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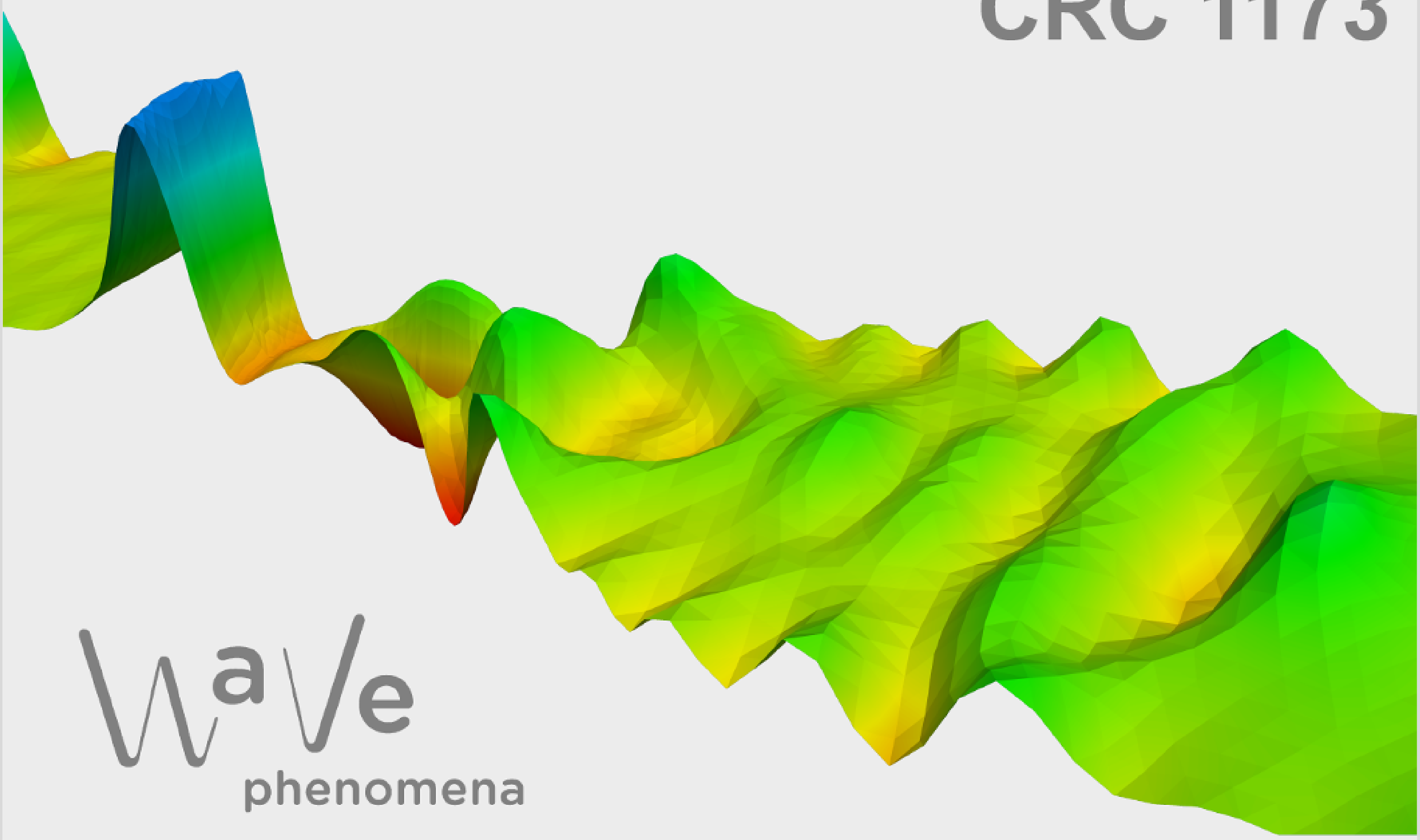

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NONLINEAR WAVE EQUATIONS WITH SLOWLY DECAYING INITIAL DATA

JAN ROZENDAAL AND ROBERT SCHIPPA*

Abstract. New local smoothing estimates in Besov spaces adapted to the half-wave group are proved via ℓ^2-decoupling. We apply these estimates to obtain new well-posedness results for the cubic nonlinear wave equation in two dimensions. The results are compared to new well-posedness results in L^p-based Sobolev spaces.

1. Introduction

Setting. We consider nonlinear wave equations with power-type nonlinearity:

$$\begin{cases} \partial_t^2 u &= \Delta_x u \pm |u|^{\alpha-1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \ d \geq 2, \\ u(0) &= f \in X, \quad \dot{u}(0) = g \in Y, \end{cases} \tag{1.1}$$

We shall analyze in detail the cubic nonlinear wave equation, where $d = 2$ and $\alpha = 3$. Moreover, we consider slowly decaying initial data, by which we mean initial data contained in L^p-based spaces for $p > 2$.

Recently, the well-posedness of the nonlinear Schrödinger equation with slowly decaying initial data has attracted attention [7, 23], in part due to the importance of such initial data for modeling signals. The well-posedness results in this article are proved via a simple contraction mapping argument; similar as in [23] by the second author. We use Duhamel's formula to write (1.1) as

$$u(t) = \Phi_{f,g}(u) = \cos(t\sqrt{-\Delta})f + \frac{\sin(t\sqrt{-\Delta})g}{\sqrt{-\Delta}} \pm \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (|u|^{\alpha-1}u)(s) ds.$$

The proof that $\Phi_{f,g}$ is a contraction in a space-time function space S hinges on linear estimates

$$\| \cos(t\sqrt{-\Delta})f \|_S \lesssim \|f\|_X, \qquad \Big\| \frac{\sin(t\sqrt{-\Delta})g}{\sqrt{-\Delta}} g \Big\|_S \lesssim \|g\|_Y,$$

and a nonlinear estimate

$$\Big\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (|u|^{\alpha-1}u)(s) ds \Big\|_S \lesssim \|u\|_S^\alpha.$$

We shall use space-time Lebesgue spaces $S = L_t^r([0,T], L^p(\mathbb{R}^2))$ as iteration spaces; possibly intersected with another function space. As spaces of initial data, for $2 < p < \infty$, we choose $X = W^{s,p}(\mathbb{R}^2)$ and $Y = W^{s-1,p}(\mathbb{R}^2)$, or we consider Besov spaces $X = \mathcal{B}^s_{p,2,2}(\mathbb{R}^2)$ and $Y = \mathcal{B}^{s-1}_{p,2,2}(\mathbb{R}^2)$ adapted to the half-wave group. The spaces $\mathcal{B}^s_{p,q,r}(\mathbb{R}^d)$, which are introduced in this article, are invariant under the half-wave group, and they satisfy Sobolev embeddings into the standard Besov scale.

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This invariance under the half-wave group is in sharp contrast with $W^{s,p}(\mathbb{R}^d)$ for $p \neq 2$, and a key motivation to consider adapted spaces.

Adapted spaces and local smoothing. The use of adapted Besov spaces builds on recent work concerning invariant spaces for Schrödinger and wave equations. Indeed, modulation spaces, invariant spaces for Schrödinger propagators, have been used extensively as spaces of initial data for nonlinear Schrödinger equations (see [1, 5, 6, 23] and references therein). On the other hand, a scale $(\mathcal{H}^p_{FIO}(\mathbb{R}^d))_{1\leq p\leq\infty}$ of Hardy spaces for Fourier integral operators (FIOs) was introduced in [10] by Hassell, Portal and the first author. This work in turn generalizes the case $p = 1$ due to Smith [26], which predates [10] by decades. The Hardy spaces for FIOs are invariant under half-wave propagators and more general FIOs, and they satisfy the Sobolev embeddings

$$W^{s(p),p}(\mathbb{R}^d) \subseteq \mathcal{H}^p_{FIO}(\mathbb{R}^d) \subseteq W^{-s(p),p}(\mathbb{R}^d) \tag{1.2}$$

for all $1 < p < \infty$, with the natural modifications involving the local Hardy space $\mathcal{H}^1(\mathbb{R}^d)$ for $p = 1$, and $\mathrm{bmo}(\mathbb{R}^d)$ for $p = \infty$. Here and throughout,

$$s(p) = \frac{d-1}{2}\Big|\frac{1}{2} - \frac{1}{p}\Big|. \tag{1.3}$$

By combining these two properties, one recovers the sharp L^p mapping properties of the half-wave group, due to Peral and Miyachi [17, 20]:

$$e^{it\sqrt{-\Delta}} : W^{2s(p),p}(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \tag{1.4}$$

for all $1 < p < \infty$ and $t \in \mathbb{R}$, and the more general L^p mapping properties of FIOs due to Seeger, Sogge and Stein [25].

The invariance of these spaces under the solution operators to Schrödinger and wave equations allows one to use iterative constructions to build parametrices, as was done for rough wave equations using $\mathcal{H}^p_{FIO}(\mathbb{R}^d)$ in [11]. It also shows that such spaces are natural for the fixed-time regularity of these equations.

On the other hand, it was observed by Sogge [27] that considering space-time Lebesgue norms of the solution to the Euclidean wave equation yields a gain of regularity over the fixed-time estimates in (1.4). More precisely, in [27] Sogge formulated the local smoothing conjecture for the Euclidean wave equation, which states that

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p_t([0,1],L^p(\mathbb{R}^d))} \lesssim_\varepsilon \|f\|_{W^{\sigma(p)+\varepsilon,p}(\mathbb{R}^d)} \tag{1.5}$$

for all $\varepsilon > 0$, where $\sigma(p) = 0$ for $2 < p \leq 2d/(d-1)$, and $\sigma(p) = 2s(p) - 1/p$ for $p > 2d/(d-1)$.

The local smoothing conjecture implies several open problems in harmonic analysis, like the Bochner–Riesz conjecture and the restriction conjecture. A breakthrough result was the proof of the sharp ℓ^2-decoupling inequality for the cone, due to Bourgain–Demeter [4]. More precisely, set

$$\overline{s}(p) := \begin{cases} 0, & 2 \leq p \leq \frac{2(d+1)}{d-1}, \\ s(p) - \frac{1}{p}, & \frac{2(d+1)}{d-1} \leq p < \infty. \end{cases}$$

For $k \in \mathbb{N}$, let $(\chi_\nu)_\nu$ be a partition of unity of $\mathbb{R}^d\backslash\{0\}$ with smooth zero-homogeneous functions, which localize to cones of aperture aproximately $2^{-k/2}$, and let $g \in \mathcal{S}(\mathbb{R})$ be such that $|g(t)| \geq 1$ for $t \in [0,1]$, and $\mathrm{supp}(\widehat{g}) \subseteq [-1,1]$. After rescaling to unit frequencies (see e.g. [3, Section 3]), using that $\|e^{it\sqrt{-\Delta}}f\|_{L^p([0,1]\times\mathbb{R}^d)} \leq$

$\|g(t)e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}\times\mathbb{R}^d)}$ and that $(t,x)\mapsto g(t)e^{it\sqrt{-\Delta}}f(x)$ has frequency support near the light cone, it then follows from the ℓ^2-decoupling inequality [4, Theorem 1.2] that

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p([0,1]\times\mathbb{R}^d)}\lesssim 2^{k(\bar{s}(p)+\varepsilon)}\Big(\sum_{\nu}\|g(t)e^{it\sqrt{-\Delta}}\chi_\nu(D)f\|^2_{L^p(\mathbb{R}\times\mathbb{R}^d)}\Big)^{1/2} \tag{1.6}$$

for any $\varepsilon>0$ and $f\in L^p(\mathbb{R}^d)$ with $\operatorname{supp}(\widehat{f}\,)\subseteq\{\xi\in\mathbb{R}^d\mid 2^{-1+k}\leq|\xi|\leq 2^{1+k}\}$. In turn, from (1.6) follow local smoothing estimates by an application of Hölder's inequality, to pass from the ℓ^2-norm to the ℓ^p-norm, and from a kernel estimate. Although (1.6) is sharp, it does not imply the local smoothing conjecture; it only yields the required bounds for $p\geq 2(d+1)/(d-1)$. The local smoothing conjecture was recently resolved for $d=2$ via a sharp (reverse) L^4-square function estimate by Guth–Wang–Zhang [9], but it is still open for $d\geq 3$.

Coming back to the nonlinear wave equation, we will use local smoothing estimates to lower the regularity of the initial data required to solve (1.1), thereby providing, to the best of the authors' knowledge, a novel approach to nonlinear wave equations with slowly decaying initial data.

Main results. Firstly, we introduce the adapted Besov spaces $\mathcal{B}^s_{p,q,r}(\mathbb{R}^d)$ and we derive some of their properties. In particular, we show the Besov counterpart of the Sobolev embeddings in (1.2):

$$B^{s+s(p)}_{p,p}(\mathbb{R}^d)\subseteq\mathcal{B}^s_{p,p,p}(\mathbb{R}^d)\subseteq B^{s-s(p)}_{p,p}(\mathbb{R}^d). \tag{1.7}$$

We also show the invariance of $\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$ under the half-wave propagators. In fact, we take this opportunity to show the sharp polynomial growth rate of the $\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$-norm under evolution of the half-wave group. This quantifies a polynomial growth result by the first author [22, Lemma 3.5], which was established for the Hardy spaces for FIOs. More precisely, in Proposition 3.1 we show that

$$\|e^{it\sqrt{-\Delta}}f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)}\lesssim(1+|t|)^{2s(p)}\|f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)} \tag{1.8}$$

for all $p\in[1,\infty]$, $q\in[1,\infty)$, $s,t\in\mathbb{R}$ and $f\in\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$. In Proposition 3.2 we show that this is sharp, by using a radial Knapp example.

Next, we obtain improved local smoothing estimates, in terms of $\mathcal{B}^s_{p,2,2}(\mathbb{R}^d)$.

Theorem 1.1. *Let $d\geq 2$, $p\in(2,\infty)$ and $\varepsilon>0$. Then there exists a $C\geq 0$ such that*

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p_t([0,1],L^p(\mathbb{R}^d))}\leq C\|f\|_{\mathcal{B}^{\bar{s}(p)+\varepsilon}_{p,2,2}(\mathbb{R}^d)} \tag{1.9}$$

for all $f\in\mathcal{B}^s_{p,2,2}(\mathbb{R}^d)$.

The exponent $\bar{s}(p)$ in (1.9) is sharp for all $2<p<\infty$, cf. Remark 4.2. In fact, the right-hand side of (1.6) is equivalent to the $\mathcal{B}^{\bar{s}(p)+\varepsilon}_{p,2,2}(\mathbb{R}^d)$-norm. Hence, when restricted to dyadic frequency annuli, $\mathcal{B}^{\bar{s}(p)+\varepsilon}_{p,2,2}(\mathbb{R}^d)$ is the largest space of initial data for which one can obtain local smoothing estimates when applying the ℓ^2-decoupling inequality in the manner in which it is typically used.

The corresponding bounds for $\mathcal{H}^{s,p}_{FIO}(\mathbb{R}^d)$, or equivalently for $\mathcal{B}^s_{p,p,p}(\mathbb{R}^d)$, are due to the first author [22]. We note that

$$W^{s+s(p)+2\varepsilon,p}(\mathbb{R}^d)\subseteq\mathcal{B}^{s+\varepsilon}_{p,p,p}(\mathbb{R}^d)\subseteq\mathcal{B}^s_{p,2,2}(\mathbb{R}^d) \tag{1.10}$$

for all $s \in \mathbb{R}$, $2 < p < \infty$ and $\varepsilon > 0$, and that the $\mathcal{B}^s_{p,p,p}(\mathbb{R}^d)$-norm of certain functions is substantially larger than their $\mathcal{B}^s_{p,2,2}(\mathbb{R}^d)$-norm (see Remark 2.5). Hence (1.9) strictly improves upon the bounds in [22], and in particular upon the local smoothing conjecture for $p \geq 2(d+1)/(d-1)$. On the other hand, it is an open question whether $\mathcal{B}^s_{p,2,2}(\mathbb{R}^d)$ is invariant under general FIOs, as $\mathcal{H}^{s,p}_{FIO}(\mathbb{R}^d)$ is. For $2 < p < 2(d+1)/(d-1)$, (1.9) neither follows from the local smoothing conjecture, nor does it imply it.

Next, we show how local smoothing estimates can be combined with nonlinear Strichartz estimates to prove well-posedness for nonlinear wave equations with slowly decaying initial data. We write $\dot{H}^s(\mathbb{R}^d) = |D|^{-s} L^2(\mathbb{R}^d)$.

Theorem 1.2. *Let $d = 2$, $\alpha = 3$, and $\varepsilon > 0$. Then, (1.1) is analytically locally well posed with initial data space*

$$X \times Y = (\mathcal{B}^{\varepsilon}_{4,2,2}(\mathbb{R}^2) + \dot{H}^{3/8}(\mathbb{R}^2)) \times (\mathcal{B}^{\varepsilon-1}_{4,2,2}(\mathbb{R}^2) + \dot{H}^{-5/8}(\mathbb{R}^2)), \tag{1.11}$$

and solution space $S_T = L^{24/7}_t([0,T], L^4(\mathbb{R}^2)) \cap C([0,T], \mathcal{B}^{\varepsilon}_{4,2,2}(\mathbb{R}^2) + \dot{H}^{3/8}(\mathbb{R}^2))$ with $T = T(\|(f,g)\|_{X\times Y})$. Moreover, (1.1) is analytically locally well posed with initial data space

$$X \times Y = (\mathcal{B}^{\varepsilon}_{6,2,2}(\mathbb{R}^2) + \dot{H}^{1/2}(\mathbb{R}^2)) \times (\mathcal{B}^{\varepsilon-1}_{6,2,2}(\mathbb{R}^2) + \dot{H}^{-1/2}(\mathbb{R}^2)), \tag{1.12}$$

and solution space $S_T = L^4_t([0,T], L^6(\mathbb{R}^2)) \cap C([0,T], \mathcal{B}^{\varepsilon}_{6,2,2}(\mathbb{R}^2) + \dot{H}^{1/2}(\mathbb{R}^2))$.

For the definition of analytic well-posedness we refer to Section 5.1. Roughly speaking, here it means that we obtain a time of existence $T(\|(f,g)\|_{X\times Y})$, for which there is a unique solution u in S_T which depnds analytically on the initial data. Moreover, it follows from the proof that, for any $T > 0$, there exists some $\varepsilon = \varepsilon(T) > 0$ such that (1.1) is well posed in S_T whenever $\|(f,g)\|_{X\times Y} < \varepsilon$.

Remark 1.3. The homogeneous Sobolev spaces $\dot{H}^{3/8}(\mathbb{R}^2)$ and $\dot{H}^{-5/8}(\mathbb{R}^2)$ in (1.11) can be replaced by the inhomogeneous spaces $H^s(\mathbb{R}^2) = W^{2,s}(\mathbb{R}^2)$ and $H^{s-1}(\mathbb{R}^2)$, for $s \geq 3/8$, and similarly for (1.12). Furthermore, the arguments from the proof yield local well-posedness for initial data in

$$X \times Y = W^{\varepsilon,4}(\mathbb{R}^2) \times W^{-1+\varepsilon,4}(\mathbb{R}^2) \tag{1.13}$$

with solutions in $S_T = L^{24/7}_t([0,T], L^4(\mathbb{R}^2))$, and for

$$X \times Y = W^{1/6+\varepsilon,6}(\mathbb{R}^2) \times W^{-5/6+\varepsilon,6}(\mathbb{R}^2),$$

with solution space $S_T = L^4_t([0,T], L^6(\mathbb{R}^2))$. By the embeddings in (1.10), we have

$$W^{1/6+3\varepsilon}(\mathbb{R}^2) \subseteq \mathcal{B}^{2\varepsilon}_{6,6,6}(\mathbb{R}^2) \subseteq \mathcal{B}^{\varepsilon}_{6,2,2}(\mathbb{R}^2),$$

which shows that a local well-posedness result with initial data in $\mathcal{B}^{s+\varepsilon}_{6,2,2}(\mathbb{R}^d)$ supersedes one involving $W^{s+1/6+\varepsilon}(\mathbb{R}^d)$. This is not quite the case for the L^4-based result because one has the sharp embeddings

$$W^{1/8+3\varepsilon,4}(\mathbb{R}^2) \subseteq \mathcal{B}^{2\varepsilon}_{4,4,4}(\mathbb{R}^2) \subseteq \mathcal{B}^{\varepsilon}_{4,2,2}(\mathbb{R}^2).$$

It appears that this mismatch of 1/8 derivatives reflects the fact that ℓ^2-decoupling does not imply the local smoothing conjecture. It would be very interesting to eventually translate this additional smoothing effect to adapted function spaces. However, it follows from Remark 4.2 that such an additional smoothing effect cannot be captured by $\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$ for $q = 2$. We do note that, since (1.9) complements the local smoothing conjecture, the well-posedness result in Theorem 1.2 neither follows from one involving (1.13), nor does it imply it.

Furthermore, we show local well-posedness for slower decaying initial data, i.e., with initial data in spaces $\mathcal{B}^{s(p)}_{p,2,2}(\mathbb{R}^d) \times \mathcal{B}^{s(p)-1}_{p,2,2}(\mathbb{R}^d)$ and $p = 4n+2$, $n \in \mathbb{Z}_{\geq 2}$, $d \in \{2,3\}$. The more technical result is stated in Theorem 5.7.

In Theorems 5.8 and 5.11 we prove global well-posedness in the defocusing case. Global well-posedness in L^2-based Sobolev spaces typically follows from conserved quantity. This does not fit well into the L^p-scale. Instead, we show global well-posedness by adapting arguments of Dodson–Soffer–Spencer [7]; see also [15, 23].

Generalizations. The presented arguments are robust in nature and allow one to treat more general nonlinearities than $\alpha = 3$ (see, e.g., Theorem 5.3). One can also consider higher dimensions $d \geq 3$, albeit in this case with a different derivative parameter s.

Moreover, the arguments transpire to the variable-coefficient case. Consider the nonlinear wave equation on a compact Riemannian manifold (M, g) with $\dim M \geq 2$:

$$\begin{cases} \partial_t^2 u &= \Delta_g u \pm |u|^{\alpha-1}u, \quad (t,x) \in \mathbb{R} \times M, \\ u(0) &= f_1 \in X, \quad \dot{u}(0) = f_2 \in Y. \end{cases} \tag{1.14}$$

For $X \times Y \in W^{s,p}(M) \times W^{s-1,p}(M)$, $\dim M = 2$, we can argue as in the proof of Theorem 1.2, because both local smoothing and Strichartz estimates remain true in the variable coefficient case. Indeed, variable-coefficient decoupling was proved by Beltran–Hickman–Sogge [3] and local-in-time Strichartz estimates remain true on compact manifolds, as proved by Kapitanskii [12, 13]. These are the key ingredients for the iteration argument in Section 5.

For an extension of the results on nonlinear equations with initial data in adapted spaces, one would have to find a suitable definition on compact manifolds. On the other hand, to prove global results it seems necessary to work with spaces of initial data which are invariant under more general FIOs. Indeed, the solution operator to the linear part of (1.14) is a Fourier integral operator, an observation which goes back to Lax [16]. This motivated the pioneering works by Seeger–Sogge–Stein [25] and Mockenhaupt–Seeger–Sogge [18] on the fixed-time and space-time mapping properties of FIOs. It is unclear whether $\mathcal{B}^s_{p,2,2}(\mathbb{R}^d)$ is invariant under more general FIOs, but $\mathcal{H}^{s,p}_{FIO}(\mathbb{R}^d)$ is. Moreover, one could solve nonlinear wave equations with initial data in $\mathcal{H}^{s,p}_{FIO}(\mathbb{R}^d)$ in the same manner as we do for $\mathcal{B}^s_{p,2,2}(\mathbb{R}^d)$.

Our goal in this article is not to develop a full theory of Besov spaces adapted to the half-wave group, as has been done for the Hardy spaces for FIOs in [8, 10, 21]. The advantage of working with Besov spaces is that it suffices to obtain estimates on dyadic frequency annuli, instead of working with square functions. On the other hand, one only recovers the sharp fixed-time regularity for wave equations in the Besov scale, cf. (1.7), as opposed to the L^p-scale, cf. (1.2).

Organization. In Section 2 we introduce the function spaces $\mathcal{B}^s_{p,q,r}(\mathbb{R}^d)$, and we determine some of their properties. In Section 3 we show that $\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$ is invariant under the action of the half-wave group, cf. (1.8), and we obtain product estimates. In Section 4 we prove the local smoothing estimates in Theorem 1.1. Using these, in Section 5 we derive local well-posedness results, and in particular Theorem 1.2. In Section 5.4 we use a blow-up alternative to prove global well-posedness in the defocusing case.

Notation. Throughout most of this article we fix a general dimension $d \geq 2$, but in Section 5 we will typically assume that $d \in \{2,3\}$.

For $\xi \in \mathbb{R}^d$ we write $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$, and $\hat{\xi} = \xi/|\xi|$ if $\xi \neq 0$. We use multi-index notation, where $\partial_\xi = (\partial_{\xi_1}, \dots, \partial_{\xi_d})$ and $\partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_d}^{\alpha_d}$ for $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$. The Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ is denoted by $\mathcal{F}f$ or $\widehat{f}$, and the Fourier multiplier with symbol $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ is denoted by $\varphi(D)$.

We write $f(s) \lesssim g(s)$ to indicate that $f(s) \leq Cg(s)$ for all s and a constant $C > 0$ independent of s, and similarly for $f(s) \gtrsim g(s)$ and $g(s) \eqsim f(s)$.

2. Function spaces

In this section we introduce the relevant function spaces for this article, and we derive some of their properties, most notably equivalent norms and embeddings.

2.1. **Definitions.** We first recall the definition of the Hardy spaces for FIOs from [8, 10, 21, 26]. Fix a non-negative radial $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi(\xi) = 0$ for $|\xi| > 1$, and $\varphi \equiv 1$ in a neighbourhood of zero. For $\omega \in \mathbb{S}^{n-1}$, $\sigma > 0$, and $\xi \in \mathbb{R}^d \setminus \{0\}$, set $\varphi_{\omega,\sigma}(\xi) := c_\sigma \varphi\big(\frac{\hat{\xi}-\omega}{\sigma^{1/2}}\big)$, where $c_\sigma := \big(\int_{\mathbb{S}^{d-1}} \varphi\big(\frac{e_1-\nu}{\sigma^{1/2}}\big)^2 d\nu\big)^{-1/2}$ for $e_1 = (1,0,\dots,0)$. Furthermore, we set $\varphi_{\omega,\sigma}(0) := 0$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative radial function such that $\psi(\xi) = 0$ if $|\xi| \notin [1/2, 2]$, with $\psi(\xi) = 0$ if $|\xi| \notin [1/2, 2]$, and

$$\int_0^\infty \psi(\sigma\xi)^2 \frac{d\sigma}{\sigma} = 1 \text{ for all } \xi \neq 0.$$

Let $\varphi_\omega(\xi) := \int_0^4 \psi(\sigma\xi)\varphi_{\omega,\sigma}(\xi)\frac{d\sigma}{\sigma}$. Recall the following properties of $\varphi_\omega \in C^\infty(\mathbb{R}^d)$ from [21, Remark 3.3]:

(1) For all $\omega \in \mathbb{S}^{d-1}$ and $\xi \neq 0$ one has $\varphi_\omega(\xi) = 0$ if $|\xi| < 1/8$ or $|\hat{\xi} - \omega| > 2|\xi|^{-1/2}$.

(2) For all $\alpha \in \mathbb{N}_0^d$ and $\beta \in \mathbb{N}_0$ there exists $C_{\alpha,\beta} \geq 0$ such that

$$|(\omega \cdot \partial_\xi)^\beta \partial_\xi^\alpha \varphi_\omega(\xi)| \leq C_{\alpha,\beta} |\xi|^{\frac{d-1}{4} - \frac{|\alpha|}{2} - \beta}$$

for all $\omega \in \mathbb{S}^{d-1}$ and $\xi \neq 0$.

(3) For all $\alpha \in \mathbb{N}_0^d$ there exists a $C_\alpha \geq 0$ such that

$$\Big|\partial_\xi^\alpha \Big(\int_{\mathbb{S}^{d-1}} \varphi_\omega(\xi) d\omega\Big)^{-1}\Big| \leq C_\alpha |\xi|^{\frac{d-1}{4} - |\alpha|}$$

for all $\xi \in \mathbb{R}^d$ with $|\xi| \geq 1/2$. Hence there is an $m \in S^{\frac{d-1}{4}}(\mathbb{R}^d)$ such that, if $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfies $\operatorname{supp}(\hat{f}) \subseteq \{\xi \in \mathbb{R}^d \mid |\xi| \geq 1/2\}$, then

$$f = \int m(D)\varphi_\nu(D) f d\nu.$$

For simplicity of notation, we write $\mathcal{H}^p(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ for $1 < p < \infty$, and $\mathcal{H}^1(\mathbb{R}^d)$ is the classical local Hardy space. Fix a $q \in C_c^\infty(\mathbb{R}^d)$ such that $q(\xi) = 1$ for $|\xi| \leq 2$.

We define the Hardy spaces for FIOs as follows.

Definition 2.1. For $p \in [1, \infty)$ and $s \in \mathbb{R}$, let $\mathcal{H}^{s,p}_{FIO}(\mathbb{R}^d)$ consist of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $q(D)f \in L^p(\mathbb{R}^d)$, $\langle D \rangle^s \varphi_\omega(D) f \in \mathcal{H}^p(\mathbb{R}^d)$ for almost all $\omega \in \mathbb{S}^{d-1}$, and

$$\|f\|_{\mathcal{H}^{s,p}_{FIO}(\mathbb{R}^d)} := \|q(D)f\|_{L^p(\mathbb{R}^d)} + \Big(\int_{\mathbb{S}^{d-1}} \|\langle D \rangle^s \varphi_\omega(D) f\|^p_{\mathcal{H}^p(\mathbb{R}^d)} d\omega\Big)^{1/p} < \infty.$$

Moreover, $\mathcal{H}^{s,\infty}_{FIO}(\mathbb{R}^d) := (\mathcal{H}^{-s,1}_{FIO}(\mathbb{R}^d))^*$.

In fact, $\mathcal{H}^p_{FIO}(\mathbb{R}^d)$ was originally defined in [10, 26] using conical square function estimates over the cosphere bundle. This includes an intrinsic definition of $\mathcal{H}^\infty_{FIO}(\mathbb{R}^d)$ in terms of Carleson measures. The equivalent characterization in Definition 2.1 was obtained in [8, 21].

In this article we consider the following Besov variant of these spaces, where $B^s_{p,q}(\mathbb{R}^d)$ is the standard Besov space on $\mathbb{R}^d$.

Definition 2.2. For $p, r \in [1,\infty]$, $q \in [1,\infty)$ and $s \in \mathbb{R}$, let $\mathcal{B}^s_{p,q,r}(\mathbb{R}^d)$ consist of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $q(D)f \in L^p(\mathbb{R}^d)$, $\varphi_\omega(D)f \in B^s_{p,q}(\mathbb{R}^d)$ for almost all $\omega \in \mathbb{S}^{d-1}$, and

$$\|f\|_{\mathcal{B}^s_{p,q,r}(\mathbb{R}^d)} := \|q(D)f\|_{L^p(\mathbb{R}^d)} + \Big(\int_{\mathbb{S}^{d-1}} \|\varphi_\omega(D)f\|^q_{B^s_{p,r}(\mathbb{R}^d)} d\omega\Big)^{1/q} < \infty.$$

We will mostly consider the case where $q = r$. Then one can use Fubini's theorem to prove the following lemma, which in turn allows one to reduce various arguments to dyadic frequency annuli. Throughout, we fix a standard Littlewood-Paley decomposition $(\psi_k)_{k=0}^\infty \subseteq C^\infty_c(\mathbb{R}^d)$, with $\operatorname{supp}(\psi_k) \subseteq \{\xi \in \mathbb{R}^d \mid 2^{-1+k} \leq |\xi| \leq 2^{1+k}\}$ for $k \in \mathbb{N}$, and $\sum_{k=0}^\infty \psi_k(\xi) = 1$ for all $\xi \neq 0$.

Lemma 2.3. *Let $p \in [1,\infty]$, $q \in [1,\infty)$ and $s \in \mathbb{R}$. Then an $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfies $f \in \mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$ if and only if $(\sum_{k=0}^\infty \|\psi_k(D)f\|^q_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)})^{1/q} < \infty$, in which case*

$$\|f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)} = \Big(\sum_{k=0}^\infty \|\psi_k(D)f\|^q_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)}\Big)^{1/q}.$$

2.2. **An equivalent norm.** For each $k \in \mathbb{N}_0$, fix a maximal collection $\Theta_k \subseteq \mathbb{S}^{d-1}$ of unit vectors such that $|\nu - \nu'| \geq 2^{-k/2}$ for all $\nu, \nu' \in \Theta_k$. Let $(\chi_\nu)_{\nu \in \Theta_k} \subseteq C^\infty(\mathbb{R}^d \setminus \{0\})$ be an associated partition of unity. That is, each χ_ν is homogeneous of order 0 and satisfies $0 \leq \chi_\nu \leq 1$ and $\operatorname{supp}(\chi_\nu) \subseteq \{\xi \in \mathbb{R}^d \mid |\hat{\xi} - \nu| \leq 2^{1-k/2}\}$. Moreover, $\sum_{\nu \in \Theta_k} \chi_\nu(\xi) = 1$ for all $\xi \neq 0$, and for all $\alpha \in \mathbb{N}_0^d$ and $\beta \in \mathbb{N}_0$ there exists a $C_{\alpha,\beta} \geq 0$ independent of N such that, if $2^{-1+k} \leq |\xi| \leq 2^{1+k}$, then

$$|(\hat{\xi} \cdot \partial_\xi)^\beta \partial^\alpha_\xi \chi_\nu(\xi)| \leq C_{\alpha,\beta} 2^{-k(|\alpha|/2+\beta)}$$

for all $\nu \in \Theta_k$. Also write $\chi^k_\nu := \chi_\nu \psi_k$ for $k \in \mathbb{N}_0$ and $\nu \in \Theta_k$, so that

$$f = \sum_{k=0}^\infty \sum_{\nu \in \Theta_k} \chi^k_\nu(D) f$$

for $f \in \mathcal{S}'(\mathbb{R}^d)$. Moreover, it follows from integration by parts that

$$\|\mathcal{F}^{-1}(\chi^k_\nu)\|_{L^p(\mathbb{R}^d)} \lesssim 2^{k\frac{n+1}{2p'}} \tag{2.1}$$

for all $p \in [1,\infty]$, with an implicit constant independent of k and ν.

We can now give a discrete description of the $\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$-norm. For $p = q$, the first statement in the following proposition is [22, Proposition 4.1].

Proposition 2.4. *Let $p \in [1,\infty]$, $q \in [1,\infty)$ and $s \in \mathbb{R}$. Then there exists a $C > 0$ such that the following holds. Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be such that $\operatorname{supp}(\widehat{f}) \subseteq \{\xi \in \mathbb{R}^d \mid 2^{-1+k} \leq |\xi| \leq 2^{1+k}\}$ for some $k \in \mathbb{N}$. Then*

$$\frac{1}{C}\|f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)} \leq 2^{k(s+\frac{d-1}{2}(\frac{1}{2}-\frac{1}{q}))}\Big(\sum_{\nu \in \Theta_k} \|\chi_\nu(D)f\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \leq C\|f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)}.$$

Hence an $f \in \mathcal{S}'(\mathbb{R}^d)$ satisfies $f \in \mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$ if and only if

$$\Big(\sum_{k=0}^{\infty} 2^{qk(s+\frac{d-1}{2}(\frac{1}{2}-\frac{1}{q}))} \sum_{\nu\in\Theta_k} \|\chi^k_\nu(D)f\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \tag{2.2}$$

is finite, and (2.2) defines an equivalent norm on $\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$.

Proof. It is straightforward to deal with the low frequencies, so we may assume that $q(D)f = 0$. Moreover, by Lemma 2.3, the second statement follows from the first.

To prove the first statement, for each $\nu \in \Theta_k$ write $\tilde{\chi}_\nu := \sum_{|\nu'-\nu|\leq 2^{2-k/2}} \chi_{\nu'}$ and $E_\nu := \{\omega \in \mathbb{S}^{d-1} \mid |\omega - \nu| \leq 2^{3-k/2}\}$. Then

$$\|\varphi_\omega(D)\chi_\nu(D)f\|_{W^{s,p}} = \|\varphi_\omega(D)\tilde{\chi}_\nu(D)\chi_\nu(D)f\|_{W^{s,p}} \lesssim 2^{k(s+\frac{d-1}{4})}\|\chi_\nu(D)f\|_{L^p}$$

for each $\omega \in E_\nu$, by a kernel estimate, and $|E_\nu| \lesssim 2^{-k(d-1)/2}$. Hence

$$\begin{aligned}\Big(\int_{\mathbb{S}^{d-1}} \|\varphi_\omega(D)f\|^q_{W^{s,p}(\mathbb{R}^d)} d\omega\Big)^{1/q} &\lesssim \Big(\sum_{\nu\in\Theta_k}\int_{E_\nu} \|\varphi_\omega(D)\chi_\nu(D)f\|^q_{W^{s,p}(\mathbb{R}^d)}\Big)^{1/q} \\ &\lesssim 2^{k(s+\frac{d-1}{4})}|E_\nu|^{1/q}\Big(\sum_{\in\Theta_k}\|\chi_\nu(D)f\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \\ &\lesssim 2^{k(s+\frac{d-1}{2}(\frac{1}{2}-\frac{1}{q}))}\Big(\sum_{\in\Theta_k}\|\chi_\nu(D)f\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q}.\end{aligned}$$

The other inequality follows by duality. □

Remark 2.5. Let $f \in \mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$ be such that

$$\operatorname{supp}(\widehat{f}\,) \subseteq \{\xi \in \mathbb{R}^d \mid 2^{-1+k} \leq |\xi| \leq 2^{1+k}, |\hat{\xi} - \nu| \leq 2^{1-k/2}\}$$

for some $k \in \mathbb{N}$ and $\nu \in \mathbb{S}^{d-1}$. Then Proposition 2.4 yields

$$\|f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)} \eqsim 2^{k(s+\frac{d-1}{2}(\frac{1}{2}-\frac{1}{q}))}\|f\|_{L^p(\mathbb{R}^d)} \eqsim \|f\|_{W^{s+\frac{d-1}{2}(\frac{1}{2}-\frac{1}{q}),p}(\mathbb{R}^d)}.$$

2.3. **Embeddings.** We first obtain Sobolev embeddings into the Besov scale. Note that (2.3) was already stated in (1.7) in the introduction, and recall the definition of $s(p)$ from (1.3).

Proposition 2.6. *Let $p \in [1,\infty)$ and $s \in \mathbb{R}$. Then*

$$B^{s+s(p)}_{p,p}(\mathbb{R}^d) \subseteq \mathcal{B}^s_{p,p,p}(\mathbb{R}^d) \subseteq B^{s-s(p)}_{p,p}(\mathbb{R}^d). \tag{2.3}$$

Moreover, one has

$$B^{s+s(p)}_{p,p'}(\mathbb{R}^d) \subseteq \mathcal{B}^s_{p,p',p'}(\mathbb{R}^d), \quad 1 < p \leq 2, \tag{2.4}$$

and

$$\mathcal{B}^s_{p,p',p'}(\mathbb{R}^d) \subseteq B^{s-s(p)}_{p,p'}(\mathbb{R}^d), \quad 2 \leq p < \infty. \tag{2.5}$$

Proof. By recalling that the Sobolev and Besov norms of a function with frequency support in a dyadic annulus are equivalent, (2.3) follows directly from (1.2).

On the other hand, by Proposition 2.4, we may prove (2.4) and (2.5) for $f \in \mathcal{S}'(\mathbb{R}^d)$ with $\operatorname{supp}(\widehat{f}\,) \subseteq \{\xi \in \mathbb{R}^d \mid 2^{-1+k} \leq |\xi| \leq 2^{1+k}\}$ for some $k \in \mathbb{N}$. Moreover, both identities follow from (2.3) for $p = 2$. Hence, by interpolation, it suffices to note that

$$\max_{\nu\in\Theta_k}\|\chi_\nu(D)f\|_{L^1(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)},$$

where we used that the kernels of the χ_ν are uniformly in $L^1(\mathbb{R}^d)$, cf. (2.1), and that

$$\|f\|_{L^\infty(\mathbb{R}^d)} = \Big\| \sum_{\nu\in\Theta_k} \chi_\nu(D)f\Big\|_{L^\infty(\mathbb{R}^d)} \leq \sum_{\nu\in\Theta_k} \|\chi_\nu(D)\|_{L^\infty(\mathbb{R}^d)}. \qquad \square$$

Remark 2.7. The Sobolev exponents in (2.3) are sharp, by Proposition 3.1 and because the half-wave propagators lose $2s(p)$ derivatives in the Besov scale.

By Proposition 2.6 and by standard embeddings for Besov spaces, one has

$$\mathcal{B}^{s(p)}_{p,p,p}(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d) \subseteq \mathcal{B}^{-s(p)}_{p,p',p'}(\mathbb{R}^d)$$

for $1 < p \leq 2$, and

$$\mathcal{B}^{s(p)}_{p,p',p'}(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d) \subseteq \mathcal{B}^{-s(p)}_{p,p,p}(\mathbb{R}^d) \tag{2.6}$$

for $2 \leq p < \infty$. These embeddings are similar to embeddings for modulation spaces.

Next, we obtain embeddings within the scales of adapted Besov spaces and Hardy spaces for FIOs. Combined with the Sobolev embeddings for $\mathcal{H}^p_{FIO}(\mathbb{R}^d)$ in (1.2), this proposition implies the embeddings in (1.10).

Proposition 2.8. *Let $p, r \in [1,\infty]$, $q \in [1,\infty)$ and $s \in \mathbb{R}$. Then*

$$\mathcal{B}^{s}_{p,q_2,r}(\mathbb{R}^d) \subseteq \mathcal{B}^{s}_{p,q_1,r}(\mathbb{R}^d) \tag{2.7}$$

and

$$\mathcal{B}^{s+\frac{d-1}{2}(\frac{1}{q_1}-\frac{1}{q_2})}_{p,q_1,q_1}(\mathbb{R}^d) \subseteq \mathcal{B}^{s}_{p,q_2,q_2}(\mathbb{R}^d) \tag{2.8}$$

for all $q_1, q_2 \in [1,\infty)$ with $q_1 \leq q_2$. Moreover, one has

$$\mathcal{B}^{s+\frac{d+1}{2}(\frac{1}{p_1}-\frac{1}{p_2})}_{p_1,q,q}(\mathbb{R}^d) \subseteq \mathcal{B}^{s}_{p_2,q,q}(\mathbb{R}^d) \tag{2.9}$$

for all $p_1, p_2 \in [1,\infty]$ with $p_1 \leq p_2$, and for all $t \in [1,\infty]$ and $\varepsilon > 0$ one has

$$\mathcal{B}^{s+\varepsilon}_{p,q,r}(\mathbb{R}^d) \subseteq \mathcal{B}^{s}_{p,q,t}(\mathbb{R}^d) \tag{2.10}$$

and

$$\mathcal{B}^{s+\varepsilon}_{p,p,r}(\mathbb{R}^d) \subseteq \mathcal{H}^{s,p}_{FIO}(\mathbb{R}^d) \subseteq \mathcal{B}^{s-\varepsilon}_{p,p,r}(\mathbb{R}^d). \tag{2.11}$$

Proof. For (2.7) one can rely on Hölder's inequality, while (2.8) follows from the inclusion $\ell^{q_1} \subseteq \ell^{q_2}$. Moreover, both (2.10) and (2.11) follow from standard embeddings between Besov and Sobolev spaces (see [28, Section 2.3.2]).

Finally, by Proposition 2.4, for (2.9) it suffices to show that

$$\|\chi^k_\nu(D)f\|_{L^{p_2}(\mathbb{R}^d)} \lesssim 2^{k\frac{d+1}{2}(\frac{1}{p_1}-\frac{1}{p_2})}\|\chi_k(D)f\|_{L^{p_1}(\mathbb{R}^d)} \tag{2.12}$$

for all $k \geq 0$, $\nu \in \Theta_k$ and $f \in \mathcal{B}^s_{p_1,q,q}(\mathbb{R}^d)$. To this end, one can construct a collection $\{\tilde{\chi}^k_\nu \mid k \geq 0, \nu \in \Theta_k\} \subseteq C^\infty_c(\mathbb{R}^d)$ of cut-offs with similar support and decay properties as the χ^k_ν, but such that $\tilde{\chi}^k_\nu \equiv 1$ on $\mathrm{supp}(\chi^k_\nu)$. Then, as in (2.1),

$$\|\mathcal{F}^{-1}(\tilde{\chi}^k_\nu)\|_{L^{p_3}(\mathbb{R}^d)} \lesssim 2^{k\frac{d+1}{2}(\frac{1}{p_1}-\frac{1}{p_2})},$$

where $\frac{1}{p_3} = 1 + \frac{1}{p_2} - \frac{1}{p_1}$. Now (2.12) follows from Young's inequality, since $\chi^k_\nu = \tilde{\chi}^k_\nu\chi^k_\nu$. $\square$

3. Invariance and product estimates

In this section we prove that $\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$ is invariant under the half-wave propagators and more general operators. We also obtain some product estimates, which are useful for solving nonlinear equations.

3.1. **Invariance.** The main result of this subsection is the following slightly more general version of (1.8).

Proposition 3.1. *Let $\phi \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be homogeneous of order 1, and let $p \in [1,\infty]$, $q \in [1,\infty)$ and $s \in \mathbb{R}$. Then there exists a $C \geq 0$ such that*

$$\|e^{it\phi(D)} f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)} \leq C(1+|t|)^{2s(p)} \|f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)}$$

for all $t \in \mathbb{R}$ and $\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$.

Proof. By Proposition 2.4, it suffices to show that

$$\|\chi^k_\nu(D) e^{it\phi(D)} f\|_{L^p(\mathbb{R}^d)} \lesssim (1+|t|)^{2s(p)} \|\chi^k_\nu(D) f\|_{L^p(\mathbb{R}^d)}$$

for all $t \in \mathbb{R}$, $f \in \mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$, $k \in \mathbb{N}_0$ and $\nu \in \Theta_k$. This is clearly true for $p = 2$. Hence, by interpolation and duality, it suffices to show the statement for $p = 1$.

To do so, we rely on a dilation argument. Let $\{\tilde{\chi}^m_\omega \mid m \in \mathbb{N}_0, \omega \in \Theta_m\} \subseteq C^\infty_c(\mathbb{R}^d)$ be a collection of cut-offs with similar support and decay properties as the χ^m_ω, but such that $\tilde{\chi}^m_\omega \equiv 1$ on $\mathrm{supp}(\chi^m_\omega)$. Then

$$\sup_{|t| \leq 4} \|\tilde{\chi}^\omega_m(D) e^{it\phi(D)} g\|_{L^1(\mathbb{R}^d)} \lesssim \|g\|_{L^1(\mathbb{R}^d)} \tag{3.1}$$

for all $g \in L^1(\mathbb{R}^d)$, as follows either from kernel bounds (see [25]), or from the boundedness of $e^{it\phi(D)}$ on $\mathcal{H}^1_{FIO}(\mathbb{R}^d)$. Either way, we may thus suppose that $|t| > 4$.

Let $l \geq 2$ be such that $2^l < |t| \leq 2^{l+1}$. Then the dilated function $\chi^k_\nu(\frac{\cdot}{|t|})$ satisfies

$$\mathrm{supp}(\chi^k_\nu(\tfrac{\cdot}{|t|})) \subseteq \{\xi \in \mathbb{R}^d \mid 2^{k+l-1} \leq |\xi| \leq 2^{k+l+2}, |\hat{\xi} - \nu| \leq 2^{1-k/2}\}.$$

Hence one has

$$\chi^k_\nu(\tfrac{\cdot}{|t|}) = \sum_{m=k+l-2}^{k+l+3} \sum_{\omega \in \tilde{\Theta}_m} \tilde{\chi}^m_\omega \chi^k_\nu(\tfrac{\cdot}{|t|}),$$

where each $\tilde{\Theta}_m$ has approximately $2^{l(n-1)/2} \eqsim |t|^{(n-1)/2}$ elements. Write $f_t(y) := |t|^d f(|t|y)$ for $y \in \mathbb{R}^d$. Then it suffices to combine (3.1) with dilation arguments:

$$\begin{aligned}
\|\chi^k_\nu(D) e^{it\phi(D)} f\|_{L^1(\mathbb{R}^d)} &= |t|^{-d} \big\| \big(\chi^k_\nu(\tfrac{D}{|t|}) e^{i\phi(D)} f_t\big)\big(\tfrac{\cdot}{|t|}\big)\big\|_{L^1(\mathbb{R}^d)} \\
&= \|\chi^k_\nu(\tfrac{D}{|t|}) e^{i\phi(D)} f_t\|_{L^1(\mathbb{R}^d)} \\
&\leq \sum_{m=k+l-2}^{k+l+3} \sum_{\omega \in \tilde{\Theta}_m} \big\|\tilde{\chi}^m_\omega(D) e^{i\phi(D)} \chi^k_\nu(\tfrac{D}{|t|}) f_t\big\|_{L^1(\mathbb{R}^d)} \\
&\lesssim |t|^{(n-1)/2} \|\chi^k_\nu(\tfrac{D}{|t|}) f_t\|_{L^1(\mathbb{R}^d)} = |t|^{(n-1)/2} \|\chi^k_\nu(D) f)\|_{L^1(\mathbb{R}^d)},
\end{aligned}$$

where we used in particular the homogeneity of ϕ. □

The bounds are sharp for $p = q$, since otherwise they would imply sharper bounds in the Besov scale than are known to be possible, by (2.3). Moreover, the following radial Knapp example shows sharpness of the polynomial growth bound for general p, q, since the $\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$ norm coincides with the $L^p(\mathbb{R}^d)$ norm for low frequencies. The analog for the Schrödinger equation was considered in [23, Corollary 1.4].

Proposition 3.2. *Let $d \geq 2$ and $p \in [1,\infty]$. Then there exist an $f \in L^p(\mathbb{R}^d)$ with* $\mathrm{supp}(\widehat{f}) \subseteq B(0,1)$, *and a $C \geq 0$, such that*

$$\|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^d)} \geq C(1+|t|)^{(d-1)|\frac{1}{2}-\frac{1}{p}|}\|f\|_{L^p(\mathbb{R}^d)}$$

for all $t \in \mathbb{R}$.

Proof. We can suppose that $t \gg 1$, and by duality it is enough to consider $p \in [1,2]$.

Let $\chi \in C_c^\infty(B(0,1)\backslash B(0,1/2))$ be a radial bump function, and let χ_0 be such that $\chi(\xi) = \chi_0(|\xi|)$. We consider as initial data f with $\widehat{f}(\xi) = \chi(\xi)$ and rewrite the linear solution by radial symmetry:

$$\begin{aligned}(e^{it\sqrt{-\Delta}} f)(x) &= \int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|)} \chi_0(|\xi|) d\xi \\ &= \int_0^\infty ds \int_{\mathbb{S}^{d-1}} d\sigma(\theta) e^{i(x\cdot s\theta + ts)} s^{d-1} \chi_0(s).\end{aligned}$$

We have

$$\int_{\mathbb{S}^{d-1}} e^{ix\cdot\theta} d\theta = |x|^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(|x|)$$

with J_ν the Bessel function of the first kind:

$$J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu+1/2)\pi^{1/2}} \int_{-1}^1 e^{irt}(1-t^2)^{\nu-\frac{1}{2}} dt, \quad \nu > -\frac{1}{2}.$$

We have the following asymptotic expansion by [19, Section 10.17]:

$$J_\nu(z) = \big(\frac{2}{\pi z}\big)^{\frac{1}{2}} \big(\cos\omega \sum_{k=0}^\infty (-1)^k \frac{a_{2k}(\nu)}{z^{2k}} - \sin\omega \sum_{k=0}^\infty (-1)^k \frac{a_{2k+1}(\nu)}{z^{2k+1}} \big)$$

with $\omega = z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi$. Hence, we can write

$$\chi_0(s)(s|x|)^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(s|x|) = \sum_{j=0}^M |x|^{-\frac{d-1}{2}-j} \sum_{\pm} e^{\pm is|x|} \chi_{j,\pm}(s) + O(|x|^{-\frac{d-1}{2}-M})$$

with $\chi_{j,\pm} \in C_c^\infty(B(0,1)\backslash B(0,1/2))$. We find

$$(e^{it\sqrt{-\Delta}} f)(x) = \sum_{j=0}^M |x|^{-\frac{d-1}{2}-j} \sum_{\pm} \int_{\mathbb{R}} e^{ist \pm is|x|} \chi_{j,\pm}(s) ds + O(|x|^{-\frac{d-1}{2}-M}).$$

Let $t \gg 1$ and $\big||x| - t\big| \leq 2^{-10}$. In this case $st + s|x|$ is a non-stationary phase, which means for $j=0$ the contribution of $\chi_{0,+}$ can be neglected against $\chi_{0,-}$. But by the explicit form of $\chi_{j,-}$, we have

$$\big| \int e^{ist - is|x|} \chi_{0,-}(s) ds \big| \gtrsim 1.$$

Thus, the higher orders can be neglected against the contribution of $\chi_{0,-}$ likewise. This shows $|e^{it\sqrt{-\Delta}} u_0|(x) \gtrsim |t|^{-\frac{d-1}{2}}$ for $\big||x| - t\big| \leq 2^{-10}$, and this concludes the proof by integration. ∎

3.2. **Product estimates.** We begin with a simple bilinear estimate.

Lemma 3.3. *Let $p_1,p_2,p\in[1,\infty]$ be such that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, and let $s>\frac{3(d-1)}{4}$. Then there exists a $C\geq 0$ such that, for all $f\in\mathcal{B}^s_{p_1,1,1}(\mathbb{R}^d)$ and $g\in\mathcal{B}^s_{p_2,1,1}(\mathbb{R}^d)$, one has $fg\in\mathcal{B}^s_{p,1,1}(\mathbb{R}^d)$ and*

$$\|fg\|_{\mathcal{B}^s_{p,1,1}(\mathbb{R}^d)}\leq C\|f\|_{\mathcal{B}^s_{p_1,1,1}(\mathbb{R}^d)}\|g\|_{\mathcal{B}^s_{p_2,1,1}(\mathbb{R}^d)}.$$

Proof. We use paraproduct analysis. More precisely, one has

$$\begin{aligned}\|f\|_{\mathcal{B}^s_{p,1,1}(\mathbb{R}^d)}&\eqsim\sum_{k=0}^{\infty}2^{k(s-\frac{d-1}{4})}\sum_{\nu\in\Theta_k}\Big\|\sum_{l,m=0}^{\infty}\sum_{\omega\in\Theta_l,\mu\in\Theta_m}\chi^k_\nu(D)(\chi^l_\omega(D)f\cdot\chi^m_\mu(D)g)\Big\|_{L^p(\mathbb{R}^d)}\\&\leq\sum_{k,l,m=0}^{\infty}2^{k(s-\frac{d-1}{4})}\sum_{\nu\in\Theta_k,\omega\in\Theta_l,\mu\in\Theta_m}\|\chi^k_\nu(D)(\chi^l_\omega(D)f\cdot\chi^m_\mu(D)g)\|_{L^p(\mathbb{R}^d)}\end{aligned}$$

by Proposition 2.4. We write the latter expression as $I_1+I_2+I_3$, where I_1 involves the sum over $m\leq l-3$, I_2 the sum over $l-2\leq m\leq l+2$, and I_3 the sum over $m\geq l+3$. Moreover, by symmetry, it suffices to consider only I_1 and I_2.

For the $High\times Low$ term I_1, we only need to consider $l-3\leq k\leq l+3$, since the low-frequency factor $\chi^m_\mu(D)g$ does not essentially change the dyadic localization. However, it can change the angular localization. For $l\geq 0$, $m\leq l-3$, $\omega\in\Theta_l$ and $\mu\in\Theta_m$, we decompose the support of χ^m_μ, which is approximately a $2^{m/2}\times\ldots\times 2^{m/2}\times 2^m$ slab, into $2^{m/2}\times\ldots\times 2^{m/2}\times 2^{\min(l/2,m)}$ slabs. Let $(\chi^{m,i}_\mu)_{i\in I}$ be a corresponding partition of unity, with $|I|\eqsim 1+2^{m-l/2}$. Then the support of the convolution of χ^ω_l with a given $\chi^{m,i}_\mu$ can only intersect the support of $O(1)$ elements of Θ_k. Hence the support of the convolution of χ^ω_l and χ^μ_m can only intersect the support of $O(1+2^{m-l/2})$ elements of Θ_k. Since $m-l/2\lesssim m/2$, we obtain

$$\begin{aligned}I_1&\eqsim\sum_{j=-3}^{3}\sum_{l=0}^{\infty}\sum_{m=0}^{l-3}2^{l(s-\frac{d-1}{4})}\sum_{\nu\in\Theta_{l+j},\omega\in\Theta_l,\mu\in\Theta_m}\|\chi^{l+j}_\nu(D)(\chi^l_\omega(D)f\cdot\chi^m_\mu(D)g)\|_{L^p(\mathbb{R}^d)}\\&\lesssim\sum_{l=0}^{\infty}\sum_{m=0}^{l-3}2^{l(s-\frac{d-1}{4})+\frac{m}{2}}\sum_{\omega\in\Theta_l,\mu\in\Theta_m}\|\chi^l_\omega(D)f\|_{L^{p_1}(\mathbb{R}^d)}\|\chi^m_\mu(D)g\|_{L^{p_2}(\mathbb{R}^d)}\\&\lesssim\|f\|_{\mathcal{B}^s_{p_1,1,1}(\mathbb{R}^d)}\|g\|_{\mathcal{B}^s_{p_2,1,1}(\mathbb{R}^d)},\end{aligned}$$

where in the final step we used Proposition 2.4 and that $s\geq(d+1)/4$.

For the $High\times High$ term I_2, all information on angular localization is lost. By trivially summing over $\nu\in\Theta_k$, using also that $s>3(d-1)/4$, we obtain

$$\begin{aligned}I_2&=\sum_{l=0}^{\infty}\sum_{m=l-2}^{l+2}\sum_{k=0}^{l+5}2^{k(s-\frac{d-1}{4})}\sum_{\nu\in\Theta_k,\omega\in\Theta_l,\mu\in\Theta_m}\|\chi^k_\nu(D)(\chi^l_\omega(D)f\cdot\chi^m_\mu(D)g)\|_{L^p(\mathbb{R}^d)}\\&\lesssim\sum_{l=0}^{\infty}\sum_{m=l-2}^{l+2}\sum_{k=0}^{l+5}2^{k(s+\frac{d-1}{4})}\sum_{\omega\in\Theta_l,\mu\in\Theta_m}\|\chi^l_\omega(D)f\|_{L^{p_1}(\mathbb{R}^d)}\|\chi^m_\mu(D)g\|_{L^{p_2}(\mathbb{R}^d)}\\&\lesssim\|f\|_{\mathcal{B}^s_{p_1,1,1}(\mathbb{R}^d)}\|g\|_{\mathcal{B}^s_{p_2,1,1}(\mathbb{R}^d)}.\end{aligned}$$

□

A trilinear estimate can be proved by similar means.

Lemma 3.4. *Let $p_1,p_2,p_3,p\in[1,\infty]$ be such that $\frac{1}{p}=\sum_{i=1}^{3}\frac{1}{p_i}$, and let $s>\max(\frac{3(d-1)}{4}-1,\frac{d-1}{4})$. Then there exists a $C\geq 0$ such that, for all $f_i\in\mathcal{B}^s_{p_i,1,1}(\mathbb{R}^d)$,*

$1 \le i \le 3$, *one has* $\prod_{i=1}^3 f_i \in \mathcal{B}^s_{p,1,1}(\mathbb{R}^d)$ *and*

$$\Big\| \prod_{i=1}^3 f_i \Big\|_{\mathcal{B}^{s-1}_{p,1,1}(\mathbb{R}^d)} \le C \prod_{i=1}^3 \|f_i\|_{\mathcal{B}^s_{p_i,1,1}(\mathbb{R}^d)}.$$

Proof. The approach to the proof is similar as in Lemma 3.3, so we only indicate how to deal with the relevant terms, involving indices $k, k_i \in \mathbb{N}_0$ for $1 \le i \le 3$.

For the $High \times Low \times Low$ term, we consider $2^k \eqsim 2^{k_1} \gg 2^{k_2} \ge 2^{k_3}$ and the term

$$I := 2^{k(s-1-\frac{d-1}{4})} \sum_{\nu \in \Theta_k} \Big\| \chi^N_\nu(D) \Big(\prod_{i=1}^3 \chi^{k_i}_{\nu_i}(D) f_i \Big) \Big\|_{L^p(\mathbb{R}^d)},$$

for $\nu_i \in \Theta_{k_i}$, $1 \le i \le 3$. We have to estimate the number of ν for which the support of χ_ν intersects the support of $\chi^{k_1}_{\nu_1} * \chi^{k_2}_{\nu_2} * \chi^{k_3}_{\nu_3}$. Note that $\chi^{k_2}_{\nu_2} * \chi^{k_3}_{\nu_3}$ is supported in a slab of dimensions approximately $2^{\max(k_3,k_2/2)} \times 2^{k_2/2} \times \ldots \times 2^{k_2/2} \times 2^{k_2}$. This we subdivide into cubes of side length no more than $2^{k_1/2}$, of which there are no more than approximately $(1 + 2^{k_3 - k_1/2})(1 + 2^{k_2 - k_1/2})$. This yields

$$I \lesssim 2^{k_1(s-1-\frac{d-1}{4})} (1 + 2^{\frac{k_3}{2}})(1 + 2^{\frac{k_2}{2}}) \prod_{i=1}^3 \|\chi^{k_i}_{\nu_i}(D) f_i\|_{L^{p_i}(\mathbb{R}^d)}.$$

Since $s \ge \frac{d-1}{4}$, this suffices for the $High \times Low \times Low$ term.

Next, suppose that $2^k \eqsim 2^{k_1} \eqsim 2^{k_2} \gg 2^{k_3}$. Then we obtain, by trivial summation,

$$\begin{aligned} &2^{k(s-1-\frac{d-1}{4})} \sum_{\nu_k \in \Theta_k} \Big\| \chi^k_\nu(D) \Big(\prod_{i=1}^3 \chi^{k_i}_{\nu_i}(D) f_i \Big) \Big\|_{L^p(\mathbb{R}^d)} \\ &\lesssim 2^{k(s-1+\frac{d-1}{4})} \prod_{i=1}^3 \|\chi^{N_i}_{\nu_i}(D) f_i\|_{L^{p_i}(\mathbb{R}^d)}. \end{aligned}$$

This suffices for the $High \times High \times Low$ term, since $s > \max(\frac{3(d-1)}{4} - 1, \frac{d-1}{4})$.

The $High \times High \times High$ term is also dealt with through trivial summation, using that $s > (d-2)/4$. By symmetry, this concludes the proof. □

4. Local smoothing in $\mathcal{B}^s_{p,2,2}(\mathbb{R}^d)$

In this section we prove Theorem 1.1. The proof is analogous to that of [22, Theorem 1.1]. In particular, the key to the proof is the following proposition, which generalizes the case $q = p$ in [22, Corollary 4.2] to arbitrary $q \in (1, \infty)$.

Proposition 4.1. *Let* $p \in [1,\infty]$, $q \in [1,\infty)$ *and* $s \in \mathbb{R}$, *and let* $0 \ne g \in \mathcal{S}(\mathbb{R})$. *Then there exists a* $C > 0$ *such that the following holds. Let* $f \in \mathcal{S}'(\mathbb{R}^d)$ *be such that* $\operatorname{supp}(\widehat{f}) \subseteq \{\xi \in \mathbb{R}^d \mid 2^{-1+k} \le |\xi| \le 2^{1+k}\}$ *for some* $k \in \mathbb{N}$. *Then*

$$\begin{aligned} \frac{1}{C} \|f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)} &\le 2^{k(s+\frac{d-1}{2}(\frac{1}{2}-\frac{1}{q}))} \Big(\sum_{\nu \in \Theta_k} \|\chi_\nu(D) g(t) e^{it\sqrt{-\Delta}} f\|^q_{L^p(\mathbb{R}\times\mathbb{R}^d)} \Big)^{1/q} \\ &\le C \|f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)}. \end{aligned}$$

Hence an $f \in \mathcal{S}'(\mathbb{R}^d)$ *satisfies* $f \in \mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$ *if and only if*

$$\Big(\sum_{k=0}^\infty 2^{qk(s+\frac{d-1}{2}(\frac{1}{2}-\frac{1}{q}))} \sum_{\nu \in \Theta_k} \|g(t) e^{it\sqrt{-\Delta}} \chi_\nu(D) \psi_k(D) f\|^q_{L^p(\mathbb{R}\times\mathbb{R}^d)} \Big)^{1/q} \tag{4.1}$$

is finite, and (4.1) *defines an equivalent norm on* $\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)$.

Proof. It is straightforward to deal with the low frequencies, so we may assume that $q(D)f = 0$. Moreover, by Lemma 2.3, the second statement follows from the first.

For the first statement, note that there exists an $N \geq 0$ such that

$$\|\chi_\nu(D)e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^d)} \lesssim (1+|t|)^N \|\chi_\nu(D)f\|_{L^p(\mathbb{R}^d)}$$

for all $\nu \in \Theta_k$ and $t \in \mathbb{R}$, as follows either from kernel bounds, or by combining Remark 2.5 and Proposition 3.1. Either way, one thus has

$$\begin{aligned}\|\chi_\nu(D)g(t)e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}\times\mathbb{R}^d)} &= \Big(\int_{\mathbb{R}} |g(t)|\|\chi_\nu(D)e^{it\sqrt{-\Delta}}f\|^p_{L^p(\mathbb{R}^d)}dt\Big)^{1/p}\\ &\lesssim \|\chi_\nu(D)f\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

It now follows from Proposition 2.4 that

$$\begin{aligned}&2^{k(s+\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q}))}\Big(\sum_{\nu\in\Theta_k}\|\chi_\nu(D)g(t)e^{it\sqrt{-\Delta}}f\|^q_{L^p(\mathbb{R}\times\mathbb{R}^d)}\Big)^{1/q}\\ &\lesssim 2^{k(s+\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q}))}\Big(\sum_{\nu\in\Theta_k}\|\chi_\nu(D)f\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q} \eqsim \|f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)}.\end{aligned}$$

On the other hand, for all $\nu \in \Theta_k$ one has

$$\|\chi_\nu(D)f\|_{L^p(\mathbb{R}^d)} = \|e^{-it\sqrt{-\Delta}}e^{it\sqrt{-\Delta}}\chi_\nu(D)f\|_{L^p(\mathbb{R}^d)} \lesssim \|g(t)e^{it\sqrt{-\Delta}}\chi_\nu(D)f\|_{L^p(\mathbb{R}^d)}$$

on any compact interval $I \subseteq \mathbb{R}$ such that $|g(t)| \gtrsim 1$ for all $t \in I$. Hence

$$\begin{aligned}\|f\|_{\mathcal{B}^s_{p,q,q}(\mathbb{R}^d)} &\eqsim 2^{k(s+\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q}))}\Big(\sum_{\nu\in\Theta_k}\|\chi_\nu(D)f\|^q_{L^p(\mathbb{R}^d)}\Big)^{1/q}\\ &\lesssim 2^{k(s+\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q}))}\Big(\sum_{\nu\in\Theta_k}\|\chi_\nu(D)g(t)e^{it\sqrt{-\Delta}}f\|^q_{L^p(\mathbb{R}\times\mathbb{R}^d)}\Big)^{1/q},\end{aligned}$$

again by Proposition 2.4. □

The proof of Theorem 1.1 is now almost immediate.

Proof of Theorem 1.1. Let $g \in \mathcal{S}(\mathbb{R})$ be such that $|g(t)| \geq 1$ for $t \in [0,1]$, and $\mathrm{supp}(\widehat{g}) \subseteq [-1,1]$. Let $\varepsilon > 0$ and $f \in \mathcal{B}^s_{p,2,2}(\mathbb{R}^d)$. We apply the Littlewood–Paley decomposition $(\psi_k)_{k=0}^\infty$ to f. Moreover, we can use a kernel estimate for the low-frequency term, so we may assume that $\psi_0(D)f = 0$. Then the ℓ^2-decoupling inequality (1.6), with ε replaced by $\varepsilon/2$, yields

$$\begin{aligned}\|e^{it\sqrt{-\Delta}}f\|_{L^p_t([0,1],L^p(\mathbb{R}^d))} &\leq \sum_{k=1}^\infty \|e^{it\sqrt{-\Delta}}\psi_k(D)f\|_{L^p_t([0,1],L^p(\mathbb{R}^d))}\\ &\lesssim \sum_{k=1}^\infty 2^{k(\overline{s}(p)+\varepsilon/2)}\Big(\sum_{\nu\in\Theta_k}\|g(t)e^{it\sqrt{-\Delta}}\chi_\nu(D)\psi_k(D)f\|^2_{L^p_{t,x}(\mathbb{R}\times\mathbb{R}^d)}\Big)^{1/2}.\end{aligned}$$

Now Proposition 4.1 implies that the final quantity is equivalent to

$$\sum_{k=1}^\infty 2^{-\varepsilon/2}\|\psi_k(D)f\|_{\mathcal{B}^{\overline{s}(p)+\varepsilon}_{p,2,2}(\mathbb{R}^d)} \lesssim \sum_{k=1}^\infty 2^{-\varepsilon/2}\|f\|_{\mathcal{B}^{\overline{s}(p)+\varepsilon}_{p,2,2}(\mathbb{R}^d)} \eqsim \|f\|_{\mathcal{B}^{\overline{s}(p)+\varepsilon}_{p,2,2}(\mathbb{R}^d)}.$$

This concludes the proof. □

Remark 4.2. For each $2 < p < \infty$ the exponent $\overline{s}(p)$ in Theorem 1.1 is sharp, in the sense that, for any $s < \overline{s}(p)$, there does not exist a $C \geq 0$ such that

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p_t([0,1],L^p(\mathbb{R}^d))} \leq C\|f\|_{\mathcal{B}^s_{p,2,2}(\mathbb{R}^d)} \tag{4.2}$$

for all $f \in \mathcal{B}^s_{p,2,2}(\mathbb{R}^d)$.

To see this, first note that (4.2) and (1.10) combine to yield

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p_t([0,1],L^p(\mathbb{R}^d))} \lesssim_\varepsilon \|f\|_{W^{s+s(p)+2\varepsilon,p}(\mathbb{R}^d)} \tag{4.3}$$

for all $f \in W^{s+s(p)+2\varepsilon,p}(\mathbb{R}^d)$ and $\varepsilon > 0$. Hence, for $p \geq 2(d+1)/(d-1)$, by choosing ε sufficiently small, (4.3) improves upon the conjectured local smoothing estimates in (1.5). Since these are known to be sharp, (4.2) cannot hold for $p \geq 2(d+1)/(d-1)$.

On the other hand, suppose $f \in \mathcal{B}^s_{p,2,2}(\mathbb{R}^d)$ is such that

$$\mathrm{supp}(\widehat{f}\,) \subseteq \{\xi \in \mathbb{R}^d \mid 2^{-1+k} \leq |\xi| \leq 2^{1+k}, |\hat{\xi} - \nu| \leq 2^{1-k/2}\}$$

for some $k \in \mathbb{N}$ and $\nu \in \mathbb{S}^{d-1}$. Then Remark 2.5, Proposition 3.1 and (4.2) yield

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^d)} &\eqsim \|f\|_{\mathcal{B}^0_{p,2,2}(\mathbb{R}^d)} \eqsim \|e^{it\sqrt{-\Delta}}f\|_{L^p_t([0,1],\mathcal{B}^0_{p,2,2}(\mathbb{R}^d))} \\ &\eqsim \|e^{it\sqrt{-\Delta}}f\|_{L^p_t([0,1],L^p(\mathbb{R}^d))} \lesssim \|f\|_{\mathcal{B}^s_{p,2,2}(\mathbb{R}^d)} \eqsim \|f\|_{W^{s,p}(\mathbb{R}^d)} \eqsim 2^{ks}\|f\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Since $\overline{s}(p) = 0$ for $2 < p < 2(d+1)/(d-1)$, this leads to a contradiction, and (4.2) cannot hold for such p.

5. Well-posedness for nonlinear wave equations

In this section we will mainly focus on the cubic nonlinear wave equation

$$\begin{cases}\partial_t^2 u - \Delta_x u = \pm|u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0) = f_1 \in X, \quad \dot{u}(0) = f_2 \in Y,\end{cases} \tag{5.1}$$

outside L^2-based Sobolev spaces.

We first collect some preliminaries. In Section 5.2 we then prove local results for slowly decaying initial data, including a theorem for the quintic nonlinear wave equation. The local results do not distinguish between focusing and defocusing nonlinearity. In Section 5.3 we then prove local results for initial data which decay even slower than in Section 5.2, and finally we prove global results for the *defocusing* equation, that is, (5.1) with a minus sign on the right hand-side.

5.1. Preliminaries.

Our notion of well-posedness is based on [2, Section 3]. We recall the key elements. We use Duhamel's formula to write (5.1) as an abstract evolution equation:

$$u = L(f, g) + N_3(u, u, u), \tag{5.2}$$

where $u \in S$, which is a space-time function space, $L : X \times Y \to S$ is a densely defined linear operator, and $N_3 : S \times S \times S \to S$ is a densely defined operator which is either linear or antilinear in each of its variables. In our case one has

$$L(f,g)(t) := \cos(t\sqrt{-\Delta})f + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g,$$

$$N_3(u_1,u_2,u_3)(t) := \pm\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1(s)\overline{u_2(s)}u_3(s)ds.$$

We say that (5.2) is *quantitatively well posed* (with initial data space $X \times Y$ and solution space S) if there exists a $C \geq 0$ such that

$$\|L(f,g)\|_S \leq C\|(f,g)\|_{X\times Y}, \tag{5.3}$$

$$\|N_3(u_1,u_2,u_3)\|_S \leq C\prod_{i=1}^{3}\|u_i\|_S, \tag{5.4}$$

for all $(f,g) \in X \times Y$ and $u_i \in S$, $1 \leq i \leq 3$.

If (5.2) is quantitatively well posed, then it follows from a fixed-point argument (see [2, Theorem 3]) that (5.2) is *analytically locally well posed*. In particular, there exist $C_0, \varepsilon_0 > 0$ such that, for all $(f,g) \in B_{(X,Y)}(0,\varepsilon_0)$, there exists a unique solution $u[f,g] \in B_S(0, C_0\varepsilon_0)$ to (5.2). Moreover, the map $(f,g) \mapsto u[f,g]$ is Lipschitz continuous from $B_{(X,Y)}(0,\varepsilon_0)$ to $B_S(0,C_0\varepsilon_0)$, and one can expand $u[f,g]$ in terms of its Picard iterates. That is, define the nonlinear maps $A_m : X\times Y \to S$ recursively:

$$\begin{aligned} &A_1(f,g) := L(f,g),\\ &A_m(f,g) := \sum_{\substack{m_1,m_2,m_3\geq 1:\\ m_1+m_2+m_3=m}} N_3(A_{m_1}(f,g),A_{m_2}(f,g),A_{m_3}(f,g)) \quad \text{for } m>1. \end{aligned} \tag{5.5}$$

Then

$$u[f,g] = \sum_{m=0}^{\infty} A_m(f,g),$$

where the series converges absolutely in S for all $(f,g) \in B_{(X,Y)}(0,\varepsilon_0)$. In the following we define solution spaces locally in time $S \to S_T$ and by improving the estimates (5.3) and (5.4) to

$$\begin{aligned} \|L(f,g)\|_{S_T} &\leq CT^{\delta}\|(f,g)\|_{X\times Y},\\ \|N_3(u_1,u_2,u_3)\|_{S_T} &\leq CT^{\delta}\prod_{i=1}^{3}\|u_i\|_{S_T} \end{aligned} \tag{5.6}$$

for some $\delta > 0$, we can find $T = T(\|(f,g)\|_{X\times Y})$, also for large data, such that we find analytic dependence on the initial data in S_T.

We use the following sharp local smoothing estimate due to Guth–Wang–Zhang [9] to prove the linear estimate (5.3) for initial data in L^p-based Sobolev spaces.

Theorem 5.1. *Let $p \in (2,\infty)$ and $s > \max(\frac{1}{2}-\frac{2}{p},0)$. Then there exists a $C \geq 0$ such that*

$$\|e^{it\sqrt{-\Delta}}f\|_{L_t^p([0,1],L^p(\mathbb{R}^2))} \leq C\|f\|_{W^{s,p}(\mathbb{R}^2)}$$

for all $f \in W^{s,p}(\mathbb{R}^2)$.

The smoothing estimate for data in $\mathcal{B}^s_{p,2,2}(\mathbb{R}^2)$ is provided by Theorem 1.1. For the proof of the nonlinear estimate (5.4), we use Strichartz estimates (cf. [14]).

Theorem 5.2. *Let $2 \leq p_i, q_i \leq \infty$, $\frac{2}{p_i}+\frac{1}{q_i}=\frac{1}{2}$, $s_i = 2(\frac{1}{2}-\frac{1}{q_i})-\frac{1}{p_i}$. Then, we find the following estimate to hold:*

$$\begin{aligned} &\|\langle D\rangle^{-s_1}u\|_{L_t^{p_1}([0,T],L^{q_1}(\mathbb{R}^2))}\\ &\lesssim \|u(0)\|_{L^2(\mathbb{R}^2)} + \big\|\langle D\rangle^{s_2}(i\partial_t+\sqrt{-\Delta})u\big\|_{L_t^{p_2'}([0,T],L^{q_2'}(\mathbb{R}^2))}. \end{aligned}$$

5.2. **Local well-posedness results.** We begin with the local well-posedness result in Theorem 1.2 and Remark 1.3.

Proof of Theorem 1.2 and Remark 1.3. As explained above, it suffices to prove (5.6). In what follows, let $T \leq 1$, which simplifies powers of T.

We first consider the linear estimate (5.3) with initial data in $W^{s,p}(\mathbb{R}^d)$, as in Remark 1.3. Theorem 5.1 and Hölder's inequality in time yield

$$\|e^{it\sqrt{-\Delta}} f\|_{L_t^{24/7}([0,T],L^4(\mathbb{R}^2))} \lesssim T^{1/24}\|f\|_{W^{\varepsilon,4}(\mathbb{R}^2)},$$
$$\|e^{it\sqrt{-\Delta}} f\|_{L_t^{4}([0,T],L^6(\mathbb{R}^2))} \lesssim T^{1/12}\|f\|_{W^{1/6+\varepsilon,6}(\mathbb{R}^2)}.$$

This yields the linear estimate for $\cos(t\sqrt{-\Delta})f$ and for the high frequencies of $\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g$. On the other hand, the low-frequency estimate holds since

$$\chi(D)\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2) \tag{5.7}$$

for some cut-off $\chi \in C_c^\infty(\mathbb{R}^2)$ near zero and all $1 < p < \infty$, due to Mikhlin's theorem.

Now consider the linear estimate (5.3) for initial data in $\mathcal{B}^s_{p,2,2}(\mathbb{R}^2) + \dot{H}^t(\mathbb{R}^2)$, cf. Theorem 1.2. Recall that in this case we consider the solution space

$$S_T = L^{24/7}([0,T], L^4(\mathbb{R}^2)) \cap C([0,T]; \mathcal{B}^\varepsilon_{4,2,2}(\mathbb{R}^2) + \dot{H}^{3/8}(\mathbb{R}^2)). \tag{5.8}$$

To obtain the linear estimate for the first space on the right-hand side, we again rely on Hölder's inequality, Theorem 1.1, and on linear Strichartz estimates as in Theorem 5.2. More precisely, let $f = f_1 + f_2$ with $f_1 \in \mathcal{B}^\varepsilon_{4,2,2}(\mathbb{R}^2)$ and $f_2 \in \dot{H}^{3/8}(\mathbb{R}^2)$. Then Theorems 1.1 and 5.2 yield

$$\|e^{it\sqrt{-\Delta}} f\|_{L_t^{24/7}([0,T],L^4(\mathbb{R}^2))} \lesssim T^{1/24}(\|f_1\|_{\mathcal{B}^\varepsilon_{4,2,2}(\mathbb{R}^2)} + \|f_2\|_{\dot{H}^{3/8}(\mathbb{R}^2)}).$$

Note that we can likewise estimate f_2 in $H^s(\mathbb{R}^2)$ for $s \geq 3/8$. By taking the infimum over all decompositions $f = f_1 + f_2$ in $\mathcal{B}^\varepsilon_{4,2,2}(\mathbb{R}^2) + \dot{H}^{3/8}(\mathbb{R}^2)$, we find

$$\|\cos(t\sqrt{-\Delta})f\|_{L_t^{24/7}([0,T],L^4(\mathbb{R}^2))} \lesssim T^{1/24}\|f\|_{\mathcal{B}^\varepsilon_{4,2,2}(\mathbb{R}^2)+\dot{H}^{3/8}(\mathbb{R}^2)}.$$

Next, write $g = g_1 + g_2$ with $g_1 \in B^{\varepsilon-1}_{4,2,2}(\mathbb{R}^2)$ and $g_2 \in \dot{H}^{-5/8}(\mathbb{R}^2)$. To obtain

$$\Big\|\frac{\sin(t\sqrt{-\Delta})}{(-\Delta)^{1/2}} g\Big\|_{L_t^{24/7}([0,T],L^4(\mathbb{R}^2))} \lesssim T^{1/24}(\|g_1\|_{\mathcal{B}^{\varepsilon-1}_{4,2,2}(\mathbb{R}^2)} + \|g_2\|_{\dot{H}^{-5/8}(\mathbb{R}^2)})$$

one proceeds in the same way when it comes to g_2 and the high frequencies of g_1, using the additional smoothing. On the other hand, for the low frequencies of g_1, one can argue as in (5.7). Indeed, one has

$$\chi(D)\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} : \mathcal{B}^{\varepsilon-1}_{p,2,2}(\mathbb{R}^2) \to \mathcal{B}^{s(p)+\varepsilon}_{p,p,p}(\mathbb{R}^2) \subseteq B^\varepsilon_{p,p}(\mathbb{R}^2) \subseteq L^p(\mathbb{R}^2). \tag{5.9}$$

Here we used Proposition 2.4, Mikhlin's theorem and trivial summation to obtain the mapping property, and (2.3) and standard embeddings from Besov spaces into $L^p(\mathbb{R}^d)$ for the inclusions. This proves the linear estimate (5.3) for the first space on the right-hand side of (5.8).

To show the linear estimate involving the solution space $C([0,T]; \mathcal{B}^\varepsilon_{4,2,2}(\mathbb{R}^2) + \dot{H}^{3/8}(\mathbb{R}^2))$, we use the invariance of $\mathcal{B}^\varepsilon_{4,2,2}(\mathbb{R}^2)$ and $\dot{H}^{3/8}(\mathbb{R}^2))$ under the half-wave group, as well as (5.9) to deal with the low frequencies of $\sin(t\sqrt{-\Delta})(-\Delta)^{-1/2}$.

Finally, by relying instead on the $L^6([0,T];L^6(\mathbb{R}^2))$ smoothing estimate in Theorem 1.1, as well as the $L_t^6([0,T],L^6(\mathbb{R}^2))$ Strichartz estimate, we obtain

$$\|\cos(t\sqrt{-\Delta})f\|_{L_t^4([0,T],L^6(\mathbb{R}^2))} \lesssim T^{1/12}\|f\|_{\mathcal{B}^{\varepsilon}_{6,2,2}(\mathbb{R}^2)+\dot{H}^{1/2}(\mathbb{R}^2)}.$$

Similarly,

$$\Big\|\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g\Big\|_{L_t^4([0,T],L^6(\mathbb{R}^2))} \lesssim T^{1/12}\|g\|_{\mathcal{B}^{\varepsilon-1}_{6,2,2}(\mathbb{R}^2)+\dot{H}^{-1/2}(\mathbb{R}^2)}.$$

Moreover, to obtain the linear estimate for the solution space $C([0,T];\mathcal{B}^{\varepsilon}_{6,2,2}(\mathbb{R}^2)+\dot{H}^{1/2}(\mathbb{R}^2))$, one argues as above. This takes care of the linear estimate (5.3) for both Theorem 1.2 and Remark 1.3.

We turn to the trilinear estimate (5.4), as a consequence of Strichartz estimates, and begin with the estimate

$$\begin{aligned}&\Big\|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}(u_1\overline{u_2}u_3)(s)ds\Big\|_{L_t^{24/7}([0,T],L^4(\mathbb{R}^2))}\\ &\lesssim T^{\frac{1}{24}}\prod_{i=1}^3\|u_i\|_{L_t^{24/7}([0,T],L^4(\mathbb{R}^2))}.\end{aligned}$$

The low frequencies $\chi(D)(u_1\overline{u_2}u_3)$ are again estimated via Mikhlin's theorem. For the high frequencies, we use Theorem 5.2 with $p_i=8$, $q_i=4$, $i=1,2$, to find

$$\begin{aligned}&\Big\|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}(1-\chi)(D)(u_1\overline{u_2}u_3)(s)ds\Big\|_{L_t^{24/7}([0,T],L^4(\mathbb{R}^2))}\\ &\lesssim T^{1/8}\Big\|\frac{|D|^{6/8}}{|D|}(1-\chi)(D)(u_1\overline{u_2}u_3)\Big\|_{L_t^{8/7}L^{4/3}} \lesssim T^{1/8}\|u_1\overline{u_2}u_3\|_{L_t^{8/7}L^{4/3}}\\ &\lesssim T^{1/8}\prod_{i=1}^3\|u_i\|^3_{L_t^{24/7}([0,T],L^4(\mathbb{R}^2))}.\end{aligned}$$

This already concludes the proof for initial data in $W^{\varepsilon,4}(\mathbb{R}^2)\times W^{-1+\varepsilon,4}(\mathbb{R}^2)$. For initial data involving the $\mathcal{B}^s_{4,2,2}(\mathbb{R}^2)$-spaces, we also need to consider the solution space $C([0,T],\dot{H}^{3/8}(\mathbb{R}^2))$. Here we estimate by Minkowski's inequality

$$\begin{aligned}&\Big\|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}\chi(D)(u_1\overline{u_2}u_3)(s)ds\Big\|_{L_t^{\infty}([0,T],\dot{H}^{3/8}(\mathbb{R}^2))}\\ &\lesssim \|u_1\overline{u_2}u_3\|_{L_t^1([0,T],L^{4/3}(\mathbb{R}^2))} \lesssim T^{\frac{3}{24}}\prod_{i=1}^3\|u_i\|_{L_t^{24/7}([0,T],L^4(\mathbb{R}^2))}\end{aligned}$$

for the low frequencies, again by Mikhlin's theorem. For the high frequencies, we use a Sobolev embedding:

$$\begin{aligned}&\Big\|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}(1-\chi)(D)(u_1\bar{u}_2u_3)(s)ds\Big\|_{L_t^{\infty}([0,T],\dot{H}^{3/8}(\mathbb{R}^2))}\\ &\lesssim \|\langle D\rangle^{-5/8}(u_1\overline{u_2}u_3)\|_{L_t^1([0,T],L^2(\mathbb{R}^2))} \lesssim T^{3/24}\prod_{i=1}^n\|u_i\|_{L_t^{24/7}([0,T],L^4(\mathbb{R}^2))}.\end{aligned}$$

This proves the required supremum norm bounds, while the continuity statements are automatic, since the half-wave group is strongly continuous on $\dot{H}^s(\mathbb{R}^2)$. This also concludes the proof for initial data as in (1.11).

Finally, we consider the trilinear estimate with parameter $p = 6$. First consider

$$\Big\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (u_1 \overline{u_2} u_3)(s) ds \Big\|_{L^4_t([0,T],L^6(\mathbb{R}^2))} \lesssim T^{1/12} \prod_{i=1}^3 \|u_i\|_{L^4_t([0,T],L^6(\mathbb{R}^2))}.$$

The estimate of low frequencies is as before, so it suffices to use Strichartz estimates with $p_1 = q_1 = 6$ and $p_2 = \infty$, $q_2 = 2$:

$$\begin{aligned} &\Big\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (1-\chi)(D)(u_1 \overline{u_2} u_3)(s) ds \Big\|_{L^4_t([0,T],L^6(\mathbb{R}^2))} \\ &\lesssim T^{1/12} \Big\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (1-\chi)(D)(u_1 \overline{u_2} u_3)(s) ds \Big\|_{L^6_t L^6_x} \\ &\lesssim T^{1/12} \|u_1 \overline{u_2} u_3\|_{L^1_t L^2_x} \lesssim T^{1/12} \prod_{i=1}^3 \|u_i\|_{L^4_t([0,T],L^6(\mathbb{R}^2))}. \end{aligned}$$

This proves the required statement for initial data in $W^{1/6+\varepsilon,6}(\mathbb{R}^2) \times W^{-5/6+\varepsilon,6}(\mathbb{R}^2)$ and concludes the proof of Remark 1.3.

On the other hand, for the local well-posedness with initial data in $\mathcal{B}^s_{6,2,2}(\mathbb{R}^2)$ we also have to consider the solution space $C([0,T],\dot{H}^{1/2}(\mathbb{R}^2))$, in the following sense:

$$\Big\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (u_1 \overline{u_2} u_3)(s) ds \Big\|_{L^\infty_t([0,T],\dot{H}^{1/2}(\mathbb{R}^2))} \lesssim T^{1/12} \prod_{i=1}^3 \|u_i\|_{L^4_t([0,T],L^6(\mathbb{R}^2))}.$$

The estimate for the low frequencies is carried out by Mikhlin's theorem, while for the high frequencies the argument is

$$\begin{aligned} \Big\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (1-\chi(D))(u_1 \bar{u}_2 u_3)(s) ds \Big\|_{L^\infty_t \dot{H}^{1/2}} &\lesssim \|u_1 \bar{u}_2 u_3\|_{L^1_t L^2_x} \\ &\lesssim T^{1/12} \prod_{i=1}^3 \|u_i\|_{L^4_t L^6_x}. \end{aligned}$$

This concludes the proof. □

We remark that there is slack in the spatial regularity in the nonlinear argument. This can be translated to solve the quintic nonlinear wave equation

(5.10)
$$\begin{cases} \partial_t^2 u - \Delta_x u = \pm |u|^4 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0) = f \in \mathcal{B}^\varepsilon_{6,2,2}(\mathbb{R}^2) + \dot{H}^{1/2}(\mathbb{R}^2), \quad \dot{u}(0) = g \in \mathcal{B}^{\varepsilon-1}_{6,2,2}(\mathbb{R}^2) + \dot{H}^{-1/2}(\mathbb{R}^2) \end{cases}$$

in the solution space $S_T = L^6_t([0,T],L^6_x) \cap C([0,T],\mathcal{B}^\varepsilon_{6,2,2}(\mathbb{R}^2) + \dot{H}^{1/2}(\mathbb{R}^2))$ for small initial data. The crucial nonlinear estimate reads

$$\begin{aligned} &\Big\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} \prod_{i=1}^5 u_i(s) ds \Big\|_{L^6_t([0,T],L^6(\mathbb{R}^2))} \\ &\lesssim \Big\| \prod_{i=1}^5 u_i \Big\|_{L^{6/5}_t([0,T],L^{6/5}(\mathbb{R}^2))} \lesssim \prod_{i=1}^5 \|u_i\|_{L^6_t([0,T],L^6(\mathbb{R}^2))} \end{aligned}$$

with Strichartz pairs $(p_i,q_i) = (6,6)$, $i = 1,2$, because inhomogeneous Strichartz pairs as in Theorem 5.2 lose exactly one derivative. Note that we cannot afford to apply Hölder's inequality in time anymore. Hence, this argument does not allow to prove well-posedness for large initial data. This is not surprising because (5.10) is

$\dot{H}^{1/2}(\mathbb{R}^2) \times \dot{H}^{-1/2}(\mathbb{R}^2)$-scaling critical. Easy variants of the above arguments yield the following theorem.

Theorem 5.3. *For any $T > 0$, there is an $\varepsilon > 0$ such that (5.10) is analytically locally well posed with $u \in S_T = L^6([0,T], L^6(\mathbb{R}^2)) \cap C([0,T], \mathcal{B}^{\varepsilon}_{6,2,2} + \dot{H}^{1/2}(\mathbb{R}^2))$ provided that*

$$\|f\|_{\mathcal{B}^{\varepsilon}_{6,2,2}(\mathbb{R}^2)+\dot{H}^{1/2}(\mathbb{R}^2)} + \|g\|_{\mathcal{B}^{\varepsilon-1}_{6,2,2}+\dot{H}^{-1/2}} \leq \varepsilon.$$

5.3. **Results for slower decaying initial data.** In the following we point out how considering higher Picard iterates allows to construct solutions for very slowly decaying initial data. The arguments are similar to [23] and [7], albeit with the difference that the Duhamel integral has a stronger smoothing effect. We consider the cubic nonlinear wave equation in d dimensions:

$$\begin{cases} \partial_t^2 u - \Delta_x u &= \pm |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \ d \geq 2, \\ \quad u(0) &= f_1 \in X, \quad \dot{u}(0) = f_2 \in Y, \end{cases}$$

although our main results concern $d \in \{2,3\}$. We write the solution abstractly:

$$u = L(f_1, f_2) + N_3(u,u,u),$$

as in Section 5.1.

For $d, n \geq 2$, we consider initial data in L^{4n+2}-based spaces, and we let

$$\begin{aligned} u^0(t) &= L(f_1, f_2), \\ u^1(t) &= N_3(u^0, u^0, u^0), \\ u^j(t) &= N_3\Big(\sum_{k=0}^{j-1} u^k, \sum_{k=0}^{j-1} u^k, \sum_{k=0}^{j-1} u^k\Big) - \sum_{k=1}^{j-1} u^k, \quad (j \geq 2). \end{aligned}$$

We will prove the existence of a

$$v \in S^0([-1,1] \times \mathbb{R}^2) := L_t^\infty([-1,1]; L_x^2(\mathbb{R}^2)) \cap L_t^4([-1,1], L_x^\infty(\mathbb{R}^2))$$

which solves

$$v = u - \sum_{j=0}^{n-1} u^j.$$

We can rewrite this as

$$v = N_3(u,u,u) - \sum_{j=1}^{n-1} u^j = N_3\Big(v + \sum_{j=0}^{n-1} u^j, v + \sum_{j=0}^{n-1} u^j, v + \sum_{j=0}^{n-1} u^j\Big) - \sum_{j=1}^{n-1} u^j \tag{5.11}$$

for $j \geq 2$. One can check that u^j contains only terms A_k with $k \geq 2j+1$, where A_k is as in (5.5) (see [24, Section 4.2]). We therefore obtain estimates for such terms.

Lemma 5.4. *Let $d \in \{2,3\}$, $n \geq 2$ and $s > \frac{d-1}{2}(1 - \frac{1}{4n+2})$. Then there exists a $C \geq 0$ such that*

$$\|A_m(f_1,f_2)\|_{L_t^\infty([-1,1]; L^{\frac{4n+2}{m}}(\mathbb{R}^d))} \leq C\big(\|f_1\|^m_{W^{s,4n+2}(\mathbb{R}^d)} + \|f_2\|^m_{W^{s-1,4n+2}(\mathbb{R}^d)}\big).$$

for all $m \in \{1, \dots, 2n-1\}$, $f_1 \in W^{s,4n+2}(\mathbb{R}^d)$ and $f_2 \in W^{s-1,4n+2}(\mathbb{R}^d)$.

Proof. First note that $A_m = 0$ if m is even. Hence we may suppose that $m = 2k+1$ for some $k \in \mathbb{N}_0$. Let $\varepsilon > 0$ and set $p := 4n+2$ and $q := (4n+2)/m$. Then, by the embeddings (2.6) and (2.8), one has

$$\|A_m(f_1,f_2)\|_{L_t^\infty L_x^q} \lesssim \|A_m(f_1,f_2)\|_{L_t^\infty \mathcal{B}^{s(q)}_{q,q',q'}} \lesssim \|A_m(f_1,f_2)\|_{L_t^\infty \mathcal{B}^{(d-1)/4}_{q,1,1}}$$
$$\lesssim \|A_m(f_1,f_2)\|_{L_t^\infty \mathcal{B}^{s(1)+\varepsilon}_{q,1,1}},$$

since $s(1) = (d-1)/4$. We can use this regularity to iterate the Duhamel integral in adapted spaces, by Lemma 3.4 and because $d \in \{2,3\}$. First, we split the Duhamel integral into low and high frequencies:

$$\Big\| \int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}}(u_1u_2u_3)(s)ds\Big\|_{L_t^\infty \mathcal{B}^{s(1)+\varepsilon}_{q,1,1}}$$
$$\le \Big\|\chi(D)\int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}}(u_1u_2u_3)(s)ds\Big\|_{L_t^\infty L_x^q}$$
$$+ \Big\|(1-\chi(D))\int_0^t e^{i(t-s)\sqrt{-\Delta}}(u_1u_2u_3)(s)ds\Big\|_{L_t^\infty \mathcal{B}^{s(1)-1+\varepsilon}_{q,1,1}}.$$

The low frequencies are estimated by Mikhlin's theorem:

$$\Big\|\chi(D)\int_0^t \frac{e^{i(t-s)\sqrt{-\Delta}}}{\sqrt{-\Delta}}(u_1u_2u_3)(s)ds\Big\|_{L^q}$$
$$\lesssim T\|u_1u_2u_3\|_{L_t^\infty L_x^q} \lesssim T\prod_{i=1}^3 \|u_i\|_{L_t^\infty L_x^{3q}} \lesssim T\prod_{i=1}^3 \|u_i\|_{L_t^\infty \mathcal{B}^{s(1)+\varepsilon}_{3q,1,1}},$$

which allows for iteration. Moreover, for the high frequencies we use the boundedness of $e^{it\sqrt{-\Delta}}$ on the adapted Besov spaces, and iterate the trilinear estimate in Lemma 3.4 k times, to obtain

$$\|A_m(f_1,f_2)\|_{L_t^\infty \mathcal{B}^{s(1)+\varepsilon}_{q,1,1}} \lesssim \|f_1\|^m_{\mathcal{B}^{s(1)+\varepsilon}_{4n+2,1,1}} + \|f_2\|^m_{\mathcal{B}^{s(1)+\varepsilon-1}_{4n+2,1,1}}$$
$$\lesssim \|f_1\|^m_{\mathcal{B}^{s(1)+2\varepsilon}_{p,p,p}} + \|f_2\|^m_{\mathcal{B}^{s(1)+2\varepsilon-1}_{p,p,p}}$$
$$\lesssim \|f_1\|^m_{W^{s(1)+s(p)+3\varepsilon,p}} + \|f_2\|^m_{W^{s(1)+s(p)-1+3\varepsilon,p}}.$$

Here we also used the embeddings (2.7), (2.10), (2.11) and (1.2). By choosing ε sufficiently small, this concludes the proof. □

Similarly, we can iterate

$$\|A_m(f_1,f_2)\|_{L_t^\infty L_x^\infty} \lesssim \|A_m(f_1,f_2)\|_{L_t^\infty \mathcal{B}^{\frac{d-1}{4}+\varepsilon}_{\infty,1,1}} \lesssim \|f_1\|^m_{\mathcal{B}^{\frac{d-1}{4}+\varepsilon}_{\infty,1,1}} + \|f_2\|^m_{\mathcal{B}^{\frac{d-1}{4}-1+\varepsilon}_{\infty,1,1}}$$

and

$$\|f\|_{\mathcal{B}^{\frac{d-1}{4}+\varepsilon}_{\infty,1,1}} \lesssim \|f\|_{\mathcal{B}^{\frac{d-1}{4}+\frac{d+1}{2p}+\varepsilon}_{p,1,1}} \lesssim \|f\|_{\mathcal{B}^{\frac{d-1}{4}+\frac{d+1}{2p}+\varepsilon}_{p,p,p}} \lesssim \|f\|_{W^{s,p}}$$

for $s > \alpha$ with

$$\alpha := \frac{d-1}{4} + \frac{d+1}{2p} + \frac{d-1}{2}\Big(\frac{1}{2}-\frac{1}{p}\Big) = \frac{d-1}{2}\Big(1-\frac{1}{p}\Big) + \frac{d+1}{2p}. \tag{5.12}$$

This shows that

$$\|A_m(f_1,f_2)\|_{L_t^\infty L_x^\infty} \lesssim \|f_1\|^m_{W^{s,p}} + \|f_2\|^m_{W^{s-1,p}}.$$

We can argue like above to find

$$\|A_m(f_1,f_2)\|_{L_t^\infty L_x^\infty} \lesssim \|f_1\|^m_{\mathcal{B}^{\frac{d-1}{4}+\varepsilon}_{\infty,1,1}} + \|f_2\|^m_{\mathcal{B}^{\frac{d-1}{4}-1+\varepsilon}_{\infty,1,1}}.$$

Now we use the embeddings

$$\|f\|_{\mathcal{B}^s_{\infty,1,1}} \lesssim \|f\|_{\mathcal{B}^{s+\frac{d+1}{2p}}_{p,1,1}} \lesssim \|f\|_{\mathcal{B}^{s+\frac{d+1}{2p}}_{p,2,2}}.$$

This shows that

$$\|A_m(f_1,f_2)\|_{L^\infty_t L^\infty_x} \lesssim \|f_1\|^m_{\mathcal{B}^s_{p,2,2}} + \|f_2\|^m_{\mathcal{B}^{s-1}_{p,2,2}}$$

for $s > \tilde{\alpha}$ with

$$\tilde{\alpha} = \frac{d-1}{4} + \frac{d+1}{2p}. \tag{5.13}$$

We have proved the following lemma, regarding the u_j from above.

Lemma 5.5. *Let $d \in \{2,3\}$, $n \geq 2$, $0 \leq j \leq n-1$. Then, for $\varepsilon > 0$, there is $\varepsilon_n \leq 1$ and $\tilde{\varepsilon}_n \leq 1$ such that*

$$\|u^j\|_{L^\infty_{t,x}([0,1]\times\mathbb{R}^d)} + \|u^j\|_{L^\infty_t L^{\frac{4n+2}{2j+1}}([0,1]\times\mathbb{R}^d)} \lesssim \|f\|_{W^{\alpha+\varepsilon,4n+2}(\mathbb{R}^d)}$$

holds true provided that $\|f\|_{W^{\alpha+\varepsilon,4n+2}(\mathbb{R}^d)} \leq \varepsilon_n$, and

$$\|u^j\|_{L^\infty_{t,x}([0,1]\times\mathbb{R}^d)} + \|u^j\|_{L^\infty_t L^{\frac{4n+2}{2j+1}}([0,1]\times\mathbb{R}^d)} \lesssim \|f\|_{\mathcal{B}^{\tilde{\alpha}+\varepsilon}_{4n+2,2,2}(\mathbb{R}^d)}$$

provided that $\|f\|_{\mathcal{B}^{\tilde{\alpha}+\varepsilon}_{4n+2,2,2}(\mathbb{R}^d)} \leq \tilde{\varepsilon}_n$.

With the estimate for the higher Picard iterates at hand, the following proposition is proved like in [23, Proposition 4.6]:

Proposition 5.6. *Let $d \in \{2,3\}$, $\varepsilon > 0$, $n \geq 2$, and ε_n, $\tilde{\varepsilon}_n \leq 1$ like in Lemma 5.4. Then, there is a unique $v \in S^0$, which solves (5.11) with $v(0) = \dot{v}(0) = 0$.*

This yields the following theorem on local well-posedness for slowly decaying initial data. We focus on the two-dimensional case with small data to simplify the Strichartz space, but there are clearly analogs available in higher dimensions.

Theorem 5.7. *Let $d = 2$, $\varepsilon > 0$, $n \geq 2$, and (f_1, f_2), ε_n, and $\tilde{\varepsilon}_n$ like in Proposition 5.6. Let $\frac{2}{p} + \frac{1}{4n+2} = \frac{1}{2}$. Then, there is $u \in L^p_t([0,1], L^{4n+2}(\mathbb{R}^2))$ which solves (5.1). Furthermore, for*

$$\|(f_1,f_2)\|_{W^{\alpha+\varepsilon,4n+2}\times W^{\alpha+\varepsilon-1,4n+2}} + \|(g_1,g_2)\|_{W^{\alpha+\varepsilon,4n+2}\times W^{\alpha+\varepsilon-1,4n+2}} \leq \varepsilon_n$$
$$\text{or } \|(f_1,f_2)\|_{\mathcal{B}^{\tilde{\alpha}+\varepsilon}_{4n+2,2,2}\times\mathcal{B}^{\tilde{\alpha}+\varepsilon}_{4n+2,2,2}} + \|(g_1,g_2)\|_{\mathcal{B}^{\tilde{\alpha}-1+\varepsilon}_{4n+2,2,2}\times\mathcal{B}^{\tilde{\alpha}-1+\varepsilon}_{4n+2,2,2}} \leq \tilde{\varepsilon}_n$$

we have for the corresponding solutions $\|u_1 - u_2\|_{L^p([0,1],L^{4n+2})} \to 0$ provided that the initial data are converging in the spaces of initial data.

5.4. **Global well-posedness results.** We prove global results for the defocusing cubic nonlinear wave equation in two dimensions:

$$\begin{cases} \partial_t^2 u - \Delta_x u &= -|u|^2 u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^2, \\ u(0) &= f_1 \in \mathcal{B}^s_{p,2}(\mathbb{R}^2), \qquad \dot{u}(0) = f_2 \in \mathcal{B}^{s-1}_{p,2}(\mathbb{R}^2). \end{cases} \tag{5.14}$$

We focus on the case $p = 6$.

The main result of this section is the following:

Theorem 5.8. *Let $s > 1/2$, and $(f_1,f_2) \in \mathcal{B}^s_{6,2,2}(\mathbb{R}^2) \times \mathcal{B}^{s-1}_{6,2,2}(\mathbb{R}^2)$. Then, for any $T > 0$, there is a global solution $u \in L^4_t([0,T], L^6(\mathbb{R}^2))$ to (5.14).*

In the following the arguments from [23] are adapted, which were previously applied to nonlinear Schrödinger equations. To avoid technicalities, we shall consider Schwartz initial data which admit global solutions and allow for integration by parts arguments. The a priori assumption can be removed later by well-posedness and limiting arguments. We denote the linear part of the solution to (5.14) by

$$w(t) = \cos(t\sqrt{-\Delta})f_1 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f_2.$$

The difference with the full solution is given by

$$v(t) = u(t) - w(t) = -\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} (|v+w|^2(v+w)) ds. \tag{5.15}$$

We have the following blow-up alternative (cf. [23, Lemma 4.9]):

Lemma 5.9. *Let $s > 0$, $(f_1, f_2) \in (\mathcal{B}^s_{6,2,2}(\mathbb{R}^2) + H^1(\mathbb{R}^2)) \times (\mathcal{B}^{s-1}_{6,2,2}(\mathbb{R}^2) + L^2(\mathbb{R}^2))$, and u be the solution to (5.14) provided by Theorem 1.2 in $L^4_t([0,T], L^6(\mathbb{R}^2))$. If T^* is maximal such that $u \in L^4_t([0,T], L^6_x(\mathbb{R}^2))$ for $T < T^*$, but $u \notin L^4_t([0,T^*], L^6_x(\mathbb{R}^2))$, then $\lim_{t \to T^*} (\|v(t)\|_{H^1(\mathbb{R}^2)} + \|\partial_t v(t)\|_{L^2}) = \infty$ with v defined like in (5.15).*

Proof. We note that for the free solution we have

$$\sup_{t \in [0,T]} \|w(t)\|_{\mathcal{B}^s_{6,2,2} + H^1} \lesssim_T \|(f,g)\|_{(\mathcal{B}^s_{6,2,2}+H^1) \times (\mathcal{B}^{s-1}_{6,2,2} + L^2)}, \tag{5.16}$$

$$\sup_{t \in [0,T]} \|\partial_t w(t)\|_{\mathcal{B}^{s-1}_{6,2,2} + L^2} \lesssim_T \|(f,g)\|_{(\mathcal{B}^s_{6,2,2}+H^1) \times (\mathcal{B}^{s-1}_{6,2,2} + L^2)}. \tag{5.17}$$

We further argue by contradiction. Suppose that there is a sequence $(t_n) \subseteq [0, T^*)$ with $t_n \uparrow T^*$ and

$$\lim_{n \to \infty} (\|v(t_n)\|_{H^1(\mathbb{R}^2)} + \|v(t_n)\|_{L^2}) \leq C. \tag{5.18}$$

But by Theorem 1.2, we can solve the nonlinear wave equation with initial data

$$w(t_n) + v(t_n) \in \mathcal{B}^s_{6,2,2} + H^1, \quad \dot{w}(t_n) + \dot{v}(t_n) \in \mathcal{B}^{s-1}_{6,2,2} + L^2$$

for times $T = T(\|w(t_n) + v(t_n)\|_{\mathcal{B}^s_{6,2,2}+H^1}, \|\dot{w}(t_n) + \dot{v}(t_n)\|_{\mathcal{B}^{s-1}_{6,2,2}+L^2})$ and by (5.16), (5.17), and (5.18), we find

$$\begin{aligned} &\|w(t_n) + v(t_n)\|_{\mathcal{B}^s_{6,2,2}+H^1} + \|\dot{w}(t_n) + \dot{v}(t_n)\|_{\mathcal{B}^{s-1}_{6,2,2}+L^2} \\ &\lesssim_{T^*} C + \|(f,g)\|_{(\mathcal{B}^s_{6,2,2}+H^1) \times (\mathcal{B}^{s-1}_{6,2,2}+L^2)}. \end{aligned}$$

This means the local existence time is bounded from below, which yields a contradiction because it means we can continue the solution beyond T^*. The proof is complete. ☐

Hence, for the proof of global well-posedness it suffices to show

$$\sup_{t \in [0,T]} (\|v(t)\|_{H^1(\mathbb{R}^2)} + \|\partial_t v(t)\|_{L^2(\mathbb{R}^2)}) \leq C(T).$$

Recall that mass and energy are conserved quantities for (smooth) solutions to (5.14):

$$M(u) = \int_{\mathbb{R}^2} |u|^2 dx, \tag{5.19}$$

$$E(u) = \int_{\mathbb{R}^2} \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{1}{4}|u|^4 dx. \tag{5.20}$$

But the quantities are not conserved for differences of solutions or v. Still we can control $M(v)+E(v)$ by Grønwall's argument for sufficiently regular initial data like in [7, 23] in the context of the defocusing nonlinear Schrödinger equation. In the proof we have to control $\|w(t)\|_{L^6(\mathbb{R}^2)}$ and $\|w(t)\|_{L^\infty(\mathbb{R}^d)}$, for which we use embeddings, namely (2.6), Propositions 2.6 and 2.8, and a standard Sobolev embedding:

$$\|w(t)\|_{L^6(\mathbb{R}^2)} \lesssim \|w(t)\|_{\mathcal{B}^{1/6}_{6,6,6}(\mathbb{R}^2)} \lesssim \|w(t)\|_{\mathcal{B}^{1/6+\varepsilon}_{6,2,2}(\mathbb{R}^2)},$$
$$\|w(t)\|_{L^\infty(\mathbb{R}^2)} \lesssim \|w(t)\|_{W^{1/3+\varepsilon,6}(\mathbb{R}^2)} \lesssim \|w(t)\|_{\mathcal{B}^{1/2+2\varepsilon}_{6,2,2}(\mathbb{R}^2)}.$$

We show the following:

Proposition 5.10. *Let $\varepsilon > 0$ and $(f_1, f_2) \in \mathcal{B}^{1/2+\varepsilon}_{6,2,2}(\mathbb{R}^2) \times \mathcal{B}^{-1/2+\varepsilon}_{6,2,2}(\mathbb{R}^2)$. With notation as above, we find the following estimate to hold:*

$$\partial_t(M(v) + E(v) + 1)(t) \lesssim_T M(v) + E(v) + 1$$

for all $0 \leq t \leq T$.

With Proposition 5.10 in place, we find by Grønwall's argument

$$(M(v) + E(v) + 1)(t) \leq e^{\int_0^t C(s)ds}$$

and hence, $M(v)$ does not blow-up. Theorem 5.8 follows.

Proof of Proposition 5.10. We introduce the notation

$$(f, g) = \Re \int_{\mathbb{R}^2} f(x)\overline{g}(x)dx.$$

For the growth of $M(v)$, we find

$$\partial_t M(v) = 2(\partial_t v, v) \lesssim E(v)^{1/2} M(v)^{1/2} \lesssim M(v) + E(v) + 1.$$

For the time-derivative of E we find

$$\begin{aligned}
\partial_t E(v) &= ((\partial_t^2 v), \partial_t v) + (\partial_t \nabla_x v, \nabla_x v) + (\partial_t v, |v|^2 v) \\
&= (\partial_t v, \partial_t^2 v - \Delta v + |v|^2 v) \\
&= (\partial_t v, -|v+w|^2(v+w) + |v|^2 v) \\
&\lesssim |(\partial_t v, |v|^2 w)| + |(\partial_t v, v|w|^2)| + |(\partial_t v, |w|^2 w)| \\
&\lesssim \|\partial_t v\|_{L^2} \|v\|_{L^4}^2 \|w\|_{L^\infty} + \|\partial_t v\|_{L^2} \|v\|_{L^2} \|w\|_{L^\infty}^2 + \|\partial_t v\|_{L^2} \|w\|_{L^6}^3 \\
&\lesssim_T E(v) + E(v)^{1/2} M(v)^{1/2} + E(v)^{1/2}.
\end{aligned}$$

This finishes the proof. ☐

We sketch the extension to slower decaying initial data:

Theorem 5.11. *Let $d = 2$, $\varepsilon > 0$, $n \geq 2$, $\tilde{\alpha}$ like in (5.13). Let $(f_1, f_2) \in \mathcal{B}^{\tilde{\alpha}+\epsilon}_{4n+2,2,2}(\mathbb{R}^2) \times \mathcal{B}^{\tilde{\alpha}+\varepsilon-1}_{4n+2,2,2}(\mathbb{R}^2)$. Let $\frac{2}{p} + \frac{1}{4n+2} = \frac{1}{2}$. Then, there is some $\tilde{p} < p$ such that for any $T > 0$ there is $u \in L_t^{\tilde{p}}([0,T], L^{4n+2}(\mathbb{R}^2))$, which solves (5.14).*

The key point is that solving (5.14) in $L_t^p L_x^q$-spaces provides us with a blow-up alternative:

Lemma 5.12. *Let $s > \tilde{\alpha}$,*

$$(f_1, f_2) \in (\mathcal{B}^s_{4n+2,2,2}(\mathbb{R}^2) + H^1(\mathbb{R}^2), \mathcal{B}^{s-1}_{4n+2,2,2}(\mathbb{R}^2) + L^2(\mathbb{R}^2)),$$

and let u be the solution to (5.14) provided by Theorem 5.7 in $L_t^{\tilde{p}}([0,T], L^{4n+2}(\mathbb{R}^2))$. If T^* is maximal such that $u \in L_t^{\tilde{p}}([0,T], L^{4n+2}(\mathbb{R}^2))$ for $T < T^*$, but we have

$u \notin L_t^{\tilde{p}}([0,T], L^{4n+2}(\mathbb{R}^2))$, then $\lim_{t \to T^*}(\|v(t)\|_{H^1} + \|\partial_t v(t)\|_{L^2}) = \infty$ with v defined like in (5.11).

The proof of Lemma 5.12 follows along the lines of the proof of Lemma 5.9. We turn to the proof of Theorem 5.11.

Proof of Theorem 5.11. By Lemma 5.12, for the proof of Theorem 5.11 it suffices to show

$$E(v) + M(v) + 1 \lesssim_T 1.$$

We use again Grønwall's argument: We have like above

$$\partial_t M(v) \lesssim M(v)^{1/2} E(v)^{1/2},$$

and we compute with $u_n = \sum_{j=0}^{n-1} u^j$

$$\begin{aligned}
\partial_t E(v) &= (\partial_t v, \partial_t^2 v) + (\partial_t \nabla_x v, \nabla_x v) + (\partial_t v, |v|^2 v) \\
&= (\partial_t v, \partial_t^2 v - \Delta v + |v|^3) \\
&= (\partial_t v, -|v + u_n|^2 (v + u_n) + |v|^3 + \big|\sum_{j=0}^{n-2} u^j\big|^2 \sum_{j=0}^{n-2} u^j).
\end{aligned}$$

We can write schematically

$$\begin{aligned}
&- |v + u_n|^2 (v + u_n) + |v|^3 + \big|\sum_{j=0}^{n-2} u^j\big|^2 \sum_{j=0}^{n-2} u^j \\
&= A_{(2,1)}(v, u_n) + A_{(1,2)}(v, u_n) + \Big[\big|\sum_{j=0}^{n-2} u^j\big|^2 \sum_{j=0}^{n-2} u^j - \big|\sum_{j=0}^{n-1} u^j\big|^2 \sum_{j=0}^{n-1} u^j\Big]
\end{aligned}$$

with $A_{(i,j)}(f,g)$ denoting terms which are homogeneous of degree i in f and of degree j in g. We can estimate $\|u_n(t)\|_{L_x^\infty} \lesssim_t 1$:

$$|(\partial_t v, A_{(2,1)}(v, u_n))| \lesssim \|\partial_t v\|_{L^2} \|v\|_{L^4}^2 \|u_n\|_{L^\infty} \lesssim_T E(v),$$

and

$$|(\partial_t v, A_{(1,2)}(v, u_n))| \lesssim \|\partial_t v\|_{L^2} \|v\|_{L^2} \|u_n\|_{L^\infty}^2 \lesssim_T E(v)^{1/2} + M(v)^{1/2}.$$

At last, we rewrite

$$\big|\sum_{j=0}^{n-2} u^j\big|^2 \sum_{j=0}^{n-2} u^j - \big|\sum_{j=0}^{n-1} u^j\big|^2 \sum_{j=0}^{n-1} u^j = -\sum_{k,m} u^{n-1} u^k u^m$$

up to complex conjugates on the right-hand side. This allows to estimate by Hölder's inequality

$$|(\partial_t v, \sum_{k,m} u^{n-1} u^k u^m)| \lesssim \sum_{k,m} \|\partial_t v\|_{L^2} \|u^{n-1}\|_{L^{\frac{4n+2}{2n-1}}} \|u^k\|_{L^{4n+2}} \|u^m\|_{L^{4n+2}} \lesssim_T E(v)^{1/2}$$

noting that $u^k \in L^{4n+2}$ for any $k \geq 0$. This shows

$$\partial_t (E(v) + M(v) + 1) \lesssim C(T)(E(v) + M(v) + 1),$$

and the proof is complete. □

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