Karlsruhe Institute of Technology

Wave phenomena
analysis and numerics

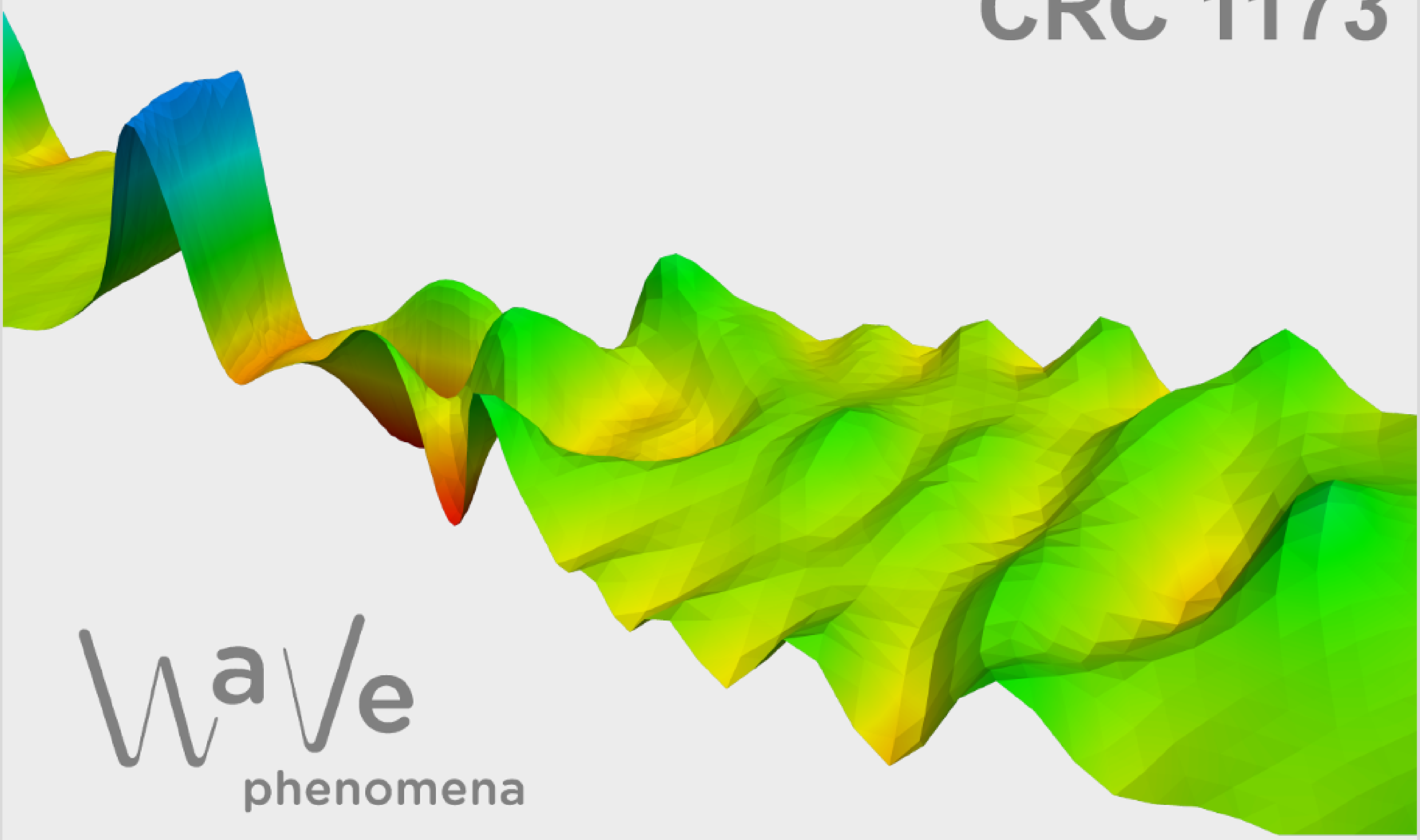
Ground states for Maxwell's equations in nonlocal nonlinear media

Rainer Mandel

CRC Preprint 2021/43, October 2021

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173



Participating universities

Universität Stuttgart

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

TU WIEN

Funded by

DFG

ISSN 2365-662X

GROUND STATES FOR MAXWELL'S EQUATIONS IN NONLOCAL NONLINEAR MEDIA

RAINER MANDEL[1]

ABSTRACT. In this paper we investigate the existence of ground states and dual ground states for Maxwell's Equations in $\mathbb{R}^3$ in nonlocal nonlinear metamaterials. We prove that several nonlocal models admit ground states in contrast to their local analogues.

1. INTRODUCTION

The existence of ground states is of central importance for a large number of linear and nonlinear time-independent models in physics. The governing idea is that the physically most relevant nontrivial solution of a given PDE is the one with lowest energy. Such a solution is called a ground state. In this paper we are interested in ground states for the nonlinear Maxwell equations

$$\partial_t \mathcal{D} - \nabla \times \mathcal{H} = 0, \qquad \partial_t \mathcal{B} + \nabla \times \mathcal{E} = 0, \qquad \nabla \cdot \mathcal{D} = \nabla \cdot \mathcal{B} = 0, \tag{1}$$

that describe the propagation of electromagnetic waves in optical media without charges and currents. The symbols $\mathcal{E}, \mathcal{D} : \mathbb{R}^3 \to \mathbb{C}^3$ denote the electric field and the electric induction whereas $\mathcal{H}, \mathcal{B} : \mathbb{R}^3 \to \mathbb{C}^3$ represent the magnetic field and the magnetic induction, respectively. This overdetermined system is accompanied with constitutive relations that provide a link between these quantities. In homogeneous and isotropic media it is usual to assume

$$\mathcal{D} = \varepsilon \mathcal{E} + \mathcal{P} \quad \text{and} \quad \mathcal{B} = \mu \mathcal{H}$$

where $\varepsilon, \mu \in \mathbb{R} \setminus \{0\}$ and $\mathcal{P}$ denotes the so-called polarization field. In [3, Section 1.3] it was shown that a time-harmonic ansatz for the electric field $\mathcal{E}(x,t) = e^{i\omega t} E(x)$ and $\mathcal{P}(x,t) = |\mathcal{E}(x,t)|^{2q-2} \mathcal{E}(x,t)$ leads, after a suitable rescaling, to a nonlinear curl-curl equation of the form

$$\nabla \times \nabla \times E + E = |E|^{2q-2} E \qquad \text{in } \mathbb{R}^3 \tag{2}$$

provided that $\varepsilon\mu\omega^2$ is negative. This assumption does not hold in natural propagation media like vacuum, glass or water, but it holds for certain artificially produced metamaterials where $\varepsilon\mu$ can be negative [24]. It is therefore reasonable to investigate the existence of ground states for (2) in order to single out physically relevant solutions of nonlinear Maxwell equations. The energy functional associated with (2) is given by

$$\mathcal{I}(E) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times E|^2 + |E|^2 \, dx - \frac{1}{2q} \int_{\mathbb{R}^3} |E|^{2q} \, dx$$

for $E \in H := H^1(\mathrm{curl}; \mathbb{R}^3) = \{E \in L^2(\mathbb{R}^3; \mathbb{R}^3) : \nabla \times E \in L^2(\mathbb{R}^3; \mathbb{R}^3)\}$. So a ground state is a nontrivial solution of the equation $\mathcal{I}'(E) = 0$ such that $\mathcal{I}(E)$ is smallest possible among all nontrivial solutions. Our first observation is that ground states for this system do not exist, which is in striking contrast to the rich theory of ground states in the context of stationary nonlinear Schrödinger equations of the form $-\Delta u + u = |u|^{2q-2} u$ in $\mathbb{R}^N$ [4,25]. Borrowing ideas from [3, Theorem 1.1], we get the following.

Proposition 1. *Assume* $1 < q < \infty$. *Then* (2) *does not have a ground state solution in* $H^1(\mathrm{curl}; \mathbb{R}^3) \cap L^{2q}(\mathbb{R}^3; \mathbb{R}^3)$.

Date: October 25, 2021.
2020 *Mathematics Subject Classification.* 35J60, 35Q61.

This is surprising given that the existence of ground state solutions can be proved for the slightly different model where $\nabla\times\nabla\times E+E$ is replaced by $\nabla\times\nabla\times E$ and the nonlinearity is $\min\{|E|^p,|E|^q\}$ for $2<p<6<q<\infty$, see [21]. One may check that the proof of Proposition 1 does not carry over to this case. We also mention the cylindrically symmetric approaches from [2, 3, 9, 12] where divergence-free solutions with higher energy are constructed. Other prototypical nonlinear Maxwell equations will be commented on in Remark 5.

Motivated by Proposition 1 our aim is to identify nonlocal variants of these nonlinear models that admit ground state solutions, thus overcoming the lack of local compactness in the gradient part of (2). In several physically relevant models, a nonlocal nonlinear effect is given in terms of a rapidly decaying convolution kernel $K:\mathbb{R}^3\to\mathbb{R}$ that quantifies the dependency of the nonlinear refractive index at a given point on the intensity of the electric field over a small neighbourhood. The kernels are typically supposed to decay rapidly at infinity. Typical choices are given by (see [16, Section IV])

$$K_1(x)=e^{-|x|^2},\qquad K_2(x)=e^{-|x|},\qquad K_3(x)=\mathbb{1}_{|x|<R},$$

while the latter is usually considered as a toy model allowing for explicit computations. In some cases oscillatory kernel functions are considered as well [22]. To study the impact of such regularizations for the existence of ground states we first consider some partially nonlocal version of the cubic Kerr nonlinearity given by $(K*|E|^2)E$ instead of $|E|^2E$. Here, $K*|E|^2$ represents a nonlocal nonlinear refractive index change of the propagation medium. This phenomenological model is particularly popular in the study of laser beams modeled by nonlinear Schrödinger equations, which in turn serve as reduced models for Maxwell's Equations [16]. It has the advantage of being variational, so it makes sense to look for ground state solutions. However, we show that this particular model admits ground states only under artificial assumptions on the kernel function K. In particular, we will see that K_1,K_2 do not admit ground states whereas K_3 has such solutions. To be more precise, we introduce the associated energy functional $I:H^1(\mathrm{curl};\mathbb{R}^3)\to\mathbb{R}$ via

$$I(E):=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\times E|^2+|E|^2\,dx-\frac{1}{4}\int_{\mathbb{R}^3}(K*|E|^2)\,|E|^2\,dx \tag{3}$$

where $K\in L^\infty(\mathbb{R}^3)$. To prove the existence of ground states the typical strategy is to minimize I over the Nehari manifold $\mathcal{N}=\{E\in H: I'(E)[E]=0, E\neq 0\}$. The following result shows that this approach fails in most cases.

Theorem 2. *Let $K\in L^\infty(\mathbb{R}^3)$ be almost everywhere continuous with $K(z)\to K(0)=\sup_{\mathbb{R}^3}K>0$ as $|z|\to 0$. Then we have $\inf_{\mathcal{N}} I=\frac{1}{4K(0)}$ and a minimizer for $\inf_{\mathcal{N}} I$ exists if and only if $\delta_K:=\sup\{\delta: K(z)=K(0)\text{ for }|z|<\delta\}$ is positive. In this case the set of minimizers consists of all gradient vector fields $E=\nabla\Phi\in\mathcal{N}$ with $\mathrm{diam}(\mathrm{supp}(E))\le\delta_K$. Every such minimizer is a ground state solution of*

$$\nabla\times\nabla\times E+E=\left(K*|E|^2\right)E\qquad\text{in }\mathbb{R}^3. \tag{4}$$

From a mathematical point of view it is interesting to see what happens for other nonlinearities in the nonlocal term. We have a look at the case of other power-type nonlinearities where $|E|^2$ is replaced by $|E|^q$. The case $q>2$ will be excluded because the assumption $E\in H:=H^1(\mathrm{curl};\mathbb{R}^3)$ does not imply $E\in L^q_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$, so the Euler functional is in general not well-defined on $H^1(\mathrm{curl};\mathbb{R}^3)$. In fact, we will see in Remark 11 (c) that for reasonable kernel functions (such as K_1,K_2,K_3) and $q>2$ ground states cannot be constructed via minimization over the Nehari manifold even if the latter is intersected with $L^q(\mathbb{R}^3;\mathbb{R}^3)$. For this reason we concentrate on the case $q\in(1,2)$ and define the corresponding energy as follows:

$$I_q(E)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\times E|^2+|E|^2\,dx-\frac{1}{2q}\int_{\mathbb{R}^3}\left(K*|E|^q\right)|E|^q\,dx$$

The Nehari manifold is then given by $\mathcal{N}_q:=\{E\in H^1(\mathrm{curl};\mathbb{R}^3): I_q'(E)[E]=0, E\neq 0\}$. Our next result shows that there are ground states for each of the kernels K_1,K_2,K_3, which is in contrast with Theorem 2. We will assume $K\in L^{1/(2-q),\infty}(\mathbb{R}^3)$ to have a well-defined functional $I_q:L^{2/q}(\mathbb{R}^3)\to\mathbb{R}$. Here, $L^{p,s}(\mathbb{R}^3)$ denotes the standard Lorentz space, which in the case $s=\infty$ is also called weak Lebesgue space or Marcinkiewicz space. Moreover, we assume K to be Schwarz-symmetric, i.e., K coincides with its spherical rearrangement [17, Chapter 3]. The common shorthand notation for this is $K = |K|^*$. These assumptions are satisfied if K is a nonnegative radially nonincreasing function satisfying $0 \leq K(z) \leq C|z|^{-3(2-q)}$ for some $C > 0$ and almost all $z \in \mathbb{R}^3$, which holds for K_1, K_2, K_3.

Theorem 3. *Assume $1 < q < 2$, $K \in L^{1/(2-q),\infty}(\mathbb{R}^3)$, $K = |K|^* \not\equiv 0$ and $K(\cdot + h) \to K$ in $L^{1/(2-q),\infty}(\mathbb{R}^3)$ as $|h| \to 0$. Then $\inf_{\mathcal{N}_q} I_q$ is attained at some ground state solution $E \in H^1(\mathrm{curl}; \mathbb{R}^3)$ of*

$$\nabla \times \nabla \times E + E = \left(K * |E|^q\right) |E|^{q-2} E \qquad \text{in } \mathbb{R}^3.$$

All ground state solutions are irrotational and one of them is given by $E(x) = t\frac{x}{|x|} f(x)^{1/q}$ where $t \neq 0$ and f is a Schwarz-symmetric maximizer of the functional $Q(f) = \int_{\mathbb{R}^3} (K * f) f \, dx$ over the unit sphere in $L^{2/q}(\mathbb{R}^3)$.

Irrotational vector fields satisfy $\nabla \times E = 0$ in $\mathbb{R}^3$, so there is a potential $\Phi : \mathbb{R}^3 \to \mathbb{R}$ such that $E = \nabla \Phi$. Such electric fields do not generate a magnetic field in view of (1). This is different for the nontrivial cylindrically symmetric solutions of these equations found in [3, 9, 12] that are divergence-free and hence do not satisfy $\nabla \times E = 0$. The ground states obtained by Mederski [21] do not have this property either, so both types of solutions come with a nontrivial magnetic field.

Given the importance of the cubic Kerr nonlinearity for nonlinear optics, we discuss another nonlocal model that admits nontrivial solutions also in the case of a cubic nonlinearity. This model originates from a description of the polarization field $\mathcal{P}(x,t) = P(x)e^{i\omega t}$ via $P = K * (|E|^2 E)$ instead of $P = E(K * |E|^2)$. In the linear case such fully nonlocal models are investigated both from a physical and mathematical point of view [10, 14, 23]. Our aim is to show that such models often admit some sort of ground state solution, which stands in contrast to Theorem 2. It even allows to treat all power-type nonlinearities $|E|^{2q-2}E$ with $1 < q < \infty$ under appropriate assumptions on the convolution kernel K. The following result applies to each of the kernels K_1, K_2, K_3 introduced above.

Theorem 4. *Assume $1 < q < \infty$, $K \in L^{q,\infty}(\mathbb{R}^3)$, $K(\cdot + h) \to K$ in $L^{q,\infty}(\mathbb{R}^3)$ as $|h| \to 0$ and $\int_{\mathbb{R}^3} (K * f) f \, dx > 0$ for some $f \in L^{(2q)'}(\mathbb{R}^3)$. Then there is a dual ground state solution $E \in L^{2q}(\mathbb{R}^3; \mathbb{R}^3)$ for*

$$\nabla \times \nabla \times E + E = K * (|E|^{2q-2} E) \qquad \text{in } \mathbb{R}^3. \tag{5}$$

Since (5) is not variational, the notion of a ground state does not make sense. A dual ground state is a function E given by $|E|^{2q-2}E = U$ where U is a ground state for the associated dual functional $J : L^{(2q)'}(\mathbb{R}^3; \mathbb{R}^3) \to \mathbb{R}$ that we will introduce in (9). This functional is defined in such a way that any critical point U of J gives rise to a distributional solution E of (5). In the Appendix we will motivate why such dual ground states may be interpreted as reasonable substitutes for ground states. In each of our results one can deduce better integrability and regularity properties of the constructed solutions under suitable assumptions on the kernel function K using classical bootstrap arguments. Clearly, it would be of interest to know about further qualitative properties of ground states like their symmetries, monotonicity and positivity properties. We believe it to be particularly interesting whether ground states are unique and irrotational.

2. Proof of Proposition 1

We have to show that the equation $\nabla \times \nabla \times E + E = |E|^{2q-2}E$ in $\mathbb{R}^3$ does not have ground state solutions. To this end define $E_j(x) := \mathbb{1}_{|x|<1/j} \frac{x}{|x|}$ for $j \in \mathbb{N}$. Then $E_j \in H^1(\mathrm{curl}; \mathbb{R}^3) \cap L^{2q}(\mathbb{R}^3; \mathbb{R}^3)$ is a weak solution of (2) satisfying $\nabla \times E_j = 0$ in the weak sense. The latter is true because of $E_j = \nabla \Phi_j$ where $\Phi_j \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is given by $\Phi_j(x) := \min\{|x|, \frac{1}{j}\}$. From $I(E_j) \to 0$ as $j \to \infty$ we deduce that a ground state can only satisfy $I(E) = 0$ if it exists. On the other hand, $I'(E) = 0, E \neq 0$ gives $I(E) = I(E) - \frac{1}{2q} I'(E)[E] = \frac{q-1}{2q} \int_{\mathbb{R}^3} |E|^{2q} \, dx > 0$, so there is no nontrivial solution at the energy level zero. Hence, a ground state cannot exist. □

Remark 5.

(a) In [3, Theorem 1.1] the authors claim that all distributional solutions $E(x) = \frac{x}{|x|}s(|x|)$ of the equation $\nabla\times\nabla\times E+E = |E|^{2q-2}E$ are given by arbitrary measurable functions $s : (0,\infty) \to \{-1, 1\}$. However, this is only a subfamily of all such distributional solutions. The correct version of this result states that all distributional solutions $E(x) = \frac{x}{|x|}s(|x|)$ are given by arbitrary measurable functions $s : (0,\infty) \to \{-1, 0, 1\}$. This small modification is important, because we exploit a shrinking of supports in our proof above by choosing $s(|x|) = \mathbb{1}_{|x|<1/j}$. In view of [3, Theorem 1.1] Proposition 1 generalizes to other nonlinear Maxwell equations like $\nabla \times \nabla \times E + E = g(|x|, |E|)E$ where shrinking solutions are given by $E_j(x) := \mathbb{1}_{\rho<|x|<\rho+1/j}a(|x|)\frac{x}{|x|}$ for any measurable function a given by $g(r, a(r)) = 1$ for almost all $r \in (\rho, \rho + \frac{1}{j}), \rho > 0$.

(b) Similarly, one obtains solutions with compact support for $\nabla\times\nabla\times E-E = g(|x|, |E|)E$. In particular, $\nabla\times\nabla\times E-E = -|E|^{2q-2}E$ with $1 < q < \infty$ has the solutions $E_j(x) := \mathbb{1}_{0<|x|<j}\frac{x}{|x|}$ and the associated energy

$$I(E_j) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \times E_j|^2 - |E_j|^2\,dx + \frac{1}{2q}\int_{\mathbb{R}^3} |E_j|^{2q}\,dx = -\frac{q-1}{2q}|\{x \in \mathbb{R}^3 : 0 < |x| < j\}|$$

tends to $-\infty$ as $j \to \infty$. So ground states do not exist for these equations. Moreover, $\nabla\times\nabla\times E+E = -|E|^{2q-2}E$ does not admit any nontrivial weak solution in $H^1(\mathrm{curl}; \mathbb{R}^3) \cap L^{2q}(\mathbb{R}^3; \mathbb{R}^3)$, which follows from testing the equation with E. The equation $\nabla \times \nabla \times E - E = |E|^{2q-2}E$ admits nontrivial cylindrically symmetric solutions for suitable exponents q [20, Theorem 3a], but these solutions do not belong to $H^1(\mathrm{curl}; \mathbb{R}^3)$ due to slow decay rates at infinity. In the related case of Helmholtz equations nonexistence results for L^2-solutions can be found in [13, Theorem 1a] or [15, Theorem 3]. Under strong extra assumptions on $|E(x)|^{2q-2}$ the absence of $H^1(\mathrm{curl}; \mathbb{R}^3)$-solutions E follows from [8, Theorem 3] choosing $\mu(x) := 1, \varepsilon(x) := 1 - |E(x)|^{2q-2}$.

(c) The existence of discontinuous and concentrating solutions illustrates that no regularity theory and no compact embeddings in whatever Lebesgue space can be exploited in the analysis of the (local) nonlinear Maxwell equation (2).

3. Proof of Theorem 2

It is convenient to split the functional I according to $I(E) = \frac{1}{2}I_L(E) - \frac{1}{4}I_{NL}(E)$ where

$$I_L(E) = \int_{\mathbb{R}^3} |\nabla \times E|^2 + |E|^2\,dx, \qquad I_{NL}(E) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} K(x-y)|E(x)|^2|E(y)|^2\,dx\,dy,$$

see (3). It is standard to show that I is continuously differentiable on $H^1(\mathrm{curl}; \mathbb{R}^3)$ under the assumptions of Theorem 2 with Fréchet derivative

$$I'(E)[\tilde{E}] = \int_{\mathbb{R}^3} (\nabla \times E)\cdot(\nabla \times \tilde{E}) + E\cdot\tilde{E}\,dx - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} K(x-y)\big(E(x)\cdot\tilde{E}(x)\big)\,|E(y)|^2\,dx\,dy$$

for $E, \tilde{E} \in H$. Moreover, critical points of I are weak solutions of (4), so a ground state solution may be obtained by minimizing I over the Nehari manifold $\mathcal{N} = \{E \in H : I'(E)[E] = 0, E \neq 0\}$. We first provide a convenient min-max characterization of the least energy level $\inf_{\mathcal{N}} I$ with the aid of the fibering map $\gamma(t) := I(tE)$ for $E \in H\setminus\{0\}$. The simple observation is that $tE \in \mathcal{N}$ holds for some $t \neq 0$ if and only if $\gamma'(t) = 0$. Given the structure of I it immediate to see that γ has a unique positive maximizer if $I_{NL}(E) > 0$. In the opposite case, γ increases to $+\infty$ and does not have any critical point. This implies

$$\inf_{\mathcal{N}} I = \inf_{E\in H\setminus\{0\}} \sup_{t>0} I(tE) = \inf_{E\in H, I_{NL}(E)>0} \frac{I_L(E)^2}{4I_{NL}(E)}. \tag{6}$$

Note that $K(z) > 0$ for small $|z|$ implies that $I_{NL}(E) > 0$ holds for E belonging to some nonempty open subset of H. In fact, one may take E with support of sufficiently small diameter. We conclude that the Nehari manifold is non-void and it remains to analyze the expression on the right of (6).

We first prove the formula for $\inf_{\mathcal{N}} J$. The lower bound is obtained as follows:

$$\begin{aligned}\frac{I_L(E)^2}{4I_{NL}(E)} &\geq \frac{(\int_{\mathbb{R}^3}|E(x)|^2\,dx)^2}{4\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(x-y)|E(x)|^2|E(y)|^2\,dx\,dy}\\ &\geq \frac{(\int_{\mathbb{R}^3}|E(x)|^2\,dx)^2}{4K(0)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|E(x)|^2|E(y)|^2\,dx\,dy}\\ &= \frac{1}{4K(0)}.\end{aligned}\tag{7}$$

Here we used $K(z)\leq K(0)$ for all $z\in\mathbb{R}^3$. On the other hand, choosing a nonconstant $\phi\in\mathring{H}^1(\mathbb{R}^3)$ such that $\nabla\phi$ has compact support and $\tilde{E}_n(x):=n^{3/2}\nabla\phi(nx)$, we observe $\nabla\times\tilde{E}_n=0$ and $\tilde{E}_n\in L^2(\mathbb{R}^3;\mathbb{R}^3)$. In particular, $\tilde{E}_n\in H=H^1(\mathrm{curl};\mathbb{R}^3)$. Moreover, the supports of $\tilde{E}_n$ shrink to $\{0\}$. Hence, $K(z)\to K(0)$ as $z\to 0$ implies

$$\begin{aligned}\frac{I_L(\tilde{E}_n)^2}{4I_{NL}(\tilde{E}_n)} &= \frac{(\int_{\mathbb{R}^3}|\tilde{E}_n(x)|^2\,dx)^2}{4\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(x-y)|\tilde{E}_n(x)|^2|\tilde{E}_n(y)|^2\,dx\,dy}\\ &= \frac{(\int_{\mathbb{R}^3}|\tilde{E}_n(x)|^2\,dx)^2}{4\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(K(0)+o(1))|\tilde{E}_n(x)|^2|\tilde{E}_n(y)|^2\,dx\,dy}\\ &= \frac{(\int_{\mathbb{R}^3}|\nabla\phi(x)|^2\,dx)^2}{4\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(0)|\nabla\phi(x)|^2|\nabla\phi(y)|^2\,dx\,dy+o(1)}\\ &= \frac{1}{4K(0)}+o(1)\quad\text{as } n\to\infty.\end{aligned}$$

This proves the formula for the infimum.

Next we show that a minimzer exists if and only if $\delta_K>0$, i.e., if and only if $K(z)=K(0)$ for almost all $z\in\mathbb{R}^3$ such that $|z|<\delta$ where $\delta>0$. In fact, choose any nonconstant $\phi\in\mathring{H}^1(\mathbb{R}^3)$ such that the support of $\tilde{E}:=\nabla\phi\in H, \tilde{E}\in L^2(\mathbb{R}^3;\mathbb{R}^3)$ has diameter $\leq\delta$. Then $\nabla\times\tilde{E}=0$ implies $\tilde{E}\in H$ and both inequalities in (7) become equality. Vice versa, if E is a minimizer, then the estimates from (7) show that $\nabla\times E=0$ holds almost everywhere and $K(x-y)=K(0)$ has to hold for almost all $(x,y)\in\mathrm{supp}(E)\times\mathrm{supp}(E)$. Since K is continuous almost everywhere, this implies $K(z)=K(0)$ for almost all $z\in\mathbb{R}^3$ with $|z|\leq\mathrm{diam}(\mathrm{supp}(E))$, which is all we had to prove. □

4. A Lemma

In this section we consider an auxiliary variational problem that will allow to deduce the existence of solutions for the nonlocal nonlinear Maxwell equations that we discuss in Theorem 3 and Theorem 4. It shares some features with Lions' application of the concentration-compactness principle presented in [18, Section II.1-2]. Let $S:=\{f\in L^p(\mathbb{R}^N;\mathbb{R}^M):\|f\|_p=1\}$ denote the unit sphere. Our aim is to solve the maximization problem

$$\sup_{f\in S}Q(f)\qquad\text{where } Q(f):=\int_{\mathbb{R}^N}(\mathcal{K}*f)\cdot f\,dx$$

under appropriate assumptions on the kernel function $\mathcal{K}$. To prove the existence of a maximizer, we will use Lions' concentration-compactness method. So we consider a maximizing sequence (f_j) in S and define the probability measures

$$\mu_j(B):=\int_B|f_j(x)|^p\,dx.$$

According to [7, Theorem 4.7.3], see [18, Lemma I.1] for the original result, this family of measures may behave in three possible ways:

(I) (Compactness) There is a sequence $(x_j)\subset\mathbb{R}^N$ such that for all $\varepsilon>0$ there is $R_\varepsilon>0$ such that $\mu_j(B_{R_\varepsilon}(x_j))\geq 1-\varepsilon$.

(II) (Vanishing) For all $R>0$ we have $\lim_{j\to\infty}\sup_{x\in\mathbb{R}^N}\mu_j(B_R(x))=0$.

(III) (Dichotomy) There are $\lambda\in(0,1)$, a sequence (R_j) with $R_j\to\infty$, a sequence $(x_j)\subset\mathbb{R}^N$ and measures μ_j^1,μ_j^2 such that $0\le\mu_j^1+\mu_j^2\le\mu_j$ and

$$\operatorname{supp}(\mu_j^1)\subset B_{R_j}(x_j),\quad \operatorname{supp}(\mu_j^2)\subset\mathbb{R}^N\setminus B_{2R_j}(x_j),\quad |\lambda-\mu_j^1(\mathbb{R}^N)|+|1-\lambda-\mu_j^2(\mathbb{R}^N)|\le\frac{1}{j}.$$

The aim is to show that (I) occurs and to derive the existence of a maximizer using some local compactness property of $f\mapsto\mathcal{K}*f$. The first step is to rule out the vanishing case (II). This is achieved with the aid of the following simple estimate.

Proposition 6. *Assume $N,M\in\mathbb{N},1<p<2$ and $\mathcal{K}\in L^{p'/2,\infty}(\mathbb{R}^N;\mathbb{R}^{M\times M})$. Then we have for all $R>0$*

$$Q(f)\le\varepsilon(R)\|f\|_p^2+C(R)\|f\|_p^p\left(\sup_{y\in\mathbb{R}^N}\|f\|_{L^p(B_R(y))}\right)^{2-p}$$

with $\varepsilon(R)\to 0$ as $R\to\infty$.

Proof. We have

$$Q(f)=\int_{\mathbb{R}^N}(\mathcal{K}*f)\cdot f\,dx=\int_{\mathbb{R}^N}((\mathcal{K}-\mathcal{K}_R)*f)\cdot f\,dx+\int_{\mathbb{R}^N}(\mathcal{K}_R*f)\cdot f\,dx.$$

where $\mathcal{K}_R(z):=\mathcal{K}(z)\mathbb{1}_{|z|+|\mathcal{K}(z)|\le R}$. Then Young's convolution inequality from [11, Theorem 1.4.25] gives due to $2<p'<\infty$

$$\int_{\mathbb{R}^N}((\mathcal{K}-\mathcal{K}_R)*f)\cdot f\,dx\le\|\mathcal{K}-\mathcal{K}_R\|_{\frac{p'}{2},\infty}\|f\|_p^2$$

and the prefactor goes to zero as $R\to\infty$ by the Dominated Convergence Theorem. On the other hand, the estimates from [1, p.109-110] yield

$$\begin{aligned}\int_{\mathbb{R}^N}(\mathcal{K}_R*f)\cdot f\,dx&\le\|\mathcal{K}_R\|_\infty\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathbb{1}_{|x-y|\le R}|f(x)||f(y)|\,dx\,dy\\&\le C(R)\|f\|_p^p\left(\sup_{y\in\mathbb{R}^N}\|f\|_{L^p(B_{3\sqrt{3}R}(y))}\right)^{2-p}\end{aligned}$$

and the claim follows. □

Proposition 7. *Assume $N,M\in\mathbb{N},1<p<2,\mathcal{K}\in L^{p'/2,\infty}(\mathbb{R}^N;\mathbb{R}^{M\times M})$ and assume that $(f_j)\subset S$ is a maximizing sequence for $m:=\sup_{f\in S}Q(f)>0$ with induced measures μ_j. Then neither (II) nor (III) occurs.*

Proof. For large j we have due to Proposition 6

$$\frac{m}{2}\le Q(f_j)\le\varepsilon(R)+C(R)\left(\sup_{y\in\mathbb{R}^N}\|f_j\|_{L^p(B_R(y))}\right)^{2-p}.$$

with $\varepsilon(R)\to 0$ as $R\to\infty$. Hence, we may choose $R>0$ so large that $\sup_{y\in\mathbb{R}^N}\mu_j(B_R(y))>0$ holds. So the case (II) cannot occur. Now assume (III) and choose $(x_j),(R_j),\lambda\in(0,1)$ accordingly. We decompose the sequence according to $f_j=f_j^1+f_j^2+f_j^3$ where $f_j^1:=f_j\mathbb{1}_{B_{R_j}(x_j)}$ and $f_j^2:=f_j\mathbb{1}_{\mathbb{R}^N\setminus B_{2R_j}(x_j)}$. Then (III) implies

$$\int_{\mathbb{R}^N}|f_j^1|^p\,dx\to\lambda,\qquad\int_{\mathbb{R}^N}|f_j^2|^p\,dx\to 1-\lambda,\qquad\int_{\mathbb{R}^N}|f_j^3|^p\,dx\to 0.\qquad(j\to\infty)\tag{8}$$

Then $\|f_j^3\|_p\to 0$ and $\operatorname{dist}(\operatorname{supp}(f_j^1),\operatorname{supp}(f_j^2))\ge R_j\to\infty$ as $j\to\infty$ yield

$$Q(f_j)=\int_{\mathbb{R}^N}(\mathcal{K}*f_j)\cdot f_j\,dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} (\mathcal{K} * f_j^1) \cdot f_j^1 \, dx + \int_{\mathbb{R}^N} (\mathcal{K} * f_j^2) \cdot f_j^2 \, dx + \int_{\mathbb{R}^N} (\mathcal{K} * f_j^3) \cdot f_j^3 \, dx \\
&+ 2 \int_{\mathbb{R}^N} (\mathcal{K} * f_j^2) \cdot f_j^1 \, dx + 2 \int_{\mathbb{R}^N} (\mathcal{K} * f_j^3) \cdot (f_j^1 + f_j^2) \, dx \\
&\leq Q(f_j^1) + Q(f_j^2) + \|\mathcal{K}\|_{\frac{p'}{2},\infty} \|f_j^3\|_p^2 \\
&+ 2\|\mathcal{K}\mathbb{1}_{\mathbb{R}^N \setminus B_{R_j}(0)}\|_{\frac{p'}{2},\infty} \|f_j^2\|_p \|f_j^1\|_p + 2\|\mathcal{K}\|_{\frac{p'}{2},\infty} \|f_j^3\|_p \|f_j^1 + f_j^2\|_p \\
&= Q(f_j^1) + Q(f_j^2) + o(1) \qquad \text{as } j \to \infty.
\end{aligned}
$$

Furthermore, since f_j^1, f_j^2 are nontrivial for large j, the definition of m implies

$$
m \geq Q(f_j^1/\|f_j^1\|_p) \overset{(8)}{=} Q(f_j^1)\lambda^{-\frac{2}{p}} + o(1), \qquad m \geq Q(f_j^2/\|f_j^2\|_p) \overset{(8)}{=} Q(f_j^2)(1-\lambda)^{-\frac{2}{p}} + o(1).
$$

Combining the previous estimates we get

$$
m = Q(f_j) + o(1) \leq Q(f_j^1) + Q(f_j^2) + o(1) \leq m \cdot \left(\lambda^{\frac{2}{p}} + (1-\lambda)^{\frac{2}{p}}\right) + o(1),
$$

which is impossible due to $\lambda \in (0,1)$ and $p < 2$. So (III) cannot occur either. □

So we are left with the compactness case (I). So we exploit some local compactness property to deduce the existence of a maximizer. This is provided next.

Proposition 8. *Assume $N, M \in \mathbb{N}, 1 < p < 2$ and that $\mathcal{K} \in L^{p'/2,\infty}(\mathbb{R}^N; \mathbb{R}^{M\times M})$ satisfies $\mathcal{K}(\cdot + h) \to \mathcal{K}$ in $L^{p'/2,\infty}(\mathbb{R}^N; \mathbb{R}^{M\times M})$ as $|h| \to 0$. Then $L^p(\mathbb{R}^N; \mathbb{R}^M) \to L^{p'}(B; \mathbb{R}^M), f \mapsto \mathcal{K} * f$ is compact for all bounded sets $B \subset \mathbb{R}^N$.*

Proof. We use the Fréchet-Kolmogorov-Riesz criterion [5, Theorem 4.26] that characterizes precompact subsets in Lebesgue spaces with exponent $< \infty$. Being given any bounded sequence (f_j) in $L^p(\mathbb{R}^N; \mathbb{R}^M)$ we have to show that $\{\mathcal{K} * f_j : j \in \mathbb{N}\}$ is precompact in $L^{p'}(\mathbb{R}^N; \mathbb{R}^M)$. The estimate $\|\mathcal{K} * f_j\|_{p'} \leq \|K\|_{p'/2,\infty} \|f_j\|_p$ shows that the family is bounded. Moreover,

$$
\|(\mathcal{K} * f_j)(\cdot + h) - (\mathcal{K} * f_j)\|_{p'} = \|(\mathcal{K}(\cdot + h) - \mathcal{K}) * f_j\|_{p'} \leq \|\mathcal{K}(\cdot + h) - \mathcal{K}\|_{p'/2,\infty} \|f_j\|_p.
$$

So the boundedness of (f_j) and $\|\mathcal{K}(\cdot + h) - \mathcal{K}\|_{p'/2,\infty} \to 0$ as $|h| \to 0$ imply the equicontinuity of $\{\mathcal{K} * f_j : j \in \mathbb{N}\}$ in $L^{p'}(\mathbb{R}^N; \mathbb{R}^M)$. By the above-mentioned criterion, this implies that $\{\mathcal{K} * f_j : j \in \mathbb{N}\}$ is precompact in $L^{p'}(B; \mathbb{R}^M)$ for all bounded sets $B \subset \mathbb{R}^N$. □

Lemma 9. *Assume $N, M \in \mathbb{N}, 1 < p < 2$ and that $\mathcal{K} \in L^{p'/2,\infty}(\mathbb{R}^N; \mathbb{R}^{M\times M})$ satisfies $\mathcal{K}(\cdot + h) \to \mathcal{K}$ in $L^{p'/2,\infty}(\mathbb{R}^N; \mathbb{R}^{M\times M})$ as $|h| \to \infty$ as well as $m := \sup_{f\in S} Q(f) > 0$. Then the functional Q has a maximizer over S. Moreover:*

(i) If each entry of $\mathcal{K}$ is a nonnegative function, then Q has a componentwise nonnegative maximizer.
(ii) If $M = 1$ and $\mathcal{K} = |\mathcal{K}|^*$, then Q has a Schwarz-symmetric maximizer.
(iii) If $M = 1$ and $\mathcal{K} = |\mathcal{K}|^*$ is radially decreasing, then each maximizer of Q is Schwarz-symmetric up to translations.

Proof. Let $(f_j) \subset S$ be a maximizing sequence for $m > 0$. Then the Concentration-Compactness Lemma (see above) and Proposition 7 imply that the sequence of measures (μ_j) induced by (f_j) satisfies alternative (I). Since Q and S are translation-invariant, we may assume $x_j = 0$ in (I). Since (f_j) is bounded in the reflexive Banach space $L^p(\mathbb{R}^N; \mathbb{R}^M)$, we may furthermore assume $f_j \rightharpoonup f$ in $L^p(\mathbb{R}^N; \mathbb{R}^M)$ where $\|f\|_p \leq \liminf_{j\to\infty} \|f_j\|_p = 1$. In order to show that f maximizes Q let $\varepsilon > 0$ be arbitrary and choose $R_\varepsilon > 0$ as in (I). Then we have

$$
\int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |f_j|^p \, dx \leq \varepsilon.
$$

Hence,

$$\begin{aligned} Q(f_j) &= \int_{\mathbb{R}^N} (\mathcal{K} * f_j) \cdot f_j \, dx \\ &\leq \int_{B_{R_\varepsilon}(0)} (\mathcal{K} * f_j) \cdot f_j \, dx + \|\mathcal{K}\|_{\frac{p'}{2}} \|f_j\|_p \|f_j \mathbb{1}_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)}\|_p \\ &\leq \int_{B_{R_\varepsilon}(0)} (\mathcal{K} * f_j) \cdot f_j \, dx + \|\mathcal{K}\|_{\frac{p'}{2}} \varepsilon^{\frac{1}{p}}. \end{aligned}$$

We have $f_j \rightharpoonup f$ in $L^p(\mathbb{R}^N)$ and Proposition 8 implies $\mathcal{K} * f_j \to \mathcal{K} * f$ in $L^{p'}(B_{R_\varepsilon}(0))$. Hence, passing to the limit $j \to \infty$ we find

$$m = \lim_{j\to\infty} Q(f_j) \leq \int_{B_{R_\varepsilon}(0)} (\mathcal{K} * f) \cdot f \, dx + \|\mathcal{K}\|_{\frac{p'}{2}} \varepsilon^{\frac{1}{p}}.$$

Using $\|f\|_p \leq \liminf_{j\to\infty} \|f_j\|_p = 1$ and sending ε to zero, we infer from the Dominated Convergence Theorem

$$0 < m = \lim_{j\to\infty} Q(f_j) \leq Q(f) = Q(f/\|f\|_p)\|f\|_p^2 \leq Q(f/\|f\|_p) \leq m.$$

In particular $f \neq 0$ is a maximizer of Q with $\|f\|_p = 1$.

The claim (i) is clear. As to (ii), note that in the case $M = 1, \mathcal{K} = |\mathcal{K}|^*$ there is even a Schwarz-symmetric maximizer. In fact, passing from f to its Schwarz-symmetric spherical rearrangement $|f|^*$ we find $Q(f) \leq Q(|f|^*)$. This follows from $\|f\|_p = \||f|^*\|_p$ [17, p.81] and Riesz' rearrangement inequality [17, Theorem 3.7]

$$Q(f) \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\mathcal{K}(x-y)||f(x)||f(y)| \, dx \, dy \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\mathcal{K}|^*(x-y)|f|^*(x)|f|^*(y) \, dx \, dy = Q(|f|^*).$$

If additionally $\mathcal{K} = |\mathcal{K}|^*$ is decreasing (i.e., strictly monotone) in the radial direction, then each minimizer of Q is Schwarz-symmetric up to translations. This follows from the sharp version of Riesz' rearrangement inequality [17, Theorem 3.9] and (iii) is proved as well. □

Remark 10.

(a) The condition $\sup_{f\in S} Q(f) > 0$ is typically easy to check. For instance, it holds for kernel functions $\mathcal{K}$ that are uniformly positive definite close to the origin because we may choose f with small support. In the case $M = 1$ it is sometimes easier to check the equivalent condition $\sup_{\mathbb{R}^3} \widehat{\mathcal{K}} > 0$. The condition $\lim_{|h|\to 0} \|\mathcal{K}(\cdot + h) - \mathcal{K}\|_{p'/2,\infty} = 0$ holds provided that $\mathcal{K} \in L^{p'/2,\infty}(\mathbb{R}^N; \mathbb{R}^{M\times M})$ is almost everywhere continuous. In fact, for $\mathcal{K}_R$ as in the proof of Proposition 6, we have

$$\|\mathcal{K}(\cdot + h) - \mathcal{K}\|_{\frac{p'}{2},\infty} \leq 2\|\mathcal{K} - \mathcal{K}_R\|_{\frac{p'}{2},\infty} + \|\mathcal{K}_R(\cdot + h) - \mathcal{K}_R\|_{\frac{p'}{2},\infty}$$

and the latter term tends to 0 by the Dominated Convergence Theorem.

(b) Lemma 9 may as well be used to give an alternative existence proof for nontrivial solutions of autonomous (possibly nonlocal) elliptic PDEs as in [6, Theorem 1]. In fact, the nonlocal PDE $P(D)u = |u|^{r-2}u$ in $\mathbb{R}^N$ with $P(i\xi) = m(\xi)$, m real-valued and positive, is equivalent to $|v|^{r'-2}v = \mathcal{K} * v$ where $v := |u|^{r-2}u$ and $\widehat{\mathcal{K}}(\xi) = m(\xi)^{-1}$. Lemma 9 provides a nontrivial solution to this problem under similar assumptions as in [6, Theorem 1], which gives a dual ground state of $P(D)u = |u|^{r-2}u$. In the Appendix we will provide the details and prove that any dual ground state is a ground state and vice versa, so we recover [6, Theorem 1] under slightly weaker assumptions.

5. Proof of Theorem 3

For $1 < q < 2$ we consider $I_q(E) = \frac{1}{2} I_L(E) - \frac{1}{2q} I_{NL}(E)$ where

$$I_L(E) := \int_{\mathbb{R}^3} |\nabla \times E|^2 + |E|^2 \, dx, \qquad I_{NL}(E) := \int_{\mathbb{R}^3} (K * |E|^q)|E|^q \, dx.$$

Then Young's convolution inequality shows that I_q is continuously differentiable provided that $K \in L^{\frac{1}{2-q}}(\mathbb{R}^3)$. In this case, the Fréchet derivative is given by

$$I_q'(E)[\tilde{E}] = \int_{\mathbb{R}^3} (\nabla \times E) \cdot (\nabla \times \tilde{E}) + E \cdot \tilde{E}\,dx - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} K(x-y)E(x)\cdot \tilde{E}(x)|E(x)|^{q-2}|E(y)|^q\,dx\,dy$$

for $E, \tilde{E} \in H$. We consider minimization over the Nehari manifold $\mathcal{N}_q = \{E \in H : I_q'(E)[E] = 0, E \neq 0\}$ and obtain as in the Proof of Theorem 2

$$\inf_{\mathcal{N}_q} I_q = \inf_{E\in H\setminus\{0\}} \sup_{t>0} I_q(tE) = \inf_{E\in H\setminus\{0\}} \frac{q-1}{2q}\left(\frac{I_L(E)^q}{I_{NL}(E)}\right)^{\frac{1}{q-1}}.$$

Note that the assumptions of Theorem 3 imply $K \geq 0, K \not\equiv 0$, so $I_{NL}(E) > 0$ holds for all $E \in H \setminus \{0\}$. Hence we obtain

$$\begin{aligned}
\inf_{\mathcal{N}_q} I_q &\geq \inf_{E\in H\setminus\{0\}} \frac{q-1}{2q}\left(\frac{(\int_{\mathbb{R}^3}|E(x)|^2\,dx)^q}{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} K(x-y)|E(x)|^q|E(y)|^q\,dx\,dy}\right)^{\frac{1}{q-1}} \\
&= \inf_{E\in H\setminus\{0\}} \frac{q-1}{2q}\left(\frac{\||E|^q\|_{2/q}^2}{Q(|E|^q)}\right)^{\frac{1}{q-1}} \\
&= \inf_{E\in H\setminus\{0\}} \frac{q-1}{2q} Q\big(|E|^q/\||E|^q\|_{2/q}^{-1}\big)^{-\frac{1}{q-1}} \\
&\geq \frac{q-1}{2q}(\max_S Q)^{-\frac{1}{q-1}}
\end{aligned}$$

where $S := \{f \in L^{2/q}(\mathbb{R}^3) : \|f\|_{2/q} = 1\}$ denotes the unit sphere in $L^{2/q}(\mathbb{R}^3)$ and $Q(f) := \int_{\mathbb{R}^3}(\mathcal{K} * f)f\,dx$. Equality holds if and only if $\nabla \times E = 0$ and $|E|^q$ is a multiple of some maximizer of Q over S. Lemma 9 shows that under the assumptions of Theorem 3 there is a Schwarz-symmetric maximizer $f(x) = f_0(|x|)$. So we may define $E_\star(x) := \frac{x}{|x|}f_0(|x|)^{1/q}$. Then $E_\star$ is irrotational because of

$$E_\star(x) = \nabla\Phi(x) \qquad \text{where } \Phi(x) := \int_0^{|x|} f_0(s)^{1/q}\,ds.$$

Moreover, $|E_\star(x)|^q = f_0(|x|) = f(x)$. Choosing $t_\star > 0$ as the maximizer of $t \mapsto I_q(tE_\star)$ we find $E^\star := t_\star E_\star \in \mathcal{N}_q$ as well as

$$I_q(E^\star) = \inf_{\mathcal{N}_q} I_q = \frac{q-1}{2q}(\max_S Q)^{-\frac{1}{q-1}},$$

so $E^\star$ minimizes I_q over $\mathcal{N}_q$. This proves that $E^\star$ is a ground state solution and the claim is proved. □

Remark 11.

(a) The uniqueness of ground states up to translations is an open question.

(b) The above proof shows that the conclusion of Theorem 3 is true as long as Q has a nonnegative radially symmetric maximizer. This may be the case for more general radially symmetric kernel functions K, possibly sign-changing ones. Our focus on nonnegative radially symmetric maximizers is motivated by the fact that any such maximizer can be written as $|E|^q$ for some irrotational vector field E. It is unclear how to link the maximizers of Q to the ground state solutions for Nonlinear Maxwell Equations in nonradial situations. Those occur if the kernel function K is nonradial or if $|E(x)|^2$ is replaced by $V(x)E(x) \cdot E(x)$ with some periodic tensor field V, which is relevant for applications in photonic crystals. In both cases, the existence of ground states and whether those are irrotational is an open problem.

(c) In the case $q > 2$ the approach presented above fails because the infimum over the Nehari manifold is zero for all relevant kernel functions. For instance, consider any kernel function K that is positive near the origin. Then choose a sequence (E_n) that is bounded in $H^1(\text{curl};\mathbb{R}^3)$, has small support and satisfies $\|E_n\|_q \to \infty$, say $E_n(x) := \frac{x}{|x|}|x|^{-\frac{3}{q}(1-\frac{1}{n})}\mathbb{1}_{|x|\leq\varepsilon}$ where $K(z) \geq \mu > 0$ for $|z| < 2\varepsilon$. Then a straightforward computation shows that the infimum over the corresponding Nehari manifold is zero because $(I_L(E_n))$ is bounded whereas $I_{NL}(E_n) \geq \mu \|E_n\|_q^{2q} \nearrow +\infty$. In particular, it does not make sense to look for nontrivial solutions using this approach.

6. Proof of Theorem 4

We present a dual variational approach for the fully nonlocal nonlinear Maxwell equation (5) given by

$$\nabla \times \nabla \times E + E = K * (|E|^{r-2}E) \quad \text{in } \mathbb{R}^3$$

where $r := 2q$ and the kernel function $K \in L^{r/2,\infty}(\mathbb{R}^3)$ satisfies $K(\cdot + h) \to K$ in $L^{r/2,\infty}(\mathbb{R}^3)$ as $|h| \to 0$. We are interested in nontrivial solutions of this problem that turn out to exist for all $r \in (2,\infty)$. In particular, the most important case of a Kerr nonlinearity $r = 4$ is covered. The above equation is not variational, so it does not make sense to look for ground states. The idea is to follow a dual variational approach, i.e., to consider the equation as a variational problem for the new unknown $U := |E|^{r-2}E$ that is obtained after inverting the linear operator $E \mapsto \nabla \times \nabla \times E + E$ in suitable Lebesgue spaces. We are looking for solutions $E \in L^r(\mathbb{R}^3;\mathbb{R}^3)$, so the dual variational approach dealing with $U = |E|^{r-2}E$ is set up in $L^{r'}(\mathbb{R}^3;\mathbb{R}^3)$. From $2 < r < \infty$ we infer $1 < r' < 2$. As mentioned above, we first need to invert the linear operator. We will need the following auxiliary result about the Helmholtz Decomposition that decomposes a vector field as a sum of its divergence-free (solenoidal) and curl-free (irrotational) part.

Proposition 12. *Assume $1 < t < \infty$. Then there is a continuous projector $\Pi : L^t(\mathbb{R}^3;\mathbb{R}^3) \to L^t(\mathbb{R}^3;\mathbb{R}^3)$ such that $E = E_1 + E_2$ with $E_1 := \Pi E, E_2 := (\mathrm{id} - \Pi)E$ implies $\nabla \cdot E_1 = 0$ and $\nabla \times E_2 = 0$ in the distributional sense. In particular,*

$$\nabla \times \nabla \times E_1 = -\Delta E_1, \qquad \nabla \times \nabla \times E_2 = 0.$$

The proof is based on the vector calculus identity $\nabla \times \nabla \times E = -\Delta E + \nabla(\nabla \cdot E)$ and the explicit definition

$$\widehat{\Pi E}(\xi) := (1 - R(\xi))\hat{E}(\xi) \qquad \text{where } R(\xi) := |\xi|^{-2}\xi\xi^T \in \mathbb{R}^{3\times 3}.$$

This operator indeed defines a projector on $L^{r'}(\mathbb{R}^3;\mathbb{R}^3)$ because of $R(\xi)^2 = R(\xi)$ and the $L^{r'} \to L^{r'}-$boundedness of Riesz transforms $f \mapsto \mathcal{F}^{-1}(\xi_j|\xi|^{-1}\hat{f})$ for $j = 1,2,3$, see [11, Corollary 5.2.8]. We now use the Helmholtz Decomposition to derive the equivalent dual formulation of (5). In fact, distributional solutions $E \in L^r(\mathbb{R}^3;\mathbb{R}^3)$ of (5) solve

$$(-\Delta + 1)\Pi E = \Pi[K * U], \qquad (\mathrm{id} - \Pi)E = (\mathrm{id} - \Pi)[K * U]$$

where $U := |E|^{r-2}E \in L^{r'}(\mathbb{R}^3;\mathbb{R}^3)$. In Fourier variables this may rewritten as

$$(|\xi|^2 + 1)(1 - R(\xi))\hat{E}(\xi) = \hat{K}(\xi)(1 - R(\xi))\hat{U}(\xi), \qquad R(\xi)\hat{E}(\xi) = \hat{K}(\xi)R(\xi)\hat{U}(\xi).$$

We stress that the symbols $|\xi|^2 + 1, \hat{K}(\xi), R(\xi)$ commute because the former two are scalar. Hence,

$$\hat{E}(\xi) = \widehat{\mathcal{K}}(\xi)\hat{U}(\xi) \quad \text{where } \mathcal{K} := \mathcal{F}^{-1}\left(\hat{K}(\cdot)\left(\frac{1 - R(\cdot)}{|\cdot|^2 + 1} + R(\cdot)\right)\right).$$

Plugging in $E = |U|^{r'-2}U$ and applying the inverse Fourier transform in these equations it remains to find a solution $U \in L^{r'}(\mathbb{R}^3;\mathbb{R}^3)$ of the integral equation

$$|U|^{r'-2}U = \mathcal{K} * U \qquad \text{in } \mathbb{R}^3.$$

Given that $\mathcal{K}$ is a real-valued and symmetric tensor field, this equation has a variational structure. The associated energy functional reads

$$J(U) := \frac{1}{r'}\int_{\mathbb{R}^3} |U|^{r'}\,dx - \frac{1}{2}\int_{\mathbb{R}^3} (\mathcal{K} * U) \cdot U\,dx. \tag{9}$$

We establish the relevant properties of $\mathcal{K}$ with regard to Lemma 9.

Proposition 13. *Assume* $2 < r < \infty$ *and* $K \in L^{r/2,\infty}(\mathbb{R}^3)$ *and* $K(\cdot + h) \to K$ *in* $L^{r/2,\infty}(\mathbb{R}^3)$ *as* $|h| \to \infty$. *Then* $\mathcal{K} \in L^{r/2,\infty}(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ *and* $\mathcal{K}(\cdot+h) \to \mathcal{K}$ *in* $L^{r/2,\infty}(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ *as* $|h| \to \infty$. *Moreover, if* $\int_{\mathbb{R}^3}(K*f)f\,dx > 0$ *holds for some* $f \in L^{r'}(\mathbb{R}^3)$, *then* $\int_{\mathbb{R}^3} \mathcal{K} * F \cdot F\,dx > 0$ *for some* $F \in L^{r'}(\mathbb{R}^3;\mathbb{R}^3)$.

Proof. We use the Mikhlin-Hörmander multiplier Theorem from [11, Theorem 6.2.7], which says that the linear operator $f \mapsto \mathcal{F}^{-1}(m\hat{f})$ is bounded on $L^t(\mathbb{R}^N;\mathbb{C}), 1 < t < \infty$ provided that $|\partial^\alpha m(\xi)| \le C(\alpha,N)|\xi|^{-|\alpha|}$ for all $\xi \in \mathbb{R}^N$ and multi-indices $\alpha \in \mathbb{N}_0^N$. We actually need a consequence of this result that $f \mapsto \mathcal{F}^{-1}(m\hat{f})$ is bounded on the Lorentz space $L^{t,s}(\mathbb{R}^3;\mathbb{C}^3), 1 < t < \infty, 1 \le s \le \infty$ provided that the tensor field $m : \mathbb{R}^3 \to \mathbb{C}^{3\times 3}$ with entries m_{ij} for $i,j \in \{1,2,3\}$ satisfies $|\partial^\alpha m_{ij}(\xi)| \le C(\alpha)|\xi|^{-|\alpha|}$ for all $\xi \in \mathbb{R}^3$ and multi-indices $\alpha \in \mathbb{N}_0^3$. This follows by real interpolation from the classical L^p-version of this theorem, see Theorem 1.6 and Example 1.27 in [19]. Using this fact for $m(\xi) = (|\xi|^2+1)^{-1}(1-R(\xi)) + R(\xi)$ we find $\|\mathcal{K}\|_{r/2,\infty} \le C\|K\|_{r/2,\infty} < \infty$. Similarly, $\|\mathcal{K}(\cdot + h) - \mathcal{K}\|_{r/2,\infty} \le C\|K(\cdot+h) - K\|_{r/2,\infty} \to 0$ as $|h| \to 0$. Finally, $\int_{\mathbb{R}^3}(K*f)f\,dx > 0$ for some $f \in L^{r'}(\mathbb{R}^3)$ implies that the vector field $F \in L^{r'}(\mathbb{R}^3;\mathbb{R}^3)$ given by $\hat{F}(\xi) = \frac{\xi}{|\xi|}\hat{f}$ satisfies $(1-R(\xi))\hat{F}(\xi) = 0$ and thus

$$\int_{\mathbb{R}^3}(\mathcal{K}*F)\cdot F\,dx = \int_{\mathbb{R}^3}\widehat{\mathcal{K}}\hat{F}\cdot\hat{F}\,d\xi = \int_{\mathbb{R}^3}\hat{K}|\hat{F}|^2\,d\xi = \int_{\mathbb{R}^3}\hat{K}|\hat{f}|^2\,d\xi = \int_{\mathbb{R}^3}(K*f)f\,dx > 0.$$

□

Proof of Theorem 4: In view of $\mathcal{K} \in L^{r/2,\infty}(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ and Young's convolution inequality the functional $J : L^{r'}(\mathbb{R}^3;\mathbb{R}^3) \to \mathbb{R}$ from (9) is continuously differentiable with Fréchet derivative

$$J'(U)[\tilde{U}] = \int_{\mathbb{R}^3}|U|^{r'-2}U\cdot\tilde{U}\,dx - \int_{\mathbb{R}^3}(\mathcal{K}*U)\cdot\tilde{U}\,dx$$

for all $U,\tilde{U} \in L^{r'}(\mathbb{R}^3;\mathbb{R}^3)$. To find a ground state we minimize J over the associated Nehari manifold $\mathcal{M} = \{U \in L^{r'}(\mathbb{R}^3;\mathbb{R}^3) : J'(U)[U] = 0, U \ne 0\}$. We find as before

$$\inf_{\mathcal{M}} J = \inf_{U\in L^{r'}(\mathbb{R}^3;\mathbb{R}^3)\setminus\{0\}}\sup_{t>0} J(tU) = \inf_{U\in L^{r'}(\mathbb{R}^3;\mathbb{R}^3)\setminus\{0\}} \frac{2-r'}{2r'}\left(\frac{(\int_{\mathbb{R}^3}|U|^{r'}\,dx)^{\frac{2}{r'}}}{\int_{\mathbb{R}^3}(\mathcal{K}*U)\cdot U\,dx}\right)^{\frac{r'}{2-r'}}.$$

Combining the assumptions of Theorem 4 with Proposition 13 we obtain that the assumptions of Lemma 9 hold. Hence, the functional $Q(U) := \int_{\mathbb{R}^3}(\mathcal{K}*U)\cdot U\,dx$ has a maximizer $U_\star \in S$ over the unit sphere $S := \{U \in L^{r'}(\mathbb{R}^3;\mathbb{R}^3) : \|U\|_{r'} = 1\}$. We thus obtain

$$\inf_{\mathcal{M}} J = \inf_{U\in L^{r'}(\mathbb{R}^3;\mathbb{R}^3)\setminus\{0\}} \frac{2-r'}{2r'}Q(U/\|U\|_{r'})^{-\frac{r'}{2-r'}} \ge \frac{2-r'}{2r'}Q(U_\star)^{-\frac{r'}{2-r'}} = J(t_\star U_\star)$$

where $t_\star > 0$ maximizes $t \mapsto J(tU_\star)$ so that $t_\star U_\star \in \mathcal{M}$. Hence, $U^\star := t_\star U_\star$ is a ground state for J and $E^\star := |U^\star|^{r'-2}U^\star$ is a dual ground state of (5). This finishes the proof of Theorem 4. □

Appendix – Ground states vs. dual ground states

We show that in several contexts the notion of a dual ground state solution coincides with the classical notion of a ground state provided that both notions make sense for the equation under investigation. In particular, dual ground states may be seen as reasonable substitutes for ground states. As a model example we consider $P(D)u = |u|^{r-2}u$ on $\mathbb{R}^N$ from Remark 10. The formally equivalent dual formulation is $|v|^{r'-2}v = P(D)^{-1}v$ for $v := |u|^{r-2}u$. The energy functional $I : H^s(\mathbb{R}^N) \to \mathbb{R}$ and the dual energy functional $J : L^{r'}(\mathbb{R}^N) \to \mathbb{R}$ are given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} m(\xi)|\hat{u}(\xi)|^2\,d\xi - \frac{1}{r}\int_{\mathbb{R}^N}|u|^r\,dx,$$
$$J(v) = \frac{1}{r'}\int_{\mathbb{R}^N}|v(x)|^{r'}\,dx - \frac{1}{2}\int_{\mathbb{R}^N} m(\xi)^{-1}|\hat{v}(\xi)|^2\,d\xi.$$

Here, $m(\xi) = P(i\xi)$ is the symbol associated with P and we shall assume $|\partial^\alpha(m(\xi)^{-1})| \leq C_\alpha |\xi|^{-|\alpha|}(1+|\xi|)^{-2s}$ for $0 < s < \frac{N}{2}$ such that $2 < r < \frac{2N}{N-2s}$ and all multi-indices $\alpha \in \mathbb{N}_0^N$. This is slightly stronger than Assumption 1 in [6]. We now show on an abstract level that a ground state solution for I is the same as a dual ground state solution.

Let Ω be a set and assume that for all $x \in \Omega$ the functions $F(x,\cdot), G(x,\cdot) \in C^2(\mathbb{R})$ are real-valued with $F_u(x,\cdot)^{-1} = G_v(x,\cdot)$ and $F(x,0) = G(x,0) = 0$ for all $x \in \Omega$. Moreover assume that there are Banach spaces X, Y consisting of real-valued functions defined on Ω and that there are continuously differentiable functionals $I : X \to \mathbb{R}, J : Y \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}Q_1(u,u) - \varphi(F(\cdot,u)), \qquad J(v) = \varphi(G(\cdot,v)) - \frac{1}{2}Q_2(v,v)$$

where $Q_1 : X \times X \to \mathbb{R}$ and $Q_2 : Y \times Y \to \mathbb{R}$ are continuous bilinear forms and φ is a linear functional acting on real-valued functions defined on Ω such as $x \mapsto F(x,u(x))$ and $x \mapsto G(x,v(x))$ for $u \in X, v \in Y$. We assume that I, J are continuously differentiable with

$$I'(u)[h_1] = Q_1(u,h_1) - \varphi(F_u(\cdot,u)h_1), \quad J'(v)[h_2] = \varphi(G_v(\cdot,v)h_2) - Q_2(v,h_2) \qquad (h_1 \in X, h_2 \in Y).$$

We moreover assume that I and J are dual to each other in the following sense: there are subsets $M \subset X$ and $N \subset Y$ such that the Euler-Lagrange equations for $I|_M, J|_N$ are equivalent, namely

$$I'(u) = 0,\ u \in M \quad \Leftrightarrow \quad J'(v) = 0,\ v \in N \qquad \text{for } v(x) = F_u(x,u(x)), u(x) = G_v(x,v(x)). \tag{10}$$

Note that the above example satisfies (10) for $M = X = H^s(\mathbb{R}^N)$ and $N = Y = L^{r'}(\mathbb{R}^N)$. In fact, the implication from left to right follows from the fractional Sobolev Embedding $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $2 \leq r \leq \frac{2N}{N-2s}$. The opposite implication, which should be seen as a regularity result, follows from bootstrapping $|v|^{r'-2}v = P(D)^{-1}v$ with the aid of $\|P(D)^{-1}f\|_q \leq C\|f\|_p$ for $0 < \frac{1}{p} - \frac{1}{q} < \frac{2s}{N}$. These estimates follows from Bessel potential estimates and Mikhlin's multiplier Theorem, which is applicable due to $|\partial^\alpha(m(\xi)^{-1})| \leq C_\alpha|\xi|^{-|\alpha|}(1+|\xi|)^{-2s}$. Here one needs the sharp inequality $r < \frac{2N}{N-2s}$ to show that $v \in L^{r'}(\mathbb{R}^N)$ implies $v \in L^2(\mathbb{R}^N)$ (via bootstrapping) and hence $u = |v|^{r'-2}v = P(D)^{-1}v \in H^s(\mathbb{R}^N)$.

We say that a ground state for $I|_M$ is a nontrivial solution of $I'(u) = 0, u \in M$ with least energy I among all nontrivial critical points of $I|_M$. A dual ground state for $I|_M$ with respect to $J|_N$ is a nontrivial solution of $I'(u) = 0, u \in M$ such that the dual function $v(x) := F_u(x,u(x))$ is a ground state for $J|_N$.

Proposition 14. *Under the assumptions from above:*

(i) A solution $u^* \in M$ is a ground state for $I|_M$ if and only if $\tilde{u} \in M, I'(\tilde{u}) = 0$ implies

$$\varphi\left(F_u(\cdot,u^*)u^* - 2F(\cdot,u^*)\right) \leq \varphi\left(F_u(\cdot,\tilde{u})\tilde{u} - 2F(\cdot,\tilde{u})\right).$$

(ii) A solution $v^* \in N$ is a ground state for $J|_N$ if and only if $\tilde{v} \in N, J'(\tilde{v}) = 0$ implies

$$\varphi\left(2G(\cdot,v^*) - G_v(\cdot,v^*)v^*\right) \leq \varphi\left(2G(\cdot,\tilde{v}) - G_v(\cdot,\tilde{v})\tilde{v}\right).$$

Proof. Part (i) follows from $2I(u) = 2I(u) - I'(u)[u] = \varphi\left(F_u(\cdot,u)u - 2F(x,u)\right)$ for all $u \in M$ such that $I'(u) = 0$. Part (ii) is proved analogously. □

Theorem 15. *Under the assumptions from above: $u^* \in M$ is a ground state for $I|_M$ if and only if it is a dual ground state for $I|_M$ with respect to $J|_N$.*

Proof. Assume that $u^* \in M$ is a ground state for $I|_M$, define $v^* := F_u(\cdot,u^*) \in N$. We have to show that v^* is a dual ground state. So take any $v \in N$ such that $J'(v) = 0$ and define $u := G_v(\cdot,v)$. By (10) we have $u \in M$ and $I'(u) = 0$. Since u^* is a ground state, we know from Proposition 14

$$\varphi\left(F_u(\cdot,u^*)u^* - 2F(\cdot,u^*)\right) \leq \varphi\left(F_u(\cdot,u)u - 2F(\cdot,u)\right).$$

Plugging in $u^* = G_v(\cdot,v^*), u = G_v(\cdot,v)$ we find

$$\varphi\left(v^*G_v(\cdot,v^*) - 2F(\cdot,G_v(\cdot,v^*))\right) \leq \varphi\left(vG_v(\cdot,v) - 2F(\cdot,G_v(\cdot,v))\right). \tag{11}$$

Since $F_u(x,\cdot)$ and $G_v(x,\cdot)$ are inverses of each other, we have for all $x \in \Omega, z \in \mathbb{R}$

$$F(x, G_v(x,z)) = \int_0^{G_v(x,z)} F_u(x,s)\,ds = \int_0^z tG_{vv}(x,t)\,dt = zG_v(x,z) - G(x,z).$$

Using this identity for $z = v^*(x), z = v(x)$, respectively, we obtain from (11)

$$\varphi\left(2G(\cdot,v^*) - v^*G_v(\cdot,v^*)\right) \le \varphi\left(2G(\cdot,v) - vG_v(\cdot,v)\right).$$

Given that $v, v^* \in N$ are critical points of J and v was arbitrary, this means that v^* is a ground state for $J|_N$. Hence, by definition, u^* is a dual ground of $I|_M$ with respect to J_N. In a similar way one shows that a dual ground state with respect to $J|_N$ yields a ground state solution. □

Acknowledgments

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

References

[1] S. Alama and Y. Y. Li. Existence of solutions for semilinear elliptic equations with indefinite linear part. *J. Differential Equations*, 96(1):89–115, 1992.

[2] A. Azzollini, V. Benci, T. D'Aprile, and D. Fortunato. Existence of static solutions of the semilinear Maxwell equations. *Ric. Mat.*, 55(2):283–297, 2006.

[3] T. Bartsch, T. Dohnal, M. Plum, and W. Reichel. Ground states of a nonlinear curl-curl problem in cylindrically symmetric media. *NoDEA Nonlinear Differential Equations Appl.*, 23(5):Art. 52, 34, 2016.

[4] H. Berestycki and P.-L. Lions. Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.*, 82(4):313–345, 1983.

[5] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[6] L. Bugiera, E. Lenzmann, A. Schikorra, and J. Sok. On symmetry of traveling solitary waves for dispersion generalized NLS. *Nonlinearity*, 33(6):2797–2819, 2020.

[7] K.-C. Chang. *Methods in nonlinear analysis.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[8] L. Cossetti and R. Mandel. A limiting absorption principle for Helmholtz systems and time-harmonic isotropic Maxwell's equations. *Journal of Functional Analysis*, 281(11):109233, 2021.

[9] T. D'Aprile and G. Siciliano. Magnetostatic solutions for a semilinear perturbation of the Maxwell equations. *Adv. Differential Equations*, 16(5-6):435–466, 2011.

[10] F. Z. Goffi, A. Khrabustovskyi, R. Venkitakrishnan, C. Rockstuhl, and M. Plum. Higher order constitutive relations and interface conditions for metamaterials with strong spatial dispersion. *Phys. Lett. A*, 412:Paper No. 127570, 11, 2021.

[11] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics.* Springer, New York, third edition, 2014.

[12] A. Hirsch and W. Reichel. Existence of cylindrically symmetric ground states to a nonlinear curl-curl equation with non-constant coefficients. *Z. Anal. Anwend.*, 36(4):419–435, 2017.

[13] T. Kato. Growth properties of solutions of the reduced wave equation with a variable coefficient. *Comm. Pure Appl. Math.*, 12:403–425, 1959.

[14] P. Kinsler. A new introduction to spatial dispersion: Reimagining the basic concepts. *Photonics and Nanostructures - Fundamentals and Applications*, 43:100897, 2021.

[15] H. Koch and D. Tataru. Carleman estimates and absence of embedded eigenvalues. *Comm. Math. Phys.*, 267(2):419–449, 2006.

[16] W. Krolikowski, O. Bang, J. J. Rasmussen, and J. Wyller. Modulational instability in nonlocal nonlinear kerr media. *Phys. Rev. E*, 64:016612, Jun 2001.

[17] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, second edition, 2001.

[18] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 1984.

[19] A. Lunardi. *Interpolation theory*, volume 16 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)].* Edizioni della Normale, Pisa, 2018. Third edition [of MR2523200].

[20] R. Mandel. Uncountably many solutions for nonlinear Helmholtz and curl-curl equations. *Adv. Nonlinear Stud.*, 19(3):569–593, 2019.

[21] J. Mederski. Ground states of time-harmonic semilinear Maxwell equations in $\mathbb{R}^3$ with vanishing permittivity. *Arch. Ration. Mech. Anal.*, 218(2):825–861, 2015.

[22] N. I. Nikolov, D. Neshev, O. Bang, and W. Z. Królikowski. Quadratic solitons as nonlocal solitons. *Phys. Rev. E*, 68:036614, Sep 2003.

[23] R. W. Schoonover, J. M. Rutherford, O. Keller, and P. S. Carney. Non-local constitutive relations and the quasi-homogeneous approximation. *Physics Letters A*, 342(5):363–367, 2005.

[24] D. R. Smith, D. C. Vier, N. Kroll, and S. Schultz. Direct calculation of permeability and permittivity for a left-handed metamaterial. *Applied Physics Letters*, 77(14):2246–2248, 2000.

[25] W. A. Strauss. Existence of solitary waves in higher dimensions. *Comm. Math. Phys.*, 55(2):149–162, 1977.

[1]Karlsruhe Institute of Technology, Institute for Analysis, Englerstrasse 2, 76131 Karlsruhe, Germany