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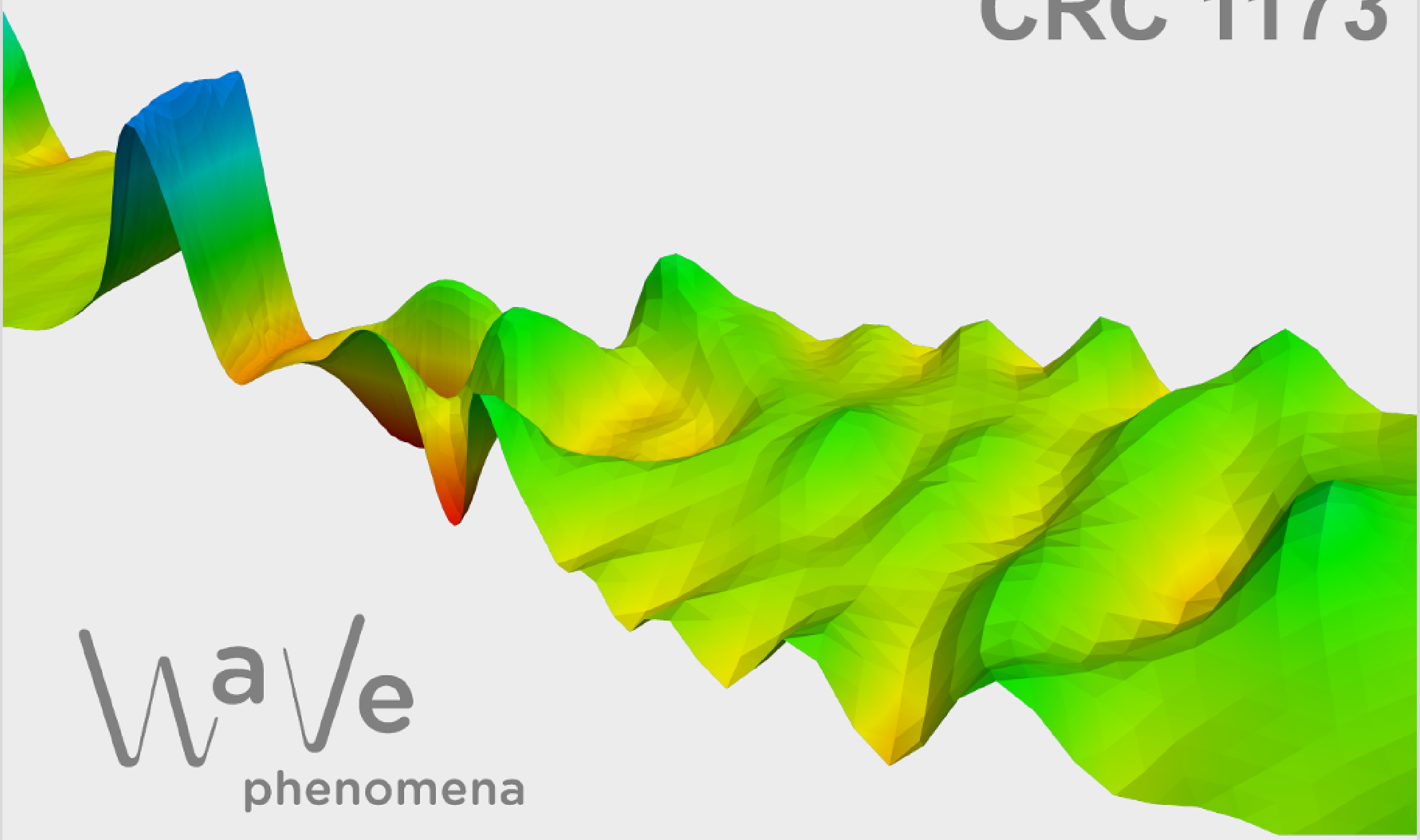
CRC Preprint 2022/11 (revised), August 2022





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ISSN 2365-662X

# OPTIMAL $W^{1,\infty}$-ESTIMATES FOR AN ISOPARAMETRIC FINITE ELEMENT DISCRETIZATION OF ELLIPTIC BOUNDARY VALUE PROBLEMS*

BENJAMIN DÖRICH†, JAN LEIBOLD†, AND BERNHARD MAIER†

**Abstract.** In this paper, we consider an elliptic boundary value problem on a domain with regular boundary and discretize it with isoparametric finite elements of order $k \geq 1$. We show optimal order of convergence of the isoparametric finite element solution in the $W^{1,\infty}$-norm. As an intermediate step, we derive stability and convergence estimates of optimal order $k$ for a (generalized) Ritz map.

**Key words.** elliptic boundary value problem, nonconforming space discretization, isoparametric finite elements, Ritz map, maximum norm error estimates, a-priori error estimates, weighted norms.

**AMS subject classifications.** 65M12, 65N15, 65N30

**1. Introduction.** In the present paper we study for an elliptic, second-order differential operator L the spatial discretization of the boundary value problem

$$\begin{aligned} \mathrm{L}u(x) &= f(x), && x \in \Omega\,, \\ u(x) &= 0, && x \in \Gamma = \partial\Omega\,, \end{aligned}$$

on a smooth domain $\Omega$ with isoparametric finite elements of order $k \geq 1$. We prove convergence of the finite element solution $u_h$ with optimal order $k$ in the $W^{1,\infty}$-norm. Since this is a nonconforming method, we also define a (generalized) Ritz map for which we show stability and convergence estimates of optimal order $k$ in the $W^{1,\infty}$-norm. We expect that with additional technical effort, one can also treat Neumann and Robin boundary conditions. For conforming discretizations of elliptic problems, the finite element solution $u_h$ is given as the Ritz projection of the exact solution $u$ and hence, estimates on the Ritz projection immediately imply convergence of $u_h$. This is no longer valid in the nonconforming case, but nevertheless, in our analysis this is also the major step in the convergence proof for the finite element discretization. Additionally, estimates on the (generalized) Ritz map can for example be exploited in the analysis of time-dependent problems, see [16] for evolving surfaces or [8] for non-autonomous wave equations.

In the conforming case, such estimates are well known for many years now. In fact, the first quasi-optimal error bounds in the maximum norm were already given in the seventies by Natterer [19], Scott [28], and Nitsche [20]. Many extensions and refinements have been achieved in the following years, see, e.g., [2,3,7,12,15,17,18,21–23,25,27].

However, none of these papers provides stability and convergence estimates in the nonconforming case. We briefly elaborate on the literature known to us in this case. In [29], Wahlbin analyzes quadratic isoparametric finite elements applied to an elliptic Dirichlet problem. The paper includes the errors introduced by numerical integration, and provides convergence in $L^\infty$ of order $h^{3-\varepsilon}$ for arbitrary $\varepsilon > 0$, where zero extensions are used order to extend the discrete solution outside the computational domain. In [26], Schatz and Wahlbin analyze $L^\infty$ error bounds for the case $\Omega_h \subseteq \Omega$ also using zero extension, and discuss possible extensions to the general case $\Omega_h \neq \Omega$.

More recently, this technique was extended in [14] to maximum norm error bounds for linear finite elements applied to an inhomogeneous Neumann problem. For (evolving) surface

*This work is funded by the German Research Foundation (DFG) – Project-ID 258734477 – SFB 1173.

†Institute for Applied and Numerical Mathematics, Karlsruhe Institute of Technology, Englerstr. 2, 76131 Karlsruhe, Germany, {benjamin.doerich, jan.leibold}@kit.edu

finite element methods, estimates on the finite element solution and the generalized Ritz map for isoparametric finite elements are considered in [6, 16].

Our analysis combines the approach in [5, Ch. 8], where stability and convergence of the Ritz projection in $W^{1,\infty}(\Omega)$ is shown, with the framework for isoparametric finite elements on smooth domains introduced in [4] and generalized in [9, 10]. Since crucial properties of the Ritz map, such as being a projection and satisfying some orthogonality condition, are not available in the nonconforming case, the proofs become significantly more technical. Following the idea of [28], for some point $z$ a regularized delta function $\delta^z$ is introduced in order to express point evaluation at $z$ as an integral and move to a variational setting. Further, considering solutions of the elliptic problem with right-hand side $\delta^z$, the $W^{1,\infty}$-norm error bounds are reduced to estimates on these solutions in $W^{1,1}(\Omega)$. To do so, several boundary perturbation terms in [9, 10] are extended to $L^p$-spaces, $p \neq 2$. In the next step, weight functions are introduced such that the estimates are further reduced to weighted $H^1$-norms. The main part of our paper is devoted to these weighted estimates. It turns out that the geometric errors introduce severe difficulties in showing the desired order of convergence, and many adaptions have to be made in the boundary perturbation estimates and the elliptic regularity results.

We point out that in this paper we do not consider convergence of optimal order $k+1$ in $L^\infty(\Omega)$. Looking at the strategy of the proof, this seems to be a straightforward generalization of the proof presented here, however, the error contribution of the geometric errors is only of order $k$. We think that showing these error bounds would be an interesting topic of future research.

The paper is organized as follows: In Section 2, we present the analytical framework and the space discretization by isoparametric Lagrange finite elements. After providing some properties of the discretized objects, we state our main results on the convergence of the finite element solution and also on the stability and convergence of the Ritz map.

In a first step towards the proofs of the main results, we reduce in Section 3 the three results to the same estimates in $W^{1,1}(\Omega)$ and several geometric errors are estimated in $L^1(\Omega)$ and $L^\infty(\Omega)$.

The proofs of the estimates in $W^{1,1}(\Omega)$ are presented in Section 4. We introduce the weight functions and move from these estimates to weighted $H^1$-norms. In the proofs several elliptic regularity results are needed. For the sake of readability, we postpone them to Appendix A.

**Notation.** In the rest of the paper we use the notation

$$a \lesssim b$$

if there is a constant $C > 0$ independent of the spatial parameter $h$ such that $a \leq Cb$. Further, for $\phi \in W^{j,p}(\Omega)$ we denote by $\nabla_j \phi$ the tensor containing all $j$-th order derivatives of $\phi$. If it is clear from the context, we write $L^p$ instead of $L^p(\Omega)$ or $L^p(\Omega_h)$.

**2. General setting and main results.** For a domain $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$, with boundary $\partial\Omega \in C^{s,1}$, $s \in \mathbb{N}$, we consider the elliptic operator

$$\mathrm{L}u = -\operatorname{div}\big(\mathcal{A}\nabla u\big) + \mathcal{B} \cdot \nabla u + \mathcal{C}u, \quad u \in H^2(\Omega),$$

with a symmetric, real, matrix-valued function $\mathcal{A} \in W^{1,\infty}(\Omega)^{N,N}$, with a constant $c_{\mathcal{A}} > 0$ such that

$$\xi \cdot \mathcal{A}(x)\xi \geq c_{\mathcal{A}}^{-1} \|\xi\|^2, \quad \xi \in \mathbb{R}^N, x \in \Omega, \tag{2.1}$$

$\mathcal{B} \in L^\infty(\Omega)^N$, and $\mathcal{C} \in L^\infty(\Omega)$. This operator induces a bilinear form for $\varphi, \psi \in H_0^1(\Omega)$

$$a(\varphi, \psi) := \int_\Omega \nabla\psi \cdot \mathcal{A}\nabla\varphi \,\mathrm{d}x + \int_\Omega \psi\mathcal{B} \cdot \nabla\varphi \,\mathrm{d}x + \int_\Omega \mathcal{C}\psi\varphi \,\mathrm{d}x \,. \tag{2.2}$$

We study for $f \in L^2(\Omega)$ the variational problem: Seek $u \in H_0^1(\Omega)$ such that

$$a(u, \psi) = (f \mid \psi)_{L^2(\Omega)}\,, \qquad \forall \psi \in H_0^1(\Omega)\,. \tag{2.3}$$

We additionally assume one of the two following settings:

(a) $\mathcal{C} \geq 0$ and $\mathcal{B} = 0$ on $\Omega$,

(b) $\mathcal{C} - |\operatorname{div}\mathcal{B}| \geq \beta > 0$ on $\Omega$ for some constant $\beta$,

and denote in the following $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. Under these assumptions, we obtain constants $c_b, c_c > 0$ such that

$$|a(\varphi, \psi)| \leq c_b \|\varphi\|_V \|\psi\|_V\,, \qquad a(\varphi, \varphi) \geq c_c \|\varphi\|_V^2\,.$$

By the Lax-Milgram theorem, we have unique solvability of (2.3), and we define the corresponding solution operator $S\colon H \to V$ by $S\colon f \mapsto u$. We also consider the dual problem: Seek $v \in H_0^1(\Omega)$ such that

$$a(\psi, v) = (f \mid \psi)_{L^2(\Omega)}\,, \qquad \forall \psi \in H_0^1(\Omega)\,,$$

and denote its solution operator by $\widetilde{S}\colon H \to V$. For the analysis, we heavily rely on the following elliptic regularity result [11, Thm. 2.4.2.5].

THEOREM 2.1 (Elliptic regularity). *Let $\partial\Omega \in C^{1,1}$, then for all $1 < p < \infty$ there is a constant $C_p > 0$ such that for all $\varphi \in L^p(\Omega)$ it holds*

$$\|\widetilde{S}\varphi\|_{W^{2,p}} + \|S\varphi\|_{W^{2,p}} \leq C_p \|\varphi\|_{L^p}\,.$$

**Space discretization.** We study the nonconforming space discretization of (2.3) based on isoparametric finite elements. For further details on this approach, we refer to [10] and assume that the boundary $\partial\Omega$ is of class $C^{k+1,1}$. We introduce a shape-regular and quasi-uniform mesh $\mathcal{T}_h$, consisting of isoparametric elements of degree $k \in \mathbb{N}$, where the subscript $h$ denotes the maximal diameter of all elements $K \in \mathcal{T}_h$. The computational domain $\Omega_h$ is given by

$$\Omega_h = \bigcup_{K \in \mathcal{T}_h} K \approx \Omega.$$

Based on the transformations $F_K$ mapping the reference element $\widehat{K}$ to $K \in \mathcal{T}_h$, we introduce the isoparametric finite element space of degree $k$

$$W_h = \{\varphi \in C_0(\overline{\Omega}_h) \mid \varphi|_K = \widehat{\varphi} \circ (F_K)^{-1} \text{ with } \widehat{\varphi} \in \mathcal{P}^k(\widehat{K}) \text{ for all } K \in \mathcal{T}_h\}\,.$$

Here, $\mathcal{P}^k(\widehat{K})$ consists of all polynomials on $\widehat{K}$ of degree at most $k$. The discrete approximation spaces are given by

$$H_h = \big(W_h, (\cdot \mid \cdot)_{L^2(\Omega_h)}\big), \qquad V_h = \big(W_h, (\cdot \mid \cdot)_{H_0^1(\Omega_h)}\big), \qquad X_h = V_h \times H_h.$$

For $h \leq h_0$ sufficiently small, following the construction in [10, Sec. 5], we introduce the lift operator $\mathcal{L}_h : C(\Omega_h) \to C(\Omega)$. In particular, for $p \in [1, \infty]$ there are constants $c_p, C_p > 0$ with

$$c_p \|\varphi_h\|_{L^p(\Omega_h)} \leq \|\mathcal{L}_h\varphi_h\|_{L^p(\Omega)} \leq C_p \|\varphi_h\|_{L^p(\Omega_h)}\,, \qquad \varphi_h \in C(\Omega_h), \tag{2.4a}$$

$$c_p \|\varphi_h\|_{W^{1,p}(\Omega_h)} \leq \|\mathcal{L}_h\varphi_h\|_{W^{1,p}(\Omega)} \leq C_p \|\varphi_h\|_{W^{1,p}(\Omega_h)}\,, \qquad \varphi_h \in W^{1,\infty}(\Omega_h), \tag{2.4b}$$

cf. [10, Prop. 5.8]. By construction, the boundary nodes of $\Omega_h$ lie on $\partial\Omega$ and zero boundary conditions are preserved by $\mathcal{L}_h$, see [10, Sec. 8.5]. Further, we denote the nodal interpolation operator by $I_h \colon C_0(\Omega) \to V_h$. Enriching the space $W_h$ by basis functions corresponding to the boundary nodes, we further define its extension $\widetilde{I}_h \colon C(\Omega) \to C(\Omega_h)$. As shown in [10, Thm. 5.9], we have for $m \in \{0,1\}$, $1 \le p \le \infty$, and $1 \le \ell \le k$ the estimates

$$\left\|(\mathrm{Id} - \mathcal{L}_h \widetilde{I}_h)\varphi\right\|_{W^{m,p}(\Omega)} \lesssim h^{\ell+1-m} \left\|\varphi\right\|_{W^{\ell+1,p}(\Omega)}, \qquad \varphi \in W^{\ell+1,p}(\Omega). \tag{2.5}$$

Further, $\ell = 0$ is allowed for $N < p \le \infty$. From now on, we assume for the coefficients

$$\mathcal{A} \in W^{k+1,\infty}(\Omega)^{N,N}, \quad \mathcal{B} \in W^{k+1,\infty}(\Omega)^{N}, \quad \mathcal{C} \in W^{k+1,\infty}(\Omega),$$

and define analogously to (2.2)

$$a_h(\varphi_h, \psi_h) \coloneqq \int_{\Omega_h} \nabla\psi_h \cdot \mathcal{A}_h \nabla\varphi_h \,\mathrm{d}x + \int_{\Omega_h} \psi_h \mathcal{B}_h \cdot \nabla\varphi_h \,\mathrm{d}x + \int_{\Omega_h} \mathcal{C}_h \psi_h \varphi_h \,\mathrm{d}x,$$

where the discrete coefficients are given by

$$\mathcal{A}_h = \widetilde{I}_h \mathcal{A}, \quad \mathcal{B}_h = \widetilde{I}_h \mathcal{B}, \quad \mathcal{C}_h = \widetilde{I}_h \mathcal{C}. \tag{2.6}$$

Then, there is some $h_0 > 0$ such that it holds

$$|a_h(\varphi_h, \psi_h)| \le \widehat{c}_b \left\|\varphi_h\right\|_{V_h} \left\|\psi_h\right\|_{V_h}, \qquad a(\varphi_h, \varphi_h) \ge \widehat{c}_c \left\|\varphi_h\right\|_{V_h}^2,$$

with constants $\widehat{c}_b, \widehat{c}_c > 0$ independent of $h \le h_0$. Under these conditions, we obtain unique solvability of the discrete variational problem

$$a_h(u_h, \psi_h) = (f_h \mid \psi_h)_{L^2(\Omega_h)} \quad \forall \psi_h \in V_h, \tag{2.7}$$

for some suitably discretized right-hand side $f_h \in L^2(\Omega_h)$. Further, we define the adjoint lift operators $\mathcal{L}_h^{H*} \colon H \to H_h$ and $\mathcal{L}_h^{V*} \colon V \to V_h$ by

$$\left(\mathcal{L}_h^{H*}\varphi \mid \psi_h\right)_{L^2(\Omega_h)} = (\varphi \mid \mathcal{L}_h \psi_h)_{L^2(\Omega)}, \qquad \varphi \in H,\ \psi_h \in H_h. \tag{2.8a}$$

$$a_h\left(\mathcal{L}_h^{V*}\varphi, \psi_h\right) = a(\varphi, \mathcal{L}_h \psi_h), \qquad \varphi \in V,\ \psi_h \in V_h. \tag{2.8b}$$

**Main results.** We are now in the position to state our main results. The proofs are given in the following sections. The first theorem is concerned with the convergence of the finite element approximation obtained by (2.7) towards the solution $u$ of (2.3). We need the following approximation estimate on the right-hand side.

ASSUMPTION 2.2. *There is a constant $C_f \ge 0$ such that the discretization of the right-hand side $f$ satisfies one of the following bounds:*

$$\left\|f - \mathcal{L}_h f_h\right\|_{L^\infty(\Omega)} \le C_f h^k, \tag{2.9a}$$

$$\left\|\mathcal{L}_h^{H*} f - f_h\right\|_{L^\infty(\Omega_h)} \le C_f h^k, \tag{2.9b}$$

*where $\mathcal{L}_h^{H*}$ is defined in (2.8a).*

The main examples are given by

$$f_h = \widetilde{I}_h f \qquad \text{or} \qquad f_h = \mathcal{L}_h^{H*} f,$$

which lead to $C_f \sim \|f\|_{W^{k,\infty}}$ or $C_f = 0$, respectively. Note that, while the interpolation can always be implemented, the evaluation of $\mathcal{L}_h^{H*}$ in the nonconforming case is more involved, since one needs to evaluate integrals over the smooth domain $\Omega$. In the conforming case $\mathcal{L}_h^{H*}$ reduces to the orthogonal $L^2$-projection onto $V_h$.

We obtain the following convergence result.

THEOREM 2.3. *Let $k \geq 1$ and $\partial\Omega \in C^{k+1,1}$. Further, let Assumption 2.2 hold and $h \leq h_0$ sufficiently small. Then, the error between the solutions $u$ of (2.3) and $u_h$ of (2.7) satisfies*

$$\|u - \mathcal{L}_h u_h\|_{W^{1,\infty}(\Omega)} \leq C h^k \big(\|u\|_{W^{k+1,\infty}(\Omega)} + C_f\big),$$

*where $C$ is independent of $h$.*

In the error analysis of evolution equations, it is often convenient to introduce the Ritz projection. In the following, we present its generalization for the nonconforming case.

DEFINITION 2.4. *Consider the adjoint lift $\mathcal{L}_h^{V*}$ given by (2.8b). We define the generalized Ritz map by*

$$\mathcal{L}_h \mathcal{L}_h^{V*} \colon V \to V\,. \tag{2.10}$$

In the conforming case, one has $\Omega_h = \Omega$, and there we call it the Ritz projection which satisfies

$$u_h = R_h u\,,$$

but this is *not* true in our case in general. In fact, one has to talk about a (generalized) Ritz map, since it is not a projection anymore. In addition, the generalized Ritz map does also not satisfy an orthogonality condition, but only an estimate of the form

$$a\left(u - \mathcal{L}_h \mathcal{L}_h^{V*} u, \mathcal{L}_h \varphi_h\right) \lesssim h^k \left\|\mathcal{L}_h^{V*} u\right\|_{V_h} \|\varphi_h\|_{V_h}\,, \qquad u \in V, \varphi_h \in V_h\,,$$

see, e.g., [10, Lem. 8.24]. This fact induces several additional error terms in the maximum norm error analysis which require a detailed inspection.

Nevertheless, the objects $u_h$ and $\mathcal{L}_h \mathcal{L}_h^{V*} u$ are quite related and, as an intermediate step towards the above theorem, we obtain the following two results on the Ritz map. These results are well known in the conforming case.

THEOREM 2.5. *Let $\partial\Omega \in C^{k+1,1}$ and $h \leq h_0$ be sufficiently small. Then the generalized Ritz map defined in (2.10) is stable in $W^{1,\infty}(\Omega)$, i.e.,*

$$\left\|\mathcal{L}_h \mathcal{L}_h^{V*} \varphi\right\|_{W^{1,\infty}(\Omega)} \lesssim \|\varphi\|_{W^{1,\infty}(\Omega)}\,, \qquad \varphi \in W^{1,\infty}(\Omega).$$

We note that by (2.4) it is sufficient to show

$$\left\|\mathcal{L}_h^{V*} \varphi\right\|_{W^{1,\infty}(\Omega_h)} \lesssim \|\varphi\|_{W^{1,\infty}(\Omega)}\,, \qquad \varphi \in W^{1,\infty}(\Omega).$$

Our third result is concerned with the approximation properties of the Ritz map.

THEOREM 2.6. *Let $k \geq 1$, $\partial\Omega \in C^{k+1,1}$ and $h \leq h_0$ be sufficiently small. Then, it holds for all $\varphi \in W^{k+1,\infty}(\Omega)$*

$$\left\|(\mathrm{Id} - \mathcal{L}_h \mathcal{L}_h^{V*})\varphi\right\|_{W^{1,\infty}(\Omega)} \leq C h^k \|\varphi\|_{W^{k+1,\infty}(\Omega)}\,,$$

*where $C$ is independent of $h$.*

The rest of the paper is dedicated to the proofs of the three theorems.

**3. Proof of the main results.** From now on, we choose an arbitrary, but fixed index $i = 1, \dots, N$ and consider $\partial_i$ instead of the full gradient $\nabla$. Following the idea of [28], let $z \in K^z$ with $K^z \in \mathcal{T}_h$. There exists $\delta^z \in C_0^1(K^z)$, see [24, Ass. A.5] for a construction, with zero extension to a function on $\Omega_h$, such that

$$(\delta^z \mid \partial_i \varphi_h)_{L^2(\Omega_h)} = \partial_i \varphi_h(z), \qquad \text{for all } \varphi_h \in V_h, \tag{3.1}$$

and

$$\|\partial^\alpha \delta^z\|_{L^\infty} \lesssim h^{-N-|\alpha|}, \quad \alpha \in \mathbb{N}_0^N, \ |\alpha| \le 1. \tag{3.2}$$

Here, we use the notation $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_N^{\alpha_N}$ and $|\alpha| = \sum_{i=1}^N \alpha_i$. Due to the stability (2.4) of $\mathcal{L}_h$, we have

$$\|\mathcal{L}_h \delta^z\|_{L^1(\Omega)} \lesssim \int_{K^z} |\delta^z| \,\mathrm{d}x \lesssim h^N h^{-N} \le C. \tag{3.3}$$

In the following we always assume that $h \le h_0$ is sufficiently small in order to apply the results of the previous section. We further introduce the solutions $g_h^z \in V_h$ and $g^z \in V$ of the elliptic variational problems

$$\begin{aligned} a_h\left(\varphi_h, g_h^z\right) &= \left(-\partial_i \delta^z \mid \varphi_h\right)_{L^2(\Omega_h)}, \quad && \varphi_h \in V_h, \\ a\left(\varphi, g^z\right) &= \left(-\partial_i \mathcal{L}_h \delta^z \mid \varphi\right)_{L^2(\Omega)}, \quad && \varphi \in V, \end{aligned} \tag{3.4}$$

which play an import role in the subsequent error analysis.

**3.1. Preliminary results and geometric errors.** The aim of this section is to reduce the maximum norm bounds to the following bounds in $W^{1,1}(\Omega)$. On the one hand, we establish a bound on the difference of the solutions of (3.4). For the conforming case, this result is shown in [5, Cor. 8.2.7].

LEMMA 3.1. *Let $g_h^z \in V_h$ and $g^z \in V$ be defined by (3.4). Then, there is a constant $C > 0$ such that*

$$\|\mathcal{L}_h g_h^z - g^z\|_{W^{1,1}(\Omega)} \le C,$$

*with $C$ independent of $h$ and $z$.*

On the other hand, we have to derive a bound on the solution $g^z \in V$. To this end, we introduce the set of boundary elements

$$\mathcal{T}_h^b := \{K \in \mathcal{T}_h \mid K \text{ has a node on the boundary } \partial\Omega_h\}$$

and also the layer around the boundary $\Gamma$ by

$$U_h := \{x \in \Omega \mid \operatorname{dist}(x, \Gamma) \le h\}.$$

By the definition of $h$, this yields that the lift $\mathcal{L}_h$ maps functions with support on the triangulation $\mathcal{T}_h^b$ to functions with support on $U_h$. The bounds on $g^z$ are shown either on this layer of width $h$ or on the whole domain with an additional factor $h$ in front of the norm.

LEMMA 3.2. *There is a constant $C > 0$ such that*

$$\|g^z\|_{W^{1,1}(U_h)} + h\,\|g^z\|_{W^{1,1}(\Omega)} \le C,$$

*with $C$ independent of $h$.*

The proofs of the two lemmas are given in Section 4. The rest of this section is concerned with the reduction of Theorems 2.3, 2.5, and 2.6 to the above lemmas. We note that we employ several times the Poincaré inequality in the form

$$\|\varphi_h\|_{L^\infty(\Omega_h)} \lesssim \|\nabla\varphi_h\|_{L^\infty(\Omega_h)} , \tag{3.5}$$

for all $\varphi_h \in V_h$. Further, we use a slight extension of the estimates in [10, Lem. 8.24] and [13, Lem. 7.3] in order to treat the errors stemming from nonconformity. We define the boundary perturbation errors for $\varphi_h, \psi_h \in W^{1,\infty}(\Omega_h)$ by

$$E^H(\varphi_h, \psi_h) := \left|(\mathcal{L}_h\varphi_h \mid \mathcal{L}_h\psi_h)_{L^2(\Omega)} - (\varphi_h \mid \psi_h)_{L^2(\Omega_h)}\right|, \tag{3.6a}$$

$$E^{H,i}(\varphi_h, \psi_h) := \left|(\partial_i\mathcal{L}_h\varphi_h \mid \mathcal{L}_h\psi_h)_{L^2(\Omega)} - (\partial_i\varphi_h \mid \psi_h)_{L^2(\Omega_h)}\right|, \tag{3.6b}$$

$$E^a(\varphi_h, \psi_h) := \left|a(\mathcal{L}_h\varphi_h, \mathcal{L}_h\psi_h) - a_h(\varphi_h, \psi_h)\right|, \tag{3.6c}$$

and state the following result

LEMMA 3.3. *Let $\varphi_h, \psi_h \in W^{1,\infty}(\Omega_h)$. The bilinear forms in (3.6) are estimated by*

$$\begin{aligned} E^H(\varphi_h, \psi_h) &\le Ch^k \|\mathcal{L}_h\varphi_h\|_{L^\infty(\Omega)} \|\mathcal{L}_h\psi_h\|_{L^1(U_h)} , \\ E^{H,i}(\varphi_h, \psi_h) &\le Ch^k \|\mathcal{L}_h\varphi_h\|_{W^{1,\infty}(\Omega)} \|\mathcal{L}_h\psi_h\|_{L^1(U_h)} , \\ E^a(\varphi_h, \psi_h) &\le Ch^{k+1} \|\mathcal{L}_h\varphi_h\|_{W^{1,\infty}(\Omega)} \|\mathcal{L}_h\psi_h\|_{W^{1,1}(\Omega)} \\ &\quad + Ch^k \|\mathcal{L}_h\varphi_h\|_{W^{1,\infty}(\Omega)} \|\mathcal{L}_h\psi_h\|_{W^{1,1}(U_h)} , \end{aligned}$$

*with $C > 0$ independent of $h$.*

*Proof.* We only prove the second claim for the dominant part of $E^a$ including $\mathcal{A}$, since the other estimates follow analogously. We expand, using the definition of $\mathcal{A}_h$ in (2.6),

$$\begin{aligned} &\int_\Omega \nabla\mathcal{L}_h\psi_h \cdot \mathcal{A}\nabla\mathcal{L}_h\varphi_h \,\mathrm{d}x - \int_{\Omega_h} \nabla\psi_h \cdot \mathcal{A}_h\nabla\varphi_h \,\mathrm{d}x \\ &= \int_\Omega \nabla\mathcal{L}_h\psi_h \cdot \big(\mathcal{A} - \mathcal{L}_h\widetilde{I}_h\mathcal{A}\big)\nabla\mathcal{L}_h\varphi_h \,\mathrm{d}x \\ &\quad + \int_\Omega \nabla\mathcal{L}_h\psi_h \cdot \mathcal{L}_h\mathcal{A}_h\nabla\mathcal{L}_h\varphi_h \,\mathrm{d}x - \int_{\Omega_h} \nabla\psi_h \cdot \mathcal{A}_h\nabla\varphi_h \,\mathrm{d}x. \end{aligned}$$

The first term is estimated using Hölder's inequality together with (2.5). For the second term, we proceed analogously to the proof of [10, Lem. 8.24]. ∎

In the following, we first prove stability and convergence of the Ritz map and then use these results in order to establish the convergence of the finite element solution.

**3.2. Proof of Theorem 2.5.** We follow the approach of [5, Sec. 8.2]. Using integration by parts as well as the definition (2.8b) of the adjoint lift operator, we obtain by (3.1)

$$\begin{aligned} \partial_i\big(\mathcal{L}_h^{V*}u\big)(z) &= \big(\mathcal{L}_h^{V*}u \mid -\partial_i\delta^z\big)_{L^2(\Omega_h)} \\ &= a_h\left(\mathcal{L}_h^{V*}u, g_h^z\right) \\ &= a\left(u, \mathcal{L}_h g_h^z\right) \\ &= a\left(u, g^z\right) + a\left(u, \mathcal{L}_h g_h^z - g^z\right) \\ &= \left(u \mid -\partial_i\mathcal{L}_h\delta^z\right)_{L^2(\Omega)} + a\left(u, \mathcal{L}_h g_h^z - g^z\right) \\ &= \left(\partial_i u \mid \mathcal{L}_h\delta^z\right)_{L^2(\Omega)} + a\left(u, \mathcal{L}_h g_h^z - g^z\right) . \end{aligned} \tag{3.7}$$

Hence, Hölder's inequality yields

$$\left|\partial_i\big(\mathcal{L}_h^{V*}u\big)(z)\right| \lesssim \big(\|\mathcal{L}_h\delta^z\|_{L^1(\Omega)} + \|\mathcal{L}_h g_h^z - g^z\|_{W^{1,1}(\Omega)}\big)\,\|u\|_{W^{1,\infty}(\Omega)} .$$

By (3.3) and Lemma 3.1, the stability estimate in Theorem 2.5 follows with the Poincaré inequality (3.5). ∎

**3.3. Proof of Theorem 2.6.** We combine the approach of the stability analysis and, in order to employ the properties of $\delta^z$, we reduce the estimate to functions on the finite element space by inserting the interpolation. We first estimate by (2.5)

$$\begin{aligned}\left\|u - \mathcal{L}_h\mathcal{L}_h^{V*}u\right\|_{W^{1,\infty}(\Omega)} &\lesssim \|u - \mathcal{L}_h I_h u\|_{W^{1,\infty}(\Omega)} + \left\|I_h u - \mathcal{L}_h^{V*}u\right\|_{W^{1,\infty}(\Omega_h)}\\ &\lesssim h^k\,\|u\|_{W^{k+1,\infty}(\Omega)} + \left\|I_h u - \mathcal{L}_h^{V*}u\right\|_{W^{1,\infty}(\Omega_h)} .\end{aligned}$$

We employ (3.7) and derive

$$\begin{aligned}\partial_i\big(I_h u - \mathcal{L}_h^{V*}u\big)(z) &= (\partial_i I_h u \mid \delta^z)_{L^2(\Omega_h)} - \big(\partial_i\mathcal{L}_h^{V*}u \mid \delta^z\big)_{L^2(\Omega_h)}\\ &= (\partial_i I_h u \mid \delta^z)_{L^2(\Omega_h)} - (\partial_i u \mid \mathcal{L}_h\delta^z)_{L^2(\Omega)} - a\left(u, \mathcal{L}_h g_h^z - g^z\right)\\ &= \big(\partial_i\big(\mathcal{L}_h I_h u - u\big) \mid \mathcal{L}_h\delta^z\big)_{L^2(\Omega_h)} + a\left(\mathcal{L}_h I_h u - u, \mathcal{L}_h g_h^z - g^z\right)\\ &\quad + \widetilde{\Delta}_1 - \widetilde{\Delta}_2\end{aligned}$$

with defects

$$\begin{aligned}\widetilde{\Delta}_1 &= (\partial_i I_h u \mid \delta^z)_{L^2(\Omega_h)} - (\partial_i\mathcal{L}_h I_h u \mid \mathcal{L}_h\delta^z)_{L^2(\Omega)} ,\\ \widetilde{\Delta}_2 &= a\left(\mathcal{L}_h I_h u, \mathcal{L}_h g_h^z - g^z\right) .\end{aligned}$$

We note that both terms vanish in the conforming case. Again, we apply the interpolation estimate (2.5) and Hölder's inequality to derive

$$\begin{aligned}\left\|\partial_i\big(I_h u - \mathcal{L}_h^{V*}u\big)\right\|_{L^\infty(\Omega_h)} &\le \|u - \mathcal{L}_h I_h u\|_{W^{1,\infty}(\Omega)}\,\|\mathcal{L}_h\delta^z\|_{L^1(\Omega)}\\ &\quad + \|u - \mathcal{L}_h I_h u\|_{W^{1,\infty}(\Omega)}\,\|\mathcal{L}_h g_h^z - g^z\|_{W^{1,1}(\Omega)}\\ &\quad + |\widetilde{\Delta}_1| + |\widetilde{\Delta}_2|\\ &\lesssim h^k\,\|u\|_{W^{k+1,\infty}(\Omega)} + |\widetilde{\Delta}_1| + |\widetilde{\Delta}_2| ,\end{aligned}$$

where we used (3.3) and Lemma 3.1 in the last step. Thus, Theorem 2.6 follows once we have employed the Poincaré inequality (3.5) and shown that

$$|\widetilde{\Delta}_1| + |\widetilde{\Delta}_2| \lesssim h^k\,\|u\|_{W^{k+1,\infty}(\Omega)} .$$

This inequality is proved in the following series of lemmas.

LEMMA 3.4. *There is a constant $C > 0$ such that*

$$|\widetilde{\Delta}_1| \le Ch^k\,\|u\|_{W^{1,\infty}(\Omega)} ,$$

*with $C$ independent of $h$.*

*Proof.* We obtain by Lemma 3.3 and (3.3)

$$|\widetilde{\Delta}_1| \le h^k\,\|\partial_i\mathcal{L}_h I_h u\|_{L^\infty(\Omega)}\,\|\mathcal{L}_h\delta^z\|_{L^1(\Omega)} \lesssim h^k\,\|u\|_{W^{1,\infty}(\Omega)} ,$$

where we used the stability of the lift (2.4) and the interpolation (2.5) in the last step. ∎

In the next lemma, we decompose the remaining defect even further into two differences of bilinear forms.

LEMMA 3.5. *The defect $\widetilde{\Delta}_2$ can be represented by*

$$\widetilde{\Delta}_2 = \widetilde{\Delta}_H + \widetilde{\Delta}_V$$

*where $\widetilde{\Delta}_H$ and $\widetilde{\Delta}_V$ are given by*

$$\begin{aligned}\widetilde{\Delta}_H &= \left(\mathcal{L}_h I_h u \mid \partial_i \mathcal{L}_h \delta^z\right)_{L^2(\Omega)} - \left(I_h u \mid \partial_i \delta^z\right)_{L^2(\Omega_h)} , \\ \widetilde{\Delta}_V &= a\left(\mathcal{L}_h I_h u, \mathcal{L}_h g_h^z\right) - a_h\left(I_h u, g_h^z\right) .\end{aligned}$$

*Proof.* Using the definitions of $g^z$ and $g_h^z$ in (3.4), we derive

$$\begin{aligned}\widetilde{\Delta}_2 &= a\left(\mathcal{L}_h I_h u, \mathcal{L}_h g_h^z - g^z\right) \\ &= a\left(\mathcal{L}_h I_h u, \mathcal{L}_h g_h^z\right) - a\left(\mathcal{L}_h I_h u, g^z\right) \\ &= a_h\left(I_h u, g_h^z\right) + \widetilde{\Delta}_V + \left(\mathcal{L}_h I_h u \mid \partial_i \mathcal{L}_h \delta^z\right)_{L^2(\Omega)} \\ &= -\left(I_h u \mid \partial_i \delta^z\right)_{L^2(\Omega_h)} + \widetilde{\Delta}_V + \left(\mathcal{L}_h I_h u \mid \partial_i \mathcal{L}_h \delta^z\right)_{L^2(\Omega)} \\ &= \widetilde{\Delta}_H + \widetilde{\Delta}_V ,\end{aligned}$$

and the decomposition is shown. ∎

The final bounds are derived in the next lemma.

LEMMA 3.6. *It holds*

$$|\widetilde{\Delta}_H| + |\widetilde{\Delta}_V| \le C h^k \|u\|_{W^{1,\infty}(\Omega)} ,$$

*with a constant $C > 0$ independent of $h$.*

*Proof.* We consider the two terms separately.

(a) Using integration by parts, Lemma 3.3, and (3.3) we obtain

$$\begin{aligned}|\widetilde{\Delta}_H| &= \left|\left(\partial_i \mathcal{L}_h I_h u \mid \mathcal{L}_h \delta^z\right)_{L^2(\Omega)} - \left(\partial_i I_h u \mid \delta^z\right)_{L^2(\Omega_h)}\right| \\ &\lesssim h^k \|\mathcal{L}_h I_h u\|_{W^{1,\infty}(\Omega)} \|\mathcal{L}_h \delta^z\|_{L^1(\Omega)} \\ &\lesssim h^k \|u\|_{W^{1,\infty}(\Omega)} ,\end{aligned}$$

where we used the stability of the lift (2.4) and the interpolation (2.5) in the last step.

(b) For the second term, we obtain by Lemma 3.3

$$\begin{aligned}|\widetilde{\Delta}_V| &= \left|a\left(\mathcal{L}_h I_h u, \mathcal{L}_h g_h^z\right) - a_h\left(I_h u, g_h^z\right)\right| \\ &\lesssim h^k \|\mathcal{L}_h I_h u\|_{W^{1,\infty}(\Omega_h)} \left(\|\mathcal{L}_h g_h^z\|_{W^{1,1}(U_h)} + h \|\mathcal{L}_h g_h^z\|_{W^{1,1}(\Omega)}\right) \\ &\lesssim h^k \|u\|_{W^{1,\infty}(\Omega)} \left(\|\mathcal{L}_h g_h^z - g^z\|_{W^{1,1}(\Omega)} + \|g^z\|_{W^{1,1}(U_h)} + h \|g^z\|_{W^{1,1}(\Omega)}\right) \\ &\lesssim h^k \|u\|_{W^{1,\infty}(\Omega)} ,\end{aligned}$$

where we used Lemmas 3.1 and 3.2 in the last inequality. ∎

Hence, the proof of Theorem 2.6 is complete. ∎

**3.4. Proof of Theorem 2.3.** Finally, we employ the already shown convergence in Theorem 2.6 and insert the Ritz map to compute

$$\begin{aligned}\|u-\mathcal{L}_h u_h\|_{W^{1,\infty}(\Omega)} &\lesssim \left\|u-\mathcal{L}_h\mathcal{L}_h^{V*}u\right\|_{W^{1,\infty}(\Omega)}+\left\|\mathcal{L}_h^{V*}u-u_h\right\|_{W^{1,\infty}(\Omega_h)}\\ &\lesssim h^k\,\|u\|_{W^{k+1,\infty}(\Omega)}+\left\|\mathcal{L}_h^{V*}u-u_h\right\|_{W^{1,\infty}(\Omega_h)}\,.\end{aligned}$$

We use the following equality established in (3.7)

$$\partial_i\big(\mathcal{L}_h^{V*}u\big)(z)=a\left(u,\mathcal{L}_h g_h^z\right)\,,$$

and derive with (3.1) and (2.3)

$$\begin{aligned}\partial_i u_h(z)&=(u_h\mid -\partial_i\delta^z)_{L^2(\Omega_h)}\\&=a_h\left(u_h,g_h^z\right)\\&=(f_h\mid g_h^z)_{L^2(\Omega_h)}\\&=(f\mid\mathcal{L}_h g_h^z)_{L^2(\Omega)}+\Delta_{f,1}+\Delta_{f,2}\\&=\partial_i\big(\mathcal{L}_h^{V*}u\big)(z)+\Delta_{f,1}+\Delta_{f,2}\,,\end{aligned}$$

with defects

$$\begin{aligned}\Delta_{f,1}&=(f_h\mid g_h^z)_{L^2(\Omega_h)}-(\mathcal{L}_h f_h\mid\mathcal{L}_h g_h^z)_{L^2(\Omega)}\,,\\ \Delta_{f,2}&=(\mathcal{L}_h f_h-f\mid\mathcal{L}_h g_h^z)_{L^2(\Omega)}\,.\end{aligned}$$

If (2.9a) holds, we employ Lemmas 3.1 to 3.3, to conclude

$$|\Delta_{f,1}+\Delta_{f,2}|\lesssim h^k\big(\|f\|_{L^\infty(\Omega)}+C_f\big)\lesssim h^k\big(\|u\|_{W^{2,\infty}(\Omega)}+C_f\big)\,. \tag{3.8}$$

If we assume (2.9b), then combining the defects as

$$\Delta_{f,1}+\Delta_{f,2}=\big(f_h-\mathcal{L}_h^{H*}f\mid g_h^z\big)_{L^2(\Omega_h)}\,,$$

where we used (2.8a), yields (3.8). Hence, we have shown for any $z\in\Omega_h$ that

$$|\partial_i\big(\mathcal{L}_h^{V*}u-u_h\big)(z)|\le Ch^k\,,$$

with a constant $C$ independent of $z$. Finally, applying the Poincaré inequality (3.5), we conclude the assertion. □

**4. Estimates in the $W^{1,1}$-norm.** The final section is concerned with the proofs of Lemma 3.1 and Lemma 3.2. We emphasize that we follow the lines of [5, Chap. 8] and add all the modifications due to the nonconformity, but give a rather complete proof for the sake of readability.

**4.1. Properties of weighted norms.** The main technical tool are weighted norms. To this end, we introduce the family $\{\sigma_z\}_{z\in\Omega}$ of weight functions with

$$\sigma_z\colon\Omega\to\mathbb{R},\qquad \sigma_z(x)=\big(|x-z|^2+\zeta^2\big)^{\frac12}\,,\qquad \zeta=\gamma h\,. \tag{4.1}$$

The parameter $\gamma>0$ is fixed below. We first state certain properties of the weight functions which are easily verified.

LEMMA 4.1. *Consider the weights defined in (4.1).*

*(a) For $\mu \in \mathbb{R}$ and $\beta \in \mathbb{N}_0^N$ there are constants $C > 0$ independent of $x, z \in \Omega$ and $h$ such that the following bounds hold:*

$$\max_{K\in\mathcal{T}_h}\Big(\sup_{x\in K}\sigma_z^{\mu}(x)/\inf_{x\in K}\sigma_z^{\mu}(x)\Big) \le C\,, \tag{4.2a}$$

$$\|\sigma_z^{\mu}\|_{L^\infty} \le C\max\{1,(\gamma h)^{\mu}\}\,, \tag{4.2b}$$

$$\big|\partial_x^{\beta}\sigma_z^{\mu}(x)\big| \le C_\beta\,\sigma_z^{\mu-|\beta|}(x)\,,\quad x\in\Omega_h\,. \tag{4.2c}$$

*(b) If $\alpha > N$, then it holds*

$$\int_\Omega \sigma_z^{-\alpha}(x)\,\mathrm{d}x \le C\max\{1,\tfrac{1}{\alpha-N}\}(\gamma h)^{-\alpha+N}\,. \tag{4.3}$$

REMARK 4.2. In the following computations, we estimate $\sigma_z$ in the maximum norm several times by

$$\|\sigma_z\|_{L^\infty} \le C\big(1+\gamma h\big)\,. \tag{4.4}$$

At the end of the proof, we fix some $\gamma$ sufficiently large which is independent of $h$. Hence, we can estimate (4.4) by some constant $C_\sigma$ which is uniformly bounded in $\gamma$ and $h$ for all $h \le h_0$.

Similar to Lemma 3.3, we need an additional extension of the estimates in [10, Lem. 8.24] and [13, Lem. 7.3] in order to treat the errors stemming from nonconformity in the analysis with weighted norms.

LEMMA 4.3. *Let $\varphi_h, \psi_h \in W^{1,\infty}(\Omega_h)$. The errors in the bilinear forms defined in (3.6) are estimated for any $\alpha \in \mathbb{R}$ by*

$$\begin{aligned}
E^H(\varphi_h,\psi_h) &\le Ch^k\Big(\int_\Omega \sigma_z^{\alpha}\,|\mathcal{L}_h\varphi_h|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega \sigma_z^{-\alpha}\,|\mathcal{L}_h\psi_h|^2\,\mathrm{d}x\Big)^{1/2},\\
E^H(\varphi_h,\psi_h) &\le Ch^{k+1/2}\Big(\int_\Omega \Big|\nabla\big(\sigma_z^{\alpha/2}\mathcal{L}_h\varphi_h\big)\Big|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega \sigma_z^{-\alpha}\,|\mathcal{L}_h\psi_h|^2\,\mathrm{d}x\Big)^{1/2},\\
E^a(\varphi_h,\psi_h) &\le Ch^k\Big(\int_\Omega \sigma_z^{\alpha}\big(|\nabla\mathcal{L}_h\varphi_h|^2+|\mathcal{L}_h\varphi_h|^2\big)\,\mathrm{d}x\Big)^{1/2}\\
&\qquad\Big(\int_\Omega \sigma_z^{-\alpha}\big(|\nabla\mathcal{L}_h\psi_h|^2+|\mathcal{L}_h\psi_h|^2\big)\,\mathrm{d}x\Big)^{1/2},
\end{aligned}$$

*with $C > 0$ independent of $h$ and $\alpha$.*

*Proof.* We proceed as in the proof of Lemma 3.3 and only discuss the inclusion of the weights writing

$$\begin{aligned}
&\big(\mathcal{L}_h\varphi_h \mid \mathcal{L}_h\psi_h\big)_{L^2(\Omega)} - \big(\varphi_h \mid \psi_h\big)_{L^2(\Omega_h)}\\
&= \int_\Omega \sigma_z^{-\alpha/2}\mathcal{L}_h\psi_h\sigma_z^{\alpha/2}\mathcal{L}_h\varphi_h\,\mathrm{d}x - \int_{\Omega_h}\big(\mathcal{L}_h^{-1}\sigma_z^{-\alpha/2}\big)\psi_h\big(\mathcal{L}_h^{-1}\sigma_z^{\alpha/2}\big)\varphi_h\,\mathrm{d}x\,.
\end{aligned}$$

Since the integrand of the first integral is the lift of the integrand in the second integral, the same proof as in [10, Lem. 8.24] gives the first inequality. Similarly, one obtains the last inequality. In order to obtain the additional order of convergence in the second inequality, we apply the narrow band inequality [9, Lem. 4.10] to $\sigma_z^{\alpha/2}\mathcal{L}_h\varphi_h$. □

With these preparations, we are in the position to prove Lemmas 3.1 and 3.2.

**4.2. Proof of Lemma 3.1.** In order to move from $W^{1,1}$ to $H^1$, we use the weight function $\sigma_z$ and obtain the following upper bound by a weighted $H^1$-norm.

LEMMA 4.4. *Let*

$$M_h \coloneqq \sup_{z\in\Omega}\Big(\int_\Omega \sigma_z^{N+\lambda}\,|\nabla(g^z - \mathcal{L}_h g_h^z)|^2 \,\mathrm{d}x\Big)^{1/2}. \tag{4.5}$$

*Then, for $\lambda \in (0,1)$ it holds*

$$\|\mathcal{L}_h g_h^z - g^z\|_{W^{1,1}} \le C M_h \lambda^{-1/2}(\gamma h)^{-\lambda/2},$$

*with a constant $C > 0$ independent of $\gamma$, $\lambda$, and $h$.*

*Proof.* By Hölder's inequality, we have

$$\big\|\nabla\big(\mathcal{L}_h g_h^z - g^z\big)\big\|_{L^1} \le M_h\Big(\int_\Omega \sigma_z^{-N-\lambda}\,\mathrm{d}x\Big)^{1/2} \le C M_h \lambda^{-1/2}(\gamma h)^{-\lambda/2}\,,$$

where we used (4.3) with $\alpha = N + \lambda$ for the last inequality. The application of the Poincaré inequality yields the assertion. ∎

From this, we see that it is sufficient to prove the following proposition from which Lemma 3.1 directly follows.

PROPOSITION 4.5. *There is a $\lambda > 0$ and $\gamma > 1$ such that for all $0 < h < h_0$ it holds for $M_h$ defined in (4.5)*

$$M_h^2 \le C_\gamma h^\lambda\,,$$

*with a constant $C_\gamma > 0$ independent of $h$.*

Before we prove Proposition 4.5, we state the following estimate on weighted norms of $\delta^z$. Later, they give the desired convergence rate $h^\lambda$.

LEMMA 4.6. *For all $\mu > 0$, the bounds*

$$\int_\Omega \sigma_z^{N+\mu}\,|\nabla\mathcal{L}_h\delta^z|^2\,\mathrm{d}x \le C_\gamma h^{\mu-2}, \qquad \int_\Omega \sigma_z^{N+\mu}\,|\mathcal{L}_h\delta^z|^2\,\mathrm{d}x \le C_\gamma h^{\mu}\,,$$

*hold with a constant $C_\gamma > 0$ independent of $h$.*

*Proof.* By the shape-regularity and the definition of the weight in (4.1), we obtain

$$\big\|\sigma_z^{N+\mu}\big\|_{L^\infty(K^z)} \lesssim (\gamma h)^{N+\mu}\,,$$

and use $\delta^z \in C_0^\infty(K^z)$ together with (3.2) to bound

$$\int_\Omega \sigma_z^{N+\mu}\,|\nabla\mathcal{L}_h\delta^z|^2\,\mathrm{d}x \lesssim h^N h^{N+\mu} h^{-2(N+1)} \lesssim h^{\mu-2},$$
$$\int_\Omega \sigma_z^{N+\mu}\,|\mathcal{L}_h\delta^z|^2\,\mathrm{d}x \lesssim h^N h^{N+\mu} h^{-2N} \lesssim h^{\mu}\,,$$

and the claim follows. ∎

In the following, we present an extension of [5, Prop. 8.3.1]. In this step, the weighted $H^1$-norm in (4.5) is replaced by a weighted $L^2$-norm and some additional error terms. We point out that in the conforming case the differences in the scalar product simply vanish.

PROPOSITION 4.7. *Let $g^z \in V$ and $g_h^z \in V_h$ be the solutions of (3.4) and define the errors $e = g^z - \mathcal{L}_h g_h^z$ and $\widehat{e} = (\mathrm{Id} - \mathcal{L}_h I_h) g^z$. Then*

$$\begin{aligned}\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2\,\mathrm{d}x \lesssim{}& \int_\Omega \sigma_z^{N+\lambda-2}\,|e|^2\,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda-2}\,|\widehat{e}|^2\,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda}\,|\nabla\widehat{e}|^2\,\mathrm{d}x\\ &+ \big|(\partial_i\mathcal{L}_h\delta^z \mid \mathcal{L}_h I_h\psi)_{L^2(\Omega)} - (\partial_i\delta^z \mid I_h\psi)_{L^2(\Omega_h)}\big|\\ &+ \big|a_h\,(g_h^z, I_h\psi) - a\,(\mathcal{L}_h g_h^z, \mathcal{L}_h I_h\psi)\big|\,,\end{aligned}$$

*with* $\psi = \sigma_z^{N+\lambda}\mathcal{L}_h(I_h g^z - g_h^z)$.

*Proof.* Let $\widetilde{e} = I_h g^z - g_h^z$, then we have $\psi = \sigma_z^{N+\lambda}\mathcal{L}_h\widetilde{e}$. We note that it holds

$$\mathcal{L}_h\widetilde{e} = e - \widehat{e} \tag{4.6}$$

and compute by (2.1) and the definition of $a(\cdot,\cdot)$ in (2.2)

$$\begin{aligned}\int_\Omega \sigma_z^{N+\lambda}|\nabla e|^2\,\mathrm{d}x &\le c_{\mathcal{A}}\int_\Omega \sigma_z^{N+\lambda}\nabla e\cdot\mathcal{A}\nabla e\,\mathrm{d}x\\ &= c_{\mathcal{A}}a\left(\sigma_z^{N+\lambda}e, e\right) - c_{\mathcal{A}}\int_\Omega(\nabla\sigma_z^{N+\lambda})e\cdot\mathcal{A}\nabla e\,\mathrm{d}x\\ &\quad - c_{\mathcal{A}}\int_\Omega e\mathcal{B}\cdot\nabla\big(\sigma_z^{N+\lambda}e\big)\,\mathrm{d}x - c_{\mathcal{A}}\int_\Omega \mathcal{C}\sigma_z^{N+\lambda}|e|^2\,\mathrm{d}x\\ &= c_{\mathcal{A}}a\left(\sigma_z^{N+\lambda}\widehat{e}, e\right) + c_{\mathcal{A}}a(\psi, e) - c_{\mathcal{A}}\int_\Omega(\nabla\sigma_z^{N+\lambda})e\cdot\mathcal{A}\nabla e\,\mathrm{d}x\\ &\quad - c_{\mathcal{A}}\int_\Omega e\mathcal{B}\cdot\nabla\big(\sigma_z^{N+\lambda}e\big)\,\mathrm{d}x - c_{\mathcal{A}}\int_\Omega \mathcal{C}\,\sigma_z^{N+\lambda}|e|^2\,\mathrm{d}x\,.\end{aligned}$$

Along the lines of the proof of [5, Prop. 8.3.1], we show using Hölder's and Young's inequality

$$\begin{aligned}\int_\Omega \sigma_z^{N+\lambda}|\nabla e|^2\,\mathrm{d}x &\lesssim \int_\Omega \sigma_z^{N+\lambda-2}|e|^2\,\mathrm{d}x\\ &\quad + \int_\Omega \sigma_z^{N+\lambda-2}|\widehat{e}|^2\,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda}|\nabla\widehat{e}|^2\,\mathrm{d}x + |a(\psi,e)|\,.\end{aligned}$$

Hence, we turn to the term

$$a(\psi, e) = a(\psi - \mathcal{L}_h I_h\psi, e) + a\left(\mathcal{L}_h I_h\psi, g^z - \mathcal{L}_h g_h^z\right)\,, \tag{4.7}$$

and note that in the conforming case the last term vanishes by orthogonality. For the first term Lemma 4.8 below shows that for any $\delta \in (0,1)$ it holds

$$\begin{aligned}a(\psi - \mathcal{L}_h I_h\psi, e) &\lesssim \delta\int_\Omega \sigma_z^{N+\lambda}\big(|\nabla e|^2 + |e|^2\big)\,\mathrm{d}x\\ &\quad + \delta^{-1}\int_\Omega \sigma_z^{-N-\lambda}\big(|\nabla(\psi - \mathcal{L}_h I_h\psi)|^2 + |\psi - \mathcal{L}_h I_h\psi|^2\big)\,\mathrm{d}x\\ &\lesssim \delta\int_\Omega \sigma_z^{N+\lambda}|\nabla e|^2\,\mathrm{d}x + \delta^{-1}\int_\Omega \sigma_z^{N+\lambda-2}\big(|e|^2 + |\widehat{e}|^2\big)\,\mathrm{d}x\end{aligned}$$

and absorption leaves the right terms. For the second term in (4.7) it remains to expand

$$\begin{aligned}a\left(\mathcal{L}_h I_h\psi, g^z - \mathcal{L}_h g_h^z\right) &= a\left(\mathcal{L}_h I_h\psi, g^z\right) - a\left(\mathcal{L}_h I_h\psi, \mathcal{L}_h g_h^z\right) \pm a_h\left(I_h\psi, g_h^z\right)\\ &= \left(\mathcal{L}_h I_h\psi \mid -\partial_i\mathcal{L}_h\delta^z\right)_{L^2(\Omega)} + \left(I_h\psi \mid \partial_i\delta^z\right)_{L^2(\Omega_h)}\\ &\quad + a_h\left(I_h\psi, g_h^z\right) - a\left(\mathcal{L}_h I_h\psi, \mathcal{L}_h g_h^z\right)\,,\end{aligned}$$

where we used (3.4) in the second inequality, and the claim follows. ∎

We state the next lemma which was already used above, since we need it several more times in the following computations. It can be found as an auxiliary result in the proof of [5, Prop. 8.3.1].

LEMMA 4.8. *Let* $\psi = \sigma_z^{N+\lambda}\mathcal{L}_h\widetilde{e}$. *Then, it holds the estimate*

$$\int_\Omega \sigma_z^{-N-\lambda}\big(|\nabla(\psi - \mathcal{L}_h I_h\psi)|^2 + |\psi - \mathcal{L}_h I_h\psi|^2\big)\,\mathrm{d}x \lesssim \int_\Omega \sigma_z^{N+\lambda-2}\big(|e|^2 + |\widehat{e}|^2\big)\,\mathrm{d}x\,.$$

The following two lemmas are devoted to control the defects stemming from the nonconformity. For the sake of presentation, we bound the two errors in two separate lemmas. We begin with the difference in the bilinear form $a(\cdot,\cdot)$.

LEMMA 4.9. *For any $\delta > 0$, it holds*

$$\begin{aligned} |a_h(I_h\psi, g_h^z) - a(\mathcal{L}_h I_h\psi, \mathcal{L}_h g_h^z)| &\lesssim (\delta^{-1}h^2)\int_\Omega \sigma_z^{N+\lambda}|\nabla e|^2\,\mathrm{d}x + C_{\gamma,\delta}h^\lambda \\ &\quad + (\delta^{-1}h^2+\delta)\int_\Omega \sigma_z^{N+\lambda-2}|e|^2\,\mathrm{d}x \\ &\quad + \delta\int_\Omega \sigma_z^{N+\lambda}|\nabla\widehat{e}|^2\,\mathrm{d}x + \delta\int_\Omega \sigma_z^{N+\lambda-2}|\widehat{e}|^2\,\mathrm{d}x, \end{aligned}$$

*with constants independent of $h$.*

*Proof.* From Lemma 4.3 we have with $k=1$, $\alpha = N+\lambda$, and Young

$$\begin{aligned} &|a_h(I_h\psi, g_h^z) - a(\mathcal{L}_h I_h\psi, \mathcal{L}_h g_h^z)| \\ &\le Ch\Big(\int_\Omega \sigma_z^{-N-\lambda}|\nabla\mathcal{L}_h I_h\psi|^2 + \sigma_z^{-N-\lambda}|\mathcal{L}_h I_h\psi|^2\,\mathrm{d}x\Big)^{1/2} \\ &\qquad \Big(\int_\Omega \sigma_z^{N+\lambda}|\nabla\mathcal{L}_h g_h^z|^2 + \sigma_z^{N+\lambda}|\mathcal{L}_h g_h^z|^2\,\mathrm{d}x\Big)^{1/2} \\ &\lesssim \delta\int_\Omega \sigma_z^{-N-\lambda}|\nabla\mathcal{L}_h I_h\psi|^2\,\mathrm{d}x + \delta\int_\Omega \sigma_z^{-N-\lambda}|\mathcal{L}_h I_h\psi|^2\,\mathrm{d}x \\ &\qquad + \delta^{-1}h^2\int_\Omega \sigma_z^{N+\lambda}|\nabla\mathcal{L}_h g_h^z|^2 + \sigma_z^{N+\lambda}|\mathcal{L}_h g_h^z|^2\,\mathrm{d}x \\ &= \Delta_\psi^1 + \Delta_\psi^2 + \Delta_g\,. \end{aligned}$$

We recall $\mathcal{L}_h g_h^z = g^z - e$, and estimate

$$\Delta_g \le \delta^{-1}h^2\int_\Omega \sigma_z^{N+\lambda}|\nabla e|^2 + \sigma_z^{N+\lambda}|e|^2\,\mathrm{d}x + \delta^{-1}h^2\int_\Omega \sigma_z^{N+\lambda}|\nabla g^z|^2 + \sigma_z^{N+\lambda}|g^z|^2\,\mathrm{d}x\,.$$

Using (4.2b) and the estimate [5, eq. (8.4.3)] with the subsequent calculations, we obtain

$$\begin{aligned} &\delta^{-1}h^2\int_\Omega \sigma_z^{N+\lambda}|\nabla g^z|^2 + \sigma_z^{N+\lambda}|g^z|^2\,\mathrm{d}x \\ &\lesssim \delta^{-1}h^2\int_\Omega \sigma_z^{N+\lambda-2}|\nabla g^z|^2\,\mathrm{d}x + \delta^{-1}h^2\int_\Omega \sigma_z^{N+\lambda-4}|g^z|^2\,\mathrm{d}x \\ &\lesssim \delta^{-1}h^2\int_\Omega \sigma_z^{N+\lambda}|\nabla\mathcal{L}_h\delta^z|^2\,\mathrm{d}x + \delta^{-1}h^2(\gamma h)^{-2}\int_\Omega \sigma_z^{N+\lambda}|\mathcal{L}_h\delta^z|^2\,\mathrm{d}x \\ &\lesssim C_{\gamma,\delta}h^\lambda\,, \end{aligned}$$

where we used Lemma 4.6 in the last line. For the first term, we expand

$$\begin{aligned}\Delta_\psi^1 &\lesssim \delta \int_\Omega \sigma_z^{-N-\lambda} |\nabla\psi|^2 \,\mathrm{d}x + \delta \int_\Omega \sigma_z^{-N-\lambda} |\nabla(\psi - \mathcal{L}_h I_h \psi)|^2 \,\mathrm{d}x \\ &\lesssim \delta \int_\Omega \sigma_z^{N+\lambda} |\nabla \mathcal{L}_h \widetilde{e}|^2 \,\mathrm{d}x + \delta \int_\Omega \sigma_z^{N+\lambda-2} |\mathcal{L}_h \widetilde{e}|^2 \,\mathrm{d}x \\ &\quad + \delta \int_\Omega \sigma_z^{-N-\lambda} |\nabla(\psi - \mathcal{L}_h I_h \psi)|^2 \,\mathrm{d}x \\ &\lesssim \delta \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x + \delta \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x + \delta \int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x \\ &\quad + \delta \int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x \,,\end{aligned}$$

where we used the definition of $\psi = \sigma_z^{N+\lambda} \mathcal{L}_h \widetilde{e}$, the representation (4.6), and Lemma 4.8. Analogously, we obtain

$$\Delta_\psi^2 \lesssim \delta \int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x + \delta \int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x \,, \tag{4.8}$$

and the assertion follows. ∎

By similar techniques, we derive the second bound.

LEMMA 4.10. *For any $\delta > 0$, there is a constant $C_{\gamma,\delta} > 0$ independent of $h$ such that*

$$\begin{aligned}&\left| (\mathcal{L}_h I_h \psi \mid \partial_i \mathcal{L}_h \delta_z)_{L^2(\Omega)} - (I_h \psi \mid \partial_i \delta_z)_{L^2(\Omega_h)} \right| \\ &\lesssim \delta \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x + C_{\gamma,\delta} h^\lambda + \delta \int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x \\ &\qquad + \delta \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x + \delta \int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x.\end{aligned}$$

*Proof.* We employ Lemmas 4.3 and 4.6 to conclude

$$\begin{aligned}&\left| (\mathcal{L}_h I_h \psi \mid \partial_i \mathcal{L}_h \delta_z)_{L^2(\Omega)} - (I_h \psi \mid \partial_i \delta_z)_{L^2(\Omega_h)} \right| \\ &\le \delta \int_\Omega \sigma_z^{-N-\lambda} |\mathcal{L}_h I_h \psi|^2 \,\mathrm{d}x + C_\delta h^2 \int_\Omega \sigma_z^{N+\lambda} |\partial_i \mathcal{L}_h \delta_z|^2 \,\mathrm{d}x \\ &\lesssim \delta \int_\Omega \sigma_z^{-N-\lambda} |\mathcal{L}_h I_h \psi|^2 \,\mathrm{d}x + C_{\gamma,\delta} h^\lambda \,,\end{aligned}$$

and the claim follows as for $\Delta_\psi^2$ in (4.8). ∎

If we combine the bounds from Proposition 4.7, Lemma 4.9, and Lemma 4.10, we have shown, for $\delta, h$ sufficiently small, that it holds

$$\begin{aligned}\int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x \lesssim \int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x + C_\gamma h^\lambda \\ + \int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x \,.\end{aligned}$$

Hence, to establish Proposition 4.5 it remains to absorb the weighted $L^2$-norm of $e$ and to obtain a factor $h^\lambda$ for the $\widehat{e}$ terms. This is done in the following two propositions. The first one estimates the interpolation error, which we state from [5] for completeness.

PROPOSITION 4.11. *For $\widehat{e} = (\mathrm{Id} - \mathcal{L}_h I_h) g^z$ it holds*

$$\int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x \le C_\gamma h^\lambda \,,$$

*with some constant $C_\gamma > 0$ independent of $h$.*

*Proof.* Using the interpolation estimate, one obtains

$$\int_\Omega \sigma_z^{N+\lambda-2} |\widehat{e}|^2 \,\mathrm{d}x + \int_\Omega \sigma_z^{N+\lambda} |\nabla \widehat{e}|^2 \,\mathrm{d}x \lesssim h^2 \int_\Omega \sigma_z^{N+\lambda} |\nabla_2 g^z|^2 \,\mathrm{d}x\,.$$

The application of [5, Lem. 8.3.11] and Lemma 4.6 yields the result. ∎

The proof of Proposition 4.5 is closed once we have shown the following bound, which extends the result of [5, Prop. 8.3.5] again due to the lack of orthogonality.

PROPOSITION 4.12. *For any $\varepsilon > 0$, there is $\gamma_0 > 1$ and $C_{\gamma,\varepsilon}$ such that*

$$\int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x \le \varepsilon \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x + C_{\gamma,\varepsilon} h^\lambda$$

*for all $\gamma \ge \gamma_0 = \gamma_0(\varepsilon, \lambda)$.*

*Proof.* We define $v \in V$ as the solution of

$$a\,(v, \phi) = \left(\sigma_z^{N+\lambda-2} e \mid \phi\right)_{L^2(\Omega)} \quad \forall \phi \in V\,,$$

and obtain

$$\int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x = a\,(v, e) = a\,(v - \mathcal{L}_h I_h v, e) + a\,(\mathcal{L}_h I_h v, e)\,.$$

Note again, that in the conforming case the second term vanishes. The first term is estimated as in the proof of [5, Prop. 8.3.5] for any $\widetilde{\varepsilon} > 0$ by

$$\begin{aligned} a\,(v - \mathcal{L}_h I_h v, e) &\le \widetilde{\varepsilon} \int_\Omega \sigma_z^{N+\lambda} \left(|\nabla e|^2 + |e|^2\right) \mathrm{d}x \\ &\quad + \frac{C}{\lambda \widetilde{\varepsilon} \gamma^2} \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 + \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x\,. \end{aligned} \tag{4.9}$$

Turning to the second term, using (3.4) we obtain

$$\begin{aligned} &a\,(\mathcal{L}_h I_h v, e) \\ &= a\,(\mathcal{L}_h I_h v, g^z) - a\,(\mathcal{L}_h I_h v, \mathcal{L}_h g_h^z) \\ &= \left(\mathcal{L}_h I_h v \mid -\partial_i \mathcal{L}_h \delta^z\right)_{L^2(\Omega)} - a_h\,(I_h v, g_h^z) + a_h\,(I_h v, g_h^z) - a\,(\mathcal{L}_h I_h v, \mathcal{L}_h g_h^z) \\ &= \left(\mathcal{L}_h I_h v \mid -\partial_i \mathcal{L}_h \delta^z\right)_{L^2(\Omega)} + \left(I_h v \mid \partial_i \delta^z\right)_{L^2(\Omega_h)} + a_h\,(I_h v, g_h^z) - a\,(\mathcal{L}_h I_h v, \mathcal{L}_h g_h^z) \\ &= \Delta_H + \Delta_a\,. \end{aligned}$$

The two terms are estimated separately in the following.

(1) We use integration by parts and apply Lemma 4.3 with $k = 1$ to obtain

$$\begin{aligned} |\Delta_H| &\le C h^{3/2} \Big(\int_\Omega \sigma_z^{N+\lambda} |\nabla \mathcal{L}_h \delta^z|^2 + \sigma_z^{N+\lambda-2} |\mathcal{L}_h \delta^z|^2 \,\mathrm{d}x\Big)^{1/2} \\ &\qquad \Big(\int_\Omega \sigma_z^{-N-\lambda} |\nabla \mathcal{L}_h I_h v|^2 \,\mathrm{d}x\Big)^{1/2} \\ &\le C h^2 \int_\Omega \sigma_z^{N+\lambda} |\nabla \mathcal{L}_h \delta^z|^2 + \sigma_z^{N+\lambda-2} |\mathcal{L}_h \delta^z|^2 \,\mathrm{d}x + C h \int_\Omega \sigma_z^{-N-\lambda} |\nabla \mathcal{L}_h I_h v|^2 \,\mathrm{d}x \\ &\le C_\gamma h^\lambda + C h \int_\Omega \sigma_z^{-N-\lambda} |\nabla v|^2 \,\mathrm{d}x + C h \int_\Omega \sigma_z^{-N-\lambda} |\nabla (v - \mathcal{L}_h I_h v)|^2 \,\mathrm{d}x\,, \end{aligned}$$

where we used Lemma 4.6 in the last step. For the interpolation term, we derive as in the proof of [5, Prop. 8.3.5] analogously to (4.9)

$$\int_\Omega \sigma_z^{-N-\lambda} |\nabla(v - \mathcal{L}_h I_h v)|^2 \,\mathrm{d}x \le \frac{C}{\lambda\gamma^2} \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 + \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x\,. \tag{4.10}$$

Finally, we employ Lemma A.1

$$\begin{aligned} h\int_\Omega \sigma_z^{-N-\lambda} |\nabla v|^2 \,\mathrm{d}x &\lesssim h(\gamma h)^{-1} \int_\Omega \sigma_z^{4-N-\lambda} \left|\nabla(\sigma_z^{N+\lambda-2} e)\right|^2 \mathrm{d}x \\ &\lesssim \gamma^{-1} \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 + \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x\,, \end{aligned}$$

and collect this to derive

$$|\Delta_H| \lesssim C_\gamma h^\lambda + \big(h(\lambda\gamma^2)^{-1} + \gamma^{-1}\big) \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 + \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x\,. \tag{4.11}$$

(2) We employ Lemma 4.3 and obtain with $k = 1$ and Young's inequality

$$\begin{aligned} |\Delta_a| \le \delta h^{3/2} &\int_\Omega \sigma_z^{N+\lambda-3/2} \big(|\nabla \mathcal{L}_h g_h^z|^2 + |\mathcal{L}_h g_h^z|^2\big) \,\mathrm{d}x \\ &+ \delta^{-1} h^{1/2} \int_\Omega \sigma_z^{-N-\lambda+3/2} \big(|\nabla \mathcal{L}_h I_h v|^2 + |\mathcal{L}_h I_h v|^2\big) \,\mathrm{d}x\,. \end{aligned}$$

For the first term we obtain as in Lemma 4.9 using $\mathcal{L}_h g_h^z = g^z - e$ and $h \le \sigma_z(x)$

$$\begin{aligned} &\delta h^{3/2} \int_\Omega \sigma_z^{N+\lambda-3/2} |\nabla \mathcal{L}_h g_h^z|^2 \,\mathrm{d}x \\ \le &\delta h^{3/2} \int_\Omega \sigma_z^{N+\lambda-3/2} |\nabla e|^2 \,\mathrm{d}x + \delta h^{3/2} \int_\Omega \sigma_z^{N+\lambda-3/2} |\nabla g^z|^2 \,\mathrm{d}x \\ \le &\delta \int_\Omega \sigma_z^{N+\lambda} |\nabla e|^2 \,\mathrm{d}x + \delta h^{3/2} \int_\Omega \sigma_z^{N+\lambda-3/2} |\nabla g^z|^2 \,\mathrm{d}x\,. \end{aligned}$$

With Lemma A.3, $\alpha = 1/2$ and $f = \mathcal{L}_h \delta^z$ we obtain

$$\begin{aligned} \delta h^{3/2} \int_\Omega \sigma_z^{N+\lambda-3/2} |\nabla g^z|^2 \,\mathrm{d}x &\le \delta h^{3/2} \int_\Omega \sigma_z^{N+\lambda+1/2} |\nabla \mathcal{L}_h \delta^z|^2 \,\mathrm{d}x \\ &\quad + \delta h^{3/2} (\gamma h)^{-3/2} \int_\Omega \sigma_z^{N+\lambda} |\mathcal{L}_h \delta^z|^2 \,\mathrm{d}x \\ &\lesssim C_\gamma h^\lambda\,, \end{aligned}$$

where we used Lemma 4.6 in the last step. Along the same lines, we deduce

$$\delta h^{3/2} \int_\Omega \sigma_z^{N+\lambda-3/2} |\mathcal{L}_h g_h^z|^2 \,\mathrm{d}x \lesssim \delta h^{3/2} \int_\Omega \sigma_z^{N+\lambda-2} |e|^2 \,\mathrm{d}x + C_\gamma h^\lambda\,.$$

Further, for the second term we expand

$$\begin{aligned} \delta^{-1} h^{1/2} \int_\Omega \sigma_z^{-N-\lambda+3/2} |\nabla \mathcal{L}_h I_h v|^2 \,\mathrm{d}x &\le \delta^{-1} h^{1/2} \int_\Omega \sigma_z^{-N-\lambda+3/2} |\nabla(v - \mathcal{L}_h I_h v)|^2 \,\mathrm{d}x \\ &\quad + \delta^{-1} h^{1/2} \int_\Omega \sigma_z^{-N-\lambda+3/2} |\nabla v|^2 \,\mathrm{d}x\,, \end{aligned}$$

and the first part is treated by an interpolation estimate as in (4.10) with (4.2b)

$$\delta^{-1}h^{1/2}\int_\Omega \sigma_z^{-N-\lambda+3/2}\,|\nabla(v-\mathcal{L}_h I_h v)|^2\,\mathrm{d}x \lesssim \frac{h^{1/2}}{\delta\lambda\gamma^2}\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2+\sigma_z^{N+\lambda-2}\,|e|^2\,\mathrm{d}x\,.$$

So it remains to bound by Lemma A.2 with $\alpha=1/2$ and $f=\sigma_z^{N+\lambda-2}e$

$$\begin{aligned}\delta^{-1}h^{1/2}\int_\Omega \sigma_z^{-N-\lambda+3/2}\,|\nabla v|^2\,\mathrm{d}x &\le Ch^{1/2}(\delta\lambda)^{-1}(\gamma h)^{-1/2}\int_\Omega \sigma_z^{4-N-\lambda}\,|\nabla f|^2\,\mathrm{d}x\\ &\le C(\delta\lambda\gamma)^{-1}\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2+\sigma_z^{N+\lambda-2}\,|e|^2\,\mathrm{d}x\,.\end{aligned}$$

By the same arguments, we show

$$\begin{aligned}\delta^{-1}h^{1/2}\int_\Omega \sigma_z^{-N-\lambda+3/2}\,|\mathcal{L}_h I_h v|^2\,\mathrm{d}x &\le \frac{h^{1/2}}{\delta\lambda\gamma^2}\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2+\sigma_z^{N+\lambda-2}\,|e|^2\,\mathrm{d}x\\ &\quad + C(\delta\lambda\gamma)^{-1}\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2+\sigma_z^{N+\lambda-2}\,|e|^2\,\mathrm{d}x\,,\end{aligned}$$

and collecting the above estimates gives

$$\Delta_a \lesssim C_\gamma h^\lambda + \big(\delta+\frac{h^{1/2}}{\delta\lambda\gamma^2}+\frac{1}{\delta\lambda\gamma}\big)\int_\Omega \sigma_z^{N+\lambda}\,|\nabla e|^2+\sigma_z^{N+\lambda-2}\,|e|^2\,\mathrm{d}x\,. \tag{4.12}$$

We close the proof using (4.9), (4.11), and (4.12), and absorb the right-hand side for $\widetilde{\varepsilon}$ and $\lambda$ fixed by first choosing some $\delta>0$ sufficiently small and then some sufficiently large $\gamma\ge\gamma_0=\gamma_0(\widetilde{\varepsilon},\lambda,\delta)$. ◻

Collecting the above lemmas, we have finally shown Proposition 4.5 and hence, by Lemma 4.4, also Lemma 3.1.

**4.3. Proof of Lemma 3.2.** We close this section by the proof of the bound on the solution $g^z$ of the regularized delta function. The key tool is the generalized version of the narrow band inequality shown for $p=2$ in [9, Lem. 4.10]. We recall $U_\delta=\{x\in\Omega\mid \operatorname{dist}(x,\Gamma)\le\delta\}$. Then for any $1\le p<\infty$, there is a constant $C_p>0$ such that for any $\varphi\in W^{1,p}(\Omega)$ it holds

$$\|\varphi\|_{L^p(U_\delta)} \le C_p\,\delta^{1/p}\,\|\varphi\|_{W^{1,p}(\Omega)}\,. \tag{4.13}$$

We apply (4.13) with $p=1$ and $\delta=h$ and obtain

$$\|g^z\|_{W^{1,1}(U_h)} \lesssim h\,\|g^z\|_{W^{2,1}(\Omega)}\,. \tag{4.14}$$

Note that the second term in the lemma is estimated against the right-hand side in (4.14) as well. Finally, we deduce by (4.3) and the elliptic regularity shown in [5, eq. (8.3.10)] the bound

$$\|\nabla_2 g^z\|_{L^1(\Omega)}^2 \lesssim h^{-\lambda}\int_\Omega \sigma_z^{N+\lambda}|\nabla_2 g^z|^2\,\mathrm{d}x \lesssim h^{-\lambda}h^{\lambda-2}\lesssim h^{-2}\,,$$

which also holds for the lower order derivatives and the assertion follows. ◻

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**Appendix A.** In this section, we collect the regularity results used in the above analysis. These are taken from [5, Chap. 8] and stated here in a slightly more general version. We recall the weight defined in (4.1) as

$$\sigma_z(x) = \left(|x-z|^2 + \zeta^2\right)^{\frac{1}{2}}, \qquad \zeta = \gamma h\,.$$

The first result is an extension of [5, Lem. 8.3.7], where the Hessian is replaced by the gradient which allows to obtain a factor $h^{-1}$ instead of $h^{-2}$.

LEMMA A.1. *Let $v \in V$ be the solution of*

$$a\left(v,\phi\right) = \left(f \mid \phi\right)_H\,, \quad \forall \phi \in V$$

*for $f \in H_0^1(\Omega)$. Then, we have for $\lambda > 0$ sufficiently small*

$$\int_\Omega \sigma_z^{-N-\lambda}\left|\nabla v\right|^2 \mathrm{d}x \le C\lambda^{-1}\zeta^{-1}\int_\Omega \sigma_z^{4-N-\lambda}\left|\nabla f\right|^2 \mathrm{d}x.$$

*Proof.* In the proof of [5, Lem. 8.3.7], one first estimates by Hölder's inequality

$$\int_\Omega \sigma_z^{-N-\lambda}\left|\nabla v\right|^2 \mathrm{d}x \lesssim \zeta^{-\lambda-N/p}\left\|\nabla v\right\|_{L^{2p}}^2\,.$$

Once, we have shown that for any $p, s > 1$

$$\left\|\nabla v\right\|_{L^{2p}} \lesssim \left\|\nabla f\right\|_{L^1} \lesssim \left\|\nabla f\right\|_{L^s}\,, \tag{A.1}$$

we conclude with $s = \frac{2pN}{N+3p} := \frac{2}{q} \in (1,2)$

$$\begin{aligned}
\left\|\nabla f\right\|_{L^s}^s &= \int_\Omega \left|\nabla f\right|^{2/q}\mathrm{d}x \\
&= \int_\Omega \sigma_z^{-\frac{4-N-\lambda}{q}}\sigma_z^{\frac{4-N-\lambda}{q}}\left|\nabla f\right|^{2/q}\mathrm{d}x \\
&\le \Big(\int_\Omega \sigma_z^{-(4-N-\lambda)\frac{q'}{q}}\,\mathrm{d}x\Big)^{1/q'}\Big(\int_\Omega \sigma_z^{4-N-\lambda}\left|\nabla f\right|^2\mathrm{d}x\Big)^{1/q}
\end{aligned}$$

and hence

$$\left\|\nabla f\right\|_{L^s}^2 = \Big(\int_\Omega \left|\nabla f\right|^{2/q}\mathrm{d}x\Big)^q \le \Big(\int_\Omega \sigma_z^{-(4-N-\lambda)\frac{q'}{q}}\,\mathrm{d}x\Big)^{q/q'}\int_\Omega \sigma_z^{4-N-\lambda}\left|\nabla f\right|^2\mathrm{d}x.$$

With $\frac{q}{q'} = q-1$ we have by (4.3)

$$\begin{aligned}
\Big(\int_\Omega \sigma_z^{-(4-N-\lambda)\frac{q'}{q}}\,\mathrm{d}x\Big)^{q/q'} &= \Big(\int_\Omega \sigma_z^{-(4-N-\lambda)/(q-1)}\,\mathrm{d}x\Big)^{q-1} \\
&\le C\zeta^{-(4-N-\lambda)+N(q-1)} \\
&= C\zeta^{-1+\lambda+\frac{N}{p}}
\end{aligned}$$

since

$$-(4-N-\lambda) + N(q-1) = -4+N+\lambda+N(\frac{N+3p}{pN}-1) = -1+\lambda+\frac{N}{p} < 0$$

for $\lambda < 1 - \frac{N}{p}$ and hence the claim follows.

It remains to prove (A.1). We employ Theorem 2.1 and [1, Thm. 4.12] to obtain

$$\|\nabla v\|_{L^{2p}} \lesssim \|v\|_{W^{2,N/2}} \lesssim \|f\|_{L^{3/2}} \lesssim \|f\|_{W^{1,1}} \lesssim \|\nabla f\|_{L^1}$$

where we use Case B ($mp = N$) for the first inequality, Case C ($m = p = 1$) for the third, and the Poincaré inequality for the last. $\square$

The next lemma is a straight forward extension of [5, Lem 8.3.7], where the case $\alpha = 2$ is derived.

LEMMA A.2. *Let $v \in V$ be the solution of*

$$a(v, \phi) = (f \mid \phi)_H, \quad \forall \phi \in V.$$

*Then for $0 < \alpha \leq 2$ and $\lambda > 0$ sufficiently small, we have*

$$\int_\Omega \sigma_z^{-N-\lambda+2-\alpha}\left(|v|^2 + |\nabla v|^2 + |\nabla_2 v|^2\right) \mathrm{d}x \leq C\lambda^{-1}\zeta^{-\alpha} \int_\Omega \sigma_z^{4-N-\lambda} |\nabla f|^2 \,\mathrm{d}x.$$

*Proof.* In order to adapt the proof, it is sufficient to guarantee the existence of a $p \in (1, \infty)$ such that the conditions

$$p > \frac{N}{2-\lambda}, \qquad p < \frac{N}{N-2},$$

and, in order to apply (4.3),

$$(-N - \lambda + 2 - \alpha)p' + N < 0 \quad \Longleftrightarrow \quad \frac{N}{p} > 2 - \alpha - \lambda$$

are all satisfied. For $2 \leq \alpha + \lambda$, the latter condition is empty. In the other cases, it is equivalent to

$$p < \frac{N}{2 - \lambda - \alpha},$$

and since $\alpha > 0$, such a $p$ can be found. $\square$

The following lemma builds upon the estimates in [5, Lem. 8.3.11], where the result is shown for $\alpha = 0$.

LEMMA A.3. *Let $v \in V$ be the solution of*

$$a(\phi, v) = (\nu \cdot \nabla f \mid \phi)_H, \quad \forall \phi \in V.$$

*Then for $\lambda > 0$ sufficiently small and $0 \leq \alpha < 1 - \lambda$, we have*

$$\int_\Omega \sigma_z^{N+\lambda-2+\alpha} |\nabla v|^2 + \sigma_z^{N+\lambda-4+\alpha} |v|^2 \,\mathrm{d}x \lesssim \int_\Omega \sigma_z^{N+\lambda+\alpha} |\nabla f|^2 \,\mathrm{d}x + \zeta^{-2+\alpha} \int_\Omega \sigma_z^{N+\lambda} |f|^2 \,\mathrm{d}x.$$

*Proof.* We use the above equation with $\phi = \sigma_z^{N+\lambda-2+\alpha} v$ and compute as in the proof of Proposition 4.7

$$\begin{aligned}
&\int_\Omega \sigma_z^{N+\lambda-2+\alpha} |\nabla v|^2 \\
&\leq C_{\mathcal{A}} a\left(\sigma_z^{N+\lambda-2+\alpha} v, v\right) - C_{\mathcal{A}} \int_\Omega (\nabla \sigma_z^{N+\lambda-2+\alpha}) v \cdot \mathcal{A} \nabla v \, \mathrm{d}x \\
&\qquad - C_{\mathcal{A}} \int_\Omega v \mathcal{B} \cdot \nabla\big(\sigma_z^{N+\lambda-2+\alpha} v\big) \, \mathrm{d}x - C_{\mathcal{A}} \int_\Omega \mathcal{C} \, \sigma_z^{N+\lambda-2+\alpha} |v|^2 \, \mathrm{d}x \\
&\lesssim \left| \big(\nu \cdot \nabla f \mid \sigma_z^{N+\lambda-2+\alpha} v\big)_H \right| + a \int_\Omega \sigma_z^{N+\lambda-2+\alpha} |\nabla v|^2 \, \mathrm{d}x + \tfrac{1}{a} \int_\Omega \sigma_z^{N+\lambda-4+\alpha} |v|^2 \, \mathrm{d}x \\
&\lesssim \int_\Omega \sigma_z^{N+\lambda+\alpha} |\nabla f|^2 \, \mathrm{d}x + a \int_\Omega \sigma_z^{N+\lambda-2+\alpha} |\nabla v|^2 \, \mathrm{d}x + \tfrac{1}{a} \int_\Omega \sigma_z^{N+\lambda-4+\alpha} |v|^2 \, \mathrm{d}x
\end{aligned}$$

and by absorption, it only remains to bound the last term. We claim that for $\alpha < 1 - \lambda$

$$\int_\Omega \sigma_z^{N+\lambda-4+\alpha} |v|^2 \leq C \zeta^{-2+\alpha} \int_\Omega \sigma_z^{N+\lambda} |f|^2 \, \mathrm{d}x \,,$$

which can be adapted from the proof of [5, Lem. 8.3.11], starting with equation (8.4.3), if one can find $r > 1$ with

$$r < \frac{2N}{2N - 2 + \lambda + \alpha} \,,$$

which is possible since $\alpha + \lambda < 1$. ◻