f_j^m\|_{M_{4,2}}^4 = 1.$$

Consequently,

$$\begin{aligned} \|P_K v\|_{L^4_{t,x}([0,1] \times \mathbb{R}^d)} &\leq \sum_m |a_m| \|P_K g_m\|_{L^4_{t,x}([0,1] \times \mathbb{R}^d)} \\ &\leq \sum_m |a_m| \left( \sum_j \|P_K e^{it\Delta} f_j^m\|_{L^4_{t,x}(I_j^m \times \mathbb{R}^d)}^4 \right)^{\frac{1}{4}} \\ &\lesssim \sum_m |a_m| \left( \sum_j \|f_j^m\|_{M_{4,2}}^4 \right)^{\frac{1}{4}} \\ &\lesssim K^s \sum_m |a_m| \lesssim K^s (1 + \varepsilon) \|P_K v\|_{U^4_{\Delta} M_{4,2}} \end{aligned}$$

for any  $\varepsilon > 0$  by choice of  $(a_m) \in \ell^1$ . Likewise, by an additional Galilean transform, we find

$$\|Q_{K'} P_N u\|_{L^4_{t,x}([0,1] \times \mathbb{R}^d)} \lesssim K^s \|Q_{K'} P_N u\|_{U^4_{\Delta} M_{4,2}}.$$

We use the embedding  $V^2_{\Delta} \hookrightarrow U^4_{\Delta}$  and carry out the square sum over  $K'$  to find

$$\begin{aligned} &\sum_{K'} \|Q_{K'} P_N u\|_{L^4_{t,x}([0,1] \times \mathbb{R}^d)}^2 \|P_K v\|_{L^4_{t,x}([0,1] \times \mathbb{R}^d)}^2 \\ &\lesssim K^{4s} \sum_{K'} \|Q_{K'} P_N u\|_{U^4_{\Delta} M_{4,2}}^2 \|P_K v\|_{U^4_{\Delta} M_{4,2}}^2 \\ &\lesssim K^{4s} \sum_{K'} \|Q_{K'} P_N u\|_{V^2_{\Delta} M_{4,2}}^2 \|P_K v\|_{V^2_{\Delta} M_{4,2}}^2 \\ &\lesssim K^{4s} \|P_N u\|_{V^2_{\Delta} M_{4,2}}^2 \|P_K v\|_{V^2_{\Delta} M_{4,2}}^2. \end{aligned}$$

The proof is complete.  $\square$

#### 4. LOCAL WELL-POSEDNESS IN MODULATION SPACES

This section is devoted to the proof of Theorems 1.3 and 1.4. We begin with the proof of Theorem 1.3, which is carried out via linear Strichartz estimates (cf. [26, Theorem 1.2]).

*Proof of Theorem 1.3.* We give the proof of (1) in detail. The key ingredients are still like in [33] smoothing and Strichartz estimates. Let  $u_0 = f_1 + f_2$  with  $f_1 \in M_{6,2}^s(\mathbb{R})$  and  $f_2 \in L^2(\mathbb{R})$ . Then, Theorem 1.1 yields

$$\|U(t)f_1\|_{L^6([0,T],L^6(\mathbb{R}))} \lesssim \langle T \rangle^{\frac{1}{6}} \|f_1\|_{M_{6,2}^s(\mathbb{R})}$$

and by Strichartz estimates we find

$$\|U(t)f_2\|_{L^6([0,T],L^6(\mathbb{R}))} \lesssim \|f_2\|_{L^2(\mathbb{R})}.$$

Furthermore, since  $U(t)(L^2(\mathbb{R}) + M_{6,2}^s(\mathbb{R})) = L^2(\mathbb{R}) + M_{6,2}^s(\mathbb{R})$ , we find

$$\|U(t)u_0\|_{L^\infty([0,T],L^2(\mathbb{R})+M_{6,2}^s(\mathbb{R}))} \lesssim \|u_0\|_{L^2(\mathbb{R})+M_{6,2}^s(\mathbb{R})}.$$

The nonlinear estimate is concluded by the inhomogeneous Strichartz estimates

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)\Delta} (|u|^4 u)(s) ds \right\|_{L^6([0,T],L^6(\mathbb{R}))} &\lesssim \| |u|^4 u \|_{L^{6/5}([0,T],L^{6/5}(\mathbb{R}))} \\ &\lesssim \|u\|_{L^6([0,T],L^6(\mathbb{R}))}^5. \end{aligned}$$

Similarly,

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)\Delta} (|u|^4 u)(s) ds \right\|_{L^\infty([0,T],L^2(\mathbb{R}))} &\lesssim \| |u|^4 u \|_{L^{6/5}([0,T],L^{6/5}(\mathbb{R}))} \\ &\lesssim \|u\|_{L^6([0,T],L^6(\mathbb{R}))}^5. \end{aligned}$$

This finishes the proof of (1). The difference with the cubic NLS on  $\mathbb{R}$  analyzed in [33] is that we cannot afford to apply Hölder's inequality in time. This gives the small data constraint. Regarding the claim (2), we note that in two dimensions,  $p = q = 4$  are sharp Strichartz indices and by Theorem 1.1 we have the smoothing estimate

$$\|U(t)f\|_{L^4([0,T],L^4(\mathbb{R}^2))} \lesssim \langle T \rangle^{\frac{1}{4}} \|f\|_{M_{4,2}^s(\mathbb{R}^2)}$$

for  $s > 0$ . □

We turn to the proof of Theorem 1.4 in earnest. As iteration space, we consider  $X^s = \ell_N^2 U_\Delta^2 M_{4,2}^s$  for  $s > 1$  (cf. Section 2). We have for the norm

$$\|u\|_{X^s} = \left( \sum_N N^{2s} \|P_N u\|_{U_\Delta^2 M_{4,2}}^2 \right)^{\frac{1}{2}}.$$

We let furthermore

$$\|v\|_{Y^s} = \left( \sum_N N^{2s} \|P_N v\|_{V_\Delta^2 M_{4,2}}^2 \right)^{\frac{1}{2}}$$

and have the embedding  $X^s \hookrightarrow Y^s$ .

With bilinear estimates in adapted function spaces like in [28] available, we can apply the arguments from [28] to prove Theorem 1.4. We have the following analog of [28, Proposition 4.1]:

**Proposition 4.1.** *Let  $d \in \{3, 4\}$ ,  $s > 1$ , and  $F(u) = \pm |u|^{\frac{4}{d-2}} u$ . Then, for any  $0 < T \leq 1$ , we find the following estimates to hold:*

$$(13) \quad \left\| \int_0^t e^{i(t-s)\Delta} F(u(s)) ds \right\|_{X^s([0,T])} \lesssim \|u\|_{X^s([0,T])}^{\frac{d+2}{d-2}}$$

and

$$(14) \quad \begin{aligned} & \left\| \int_0^t e^{i(t-s)\Delta} [F(u+w)(s) - F(u(s))] ds \right\|_{X^s([0,T])} \\ & \lesssim \|w\|_{X^s([0,T])} (\|u\|_{X^s([0,T])} + \|w\|_{X^s([0,T])})^{\frac{4}{d-2}}. \end{aligned}$$

The implicit constants do not depend on  $T$ .

*Proof.* We only have to prove (14) because (13) is a special case. By duality, it is enough to show

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^d} [F(u+w)(t) - F(u)(t)] v(t, x) dx dt \right| \\ & \lesssim \|v\|_{Y^{-s}([0,T])} \|u\|_{X^s([0,T])} (\|u\|_{X^s([0,T])} + \|w\|_{X^s([0,T])})^{\frac{4}{d-2}}. \end{aligned}$$

For the above display, it is enough to show

$$(15) \quad \begin{aligned} & \sum_{N_0 \geq 1} \sum_{N_1 \geq \dots \geq N_{\frac{d+2}{d-2}} \geq 1} \left| \int_0^T \int_{\mathbb{R}^d} v_{N_0}(t, x) \prod_{j=1}^{\frac{d+2}{d-2}} u_{N_j}^{(j)}(t, x) dx dt \right| \\ & \lesssim \|v\|_{Y^{-s}} \prod_{j=1}^{\frac{d+2}{d-2}} \|u^{(j)}\|_{X^s([0,T])}. \end{aligned}$$

The proof of (15) follows from linear and bilinear Strichartz estimates combined with Bernstein's inequality. We shall only show the variant of the Killip–Viřan argument for  $d = 3$  to avoid redundancy. In the following let  $\varepsilon = s - 1 > 0$ .

**Case I:**  $d = 3$ . By Littlewood–Paley theory, the two highest frequencies have to be comparable.

**Case I.1:**  $N_0 \sim N_1 \geq \dots \geq N_5$ : We apply Proposition 3.1 to  $v_{N_0} u_{N_2}^{(2)}$  and  $u_{N_1}^{(1)} u_{N_3}^{(3)}$  and estimate the remaining factors in  $L_{t,x}^\infty$ . The estimate in  $L_{t,x}^\infty$  is not a local smoothing estimate, but due to Bernstein's and the Cauchy-Schwarz inequality:

$$\|P_N f\|_{L_x^\infty} \lesssim \|P_N f\|_{M_{\infty,1}} \lesssim \|P_N f\|_{M_{4,1}} \lesssim N^{\frac{3}{2}} \|P_N f\|_{M_{4,2}}.$$

We write  $\mathcal{N}_1 = \{(N_0, N_1, \dots, N_5) : N_0 \sim N_1 \geq \dots \geq N_5\}$  for brevity. The estimates yield

$$\begin{aligned} & \sum_{\mathcal{N}_1} \left| \int_0^T \int_{\mathbb{R}^d} v_{N_0}(t, x) u_{N_1}^{(1)}(t, x) \dots u_{N_5}^{(5)}(t, x) dx dt \right| \\ & \lesssim \sum_{\mathcal{N}_1} \|v_{N_0} u_{N_2}^{(2)}\|_{L_{t,x}^2} \|u_{N_1}^{(1)} u_{N_3}^{(3)}\|_{L_{t,x}^2} \|u_{N_4}^{(4)}\|_{L_{t,x}^\infty} \|u_{N_5}^{(5)}\|_{L_{t,x}^\infty} \\ & \lesssim \sum_{\mathcal{N}_1} N_2^{\frac{1}{2} + \frac{\varepsilon}{2}} N_3^{\frac{1}{2} + \frac{\varepsilon}{2}} N_4^{\frac{3}{2}} N_5^{\frac{3}{2}} \|v_{N_0}\|_{V_{\Delta}^2 M_{4,2}} \prod_{i=1}^5 \|u_{N_i}^{(i)}\|_{V_{\Delta}^2 M_{4,2}} \\ & \lesssim \|v\|_{Y^{-s}} \prod_{i=1}^5 \|u^{(i)}\|_{Y^s}. \end{aligned}$$

By the embedding  $X^s \hookrightarrow Y^s$  the proof of Case I.1 is complete.

**Case I.2:**  $N_0 \lesssim N_1 \sim N_2 \geq N_3 \geq N_4 \geq N_5$ . Denote the summation set with  $\mathcal{N}_2$ .

We apply two bilinear estimates to  $v_{N_0} u_{N_1}^{(1)}$  and  $u_{N_2}^{(2)} u_{N_3}^{(3)}$  and  $L_{t,x}^\infty$ -estimates to the other factors to find

$$\begin{aligned}
& \sum_{N_2} \left| \int_0^T \int_{\mathbb{R}^d} v_{N_0}(t,x) u_{N_1}^{(1)}(t,x) \dots u_{N_5}^{(5)}(t,x) dx dt \right| \\
& \lesssim \sum_{N_2} \|v_{N_0} u_{N_1}^{(1)}\|_{L_{t,x}^2} \|u_{N_2}^{(2)} u_{N_3}^{(3)}\|_{L_{t,x}^2} \|u_{N_4}^{(4)}\|_{L_{t,x}^\infty} \|u_{N_5}^{(5)}\|_{L_{t,x}^\infty} \\
& \lesssim \sum_{N_2} N_0^{\frac{1}{2}+\frac{\varepsilon}{2}} N_3^{\frac{1}{2}+\frac{\varepsilon}{2}} N_4^{\frac{3}{2}} N_5^{\frac{3}{2}} \|v_{N_0}\|_{V_{\Delta}^2 M_{4,2}} \prod_{i=1}^5 \|u_{N_i}^{(i)}\|_{V_{\Delta}^2 M_{4,2}} \\
& \lesssim \sum_{N_2} \frac{N_0^{\frac{3}{2}+\frac{3\varepsilon}{2}} N_4^{\frac{1}{2}-\varepsilon} N_5^{\frac{1}{2}-\varepsilon}}{N_1^s N_2^s N_3^{\frac{s}{2}}} \|v_{N_0}\|_{Y^{-s}} \prod_{i=1}^5 \|u_{N_i}^{(i)}\|_{Y^s} \\
& \lesssim \|v\|_{Y^{-s}} \prod_{i=1}^5 \|u^{(i)}\|_{Y^s}.
\end{aligned}$$

This finishes the proof of Case *I*. For the details of the proof of Case *II* for  $d = 4$  we refer to [28].  $\square$

We can complete the proof of Theorem 1.4 along the lines of [23, 28] with Proposition 4.1 at hand.

*Proof of Theorem 1.4.* For small initial data we can construct a solution on  $[0, 1]$  by showing that

$$\Phi(u)(t) := e^{it\Delta} u_0 \mp i \int_0^t e^{i(t-s)\Delta} F(u(s)) ds$$

is a contraction mapping within

$$B = \{u \in X^s([0, 1]) \cap C_t([0, 1], M_{4,2}^s(\mathbb{R}^d)) : \|u\|_{X^s} \leq 2\eta\}$$

endowed with  $d(u, v) := \|u - v\|_{X^s([0,1])}$ . This is a consequence of Proposition 4.1 by observing that  $\Phi$  maps  $B$  into itself by (13) and  $\Phi$  is indeed contracting by (14). This proves Theorem 1.4 for small data.

For large initial data, we argue with a frequency cutoff. Let  $u_0 \in M_{4,2}^s(\mathbb{R}^d)$  with

$$\|u_0\|_{M_{4,2}^s(\mathbb{R}^d)} \leq A$$

for some  $0 < A < \infty$ . We consider

$$B = \{u \in X^s([0, T]) \cap C_t([0, T], M_{4,2}^s(\mathbb{R}^d)) : \|u\|_{X^s([0,T])} \leq 2A, \quad \|u_{>N}\|_{X^s([0,T])} \leq 2\delta\}$$

under the metric  $d(u, v) := \|u - v\|_{X^s([0, T])}$ .  
First, we see that  $\Phi$  indeed maps  $B$  to itself:

$$\begin{aligned}
\|\Phi(u)\|_{X^s} &\leq \|e^{it\Delta}u_0\|_{X^s} + \left\| \int_0^t e^{i(t-s)\Delta} F(u_{\leq N}(s)) ds \right\|_{X^s} \\
&\quad + \left\| \int_0^t e^{i(t-s)\Delta} [F(u)(s) - F(u_{\leq N})(s)] ds \right\|_{X^s} \\
&\leq \|u_0\|_{M_{4,2}^s} + C\|F(u_{\leq N})\|_{L_t^1 M_{4,2}^s} + C\|u_{\geq N}\|_{X^s} \|u\|_{X^s}^{\frac{4}{d-2}} \\
&\leq A + CT\|u_{\leq N}\|_{L_t^\infty M_{4,2}^s} \|u_{\leq N}\|_{L_t^\infty M_{\infty,1}^s}^{\frac{4}{d-2}} + C(2\delta)(2A)^{\frac{4}{d-2}} \\
&\leq A + CTN^{\frac{6}{d-2}}(2A)^{\frac{d+2}{d-2}} + C(2\delta)(2A)^{\frac{4}{d-2}} \leq 2A
\end{aligned}$$

provided  $\delta$  is chosen small enough depending on  $A$ , and  $T$  is chosen small enough depending on  $A$  and  $N$ .

Next, we decompose  $F(u) = F_1(u) + F_2(u)$ , where

$$F_1(u) = O(u_{>N}^2 u^{\frac{6-d}{d-2}}) \text{ and } F_2(u) = O(u_{\leq N}^{\frac{4}{d-2}} u).$$

We estimate with the Hölder-like inequality for modulation spaces (cf. [12, Theorem 4.3])

$$\begin{aligned}
&\|P_{>N}\Phi(u)\|_{X^s} \\
&\leq \|e^{it\Delta}P_{>N}u_0\|_{X^s} + \left\| \int_0^t e^{i(t-s)\Delta} F_1(u(s)) ds \right\|_{X^s} \\
&\quad + \left\| \int_0^t e^{i(t-s)\Delta} F_2(u(s)) ds \right\|_{X^s} \\
&\leq \|P_{>N}u_0\|_{M_{4,2}^s(\mathbb{R}^d)} + C\|u_{>N}\|_{X^s}^2 \|u\|_{X^s}^{\frac{6-d}{d-2}} + C\|F_2(u)\|_{L_t^1 M_{4,2}^s} \\
&\leq \delta + C(2\delta)(2A)^{\frac{6-d}{d-2}} + CT\|u\|_{L_t^\infty M_{4,2}^s} \|u_{\leq N}\|_{L_t^\infty M_{\infty,1}^s}^{\frac{2d}{d-2}} \\
&\leq \delta + C(2\delta)(2A)^{\frac{6-d}{d-2}} + CTN^{\frac{2d}{d-2}}(2A)^{\frac{d+2}{d-2}}.
\end{aligned}$$

We can bound the above by  $2\delta$  provided that  $\delta$  is chosen small enough depending on  $A$ , and  $T$  is chosen small enough depending on  $A$ ,  $\delta$ , and  $N$ .

Next, we prove that  $\Phi$  is a contraction. We decompose like above  $F = F_1 + F_2$  and observe

$$F_1(u) - F_1(v) = O((u - v)(u_{>N} - v_{>N})(u_{>N}^{\frac{6-d}{d-2}} + v_{>N}^{\frac{6-d}{d-2}}))$$

and

$$F_2(u) - F_2(v) = O((u - v)(u_{\leq N} + v_{\leq N})^{\frac{4}{d-2}}) + O((u_{\leq N} - v_{\leq N})(u + v)(u_{\leq N} + v_{\leq N})^{\frac{6-d}{d-2}}).$$

By the above arguments for  $u, v \in B$ :

$$\begin{aligned}
& d(\Phi(u), \Phi(v)) \\
& \lesssim \|u - v\|_{X^s} (\|u_{>N}\|_{X^s} + \|v_{>N}\|_{X^s}) (\|u\|_{X^s} + \|v\|_{X^s})^{\frac{6-d}{d-2}} \\
& \quad + \|F_2(u) - F_2(v)\|_{L_t^1 M_{4,2}^s} \\
& \lesssim (4\delta)(4A)^{\frac{6-d}{d-2}} d(u, v) + T \|u - v\|_{L_t^\infty M_{4,2}^s} (\|u_{\leq N}\|_{L_t^\infty M_{\infty,1}^s} + \|v_{\leq N}\|_{L_t^\infty M_{\infty,1}^s})^{\frac{4}{d-2}} \\
& \quad + T (\|u\|_{L_t^\infty M_{4,2}^s} + \|v\|_{L_t^\infty M_{4,2}^s}) \|u_{\leq N} - v_{\leq N}\|_{L_t^\infty M_{\infty,1}^s} \\
& \quad \times (\|u_{\leq N}\|_{L_t^\infty M_{\infty,1}^s} + \|v_{\leq N}\|_{L_t^\infty M_{\infty,1}^s})^{\frac{6-d}{d-2}} \\
& \lesssim [(4\delta)(4A)^{\frac{6-d}{d-2}} + TN^{\frac{4d}{d-2}} (4A)^{\frac{4}{d-2}}] d(u, v) \leq \frac{1}{2} d(u, v),
\end{aligned}$$

provided  $\delta$  is chosen small enough depending on  $A$ , and  $T$  is chosen small enough depending on  $A$  and  $N$ . This yields uniqueness and analytic dependence of the data-to-solution mapping within  $B$ . To infer uniqueness in  $X^s([0, T]) \cap C_t([0, T], M_{4,2}^s(\mathbb{R}^d))$ , we can compare two solutions for the same initial data through a common frequency cutoff chosen high enough. Then, we find these to be coinciding in a ball and hence in  $X^s([0, T]) \cap C_t([0, T], M_{4,2}^s(\mathbb{R}^d))$ .  $\square$

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