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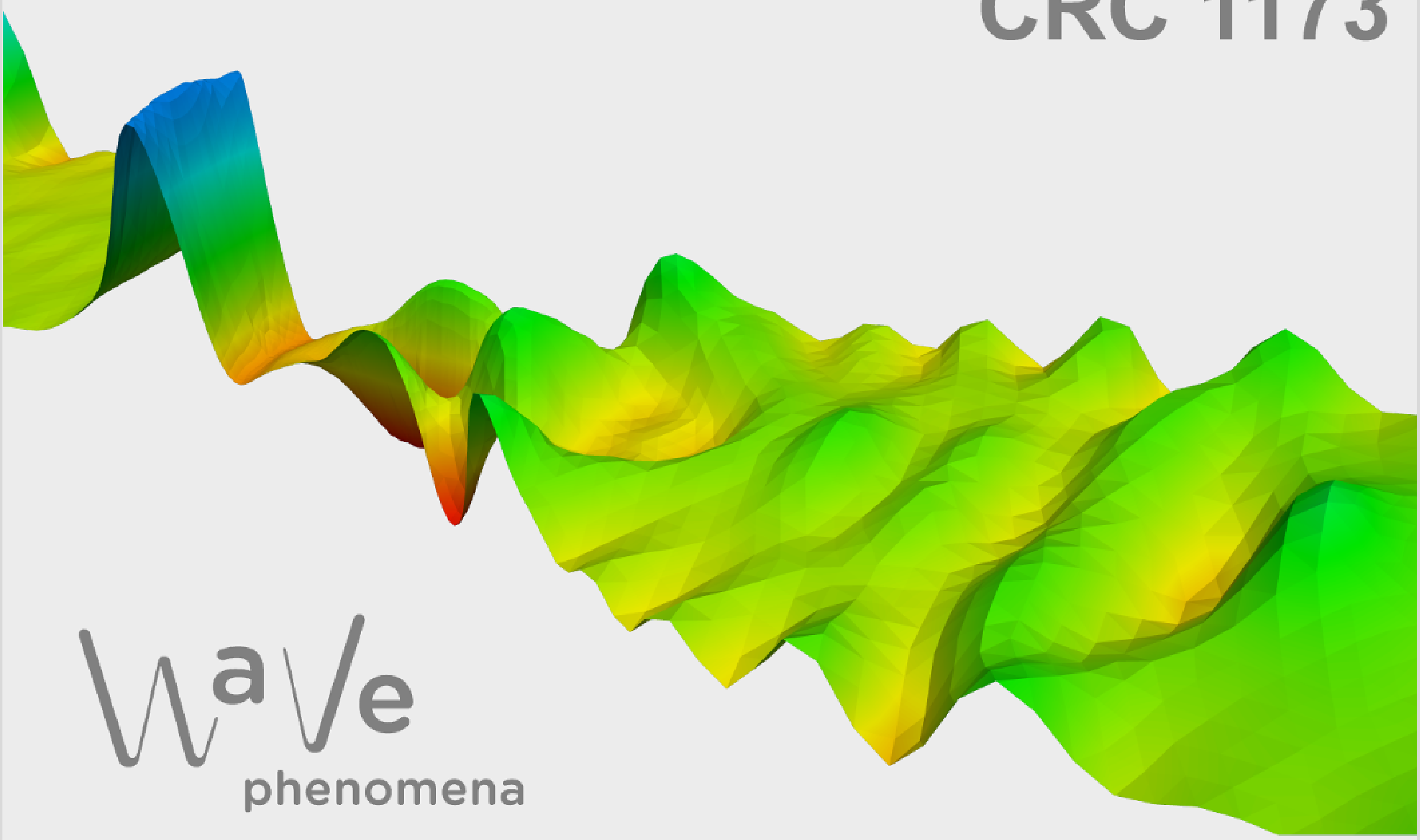

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A UNIQUENESS RESULT FOR THE SINE-GORDON BREATHER

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Abstract. In this note we prove that the sine-Gordon breather is the only quasimonochromatic breather in the context of nonlinear wave equations in $\mathbb{R}^N$.

1. Introduction

Breathers are time-periodic and spatially localized patterns that describe the propagation of waves. The most impressive solution of this kind is the so-called sine-Gordon breather for the 1D sine-Gordon equation

$$\partial_{tt}u - \partial_{xx}u + \sin(u) = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}.$$

It is given by the explicit formula

$$u^*(x,t) = 4\arctan\left(\frac{m\sin(\omega t)}{\omega\cosh(mx)}\right) \qquad \text{for } (x,t)\in\mathbb{R}\times\mathbb{R}, \tag{1}$$

where the parameters $m,\omega>0$ satisfy $m^2+\omega^2=1$. It is natural to ask if other real-valued breather solutions exist. We shall address this question in the broader context of more general nonlinear wave equations of the form

$$\partial_{tt}u - \Delta u = g(u) \quad \text{in } \mathbb{R}^N\times\mathbb{R}, \tag{2}$$

where the space dimension $N\in\mathbb{N}$ and the nonlinearity $g:\mathbb{R}\to\mathbb{R}$ are arbitrary.

The existence of radially symmetric breather solutions for the cubic Klein-Gordon equation $g(z)=-m^2z+z^3$, $m>0$ in three spatial dimensions was established in [10]. These real-valued solutions are only weakly localized in the sense that they satisfy $u(\cdot,t)\in L^q(\mathbb{R}^N)$ for some $q\in(2,\infty)$ but $u(\cdot,t)\notin L^2(\mathbb{R}^N)$. In [8] infinitely many weakly localized breathers were found for nonlinearities $Q(x)|u|^{p-2}u$ where Q lies in a suitable Lebesgue space and $p>2$ is chosen suitably depending on Q as well as the space dimension $N\geq 2$. Up to now, nothing is known about the existence of strongly localized breathers of (2) satisfying $u(\cdot,t)\in L^2(\mathbb{R}^N)$ for almost all $t\in\mathbb{R}$ and $N\geq 2$, see however [9] for a an existence result for semilinear curl-curl equations for $N=3$. In the case $N=1$ strongly localized breather solutions different from the sine-Gordon breather have been found for nonlinear wave equations of the form

$$s(x)\partial_{tt}u - u_{xx} + q(x)u = f(x,u) \qquad (x\in\mathbb{R})$$

where the coefficient functions s,q are discontinuous and periodic, see [5, Theorem 1.3] and [1, Theorem 1.1]. Given the discontinuity of s,q it must be expected that these breathers are not twice continuously differentiable. To sum up, the existence of smooth and strongly

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localized breather solutions of (2) different from the sine-Gordon breather is not known. Still for $N = 1$ there are nonexistence results by Denzler [2] and Kowalczyk, Martel, Muñoz [7] dealing with small perturbations of the sine-Gordon equation respectively small odd breathers (not covering the even sine-Gordon breather). We are not aware of any other mathematically rigorous existence or nonexistence results for (2).

One of the main obstructions for the construction of localized breathers is polychromaticity. Indeed, plugging in an ansatz of the form $u(x,t) = \sum_{k\in\mathbb{Z}} u_k(x)e^{ikt}$ with $u_k = \overline{u_{-k}}$ one ends up with infinitely many equations of nonlinear Helmholtz type that typically do not possess strongly localized solutions, see for instance [6, Theorem 1a]. For this reason the solutions obtained in [8, 10] are only weakly localized. On the other hand, a purely monochromatic ansatz like $u(x,t) = \sin(\omega t)p(x)$ cannot be successful either provided that g is not a linear function. In view of the formula (1) for the sine-Gordon breather we investigate whether quasimonochromatic breathers exist.

Definition 1. *We call the function $u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ a quasimonochromatic breather if*

$$u(x,t) = F(\sin(\omega t)p(x)) \qquad (x \in \mathbb{R}^N, t \in \mathbb{R})$$

for some $\omega \in \mathbb{R} \setminus \{0\}$ and nontrivial functions $F \in C^2(\mathbb{R}), p \in C^2(\mathbb{R}^N)$ such that $F(0) = 0$ and $p(x) \to 0$ as $|x| \to \infty$.

We show that in one spatial dimension the sine-Gordon breather is, up to translation and dilation, the only one for (2) and that no such breathers exist in higher dimensions.

Theorem 1. *Assume $N \in \mathbb{N}$ and that $g : \mathbb{R} \to \mathbb{R}$ is not a linear function.*

(i) In the case $N \geq 2$ there is no quasimonochromatic breather solution of (2).

(ii) In the case $N = 1$ each quasimonochromatic breather solution of (2) is of the form $u(x,t) = \kappa u^*(x-x_0,t)$ for $x_0 \in \mathbb{R}$, $m, \omega, \kappa \in \mathbb{R}\setminus\{0\}$ and u^* as in (1). The nonlinearity then satisfies $g(z) = -(m^2+\omega^2)\kappa\sin(\kappa^{-1}z)$ whenever $|z| < 2\pi|\kappa|$.

We stress that our result holds regardless of any smoothness assumption on g nor any kind of growth condition at 0 or infinity. Moreover, our considerations are not limited to small perturbations of u^* or small breathers in whatever sense. Following the proof of Theorem 1 one also finds that quasimonochromatic breathers of wave equations on any open set $\Omega \subsetneq \mathbb{R}^N$ with homogeneous Dirichlet conditions

$$\partial_{tt}u - \Delta u = g(u) \quad \text{in } \Omega \times \mathbb{R}, \qquad u = 0 \text{ on } \partial\Omega \times \mathbb{R} \tag{3}$$

with profile functions $p \in C^2(\overline{\Omega})$ do not exist either (even if $N = 1$) provided that g is not linear. We will comment on this fact at the end of this paper.

For completeness we briefly comment on the linear case $g(z) = \beta z$, $\beta \in \mathbb{R}$. Then the profile function p of any given quasimonochromatic breather of (2) satisfies the linear elliptic PDE $-\Delta p - (\omega^2 + \beta)p = 0$ in $\mathbb{R}^N$. For $\beta < -\omega^2$ there are positive, radially symmetric and exponentially decaying solutions p, see [3, Theorem 2]. In the case $\beta > \omega^2, N \geq 2$ one can find radial as well as non-radial solutions of the associated Helmholtz equation all of which have infinitely many nodal domains and satisfy $|p(x)| + |\nabla p(x)| \gtrsim |x|^{\frac{1-N}{2}}$ in a suitable integrated sense, see [11, Theorem 1] respectively [6, Theorem 1a]. For $\beta > -\omega^2, N = 1$ all solutions are linear combinations of sin and cos so that breather solutions do not exist. So we see that the picture is already quite complete in the case of linear wave equations.

2. Proof of Theorem 1

In the following let $u(x,t) = F(\sin(\omega t)p(x))$ be a solution of (2) as in (1) with g as in the Theorem. Plugging in this ansatz we get for all $x \in \mathbb{R}^N$ such that $p(x) \neq 0$,

$$\begin{aligned}\partial_{tt}u(x,t) &= -\omega^2 \sin(\omega t)p(x)F'(\sin(\omega t)p(x)) + \omega^2\cos(\omega t)^2 p(x)^2 F''(\sin(\omega t)p(x)) \\ &= -\omega^2 zF'(z) + \omega^2(p(x)^2 - z^2)F''(z), \\ \Delta u(x,t) &= \sin(\omega t)\Delta p(x)F'(\sin(\omega t)p(x)) + \sin(\omega t)^2|\nabla p(x)|^2 F''(\sin(\omega t)p(x)) \\ &= \frac{\Delta p(x)}{p(x)} zF'(z) + \frac{|\nabla p(x)|^2}{p(x)^2} z^2 F''(z),\end{aligned}$$

where $z = \sin(\omega t)p(x) \in [-\|p\|_\infty, +\|p\|_\infty]$. This and (2) imply for $x \in \mathbb{R}^N, z \in \mathbb{R}$ such that $p(x) \neq 0, z \in [-\|p\|_\infty, +\|p\|_\infty]$

$$g(F(z)) + \omega^2 zF'(z) + \omega^2 z^2 F''(z) = p(x)^2\omega^2 F''(z) - \frac{\Delta p(x)}{p(x)} zF'(z) - \frac{|\nabla p(x)|^2}{p(x)^2} z^2 F''(z). \tag{4}$$

If F was linear on $[-\|p\|_\infty, +\|p\|_\infty]$, then g would be linear as well. Since the latter is not true by assumption, we know that $z \mapsto z^2 F''(z)$ does not vanish identically on that interval. Multiplying (4) with $p(x)$ and choosing z according to $F''(z)z \neq 0$ we find that p does not change sign. Indeed, if $p(x^*) \neq 0$ and $R > 0$ is the smallest radius such that p is a fixed sign in the open ball $B_R(x^*)$, then Hopf's Lemma [4, Lemma 3.4] implies $|\nabla p| > 0$ on $\partial B_R(x^*)$. But then (4) implies that Δp is unbounded on $\partial B_R(x^*)$, which contradicts $p \in C^2(\mathbb{R}^N)$. Hence, p does not change sign and we will without loss of generality assume that p is positive. So (4) holds for all $x \in \mathbb{R}^N$ and all $z \in [-\|p\|_\infty, \|p\|_\infty]$ and standard elliptic regularity theory gives $p \in C^\infty(\mathbb{R}^N)$.

Differentiating (4) with respect to x_i we get

$$\partial_i(p(x)^2)\omega^2 F''(z) - \partial_i\left(\frac{\Delta p(x)}{p(x)}\right) zF'(z) - \partial_i\left(\frac{|\nabla p(x)|^2}{p(x)^2}\right) z^2 F''(z) = 0. \tag{5}$$

Since p^2 is non-constant, we infer that F satisfies an ODE of the form

$$F''(z) = \frac{-\mu_2 z}{\omega^2 + \mu_1 z^2} F'(z) \qquad (|z| \leq \|p\|_\infty,\ \mu_1 \in \mathbb{R}, \mu_2 \in \mathbb{R}\setminus\{0\}). \tag{6}$$

Here, $\mu_2 \neq 0$ is due to the fact that F is not a linear function. Each nontrivial solution of such an ODE satisfies $F'(z) \neq 0$ for almost all $z \in [-\|p\|_\infty, \|p\|_\infty]$. Combining (5) and (6) we thus infer

$$-\partial_i(p(x)^2)\frac{\mu_2\omega^2 z}{\omega^2 + \mu_1 z^2} - \partial_i\left(\frac{\Delta p(x)}{p(x)}\right) z + \partial_i\left(\frac{|\nabla p(x)|^2}{p(x)^2}\right)\frac{\mu_2 z^3}{\omega^2 + \mu_1 z^2} = 0.$$

Since (6) holds for all $i \in \{1, \dots, N\}$ and $z \in [-\|p\|_\infty, \|p\|_\infty]$, we get

$$-\mu_1 \partial_i \left(\frac{\Delta p(x)}{p(x)} \right) + \mu_2 \partial_i \left(\frac{|\nabla p(x)|^2}{p(x)^2} \right) = 0,$$
$$-\mu_2 \partial_i (p(x)^2) - \partial_i \left(\frac{\Delta p(x)}{p(x)} \right) = 0.$$

Since $\mu_2 \neq 0$ we can find $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$-\mu_1 \frac{\Delta p}{p} + \mu_2 \frac{|\nabla p|^2}{p^2} = -\lambda_2 \mu_1 + \lambda_1 \mu_2, \qquad -\mu_2 p^2 - \frac{\Delta p}{p} = -\lambda_2.$$

This implies

$$|\nabla p|^2 = \lambda_1 p^2 - \mu_1 p^4, \qquad -\Delta p + \lambda_2 p = \mu_2 p^3. \tag{7}$$

We now use (7) and the positivity of p to show that p is radially symmetric about its maximum point $x_0 \in \mathbb{R}^N$. We concentrate on the case $N \geq 2$ since the claim for $N = 1$ follows from the fact that $x \mapsto u(x_0 + x)$ and $x \mapsto u(x_0 - x)$ solve the same initial value problem. Since p vanishes at infinity, we must have $\lambda_1 \geq 0$ and, since p does not change sign, $\lambda_2 \geq 0$, see [11, Theorem 1]. Moreover, p attains its maximum at some point $x_0 \in \mathbb{R}^N$ with $p(x_0) > 0, |\nabla p(x_0)| = 0, \Delta p(x_0) \leq 0$. This and (7) implies $\lambda_1, \mu_1 > 0$ as well as $\mu_2 \geq 0$. So we know that (7) holds for

$$\lambda_1, \mu_1 > 0, \qquad \lambda_2, \mu_2 \geq 0.$$

In the case $\lambda_2 > 0$ Theorem 2 from [3] implies the radial symmetry about x_0, so we are left with the case $\lambda_2 = 0$.

So let use assume $\lambda_2 = 0$. Liouville's Theorem implies that $\mu_2 = 0$ is impossible, so we have $\mu_2 > 0$ in this case. Define $\alpha := 1 - \frac{\mu_2}{\mu_1} \in (-\infty, 1)$. In the case $\alpha \in (0, 1)$ the function $\psi(x) := p(x)^\alpha$ satisfies

$$-\Delta \psi = -\alpha (\Delta p) p^{\alpha - 1} - \alpha(\alpha - 1) |\nabla p|^2 p^{\alpha - 2} \overset{(7)}{=} \alpha(1 - \alpha) \lambda_1 \psi.$$

In view of $\alpha(1 - \alpha)\lambda_1 > 0$ Theorem 1 from [11] implies that ψ has infinitely many nodal domains, which contradicts the positivity of ψ. So this case cannot occur. In the case $\alpha \in (-\infty, 0)$ radial symmetry about x_0 follows once more from [3, Theorem 2], so it remains to discuss the case $\alpha = 0$, i.e., $\mu_1 = \mu_2$. Then $\psi(x) := \log(p(x))$ satisfies

$$-\Delta \psi = -(\Delta p) p^{-1} + |\nabla p|^2 p^{-2} \overset{(7)}{=} \lambda_1 \psi$$

and we find as above that ψ has to change sign, which is a contradiction. So we have shown that p is radially symmetric about x_0 also in the case $\lambda_2 = 0$.

So we have

$$p(x) = p_0(|x - x_0|) \qquad \text{where } p_0'(r)^2 = \lambda_1 p_0(r)^2 - \mu_1 p_0(r)^4, \quad p_0'(0) = 0.$$

Solving this ODE gives

$$p_0(r) = \frac{A}{\cosh(mr)} \qquad \text{where } \lambda_1 = m^2, \mu_1 = m^2 A^{-2}$$

for some $A > 0, m \neq 0$. So $-\Delta p + \lambda_2 p = \mu_2 p^3$ can only hold for $N = 1$ as well as $\lambda_2 = m^2$, $\mu_2 = 2m^2 A^{-2}$. Plugging these values into (6) and solving the ODE we get from $F(0) = 0, F \not\equiv 0$

$$F(z) = 4\kappa \arctan\left(\frac{mz}{A\omega}\right) \qquad \text{for some } \kappa \in \mathbb{R} \setminus \{0\}.$$

This implies that the breather solution is given by

$$u(x,t) = F(\sin(\omega t)p(x)) = F(\sin(\omega t)p_0(|x - x_0|)) = \kappa u^*(x - x_0, t)$$

for u^* as in (1). So have proved the nonexistence of such breathers for $N \geq 2$ from claim (i) and the uniqueness statement from claim (ii). To see that this solution formula determines the nonlinearity g, we combine (6) and (7) to get

$$p(x)^2\omega^2 F''(z) - \frac{\Delta p(x)}{p(x)} zF'(z) - \frac{|\nabla p(x)|^2}{p(x)^2} z^2 F''(z) = \frac{m^2(m^2z^2 - A^2\omega^2)}{m^2z^2 + A^2\omega^2} F'(z)z.$$

So (4) implies

$$\begin{aligned} g(F(z)) &= -\omega^2 zF'(z) - \omega^2 z^2 F''(z) + \frac{m^2(m^2z^2 - A^2\omega^2)}{m^2z^2 + A^2\omega^2} F'(z)z \\ &= \frac{(m^2 + \omega^2)(m^2z^2 - A^2\omega^2)}{m^2z^2 + A^2\omega^2} zF'(z) \\ &= \frac{4Am\kappa\omega(m^2 + \omega^2)(m^2z^2 - A^2\omega^2)z}{(m^2z^2 + A^2\omega^2)^2} \end{aligned}$$

Plugging in $z = \frac{A\omega}{m}\tan(\frac{y}{4\kappa})$ for $|y| < 2\pi|\kappa|$ we get $F(z) = y$ and hence

$$\begin{aligned} g(y) &= \frac{4A^2\omega^2\kappa(m^2 + \omega^2)(A^2\omega^2\tan(\frac{y}{4\kappa})^2 - A^2\omega^2)\tan(\frac{y}{4\kappa})}{(A^2\omega^2\tan(\frac{y}{4\kappa})^2 + A^2\omega^2)^2} \\ &= \frac{4\kappa(m^2 + \omega^2)(\tan(\frac{y}{4\kappa})^2 - 1)\tan(\frac{y}{4\kappa})}{(\tan(\frac{y}{4\kappa})^2 + 1)^2} \\ &= 4\kappa(m^2 + \omega^2)(\sin(\frac{y}{4\kappa})^2 - \cos(\frac{y}{4\kappa})^2)\sin(\frac{y}{4\kappa})\cos(\frac{y}{4\kappa}) \\ &= -2\kappa(m^2 + \omega^2)\cos(\frac{y}{2\kappa})\sin(\frac{y}{2\kappa}) \\ &= -\kappa(m^2 + \omega^2)\sin(\frac{y}{\kappa}). \end{aligned}$$

□

Remark 1. *We explain why nonlinear quasimonochromatic breathers of* (3) *with profile functions* $p \in C^2(\overline{\Omega})$ *do not exist on open sets* $\Omega \subsetneq \mathbb{R}^N$. *The arguments presented above reveal that any such breather is given by functions* F, p *as in Definition 1 such that for all* $x \in \Omega, p(x) \neq 0, |z| \leq \|p\|_\infty$ *we have as in* (4)

$$g(F(z)) + \omega^2 zF'(z) + \omega^2 z^2 F''(z) = p(x)^2\omega^2 F''(z) - \frac{\Delta p(x)}{p(x)} zF'(z) - \frac{|\nabla p(x)|^2}{p(x)^2} z^2 F''(z).$$

Again, multiplying this equation with $p(x)$ *and choosing* z *according to* $F''(z)z \neq 0$ *we deduce as above that* p *does not change sign inside* Ω. *Choose a sequence* $x_n \to x^* \in \partial\Omega$. *Fix* $x^* \in \Omega$ *with* $p(x^*) \neq 0$ *and choose* $R > 0$ *(as in the above proof) to be the smallest radius such that* $|p|$ *is positive in the open ball* $B_R(x^*) \subset \Omega$. *Notice that* R *indeed exists and* $0 < R \leq \operatorname{dist}(x^*, \partial\Omega)$. *The homogeneous Dirichlet boundary conditions imply that* p *has a zero on* $\partial B_R(x^*)$. *So the same argument as in the above proof (Hopf's Lemma) shows that* $|\Delta p|$ *is unbounded on* $B_R(x^*)$, *a contradiction. As a consequence, such a profile function cannot exist and we obtain the nonexistence of breathers of* (3).

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