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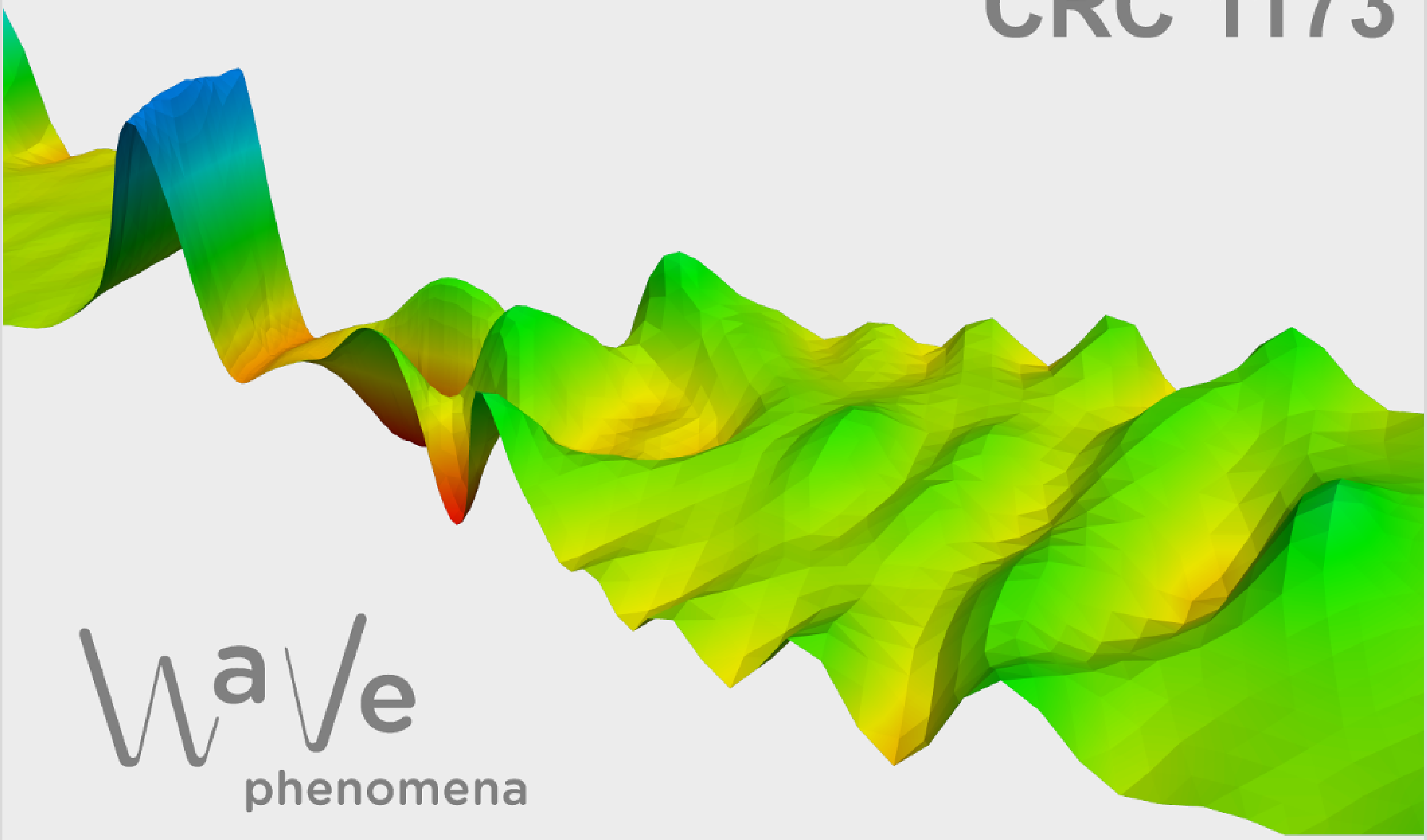

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OSCILLATORY INTEGRAL OPERATORS WITH HOMOGENEOUS PHASE FUNCTIONS

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Abstract. Oscillatory integral operators with 1-homogeneous phase functions satisfying a convexity condition are considered. For these we show the $L^p - L^p$-estimates for the Fourier extension operator of the cone due to Ou–Wang via polynomial partitioning. For this purpose, we combine the arguments of Ou–Wang with the analysis of Guth–Hickman–Iliopoulou, who previously showed sharp $L^p - L^p$-estimates for non-homogeneous phase functions with variable coefficients under a convexity assumption. The estimates are supplemented by examples exhibiting Kakeya compression. We apply the estimates to show new local smoothing estimates for wave equations on compact Riemannian manifolds (M,g) with $\dim M \geq 3$.

1. Introduction

We consider operators with $\lambda \geq 1$,

$$T^\lambda f(x) = \int e^{i\phi^\lambda(x;\omega)} a^\lambda(x;\omega) f(\omega) d\omega \tag{1}$$

and $a \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1}, \mathbb{R})$, $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1} \backslash 0; \mathbb{R})$, $\phi^\lambda(x;\omega) = \lambda \phi(x/\lambda;\omega)$, $a^\lambda(x;\omega) = a(x/\lambda;\omega)$. We suppose that ϕ is 1-homogeneous in ω, i.e.,

$$\phi(x;\mu\omega) = \mu\phi(x;\omega) \tag{2}$$

for $\mu > 0$. For the support of a we suppose that

$$\mathrm{supp}(a) \subseteq A^{n-1} = B_{n-1}(0,2) \backslash B_{n-1}(0,1/2).$$

We write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and impose the following conditions on ϕ in $\mathrm{supp}(a)$:

$C1)$ $\mathrm{rank}(\partial^2_{x\omega}\phi) = n-1$,

$C2^+)$ $\partial^2_{\omega\omega} \langle \partial_x \phi, G(x;\omega_0)\rangle\big|_{\omega=\omega_0}$ has $n-2$ non-vanishing eigenvalues of the same sign,

where G denotes the Gauss map

$$G_0(x;\omega) = \bigwedge_{j=1}^{n-1} \partial^2_{x\omega_j}\phi(x;\omega), \qquad G = G_0/|G_0| \tag{3}$$

of the embedded surface $\omega \mapsto \partial_x \phi(x;\omega)$. We identify $\bigwedge^{n-1} \mathbb{R}^n \simeq \mathbb{R}^n$.

The operators defined in (1) naturally extend the adjoint Fourier restriction operator for the cone

$$\mathcal{E} f(x) = \int_{A^{n-1}} e^{i(x'.\omega + x_n|\omega|)} f(\omega) d\omega. \tag{4}$$

In this note we prove new estimates

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim_{\varepsilon,\phi,a} \lambda^\varepsilon \|f\|_{L^p(A^{n-1})} \tag{5}$$

for operators (1) like described above. Firstly, we recall the conjectured range of L^p-estimates

$$\|\mathcal{E} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(A^{n-1})}. \tag{6}$$

is given by $p > \frac{2(n-1)}{n-2}$. This prominent open problem is known as *restriction conjecture for the cone* and goes back to Stein. The conjecture was solved for $n = 3$ by Barcelo [32], for $n = 4$ by Wolff [38] via bilinear estimates, and for $n = 5$ by Ou–Wang [25] via polynomial partitioning. Let

$$p_n = \begin{cases} 4, & n = 3, \\ 2 \cdot \frac{3n+1}{3n-3}, & n > 3 \text{ odd}, \\ 2 \cdot \frac{3n}{3n-4}, & n > 3 \text{ even}. \end{cases} \tag{7}$$

Ou–Wang showed (6) for $p > p_n$, which is also currently the widest range in higher dimensions to the best of the author's knowledge. Notably, in the case of Carleson-Sjölin phase functions (cf. [7, 18]), which are not 1-homogeneous anymore, where $C2^+$) is replaced with

$H2^+$) $\quad \partial^2_{\omega\omega} \langle \partial_x \phi(x;\omega), G(x;\omega_0)\rangle \big|_{\omega=\omega_0}$ has $n-1$ eigenvalues of the same sign,

Guth–Hickman–Iliopoulou [13] showed the sharp range of $L^p - L^p$ estimates, in the sense that there are phase functions for which the estimate fails for lower values of p. The deviation from the corresponding generalized restriction conjecture for the paraboloid occurs due to Kakeya compression. This was initially observed by Bourgain [4], see also Wisewell [36] and Bourgain–Guth [6]. Related phenomena were discussed by Minicozzi–Sogge [22] and Sogge [29]. In this note we point out Kakeya compression for 1-homogeneous phases with variable coefficients, which shows that the following L^p-estimates are sharp up to endpoints:

Theorem 1.1. *Let $\phi : \mathbb{R}^n \times \mathbb{R}^{n-1} \backslash 0 \to \mathbb{R}$ be a 1-homogeneous phase satisfying $C1$) and $C2^+$) and $a \in C^\infty_c(A^{n-1})$ be an amplitude. Then, we find the estimate (5) to hold for $p \geq p_n$ with p_n as in (7).*

We remark that for $p > p_n$ the λ^ε-factor can be dropped. Guth–Hickman–Iliopoulou showed the ε-removal lemma for oscillatory integral operators in [13, Section 12], albeit with a stronger non-degeneracy hypothesis than presently considered. The idea goes back to Tao [34, 35]. In Section 9 we prove the following global estimates for $p > p_n$ by a small variation of the argument in [13]:

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim_{\phi,a} \|f\|_{L^p(A^{n-1})}.$$

The proof of Theorem 1.1 combines ideas from the case of constant-coefficient homogeneous phases due to Ou–Wang [25] and Gao–Liu–Miao–Xi [9] and variable-coefficient non-homogeneous phases due to Guth–Hickman–Iliopoulou [13]. We digress for a moment to describe the tools we will use and put them into context. Bennett–Carbery–Tao [3] delivered an important contribution with sharp n-multilinear restriction estimates. We note that the multilinear estimates were shown as well for constant-coefficient phase functions as smooth perturbations thereof. Bourgain–Guth [6] devised an iteration to deduce linear estimates from multilinear estimates. Guth [11] observed that the full strength of k-multilinear estimates is not required, but a slightly weaker variant given by k-broad norms suffices to run the iteration. He used polynomial partitioning to improve on the previous results in [11, 12]. The idea is to equipartition the broad norm with polynomials of controlled degree: After wave packet decomposition, one finds that either the broad norm is concentrated on "cells" or on the "wall", which is a neighbourhood of a variety. To oversimplify matters for a moment, if the broad norm is concentrated on the cells, then sharp bounds follow from induction on scales. If the broad norm is concentrated along the wall, then we are morally dealing with a restriction problem in lower dimensions, which is amenable to another induction hypothesis.

We introduce the k-broad norms in the present context: For its definition decompose A^{n-1} into finitely overlapping sectors τ of aperture $\sim K^{-1}$ and length ~ 1, where K is a large constant. Given $f : A^{n-1} \to \mathbb{C}$, write $f = \sum f_\tau$, where f_τ is supported in τ. In view of the rescaling ϕ^λ of the phase, we define the rescaled Gauss map

$$G^\lambda(x;\omega) = G(\frac{x}{\lambda};\omega) \text{ for } (x;\omega) \in \text{supp } (a^\lambda).$$

For each $x \in B(0,\lambda)$

$$G^\lambda(x;\tau) = \{G^\lambda(x;\omega) \; : \; \omega \in \tau \text{ and } (x;\omega) \in \text{supp} a^\lambda\}.$$

For $V \subseteq \mathbb{R}^n$ a linear subspace, let $\angle(G^\lambda(x;\tau), V)$ denote the smallest angle between any non-zero vector $v \in V$ and $G^\lambda(x;\tau)$.

The spatial ball $B(0,\lambda)$ is decomposed into relatively small balls B_{K^2} of radius K^2. We fix $\mathcal{B}_{K^2}$ a collection of finitely-overlapping K^2-balls, which are centred in and cover $B(0,\lambda)$. For $B_{K^2} \in \mathcal{B}_{K^2}$ centred at $\bar{x} \in B(0,\lambda)$, define

$$\mu_{T^\lambda f}(B_{K^2}) = \min_{V_1,\dots,V_A \in Gr(k-1,n)} \big(\max_{\tau : \angle(G^\lambda(\bar{x};\tau),V_a) > K^{-2} \; \forall a} \|T^\lambda f_\tau\|^p_{L^p(B_{K^2})} \big), \tag{8}$$

where $Gr(k-1,n)$ denotes the Grassmannian manifold of $(k-1)$-dimensional subspaces in $\mathbb{R}^n$. We stress the deviation from [13], in which the angle threshold K^{-1} was considered. In case of the Fourier extension operator associated with the cone, we have to strengthen the angle condition to K^{-2} to further confine the narrow part.

We write $\tau \notin V_a$ as shorthand for $\angle(G^\lambda(\bar{x};\tau), V_a) > K^{-2}$ provided that $\bar{x}$ is clear from context. Thus, we can write as well

$$\mu_{T^\lambda f}(B_{K^2}) = \min_{V_1,\dots,V_A \in Gr(k-1,n)} \big(\max_{\substack{\tau : \tau \notin V_a, \\ \text{for } 1 \le a \le A}} \|T^\lambda f_\tau\|^p_{L^p(B_{K^2})} \big).$$

For $U \subseteq \mathbb{R}^n$ the k-broad norm is defined as

$$\|T^\lambda f\|_{BL^p_{k,A}}(U) = \big(\sum_{\substack{B_{K^2} \in \mathcal{B}_{K^2}, \\ B_{K^2} \cap U \neq \emptyset}} \mu_{T^\lambda f}(B_{K^2}) \big)^{1/p}.$$

A key step in the proof of the L^p-L^p-estimate is the proof of k-broad estimates:

Theorem 1.2. *For $2 \le k \le n$ and all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 1$ and an integer A such that, whenever T^λ is an oscillatory integral operator with reduced 1-homogeneous phase satisfying $C1)$ and $C2^+)$, the estimate*

$$\|T^\lambda f\|_{BL^p_{k,A}(\mathbb{R}^n)} \lesssim_\varepsilon K^{C_\varepsilon} \lambda^\varepsilon \|f\|_{L^2(A^{n-1})} \tag{9}$$

holds for all $\lambda \ge 1$ and $K \ge 1$ whenever

$$p \ge \bar{p}(k,n) = \frac{2(n+k)}{n+k-2}. \tag{10}$$

Reduced phase functions are introduced in Section 3.1. These phases are basically small C^N perturbations of 1-homogeneous phases with constant coefficients. These reduced phase functions were previously used by Beltran–Hickman–Sogge [1] to derive decoupling estimates. As in [1], general phase functions satisfying $C1$) and $C2^+$) are transformed by partitioning the support of the amplitude and Lorentz rescaling to reduced phases.

The arguments to deduce Theorem 1.1 from Theorem 1.2 are essentially due to Bourgain–Guth and we give a sketch in the following: As mentioned previously, it is enough to work with reduced phases and amplitudes.

Firstly, we write

$$\|T^\lambda f\|^p_{L^p(B(0,\lambda))} \lesssim \sum_{B_{K^2} \in \mathcal{B}_{K^2}} \|T^\lambda f\|^p_{L^p(B_{K^2})}.$$

Fixing one K^2-ball, there is a collection of $(k-1)$-dimensional subspaces $V_1, \dots, V_A$ such that

$$\mu_{T^\lambda f}(B_{K^2}) = \max_{\substack{\tau \notin V_a \\ \text{for } 1 \leq a \leq A}} \|T^\lambda f_\tau\|^p_{L^p(B_{K^2})}.$$

Note that there are only $K^{O(1)}$ K^{-1}-sectors. Writing $T^\lambda f = \sum_\tau T^\lambda f_\tau$, sectors $\tau \notin V_a$ for $1 \leq a \leq A$ are estimated by $\mu_{T^\lambda f}(B_{K^2})$. The remaining sectors are isolated. This yields

$$\|T^\lambda f\|^p_{L^p(B_{K^2})} \lesssim_A K^{O(1)} \mu_{T^\lambda f}(B_{K^2}) + \sum_{a=1}^{A} \| \sum_{\tau \in V_a} T^\lambda f_\tau \|^p_{L^p(B_{K^2})}. \tag{11}$$

The first term is captured by the broad estimate; the second term is estimated by ℓ^p-decoupling (cf. [5, 1]) and induction on scales [25, 9].

Very recently, Gao–Liu–Miao–Xi [9] proved an extension of Ou–Wang's result for the circular cone $\phi(x,\omega) = x'.\omega + x_n|\omega|$ for more general conic surfaces, but still with constant coefficients. For these constant coefficient phase functions, the Kakeya compression described in the present work cannot happen. Gao *et al.* [9] used k-broad estimates to derive new local smoothing estimates for the wave equation in Euclidean space. At small spatial scales, the variable coefficient phases are approximated with extension operators for conic surfaces. Then we can use arguments from [9]. Furthermore, in [17] Hickman and Iliopoulou showed sharp L^p-estimates for non-homogeneous phases with indefinite signature. This suggests to study also homogeneous phase functions with indefinite signature with the methods of the paper.

Notably, we do not use the usual wave packet decomposition for the cone as e.g. in [25] or [9] to prove the broad estimate. Instead, we stick to the wave packet decomposition commonly used for the Fourier extension operator of the paraboloid or its variable coefficient counterpart [13]. This allows to use many arguments from [13] without change and hints at the possibility of a unified approach. A major change happens for the transverse equidistribution estimates, to be analyzed in Section 5. Secondly, the narrow decoupling requires additional considerations, see Section 8. Since the polynomial partitioning approach is involved, we elected to elaborate on the argument in Sections 3 - 8.

We remark that the idea to use the same wave packet decomposition for homogeneous and inhomogeneous phase functions in the variable coefficient context is not new: In [20] Lee considered linear and bilinear estimates for oscillatory integral operators and could treat variable coefficient versions of the Fourier extension operator of the paraboloid and the cone with the same wave packet decomposition. He generalized bilinear estimates due to Tao [33] and Wolff [38] to variable coefficient phases. Notably, in [20] was pointed out for the first time that a convexity condition as $H2^+$) or $C2^+$) allows to go beyond Tomas–Stein $L^2 - L^p$-estimates, which are sharp for phases without convexity condition. Bourgain [4] showed in the context of non-homogeneous phases without convexity conditions that the Tomas-Stein range is sharp (see also [6]). In the present work, the L^p-L^p-estimates for general oscillatory integral operators with phase satisfying $C1$) and $C2^+$) due to Lee [20] are improved to the sharp range up to the endpoint for $n \geq 5$.

In Section 10 we apply the new estimates for oscillatory integral operators to prove new local smoothing estimates for solutions to wave equations on compact Riemannian manifolds with $\dim(M) \geq 3$. Local parametrices are given by

$$\mathcal{F}f(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{i\phi(x', x_n; \omega)} a(x; \omega) \hat{f}(\omega) d\omega$$

with $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1} \backslash 0)$ a phase function satisfying $C1$) and $C2^+$) and $a \in S^0(\mathbb{R}^{2d+1})$ with compact support in x. Hence, it suffices to prove local smoothing estimates of rescaled Fourier integral operators $\mathcal{F}^\lambda$. In Theorem 10.1 and Corollary 10.2 we extend the recent results due to Gao *et al.* [9] for the Euclidean wave equation to wave equations on compact Riemannian manifolds. This improves on the previously best local smoothing estimates due to Beltran–Hickman–Sogge [1] for wave equations on compact manifolds.

Outline of the paper: In Section 2 we show the necessary conditions for L^p-estimates for variable-coefficient 1-homogeneous phases. Preliminaries for the polynomial partitioning argument to show Theorem 1.2 are given in Section 3. In this section we introduce the notion of a reduced homogeneous phase function and collect geometric consequences. This will simplify the proof of Theorem 1.2. We recall the wave packet analysis in the context of variable coefficients [20, 13] and collect facts on the k-broad norms. In Section 4 we recall the polynomial partitioning tools. In Section 5 transverse equidistribution estimates are proved. These differ from the transverse equidistribution estimates shown in [13] for Carleson–Sjölin phase functions. In Section 6 we compare wave packet decompositions at different scales, which is necessary to run the induction on scales in Section 7. In this section we deduce Theorem 1.2 from Theorem 7.1, which is suitable for induction on dimension and radius. In Section 8 we show how Theorem 1.2 implies Theorem 1.1. In Section 9 we show how the λ^ε-factor can be removed away from the endpoint.In Section 10 we apply the oscillatory integral estimates to show new local smoothing estimates for solutions to wave equations on compact Riemannian manifolds.

2. Kakeya compression

In the following we modify the example due to Guth–Hickman–Iliopoulou [13, Section 2] (see also [6]) for homogeneous phase functions. This yields the necessary conditions:

Proposition 2.1. *Necessary for the estimate (5) to hold for $n \geq 5$ is*

$$p \geq \begin{cases} 2 \cdot \frac{3n}{3n-4}, & n \text{ even}, \\ 2 \cdot \frac{3n+1}{3n-3}, & n \text{ odd}. \end{cases}$$

We only consider $n \geq 5$ because the bilinear estimates due to Wolff [38] and Lee [20] solve the cone restriction conjecture for $3 \leq n \leq 4$. Let

$$x = (\underbrace{x'', x_{n-1}}_{x'}, x_n) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R} \text{ and } \omega = (\omega', \omega_{n-1}) \in \mathbb{R}^{n-2} \times \mathbb{R}.$$

We consider the phase functions

$$\phi(x;\omega) = x'.\omega + \frac{\langle A(x_n)\omega', \omega' \rangle}{2\omega_{n-1}}, \quad \omega_{n-1} \in (1/2, 1). \tag{12}$$

$A(x_n)$ denotes the $(n-2) \times (n-2)$–positive definite matrix

$$A(x_n) = \begin{cases} \bigoplus_{i=1}^{\frac{n-2}{2}} \begin{pmatrix} x_n & x_n^2 \\ x_n^2 & x_n + x_n^3 \end{pmatrix}, & n-2 \text{ even}, \\ \bigoplus_{i=1}^{\frac{n-3}{2}} \begin{pmatrix} x_n & x_n^2 \\ x_n^2 & x_n + x_n^3 \end{pmatrix} \oplus (x_n), & n-2 \text{ odd}. \end{cases}$$

The main idea is to construct many wave packets which are concentrated in the neighbourhood of a lower dimensional algebraic variety. Whereas the direction governed by the frequency ω_θ below varies, for fixed ω_θ we consider precisely one starting position v_θ. This concentration in a low dimensional algebraic variety does not happen in the linear case (4).

We consider wave packets adapted to ϕ as follows: $\Xi = (B^{n-2}(0,c) \times (1/2, 1)$ is covered by essentially disjoint elongated caps

$$\Xi_\theta = \{(\omega', \omega_{n-1}) \in \Xi : |\omega'/\omega_{n-1} - \omega_\theta| \lesssim \lambda^{-\frac{1}{2}}\}$$

with $\omega_\theta \in B^{n-2}(0,c)$ for $|c| \ll 1$. Apparently, Ξ can be covered by $\sim \lambda^{\frac{n-2}{2}}$ finitely overlapping sets Ξ_θ. We consider a corresponding smooth partition of unity $(\psi_\theta)_{\omega_\theta \in \Xi}$ and wave packets

$$f_{\theta,v}(\omega) = e^{-i\lambda \langle v, \omega' \rangle} \psi_\theta(\omega), \quad v = (v_1, \dots, v_{n-2}) \in \mathbb{R}^{n-2}.$$

We have by non-oscillation of the phase

$$|T^\lambda f_{\theta,v}(x'', x_{n-1}, x_n)| \gtrsim \lambda^{-\frac{n-2}{2}} \chi_{T_{\theta,v}}(x).$$

$\chi_{T_{\theta,v}}$ denotes the characteristic function of $T_{\theta,v}$. The $T_{\theta,v}$ are curved slabs of size $(1 \times \underbrace{\lambda^{1/2} \times \dots \times \lambda^{1/2}}_{n-2 \text{ times}} \times \lambda)$ with

$$T_{\theta,v} \subseteq \{x \in B(0,\lambda) : |x'' - \lambda \gamma_{\theta,v}\left(\frac{x_n}{\lambda}\right)| < c\lambda^{\frac{1}{2}+\varepsilon} \text{ and } |x_{n-1} - \lambda \gamma'_\theta(x_n/\lambda)| < c\lambda^{\frac{1}{2}+\varepsilon}\},$$

for any $\varepsilon > 0$, which follows from non-stationary phase; c denotes a small constant and $\gamma_{\theta,v}$, γ'_θ denote curves:

$$\gamma_{\theta,v}(x_n) = v - A(x_n)\omega_\theta, \quad \gamma'_\theta(x_n) = \frac{1}{2}\langle A(x_n)\omega_\theta, \omega_\theta \rangle.$$

Furthermore, note that the condition

$$\left|\frac{\omega'}{\omega_{n-1}} - \omega_\theta\right| \lesssim \lambda^{-\frac{1}{2}}, \ \omega_{n-1} \in (1,2), \ \omega' \in B(\omega, c\lambda^{-\frac{1}{2}})$$

corresponds to considering $\lambda^{-\frac{1}{2}}$-sectors into direction $(\omega_\theta, 1)$. The degeneracy of $\partial^2 \phi$ into radial direction gives the localization of tubes to size λ^ε into this direction:

We have

$$\partial_\omega \phi(x;\omega).\frac{(\omega_\theta,1)}{|(\omega_\theta,1)|} = \partial_\omega \phi(x;(\omega_\theta,1)).\frac{(\omega_\theta,1)}{|(\omega_\theta,1)|} + O(\lambda^{-1}) \text{ for } \Big|\frac{\omega}{|\omega|} - \frac{(\omega_\theta,1)}{|(\omega_\theta,1)|}\Big| \lesssim \lambda^{-\frac{1}{2}}.$$

The non-degeneracy of $\partial^2 \phi$ gives localization to size $\lambda^{1/2+\varepsilon}$ into the remaining directions. We argue in the following why the curved tubes $\chi_{T_{\theta,v}}$ are in fact of size $1 \times \lambda^{\frac{1}{2}} \times \ldots \times \lambda^{\frac{1}{2}} \times \lambda$: Consider the oscillatory integral

$$F(x') = \int e^{i(x'.\omega + \lambda \tilde{\phi}(x_n/\lambda,\omega))} \psi_\theta(\omega) d\omega$$

with $\psi_\theta \in C^\infty_c(A^{n-1})$ localizing to a slab into direction $\theta \in \mathbb{S}^{n-2}$ and

$$\tilde{\phi}(x_n,\mu\omega) = \mu \tilde{\phi}(x_n,\omega) \text{ for } \mu > 0.$$

We use Taylor expansion in ω to write

$$\begin{aligned} \lambda \tilde{\phi}(x_n/\lambda,\omega) &= |\omega| (\lambda \tilde{\phi}(x_n/\lambda, \omega/|\omega|)) \\ &= |\omega| (\lambda \tilde{\phi}(x_n/\lambda,\theta) + \lambda \nabla_\omega \tilde{\phi}(x_n/\lambda,\theta)(\frac{\omega}{|\omega|} - \theta) + O(c)). \end{aligned}$$

For $\omega \in \operatorname{supp}(\psi_\theta)$ we have

$$|\omega| = \omega.\theta + O(c\lambda^{-1}).$$

Hence, we can write

$$\lambda \tilde{\phi}(x_n/\lambda,\omega) = \lambda \tilde{\phi}(x_n/\lambda,\theta)(\omega.\theta) + \lambda \nabla_\omega \tilde{\phi}(x_n/\lambda,\theta)(\omega - (\omega.\theta)\theta) + O(c).$$

Let $\{\theta^1_\perp, \ldots, \theta^{n-2}_\perp, \theta\}$ be an orthonormal basis of $\mathbb{R}^{n-1}$. Then,

$$\lambda \tilde{\phi}(x_n/\lambda,\omega) = \lambda \tilde{\phi}(x_n/\lambda,\theta)(\omega.\theta) + \lambda \sum_{i=1}^{n-2} (\nabla_\omega \tilde{\phi}(x_n/\lambda,\theta).\theta^i_\perp)(\omega.\theta^i_\perp) + O(c).$$

Consequently,

$$\begin{aligned} x'.\omega + \lambda \tilde{\phi}(x_n/\lambda,\omega) &= (x'.\theta + \lambda \tilde{\phi}(x_n/\lambda,\theta))(\omega.\theta) \\ &\quad + \sum_{i=1}^{n-2} (x'.\theta^i_\perp + \lambda \nabla_\omega \tilde{\phi}(x_n/\lambda,\theta).\theta^i_\perp)(\omega.\theta^i_\perp) + O(c). \end{aligned}$$

And for $|x'.\theta + \lambda \tilde{\phi}(x_n/\lambda,\theta)| \ll c$ and $|x'.\theta^i_\perp + \lambda \nabla_\omega \tilde{\phi}(x_n/\lambda,\theta).\theta^i_\perp| \ll c\lambda^{1/2}$, we see that the whole phase is $O(c)$. Hence, there is no oscillation within $\operatorname{supp}(\psi_\theta)$ and for fixed x_n this defines a region A_{x_n} for x' of size $1 \times \lambda^{1/2} \times \ldots \times \lambda^{1/2}$. Taking $T_\theta = \bigcup_{x_n} A_{x_n}$ yields the $1 \times \lambda^{1/2} \times \ldots \times \lambda^{1/2} \times \lambda$-tube. Note that the factor $e^{-i\lambda \langle v,\omega' \rangle}$ amounts to a shift in x' by λv, but does not change the size of the tube.

We prepare the initial data with randomized signs:

$$f = \sum_\theta \varepsilon_\theta f_{\theta,v}.$$

By Khintchine's theorem, the expected value of $|T^\lambda f(x)|$ is given by the square sum:

$$\mathbf{E}[|T^\lambda f(x)|] \sim \big(\sum_\theta |T^\lambda f_{\theta,v_\theta}|^2 \big)^{1/2} \gtrsim \lambda^{-\frac{n-2}{2}} \big(\sum_\theta \chi_{T_{\theta,v_\theta}}(x) \big)^{1/2}.$$

Taking L^p-norms yields by Minkowski's inequality

$$\lambda^{-\frac{n-2}{2}} \big(\int \big(\sum_\theta \chi_{T_{\theta,v_\theta}} \big)^{\frac{p}{2}} \big)^{\frac{1}{p}} \lesssim \mathbf{E}[\| T^\lambda f \|_{L^p}].$$

Next, we find by applying Hölder's inequality

$$\lambda^{-\frac{n-2}{2}} \Big(\int \sum_\theta \chi_{T_{\theta,v_\theta}} \Big)^{1/2} \lesssim \Big| \bigcup_\theta T_{\theta,v_\theta} \Big|^{1/2-1/p} \mathbf{E}[\|T^\lambda f\|_{L^p}]$$
$$\lesssim \Big| \bigcup_\theta T_{\theta,v_\theta} \Big|^{1/2-1/p} \|f\|_p \lesssim \Big| \bigcup_\theta T_{\theta,v_\theta} \Big|^{1/2-1/p}.$$

The penultimate estimate is by hypothesis, and the final estimate follows from $|f| = 1$ and $|\text{supp } f| \sim 1$. Since the tubes T_{θ,v_θ} are $(1 \times \lambda^{1/2} \times \ldots \times \lambda^{1/2} \times \lambda)$-slabs, $\int \chi_{T_{\theta,v_\theta}} \sim \lambda^{\frac{n}{2}}$. Moreover, there are about $\lambda^{\frac{n-2}{2}}$ slabs. Hence,

$$\lambda^{-\frac{n-2}{2}} \Big(\int \sum \chi_{T_{\theta,v_\theta}} \Big)^{1/2} \sim \lambda^{\frac{1}{2}}.$$

Thus, we arrive at

$$1 \sim \|f\|_{L^p(B^{n-1})} \lesssim \Big| \bigcup_\theta T_{\theta,v_\theta} \Big|^{1/2-1/p} \lambda^{-\frac{1}{2}} \|f\|_{L^p(B^{n-1})}. \tag{13}$$

Next, we shall see how to choose v_θ such that the curved slabs are concentrated in a neighbourhood of a low-dimensional algebraic variety inspired by [13].
For Ξ_θ, we set

$$v_{\theta,2j-1} = -(\omega_\theta)_{2j} \text{ and } v_{\theta,2j} = v_{\theta,n-2} = 0 \text{ for } 1 \leq j \leq \lfloor \frac{n-2}{2} \rfloor. \tag{14}$$

Let $d = n - 1 - \lfloor \frac{n-2}{2} \rfloor$ and $Z = Z(P_1, \ldots, P_{n-1-d})$ be the common zero set of the polynomials

$$P_j(x_1, \ldots, x_{n-2}, x_n) = \lambda x_{2j} - x_{2j-1} x_n \text{ for } 1 \leq j \leq \lfloor \frac{n-2}{2} \rfloor. \tag{15}$$

It is straight-forward to show that $x_n \mapsto (\lambda \gamma_{\theta,v_\theta}(x_n/\lambda), x_n)$ is contained in $Z(P_1, \ldots, P_{n-1-d})$. Z is an algebraic variety of dimension

$$d = (n-1) - \lfloor \frac{n-2}{2} \rfloor \tag{16}$$

in $\mathbb{R}^{n-1}$ and of degree $O_n(1)$. Thus, Wongkew's theorem (cf. Theorem 5.8) on the size of neighbourhoods of algebraic varieties applies, and we find

$$|N_{\lambda^{1/2}}(Z) \cap B^{n-1}(0,\lambda)| \lesssim \lambda^{d + \frac{n-1-d}{2}}. \tag{17}$$

We find by (16) and (17)

$$|N_{\lambda^{\frac{1}{2}}}(Z)| \lesssim \begin{cases} \lambda^{\frac{3n-2}{4}}, & n \text{ even}, \\ \lambda^{\frac{3n-1}{4}}, & n \text{ odd}. \end{cases} \tag{18}$$

Moreover, for $(x_1, \ldots, x_n) \in T_{\theta,v_\theta}$ we have $x_{n-1} \in B(\lambda \gamma'_\theta(x_n/\lambda), \lambda^{\frac{1}{2}+\varepsilon})$.
This yields

$$\Big| \bigcup_\theta T_{\theta,v_\theta} \Big|^{1/2-1/p} \lesssim |N_{\lambda^{1/2}}(Z) \cdot \lambda^{\frac{1}{2}}|^{1/2-1/p}. \tag{19}$$

Plugging (19) into (13) with the estimate from (18), we find

$$p \geq \begin{cases} 2 \cdot \frac{3n}{3n-4}, & n \text{ even}, \\ 2 \cdot \frac{3n+1}{3n-3}, & n \text{ odd}. \end{cases}$$

This finishes the proof of Proposition 2.1. □

3. Preliminaries

3.1. **Basic reductions of the phase function.** In this paragraph we shall reduce 1-homogeneous phase functions satisfying the above assumptions to a form, which highlights that the class of considered phase functions are indeed smooth perturbations of the translation-invariant case

$$\phi_*(x;\omega) = \langle x', \omega \rangle + \frac{t(\omega')^2}{\omega_{n-1}}, \quad \omega' \in B(0,c),\ \omega_{n-1} \in (1,2).$$

Constant-coefficient perturbations were analyzed in [9].

The arguments were provided in [1, Section 2] and details are omitted here (see also [20]). After localisation and translation, we may assume that a is supported inside $X \times \Xi$, where $X = X' \times T$ for $X' \subseteq B(0,1) \subseteq \mathbb{R}^{n-1}$ and $T \subseteq (-1,1) \subseteq \mathbb{R}$ are small open neighbourhoods of the origin and $\Xi \subseteq A^{n-1}$ is a small sector around $e_{n-1} = (0,\dots,0,1) \in \mathbb{R}^{n-1}$.
Firstly, we can suppose that

$C1'$) $\det \partial^2_{\omega x'} \phi(x;\omega) \neq 0$ for all $(x,\omega) \in X \times \Xi$,

$C2'$) $\partial^2_{\omega'\omega'} \partial_{x_n} \phi(x,\omega)$ has eigenvalues of the same sign for all $(x,\omega) \in X \times \Xi$.

This follows as in [1]. By rotation in the x-variables, we can also suppose that

$$G(0;e_{n-1}) = e_n \text{ and } \partial^2_{x_n\omega}\phi(0;e_{n-1}) = 0.$$

Hence, by making Ξ small enough, we find that

$$|\partial^2_{x_n\omega}\phi(x;\omega)| \leq c_{par} \text{ for } (x,\omega) \in X \times \Xi. \tag{20}$$

By non-degeneracy $C1'$), we find a smooth locally inverse mapping $\Phi_{x_n,\omega} : X' \to \mathbb{R}^{n-1}$ such that

$$\partial_\omega \phi(\Phi_{x_n,\omega}(x'), x_n; \omega) = x'.$$

We shall also write $\Phi_{x_n,\omega}(x') = \Phi(x', x_n; \omega)$. There is also a smooth mapping $\Psi(x,\cdot)$ with

$$\partial_{x'}\phi(x;\Psi(x;\omega)) = \omega.$$

For $\lambda \geq 1$, we consider the rescaled versions $\Phi^\lambda(x;\omega) = \lambda \Phi(x/\lambda;\omega)$ and $\Psi^\lambda(x;\omega) = \Psi(x/\lambda;\omega)$. We assume that X and Ξ are such that the above mappings are defined on the whole support of a.

In the following we shall quantify the deviation from ϕ_* further, by restricting the values of second and third derivatives and bounding higher derivatives: Let $c_{par} > 0$ denote a small constant. Firstly note that there are (possibly large) constants $A_1, A_2, A_3 \geq 1$ such that

$C1''$) $|\partial^2_{\omega x'}\phi(x;\omega) - I_{n-1}| \leq c_{par} A_1$ for $(x;\omega) \in X \times \Xi$,

$C2''$) $|\partial^2_{\omega'\omega'}\partial_{x_n}\phi(x;\omega) - \dfrac{I_{n-1}}{\omega_{n-1}}| \leq c_{par} A_2$ for $(x;\omega) \in X \times \Xi$.

For the higher derivatives, we suppose that

$D1)$ $\|\partial_\omega^\beta \partial_{x_k}\phi\|_{L^\infty(Z\times\Xi)} \leq c_{par}A_1$ for $1 \leq k \leq n-1$ and $\beta \in \mathbb{N}_0^{n-1}$ with $2 \leq |\beta| \leq 3$ such that $|\beta'| \geq 2$;

$$\|\partial_{\omega'}^{\beta'}\partial_{x_n}\phi\|_{L^\infty(Z\times\Xi)} \leq \frac{c_{par}}{2n}A_1 \text{ for all } \beta' \in \mathbb{N}_0^{n-2} \text{ with } |\beta'| = 3,$$

$D2)$ For some large integer $N \in \mathbb{N}$, one has

$$\|\partial_\omega^\beta \partial_x^\alpha \phi\|_{L^\infty(X\times\Xi)} \leq \frac{c_{par}}{2n}A_3 \text{ for all } (\alpha,\beta) \in \mathbb{N}_0^n \times \mathbb{N}_0^{n-1} \text{ with } 2 \leq |\alpha| \leq 4N$$

and $1 \leq |\beta| \leq 2N+2$ satisfying $1 \leq |\beta| \leq 2N$ or $|\beta'| \geq 2$.

A phase ϕ satisfying the above four conditions with constants A_1, A_2, A_3 is said to be of type (A_1, A_2, A_3). By parabolic rescaling (cf. Lemma 8.7), we see that we can reduce to initial data with $A_i = 1$; these phases are said to be reduced.

3.2. **Geometric consequences.** Let ϕ be a reduced phase function in the above sense. We shall see how the corresponding hypersurfaces Σ_x parametrized by $\omega \mapsto \partial_x\phi(x;\omega)$ resemble the ones from ϕ_*. To see this, recall that $\Psi : U \to \Xi$ satisfies $\partial_{x'}\phi(x;\Psi(x;\omega)) = \omega$. Hence, Σ_x is the graph of the function $h_x(\omega) = \partial_{x_n}\phi(x;\Psi(x;\omega))$ over the fibre U_x.
Each h_x is a perturbation of the translation invariant case in the following sense:

Lemma 3.1. *The following estimate holds true for all $\omega \in U_x$:*

$$\|\partial^2_{\omega'\omega'}h_x(\omega) - I_{n-1}/\omega_{n-1}\|_{L^\infty} \lesssim c_{par}. \tag{21}$$

Here $c_{par} > 0$ denotes the constant from the definition of a reduced phase function.

Proof. This is a consequence of properties of Ψ. Firstly, we record that $\Psi(x;e_{n-1}) = 1$. By the implicit function theorem and non-degeneracy of ϕ, we find

$$\partial_\omega\Psi(x;\omega) = \partial^2_{x'\omega}\phi(x;\Psi(x;\omega))^{-1}.$$

Hence,

$$\|\partial_\omega\Psi(x;\omega) - I_{n-1}\|_{L^\infty} = O(c_{par}).$$

As a consequence of this identity (and choosing c_{par} to be sufficiently small),

$$|\Psi(x;\omega) - \Psi(x;\omega')| \sim |\omega - \omega'| \text{ for all } \omega,\omega' \in U_x$$

with implicit constant only depending on n.
Additionally, if $1 \leq k \leq n-1$, then by twice differentiating the identity

$$\partial_{x_k}\phi(x;\Psi(x;\omega)) = \omega_k$$

in the ω-variables, it follows that

$$\|\partial^2_{\omega\omega}\Psi_k(x;\omega)\|_{L^\infty} = O(c_{par}).$$

By the previous estimate, (21) follows from $C2'')$. ☐

By similar means, we infer estimates for the generalized Gauss map associated with T^λ. To give the results, let

$$X^\lambda = \{x \in \mathbb{R}^n \mid \frac{x}{\lambda} \in X\}$$

denote the λ-dilate of X, so that a^λ is supported in $X^\lambda \times \Xi$.

Lemma 3.2. *For all $x, \bar{x} \in X^\lambda$ and $\omega, \bar{\omega} \in \Xi$, the estimates*

$$\begin{aligned} &\angle(G^\lambda(x;\omega), G^\lambda(x;\bar{\omega})) \sim |\frac{\omega}{|\omega|} - \frac{\bar{\omega}}{|\bar{\omega}|}| \sim \angle(\omega,\bar{\omega}), \\ &\angle(G^\lambda(x;\omega), G^\lambda(\bar{x};\omega)) \lesssim \lambda^{-1}|x-\bar{x}| \end{aligned} \tag{22}$$

hold true.

This will be helpful to understand the wave packet analysis in the following sections.

3.3. **Wave packet decomposition.** We carry out the wave packet decomposition with respect to some spatial parameter $1 \ll R \ll \lambda$. For this purpose, we follow [13] and use that the construction only depends on the non-degeneracy condition $C1$). We do not use the usual wave packet decomposition for the cone as e.g. in [25], but adapt the parabolic case, as previously done by Lee [20]. The reason is that in Section 6 we would sort the smaller cone tubes into larger tubes anyway. It appears that the present choice of wave packet decomposition allows to transfer arguments from [13] to the homogeneous setting more directly. In the following we introduce notations from [13].

Cover A^{n-1} by finitely overlapping balls of radius $R^{-1/2}$, and let ψ_θ be a smooth partition of unity adapted to this cover. These θ will frequently be referred to as $R^{-1/2}$-balls. For a ball θ, cover $\mathbb{R}^{n-1}$ with finitely overlapping balls of size $R^{\frac{1+\delta}{2}} \times \ldots \times R^{\frac{1+\delta}{2}}$ with center $v \in R^{\frac{1+\delta}{2}}\mathbb{Z}^{n-1}$. Let $\eta_v = \eta(\cdot - v)$ denote a bump function adapted to $B(v, R^{\frac{1+\delta}{2}})$ such that

$$\sum_{v \in \mathbb{Z}^{n-1}} \eta_v = 1$$

with $\hat{\eta}_v$ essentially supported in $B(0, CR^{-\frac{1+\delta}{2}})$. This is possible by the Poisson summation formula.

Let $\mathbb{T}$ denote the collection of all pairs (θ, v). Then, for $f : \mathbb{R}^{n-1} \to \mathbb{C}$ with support in A^{n-1} and sufficiently regular, we find

$$f = \sum_{(\theta,v)\in\mathbb{T}} (\eta_v(\psi_\theta f)\check{\ })\hat{\ } = \sum_{(\theta,v)\in\mathbb{T}} \hat{\eta}_v * (\psi_\theta f).$$

For each $R^{-1/2}$-ball θ, let ω_θ denote its centre. Choose a real-valued smooth function $\tilde{\psi}$ so that $\tilde{\psi}_\theta$ is supported in θ, and $\tilde{\psi}_\theta(\omega) = 1$ whenever ω belongs to a $cR^{-1/2}$-neighbourhood of the support of ψ_θ for some small $c > 0$. Finally, define

$$f_{\theta,v} = \tilde{\psi}_\theta \cdot [\hat{\eta}_v * (\psi_\theta f)].$$

The function $\hat{\eta}_v$ is rapidly decaying outside $B(0, CR^{-\frac{1+\delta}{2}})$ and, consequently,

$$\|f_{\theta,v} - (\hat{\eta}_v * (\psi_\theta f))\|_{L^\infty(\mathbb{R}^{n-1})} \leq \mathrm{RapDec}(R)\|f\|_{L^2(A^{n-1})}.$$

The functions $f_{\theta,v}$ are almost orthogonal: if $\mathbb{S} \subseteq \mathbb{T}$, then

$$\|\sum_{(\theta,v)\in\mathbb{S}} f_{\theta,v}\|^2_{L^2(\mathbb{R}^{n-1})} \sim \sum_{(\theta,v)\in\mathbb{S}} \|f_{\theta,v}\|^2_{L^2(\mathbb{R}^{n-1})}.$$

Let T^λ be an oscillatory integral operator with reduced phase ϕ satisfying $C1'$) and amplitude a supported in $X \times \Xi$. For $(\theta, v) \in \mathbb{T}$ define the curve $\gamma^1_{\theta,v} : I^1_{\theta,v} \to \mathbb{R}^{n-1}$ by setting $\gamma^1_{\theta,v}(t) = \Phi(v, t; \omega_\theta)$, where Φ is the function introduced above and

$$I^1_{\theta,v} = \{x_n \in T \,|\, \partial_\omega \phi(x', x_n; \omega_\theta) = v \text{ for some } x' \in X'\}.$$

Hence, $\partial_\omega \phi(\gamma^1_{\theta,v}(x_n), x_n; \omega_\theta) = v$ for all $x_n \in I^1_{\theta,v}$. For the rescaled curve

$$\gamma^\lambda_{\theta,v}(t) = \lambda \gamma^1_{\theta,v/\lambda}(t/\lambda),$$

we find

$$\partial_\omega \phi^\lambda(\gamma^\lambda_{\theta,v}(x_n), x_n; \omega_\theta) = v \text{ for all } t \in I^\lambda_{\theta,v} = \{t \in \mathbb{R} : \frac{t}{\lambda} \in I^1_{\theta,v}\}.$$

Let $\Gamma^\lambda_{\theta,v} : I^\lambda_{\theta,v} \to \mathbb{R}^n$ denote the graph mapping $\Gamma^\lambda_{\theta,v}(x_n) = (\gamma^\lambda_{\theta,v}(x_n), x_n)$; for the sake of brevity, the image of this mapping is denoted by $\Gamma^\lambda_{\theta,v}$, too.

Lemma 3.3 ([13, Lemma 5.2]). *The tangent space $T_{\Gamma^\lambda_{\theta,v}(x_n)} \Gamma^\lambda_{\theta,v}$ lies in the direction of the unit vector $G^\lambda(\Gamma^\lambda_{\theta,v}(x_n); \omega_\theta)$ for all $x_n \in I^\lambda_{\theta,v}$.*

We consider curved tubes

$$T_{\theta,v} = \{(x', x_n) \in B(0,R) : x_n \in I^\lambda_{\theta,v} \text{ and } x' \in B(\gamma^\lambda_{\theta,v}(x_n), R^{\frac{1}{2}+\delta})\}.$$

We refer to the curve $\Gamma^\lambda_{\theta,v}$ as the core of $T_{\theta,v}$. Since ϕ is of reduced form, we find by the diffeomorphism property of Φ (writing $x' = \Phi^{-1}_{x_n,\omega_\theta} \circ \Phi_{x_n,\omega_\theta}(x')$)

$$|x' - \gamma^\lambda_{\theta,v}| \sim |\partial_\omega \phi^\lambda(x; \omega_\theta) - v|,$$

for all $x = (x', x_n) \in X_\lambda$ with $x_n \in I^\lambda_{\theta,v}$ uniformly in λ. This has the following consequence:

Lemma 3.4 ([13, Lemma 5.4]). *If $1 \ll R \ll \lambda$ and $x \in B(0,R) \backslash T_{\theta,v}$, then*

$$|T^\lambda f_{\theta,v}(x)| \leq (1 + R^{-1/2} |\partial_\omega \phi^\lambda(x;\omega_\theta) - v|)^{-(n+1)} RapDec(R) \|f\|_{L^2(A^{n-1})}.$$

3.4. **L^2-L^2-estimate.** We recall the following generalization of Parseval's theorem, only depending on non-degeneracy $C1')$ of the phase function (cf. [30, Section 2.1]):

Lemma 3.5 ([13, Lemma 5.5]). *If $1 \leq R \leq \lambda$ and B_R is any ball of radius R, then*

$$\|T^\lambda f\|_{L^2(B_R)} \lesssim R^{1/2} \|f\|_{L^2(A^{n-1})}. \tag{23}$$

This follows from the following estimate:

Lemma 3.6 ([13, Lemma 5.6]). *For any fixed $x_n \in \mathbb{R}$, we find the estimate*

$$\|T^\lambda f\|_{L^2(\mathbb{R}^{n-1} \times \{x_n\})} \lesssim \|f\|_{L^2(A^{n-1})}. \tag{24}$$

3.5. **k-broad norms.** Here we recall basic properties of the k-broad norms. Although the naming is misleading as k-broad norms are strictly speaking no norms, the properties are similar enough to make the following arguments work. We shall also see that $U \mapsto \|T^\lambda f\|^p_{BL^p_{k,A}(U)}$ behaves as a measure.

Lemma 3.7 (Finite (sub-)additivity, [13, Lemma 6.1]). *Let $U_1, U_2 \subseteq \mathbb{R}^n$ and $U = U_1 \cup U_2$. If $1 \leq p < \infty$ and A is a non-negative integer, then*

$$\|T^\lambda f\|^p_{BL^p_{k,A}(U)} \leq \|T^\lambda f\|^p_{BL^p_{k,A}(U_1)} + \|T^\lambda f\|^p_{BL^p_{k,A}(U_2)} \tag{25}$$

holds for all integrable $f : A^{n-1} \to \mathbb{C}$.

Secondly, we have the following variant of the triangle inequality:

Lemma 3.8 (Triangle inequality, [13, Lemma 6.2]). *If $U \subseteq \mathbb{R}^n$, $1 \leq p < \infty$ and $A = A_1 + A_2$ for A_1 and A_2 non-negative integers, then*

$$\|T^\lambda (f_1 + f_2)\|_{BL^p_{k,A}(U)} \lesssim \|T^\lambda f_1\|_{BL^p_{k,A_1}(U)} + \|T^\lambda f_2\|_{BL^p_{k,A_2}(U)} \tag{26}$$

holds for all integrable $f_1, f_2 : A^{n-1} \to \mathbb{C}$.

We further have the following variant of Hölder's inequality:

Lemma 3.9 (Logarithmic convexity, [13, Lemma 6.3])**.** *Suppose that $U \subseteq \mathbb{R}^n$, $1 \leq p, p_1, p_2 < \infty$ and $0 \leq \alpha_1, \alpha_2 \leq 1$ satisfy $\alpha_1 + \alpha_2 = 1$ and*

$$\frac{1}{p} = \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2}.$$

If $A = A_1 + A_2$ for A_1, A_2 non-negative integers, then

$$\|T^\lambda f\|_{BL^p_{k,A}(U)} \lesssim \|T^\lambda f\|^{\alpha_1}_{BL^{p_1}_{k,A_1}(U)} \|T^\lambda f\|^{\alpha_2}_{BL^{p_2}_{k,A_2}(U)}.$$

Later on, we shall only consider $A \gg 1$, which will allow for the use of Hölder's and Minkowski's inequality for k-broad norms.

4. Polynomial partitioning

A key tool in the proof will be polynomial partitioning following previous work by Guth [11, 12] (see also Guth–Katz [14]) and in the variable coefficient case Guth–Hickman–Iliopoulou [13]. The idea is to divide the ball B_R through the zero set of a polynomial into cells, which equidistribute the broad norm. Either $\mu_{T^\lambda f}$ will be concentrated in the cells or at the wall, i.e., neighbourhood of the zero locus of the polynomial. Both cases will be handled by induction. We recall some facts from [13], which we will rely on in the following.

4.1. **Tools from algebraic geometry.** Given a polynomial P in $\mathbb{R}^n$, its zero set is denoted by $Z(P)$. To make the varieties $Z(P_1, \ldots, P_{n-m})$ smooth m-dimensional manifolds, we consider transverse complete intersections:

Definition 4.1. Let $m \in \mathbb{N}$, $m \leq n$, and let $P_1, \ldots, P_{n-m}$ be polynomials on $\mathbb{R}^n$ whose common zero set is denoted by $Z(P_1, \ldots, P_{n-m})$. The variety $Z(P_1, \ldots, P_{n-m})$ is called a transverse complete intersection if

$$\nabla P_1(x) \wedge \ldots \wedge \nabla P_{n-m}(x) \neq 0 \qquad \forall x \in Z(P_1, \ldots, P_{n-m}).$$

The degree of the transverse complete intersection $\overline{\deg} Z$ is defined as $\max_{j=1,\ldots,n-m} \deg P_j$.

We have the following partitioning argument:

Theorem 4.2 ([13, Theorem 7.3])**.** *Suppose that $W \geq 0$ is a non-zero L^1-function on $\mathbb{R}^n$. Then, for any degree $D \in \mathbb{N}$, there exists a non-zero polynomial P of degree $\deg P \lesssim D$ such that the following holds:*

- *The set $Z(P)$ is a finite union of $\log D$ transverse complete intersection.*
- *If $(O_i)_{i \in \mathcal{I}}$ denotes the set of connected components of $\mathbb{R}^n \backslash Z(P)$, then $\# \mathcal{I} \lesssim D^n$ and*

$$\int_{O_i} W \sim D^{-n} \int_{\mathbb{R}^n} W \text{ for all } i \in \mathcal{I}. \tag{27}$$

The connected components are called *cells*.

We further need the following lemma on transverse intersections of tubes with varieties:

Lemma 4.3 ([12, Lemma 5.7])**.** *Let T be a cylinder of radius r with central line ℓ and suppose that $Z = Z(P_1, \ldots, P_{n-m}) \subseteq \mathbb{R}^n$ is a transverse complete intersection, where the polynomials P_j have degree at most D. For $\alpha > 0$, let*

$$Z_{>\alpha} = \{z \in Z : \mathit{Angle}(T_z Z, \ell) > \alpha\}.$$

Then $Z_{>\alpha} \cap T$ is contained in a union of $\lesssim D^n$ balls of radius $\lesssim r\alpha^{-1}$.

For the application, we are interested in $r = R^{(1+\delta)/2}$, as this will be the radius of the (thin) tubes and $\alpha = R^{-\frac{1}{2}+\delta}$.

4.2. **Polynomial approximation.** However, with smooth core curves, Lemma 4.3 is not applicable directly. We approximate the core curves by polynomials such that algebraic methods can still be applied to the curved tubes. We follow [13, Section 7.2]. Let $\varepsilon > 0$ be a small parameter and let $N = N_\varepsilon := \lceil 1/(2\varepsilon) \rceil \in \mathbb{N}$. Suppose that $\Gamma : (-1,1) \to \mathbb{R}^n$ is a smooth curve with

$$\|\Gamma\|_{C^{N+1}(-1,1)} = \max_{0 \leq k \leq N+1} \sup_{|t|<1} |\Gamma^{(k)}(t)| \lesssim 1.$$

After reductions of Section 3.1, we find the following estimates:

Lemma 4.4. *The curves $\Gamma^1_{\theta,v}$ satisfy*

$$|(\Gamma^1_{\theta,v})'(t)| \sim 1 \text{ for all } t \in I^1_{\theta,v},$$

and

$$\sup_{t \in I^1_{\theta,v}} |(\Gamma^1_{\theta,v})^{(k)}(t)| \lesssim c_{par} \text{ for } 2 \leq k \leq N.$$

The proof from [13, Lemma 7.4] applies verbatim although the phase functions are from different classes, but because of bounds (20) and $D2$) from Subsection 3.1.

We denote by $[\Gamma]_\varepsilon : \mathbb{R} \to \mathbb{R}^n$ the polynomial curve given by the degree-N Taylor approximation of Γ around zero. Observe that

$$\|[\Gamma]_\varepsilon\|_{C^\infty(-2,2)} \leq e^2 \|\Gamma\|_{C^N(-1,1)} \lesssim 1.$$

Furthermore, for $\lambda \gg 1$, noting that $\lambda^{-\varepsilon N} \leq \lambda^{-1/2}$, Taylor's theorem yields

$$|\Gamma^{(i)}(t) - [\Gamma]^{(i)}_\varepsilon(t)| \lesssim_\varepsilon \lambda^{-\frac{1}{2}} |t|^{1-i} \text{ for all } |t| \lesssim_\varepsilon \lambda^{-\varepsilon} \text{ and } i = 0, 1.$$

Letting $\Gamma^\lambda : (-\lambda, \lambda) \to \mathbb{R}^n$ denote the rescaled curve $\Gamma^\lambda(t) = \lambda \Gamma(t/\lambda)$, the above inequalities imply that

$$\|[\Gamma^\lambda]'_\varepsilon\|_{C^\infty(-2\lambda,2\lambda)} \lesssim 1 \text{ and } \|[\Gamma^\lambda]''_\varepsilon\|_{C^\infty(-2\lambda,2\lambda)} \lesssim \lambda^{-1}, \tag{28}$$

and

$$|(\Gamma^\lambda)^{(i)}(t) - ([\Gamma^\lambda]_\varepsilon)^{(i)}(t)| \lesssim_\varepsilon \lambda^{-\frac{1}{2}} |t|^{1-i} \text{ for all } |t| \lesssim_\varepsilon \lambda^{1-\varepsilon} \text{ and } i = 0, 1.$$

As a consequence of $|(\Gamma^\lambda)'(t)| \sim |[\Gamma^\lambda]'_\varepsilon(t)| \sim 1$, the tangent spaces to the curves Γ^λ and $[\Gamma^\lambda]_\varepsilon$ have a small angular separation, i.e.,

$$\angle(T_{\Gamma^\lambda(t)} \Gamma^\lambda, T_{[\Gamma^\lambda]_\varepsilon(t)} [\Gamma^\lambda]_\varepsilon) \lesssim_\varepsilon \lambda^{-\frac{1}{2}} \text{ for all } |t| \lesssim_\varepsilon \lambda^{1-\varepsilon}. \tag{29}$$

4.3. **Transverse interactions between curved tubes and varieties.** Next, we generalize the transverse interaction of straight lines and varieties as in Lemma 4.3 to curved tubes, which are approximated by polynomials. Let $Z = Z(P_1, \ldots, P_{n-m})$ be a transverse complete intersection and $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be a polynomial curve. Given $\alpha, r > 0$, the problem is to estimate the size of the set

$$Z_{>\alpha,r,\Gamma} := \{z \in Z : \text{there exists } x \in \Gamma \text{ with } |x - z| < r \text{ and } \angle(T_z Z, T_x \Gamma) > \alpha\}.$$

We further assume that Γ is a *polynomial graph*, which means it can be rotated so that it is given by $\Gamma(t) = (\gamma(t), t)$ for some polynomial mapping $\gamma : \mathbb{R} \to \mathbb{R}^{n-1}$. This is the case considered in the present context. We have the following generalization of Lemma 4.3:

Lemma 4.5 ([13, Lemma 7.5]). *Let $n \geq 2$, $1 \leq m \leq n$ and $Z = Z(P_1, \ldots, P_{n-m}) \subseteq \mathbb{R}^n$ be a transverse complete intersection. Suppose that $\Gamma : \mathbb{R} \to \mathbb{R}^n$ is a polynomial graph satisfying*

$$\|\Gamma'\|_{L^\infty(-2\lambda,2\lambda)} \lesssim 1 \text{ and } \|\Gamma''\|_{L^\infty(-2\lambda,2\lambda)} \leq \delta \tag{30}$$

for some $\lambda, \delta > 0$. There exists a dimensional constant $\bar{C} > 0$ such that, for all $\alpha > 0$ and $0 < r < \lambda$ satisfying $\alpha \geq \bar{C}\delta r$, the set $Z_{>\alpha,r,\Gamma} \cap B(0,\lambda)$ is contained in a union of

$$O((\overline{\deg} Z \cdot \deg \Gamma)^n)$$

balls of radius r/α.

In the present context, we consider the polynomial approximant $\Gamma = [\Gamma^\lambda_{\theta,v}]_\varepsilon$ of the curve $\Gamma^\lambda_{\theta,v}$ as defined in Subsection 4.2. Then, $\deg \Gamma \lesssim_\varepsilon 1$ and by (28) we find (30) to hold with $\delta \sim_\varepsilon 1/\lambda$. Consequently, for $\alpha > 0$ and $0 < r < \lambda$ with $\alpha \gtrsim r/\lambda$, the set $Z_{>\alpha,r,\Gamma} \cap B(0,\lambda)$ can be covered by $O_\varepsilon((\overline{\deg} Z)^n)$ balls of radius r/α.

5. Transverse equidistribution estimates

5.1. **Linearizing the phase function.** In this section transverse equidistribution estimates for wave packets tangential to varieties will be examined. This is a key point in the main induction argument. Contrary to [25] or [9], however, we stick to the wave packet decomposition used in [13].

Definition 5.1. Let $Z = Z(P_1, \ldots, P_{n-m})$ be a transverse complete intersection. A wave packet (θ, v) is said to be $R^{-\frac{1}{2}+\delta_m}$-tangent to Z in $B(0,R)$ if

$$T_{\theta,v} \cap B_R \subseteq N_{R^{\frac{1}{2}+\delta_m}}(Z) \tag{31}$$

and

$$\angle(G^\lambda(x;\omega_\theta), T_z Z) \leq \bar{c}_{tang} R^{-\frac{1}{2}+\delta_m} \tag{32}$$

for any $x \in T_{\theta,v}$ and $z \in Z \cap B(0,2R)$ with $|x - z| \leq \bar{C}_{tang} R^{\frac{1}{2}+\delta_m}$.

We want to study functions concentrated on the collection of wave packets

$$\mathbb{T}_Z = \{(\theta, v) \in \mathbb{T} : T_{\theta,v} \text{ in } R^{-\frac{1}{2}+\delta_m} - \text{tangent to } Z \text{ in } B(0,R)\}.$$

Precisely, we make the following definition:

Definition 5.2. If $\mathbb{S} \subseteq \mathbb{T}$, then f is said to be concentrated on wave packets from $\mathbb{S}$ if

$$f = \sum_{(\theta,v)\in\mathbb{S}} f_{\theta,v} + \mathrm{RapDec}(R)\|f\|_{L^2}.$$

Let $B \subseteq \mathbb{R}^n$ be a ball of radius $R^{\frac{1}{2}+\delta_m}$ with centre $\bar{x} \in B(0,R)$. We study $\eta_B \cdot T^\lambda g$, where η_B is a suitable choice of Schwartz function adapted to B. A stationary phase argument yields that $\eta_B \cdot T^\lambda g_{\theta,v}$ is concentrated near the surface $\Sigma = \{\Sigma(\omega) : \omega \in A^{n-1}\}$, where $\Sigma(\omega) = \partial_x \phi^\lambda(\bar{x};\omega)$. This leads to the refined set of wave packets

$$\mathbb{T}_{Z,B} = \{(\theta, v) \in \mathbb{T}_Z : T_{\theta,v} \cap B \neq \emptyset\}.$$

For $(\theta, v) \in \mathbb{T}_{Z,B}$, the direction $G^\lambda(\bar{x};\omega_\theta)$ of $T_{\theta,v}$ on the ball B must make a small angle with each of the tangent spaces $T_z Z$ for all $z \in Z \cap B$. This constrains $\Sigma(\omega_\theta)$ to lie in a small neighbourhood of some typically m-dimensional manifold S_ξ. But in the homogeneous case, S_ξ might only be one-dimensional, or "close" to a one-dimensional manifold. This will be quantified below. This case does not contribute essentially in the broad norm. To linearize S_ξ, if it is not a "thin", essentially one-dimensional set, let $R^{\frac{1}{2}} < \rho \ll R$ and for the remainder of this section, let $\tau \subseteq A^{n-1}$ denote a sector of radius $O(\rho^{-\frac{1}{2}+\delta_m})$. We define

$$\mathbb{T}_{Z,B,\tau} = \{(\theta, v) \in \mathbb{T}_Z : \theta \cap \tau \neq \emptyset \text{ and } T_{\theta,v} \cap B \neq \emptyset\}.$$

We recall the constant-coefficient examples. Suppose that Z is an m-dimensional affine plane so that $T_z Z = V$ for all $z \in Z$, where $V \parallel Z$. The extension operator for the paraboloid has the unnormalized Gauss map $G_0(\omega) = (-\omega, 1)$. Consequently,

$$A_\omega = \{\omega \in \mathbb{R}^{n-1} : G_0(\omega) \in V\}$$

is an affine subspace of $\mathbb{R}^{n-1}$ of dimension $m-1$. If $G_0(\omega) \in V$, then $\Sigma(\omega) \in A_\xi = A_u \times \mathbb{R}$. Due to the frequency localization and the uncertainty principle, $\eta_B \cdot T^\lambda g$ will decay little in directions transverse to A_ξ. This was exploited in [12].
The situation for the cone is a little different. Here, $G_0(\omega) = \big(-\frac{\omega}{|\omega|}, 1\big)$ is not an affine map. Let $V^+ = \{\omega : G_0(\omega) \in V\}$. By a crucial observation due to Ou–Wang [25], if V^+ is tangent to $\mathcal{C} = \{(\omega, \frac{\omega}{|\omega|}) : \omega \in A^{n-1}\}$ up to an angle $R^{-\delta_m}$, then $N_{R^{-\frac{1}{2}+\delta_m}} V^+ \cap \mathcal{C}$ is a $O(R^{-\delta_m})$-neighbourhood of $O(1)$ radial lines.

In the variable coefficient case, we see that if V^+ is tangent to $\mathcal{C}_x = \{\partial_x \phi^\lambda(\bar{x}; \omega)\}$ up to an angle $R^{-\delta_m}$, then $N_{R^{-\frac{1}{2}+\delta_m}} V^+ \cap C$ is a $O(R^{-\delta_m})$-neighbourhood of $O(1)$ radial lines. A contribution like this can be neglected in the K-broad norm. Otherwise, we shall see that we find quantitative transversality to hold and can deduce transverse equidistribution estimates similar to the paraboloid case (or its variable coefficient counterpart). In the constant coefficient case, but for arbitrary cones, this was recently investigated in [9]. We shall see how the arguments adapt to the variable coefficient case.

Consider an m-dimensional linear subspace $V = \{\sum_{j=1}^n a_{i,j} x_j = 0,\ i = 1, \ldots, n-m\}$ and let $V^- = \{\sum_{j=1}^{n-1} a_{i,j} x_j = 0\}$. We change to u-frequencies via Ψ^λ, which recall is defined by

$$\partial_{x'} \phi^\lambda(\bar{x}; \Psi^\lambda(\bar{x}; u)) = u.$$

We use short-hand notation $\Psi(u) := \Psi^\lambda(\bar{x}; u)$. It is easy to see that Ψ like $\partial_{x'}\phi^\lambda$ and the identity mapping is 1-homogeneous because

$$\partial_{x'} \phi^\lambda(\bar{x}; \Psi(\mu u)) = \mu u = \mu \partial_{x'} \phi^\lambda(\bar{x}; \Psi(u)) = \partial_{x'} \phi^\lambda(\bar{x}; \mu \Psi(u)).$$

By substituting $\tilde{\phi}(u) = h_{\bar{x}}(u) = \partial_{x_n} \phi^\lambda(\bar{x}; \Psi(u))$ the arguments from [9] apply. We define a set

$$L = \{u \in A^{n-1} : \sum_{j=1}^{n-1} a_{i,j} \partial_j \tilde{\phi}(u) - a_{i,n} = 0; \quad i = 1, \ldots, n-m\}.$$

The set $\{(u, \tilde{\phi}(u)) : u \in L\}$ describes the points on the generalized cone, which have a normal in V. The *tangential case* gives a negligible contribution to the broad norm:

Lemma 5.3 ([9, Lemma 4.5]). *Let $\eta \in \mathbb{S}^{n-2} \subseteq \mathbb{R}^{n-1}$. If $\eta \in L$ and $\angle(\eta, V^-) > \frac{\pi}{2} - K^{-2}$, then L is contained in the set $\{\xi \in \mathbb{R}^{n-1} \backslash 0 : \angle(\xi, \eta) \lesssim K^{-2}\}$.*

It is important to note that, contrary to the transverse case analyzed below, the lemma does not hinge on a stronger localization of η. For later purposes, note that for the suitably defined k-broad norm balls $B(\bar{x}; R^{\frac{1}{2}+\delta_m})$, for which Lemma 5.3 applies, make a negligible contribution.

We turn to the more involved *transverse case*: In general $\{(u,\tilde{\phi}(u)) : u \in L\}$ may not lie in an affine subspace because L may have curvature. We start by linearizing L. By taking the orthogonal complement of a suitable extension of the tangent space we shall construct W, which is quantitatively transverse to V. Let $\eta \in L \cap \mathbb{S}^{n-2}$ with $\mathrm{Ang}(\eta, V^-) < \frac{\pi}{2} - K^{-2}$. Define $\tilde{V}$ to be the $n-m$-dimensional linear subspace spanned by $\gamma_1, \dots, \gamma_{n-m}$ given by

$$\gamma_i = \partial^2 \tilde{\phi}(\eta)\alpha_i, \quad \alpha_i = (a_{i,1}, \dots, a_{i,n}), \quad i = 1, \dots, n-m.$$

The angle condition $\angle(\eta, V^-) < \frac{\pi}{2} - K^{-2}$ ensures that γ_i, $i = 1, \dots, n-m$ are linearly independent. Indeed, the Hessian is degenerate in the direction of η, but α_i is orthogonal to V^-. Let $\bar{V}^-$ be the orthogonal complement of $\tilde{V}$ in $\mathbb{R}^{n-1}$, i.e.,

$$\mathbb{R}^{n-1} = \tilde{V} \oplus \bar{V}^-.$$

Note that $\bar{V}^-$ denotes the tangent space of L: Starting from the equations

$$\sum_{j=1}^{n-1} a_{i,j} \partial_j \tilde{\phi}(\eta + \xi') - a_{i,n} = 0,$$

linearizing yields for $\xi' \in T_\eta L$

$$\sum_{j=1}^{n-1} a_{i,j} \sum_{k=1}^{n-1} \partial^2_{jk} \tilde{\phi}(\eta) \xi'_k = (\partial^2 \tilde{\phi}(\eta)\alpha_i, \xi') = 0.$$

Let $\bar{V}$ be the linear subspace spanned by $\bar{V}^-$ and e_n. Define W to be the orthogonal complement of $\bar{V}$ in $\mathbb{R}^n$, i.e.,

$$\mathbb{R}^n = \bar{V} \oplus W.$$

As pointed out in [9], all the linear subspaces depend on the choice of η. We have the following quantitative transversality:

Lemma 5.4 ([9, Lemma 4.6]). *Let $\eta \in \mathbb{S}^{n-2} \cap L$. If $\angle(\eta, V^-) \leq \frac{\pi}{2} - K^{-2}$, then W and V are transverse in the sense that $\angle(V, W) \gtrsim K^{-4}$.*

5.2. **Verifying the transverse equidistribution estimate.** We turn to the key equidistribution estimate. In the following let $\tau \subseteq A^{n-1}$ denote a sector of aperture $O(\rho^{-\frac{1}{2}+\delta_m})$ and B a ball of radius $R^{\frac{1}{2}+\delta_m}$. Moreover, we suppose that Z is K-flat, i.e., for any $z, z' \in Z \cap 2B$ we have

$$\angle(T_z Z, T_{z'} Z) \lesssim K^{-5}$$

with $K \lesssim R^\delta \ll \rho^{\delta_m}$.

Lemma 5.5. *Let Z be a K-flat, transverse complete intersection with $\dim Z = m$, $\overline{deg}\, Z \lesssim_\varepsilon 1$, $B = B(\bar{x}, R^{\frac{1}{2}+\delta_m})$ a ball of radius $R^{\frac{1}{2}+\delta_m}$, and let g be concentrated on wave packets in $\mathbb{T}_{Z,B,\tau}$. Suppose that with the notations of Subsection 5.1, with $\tilde{\phi} = h_{\bar{x}}$, and for some $\eta \in \Psi^{-1}(\tau) \cap \mathbb{S}^{n-2}$ we are in the situation of Lemma 5.4. Then, for any $\rho \leq R$,*

$$\int_{B \cap N_{\rho^{\frac{1}{2}+\delta_m}}(Z)} |T^\lambda g|^2 \lesssim R^{\frac{1}{2}+O(\delta_m)} \Big(\frac{\rho}{R}\Big)^{\frac{n-m}{2}} \|g\|^2_{L^2}. \tag{33}$$

For the proof, we recall the following quantifications of the uncertainty principle from [13, Subsection 8.2]: Let $G : \mathbb{R}^n \to \mathbb{C}$ be frequency supported on a ball of radius $r > 0$. Then we have the moral estimate due to lack of L^2-norm concentration:

$$\fint_{B(x_0,\rho)} |G|^2 \lesssim \fint_{B(x_0,r^{-1})} |G|^2.$$

We use the below manifestation with $\hat{G}$ essentially supported in a ball of radius $r > 0$.

Lemma 5.6. *If $r^{-\frac{1}{2}} \leq \rho \leq r^{-1}$, then for any $B(x_0, \rho)$, $\xi_0 \in \mathbb{R}^n$ and $\rho > 0$ one has*

$$\fint_{B(x_0,\rho)} |G|^2 \lesssim_\delta \|\hat{G} w^{-1}_{B(\xi_0,r)}\|_\infty^{\frac{2\delta}{1+\delta}} \frac{1}{|B(r)|^{-1}} \big(\int_{\mathbb{R}^n} |G|^2 \big)^{\frac{1}{1+\delta}}.$$

Above, $w_{B(\xi_0,r)}$ is a weight concentrated on $B(\xi_0, r)$ given by

$$w_{B(\xi_0,r)}(\xi) = (1 + r^{-1}|\xi - \xi_0|)^{-N} \text{ for some large } N = N_\delta \in \mathbb{N}.$$

As first step in the proof of Lemma 5.5, we consider wave packets tangential to linear subspaces: In the following transverse equidistribution estimates are considered with respect to some fixed linear subspace $V \subseteq \mathbb{R}^n$. Let B be a ball of radius $R^{\frac{1}{2}+\delta_m}$ with centre $\bar{x} \in \mathbb{R}^n$, and define

$$\mathbb{T}_{V,B} = \{(\theta, v) : \angle(G^\lambda(\bar{x}; \omega_\theta), V) \lesssim R^{-\frac{1}{2}+\delta_m} \text{ and } T_{\theta,v} \cap B \neq \emptyset\}.$$

Let $R^{\frac{1}{2}} < \rho < R$ and, for $\tau \subseteq \mathbb{R}^{n-1}$ be a sector of aperture $O(\rho^{-\frac{1}{2}+\delta_m})$ centred around a point in A^{n-1}, define

$$\mathbb{T}_{V,B,\tau} = \{(\theta, v) \in \mathbb{T}_{V,B} : \theta \cap \big(\frac{\tau}{10}\big) \neq \emptyset\}.$$

Lemma 5.7. *If $V \subseteq \mathbb{R}^n$ is a linear subspace, then there exists a linear subspace W with the following properties:*

(1) $\dim V + \dim W = n$;
(2) *V and W are quantitatively transverse with $\angle(v, w) \gtrsim K^{-4}$ for any $v \in V$, $w \in W$, $v, w \neq 0$;*
(3) *if g is concentrated on wave packets from $\mathbb{T}_{V,B,\tau}$ and there is $\eta \in \Psi^{-1}(\tau)$ such that for $\tilde{\phi} = h_{\bar{x}}$ the assumptions of Lemma 5.4 are valid, Π is any plane parallel to W and $x_0 \in \Pi \cap B$, then the inequality*

$$\int_{\Pi \cap B(x_0,\rho^{\frac{1}{2}+\delta_m})} |T^\lambda g|^2 \lesssim_\delta R^{O(\delta_m)} \big(\frac{\rho}{R}\big)^{\frac{\dim W}{2}} \|g\|_{L^2}^{2\delta/(1+\delta)} \big(\int_{\Pi \cap 2B} |T^\lambda g|^2 \big)^{\frac{1}{1+\delta}}$$

holds, up to inclusion of $RapDec(R)\|g\|_{L^2}$ on the right-hand side.

Proof. **Constructing the subspace W:** We choose W after linearizing as in Lemma 5.4. Recall that $h_{\bar{x}}(u) = \partial_{x_n} \phi^\lambda(\bar{x}; \Psi(u))$ with $\partial_{x'} \phi^\lambda(\bar{x}; \Psi(u)) = u$ such that $(u, h_{\bar{x}}(u))$ is a graph parametrization of $\partial_x \phi^\lambda(\bar{x}; \cdot)$ in u-frequencies with $\omega = \Psi(u)$. Fix some $\eta \in L \cap \Psi^{-1}(\tau)$ and construct W as in Lemma 5.4. Note that

$$\partial^2_{xu_1} \phi^\lambda(\bar{x}; \Psi(u)) \wedge \ldots \wedge \partial^2_{xu_{n-1}} \phi^\lambda(\bar{x}; \Psi(u)) = G_0(\bar{x}; \omega) \cdot \det J\Psi(u).$$

If $L \cap \Psi^{-1}(\tau) = \emptyset$, then $\Psi(L) \cap \tau = \emptyset$, but then, by Lemma 3.2 we had $\mathbb{T}_{V,B,\tau} = \emptyset$ and there is nothing to show. Hence, we can construct W around $\eta \in L \cap \Psi^{-1}(\tau)$ as in Subsection 5.1. W and V are quantitatively transverse as in (2) by Lemma 5.4.

Verifying the transverse equidistribution estimate: Recall that g is concentrated on wave packets $\mathbb{T}_{V,B,\tau}$, B is a $R^{\frac{1}{2}+\delta_m}$-ball, and τ is a $O(\rho^{-\frac{1}{2}+\delta_m})$-sector.

W is constructed as above. Let $\eta_B(x) = \eta((x - \bar{x})/R^{\frac{1}{2}+\delta_m})$ denote a Schwartz cutoff, which satisfies $\eta(x) = 1$ for $x \in B(0,2)$. We have

$$(\eta_B T^\lambda g_{\theta,v}|_\Pi)\widehat{}(\xi) = e^{-2\pi i x_0.\xi} R^{\dim W\left(\frac{1}{2}+\delta_m\right)} \int_{A^{n-1}} K^{\lambda,R}(\xi;\omega) g_{\theta,v}(\omega) d\omega \tag{34}$$

with $K^{\lambda,R}$ given by

$$K^{\lambda,R}(\xi;\omega) = \int_W e^{2\pi i \phi_\omega^{\lambda,R}(z)} a_\omega^{\lambda,R}(z) dz$$

for the phase and amplitude function

$$\phi_\omega^{\lambda,R}(z) = \phi^\lambda(x_0 + R^{\frac{1}{2}+\delta_m} z;\omega) - R^{\frac{1}{2}+\delta_m}\langle z;\xi\rangle,$$
$$a_\omega^{\lambda,R}(z) = a^\lambda(x_0 + R^{\frac{1}{2}+\delta_m} z;\omega)\tilde{\eta}(z),$$

and

$$\tilde{\eta}(z) = \eta\Big(z + \frac{x_0 - \bar{x}}{R^{\frac{1}{2}+\delta_m}}\Big).$$

Let $\Sigma(\omega) = \partial_x \phi^\lambda(\bar{x};\omega)$. Fixing $\omega \in \Xi$, $\xi \in \hat{\mathbb{R}}^n$ such that $|\xi - \mathrm{proj}_W \Sigma(\omega)| \gtrsim R^{-\frac{1}{2}+\delta_m}$ and $R \gg 1$, the following estimates hold on $\mathrm{supp}(a_z^{\lambda,R})$:

(i) $|\partial_z \phi_\omega^{\lambda,R}(z)| \sim R^{\frac{1}{2}+\delta_m}|\xi - \mathrm{proj}_W \Sigma(\omega)| \gtrsim R^{2\delta_m}$,
(ii) $|\partial_z^\alpha \phi_\omega^{\lambda,R}(z)| \lesssim |\partial_z \phi_\omega^{\lambda,R}|$ for $2 \le |\alpha| \le N_{par}$,
(iii) $|\partial_z^\alpha a_\omega^{\lambda,R}| \lesssim_\varepsilon 1$.

This is verified as in [13, Claim 2, p. 308].

Furthermore, $\omega \in \mathrm{supp}(g_{\theta,v})$, then $|\omega - \omega_\theta| < R^{-\frac{1}{2}}$, and so $|\Sigma(\omega) - \xi_\theta| \lesssim R^{-\frac{1}{2}}$, where $\xi_\theta = \Sigma(\omega_\theta)$. Consequently, by non-stationary phase,

$$|(\eta_B \cdot T^\lambda g_{\theta,v})|_\Pi\widehat{}(\xi)| \lesssim_N R^{O(1)} w_{B(\mathrm{proj}_W \xi_\theta, R^{-\frac{1}{2}})}(\xi) \|g_{\theta,v}\|_{L^2}.$$

Let

$$L = \{u \in A^{n-1} : (-\nabla h_{\bar{x}}(u), 1) \in V\},$$
$$S_\omega = \{\omega \in A^{n-1} : G^\lambda(\bar{x};\omega) \in V\}.$$

Let $A_u = T_\eta L^{\mathrm{aff}}$ denote the affine variant of the linear subspace $T_\eta L$, and $A_\xi = A_u \times \mathbb{R}$. With V_ξ denoting the linear subspace associated with A_ξ, we have $V_\xi^\perp = W$.

Next, we shall show that

$$\mathrm{dist}(\xi_\theta, A_\xi) \lesssim R^{-\frac{1}{2}+\delta_m}. \tag{35}$$

Then it follows

$$w_{B(\mathrm{proj}_W \xi_\theta, R^{-\frac{1}{2}})} \lesssim_\delta w_{B(\xi_*, R^{-\frac{1}{2}+\delta_m})}.$$

Here ξ_* denotes the centre of a ball of radius $O(R^{-\frac{1}{2}+\delta_m})$ containing $\mathrm{proj}_W(\xi_\theta)$ for $(\theta, v) \in \mathbb{T}_{B,\tau,V}$. We can again refer to [13] for details. The claim follows then by estimating

$$\|(\eta_B \cdot T^\lambda g|_\Pi)\widehat{}\, w^{-1}_{B(\xi_*, R^{-\frac{1}{2}+\delta_m})}\|_\infty$$

via Lemma 5.6.

We turn to the proof of (35): Fix $(\theta, v) \in \mathbb{T}_{B,\tau,V}$ and let

$$u_\theta = \mathrm{proj}_{x_n^\perp}(\Sigma(\omega_\theta)).$$

We compute by triangle inequality

$$\mathrm{dist}(\xi_\theta, A_\xi) = \mathrm{dist}(u_\theta, A_u) \le \mathrm{dist}(u_\theta, L \cap \Psi^{-1}(\tau)) + \sup_{u_* \in L \cap \Psi^{-1}(\tau)} \mathrm{dist}(u_*, A_u).$$

Furthermore, by Lemma 3.2,

$$\mathrm{dist}(u_\theta, L \cap \Psi^{-1}(\tau)) \sim \mathrm{dist}(\omega_\theta, S_\omega \cap \tau) \lesssim \angle(G^\lambda(\bar{x};\omega_\theta), V) \lesssim R^{-\frac{1}{2}+\delta_m},$$

where the last inequality is by the definition of $\mathbb{T}_{V,B,\tau}$.

We turn to the estimate of the second term: Fix $u_* \in L \cap \Psi^{-1}(\tau)$. We note that $\mathrm{dist}(u_*, A_u) = \mathrm{dist}(u_*, A_{\bar{u}})$ for $\bar{u} = \frac{\|u_*\|}{\|u\|} u$ by null direction. Let $A_{\bar{u}} = u_0 + V_u$ for some linear subspace V_u. Now we note that the surface $L \cap \Psi^{-1}(\tau) \cap \{\|u_*\| \mathbb{S}^{n-2}\}$, provided ρ is large enough, can be written as subset of graph of a function $\psi : \mathcal{W} \to V_u^\perp$, where $\mathcal{W} \subseteq V_u$ is a subset around the origin of size $O(\rho^{-\frac{1}{2}+\delta_m})$. More precisely, we may write

$$L \cap \Psi^{-1}(\tau) \cap \{\|u_*\| \mathbb{S}^{n-2}\} \subseteq \{w + \psi(w) : w \in \mathcal{W}\} + u_0$$

with $\psi(0) = 0$ and $\nabla \psi(0) = 0$. The estimate now follows from Taylor expansion as in [13, p. 310].

$\square$

For proof of the transverse equidistribution estimate in Lemma 5.5 requires we have to extend the estimate from fixed vector space to variety. The argument follows [13, Section 8.4] with the difference that the quantitative transversality mildly depends on the scale. We use the following result of Wongkew [39] to control the size of neighbourhoods of varieties.

Theorem 5.8 ([39])**.** *Suppose $Z = Z(P_1, \dots, P_{n-m})$ is an m-dimensional transverse complete intersection in $\mathbb{R}^n$ with $\overline{\deg} Z \leq D$. For any $0 < \rho \leq R$ and R-ball B_R, $N_\rho(Z \cap B_R)$ can be covered by $O_D((R/\rho)^m)$ balls of radius ρ.*

Next, we consider planar slices of neighbourhoods of varieties. We recall the following from [13]: Any m-dimensional linear subspace V can be expressed as a transverse complete intersection $V = Z(P_{N_1}, \dots, P_{N_{n-m}})$ with $\{N_1, \dots, N_{n-m}\}$ an orthonormal basis of $V^\perp$ and $P_{N_j}(x) = \langle x, N_j \rangle$. Let V_1, V_2 be linear subspaces in $\mathbb{R}^n$ and suppose that

$$\dim V_1 + \dim V_2 \geq n. \tag{36}$$

$V_1 \cap V_2$ is a transverse complete intersection if and only if

$$\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - n.$$

This means that $V_1 \cap V_2$ is as small as possible.

Definition 5.9. A pair (V_1, V_2) of linear subspaces $\mathbb{R}^n$ satisfying (36) is said to be *quantitatively transverse* if the following hold:

- $\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - n$;
- $\angle(v_1, v_2) \geq c_{\text{trans}}$ for all non-zero vectors $v_j \in (V_1 \cap V_2)^\perp \cap V_j$, $j = 1, 2$.

In the main argument the constant c_{trans} will not be fixed, contrary to [13]. We need to quantify the dependence on c_{trans} in [13, Lemma 8.13]:

Lemma 5.10. *There exists some dimensional constant $C > 0$ such that the following holds. Let $B_r \subseteq \mathbb{R}^n$ be an r-ball, $V \subseteq \mathbb{R}^n$ be a linear subspace, Z be a transverse complete intersection and suppose that $\dim Z + \dim V \geq n$ and $(T_z Z, V)$ is a quantitatively transverse pair for all $z \in Z \cap 2B_r$. Then, the following inclusion holds*

$$V \cap B_r \cap N_\rho(Z) \subseteq N_{2\bar{C}\rho}(V \cap Z).$$

for all $0 < \rho \ll r$ with $\bar{C} = \sin(c_{trans})^{-1}$.

Proof. This follows from the proof of [13, Lemma 8.13]. □

We are ready for the proof of Lemma 5.5:

Proof of Lemma 5.5. First, by

$$|G^\lambda(\bar{x};\theta) - G^\lambda(x;\theta)| \lesssim |x-\bar{x}|/\lambda \lesssim R^{-\frac{1}{2}+\delta_m},$$

we infer that

$$\angle(G^\lambda(\bar{x};\theta), T_z Z) \lesssim R^{-\frac{1}{2}+\delta_m} \text{ for all } z \in Z \cap 2B.$$

Letting $V = T_z Z$, we have

$$\mathbb{T}_{Z,B,\tau} \subseteq \mathbb{T}_{V,B,\tau}.$$

We can apply Lemma 5.7 to find a subspace W such that

$$\angle(V,W) \gtrsim K^{-4} \tag{37}$$

and

$$\int_{\Pi\cap B(x_0,\rho^{\frac{1}{2}+\delta_m})} |T^\lambda g|^2 \lesssim_\delta R^{O(\delta_m)} \big(\frac{\rho}{R}\big)^{\frac{\dim W}{2}} \|g\|_{L^2}^{\frac{2\delta}{1+\delta}} \big(\int_{\Pi\cap 2B} |T^\lambda g|^2\big)^{\frac{1}{1+\delta}} \tag{38}$$

for every affine subspace Π parallel to W. By K-flatness of Z, we have that $(T_z Z, W)$ is a quantitatively transverse pair for all $z \in Z \cap 2B$. By Lemma 5.10, we have

$$\Pi \cap N_{\rho^{\frac{1}{2}+\delta_m}}(Z) \cap B \subseteq N_{CK^4\rho^{\frac{1}{2}+\delta_m}}(\Pi \cap Z) \cap 2B.$$

Since $\Pi \cap Z$ is a transverse complete intersection of dimension $\dim V' + \dim Z - n$, Theorem 5.8 yields that $\Pi \cap N_{\rho^{\frac{1}{2}+\delta_m}}(Z) \cap B$ can be covered by

$$O\big(R^{O(\delta_m)}\big(\frac{R}{\rho}\big)^{(\dim V + \dim Z - n)/2}\big) = O(R^{O(\delta_m)})$$

balls of radius $\rho^{\frac{1}{2}+\delta_m}$ because $K \lesssim R^\delta \ll \rho^{\delta_m}$. Applying (38) to each of these balls and summing, one deduces that

$$\int_{\Pi\cap N_{\rho^{\frac{1}{2}+\delta_m}}(Z)\cap B} |T^\lambda g|^2 \lesssim_\delta R^{O(\delta_m)} \big(\frac{\rho}{R}\big)^{\frac{n-m}{2}} \|g\|_{L^2}^{2\delta/(1+\delta)} \big(\int_{\Pi\cap 2B} |T^\lambda g|^2\big)^{\frac{1}{1+\delta}}.$$

Following the steps from [13, p. 318] completes the proof. □

6. Comparing wave packets at different spatial scales

For the induction on scales, we shall compare wave packet decompositions at different radii. Let $1 \ll R \ll \lambda$, and

$$T^\lambda f(x) = \sum_{(\theta,v)\in\mathbb{T}} T^\lambda f_{\theta,v}(x) + \mathrm{RapDec}(R)\|f\|_{L^2(A^{n-1})}.$$

In this section we recall the results from [13, Section 9], which again did not hinge on $H2^+$), but on non-degeneracy.

6.1. **Wave packets at smaller scale.** Let $R^{\frac{1}{2}} \leq \rho \leq R$ and fix $B(y,\rho) \subseteq B(0,R)$. $T^\lambda f_{\theta,v}$ can be decomposed into wave packets at scale ρ over $B(y,\rho)$. For $g : A^{n-1} \to \mathbb{C}$, define $\tilde{g} = e^{i\phi^\lambda(y;\cdot)} g$, so that

$$T^\lambda g(x) = \tilde{T}^\lambda \tilde{g}(\tilde{x}) \text{ for } \tilde{x} = x - y,$$

where $\tilde{T}^\lambda$ is the oscillatory integral operator with phase $\tilde{\phi}^\lambda$ and amplitude $\tilde{a}^\lambda$ given by

$$\tilde{\phi}(x;\omega) = \phi(x + \frac{y}{\lambda};\omega) - \phi(\frac{y}{\lambda};\omega) \text{ and } \tilde{a}(x;\omega) = a(x + \frac{y}{\lambda};\omega). \tag{39}$$

This yields by linearity

$$T^\lambda f(x) = \sum_{(\theta,v)\in\mathbb{T}} \tilde{T}^\lambda((f_{\theta,v})\tilde{})(\tilde{x}) + \mathrm{RapDec}(R)\|f\|_{L^2(A^{n-1})}.$$

Each $T^\lambda f_{\theta,v}$ is (spatially) concentrated on the curve $R^{\frac{1}{2}+\delta}$-tube $T_{\theta,v}$ and, consequently, each $\tilde{T}^\lambda (f_{\theta,v})\tilde{}$ is concentrated on $T_{\theta,v} - y$. Since

$$\partial_\omega \tilde{\phi}^\lambda((\gamma^\lambda_{\omega,v}(t),t) - y;\omega) = v - \partial_\omega \phi^\lambda(y;\omega), \tag{40}$$

the core curve $\Gamma^\lambda_{\theta,v} - y$ of $T_{\theta,v} - y$ is equal to $\Gamma^\lambda_{\theta,v-\bar{v}(y;\omega_\theta)}$, where

$$\bar{v}(y;\omega) = \partial_\omega \phi^\lambda(y;\omega).$$

We repeat the construction of wave packets for each $\tilde{T}^\lambda (f_{\theta,v})\tilde{}$ at scale ρ. Cover A^{n-1} by finitely overlapping balls $\tilde{\theta}$ of radius $\rho^{-\frac{1}{2}}$, and $\mathbb{R}^{n-1}$ by finitely-overlapping balls of radius $\rho^{\frac{1+\delta}{2}}$ centered at vectors $\tilde{v} \in \rho^{\frac{1+\delta}{2}}\mathbb{Z}^{n-1}$. Let $\tilde{\mathbb{T}}$ denote the set of all pairs $(\tilde{\theta},\tilde{v})$. For each $(\theta,v) \in \mathbb{T}$ one may decompose

$$(f_{\theta,v})\tilde{} = \sum_{(\tilde{\theta},\tilde{v})\in\tilde{\mathbb{T}}} (f_{\theta,v})\tilde{}_{\tilde{\theta},\tilde{v}} + \mathrm{RapDec}(R)\|f\|_{L^2(A^{n-1})}.$$

The significant contributions to this sum arise from pairs $(\tilde{\theta},\tilde{v})$ belonging to

$$\tilde{\mathbb{T}}_{\theta,v} = \{(\tilde{\theta},\tilde{v}) \in \tilde{\mathbb{T}} : \mathrm{dist}(\theta,\tilde{\theta}) \lesssim \rho^{-\frac{1}{2}} \text{ and } |v - \bar{v}(y;\omega_\theta) - \tilde{v}| \lesssim R^{\frac{1+\delta}{2}}\}.$$

Lemma 6.1 ([13, Lemma 9.1])**.** *The following holds:*

$$(f_{\theta,v})\tilde{} = \sum_{(\tilde{\theta},\tilde{v})\in\tilde{\mathbb{T}}_{\theta,v}} (f_{\theta,v})\tilde{}_{\tilde{\theta},\tilde{v}} + \mathit{RapDec}(R)\|f\|_{L^2}.$$

6.2. **Tangency properties.** In this subsection we recall how tangency properties of the large wave packets are inherited by the small wave packets (cf. [13, Section 9.2]). On the other hand, recall that a small wave packet coming from a large packet, which is tangential to Z, need not be contained in a neighbourhood of Z on a small scale. The small wave packet is located too far away from Z. However, with the angle condition inherited, we shall see that the small wave packet is contained in a small neighbourhood of a translate of the variety.

We analyze functions h concentrated on wave packets from

$$\mathbb{T}_{Z,B(y,\rho)} = \{(\theta,v) \in \mathbb{T}_Z : T_{\theta,v} \cap B(y,\rho) \neq \emptyset\}.$$

For this purpose, we consider the core of a small tube:

$$\partial_\omega \tilde{\phi}^\lambda(\tilde{\gamma}^\lambda_{\omega,v}(t),t;\omega) = v$$

for $t \in (-\rho,\rho)$. By (40), we have the identity:

$$\gamma^\lambda_{\omega,v}(t) = \tilde{\gamma}^\lambda_{\omega,v-\bar{v}(y;\omega)}(t - y_n) + y'. \tag{41}$$

Let $\tilde{T}_{\omega,v}$ be the $\rho^{\frac{1}{2}+\delta}$-tube with core curve $\tilde{\Gamma}^{\lambda}_{\omega,v} = (\tilde{\gamma}^{\lambda}_{\omega,v}(t), t)$. We have the following:

Lemma 6.2 ([13, Lemma 9.3])**.** *If $(\theta, v) \in \mathbb{T}$ and $(\tilde{\theta}, \tilde{v}) \in \tilde{\mathbb{T}}_{\theta,v}$, then*

$$|\tilde{\Gamma}^{\lambda}_{\tilde{\theta},\tilde{v}}(t) - (\Gamma^{\lambda}_{\theta,v}(t + y_n) - y)| \lesssim R^{\frac{1+\delta}{2}}$$

for all $t \in (-\rho, \rho)$.

Fix $(\theta, v) \in \mathbb{T}_Z$ and $(\tilde{\theta}, \tilde{v}) \in \tilde{\mathbb{T}}_{\theta,v}$. Next, we show that for $x \in \tilde{T}_{\tilde{\theta},\tilde{v}}$ and $z \in Z$ and $b \in B(0, 2R^{\frac{1}{2}+\delta_m})$ are such that

$$z - y + b \in B(0, 4\rho) \text{ and } |x - (z - y + b)| \leq \bar{C}_{tang}\rho^{\frac{1}{2}+\delta_m},$$

then we find the following estimate to hold:

$$\angle(\tilde{G}^{\lambda}(x; \omega_{\tilde{\theta}}), T_{z-y+b}(Z - y + b)) \leq \bar{c}_{tang}\rho^{-\frac{1}{2}+\delta_m}, \tag{42}$$

where $\tilde{G}^{\lambda}$ is the generalized Gauss map associated with $\tilde{\phi}^{\lambda}$. We have $\tilde{G}^{\lambda}(x; \omega) = G^{\lambda}(x + y; \omega)$ and $T_{z-y+b}(Z - y + b) = T_z Z$, so it is equivalent to check that

$$\angle(G^{\lambda}(x + y; \omega_{\tilde{\theta}}), T_z Z) \leq \bar{c}_{tang}\rho^{-\frac{1}{2}+\delta_m}.$$

By Lemma 6.2, the definition of $\tilde{T}_{\tilde{\theta},\tilde{v}}$, and assuming that $\rho \leq R^{1-\delta}$, it follows

$$|x + y - \Gamma^{\lambda}_{\theta,v}(x_n + y_n)| \lesssim R^{\frac{1+\delta}{2}}.$$

By expanding the Gauss map, we find

$$\angle(G^{\lambda}(x + y; \omega_{\tilde{\theta}}), T_z Z) \lesssim \angle(G^{\lambda}(\Gamma^{\lambda}_{\theta,v}(x_n + y_n); \omega_{\theta}), T_z Z) + \rho^{-\frac{1}{2}}.$$

Finally, $\Gamma^{\lambda}_{\theta,v}(x_n + y_n) \in T_{\theta,v}$, which is $R^{-\frac{1}{2}+\delta_m}$-tangent to Z. Hence,

$$\angle(G^{\lambda}(\Gamma^{\lambda}_{\theta,v}(x_n + y_n); \omega_{\theta}), T_z Z) \leq \bar{c}_{tang}R^{-\frac{1}{2}+\delta_m}.$$

Likewise the argument in [13, pp. 326f] shows that a $\rho^{\frac{1}{2}+\delta}$-tube, which intersects $N_{\rho^{\frac{1}{2}+\delta_m}/2}(Z - y + b) \cap B(0, \rho)$, is actually contained in $N_{\rho^{\frac{1}{2}+\delta_m}}(Z - y + b)$ by virtue of (42).

We arrive at the following proposition (cf. [13, Proposition 9.2]):

Proposition 6.3. *Let $R^{\frac{1}{2}} \leq \rho \leq R^{1-\delta}$ and $Z \subseteq \mathbb{R}^n$ be a transverse complete intersection.*

(1) Let $(\theta, v) \in \mathbb{T}_Z$ and $b \in B(0, 2R^{\frac{1}{2}+\delta_m})$. If $(\tilde{\theta}, \tilde{v}) \in \tilde{\mathbb{T}}_{\theta,v}$ satisfies

$$\tilde{T}_{\tilde{\theta},\tilde{v}} \cap N_{\rho^{\frac{1}{2}+\delta_m}/2}(Z - y + b) \neq \emptyset,$$

then $(\tilde{\theta}, \tilde{v}) \in \tilde{\mathbb{T}}_{Z-y+b}$.

(2) If h is concentrated on wave packets in $\mathbb{T}_{Z,B(y,\rho)}$, then $\tilde{h}$ is concentrated on wave packets in

$$\bigcup_{|b| \lesssim R^{\frac{1}{2}+\delta_m}} \tilde{\mathbb{T}}_{Z-y+b}.$$

We also make the following definition:

$$\tilde{\mathbb{T}}_b = \{(\tilde{\theta}, \tilde{v}) : (\tilde{\theta}, \tilde{v}) \in \bigcup_{(\theta,v) \in \mathbb{T}_{Z,B(y,\rho)}} \tilde{\mathbb{T}}_{\theta,v} : \tilde{T}_{\tilde{\theta},\tilde{v}} \cap N_{\rho^{\frac{1}{2}+\delta_m}/2}(Z - y + b) \neq \emptyset\}.$$

By the above, we have $\tilde{\mathbb{T}}_b \subseteq \tilde{\mathbb{T}}_{Z-y+b}$. For a function h concentrated on wave packets in $\mathbb{T}_{Z,B(y,\rho)}$, we consider a function of the form

$$\tilde{h}_b = \sum_{(\tilde{\theta},\tilde{v})\in\tilde{\mathbb{T}}_b} \tilde{h}_{\tilde{\theta},\tilde{v}}.$$

Since $\tilde{\mathbb{T}}_b \subseteq \tilde{\mathbb{T}}_{Z-y+b}$ by Proposition 6.3, we have

$$\tilde{T}^\lambda \tilde{h}_b(\tilde{x}) = T^\lambda h_b(x)\chi_{N_{\rho^{\frac{1}{2}+\delta_m}}(Z+b)}(x) + \mathrm{RapDec}(R)\|h\|_{L^2}$$

for all $x = \tilde{x} + y \in B(y,\rho)$.

6.3. **Sorting wave packets.** In this section we recall sorting the large wave packets by "medium tubes". This was carried out in[13, Section 9.3]. Given a ball $B(y,\rho)$, many large wave packets (θ, v) might give rise to essentially the same set $\tilde{\mathbb{T}}_{\theta,v}$. Medium tube segments of T_ρ of length ρ and radius $R^{\frac{1}{2}+\delta}$ allow for a grouping into large and small wave packets.

We give the details: Let $\mathcal{T}$ denote the collection of all pairs $(\tilde{\theta}, w)$ formed by a $\rho^{-\frac{1}{2}}$-ball $\tilde{\theta}$ and $w \in R^{(1+\delta)/2}\mathbb{Z}^{n-1}$. For each $(\tilde{\theta}, w) \in \mathcal{T}$, choose some

$$\mathcal{T}_{\tilde{\theta},w} \subseteq \{(\theta, v) \in \mathbb{T} : \mathrm{dist}(\theta,\tilde{\theta}) \lesssim \rho^{-\frac{1}{2}} \text{ and } |v - \bar{v}(y;\omega_\theta) - w| \lesssim R^{(1+\delta)/2}\}$$

so that the family $\{\mathcal{T}_{\tilde{\theta},w} : (\tilde{\theta}, w) \in \mathcal{T}\}$ forms a covering of $\mathbb{T}$ by disjoint sets. The medium tubes are given by

$$T_{\tilde{\theta},w} = \bigcup_{(\theta,v)\in\mathcal{T}_{\tilde{\theta},w}} T_{\theta,v} \cap B(y,\rho).$$

If $(\tilde{\theta}, w) \in \mathcal{T}$ and $(\theta, v) \in \mathcal{T}_{\tilde{\theta},w}$, then (cf. [13, Cor. 9.4])

$$\mathrm{dist}_H(T_{\theta,v} \cap B(y,\rho), T_{\tilde{\theta},w}) \lesssim R^{\frac{1}{2}+\delta}.$$

Let $g : A^{n-1} \to \mathbb{C}$ be integrable and define

$$g_{\tilde{\theta},w} = \sum_{(\theta,v)\in\mathcal{T}_{\tilde{\theta},w}} g_{\theta,v}$$

for all $(\tilde{\theta}, w) \in \mathcal{T}$. Since $\mathcal{T}_{\tilde{\theta},w}$ cover $\mathbb{T}$ and are disjoint, it follows that

$$g = \sum_{(\tilde{\theta},w)\in\mathcal{T}} g_{\tilde{\theta},w} + \mathrm{RapDec}(R)\|g\|_{L^2}.$$

The functions $g_{\tilde{\theta},w}$ are almost orthogonal and, consequently,

$$\|g\|^2_{L^2} \sim \sum_{(\tilde{\theta},w)\in\mathcal{T}} \|g_{\tilde{\theta},w}\|^2_{L^2}.$$

$(g_{\tilde{\theta},w})\tilde{}$ is concentrated on scale ρ wave packets belonging to $\bigcup_{(\theta,v)\in\mathcal{T}_{\tilde{\theta},w}} \tilde{\mathbb{T}}_{\theta,v}$. This union is contained in

$$\tilde{\mathcal{T}}_{\tilde{\theta},w} = \{(\tilde{\theta}', \tilde{v}) \in \tilde{\mathbb{T}} : \mathrm{dist}(\tilde{\theta}',\tilde{\theta}) \lesssim \rho^{-1/2} \text{ and } |\tilde{v} - w| \lesssim R^{\frac{1+\delta}{2}}\}.$$

The family $\{\tilde{\mathcal{T}}_{\tilde{\theta},w} : (\tilde{\theta}, w) \in \mathcal{T}\}$ forms a covering of $\tilde{\mathbb{T}}$ by almost disjoint sets. Hence, we have almost orthogonality between the scale ρ wave packets of the different functions $(g_{\tilde{\theta},w})\tilde{}$ (cf. [13, Eq. (9.17)]):

$$\| \sum_{(\tilde{\theta},w)\in\mathcal{T}} (g_{\tilde{\theta},w})_b\tilde{}\|^2_{L^2} \sim \sum_{(\tilde{\theta},w)\in\mathcal{T}} \|(g_{\tilde{\theta},w})_b\tilde{}\|^2_{L^2}.$$

6.4. **Reverse Hörmander L^2-estimate.** In the following we record a reverse Hörmander L^2-estimate. This will imply transverse equidistribution estimates for functions concentrated on wave packets, which are sorted as above. This was previously done in [13, Subsection 9.4], whose statements carry over. Thus, the proofs are omitted. We collect the relevant estimates here for future reference.

Let Z be an m-dimensional transverse complete intersection, $(\tilde{\theta},w)\in\mathcal{T}$ and h be a function concentrated on $\mathbb{T}_{Z\cap B(y,\rho)}\cap\mathcal{T}_{\tilde{\theta},w}$. By the above, every scale R wave packet of h intersects $B(y,\rho)$ on the set $T_{\tilde{\theta},w}$, which has a Hausdorff distance $\lesssim R^{\frac{1}{2}+\delta_m}$ to $T_{\theta,v}\cap B(y,\rho)$ for any $(\theta,v)\in\mathcal{T}_{\tilde{\theta},w}$. Moreover, the scale ρ wave packets of $\tilde{h}$ will intersect $B(x_0-y,CR^{\frac{1}{2}+\delta_m})$. In this case, the following reverse of Hörmander's L^2-estimate holds:

Lemma 6.4 ([13, Lemma 9.5, p. 329])**.** *Let T^λ be an oscillatory integral operator with phase ϕ^λ given by a translate of a reduced phase in the sense of (39) and $1\le R^{\frac{1}{2}+\delta}\le r\lesssim\lambda^{\frac{1}{2}}$. There exists a family of oscillatory integral operators $\mathbf{T}^\lambda$ all with phase ϕ^λ such that the following hold:*

- *(i) each $T^\lambda\in\mathbf{T}^\lambda$ is again an operator with phase given by a translate of a reduced phase as in (39),*
- *(ii) $\#\mathbf{T}^\lambda=O(1)$;*
- *(iii) if f is concentrated on wave packets to (with respect to T^λ) which intersect some $B(\bar{x},r)\subseteq B(0,R)$, then*

$$\|f\|^2_{L^2}\lesssim r^{-1}\|T^\lambda_* f\|^2_{L^2(B(\bar{x};Cr))}$$

holds for some $T^\lambda_*\in\mathbf{T}^\lambda$.

Lemma 6.4 is proved in [13] via Fourier series expansion and Plancherel's theorem. The proof hinges only on the non-degeneracy $C1$) of the phase function. Hence, it applies to the homogeneous phase functions presently considered as well. For h as above, $x_0\in T_{\tilde{\theta},w}$ and $|b|\lesssim R^{\frac{1}{2}+\delta_m}$, this implies that $\tilde{h}_b$, as defined above, is a sum of wave packets which intersect

$$B(x_0-y,CR^{\frac{1}{2}+\delta_m}).$$

Applying Lemma 6.4 at scale ρ with $r\sim R^{\frac{1}{2}+\delta_m}$ to $\tilde{h}_b$ yields

$$\|\tilde{h}_b\|^2_{L^2}\lesssim R^{-\frac{1}{2}-\delta_m}\|\tilde{T}^\lambda_*\tilde{h}_b\|^2_{L^2(B(x_0-y,CR^{\frac{1}{2}+\delta_m})}.$$

Since the tangency properties of T^λ are inherited by T^λ_*, we infer

$$\|\tilde{h}_b\|^2_{L^2}\lesssim R^{-\frac{1}{2}-\delta_m}\|T^\lambda_* h_b\|^2_{L^2(N_{\rho^{\frac{1}{2}+\delta_m}}(Z+b)\cap B(x_0,CR^{\frac{1}{2}+\delta_m}))}.$$

Applying Hörmander's L^2-bound yields the following lemma:

Lemma 6.5. [13, Lemma 9.6] *Let h be concentrated on wave packets from $\mathbb{T}_{Z\cap B(y,\rho)}\cap\mathcal{T}_{\tilde{\theta},w}$ for some $(\tilde{\theta},w)\in\mathcal{T}$. Let $\mathfrak{B}\subseteq B(0,CR^{\frac{1}{2}+\delta_m})$ be such that the sets*

$$N_{\rho^{\frac{1}{2}+\delta_m}}(Z+b)\cap B(x_0,CR^{\frac{1}{2}+\delta_m})$$

are essentially disjoint over $b \in \mathfrak{B}$. Then,

$$\sum_{b \in \mathfrak{B}} \|\tilde{h}_b\|_{L^2}^2 \lesssim \|h\|_{L^2}^2.$$

7. Main inductive argument

The k-broad estimate is a consequence of the following claim, which is suitable for induction. Let

$$\bar{p}(k,n) = 2 \cdot \frac{n+k}{n+k-2}.$$

Theorem 7.1. *For $\varepsilon > 0$, sufficiently small, there are*

$$0 < \delta \ll \delta_{n-1} \ll \ldots \ll \delta_1 \ll \delta_0 \ll \varepsilon$$

and large dyadic parameters $\bar{A}_\varepsilon$, $\bar{C}_\varepsilon$, $D_{m,\varepsilon} \lesssim_\varepsilon 1$ and $\theta_m < \varepsilon$ such that the following holds. Suppose $Z = Z(P_1, \ldots, P_{n-m})$ is a transverse complete intersection with $\overline{\deg} Z \le D_{m,\varepsilon}$. For all $2 \le k \le n$, $1 \le A \le \bar{A}_\varepsilon$ dyadic and $1 \le K \le R \le \lambda$, the inequality

$$\|T^\lambda f\|_{BL^p_{k,A}(B(0,R))} \lesssim_\varepsilon K^{\bar{C}_\varepsilon} R^{\theta_m + \delta(\log \bar{A}_\varepsilon - \log A) - e_{k,n}(p) + \frac{1}{2}} \|f\|_{L^2(A^{n-1})} \tag{43}$$

holds whenever f is concentrated on wave packets from $\mathbb{T}_Z$ and

$$2 \le p \le \bar{p}_0(k,m) = \begin{cases} \bar{p}(k,m), & \text{if } k < m, \\ \bar{p}(m,m) + \delta, & \text{if } k = m. \end{cases} \tag{44}$$

Above,

$$e_{k,n}(p) = \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{p}\Big)(n+k).$$

In the first step we reduce to $R \lesssim_\varepsilon \lambda^{1-\varepsilon}$ by covering $B(0,\lambda)$ with balls of radius $\lambda^{1-\varepsilon}$. The technical details are provided in [13, Lemma 10.2]. This reduction is necessary to allow for polynomial approximation of the core curve $\gamma^\lambda_{\omega,v}$ uniformly in R.

Next, we set up the induction argument for $1 \le R \lesssim_\varepsilon \lambda^{1-\varepsilon}$. For $\varepsilon > 0$ sufficiently small, it is enough to consider $K \lesssim R^\delta$ by choosing $\bar{C}_\varepsilon$ sufficiently large (as the claim then follows from the trivial L^1-L^∞-estimate and crude summation). We let furthermore

$$\begin{gathered} D_{m,\varepsilon} = \varepsilon^{-\delta^{-(2n-m)}}, \quad \theta(\varepsilon) = \varepsilon - c_n \delta_m, \quad \bar{A}_\varepsilon = \lceil e^{\frac{10n}{\delta}} \rceil, \\ \delta_i = \delta_i(\varepsilon) = \varepsilon^{2i+1} \text{ for all } i = 1, \ldots, n-1, \text{ and } \delta = \delta(\varepsilon) \ll \delta_{n-1}. \end{gathered} \tag{45}$$

The base case is given by $m \le k-1$, and $A \ge 2^{10}$. For details we refer to [13, Subsection 10.3].

7.1. Inductive step.

Let $2 \le k \le n-1$, $k \le m \le n$, and $K \lesssim_\varepsilon R^\delta$. Assume, by way of induction hypothesis, that (43) holds whenever $\dim Z \le m-1$ or the radial parameter is at most $\frac{R}{2}$. Fix $\varepsilon > 0$, $1 < A \le \bar{A}_\varepsilon$ and a transverse complete intersection $Z = Z(P_1, \ldots, P_{n-m})$ with $\overline{\deg} Z \le D_{m,\varepsilon}$, where $\bar{A}_\varepsilon$ and $D_{m,\varepsilon}$ are as in (45). Let f be concentrated on wave packets from $\mathbb{T}_Z$. It suffices to show (43) for $p = \bar{p}_0(k,m)$ by interpolation with the trivial L^2-bound. We recall the two cases to be analyzed:

The algebraic case: There exists a transverse complete intersection $Y^l \subseteq Z$ of dimension $1 \leq l \leq m-1$ of maximum degree at most $(D_{m,\varepsilon})^n$ such that

$$\|T^\lambda f\|^p_{BL^p_{k,A}(N_{R^{\frac{1}{2}+\delta_m}/4}(Y^l)\cap B(0,R))} \geq c_{alg}\|T^\lambda f\|^p_{BL^p_{k,A}(B(0,R))}. \tag{46}$$

Here $c_{alg} > 0$ depends on n and ε.

The cellular case: For any transverse complete intersection $Y^l \subseteq Z$ of dimension $1 \leq l \leq m-1$ and maximum degree at most $(D_{m,\varepsilon})^n$, the inequality

$$\|T^\lambda f\|^p_{BL^p_{k,A}(N_{R^{\frac{1}{2}+\delta_m}/4}(Y^l))\cap B(0,R)} < c_{alg}\|T^\lambda f\|^p_{BL^p_{k,A}(B(0,R))} \tag{47}$$

holds.

The cellular case is as usually treated by induction on the radius. Via polynomial partitioning the $BL^p_{k,A}$-norm is equidistributed among the cells and the induction closes. The algebraic case is more involved: $T^\lambda f$ can be regarded as concentrated near a low-dimensional and low degree variety Y^l (for an oversimplification, think of a hyperplane). If the wave packets from f are also tangential to this variety, then we can use induction on the dimension to conclude. If this is not the case and many wave packets are transverse to Y^l, we conclude via transverse equidistribution estimates.

7.1.1. *Cellular case.* This case is handled as in [13, Section 10.5]. We omit the details.

7.1.2. *Algebraic case.* In this case transverse equidistribution estimates become important at one step. This is different than [13, Section 10.6], and we turn to the details. Fix a transverse complete intersection Y^l of dimension $1 \leq l \leq m-1$ of maximum degree $\overline{\deg} Y^l \leq (D_{m,\varepsilon})^n$, which satisfies (43). Let $R^{\frac{1}{2}} \ll \rho \ll R$ be such that $\rho^{\frac{1}{2}+\delta_l} = R^{\frac{1}{2}+\delta_m}$, and note that

$$R \leq R^{2\delta_l}\rho \text{ and } \rho \leq R^{-\delta_l/2}R.$$

Let $\mathcal{B}_\rho$ be a finitely overlapping cover of $B(0,R)$ by ρ-balls, and for each $B \in \mathcal{B}_\rho$ define

$$\mathbb{T}_B = \{(\theta,v) \in \mathbb{T} : T_{\theta,v} \cap N_{R^{\frac{1}{2}+\delta_m}/4}(Y^l) \cap B \neq \emptyset\}$$

and

$$f_B := \sum_{(\theta,v)\in\mathbb{T}_B} f_{\theta,v}.$$

We have by the triangle inequality for broad norms

$$\|T^\lambda f\|^p_{BL^p_{k,A}(B(0,R))} \lesssim \sum_{B\in\mathcal{B}_\rho} \|T^\lambda f_B\|^p_{BL^p_{k,A}(N_{R^{\frac{1}{2}+\delta_m}/4}(Y^l)\cap B))}$$

up to $\mathrm{RapDec}(R)\|f\|^p_{L^2}$ on the right-hand side by the rapid decay off the wave packets.

For $B = B(y,\rho) \in \mathcal{B}_\rho$, let $\mathbb{T}_{B,tang}$ denote the set of all $(\theta,v) \in \mathbb{T}_B$ with the property that, whenever $x \in T_{\theta,v}$ and $z \in Y^l \cap B(y,2\rho)$ satisfy $|x-z| \leq 2\bar{C}_{tang}\rho^{\frac{1}{2}+\delta_l}$, it follows that

$$\angle(G^\lambda(x;\omega_\theta), T_zY^l) \leq \frac{1}{2}\bar{c}_{tang}\rho^{-\frac{1}{2}+\delta_l},$$

where $\bar{C}_{tang}$ and $\bar{c}_{tang}$ are the constants appearing in the definition of tangency. Furthermore, let $\mathbb{T}_{B,trans} = \mathbb{T}_B \backslash \mathbb{T}_{B,tang}$ and define

$$f_{B,tang} = \sum_{(\theta,v)\in\mathbb{T}_{B,tang}} f_{\theta,v} \text{ and } f_{B,trans} = \sum_{(\theta,v)\in\mathbb{T}_{B,trans}} f_{\theta,v}.$$

It follows that $f_B = f_{B,tang} + f_{B,trans}$ and, by the triangle inequality for broad norms, one concludes that

$$\|T^\lambda f\|^p_{BL^p_{k,A}(B(0,R))} \lesssim \sum_{B\in\mathcal{B}_\rho} \|T^\lambda f_{B,tang}\|^p_{BL^p_{k,A/2}(B)} + \sum_{B\in\mathcal{B}_\rho} \|T^\lambda f_{B,trans}\|^p_{BL^p_{k,A/2}(B)}.$$

Either the tangential or transverse contribution to the above sum dominates, and each case is treated separately.

Tangential subcase: Suppose that the tangential term dominates and we have

$$\|T^\lambda f\|^p_{BL^p_{k,A}(B(0,R))} \lesssim \sum_{B\in\mathcal{B}_\rho} \|T^\lambda f_{B,tang}\|^p_{BL^p_{k,A/2}(B)}. \tag{48}$$

This case can be handled as in [13, pp.345-346], and we skip the details.

Transverse sub-case: In this case, we have

$$\|T^\lambda f\|^p_{BL^p_{k,A}(B(0,R))} \lesssim \sum_{B\in\mathcal{B}_\rho} \|T^\lambda f_{B,trans}\|^p_{BL^p_{k,A/2}(B)}.$$

Following [13], we use an argument similar to the cellular case. In the transverse case the number of cells a given tube can enter is controlled by transversality as follows:

Lemma 7.2 ([13, Eq. (10.23)]). *We find the following estimate to hold:*

$$\sum_{B\in\mathcal{B}_\rho} \|f_{B,trans}\|^2_{L^2(A^{n-1})} \lesssim_\varepsilon \|f\|^2_{L^2(A^{n-1})}. \tag{49}$$

The strategy in the transverse case is to use induction on radius to show that for some $\bar{c}_\varepsilon > 0$ one has (redenoting f_j for $f_{B_j,trans}$)

$$\|T^\lambda f_j\|_{BL^p_{k,A/2}(B)} \leq \bar{c}_\varepsilon E_{m,A}(R) \|f_j\|_{L^2(A^{n-1})} \tag{50}$$

for all $B_j \in \mathcal{B}_\rho$.

Provided $\bar{c}_\varepsilon > 0$ is chosen sufficiently small, depending only on n and ε, (50) can be combined with (49) and the estimate

$$\|f_{B,trans}\|_{L^2(A^{n-1})} \lesssim \|f\|_{L^2}$$

to yield

$$\|T^\lambda f\|_{BL^p_{k,A/2}(B(0,R))} \lesssim_\varepsilon \bar{c}_\varepsilon E_{m,A}(R) \|f\|^{1-\frac{2}{p}}_{L^2} \big(\sum_{B\in\mathcal{B}_\rho} \|f_{B,trans}\|^2_{L^2} \big)^{\frac{1}{p}} \leq E_{m,A}(R)\|f\|_{L^2}.$$

The main obstacle is that f_j do not, in general, satisfy the hypothesis of Theorem 7.1 at scale ρ. The remedy is to break f_j into pieces $f_{j,b}$, which are $\rho^{\frac{1}{2}+\delta_m}$-tangent to a translated variety of $Z+b$.

Flattening the variety: In the first step we flatten the variety up to K^{-5} (cf. [25, p. 25]). This requires to estimate $K^{5(n-1)}$-expressions of the form $\|T^\lambda f_j\|_{BL^p_{k,A/2}(B)}$ with Z flat up to angles K^{-5}. The factor of $K^{O(n)}$ is admissible (see (55) below).

Separating essentially and non-essentially contributing $R^{\frac{1}{2}+\delta_m}$-balls: Cover B_j by finitely overlapping $R^{\frac{1}{2}+\delta_m}$-balls $B_{j,k}$. Let $(\theta,v) \in \mathbb{T}_{Z,B_{j,k}}$ and $x \in T_{\theta,v} \cap N_{R^{\frac{1}{2}+\delta_m}}(Z) \cap B_{j,k}$, $z \in Z$ with $|x-z| \leq \bar{C}_{tang} R^{\frac{1}{2}+\delta_m}$. By definition of tangency, we have

$$\angle(G^\lambda(x;\omega_\theta), T_z Z) \lesssim R^{-\frac{1}{2}+\delta_m}.$$

Let $V = T_z Z$. By Lemma 3.2, we have

$$\angle(G^\lambda(\bar{x};\omega_\theta), V) \lesssim R^{-\frac{1}{2}+\delta_m}.$$

Now we consider the linearization $\tilde{\phi}_{\bar{x}}(u) = \partial_{x_n} \phi^\lambda(\bar{x};\Psi^\lambda(u))$ around $\bar{x}$, the centre of $B_{j,k}$. We consider as in Section 5.1

$$V = \{\sum_{j=1}^n a_{i,j} x_j = 0, \quad i = 1,\dots,n-m\}, \quad L_{\bar{x}} = \{u \in A^{n-1} : \sum_{j=1}^{n-1} a_{i,j} \partial_j \tilde{\phi}_{\bar{x}}(u) - a_{i,n} = 0\}$$

such that $L_{\bar{x}}$ denotes the u-frequencies with normal in V.
We apply the dichotomy of Section 5.1: Either $L_{\bar{x}}$ is contained in $O(1)$ slabs of size $1 \times K^{-2} \times \ldots \times K^{-2}$ by Lemma 5.3. This is referred to as Case I. Note that if $L_{\bar{x}}$ is contained in $O(1)$ $1 \times K^{-2} \times \ldots \times K^{-2}$-slabs, then so is $\bigcup \theta$ with $\angle(G^\lambda(\bar{x};\omega_\theta), V) \lesssim R^{-\frac{1}{2}+\delta_m}$ by Lemma 3.2. Consequently, Case I-balls can be neglected in the k-broad norm (see (51) below).

Otherwise, we consider the further refinement $\mathbb{T}_{V,B_{j,k},\tau}$ with τ a $\rho^{-\frac{1}{2}}$-sector. By Lemma 5.4, there is a quantitatively transverse subspace W with $\bar{V} \oplus W = \mathbb{R}^n$ and

$$\angle(V,W) \gtrsim K^{-4}.$$

$\bar{V}$ denotes a suitable extension of a tangent space of $L_{\bar{x}}$ from Subsection 5.1 (Case II). We let X_I and X_{II} denote the union of balls $B_{j,k}$ from Cases I and II.

Next, we use the sorting into medium tubes as in Section 5.1. Recall notations $\mathcal{T}_{\tilde{\theta},w}$ and $\tilde{\mathcal{T}}_{\tilde{\theta},w}$ with $\tilde{\theta}$ a $\rho^{-\frac{1}{2}}$-cap and $w \in R^{\frac{1+\delta}{2}} \mathbb{Z}^{n-1}$ for sortings, which relate ρ-wave packets on B_j with the large R-wave packets. For the sake of brevity let $g = f_{j,trans}$. We define as in [25, p. 25]:

$$g_{ess} = \sum_{(\tilde{\theta},w) \in \mathcal{T}_{ess}} g_{\tilde{\theta},w} = g - \sum_{(\tilde{\theta},w) \in \mathcal{T}_{tail}} g_{\tilde{\theta},w},$$

where

$$\mathcal{T}_{ess} = \{(\tilde{\theta},w) : \exists (\theta,v) \in \mathcal{T}_{\tilde{\theta},w} \text{ so that } T_{\theta,v} \cap X_{II} \neq \emptyset\},$$
$$\mathcal{T}_{tail} = \{(\tilde{\theta},w) : \forall (\theta,v) \in \mathcal{T}_{\tilde{\theta},w} : T_{\theta,v} \cap X_{II} = \emptyset\}.$$

Like in [25], we infer that

$$\|T^\lambda g\|_{BL^p_{k,A}(B_j)} \leq \|T^\lambda g_{ess}\|_{BL^p_{k,A/2}(B_j)} + \text{RapDec}(R) \|f\|_{L^2}. \tag{51}$$

As in [13], we choose a set of translates $\mathcal{B}$, so that we can write

$$\|T^\lambda g_{ess}\|_{BL^p_{k,A/2}(B_j)} \lesssim \big(\sum_{b \in \mathcal{B}} \|T^\lambda g_{ess,b}\|^p_{BL^p_{k,A/2}(B_j)} \big)^{\frac{1}{p}}, \tag{52}$$

where each piece $g_{ess,b}$ is defined so that it is concentrated on scale ρ wave packets, which are tangential to some translate $Z - y + b$ of Z. At this point, we can use transverse equidistribution and infer that $g_{ess,b}$ satisfy favorable L^2-estimates.

Moreover, the radial induction hypothesis is applied to each of the $T^\lambda g_{ess,b}$. To close the induction, one must estimate

$$\big(\sum_{b \in \mathcal{B}} \| g_{ess,b} \|_{L^2}^p \big)^{\frac{1}{p}}$$

in terms of $\| g_{ess} \|_{L^2}$. The gain in (ρ / R) stemming from transverse equidistribution is crucial. We can sum the contributions from the individual pieces $g_{ess,b}$ without any (significant) loss in R.

To ensure that $g_{ess,b}$ form a reasonable decomposition of g_{ess} so that (52) holds up to logarithmic factors, the set of translates $\mathcal{B}$ must be chosen so that $\bigcup_{b \in \mathcal{B}} N_{\rho^{\frac{1}{2}+\delta_m}}(Z - y + b)$ covers $N_{R^{\frac{1}{2}+\delta_m}}(Z)$ (where the mass of $T^\lambda g_{ess,b}$ is concentrated) and so that the $N_{\rho^{\frac{1}{2}+\delta_m}}(Z-y+b)$ are essentially disjoint. This was achieved in [13] using a probabilistic construction: Fix $B = B(y,\rho) \in \mathcal{B}_\rho$, one may show the following:

Lemma 7.3 ([13, Lemma 10.5])**.** *There exists a finite set $\mathcal{B} \subseteq B(0, 2R^{\frac{1}{2}+\delta_m})$ and a collection*

$$\mathcal{B}' \subseteq \{ B_{K^2} \in \mathcal{B}_{K^2} : B_{K^2} \cap B(y,\rho) \neq \emptyset \}$$

such that, up to inclusion of a rapidly decreasing error term,

$$\| T^\lambda f_{B,trans} \|_{BL^p_{k,A/2}(B)} \lesssim (\log R)^2 \big(\sum_{B_{K^2} \in \mathcal{B}'} \mu_{T^\lambda f_{B,trans}}(B_{K^2}) \big)^{\frac{1}{p}} \tag{53}$$

and for each $B_{K^2} \in \mathcal{B}'$ the following holds:

(i) there exists some $b \in \mathcal{B}$ such that

$$B_{K^2} \subseteq N_{\rho^{\frac{1}{2}+\delta_m}/2}(Z+b); \tag{54}$$

(ii) there exist at most $O(1)$ vectors $b \in \mathcal{B}$ for which

$$B_{K^2} \cap N_{\rho^{\frac{1}{2}+\delta_m}}(Z+b) \neq \emptyset.$$

By the lemma, we may argue as follows: For each $b \in \mathcal{B}$, let $\mathcal{B}'_b$ denote the collection of all $B_{K^2} \in \mathcal{B}'$ for which (54) holds. Then, by (53) and property (i) in the lemma,

$$\| T^\lambda f_{B,trans} \|_{BL^p_{k,A/2}(B)} \lesssim (\log R)^2 \big(\sum_{b \in \mathcal{B}} \sum_{B_{K^2} \in \mathcal{B}'_b} \mu_{\tilde{T}^\lambda (f_{B,trans})\tilde{}}(B_{K^2} - y) \big)^{\frac{1}{p}}.$$

Define the collection of wave packets

$$\tilde{\mathbb{T}}'_b = \{ (\tilde{\theta}, \tilde{v}) \in \bigcup_{(\theta,v) \in \mathbb{T}_{ess}} \tilde{\mathbb{T}}_{\theta,v} : \tilde{T}_{\tilde{\theta},\tilde{v}} \cap \big(\bigcup_{B_{K^2} \in \mathcal{B}'_b} (B_{K^2} - y) \big) \neq \emptyset \}.$$

If $g_{ess,b}$ is defined by

$$(g_{ess,b})\tilde{} = \sum_{(\tilde{\theta},\tilde{v}) \in \tilde{\mathbb{T}}'_b} (g_{ess})\tilde{}_{\tilde{\theta},\tilde{v}},$$

then $(g_{ess,b})\tilde{}$ is concentrated on wave packets that are $\rho^{-\frac{1}{2}+\delta_m}$-tangent to $Z - y + b$. Furthermore, again up to a rapidly decreasing error term, one has

$$\| T^\lambda g_{ess,b} \|_{BL^p_{k,A/4}(B_j)} \lesssim (\log R)^2 \big(\sum_{b \in \mathcal{B}} \| \tilde{T}^\lambda (g_{ess,b})\tilde{} \|^p_{BL^p_{k,A/4}(B(0,\rho))} \big)^{\frac{1}{p}}.$$

The function $(g_{ess,b})\check{}$ satisfies the hypotheses of Theorem 7.1 at scale ρ and therefore the radial induction hypothesis yields

$$\Big(\sum_{b\in\mathcal{B}} \|\tilde{T}^\lambda (g_{ess,b})\tilde{}\|^p_{BL^p_{k,A/4}(B(0,\rho))}\Big)^{\frac{1}{p}} \le E_{m,A/4}(\rho)\Big(\sum_{b\in\mathcal{B}} \|g_{ess,b}\|^p_{L^2}\Big)^{\frac{1}{p}}.$$

We claim that

$$\Big(\sum_{b\in\mathcal{B}} \|g_{ess,b}\|^p_{L^2}\Big)^{\frac{1}{p}} \lesssim R^{O(\delta_m)}\Big(\frac{\rho}{R}\Big)^{(n-m)\left(\frac{1}{4}-\frac{1}{2p}\right)}\|g_{ess}\|_{L^2}. \tag{55}$$

We show this via interpolation between $p=2$ and $p=\infty$.

For $p=2$ this follows from orthogonality of the wave packets and property (ii) of Lemma 7.3.

For $p=\infty$ we use transverse equidistribution. By almost orthogonality of $(\tilde{\theta},w)\in\mathcal{T}$ and the definition of $\mathcal{T}_{ess}$ we have

$$\|\tilde{g}_{ess,b}\|^2_{L^2} \sim \sum_{(\tilde{\theta},w)\in\mathcal{T}_{ess}} \|\tilde{g}_{ess,b,\tilde{\theta},w}\|^2_{L^2}.$$

By construction of $g_{ess,b,\tilde{\theta},w}$ there is $(\theta,v)\in\mathcal{T}_{\tilde{\theta},w}$ such that $T_{\theta,v}$ for (θ,v) intersects X_{II}. Let $B=B(\bar{x};R^{\frac{1}{2}+\delta_m})$ denote the corresponding ball in X_{II}. Since the Hausdorff distance between T_{θ_1,v_1} for further $(\theta_1,v_1)\in\mathcal{T}_{\tilde{\theta},w}$ is $\lesssim R^{\frac{1}{2}+\delta}$, we can apply Lemma 6.4 at scale ρ with $r\sim R^{\frac{1}{2}+\delta_m}$ to find that

$$\|\tilde{g}_{ess,b,\tilde{\theta},w}\|^2_{L^2} \lesssim R^{-\frac{1}{2}-\delta_m}\|\tilde{T}^\lambda_* \tilde{g}_{ess,b,\tilde{\theta},w}\|^2_{L^2(10B)}. \tag{56}$$

Next, we can apply Lemma 5.5 to find that

$$\begin{aligned}\|T^\lambda g_{ess,b,\tilde{\theta},w}\|^2_{L^2(10B\cap N_{\rho^{\frac{1}{2}+\delta_m}}(Z+b))} &\lesssim R^{\frac{1}{2}+O(\delta_m)}\Big(\frac{\rho}{R}\Big)^{\frac{n-m}{2}}\|g_{ess,b,\tilde{\theta},w}\|^2_{L^2}\\ &\lesssim R^{\frac{1}{2}+O(\delta_m)}\Big(\frac{\rho}{R}\Big)^{\frac{n-m}{2}}\|g_{ess,b}\|^2_{L^2}.\end{aligned} \tag{57}$$

Taking (56) and (57) together, we find

$$\|\tilde{g}_{ess,b,\tilde{\theta},w}\|^2_{L^2} \lesssim R^{O(\delta_m)}\Big(\frac{\rho}{R}\Big)^{\frac{n-m}{2}}\|g_{ess,b}\|^2_{L^2},$$

which is the claimed $p=\infty$ estimate for (55).

At this point, the computation to close the induction follows [13, p. 351] verbatim. The proof of Theorem 7.1 is complete. □

8. From k-broad to linear estimates

In this section we deduce the linear estimates from the k-broad estimates by applying the Bourgain–Guth argument [6]. We show the following proposition:

Proposition 8.1. *Suppose that for all $K\ge 1$ and all $\varepsilon>0$ any oscillatory integral operator T^λ with reduced 1-homogeneous phase satisfying $C1)$ and $C2^+)$ obeys the k-broad inequality*

$$\|T^\lambda f\|_{BL^p_{k,A}(B(0,R))} \lesssim_\varepsilon K^{C_\varepsilon}R^\varepsilon\|f\|_{L^p(A^{n-1})} \tag{58}$$

for some fixed k, A, p, C_ε, and all $R\ge 1$. If

$$p(k,n)\le p\le \frac{2n}{n-2}, \qquad p(k,n)=\begin{cases}2\cdot\frac{n-1}{n-2} & \text{if } 2\le k\le 3,\\ 2\cdot\frac{2n-k+1}{2n-k-1} & \text{if } k>3,\end{cases} \tag{59}$$

then any oscillatory integral operator with $C1)$ and $C2^+)$ phase ϕ and amplitude a satisfies

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim_{\phi,\varepsilon,a} \lambda^\varepsilon \|f\|_{L^p(A^{n-1})}. \tag{60}$$

From this proposition Theorem 1.1 is immediate by choosing $k = \frac{n+1}{2}$ for n odd and $k = \frac{n}{2}+1$ for n even as $\max(p(k,n), \bar{p}(k,n))$ gives the lower bound for p in Theorem 1.1. For the proof we use induction on scales: $Q_{p,\delta}(R)$ will denote the infimum over all constants C for which the estimate

$$\|T^\lambda f\|_{L^p(B(0,r))} \leq C\|f\|_{L^p(A^{n-1})}$$

holds for $1 \leq r \leq R$ and all oscillatory integral operators built from a suitable class of phase functions, which is invariant under rescaling and amenable to narrow decoupling, which is explained below.

With this definition, it remains to prove that for p as in Proposition 8.1

$$Q_{p,\delta}(R) \lesssim_\varepsilon R^\varepsilon$$

for all $\varepsilon > 0$ and $1 \leq R \leq \lambda$.
For this purpose, we decompose $B(0,R)$ into finitely overlapping balls B_{K^2} of radius K^2 and estimate $\|T^\lambda f\|_{L^p(B_{K^2})}$. f is decomposed into "broad" and "narrow" term. The narrow term is of the form

$$\sum_{\substack{\tau \in V_a \\ \text{for some } a}} f_\tau, \tag{61}$$

consisting of contributions to f from sectors for which $G^\lambda(\bar{x};\tau)$ makes a small angle with some member of a family of $(k-1)$-planes. Here $\bar{x}$ denotes the centre of B_{K^2}. The broad term consists of contributions to f from the remaining sectors. One may choose the planes $V_1, \ldots, V_A$ so that the broad term can be bounded by the k-broad inequality. Thus, f of the form (61) has to be analyzed. This is accomplished by narrow ℓ^p-decoupling and rescaling. We use the following decoupling result:

Proposition 8.2 (Variable coefficient decoupling)**.** *Suppose that T^λ is an oscillatory integral operator with reduced $C1)$ and $C2^+)$ phase, which is K-flat and let $B_{K^2} \subseteq \lambda^{1-\delta}$ with $1 \leq K^2 \leq \lambda^{\frac{1}{2}-\delta}$, $0 < \delta \leq 1/2$. If $V \subseteq \mathbb{R}^n$ is an m-dimensional linear subspace, then for $2 \leq p \leq \frac{2n}{n-2}$ and any $\delta > 0$ one has*

$$\begin{aligned} \big\| \sum_{\tau \in V} T^\lambda g_\tau \big\|_{L^p(B_{K^2})} &\lesssim_\delta \max(1, K^{(m-2)\left(\frac{1}{2}-\frac{1}{p}\right)}) K^\delta \big(\sum_{\tau \in V} \|T^\lambda g_\tau\|^p_{L^p(w_{B_{K^2}})} \big)^{\frac{1}{p}} \\ &\quad + \lambda^{-\frac{\delta N}{2}} \|g\|_{L^2}. \end{aligned}$$

Here, the sum ranges over sectors τ for which $\angle(G^\lambda(\bar{x};\tau), V) \leq K^{-2}$, where $\bar{x}$ is the centre of B_{K^2} and $w_{B_{K^2}} = (1+|x-\bar{x}|)^{-N}$ is a rapidly decaying weight off B_{K^2} with N the same as in the notion of K-flatness.

We remark that on the right-hand side there are strictly speaking slightly different amplitude functions involved. If we choose $B_{K^2} \subseteq B(0, \lambda^{1-\delta})$ for λ large enough however, the amplitude functions satisfy the uniform bounds

$$|\partial_\omega^\alpha a(x;\omega)| \lesssim_N 1$$

for $0 \leq |\alpha| \leq N$, N being the parameter from K-flatness. This technicality of dealing with different amplitude functions is handled by appropriate definition of the induction quantity.

In the translation-invariant case, e.g., with $\mathcal{E}$ as in (4), this follows from the ℓ^2-decoupling

$$\big\| \sum_{\tau} \mathcal{E} g_\tau \big\|_{L^p(B_{K^2})} \lesssim_\delta K^\delta \big(\sum_{\tau \in V} \|\mathcal{E} g_\tau\|^2_{L^p(w_{B_{K^2}})} \big)^{\frac{1}{2}}$$

for $2 \leq p \leq \frac{2n}{n-2}$ and by counting the sectors τ such that $\angle(G(\tau), V) \leq K^{-2}$. This is carried out in [25]; see also [16, Lemma 2.2]. The error term $\lambda^{-\frac{\delta N}{2}} \|f\|_{L^2}$ comes from approximation with constant coefficient operators. Gao *et al.* [9] used K-flatness to prove narrow decoupling of general homogeneous phases in the constant coefficient case.

Definition 8.3. We say that a 1-homogeneous smooth $\phi : \mathbb{R}^{n-1} \backslash 0 \to \mathbb{R}$ supported in Ξ is K-flat if

$$\begin{aligned} \phi(\omega', \omega_{n-1}) &= \omega_{n-1} \phi(\omega'/\omega_{n-1}, 1) \\ &= \omega_{n-1} \phi(e_{n-1}) + \partial_{\omega'} \phi(e_{n-1}) \omega' + \frac{\langle \partial^2_{\omega' \omega'} \phi(e_{n-1}) \omega', \omega' \rangle}{2 \omega_{n-1}} + K^{-4} E(\omega) \end{aligned}$$

with $E(\omega)$ 1-homogeneous, satisfying $|\partial^\alpha E_R| \lesssim_\alpha 1$ for $0 \leq |\alpha| \leq N$.

In the course of the argument, we will need to consider higher derivatives; above unspecified as N. These are needed for approximation with constant-coefficient operators. In the end, we choose $N = N(\varepsilon)$ (since $\delta = \delta(\varepsilon)$) large enough such that the error term $\lambda^{-\frac{\delta N}{2}} \|f\|_{L^2}$ propagates through the argument. Note that by comparison with Taylor's formula we have

$$K^{-4} E(\omega) = \sum_{|\alpha| = 3} \frac{3}{\alpha!} \int_0^1 (1-s)^2 (\partial^\alpha_{\omega'} \phi)\big(\frac{s\omega'}{\omega_{n-1}}, 1\big) ds \frac{(\omega')^\alpha}{\omega^2_{n-1}}.$$

For these constant-coefficient operators, Harris's argument [16, Lemma 2.2] of sector counting applies. To apply narrow decoupling for the variable coefficient operator on a small K^2-ball with $K^2 \lesssim \lambda^{\frac{1}{2} - \varepsilon}$, we approximate the variable coefficient phase with a constant coefficient phase. Beltran–Hickman–Sogge [1] worked out that this is possible by Taylor expansion.
We need the following notations: Let ϕ be a reduced phase and $\bar{x} \in \mathbb{R}^n$, which will be the centre of the small ball on which we want to apply decoupling. Recall that $u \mapsto \partial_x \phi^\lambda(\bar{x}; \Psi^\lambda(\bar{x}; u))$ is a graph parametrization of the hypersurface $\Sigma_{\bar{u}}$. We have

$$\langle x, (\partial_x \phi^\lambda)(\bar{x}; \Psi^\lambda(\bar{x}; u)) \rangle = \langle x', u \rangle + x_n h_{\bar{x}}(u)$$

for all $x = (x', x_n) \in \mathbb{R}^n$ with $h_{\bar{z}}(u) = (\partial_{x_n} \phi^\lambda)(\bar{z}; \Psi^\lambda(\bar{z}; u))$. We suppose for technical reasons that $a(x; \omega) = a_1(x) a_2(\omega)$; the general case is reduced to this by Fourier series expansion. Let $E_{\bar{x}}$ denote the extension operator associated to $\Sigma_{\bar{x}}$, given by

$$E_{\bar{x}} g(x) = \int_{\mathbb{R}^{n-1}} e^{i(\langle x', u \rangle + x_n h_{\bar{x}}(u))} a_{\bar{x}}(u) g(u) du \text{ for all } x \in \mathbb{R}^n,$$

where $a_{\bar{x}}(u) = a_2 \circ \Psi^\lambda(\bar{x}; u) |\det \partial_u \Psi^\lambda(\bar{x}; u)|$. We recall how T^λ is approximated by $E_{\bar{x}}$: Let $x \in B(\bar{x}; K^2) \subseteq B(0, 3\lambda/4)$. By change of variables $\omega = \Psi^\lambda(\bar{x}; u)$ and a Taylor expansion of ϕ^λ around $\bar{x}$, we have

$$T^\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{i(\langle x - \bar{x}, (\partial_x \phi^\lambda)(\bar{x}; \Psi^\lambda(\bar{z}; u)) \rangle + \mathcal{E}^\lambda_{\bar{x}}(x - \bar{x}; u))} a_1^\lambda(x) a_{\bar{z}}(u) f_{\bar{z}}(u) du$$

with $f_{\bar{x}} = e^{i\phi^\lambda(\bar{x};\Psi^\lambda(\bar{x};\cdot))} f \circ \Psi^\lambda(\bar{x};\cdot)$ and by Taylor expansion

$$\mathcal{E}^\lambda_{\bar{x}}(v;u) = \frac{1}{\lambda} \int_0^1 (1-r) \langle (\partial_{xx}\phi)((\bar{x}+rv)/\lambda; \Psi^\lambda(\bar{x};u))v, v \rangle dr.$$

By the derivative bounds

$$\sup_{(v;u) \in B(0,K^2) \times \operatorname{supp} a_{\bar{x}}} |\partial^\beta_\omega \mathcal{E}^\lambda_{\bar{x}}(v;u)| \lesssim_N 1$$

and Fourier series expansion, the oscillation of $\mathcal{E}^\lambda_{\bar{x}}$ can be neglected. This yields the following lemma:

Lemma 8.4 ([1, Lemma 2.6])**.** *Let T^λ be an oscillatory integral operator with reduced $C1$) and $C2^+$) phase. Let $0 < \delta \leq 1/2$, $1 \leq K^2 \leq \lambda^{\frac{1}{2}-\delta}$ and $\bar{x}/\lambda \in X$ so that $B(\bar{x};K^2) \subseteq B(0,3\lambda/4)$.*

- *Then*

$$\|T^\lambda f\|_{L^p(w_{B(\bar{x};K^2)})} \lesssim_N \|E_{\bar{x}} f_{\bar{x}}\|_{L^p(w_{B(0;K^2)})} + \lambda^{-\frac{\delta N}{2}} \|f\|_{L^2} \tag{62}$$

  holds provided that N is sufficiently large depending on n, δ, and p.
- *Suppose that $|\bar{x}| \leq \lambda^{1-\delta'}$. There exists a family of operators $\mathbf{T}^\lambda$ all with phase ϕ and of type $(1,1,C)$ data such that*

$$\|E_{\bar{x}} f_{\bar{x}}\|_{L^p(w_{B(0,K^2)})} \lesssim_N \|T^\lambda_* f\|_{L^p(w_{B(\bar{x};K^2)})} + \lambda^{-\frac{N\min(\delta,\delta')}{2}} \|f\|_2 \tag{63}$$

  holds for some $T^\lambda_* \in \mathbf{T}^\lambda$. The family $\mathbf{T}^\lambda$ has cardinality $O_N(1)$ and is independent of $B(\bar{x};K^2)$.

To apply the narrow decoupling to $E_{\bar{x}} f_{\bar{x}}$, we need that the constant coefficient phase

$$h_{\bar{x}}(u) = \partial_{x_n} \phi^\lambda(\bar{x};\Psi^\lambda(\bar{x};u))$$

is K-flat.

Definition 8.5. Let $K \gg 1$. We say that a reduced homogeneous phase $\phi : \mathbb{R}^n \times \mathbb{R}^{n-1}\backslash 0 \to \mathbb{R}$ is K-flat, if all its constant-coefficient approximations $h_{\bar{x}}$ are K-flat and

$$\begin{aligned} |\partial_{x'} \partial^\alpha_\xi \phi| &\lesssim K^{-4} \qquad 2 \leq |\alpha'| \leq N, \\ |\partial_{x_n} \partial^\alpha_\xi \phi| &\lesssim K^{-4} \qquad 3 \leq |\alpha'| \leq N. \end{aligned}$$

The derivative bounds are required to control a change of variables in frequencies. We remark that with this definition, Proposition 8.2 now follows from the constant-coefficient decoupling and the approximation by constant-coefficient operators provided by the previous lemma.

We can give the definition of the inductive quantity now:

Definition 8.6. For $1 \leq p \leq \infty$ and $R \geq 1$ let $Q_{p,\delta}(R)$ denote the infimum over all constants C for which the estimate

$$\|T^\lambda f\|_{L^p(B(0,r))} \leq C \|f\|_{L^p(A^{n-1})}$$

holds for $1 \leq r \leq R$ and all oscillatory integral operators T^λ with reduced $C1$) and $C2^+$) 1-homogeneous phase, which is λ^δ-flat, and all $\lambda \geq R$. Furthermore, we require estimates

$$|\partial^\alpha_\omega a(x;\omega)| \lesssim_N 1$$

for the amplitude function with $0 \leq |\alpha| \leq N$.

Before we turn to the parabolic rescaling, note that by homogeneity,

$$\partial_{x'}\phi(x,\omega) = \sum_{j=1}^{n-1} \omega_j \cdot \partial_{\omega_j}\partial_{x'}\phi(x,\omega). \tag{64}$$

Thus, for each $t \in (-1,1)$ and $\omega \in \mathbb{R}^{n-1}$ the Jacobian determinant of the map $x' \mapsto ((\partial_{\omega'}\phi)(x;\omega), \phi(x;\omega))$ is given by $\omega_{n-1} \cdot \det \partial^2_{\omega x'}\phi(x;\omega)$ and hence, non-vanishing. Let $x = (x'', x_{n-1}, x_n) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}$. The implicit function theorem yields a smooth local inverse mapping $\Upsilon(\cdot, x_n; \omega)$, which satisfies

$$\partial_\omega \phi(\Upsilon(x', x_n;\omega), x_n;\omega) = x'' \text{ and } \phi(\Upsilon(x;\omega), x_n;\omega) = x_{n-1}.$$

Lemma 8.7 (Parabolic rescaling)**.** *Let supp$(f) \subseteq \Xi$ be supported in a ρ^{-1}-plate and ϕ be a reduced phase, that is λ^δ-flat. Then, for any $1 \le \rho \le R \le \lambda$:*

$$\|T^\lambda f\|_{L^p(B(0,R))} \lesssim_{\delta'} R^{\delta'} Q_{p,\delta}(R/\rho)\rho^{\frac{2(n-1)}{p}-(n-2)}\|f\|_{L^p}. \tag{65}$$

The proof combines arguments from [1] and [13]. In [1] the phase after parabolic rescaling was computed, and it was shown how after rescaling we find the bounds for higher derivatives introduced in Section 3.1 to hold, even for arbitrary phases. We shall also see that these phases are λ^δ-flat. Since we need expressions from the computations in [1], some details are repeated.

Proof. Let $\omega \in B_{n-2}(0,1)$ with $(\omega, 1)$ the centre of the ρ^{-1}-plate encasing the support of g:

$$\operatorname{supp}(g) \subseteq \{(\xi', \xi_{n-1}) \in \mathbb{R}^{n-1} : 1/2 \le \xi_{n-1} \le 2 \text{ and } \Big|\frac{\xi'}{\xi_{n-1}} - \omega\Big| \le \rho^{-1}\}.$$

We perform the change of variables

$$(\xi', \xi_{n-1}) = (\eta_{n-1}\omega + \rho^{-1}\eta', \eta_{n-1}),$$

after which follows

$$T^\lambda g(x) = \int_{\mathbb{R}^{n-1}} e^{i\phi^\lambda(x;\eta_{n-1}\omega+\rho^{-1}\eta',\eta_{n-1})} a^\lambda(x;\eta_{n-1}\omega + \rho^{-1}\eta', \eta_{n-1})\tilde{g}(\eta)d\eta,$$

where $\tilde{g}(\eta) = \rho^{-(n-2)}g(\eta_{n-1}\omega+\rho^{-1}\eta', \eta_{n-1})$ and $\operatorname{supp}(\tilde{g}) \subseteq \Xi$. By Taylor expansion and homogeneity of the phase, we find

$$\begin{aligned}\phi(x;\eta_{n-1}\omega + \rho^{-1}\eta', \eta_{n-1}) &= \phi(x;\omega,1)\eta_{n-1} + \rho^{-1}\langle \partial_{\omega'}\phi(x;\omega,1), \eta'\rangle \\ &\quad + \rho^{-2}\int_0^1 (1-r)\langle \partial^2_{\omega'\omega'}\phi(x;\eta_{n-1}\omega + r\rho^{-1}\eta', \eta_{n-1})\eta', \eta'\rangle dr.\end{aligned}$$

Let $\Upsilon_\omega(y', y_n) = (\Upsilon(y', y_n;\omega,1), y_{n-1})$ and $\Upsilon^\lambda_\omega(y', y_n) = \lambda\Upsilon_\omega(y'/\lambda, y_n/\lambda)$ and consider anisotropic dilations

$$D_\rho(y'', y_{n-1}, y_n) = (\rho y'', y_{n-1}, \rho^2 y_n) \text{ and } D'_{\rho^{-1}}(y'', y_{n-1}) = (\rho^{-1}y'', \rho^{-2}y_{n-1})$$

on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. By definition of Υ, we find

$$T^\lambda g \circ \Upsilon^\lambda_\omega \circ D_\rho = \tilde{T}^{\lambda/\rho^2}\tilde{g}$$

where

$$\tilde{T}^{\lambda/\rho^2}\tilde{g}(y) = \int_{\mathbb{R}^{n-1}} e^{i\tilde{\phi}^{\lambda/\rho^2}(y;\eta)}\tilde{a}^\lambda(y;\eta)\tilde{g}(\eta)d\eta$$

for the phase $\tilde{\phi}(y;\eta)$ given by

$$\langle y',\eta\rangle + \int_0^1 (1-r)\langle \partial^2_{\xi'\xi'}\phi(\Upsilon_\omega(D'_{\rho^{-1}}y',y_n);\eta_{n-1}\omega + r\rho^{-1}\eta',\eta_{n-1})\eta',\eta'\rangle dr$$

and the amplitude $\tilde{a}(y;\eta) = a(\Upsilon_\omega(D'_{\rho^{-1}}y',y_n);\eta_{n-1}\omega + \rho^{-1}\eta',\eta_{n-1})$. By a change of spatial variables, we find

$$\|T^\lambda g\|_{L^p(B_R)} \lesssim \rho^{\frac{n}{p}} \|\tilde{T}^{\lambda/\rho^2}\tilde{g}\|_{L^p((\Upsilon^\lambda_\omega \circ D_\rho)^{-1}(B_R))}.$$

We want to apply the induction hypothesis at scale R/ρ^2. First, we make a harmless linear change of variables: Let $L \in GL(n-1;\mathbb{R})$ be such that $Le_{n-1} = e_{n-1}$ and

$$\partial^2_{\eta'\eta'}\partial_{y_n}\tilde{\phi}_L(0,0;e_{n-1}) = I_{n-1},$$

where

$$\tilde{\phi}_L(y;\eta) = \tilde{\phi}(L^{-1}y',y_n;L\eta).$$

It suffices to analyze $\tilde{T}_L^{\lambda/\rho^2}\tilde{g}_L$ with $\tilde{T}_L^{\lambda/\rho^2}$ defined with respect to the datum $(\tilde{\phi}_L,\tilde{a}_L)$ for $\tilde{\phi}_L$ as above, $\tilde{a}_L(y;\eta) = \tilde{a}(L^{-1}y',y_n;L\eta)$ and $\tilde{g}_L = |\det L|\tilde{g}\circ L$.
To see that $\tilde{\phi}_L$ is still a reduced phase, note the representations

$$\tilde{\phi}_L(y;\eta) = \rho^2\phi(\Upsilon_\omega(D'_{\rho^{-1}}\circ L^{-1}y',y_n),y_n;\eta_n\omega + \rho^{-1}L'\eta',\eta_n)$$

and

$$\langle y',\eta\rangle + \int_0^1 (1-r)\langle \partial^2_{\omega'\omega'}\phi(\Upsilon_\omega(D'_{\rho^{-1}}\circ L^{-1}y',y_n);\eta_{n-1}\omega + r\rho^{-1}L'\eta',\eta_{n-1})L'\eta',L'\eta'\rangle dr,$$

where L' denotes the $(n-2)\times(n-2)$-submatrix of L, containing the first $n-2$ rows and columns. In [1] was then shown that, starting with a reduced phase ϕ, that $\tilde{\phi}_L$ is again a reduced phase. For sake of simplicity, suppose that $L = 1$ in the following as taking derivatives only gives additional components of L. For reduced phase functions the components are bounded. We still have to show that it is still λ^δ-flat: Consider the formula

$$\tilde{\phi} = \langle y',\eta\rangle + \int_0^1 (1-r)\langle \partial^2_{\omega'\omega'}\phi(\Upsilon_\omega(y',y_n);\eta_{n-1}\omega + r\rho^{-1}\eta',\eta_{n-1})\eta',\eta'\rangle dr$$

Hence, we find

$$\partial_{x_n}\tilde{\phi} = \int_0^1 (1-r)\sum_{i,j,k}\partial^2_{\omega'_j\omega'_k}\partial_{x_i}\phi(\Upsilon_\omega(D'_{\rho^{-1}}y',y_n);\eta_{n-1}\omega + r\rho^{-1}\eta',\eta_{n-1})\cdot\frac{\partial\Upsilon^i_\omega}{\partial x_n}(\eta'_j)(\eta'_k)dr \tag{66}$$

and for $\Psi(\bar{x};u) = u$, it is straight-forward from taking additional derivatives in η' that the resulting extension operator is indeed λ^δ-flat. Next, we consider

$$h_{\bar{x}}(u) = \partial_{x_n}\tilde{\phi}(\bar{x};\Psi(\bar{x};u)).$$

By definition of $\Psi(\bar{x};u) = (\Psi'(\bar{x};u),\Psi_{n-1}(\bar{x};u))$, we find

$$\begin{aligned}
u_i &= \partial_{x'_i}\tilde{\phi}(\bar{x};\Psi(\bar{x};u)) \\
&= \Psi'_i(\bar{x};u) + \rho^{-1}\int_0^1 (1-r)\sum_{\ell,j,k}\partial_{x'_\ell}\partial^2_{\omega'_j\omega'_k}\phi(\Upsilon_\omega(D'_{\rho^{-1}}y',y_n); \\
&\qquad \Psi_{n-1}\omega + r\rho^{-1}\Psi'(\bar{x};u)\Psi_{n-1}(\bar{x};u))\frac{\partial\Upsilon^\ell_\omega}{\partial x'_i}(D'_{\rho^{-1}}y',y_n)\Psi'_j(\bar{x};u)\Psi'_k(\bar{x};u)dr.
\end{aligned} \tag{67}$$

We find

$$\Psi(\bar{x}; u) = u + f(u), \quad |f(u)| \lesssim \rho^{-1}.$$

This also yields bounds for the derivatives of f in (67) and taking the bounds of Ψ and we see that $h_{\bar{x}}(u)$ is ρ^{-1}-flat. The argument also shows that, if ϕ was already $\tilde{\rho}^{-1}$-flat, in particular, $|\partial_{x'} \partial^2_{\omega'\omega'} \phi| \lesssim \rho^{-1}$ and bounds for higher derivatives, then $\tilde{\phi}$ is $\rho^{-1}\tilde{\rho}^{-1}$-flat. This matches the heuristic that rescaling makes the phase more resemble the translation-invariant case.

Hence, it suffices to show that

$$\|T^\lambda f\|_{L^p(D_{\mathbf{R}})} \lesssim_{\delta'} Q_{p,\delta}(R) R^{\delta'} \|f\|_{L^p}$$

for an ellipse

$$D_{\mathbf{R}} = \{x \in \mathbb{R}^n : \big(\frac{|x'|}{R'}\big)^2 + \big(\frac{|x_n|}{R}\big)^2 \leq 1\}$$

and an oscillatory integral operator with λ^δ-flat phase. This can be argued as in [13, Section 11.2]. □

The narrow decoupling allows to separate the contribution of $T^\lambda f_\tau$ and it remains to estimate $\|T^\lambda f_\tau\|_{L^p(B_R)}$. We are ready for the proof of Proposition 8.1:

Proof of Proposition 8.1. It suffices to prove the linear estimate for p satisfying the additional constraint

$$p(k,n) < p$$

by interpolation. In the first step, for $\lambda \gg 1$, we carry out a parabolic rescaling depending on the phase such that it is enough to consider $\lambda^{\tilde{\delta}}$-flat phase functions. This loses a factor $C_\phi \lambda^{O(n)\tilde{\delta}}$ by partitioning Ξ into sectors, which will be admissible provided that

$$\lambda^{O(n)\tilde{\delta}} \leq \lambda^{\varepsilon}. \tag{68}$$

Let $\varepsilon > 0$. By the assumed k-broad estimate, we find

$$\sum_{\substack{B_{K^2} \in \mathcal{B}_{K^2}, \\ B_{K^2} \cap B(0,R) \neq \emptyset}} \min_{V_1,\dots,V_A} \max_{\tau \notin V_a} \int_{B_{K^2}} |T^\lambda f_\tau|^p \leq \tilde{C}_\varepsilon K^{C_\varepsilon} R^{\frac{p\varepsilon}{2}} \|f\|^p_{L^p(A^{n-1})}, \tag{69}$$

where V_1,...,V_A are $(k-1)$-planes and $\tau \notin V_a$ is short-hand for

$$\angle(G^\lambda(\bar{x};\tau), V_a) > K^{-2},$$

with $\bar{x}$ being centre of B_{K^2}.

We choose V_1,...,V_A for each B_{K^2}, which attains the minimum in (69). By this, we may write

$$\int_{B_{K^2}} |T^\lambda f|^p \lesssim K^{O(n)} \max_{\tau \notin V_a} \int_{B_{K^2}} |T^\lambda f_\tau|^p + \sum_{a=1}^A \int_{B_{K^2}} \big| \sum_{\tau \in V_a} T^\lambda f_\tau \big|^p.$$

By summing over B_{K^2} and using (69), we find

$$\int_{B(0,R)} |T^\lambda f|^p \lesssim K^{O(n)} \tilde{C}_\varepsilon K^{C_\varepsilon} R^{p\varepsilon/2} \|f\|^p_{L^p} + \sum_{\substack{B_{K^2} \in \mathcal{B}_{K^2}, \\ B_{K^2} \cap B(0,R) \neq \emptyset}} \sum_{a=1}^A \int_{B_{K^2}} \big| \sum_{\tau \in V_a} T^\lambda f_\tau \big|^p.$$

By the decoupling result Proposition 8.2, we find for any $\delta' > 0$, provided that $K \leq \lambda^{\tilde{\delta}}$,

$$\int_{B_{K^2}} \big|\sum_{\tau \in V_a} T^\lambda f_\tau\big|^p \lesssim_{\delta'} K^{\max((k-3)(\frac{p}{2}-1),0)+\delta'} \sum_{\tau \in V_a} \int_{\mathbb{R}^n} |T^\lambda f_\tau|^p w_{B_{K^2}}$$

and summing over a and B_{K^2}, we find

(70)
$$\sum_{B_{K^2} \in \mathcal{B}_{K^2}} \sum_{a=1}^{A} \int_{B_{K^2}} \big|\sum_{\tau \in V_a} T^\lambda f_\tau\big|^p \lesssim_{\delta'} K^{\max((k-3)(p/2-1),0)+\delta'} \sum_{\tau : K^{-1}} \int_{B(0,2R)} |T^\lambda f_\tau|^p.$$

The separated expressions $T^\lambda f_\tau$ are amenable to Lemma 8.7 and an application gives

$$\int_{B(0,2R)} |T^\lambda f_\tau|^p \lesssim_\delta (Q_{p,\tilde{\delta}}(R))^p R^\delta K^{2(n-1)-(n-2)p} \|f_\tau\|^p_{L^p}. \tag{71}$$

Plugging (71) into (70), we find

$$\int_{B(0,R)} |T^\lambda f|^p \leq (K^{O(n)} \tilde{C}_\varepsilon K^{C_\varepsilon} R^{p\varepsilon/2} + C_{\delta,\delta'} (Q_{p,\tilde{\delta}}(R))^p R^\delta K^{-e(k,p)+\delta'}) \|f\|^p_{L^p(A^{n-1})}.$$

This yields

$$(Q_{p,\tilde{\delta}}(R))^p \leq K^{O(n)} \tilde{C}_\varepsilon K^{C_\varepsilon} R^{p\varepsilon/2} + C_{\delta,\delta'} (Q_{p,\tilde{\delta}}(R))^p R^\delta K^{-e(k,p)+\delta'}.$$

Since p is as in (59), we find $e(k,p) > 0$, and may choose $\delta' = e(k,p)/2$, so that the K exponent in the second term on the right-hand side is negative. Moreover, we can choose δ small enough such that $\frac{2\delta}{e(k,p)} C_\varepsilon \leq \frac{p\varepsilon}{8}$ and $\frac{O(n)\delta}{2e(k,p)} \leq \frac{\varepsilon}{8}$. This ensures for the first term on the right-hand side:

$$K^{O(n)} K^{C_\varepsilon} \leq \tilde{D}_\varepsilon R^{\frac{3p\varepsilon}{4}}.$$

Thus, if $K = K_0 R^{\frac{2\delta}{e(k,p)}}$ for a sufficiently large K_0, depending on ε, $\delta = \delta(\varepsilon)$, p and n, it follows that

$$(Q_{p,\tilde{\delta}}(R))^p \leq \tilde{D}_\varepsilon R^{\frac{3p\varepsilon}{4}} + \frac{1}{2} (Q_{p,\tilde{\delta}}(R))^p.$$

By choosing $\tilde{\delta} = \frac{3\delta}{e(k,p)}$ and $\lambda \geq E(\varepsilon)$ such that $\lambda^{\tilde{\delta}} \geq K_0 \lambda^{\frac{2\delta}{e(k,p)}}$, the proof is complete because (68) is ensured by $\frac{O(n)\delta}{2e(k,p)} \leq \frac{\varepsilon}{8}$. □

9. ε-removal away from the endpoint

In the following we prove the estimate

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim_{\phi,a} \|f\|_{L^p(A^{n-1})} \tag{72}$$

for $p > p_n$ with p_n defined in (7). The argument is essentially well-known in the literature [34, 35, 13] and we shall be brief. The detailed argument from [13] cannot be applied directly because it relies on non-degenerate curvature properties $H2)$ of the phase function. However, we shall see that the partial non-degeneracy

$$\exists \text{non-vanishing eigenvalue of } \partial^2_{\omega\omega} \langle \partial_x \phi^\lambda(x;\omega), G^\lambda(x;\omega_0)\rangle|_{\omega=\omega_0} \tag{73}$$

suffices for the argument. In the following we suppose that the phase ϕ satisfies the non-degeneracy $C1$) and (73). We shall prove that, if for $\bar{p} \geq 2$ and for all $\varepsilon > 0$ the estimate

$$\|T^\lambda f\|_{L^p(B_R)} \lesssim_{\varepsilon,\phi,a} R^\varepsilon \|f\|_{L^p(A^{n-1})} \tag{74}$$

holds for all $p \geq \bar{p}$, all R-balls B_R, and any amplitude, then we find the global estimate (72) to hold for all $p > \bar{p}$. The following notion plays an important role in the argument:

Definition 9.1 (Tao [35])**.** Let $R \geq 1$. A collection $\{B(x_j, R)\}_{j=1}^N$ of R-balls in $\mathbb{R}^d$ is sparse if $\{x_1, \dots, x_N\}$ are $(RN)^{\bar{C}}$-separated. Here $\bar{C} \geq 1$ is a fixed constant, chosen large enough to satisfy the requirements of the forthcoming argument.

Like in previous instances of the argument, we are reduced to the analysis of sparse families of balls.

Lemma 9.2 ([13, Lemma 12.2])**.** *To prove* (72) *for all $p > \bar{p}$, it suffices to show that for all $\varepsilon > 0$ the estimate*

$$\|T^\lambda f\|_{L^{\bar{p}}(S)} \lesssim_{\varepsilon,\phi,a} R^\varepsilon \|f\|_{L^{\bar{p}}(A^{n-1})} \tag{75}$$

holds whenever $R \geq 1$ and $S \subseteq \mathbb{R}^n$ is a union of R-balls belonging to a sparse collection, for any choice of amplitude function.

The key ingredient in the proof of Lemma 9.2 is the following covering lemma due to Tao [34]:

Lemma 9.3 (Covering lemma, [34, 35])**.** *Suppose that $E \subseteq \mathbb{R}^n$ is a finite union of 1-cubes and $N \geq 1$. Define the radii R_j inductively by $R_0 = 1$ and $R_j = (R_{j-1}|E|)^{\bar{C}}$ for $1 \leq j \leq N-1$. Then, for each $0 \leq j \leq N-1$, there exists a family of sparse collections $(\mathcal{B}_{j,\alpha})_{\alpha \in A_j}$ of balls of radius R_j such that the index sets A_k have cardinality $O(|E|^{1/N})$ and*

$$E \subseteq \bigcup_{j=0}^{N-1} \bigcup_{\alpha \in A_j} S_{j,\alpha},$$

where $S_{j,\alpha}$ is the union of all the balls belonging to the family $\mathcal{B}_{j,\alpha}$.

With Lemma 9.3 at hand, the proof of Lemma 9.2 from [13] applies. It remains to establish the estimates for T^λ over sparse collections of R-balls.

Lemma 9.4. *Under the above hypotheses, if $p \geq \bar{p}$, then the estimate*

$$\|T^\lambda f\|_{L^p(S)} \lesssim_{\varepsilon,\phi,a} R^\varepsilon \|f\|_{L^p}$$

holds for all $\varepsilon > 0$ whenever $S \subseteq \mathbb{R}^n$ is a union of R-balls belonging to a sparse collection.

Proof. Let $(B(x_j, R))_{j=1}^N$ be a sparse collection of balls. We can suppose that $R \ll \lambda$ and that all $B(x_j, R)$ intersect the x-support of a^λ. Furthermore, letting $c_{diam} > 0$ be a small constant $\mathrm{diam} X < c_{diam}$ so that

$$\frac{|x_{j_1} - x_{j_2}|}{\lambda} \lesssim c_{diam} \text{ for all } 1 \leq j_1, j_2 \leq N.$$

Fix $\eta \in C_c^\infty(\mathbb{R}^{n-1})$ satisfying $0 \leq \eta \leq 1$, $\mathrm{supp}(\eta) \subseteq B^{n-1}$ and $\eta(z) = 1$ for all $z \in B(0, 1/2)$. For $R_1 := CNR$, where $C \geq 1$ is a large constant, define

$\eta_R(z) = \eta(z/R_1)$. Let $\psi \in C_c^\infty(\mathbb{R}^{n-1})$ satisfy $0 \leq \psi \leq 1$, $\mathrm{supp}(\psi) \subseteq \Omega$ and $\psi(\omega) = 1$ for ω belonging to the ω-support of a^λ. Fix $1 \leq j \leq N$ and write

$$e^{i\phi^\lambda(x_j;\cdot)}\psi f = P_j f + (e^{i\phi^\lambda(x_j;\cdot)}\psi f - P_j f) =: P_j f + f_{j,\infty},$$

where $P_j f = \hat{\eta}_{R_1} * [e^{i\phi^\lambda(x_j;\cdot)}\psi f]$. If one defines

$$\mathrm{Err}(x) = \int_{\mathbb{R}^{n-1}} e^{i(\phi^\lambda(x;\omega)-\phi^\lambda(x_j;\omega))} a^\lambda(x;\omega) f_{j,\infty}(\omega) d\omega,$$

then it follows that

$$T^\lambda f(x) = T^\lambda[e^{-i\phi^\lambda(x_j;\cdot)} P_j f](x) + \mathrm{Err}(x).$$

For $x \in B(x_j;R)$, the term $\mathrm{Err}(x)$ is negligible. By Plancherel's theorem,

$$\mathrm{Err}(x) = \int \overline{G_x^\vee(z)} \cdot (1-\eta_{R_1}(z))[e^{i\phi^\lambda(x_j;\cdot)}\psi f]^\vee(z) dz,$$

where

$$G_x^\vee(z) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i(\langle z,\omega\rangle - \phi^\lambda(x;\omega)+\phi^\lambda(x_j;\omega))} a^\lambda(x;\omega) d\omega.$$

Taking the ω-derivatives of the phase of $G_x^\vee(z)$, one obtains

$$\begin{aligned} &z - \lambda\big(\partial_\omega \phi(\frac{x}{\lambda};\omega) - \partial_\omega \phi(\frac{x_j}{\lambda};\omega)\big) = z + O(R) \\ &\Rightarrow -\lambda\big(\partial_\omega^\alpha \phi(\frac{x}{\lambda};\omega) - \partial_\omega^\alpha \phi(\frac{x_j}{\lambda};\omega)\big) = O(R) \text{ for } |\alpha| \geq 2. \end{aligned}$$

Hence, if z belongs to the support of $1-\eta_{R_1}$, then integration by parts shows that $G_x(z)$ is rapidly decaying in R_1, and we find

$$|\mathrm{Err}(x)| \leq \mathrm{RapDec}(R_1)\|f\|_{L^p}.$$

By applying the estimate for T^λ with R^ε-loss over each ball $B(x_j;R)$, one obtains

$$\begin{aligned} \|T^\lambda f\|_{L^p(S)} &\leq \big(\sum_{j=1}^N \|T^\lambda[e^{-i\phi^\lambda(x_j;\cdot)}P_j f]\|^p_{L^p(B(x_j;R))}\big)^{\frac{1}{p}} \\ &\lesssim_{\varepsilon,a,\phi} R^\varepsilon \big(\sum_{j=1}^N \|P_j f\|^p_{L^p(A^{n-1})}\big)^{\frac{1}{p}} + \|f\|_{L^p}. \end{aligned}$$

Thus, it suffices to show that

$$\big(\sum_{j=1}^N \|P_j f\|^p_{L^p}\big)^{\frac{1}{p}} \lesssim \|f\|_{L^p}.$$

This follows via interpolation between $p=2$ and $p=\infty$. For $p=\infty$, this is a consequence of Young's inequality. The estimate for $p=2$ is by duality equivalent to

$$\big\|\sum_{j=1}^N e^{-2\pi i\phi^\lambda(x_j;\cdot)}\psi \cdot [\hat{\eta}_{R_1} * g_j]\big\|_{L^2(\mathbb{R}^{d-1})} \lesssim \big(\sum_{j=1}^N \|g_j\|^2_{L^2}\big)^{\frac{1}{2}}.$$

By squaring the left-hand side, we find

$$\sum_{j_1,j_2=1}^N \int_{\mathbb{R}^{d-1}} \overline{G_{j_1,j_2}(\omega)} \hat{\eta}_{R_1} * g_{j_1}(\omega) \overline{\hat{\eta}_{R_1} * g_{j_2}(\omega)} d\omega,$$

where

$$G_{j_1,j_2}(\omega) = e^{i(\phi^\lambda(x_{j_1};\omega)-\phi^\lambda(x_{j_2};\omega))}\psi^2(\omega).$$

Plancherel's theorem yields

(76)
$$\int_{\mathbb{R}^{n-1}} \overline{G_{j_1,j_2}(\omega)}\hat{\eta}_{R_1} * g_{j_1}(\omega)\overline{\hat{\eta}_{R_1} * g_2(\omega)}d\omega = \int_{\mathbb{R}^{n-1}} \overline{G^{\vee}_{j_1,j_2}(z)}(\eta_{R_1}\check{g}_{j_1}) * (\eta_{R_1}\check{g}_{j_2})\tilde{\ }(z)dz.$$

Here $(\eta_{R_1}\check{g}_{j_2})\tilde{\ }(z) = \overline{\eta_{R_1}\check{g}_{j_2}(-z)}$. Fix $1 \leq j_1, j_2 \leq N$ with $j_1 \neq j_2$, let $z \in \mathbb{R}^{n-1}$ with $|z| \lesssim R_1 < |x_{j_2} - x_{j_1}|$ and consider

$$G^{\vee}_{j_1,j_2}(z) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i(\langle z,\omega\rangle + \phi^\lambda(x_{j_1};\omega) - \phi^\lambda(x_{j_2};\omega))}\psi^2(\omega)d\omega.$$

For $\alpha \in \mathbb{N}^{n-1}$ with $|\alpha| \leq 2$, consider

$$\partial^\alpha_\omega[\phi^\lambda(x_{j_1};\omega) - \phi^\lambda(x_{j_2};\omega)] = \partial^\alpha_\omega\langle\partial_x\phi^\lambda(x_{j_1};\omega), x_{j_2} - x_{j_1}\rangle + O(c_{\text{diam}}|x_{j_2} - x_{j_1}|).$$

Let $c_{crit} > 0$ be a small constant, chosen to satisfy the further needs of the arguments, and $\omega_0 \in \Omega$. Suppose that

$$\big| \pm \frac{x_{j_2} - x_{j_1}}{|x_{j_2} - x_{j_1}|} - G^\lambda(x_{j_1};\omega_0)\big| \geq c_{crit}. \tag{77}$$

The non-degeneracy $C1$) implies that the vector $G^\lambda(x;\omega_0)$ spans the kernel of $\partial^2_{\omega x}\phi^\lambda(x;\omega_0)$. This yields, in case of (77),

$$|\partial_\omega[\langle\partial_x\phi^\lambda(x_{j_1};\omega), x_{j_2} - x_{j_1}\rangle|_{\omega=\omega_0}| \gtrsim |x_{j_2} - x_{j_1}|$$

and consequently,

$$|\partial_\omega[\phi^\lambda(x_{j_1};\omega) - \phi^\lambda(x_{j_2};\omega)]|_{\omega=\omega_0}| \gtrsim |x_{j_2} - x_{j_1}|.$$

Then, rapid decay of $\check{G}_{j_1,j_2}$ follows by integration by parts. If (77) fails, then

$$\partial^\alpha_\omega\langle\partial_x\phi^\lambda(x_{j_1};\omega), \frac{x_{j_2} - x_{j_1}}{|x_{j_2} - x_{j_1}|}\rangle|_{\omega=\omega_0} = \partial^\alpha_\omega\langle\partial_x\phi^\lambda(x_{j_1};\omega), G^\lambda(x_{j_1};\omega_0)\rangle|_{\omega=\omega_0} + O(c_{crit}).$$

Hence, by a Van der Corput-argument [31, Proposition 5, p. 342] we still find due to (73)

$$|\check{G}_{j_1,j_2}(z)| \lesssim |x_{j_2} - x_{j_1}|^{-\frac{1}{2}}.$$

This yields the estimate for the absolute value of (76):

$$\begin{aligned}|(76)| \lesssim R_1^{-\bar{C}/2}\|(\eta_{R_1}\check{g}_{j_1}) * (\eta_{R_1}\check{g}_{j_2})\tilde{\ }\|_{L^1(\mathbb{R}^{n-1})} &\lesssim R_1^{-\bar{C}/2}\prod_{j=1}^2 \|\eta_{R_1}\check{g}_{j_i}\|_{L^1(\mathbb{R}^{n-1})} \\ &\lesssim R_1^{-\bar{C}/2+n-1}\prod_{j=1}^2 \|g_{j_i}\|_{L^2(\mathbb{R}^{n-1})}.\end{aligned}$$

Since there are only $O(N^2)$ choices of indices j_1 and j_2 and $R_1 = CRN$, the trivial estimate

$$\prod_{i=1}^2 \|g_{j_i}\|_{L^2} \lesssim \sum_{j=1}^N \|g_j\|^2_{L^2}$$

suffices to sum the off-diagonal terms. The diagonal contribution is estimated by

$$\big(\sum_{j=1}^N \|\hat{\eta}_{R_1} * g_j\|^2_{L^2(A^{n-1})}\big)^{1/2} \lesssim \big(\sum_{j=1}^N \|g_j\|^2_{L^2(A^{n-1})}\big)^{1/2}.$$

The proof is complete. $\square$

10. Improved local smoothing for Fourier integral operators

In this section we improve L^p-smoothing estimates for solutions to wave equations on compact Riemannian manifolds (M,g) with $\dim(M) \geq 3$: We consider

$$\begin{cases} \partial_t^2 u - \Delta_g u &= 0, \quad (t,x) \in \mathbb{R} \times M, \\ u(\cdot,0) &= f_0, \quad \dot{u}(\cdot,0) = f_1 \end{cases} \tag{78}$$

with the solution u to (78) given by

$$u(t) = \cos(t\sqrt{-\Delta_g}) f_0 + \frac{\sin(t\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}} f_1.$$

Parametrices for the half-wave equation are provided by Fourier integral operators (FIOs); see below. By results due to Seeger–Sogge–Stein [27] relying on the parametrix representation (see also [26, 23] in the Euclidean case), it is known that the fixed-time estimate

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)} \lesssim \|f_0\|_{L^p_{\bar{s}_p}(\mathbb{R}^d)} + \|f_1\|_{L^p_{\bar{s}_p - 1}(\mathbb{R}^d)}$$

with

$$\bar{s}_p = (d-1)\big|\frac{1}{2} - \frac{1}{p}\big| \tag{79}$$

is sharp for all $1 < p < \infty$ provided that t avoids a discrete set. The local smoothing conjecture due to C. Sogge [28] for the Euclidean wave equation, i.e., $(M,g) = (\mathbb{R}^d, (\delta^{ij}))$ in (78), states that

$$\big(\int_1^2 \|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \big)^{\frac{1}{p}} \lesssim \|f_0\|_{L^p_{\bar{s}_p - \sigma}(\mathbb{R}^d)} + \|f_1\|_{L^p_{\bar{s}_p - 1 - \sigma}(\mathbb{R}^d)} \tag{80}$$

for $\sigma < \frac{1}{p}$ and $\frac{2d}{d-1} \leq p < \infty$. (Note that $\bar{s}_p - \frac{1}{p} = 0$ for $p = \frac{2d}{d-1}$.) This conjecture stands on top of prominent open problems in Harmonic Analysis as it implies as well the restriction conjecture as the Bochner–Riesz conjecture. Initial progress was due to Sogge [28] and Mockenhaupt–Seeger–Sogge [24]. Wolff identified decoupling inequalities [37] to yield sharp local smoothing estimates. Further progress in this direction was made in [10, 19, 21]. Bourgain–Demeter [5] covered the sharp range for decoupling inequalities, which implies sharp local smoothing estimates for $p \geq \frac{2(d+1)}{d-1}$. We refer to the survey by Beltran–Hickman–Sogge [2] for local smoothing estimates for FIOs. Guth–Wang–Zhang [15] verified the Euclidean local smoothing conjecture for $d = 2$ by a sharp L^4-square function estimate. Gao *et al.* [8] extended this to compact Riemannian surfaces. We remark that for $d \geq 3$, counterexamples due to Minicozzi–Sogge [22] show that (80) fails if one replaces $\mathbb{R}^d$ with general compact Riemannian manifolds for $\sigma < 1/p$, if $p < p_{d,+}$ with

$$p_{d,+} = \begin{cases} \frac{2\cdot(3d+1)}{3d-3}, & \text{if } d \text{ is odd}, \\ \frac{2\cdot(3d+2)}{3d-2}, & \text{if } d \text{ is even}. \end{cases} \tag{81}$$

Hence, local smoothing estimates for solutions to wave equations on compact Riemannian manifolds are only conjectured for $p \geq p_{d,+}$ with $\sigma < 1/p$.

Gao *et al.* [9] also improved the Euclidean local smoothing estimates for $d \geq 3$ and $2 \leq p \leq \frac{2(d+1)}{d-1}$ due to Bourgain–Demeter by a broad–narrow iteration.

Presently, we extend their arguments to the variable coefficient case. Let $d \geq 3$ and

$$p_d = \begin{cases} 2 \cdot \frac{3d+5}{3d+1} \text{ for } d \text{ odd,} \\ 2 \cdot \frac{3d+6}{3d+2} \text{ for } d \text{ even.} \end{cases} \tag{82}$$

We show the following:

Theorem 10.1 (Improved local smoothing on compact manifolds)**.** *Let (M,g) be a compact Riemannian manifold with $\dim(M) \geq 3$. Let $\bar{s}_p$ be as in (79), $p_d \leq p < \infty$ with p_d as in (82) and $\sigma < \frac{2}{p} - \frac{1}{2}$. Let u be a solution to (78). Then, we find the following estimate to hold:*

$$\|u\|_{L^p_t([1,2],L^p_x(M))} \lesssim_{M,g,p,\sigma} \|f_0\|_{L^p_{\bar{s}_p-\sigma}(M)} + \|f_1\|_{L^p_{\bar{s}_p-\sigma+1}(M)}. \tag{83}$$

We can interpolate with the trivial L^2-estimate and the sharp local smoothing estimates for $p \geq \frac{2(d+1)}{d-1}$ due to Beltran–Hickman–Sogge [1] to find a broader range of estimates (cf. [9, Corollary 1.3]):

Corollary 10.2. *Let $d \geq 3$ and u be a solution to (78). Then, (83) holds true for $\sigma < \sigma_p$, where, if $d \geq 3$ is odd,*

$$\sigma_p = \begin{cases} \frac{3d-3}{4}\big(\frac{1}{2} - \frac{1}{p}\big), \quad 2 < p \leq p_d, \\ \frac{d-1}{d+3}\big(\frac{3d+1}{6d+10} - \frac{1}{p}\big) + \frac{3d-3}{6d+10}, \quad p_d < p \leq \frac{2(d+1)}{d-1}, \end{cases}$$

and, if $d \geq 3$ is even,

$$\sigma_p = \begin{cases} \frac{3d-2}{4}\big(\frac{1}{2} - \frac{1}{p}\big), \quad 2 < p \leq p_d, \\ \frac{d-2}{d+4}\big(\frac{3d+2}{6d+12} - \frac{1}{p}\big) + \frac{3d-2}{6d+12}, \quad p_d < p \leq \frac{2(d+1)}{d-1}. \end{cases}$$

It is well-known (cf. [30, Chapter 4], [22, p. 224]) that local parametrices for (78) take the form of FIOs

$$(\mathcal{F}f)(x,t) = \int_{\mathbb{R}^d} e^{i\phi(x,t;\xi)} a(x,t;\xi) \hat{f}(\xi) d\xi \tag{84}$$

with phase functions $\phi \in C^\infty(\mathbb{R}^{d+1} \times \mathbb{R}^d \backslash 0)$, which are 1-homogeneous in ξ and satisfy $C1$) and $C2^+$). $a \in S^0(\mathbb{R}^{2d+1})$ is a symbol of order zero, compactly supported in (x,t).

It turns out that for the proof of Theorem 10.1, it suffices to prove bounds for rescaled operators

$$(\mathcal{F}^\lambda f)(x,t) = \int_{\mathbb{R}^d} e^{i\phi^\lambda(x,t;\xi)} a^\lambda(x,t;\xi) \hat{f}(\xi) d\xi \tag{85}$$

with a^λ and ϕ^λ defined like in previous sections. Theorem 10.1 is a consequence of the following (cf. [1, Section 3]):

Proposition 10.3. *Let $\mathcal{F}$ be an FIO as in (84) and p_d as in (82). Then, we find the following local smoothing estimate to hold for $p_d \leq p < \infty$:*

$$\|\mathcal{F}^\lambda f\|_{L^p_{t,x}(\mathbb{R}^{d+1})} \lesssim_{\varepsilon,\phi,a} \lambda^{d\left(\frac{1}{2}-\frac{1}{p}\right)+\varepsilon} \|f\|_{L^p(\mathbb{R}^d)}. \tag{86}$$

Proposition 10.3 improves on the previously best estimates due to Beltran–Hickman–Sogge [1], which read

$$\|\mathcal{F}^\lambda f\|_{L^p_{t,x}(\mathbb{R}^{d+1})} \lesssim_{\varepsilon,\phi,a} \lambda^{\frac{(d-1)}{2}\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{1}{p}+\varepsilon} \|f\|_{L^p(\mathbb{R}^d)}$$

for $2 \leq p \leq \frac{2(d+1)}{d-1}$. Beltran–Hickman–Sogge [1] extended the decoupling inequalities in the constant coefficient case [5] to variable coefficients. This argument also yields local smoothing estimates for FIOs, which do not satisfy the convexity condition $C2^+$). Indeed, the FIOs, for which decoupling yields the sharp smoothing estimates (cf. [1, Section 4]), are the ones with d odd, and

$$\partial^2_{\xi\xi} \langle \partial_x \phi(x,t;\xi), G_0(x,t;\xi) \rangle$$

having $\frac{d-1}{2}$ positive and $\frac{d-1}{2}$ negative eigenvalues.

For the proof of Proposition 10.3, we run almost the same iteration as in the proof of Theorem 1.1. The following lemma based on finite speed of propagation allows to convert L^2-estimates for T^λ into L^p-estimates for $\mathcal{F}^\lambda$:

Lemma 10.4. *Let (ϕ, a) be reduced data and $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $supp(\hat{\psi}) \subseteq B(0,1)$, $\sum_{\ell \in \mathbb{Z}^d} \psi(x - \ell) \equiv 1$ for any $x \in \mathbb{R}^d$. Assume $supp(\hat{f}) \subseteq A^d$. Then, for any $\varepsilon > 0$, the following estimate holds true:*

$$\begin{aligned} |\mathcal{F}^\lambda f(x,t)| &\lesssim_\varepsilon |\mathcal{F}^\lambda (\psi_{R^{1+\varepsilon}(x_0)} f)(x,t)| \\ &\quad + RapDec(R) \sum_{|\ell| > R^\varepsilon} (1+|\ell|)^{-M} \| f |\psi_\ell(\cdot - x_0)|^{\frac{1}{2}} \|_{L^p(w_{B^d_R(x_0)})} \end{aligned} \tag{87}$$

for $(x,t) \in B(x_0,R) \times [-R,R]$, $1 < p < \infty$, where

$$\psi_{R^{1+\varepsilon}(x_0)}(x) = \sum_{|\ell| < R^\varepsilon} \psi(R^{-1}(x - x_0) - \ell).$$

Proof. The claim follows from a kernel estimate. We have

$$\begin{aligned} \mathcal{F}^\lambda(x,t) &= \int e^{i\phi^\lambda(x,t;\xi)} a^\lambda(x,t;\xi) \hat{f}(\xi) d\xi \\ &= \frac{1}{(2\pi)^d} \int e^{i(\phi^\lambda(x,t;\xi) - \langle y, \xi \rangle)} a^\lambda(x,t;\xi) f(y) dy d\xi. \end{aligned}$$

We set $K^\lambda(x,t;y) = \int e^{i(\phi^\lambda(x,t;\xi) - \langle y,\xi \rangle)} a^\lambda(x,t;\xi) d\xi$. Let $\Phi^\lambda(x,y,\xi,t) = \phi^\lambda(x,t;\xi) - \langle y, \xi \rangle$. We have

$$\nabla_\xi \Phi^\lambda(x,y,\xi,t) = \nabla_\xi \phi^\lambda(x,t;\xi) - y.$$

For a reduced phase function, we have

$$\nabla_\xi \phi^\lambda(x,0;\xi) = x, \quad \nabla_\xi \phi^\lambda(x,t;\xi) = \frac{1}{\lambda} \nabla_\xi \int_0^{\lambda t} \partial_t \phi^\lambda(x,s;\xi) ds + x.$$

By $|\nabla_\xi \partial_t \phi(x,t;\xi)| \lesssim 1$ for a reduced phase function, we find for $|t| \leq R$ and $|x - y| \geq R^{1+\varepsilon}$ rapid decay by non-stationary phase. We have the estimate

$$|K^\lambda(x,t;y)| \leq C_N (1 + R|x-y|)^{-N}.$$

Provided that $|\partial^\alpha_\xi a| \leq c_{par}$ for $0 \leq |\alpha| \leq N$ and reduced phase functions, C_N can be chosen uniformly. By this, we find (87) to hold. $\square$

By the same arguments as in Section 8, we can show the following narrow decoupling:

Proposition 10.5. *Let $B_{K^2} \subseteq B(0, \lambda^{1-\delta'})$ be a K^2-ball. Let (ϕ, a) be a K-flat datum. Let $k \geq 3$ and V be a $(k-1)$-dimensional vector space. Suppose that*

$supp(f) \subseteq \bigcup_\nu S_\nu$ be a union of K^{-1}-slabs such that $\angle(G^\lambda(\bar{x};\tau),V) \leq K^{-2}$. Then, we find the following estimate to hold:

$$\|T^\lambda f\|_{L^p(B^{d+1}_{K^2})} \lesssim_\delta K^\delta \big(\sum_\nu \|T^\lambda f_\nu\|^2_{L^p(w_{B^{d+1}_{K^2}})} \big)^{1/2} + \lambda^{-\frac{\min(\delta,\delta')N}{2}} \|f\|_{L^2} \tag{88}$$

for $2 \leq p \leq \frac{2(k-1)}{k-3}$.

Like in Proposition 8.2, the amplitude functions on the right-hand side are slightly different, but satisfy uniform bounds. The discrepancy will be hidden in the induction hypothesis again.

As further ingredient we use the following Lorentz rescaling for FIOs. Let $Q_{p,\delta}(R)$ be the infimum over all constants such that

$$\|\mathcal{F}^\lambda f\|_{L^p(B(0,R))} \leq Q_{p,\delta}(R) R^{d\left(\frac{1}{2}-\frac{1}{p}\right)} \|f\|_{L^p}$$

for $1 \leq R \leq \lambda$ and all FIOs $\mathcal{F}$ with λ^δ-flat phase functions and amplitude functions, which satisfy

$$|\partial^\alpha_\xi a(x,t;\xi)| \leq c_{par}$$

for $0 \leq |\alpha| \leq N$.

Lemma 10.6 (Lorentz rescaling for FIOs)**.** *Let (ϕ,a) be reduced data with ϕ a λ^δ-flat phase function and $\hat{f}$ supported in a ρ^{-1}-slab in A^d. Then, for any $1 \leq \rho \leq R \leq \lambda$:*

$$\|\mathcal{F}^\lambda g\|_{L^p(B(0,R))} \lesssim_{\delta'} R^{\delta'} Q_{p,\delta}(R/\rho^2) R^{d\left(\frac{1}{2}-\frac{1}{p}\right)} \rho^{\frac{2(d+1)}{p}-d} \|g\|_{L^p}. \tag{89}$$

Proof. The proof has much in common with the proof of Lemma 8.7. However, after rescaling, we use almost orthogonality in space-time, which comes from finite speed of propagation (cf. Lemma 10.4). Let $\omega \in B_{d-1}(0,1)$ with $(\omega,1)$ the centre of the ρ^{-1}-slab encasing the support of $\hat{g}$:

$$\text{supp}(\hat{g}) \subseteq \{(\xi',\xi_d) \in \mathbb{R}^d \, : \, 1/2 \leq \xi_d \leq 2 \text{ and } \big|\frac{\xi'}{\xi_d} - \omega\big| \leq \rho^{-1}\}.$$

We perform the change of variables:

$$(\xi',\xi_d) = (\eta_d\omega + \rho^{-1}\eta', \eta_d),$$

after which follows

$$(\mathcal{F}^\lambda g)(x,t) = \int_{\mathbb{R}^d} e^{i\phi^\lambda(x,t;\eta_d\omega+\rho^{-1}\eta',\eta_d)} a^\lambda(x,t;\eta_d\omega+\rho^{-1}\eta',\eta_d)\hat{\tilde{g}}(\eta) d\eta,$$

where $\hat{\tilde{g}}(\eta) = \rho^{-(d-1)}\hat{g}(\eta_d\omega+\rho^{-1}\eta',\eta_d)$ and $\text{supp}(\hat{\tilde{f}}) \subseteq \Xi$. By Taylor expansion and homogeneity of the phase, we find

$$\begin{aligned}\phi(x,t;\eta_d\omega+\rho^{-1}\eta',\eta_d) &= \phi(x,t;\omega,1)\eta_d + \rho^{-1}\langle \partial_{\xi'}\phi(x,t;\omega,1),\eta'\rangle \\ &\quad + \rho^{-2}\int_0^1 (1-r)\langle \partial^2_{\xi'\xi'}\phi(x,t;\eta_d\omega+r\rho^{-1}\eta',\eta_d)\eta',\eta'\rangle dr.\end{aligned}$$

Let $\Upsilon_\omega(x,t) = (\Upsilon(x,t;\omega,1),x_d)$ and $\Upsilon^\lambda_\omega(x,t) = \lambda\Upsilon_\omega(x/\lambda,t/\lambda)$ and consider anisotropic dilations

$$D_\rho(x',x_d,t) = (\rho x', x_d, \rho^2 t) \text{ and } D'_{\rho^{-1}}(x',x_d) = (\rho^{-1}x', \rho^{-2}x_d)$$

on $\mathbb{R}^{d+1}$ and $\mathbb{R}^d$, respectively. By definition of Υ, we find

$$\mathcal{F}^\lambda g \circ \Upsilon^\lambda_\omega \circ D_\rho = \tilde{\mathcal{F}}^{\lambda/\rho^2}\tilde{g},$$

where

$$\tilde{\mathcal{F}}^{\lambda/\rho^2}\tilde{g}(y,\tau) = \int_{\mathbb{R}^d} e^{i\tilde{\phi}^{\lambda/\rho^2}(y,\tau;\eta)}\tilde{a}^{\lambda/\rho^2}(y,\tau;\eta)\hat{\tilde{g}}(\eta)d\eta$$

for the phase $\tilde{\phi}(y,\tau;\eta)$ given by

$$\langle y,\eta\rangle + \int_0^1 (1-r)\langle \partial^2_{\xi'\xi'}\phi(\Upsilon_\omega(D'_{\rho^{-1}}y,y_d);\eta_d\omega + r\rho^{-1}\eta',\eta_d)\eta',\eta'\rangle dr$$

and the amplitude

$$\tilde{a}(y,\tau;\eta) = a(\Upsilon_\omega(D'_{\rho^{-1}}y;\tau);\eta_d\omega + \rho^{-1}\eta',\eta_d).$$

By change of space-time variables, we find

$$\|\mathcal{F}^\lambda g\|_{L^p(B_R)} \lesssim \rho^{\frac{d+1}{p}}\|\tilde{\mathcal{F}}^{\lambda/\rho^2}\tilde{g}\|_{L^p((\Upsilon^\lambda_\omega\circ D_\rho)^{-1}(B_R))}. \tag{90}$$

Note that $(\Upsilon^\lambda_\omega\circ D_\rho)^{-1}(B_R) = D_R$ is roughly a set of size $R/\rho\times\ldots\times R/\rho\times R\times R/\rho^2$. We want to apply the induction hypothesis, to which end we use finite speed of propagation: Since the time-scale is R/ρ^2, the localization by Lemma 10.4 yields

$$\|\tilde{\mathcal{F}}^{\lambda/\rho^2}\tilde{g}\|_{L^p(D_R)} \lesssim_{\delta'} R^{\delta'}Q_{p,\delta}(R/\rho^2)(R/\rho^2)^{d\left(\frac12-\frac1p\right)}\|\tilde{g}\|_{L^p}. \tag{91}$$

Since $\tilde{g}(x) = g(\rho x', x_d - \omega x')$, we find

$$\|\tilde{g}\|_{L^p} = \rho^{-\frac{d-1}{p}}\|g\|_{L^p}. \tag{92}$$

Taking (90), (91), and (92) together, we find (89) to hold. $\square$

We are ready for the proof of the following proposition:

Proposition 10.7. *Let $d\geq 3$, $2\leq k\leq d$, and $\lambda\geq 1$. If for all $\varepsilon>0$ and*

$$\bar{p}(k,d)\leq p\leq \begin{cases}\infty, & 2\leq k\leq 3,\\ 2\frac{k-1}{k-3}, & k\geq 4,\end{cases} \qquad \textit{with } \bar{p}(k,d) = \begin{cases}\frac{2(d+1)}{d}, & k=2,\\ 2\cdot\frac{2d-k+5}{2d-k+3}, & k\geq 3,\end{cases}$$

FIOs with reduced data (ϕ,a) obey the k-broad estimate for all $1\leq K\leq R\leq\lambda$ and some fixed choice of A

$$\|\mathcal{F}^\lambda f\|_{BL^p_{k,A}(B_R^{d+1})} \leq \bar{C}_\varepsilon K^{C_\varepsilon}R^\varepsilon R^{d\left(\frac12-\frac1p\right)}\|f\|_{L^p},$$

then we have for some $D_{\varepsilon,\phi,a}$

$$\|\mathcal{F}^\lambda f\|_{L^p(\mathbb{R}^{d+1})} \leq D_{\varepsilon,\phi,a}\lambda^{d\left(\frac12-\frac1p\right)+\varepsilon}\|f\|_{L^p(\mathbb{R}^d)}. \tag{93}$$

Proposition 10.3 follows from Proposition 10.7 by choosing $k=\frac{d+5}{2}$ for d odd and $k=\frac{d+4}{2}$ for d even. This will complete the proof of Theorem 10.1.

Proof of Proposition 10.7. The proof has many similarities with the proof of Proposition 8.1, and we shall be brief. By one parabolic rescaling depending on the phase as in the beginning of the proof of Proposition 8.1, we can suppose that (ϕ,a) is λ^δ-flat, and $R\leq\lambda^{1-\frac{\varepsilon}{10d}}$.

In the following $Q_{p,\tilde{\delta}}(R)$ denotes the smallest constant such that for all $\lambda^{\tilde{\delta}}$-flat phase functions and normalized amplitude functions, we have

$$\|\mathcal{F}^\lambda f\|_{L^p(B(0,R))} \leq R^{d\left(\frac12-\frac1p\right)}Q_{p,\tilde{\delta}}(R)\|f\|_{L^p}.$$

It suffices to prove that for any $\varepsilon>0$ there is $C_\varepsilon>0$ such that $Q_{p,\tilde{\delta}}(R)\leq C_\varepsilon R^\varepsilon$ for any $R\leq\lambda^{1-\frac{\varepsilon}{10d}}$.

For a given ball $B_{K^2} \subseteq B(0,R)$, let $V_1, \ldots, V_A$ be $(k-1)$-dimensional linear subspaces, which achieve the minimum in the definition of the k-broad norm, such that

$$\int_{B^{d+1}_{K^2}} |\mathcal{F}^\lambda f(x,t)|^p dxdt \lesssim K^{O(1)} \max_{\tau \notin V_\ell} \int_{B^{d+1}_{K^2}} |\mathcal{F}^\lambda f^\tau(x,t)|^p dxdt + \sum_{\ell=1}^{A} \int_{B^{d+1}_{K^2}} \big| \sum_{\tau \in V_\ell} \mathcal{F}^\lambda f^\tau(x,t) \big|^p dxdt.$$

Summing over a finitely overlapping family $(B_{K^2}) = \mathcal{B}_{K^2}$ covering $B(0,R)$ yields

$$\int_{B(0,R)} |\mathcal{F}^\lambda f(x,t)|^p dxdt \lesssim K^{O(1)} \sum_{B_{K^2} \in \mathcal{B}_{K^2}} \min_{V_1,\ldots,V_A} \max_{\tau \notin V_\ell} \int_{B_{K^2}} |\mathcal{F}^\lambda f^\tau(x,t)|^p dxdt + \sum_{B_{K^2} \in \mathcal{B}_{K^2}} \sum_{\ell=1}^{A} \int_{B_{K^2}} \big| \sum_{\tau \in V_\ell} \mathcal{F}^\lambda f^\tau(x,t)|^p dxdt.$$

By the broad norm estimate, we find

$$\sum_{B_{K^2} \in \mathcal{B}_{K^2}} \min_{V_1,\ldots,V_A} \max_{\tau \notin V_\ell} \int_{B_{K^2}} |\mathcal{F}^\lambda f^\tau(x,t)|^p dxdt \leq \bar{C}_\varepsilon K^{C_\varepsilon} R^{\frac{\varepsilon p}{2}} R^{dp\left(\frac{1}{2}-\frac{1}{p}\right)} \|f\|^p_{L^p}.$$

The narrow contribution is estimated by Proposition 10.5:

$$\sum_{\ell=1}^{A} \int_{B_{K^2} \in \mathcal{B}_{K^2}} \big| \sum_{\tau \in V_\ell} \mathcal{F}^\lambda f^\tau(x,t) \big|^p dxdt \leq C_\delta K^\delta K^{\max((k-3)\left(\frac{1}{2}-\frac{1}{p}\right)p,0)} \times \sum_\tau \int_{\mathbb{R}^{d+1}} w_{B_{K^2}} |\mathcal{F}^\lambda f^\tau(x,t)|^p dxdt, \tag{94}$$

where we have used the sector counting estimate

$$\#\{\tau : \tau \in V_\ell\} \lesssim \max(1, K^{k-3}).$$

Summing over $B_{K^2} \in \mathcal{B}_{K^2}$ in (94), we find

$$\sum_{B_{K^2} \in \mathcal{B}_{K^2}} \sum_{\ell=1}^{A} \int_{B^{d+1}_{K^2}} \big| \sum_{\tau \in V_\ell} \mathcal{F}^\lambda f^\tau(x,t) \big|^p dxdt \leq C_\delta K^\delta K^{\max((k-3)\left(\frac{1}{2}-\frac{1}{p}\right)p,0)} \times \sum_\tau \int_{\mathbb{R}^{d+1}} w_{B(0,2R)} |\mathcal{F}^\lambda f^\tau(x,t)|^p dxdt.$$

By Lemma 8.7, we find

$$\int_{B(0,R)} |\mathcal{F}^\lambda f^\tau(x,t)|^p dxdt \lesssim_{\delta_1} K^{-2d\left(\frac{1}{2}-\frac{1}{p}\right)p+2} Q^p_{p,\tilde{\delta}}(R/K^2) R^{dp\left(\frac{1}{2}-\frac{1}{p}\right)+\delta_1} \|f^\tau\|^p_p + \mathrm{RapDec}(R)\|f\|^p_p. \tag{95}$$

Note the following by Plancherel's theorem for $p=2$, the kernel estimate for $p=\infty$, and interpolation:

$$\big(\sum_\tau \|f^\tau\|^p_p\big)^{\frac{1}{p}} \lesssim \|f\|_p. \tag{96}$$

Hence, summing (95) over τ yields by (96)

$$\int_{B_R^{d+1}} |\mathcal{F}^\lambda f(x,t)|^p dxdt \leq C_\varepsilon R^{dp\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{\varepsilon p}{2}} \|f\|_{L^p}^p + C_{\delta,\delta_1} K^\delta R^{dp\left(\frac{1}{2}-\frac{1}{p}\right)+\delta_1} K^{-e(p,k,d)} Q^p_{p,\tilde{\delta}}(R/K^2) \|f\|^p_{L^p}$$

with

$$e(p,k,d) = \max\{2d\big(\frac{1}{2}-\frac{1}{p}\big)p-2, 2d\big(\frac{1}{2}-\frac{1}{p}\big)p-2-(k-3)\big(\frac{1}{2}-\frac{1}{p}\big)p\}.$$

We find $e(p,k,d) \geq 0$, if

$$p \geq \begin{cases} \frac{2(d+1)}{d}, & k=2, \\ 2\cdot\frac{2d-k+5}{2d-k+3}, & k \geq 3. \end{cases}$$

By the induction hypothesis, we have

$$Q^p_{p,\tilde{\delta}}(R) \leq K^{O(1)} \tilde{C}_\varepsilon K^{\bar{C}_\varepsilon} R^{\frac{\varepsilon p}{2}} + C_{\delta,\delta_1} R^{\delta_1} Q^p_{p,\tilde{\delta}}(R) K^{-e(p,k,d)+\delta}.$$

We can choose $\delta(\varepsilon)$, $\delta_1(\varepsilon)$, and $K = K_0 R^{\tilde{\delta}}$ similarly as at the end of the proof of Proposition 8.1 to close the induction. $\square$

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