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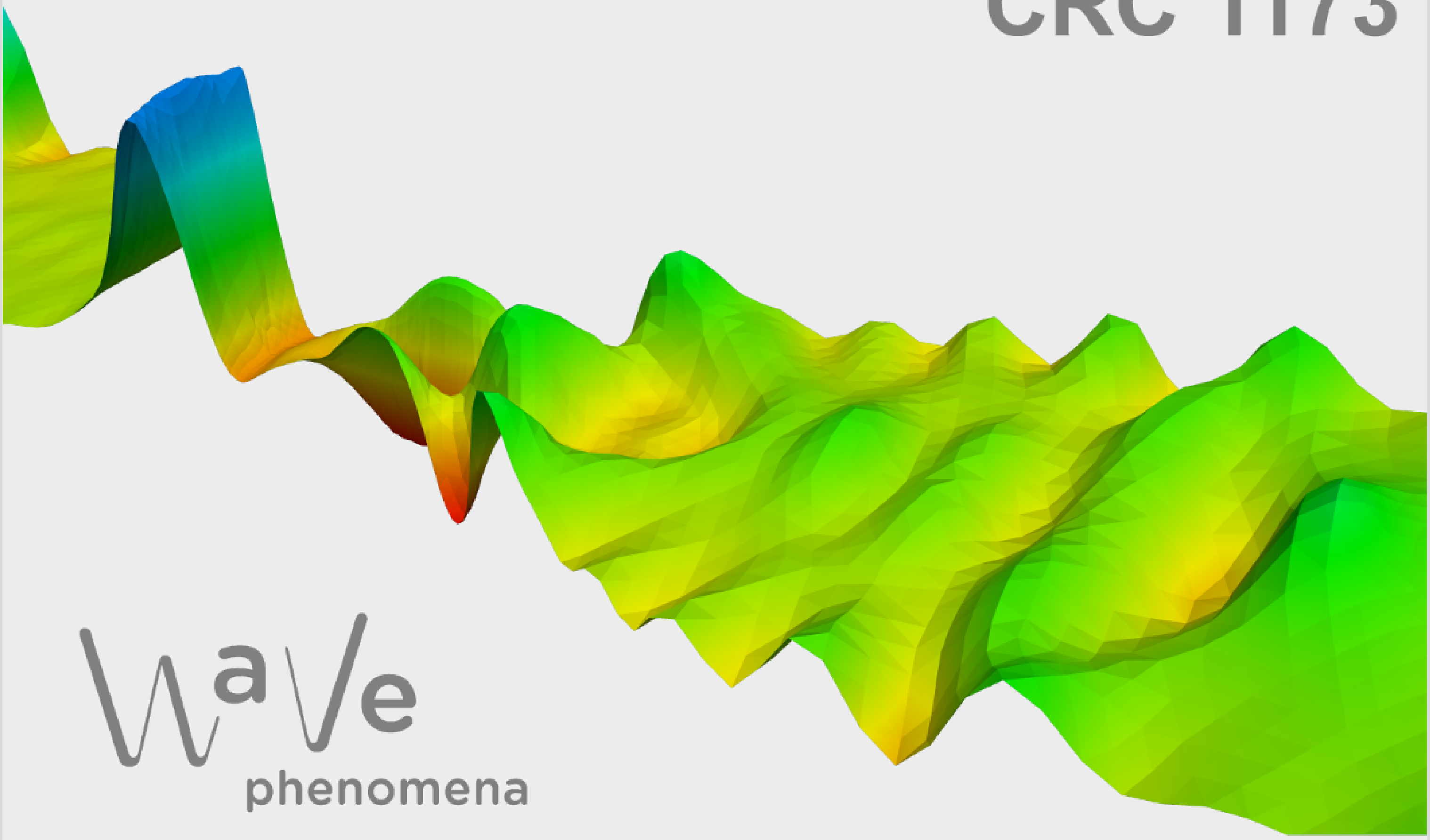

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Monotonicity in inverse scattering for Maxwell's equations

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Abstract

A recent area of research in inverse scattering theory has been the study of monotonicity relations for the eigenvalues of far field operators and their use in shape reconstruction for inverse scattering problems. We develop such monotonicity relations for an electromagnetic inverse scattering problem governed by Maxwell's equations, and we apply them to establish novel rigorous characterizations of the shape of scattering objects in terms of the corresponding far field operators. Along the way we establish the existence of electromagnetic fields that have arbitrarily large energy in some prescribed region, while at the same time having arbitrarily small energy in some other prescribed region. These localized vector wave functions not only play an important role in the proofs of the novel monotonicity based shape characterizations but they are also of independent interest. We conclude with some simple numerical demonstrations of our theoretical results.

Mathematics subject classifications (MSC2010): 35R30, (65N21)

Keywords: Inverse scattering, Maxwell's equations, monotonicity principles, far field operator, inhomogeneous medium

Short title: Monotonicity in inverse electromagnetic scattering

1 Introduction

We discuss an inverse scattering problem for time-harmonic Maxwell's equations, where the goal is to determine the position and the shape of a collection of compactly supported scattering objects from far field observations of scattered electromagnetic waves. We extend the monotonicity based approach to shape reconstruction for inverse scattering problems that was established in [18, 30, 31] from scalar wave propagation described by the Helmholtz equation to electromagnetic wave propagation governed by Maxwell's equations. The main outcome of this work is a new rigorously justified shape reconstruction technique for the electromagnetic inverse scattering problem. This monotonicity method is formulated in terms of far field operators that map superpositions of incident plane waves to the far field patterns of the corresponding scattered waves. Throughout we assume that the scattering objects are penetrable, non-magnetic and non-absorbing, i.e., the magnetic permeability μ is constant throughout $\mathbb{R}^3$, while the real-valued electric permittivity ε is constant outside the support of the scatterer but may be inhomogeneous inside the scattering objects.

The first result of this work is a novel monotonicity property for the eigenvalues of the difference of two far field operators corresponding to two different scattering objects in terms of the difference of the corresponding relative electric permittivities. This monotonicity property

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is interesting because it immediately yields an idea and a partial justification for a new characterizations of the shape of an unknown scattering object in terms of its far field operator. The resulting monotonicity based shape reconstruction technique consists in comparing a given (observed) far field operator to certain probing operators to decide whether some probing domains $B \subseteq \mathbb{R}^3$ corresponding to the probing operators are contained inside the support $D \subseteq \mathbb{R}^3$ of the unknown scatterer, or whether they contain the support of the scatterer. These probing operators can either be simulated far field operators corresponding to the probing domains B, or simulated linearizations of such far field operators.

To establish a complete theoretical justification of this monotonicity based shape characterization we require another theoretical tool, which constitutes the second result of this work. We show the existence of solutions to the direct scattering problem corresponding to suitable incident fields that have arbitrarily large energy in some prescribed region $B \subseteq \mathbb{R}^3$, while at the same time having arbitrarily small energy in a different prescribed region $\Omega \subseteq \mathbb{R}^3$, assuming that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected and $B \not\subseteq \Omega$. Similar classes of solutions have recently, e.g., been studied for the Laplace equation [16], for the Helmholtz equation on bounded domains [31] and on unbounded domains [18], and for Maxwell's equations on bounded domains [27].

Combining the novel monotonicity relations with the newly developed localized vector wave functions we give rigorous proofs for the new monotonicity based shape characterizations. This is the main result of this work.

Monotonicity based shape reconstruction techniques have first been analyzed for the inverse conductivity problem in [16, 32], extending an earlier monotonicity based reconstruction scheme that has been proposed in [48]. The method is related to monotonicity principles for the Laplace equation which have been established in [34, 35]. It has been further developed in [28, 29, 33], and its numerical implementation has been considered in [13, 14, 15]. More recently, an extension to impenetrable conductivity inclusions has been provided in [9]. The results from [32] have been extended to an inverse coefficient problem for the Helmholtz equation on bounded domains in [30, 31], and in [18] the approach has been generalized to an inverse medium scattering problem for the Helmholtz equation on unbounded domains. Inverse obstacle scattering problems have been considered in [1, 12], and an inverse crack detection problem has been studied in [11]. For further recent contributions on monotonicity based reconstruction methods for inverse problems for various other partial differential equations we refer, e.g., to [5, 6, 25, 26, 43, 47, 49]. In this work we generalize the concepts and the rigorous analysis from [18] to an inverse medium scattering problem governed by time-harmonic Maxwell's equations.

The theoretical underpinning of the new monotonicity based shape characterization is somewhat related to the linear sampling method (see, e.g., [8, 24]) and to the factorization method (see, e.g., [36, 37, 38]). However, in contrast to these methods the monotonicity based characterization of the support of the scattering objects is independent of so-called transmission eigenvalues (see, e.g., [7, 10] for an account on the latter). On the other hand, there are connections between the monotonicity relation for far field operators developed in this work and monotonicity principles for the phases of the eigenvalues of the scattering operator, which have been discussed in [42] to describe transmission eigenvalues in terms of far field operators (see also [3, 40, 41] for further results in this direction). Another advantage of the monotonicity based approach is that it also applies to a large class of indefinite scattering objects, i.e., when the relative electric permittivity of the scattering object takes values larger and smaller than 1 inside the scattering objects.

Although the main focus of this work is on the rigorous theoretical justification of the monotonicity based approach for the electromagnetic inverse scattering problem, we also present numerical demonstrations of our findings. In particular we discuss an explicit example for the radially symmetric case, we consider a sampling strategy for sign-definite scattering objects (i.e., when the relative electric permittivity is either strictly larger or strictly smaller than 1 everywhere inside the scatterer), and we discuss a sampling scheme to distinguish well separated components of indefinite scattering configurations, where the relative electric permittivity is either strictly larger or strictly smaller than 1.

The outline of this work is as follows. After introducing some notation in the next section, we briefly recall the mathematical formulation of the scattering problem in Section 3. In Section 4 we discuss a monotonicity principle for the far field operator, and in Section 5 we establish the existence of localized vector wave functions for Maxwell's equations on unbounded domains. We combine the monotonicity principle and the localized vector wave functions to develop rigorous characterizations of the support of sign-definite scattering objects in terms of the far field operator in Section 6. In Sections 7 and 8 we establish corresponding results for the indefinite case, and in Section 9 we present numerical results.

2 Preliminaries

We start by introducing some notation (see, e.g., [10, 39, 44] for details). The boldface Latin letters $\boldsymbol{x}, \boldsymbol{y}$ refer to generic points in $\mathbb{R}^3$, $\boldsymbol{x}\cdot\boldsymbol{y}$ and $\boldsymbol{x}\times\boldsymbol{y}$ denote the inner product and the vector product of $\boldsymbol{x}$ and $\boldsymbol{y}$, and $|\boldsymbol{x}|$ is the Euclidean norm of $\boldsymbol{x}$. By $B_R(0)\subseteq\mathbb{R}^3$ we denote the ball of radius $R>0$ centered at the origin.

For a bounded smooth domain $\Omega\subseteq\mathbb{R}^3$ we define

$$\begin{aligned} H(\mathbf{curl};\Omega) &:= \bigl\{\boldsymbol{u}\in L^2(\Omega,\mathbb{C}^3)\;\big|\;\mathbf{curl}\,\boldsymbol{u}\in L^2(\Omega,\mathbb{C}^3)\bigr\}\,,\\ H_{\mathrm{loc}}(\mathbf{curl};\mathbb{R}^3\setminus\overline{\Omega}) &:= \bigl\{\boldsymbol{u}\in L^2_{\mathrm{loc}}(\mathbb{R}^3\setminus\overline{\Omega},\mathbb{C}^3)\;\big|\;\mathbf{curl}\,\boldsymbol{u}\in L^2_{\mathrm{loc}}(\mathbb{R}^3\setminus\overline{\Omega},\mathbb{C}^3)\bigr\}\,, \end{aligned}$$

where $L^2_{\mathrm{loc}}(\mathbb{R}^3\setminus\overline{\Omega},\mathbb{C}^3)$ is the space of complex-valued locally square integrable vector fields on $\mathbb{R}^3\setminus\overline{\Omega}$. The unit outward normal vector field on $\partial\Omega$ is denoted by $\boldsymbol{\nu}$, and for smooth functions on $\partial\Omega$ the *surface gradient* **Grad** and the *surface vector curl* **Curl** may be defined in the usual way via parametric representation. The dual operators of $-$ **Grad** and **Curl** (with respect to the duality pairing given by the L^2 bilinear forms) are the *surface divergence* Div and the *surface scalar curl* Curl. Denoting by $H_t^{-1/2}(\partial\Omega,\mathbb{C}^3)$ the Hilbert space of tangential $H^{-1/2}(\partial\Omega,\mathbb{C}^3)$-vector fields, let

$$\begin{aligned} H^{-1/2}(\mathrm{Div};\partial\Omega) &:= \bigl\{\boldsymbol{\phi}\in H_t^{-1/2}(\partial\Omega,\mathbb{C}^3)\;\big|\;\mathrm{Div}\,\boldsymbol{\phi}\in H^{-1/2}(\partial\Omega,\mathbb{C})\bigr\}\,,\\ H^{-1/2}(\mathrm{Curl};\partial\Omega) &:= \bigl\{\boldsymbol{\phi}\in H_t^{-1/2}(\partial\Omega,\mathbb{C}^3)\;\big|\;\mathrm{Curl}\,\boldsymbol{\phi}\in H^{-1/2}(\partial\Omega,\mathbb{C})\bigr\}\,. \end{aligned}$$

Then the space $H^{-1/2}(\mathrm{Div};\partial\Omega)$ is naturally identified with the dual space of $H^{-1/2}(\mathrm{Curl};\partial\Omega)$. Throughout we write the dual pairing between $H^{-1/2}(\mathrm{Div};\partial\Omega)$ and $H^{-1/2}(\mathrm{Curl};\partial\Omega)$ as an integral for notational convenience.

For any regular vector field $\boldsymbol{u}$ on Ω we define the tangential traces $\gamma_t(\boldsymbol{u}) := \boldsymbol{\nu}\times\boldsymbol{u}|_{\partial\Omega}$ and $\pi_t(\boldsymbol{u}) := (\boldsymbol{\nu}\times\boldsymbol{u}|_{\partial\Omega})\times\boldsymbol{\nu}$. These can be extended to continuous linear, surjective operators

$$\gamma_t:\;H(\mathbf{curl};\Omega)\to H^{-1/2}(\mathrm{Div};\partial\Omega)\,,\qquad \pi_t:\;H(\mathbf{curl};\Omega)\to H^{-1/2}(\mathrm{Curl};\partial\Omega)\,,\tag{2.1}$$

and for all $\boldsymbol{u},\boldsymbol{w}\in H(\mathbf{curl};\Omega)$ we have the integration by parts formula

$$\int_\Omega(\mathbf{curl}\,\boldsymbol{u})\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}-\int_\Omega\boldsymbol{u}\cdot(\mathbf{curl}\,\boldsymbol{w})\,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega}(\boldsymbol{\nu}\times\boldsymbol{u})\cdot\bigl((\boldsymbol{\nu}\times\boldsymbol{w})\times\boldsymbol{\nu}\bigr)\,\mathrm{d}s\,.\tag{2.2}$$

Similarly, the map r, which is given by $r(\boldsymbol{\phi}) := \boldsymbol{\nu}\times\boldsymbol{\phi}$ for any smooth vector field $\boldsymbol{\phi}$ on $\partial\Omega$, can be extended to an isomorphism $r:H^{-1/2}(\mathrm{Div};\partial\Omega)\to H^{-1/2}(\mathrm{Curl};\partial\Omega)$. For the matter of readability, we will use the classical notation $\boldsymbol{\nu} \times \cdot$ and $(\boldsymbol{\nu} \times \cdot\,) \times \boldsymbol{\nu}$ for the trace operators in (2.1), and for the isomorphism r throughout this work.

The subspace of $H(\mathbf{curl};\Omega)$-functions with vanishing tangential traces is denoted by

$$H_0(\mathbf{curl};\Omega) \;:=\; \bigl\{ \boldsymbol{u} \in H(\mathbf{curl};\Omega) \;\big|\; \boldsymbol{\nu} \times \boldsymbol{u}|_{\partial\Omega} = 0 \bigr\} \,.$$

We also write $\boldsymbol{\nu}$ for the outward normal vector field on the unit sphere S^2, and accordingly we define

$$L^2_t(S^2,\mathbb{C}^3) \;:=\; \bigl\{ \boldsymbol{u} \in L^2(S^2,\mathbb{C}^3) \;\big|\; \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \;\text{ a.e. on } S^2 \bigr\} \,.$$

3 Scattering by an inhomogeneous medium

We consider the propagation of time-harmonic electromagnetic waves in non-magnetic media in $\mathbb{R}^3$. Let $k = \omega\sqrt{\varepsilon_0\mu_0}$ be the *wave number* at an *angular frequency* $\omega > 0$ in free space with *electric permittivity* $\varepsilon_0 > 0$ and *magnetic permeability* $\mu_0 > 0$. An *incident field* $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ is an entire solution to Maxwell's equations

$$\mathbf{curl}\,\boldsymbol{E}^i - \mathrm{i}\omega\mu_0\boldsymbol{H}^i \;=\; 0\,, \quad \mathbf{curl}\,\boldsymbol{H}^i + \mathrm{i}\omega\varepsilon_0\boldsymbol{E}^i \;=\; 0 \qquad \text{in } \mathbb{R}^3\,. \tag{3.1}$$

We suppose that such an incident field is scattered by an inhomogeneous medium with space dependent electric permittivity ε, and constant magnetic permeability $\mu = \mu_0$. We denote by $\varepsilon_r := \varepsilon/\varepsilon_0$ the *relative electric permittivity* of the inhomogeneous medium, and we assume that $\varepsilon_r^{-1} = 1 - q$ for some real-valued contrast function

$$q \in \mathcal{Y}_D \;:=\; \bigl\{ f \in L^\infty(\mathbb{R}^3) \;\big|\; f|_D \in W^{1,\infty}(D,\mathbb{R})\,,\; \mathrm{supp}(f) = \overline{D}\,,\; \operatorname{ess\,inf}(1-f) > 0 \bigr\}\,,$$

where $D \subseteq \mathbb{R}^3$ is open and bounded of class C^0. The *total field* $(\boldsymbol{E}_q, \boldsymbol{H}_q)$ excited by an incident field $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ in the inhomogeneous medium satisfies

$$\mathbf{curl}\,\boldsymbol{E}_q - \mathrm{i}\omega\mu_0\boldsymbol{H}_q \;=\; 0\,, \quad \mathbf{curl}\,\boldsymbol{H}_q + \mathrm{i}\omega\varepsilon\boldsymbol{E}_q \;=\; 0 \qquad \text{in } \mathbb{R}^3\,. \tag{3.2}$$

Rewriting

$$(\boldsymbol{E}_q, \boldsymbol{H}_q) \;=\; (\boldsymbol{E}^i, \boldsymbol{H}^i) + (\boldsymbol{E}^s_q, \boldsymbol{H}^s_q) \tag{3.3}$$

as a superposition of the incident field $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ and the *scattered field* $(\boldsymbol{E}^s_q, \boldsymbol{H}^s_q)$, we assume that the scattered field satisfies the Silver-Müller radiation condition

$$\lim_{|\boldsymbol{x}|\to\infty} \bigl(\sqrt{\varepsilon_0}\, \boldsymbol{x} \times \boldsymbol{E}^s_q(\boldsymbol{x}) - |\boldsymbol{x}| \sqrt{\mu_0} \boldsymbol{H}^s_q(\boldsymbol{x}) \bigr) \;=\; 0 \tag{3.4}$$

uniformly with respect to all directions $\widehat{\boldsymbol{x}} := \boldsymbol{x}/|\boldsymbol{x}| \in S^2$.

It will often be convenient to eliminate either the electric field or the magnetic field from (3.1)–(3.4) and to work with one of the second order formulations given by

$$\begin{aligned}
&\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}^i - k^2\boldsymbol{E}^i \;=\; 0 \quad \text{in } \mathbb{R}^3\,, && \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{H}^i - k^2\boldsymbol{H}^i \;=\; 0 \quad \text{in } \mathbb{R}^3\,, && \text{(3.5a)}\\
&\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E}_q - k^2\varepsilon_r\boldsymbol{E}_q \;=\; 0 \quad \text{in } \mathbb{R}^3\,, && \mathbf{curl}\bigl(\varepsilon_r^{-1}\,\mathbf{curl}\,\boldsymbol{H}_q\bigr) - k^2\boldsymbol{H}_q \;=\; 0 \quad \text{in } \mathbb{R}^3\,, && \text{(3.5b)}\\
&\boldsymbol{E}_q \;=\; \boldsymbol{E}^i + \boldsymbol{E}^s_q \qquad \text{in } \mathbb{R}^3\,, && \boldsymbol{H}_q \;=\; \boldsymbol{H}^i + \boldsymbol{H}^s_q \qquad \text{in } \mathbb{R}^3\,, && \text{(3.5c)}\\
&\lim_{|\boldsymbol{x}|\to\infty} \bigl(\boldsymbol{x} \times \mathbf{curl}\,\boldsymbol{E}^s_q(\boldsymbol{x}) + \mathrm{i}k|\boldsymbol{x}|\boldsymbol{E}^s_q(\boldsymbol{x}) \bigr) \;=\; 0\,, && \lim_{|\boldsymbol{x}|\to\infty} \bigl(\boldsymbol{x} \times \mathbf{curl}\,\boldsymbol{H}^s_q(\boldsymbol{x}) + \mathrm{i}k|\boldsymbol{x}|\boldsymbol{H}^s_q(\boldsymbol{x}) \bigr) \;=\; 0\,, && \text{(3.5d)}
\end{aligned}$$

respectively.

Remark 3.1. Throughout this work, Maxwell's equations are always to be understood in a weak sense. For instance, $\boldsymbol{E}_q, \boldsymbol{H}_q \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ are solutions to (3.2) (or equivalently to (3.5b)) if and only if either

$$\int_{\mathbb{R}^3} \bigl(\mathbf{curl}\, \boldsymbol{E}_q \cdot \mathbf{curl}\, \boldsymbol{\psi} - k^2 \varepsilon_r \boldsymbol{E}_q \cdot \boldsymbol{\psi}\bigr) \,\mathrm{d}\boldsymbol{x} \;=\; 0 \qquad \text{for all } \boldsymbol{\psi} \in H_0(\mathbf{curl}; \mathbb{R}^3)\,,$$

or

$$\int_{\mathbb{R}^3} \bigl(\varepsilon_r^{-1}\, \mathbf{curl}\, \boldsymbol{H}_q \cdot \mathbf{curl}\, \boldsymbol{\psi} - k^2 \boldsymbol{H}_q \cdot \boldsymbol{\psi}\bigr) \,\mathrm{d}\boldsymbol{x} \;=\; 0 \qquad \text{for all } \boldsymbol{\psi} \in H_0(\mathbf{curl}; \mathbb{R}^3)\,,$$

respectively. Standard regularity results (see, e.g., [50]) yield smoothness of $(\boldsymbol{E}_q, \boldsymbol{H}_q)$ and $(\boldsymbol{E}_q^s, \boldsymbol{H}_q^s)$ in $\mathbb{R}^3 \setminus \overline{B_R(0)}$, whenever $B_R(0)$ contains the scatterer D, and similarly the entire solution $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ is smooth throughout $\mathbb{R}^3$. In particular the Silver-Müller radiation condition (3.4) is well defined.

Suppose that the incident field $(\boldsymbol{E}^i, \boldsymbol{H}^i) \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3) \times H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ satisfies (3.1). Using either a volume integral equation approach (see [38, pp. 113–118]) or a variational formulation on $B_R(0)$ involving the exterior Calderon operator (see, e.g., [44, pp. 262–272]), Riesz–Fredholm theory can be applied to show existence of a solution to (3.2)–(3.4), provided uniqueness holds. Under our assumptions on the coefficients, uniqueness of solutions to (3.2)–(3.4) follows, e.g., from [4, Thm. 2.1].

Throughout this work we call a solution to Maxwell's equations on an unbounded domain that satisfies the Silver-Müller radiation condition a *radiating solution*. $\Diamond$

The scattered field $(\boldsymbol{E}_q^s, \boldsymbol{H}_q^s)$ has the asymptotic behavior

$$\boldsymbol{E}_q^s(\boldsymbol{x}) \;=\; \frac{e^{\mathrm{i}k|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}\bigl(\boldsymbol{E}_q^\infty(\widehat{\boldsymbol{x}}) + \mathcal{O}(|\boldsymbol{x}|^{-1})\bigr)\,, \quad \boldsymbol{H}_q^s(\boldsymbol{x}) \;=\; \frac{e^{\mathrm{i}k|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}\bigl(\boldsymbol{H}_q^\infty(\widehat{\boldsymbol{x}}) + \mathcal{O}(|\boldsymbol{x}|^{-1})\bigr) \tag{3.6}$$

as $|\boldsymbol{x}| \to \infty$, uniformly in $\widehat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|$ (see, e.g., [44, Cor. 9.5]). The *electric* and *magnetic far field patterns* $\boldsymbol{E}_q^\infty, \boldsymbol{H}_q^\infty \in L_t^2(S^2, \mathbb{C}^3)$ are given by

$$\boldsymbol{H}_q^\infty(\widehat{\boldsymbol{x}}) \;=\; \widehat{\boldsymbol{x}} \times \int_{\partial B_R(0)} \bigl(\mathrm{i}k(\boldsymbol{\nu} \times \boldsymbol{H}_q^s)(\boldsymbol{y}) + (\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}_q^s)(\boldsymbol{y}) \times \widehat{\boldsymbol{x}}\bigr) e^{-\mathrm{i}k\widehat{\boldsymbol{x}}\cdot\boldsymbol{y}} \,\mathrm{d}s(\boldsymbol{y})\,, \tag{3.7a}$$

$$\boldsymbol{E}_q^\infty(\widehat{\boldsymbol{x}}) \;=\; \widehat{\boldsymbol{x}} \times \int_{\partial B_R(0)} \bigl(\mathrm{i}k(\boldsymbol{\nu} \times \boldsymbol{E}_q^s)(\boldsymbol{y}) + (\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{E}_q^s)(\boldsymbol{y}) \times \widehat{\boldsymbol{x}}\bigr) e^{-\mathrm{i}k\widehat{\boldsymbol{x}}\cdot\boldsymbol{y}} \,\mathrm{d}s(\boldsymbol{y})\,, \tag{3.7b}$$

(see, e.g., [38, p. 121]). In particular, $\boldsymbol{H}_q^\infty(\widehat{\boldsymbol{x}}) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\widehat{\boldsymbol{x}} \times \boldsymbol{E}_q^\infty(\widehat{\boldsymbol{x}})$ for all $\widehat{\boldsymbol{x}} \in S^2$.

For the special case of a *plane wave incident field*

$$\boldsymbol{E}^i(\boldsymbol{x}; \boldsymbol{\theta}, \boldsymbol{p}) \;:=\; -\sqrt{\frac{\mu_0}{\varepsilon_0}}(\boldsymbol{\theta} \times \boldsymbol{p})\, e^{\mathrm{i}k\boldsymbol{\theta}\cdot\boldsymbol{x}}\,, \quad \boldsymbol{H}^i(\boldsymbol{x}; \boldsymbol{\theta}, \boldsymbol{p}) \;:=\; \boldsymbol{p}\, e^{\mathrm{i}k\boldsymbol{\theta}\cdot\boldsymbol{x}}\,, \qquad \boldsymbol{x} \in \mathbb{R}^3\,,$$

we explicitly indicate the dependence on the *direction of propagation* $\boldsymbol{\theta} \in S^2$ and on the *polarization* $\boldsymbol{p} \in \mathbb{C}^3$, which must satisfy $\boldsymbol{p} \cdot \boldsymbol{\theta} = 0$. Accordingly we write $(\boldsymbol{E}_q(\cdot; \boldsymbol{\theta}, \boldsymbol{p}), \boldsymbol{H}_q(\cdot; \boldsymbol{\theta}, \boldsymbol{p}))$, $(\boldsymbol{E}_q^s(\cdot; \boldsymbol{\theta}, \boldsymbol{p}), \boldsymbol{H}_q^s(\cdot; \boldsymbol{\theta}, \boldsymbol{p}))$, and $(\boldsymbol{E}_q^\infty(\cdot; \boldsymbol{\theta}, \boldsymbol{p}), \boldsymbol{H}_q^\infty(\cdot; \theta, \boldsymbol{p}))$ for the corresponding scattered field, the total field, and the far field pattern, respectively.

The *magnetic far field operator* is defined as

$$F_q : L_t^2(S^2, \mathbb{C}^3) \to L_t^2(S^2, \mathbb{C}^3)\,, \quad (F_q \boldsymbol{p})(\widehat{\boldsymbol{x}}) \;:=\; \int_{S^2} \boldsymbol{H}_q^\infty(\widehat{\boldsymbol{x}}; \boldsymbol{\theta}, \boldsymbol{p}(\boldsymbol{\theta})) \,\mathrm{d}s(\boldsymbol{\theta})\,, \tag{3.8}$$

and it is compact and normal (see, e.g., [38, Thm. 5.7]). Moreover, the *magnetic scattering operator*

$$\mathcal{S}_q : L^2_t(S^2,\mathbb{C}^3) \to L^2_t(S^2,\mathbb{C}^3)\,, \quad \mathcal{S}_q\boldsymbol{p} := \Big(I + \frac{\mathrm{i}k}{8\pi^2}F_q\Big)\boldsymbol{p}\,, \tag{3.9}$$

is unitary. Consequently the eigenvalues of F_q lie on the circle of radius $8\pi^2/k$ centered in $8\pi^2\,\mathrm{i}/k$ in the complex plane (cf., e.g., [38, Thm. 5.7]).

For any given $\boldsymbol{p} \in L^2_t(S^2,\mathbb{C}^3)$ the tangential vector field $F_q\boldsymbol{p} \in L^2_t(S^2,\mathbb{C}^3)$ is the far field pattern of the scattered magnetic field $\boldsymbol{H}^s_{\boldsymbol{p}}$ due to the incident field

$$\boldsymbol{E}^i_{\boldsymbol{p}}(\boldsymbol{x}) := -\sqrt{\frac{\mu_0}{\varepsilon_0}}\int_{S^2}\big(\boldsymbol{\theta}\times\boldsymbol{p}(\boldsymbol{\theta})\big)e^{\mathrm{i}k\boldsymbol{\theta}\cdot\boldsymbol{x}}\,\mathrm{d}s(\boldsymbol{\theta})\,,\ \boldsymbol{H}^i_{\boldsymbol{p}}(\boldsymbol{x}) := \int_{S^2}\boldsymbol{p}(\boldsymbol{\theta})e^{\mathrm{i}k\boldsymbol{\theta}\cdot\boldsymbol{x}}\,\mathrm{d}s(\boldsymbol{\theta})\,,\quad \boldsymbol{x}\in\mathbb{R}^3\,. \tag{3.10}$$

The latter is called a *Herglotz wave pair* with density $\boldsymbol{p}$. We write $(\boldsymbol{E}_{q,\boldsymbol{p}},\boldsymbol{H}_{q,\boldsymbol{p}})$ and $(\boldsymbol{E}^s_{q,\boldsymbol{p}},\boldsymbol{H}^s_{q,\boldsymbol{p}})$ for the corresponding total field and the scattered field, respectively. By linearity we have

$$\boldsymbol{E}_{q,\boldsymbol{p}}(\boldsymbol{x}) = \int_{S^2}\boldsymbol{E}_q(\boldsymbol{x};\boldsymbol{\theta},\boldsymbol{p}(\boldsymbol{\theta}))\,\mathrm{d}s(\boldsymbol{\theta})\,,\quad \boldsymbol{H}_{q,\boldsymbol{p}}(\boldsymbol{x}) = \int_{S^2}\boldsymbol{H}_q(\boldsymbol{x};\boldsymbol{\theta},\boldsymbol{p}(\boldsymbol{\theta}))\,\mathrm{d}s(\boldsymbol{\theta})\,,\quad \boldsymbol{x}\in\mathbb{R}^3\,. \tag{3.11}$$

4 A monotonicity relation for the magnetic far field operator

The following extension of the Loewner order will be used to describe relative orderings of compact self-adjoint operators. Given two compact self-adjoint linear operators $A, B : X \to X$ on a Hilbert space X, we say that

$$A \le_r B \qquad \text{for some } r\in\mathbb{N}\,,$$

if $B - A$ has at most r negative eigenvalues. Similarly, we write $A \le_{\text{fin}} B$ if $A \le_r B$ holds for some $r \in \mathbb{N}$, and the notations $A \ge_r B$ and $A \ge_{\text{fin}} B$ are defined accordingly.

The next lemma was shown in [31, Cor. 3.3].

Lemma 4.1. *Let $A, B : X \to X$ be two compact self-adjoint linear operators on a Hilbert space X with inner product $\langle\cdot,\cdot\rangle$, and let $r \in \mathbb{N}$. Then the following statements are equivalent:*

(a) $A \le_r B$

(b) There exists a finite-dimensional subspace $V \subseteq X$ with $\dim(V) \le r$ such that

$$\langle (B-A)v, v\rangle \ge 0 \qquad \textit{for all } v\in V^\perp\,.$$

Lemma 4.1 implies that $\le_{\text{fin}}$ and $\ge_{\text{fin}}$ are transitive relations (see [31, Lem. 3.4]). The theorem below gives a *monotonicity relation* for the magnetic far field operator in terms of this modified Loewner order. As usual the real part of a linear operator $A : X \to X$ on a Hilbert space X is the self-adjoint operator given by $\mathrm{Re}(A) := \frac{1}{2}(A + A^*)$.

Theorem 4.2. *Let $D_1, D_2 \subseteq \mathbb{R}^3$ be open and bounded of class C^0, and let $q_1 \in \mathcal{Y}_{D_1}$ and $q_2 \in \mathcal{Y}_{D_2}$. Then there exists a finite-dimensional subspace $V \subseteq L^2_t(S^2,\mathbb{C}^3)$ such that*

$$\mathrm{Re}\Big(\int_{S^2}\boldsymbol{p}\cdot\overline{\mathcal{S}^*_{q_1}(F_{q_2}-F_{q_1})\boldsymbol{p}}\,\mathrm{d}s\Big) \ge \int_{\mathbb{R}^3}(q_2-q_1)\,|\,\mathbf{curl}\,\boldsymbol{H}_{q_1,\boldsymbol{p}}|^2\,\mathrm{d}\boldsymbol{x} \qquad \textit{for all } \boldsymbol{p}\in V^\perp\,. \tag{4.1}$$

In particular

$$q_1 \le q_2 \qquad \textit{implies that} \qquad \mathrm{Re}(\mathcal{S}^*_{q_1}F_{q_1}) \le_{\text{fin}} \mathrm{Re}(\mathcal{S}^*_{q_1}F_{q_2})\,. \tag{4.2}$$

Remark 4.3. Recalling (3.9) and using that $\mathcal{S}_1$ and $\mathcal{S}_2$ are unitary operators, we find that

$$\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1}) = \frac{8\pi^2}{\mathrm{i}k}\mathcal{S}_{q_1}^*(\mathcal{S}_{q_2} - \mathcal{S}_{q_1}) = \Bigl(\frac{8\pi^2}{\mathrm{i}k}\mathcal{S}_{q_2}^*(\mathcal{S}_{q_2} - \mathcal{S}_{q_1})\Bigr)^* = \bigl(\mathcal{S}_{q_2}^*(F_{q_2} - F_{q_1})\bigr)^* .$$

Accordingly $\mathrm{Re}(\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})) = \mathrm{Re}(\mathcal{S}_{q_2}^*(F_{q_2} - F_{q_1}))$, and therefore the monotonicity relations (4.1)–(4.2) remain valid, if we replace $\mathcal{S}_{q_1}^*$ by $\mathcal{S}_{q_2}^*$ in these formulas. ♢

Applying Remark 4.3 we may interchange the roles of q_1 and q_2 in Theorem 4.2, except for $\mathcal{S}_{q_1}^*$, to obtain the following result.

Corollary 4.4. *Let $D_1, D_2 \subseteq \mathbb{R}^3$ be open and bounded of class C^0, and let $q_1 \in \mathcal{Y}_{D_1}$ and $q_2 \in \mathcal{Y}_{D_2}$. Then there exists a finite-dimensional subspace $V \subseteq L^2_t(S^2, \mathbb{C}^3)$ such that*

$$\mathrm{Re}\Bigl(\int_{S^2} \boldsymbol{p} \cdot \overline{\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})\boldsymbol{p}} \,\mathrm{d}s\Bigr) \le \int_{\mathbb{R}^3} (q_2 - q_1)\,|\,\mathbf{curl}\,\boldsymbol{H}_{q_2,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \boldsymbol{p} \in V^\perp . \tag{4.3}$$

Before we establish the proof of Theorem 4.2, we discuss three preparatory lemmas. In the first lemma we collect some integral identities for the magnetic field.

Lemma 4.5. *Let $D \subseteq B_R(0)$ be open and of class C^0, and let $q \in \mathcal{Y}_D$. Then,*

$$\int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s = \int_{B_R(0)} q \,\mathbf{curl}\,\boldsymbol{H}_{\boldsymbol{p}}^i \cdot \overline{\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}} \,\mathrm{d}\boldsymbol{y} \qquad \text{for all } \boldsymbol{p} \in L^2_t(S^2, \mathbb{C}^3) , \tag{4.4}$$

and, for any $\boldsymbol{\psi} \in H(\mathbf{curl}; B_R(0))$,

$$\int_{B_R(0)} \bigl(\varepsilon_r^{-1}\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}^s \cdot \mathbf{curl}\,\boldsymbol{\psi} - k^2 \boldsymbol{H}_{q,\boldsymbol{p}}^s \cdot \boldsymbol{\psi}\bigr) \,\mathrm{d}\boldsymbol{x} + \int_{\partial B_R(0)} \bigl(\boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}^s\bigr) \cdot \boldsymbol{\psi} \,\mathrm{d}s = \int_{B_R(0)} q \,\mathbf{curl}\,\boldsymbol{H}_{\boldsymbol{p}}^i \cdot \mathbf{curl}\,\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} . \tag{4.5}$$

Moreover, if $q_1 \in \mathcal{Y}_{D_1}$ and $q_2 \in \mathcal{Y}_{D_2}$ for some $D_1, D_2 \subseteq B_R(0)$ that are open and of class C^0, then

$$\int_{\partial B_R(0)} \Bigl(\boldsymbol{H}_{q_j,\boldsymbol{p}}^s \cdot \bigl(\boldsymbol{\nu} \times \overline{\mathbf{curl}\,\boldsymbol{H}_{q_l,\boldsymbol{p}}^s}\bigr) - \overline{\boldsymbol{H}_{q_l,\boldsymbol{p}}^s} \cdot \bigl(\boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{H}_{q_j,\boldsymbol{p}}^s\bigr)\Bigr) \,\mathrm{d}s = \frac{\mathrm{i}k}{8\pi^2} \int_{S^2} F_{q_j}\boldsymbol{p} \cdot \overline{F_{q_l}\boldsymbol{p}} \,\mathrm{d}s \tag{4.6}$$

for any $j, l \in \{1, 2\}$.

Proof. Let $\boldsymbol{p} \in L^2_t(S^2, \mathbb{C}^3)$. Then the scattered field $\boldsymbol{H}_{q,\boldsymbol{p}}^s \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ satisfies

$$\mathbf{curl}\bigl(\varepsilon_r^{-1}\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}^s\bigr) - k^2 \boldsymbol{H}_{q,\boldsymbol{p}}^s = -\,\mathbf{curl}\bigl(\varepsilon_r^{-1}\,\mathbf{curl}\,\boldsymbol{H}_{\boldsymbol{p}}^i\bigr) + k^2 \boldsymbol{H}_{\boldsymbol{p}}^i = \mathbf{curl}\bigl(q \,\mathbf{curl}\,\boldsymbol{H}_{\boldsymbol{p}}^i\bigr) \tag{4.7}$$

in $\mathbb{R}^3$. Multiplying (4.7) by $\boldsymbol{\psi} \in H(\mathbf{curl}; B_R(0))$ and integrating by parts using (2.2) yields

$$\int_{B_R(0)} \varepsilon_r^{-1}\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}^s \cdot \mathbf{curl}\,\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} = \int_{B_R(0)} q \,\mathbf{curl}\,\boldsymbol{H}_{\boldsymbol{p}}^i \cdot \mathbf{curl}\,\boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} + k^2 \int_{B_R(0)} \boldsymbol{H}_{q,\boldsymbol{p}}^s \cdot \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} - \int_{\partial B_R(0)} \bigl(\boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}^s\bigr) \cdot \boldsymbol{\psi} \,\mathrm{d}s . \tag{4.8}$$

This implies (4.5).

Likewise,

$$\int_{B_R(0)} \varepsilon_r^{-1}\, \mathbf{curl}\, \boldsymbol{H}_{\boldsymbol{p}}^i \cdot \mathbf{curl}\, \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} \;=\; -\int_{B_R(0)} q\, \mathbf{curl}\, \boldsymbol{H}_{\boldsymbol{p}}^i \cdot \mathbf{curl}\, \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} \\ + k^2 \int_{B_R(0)} \boldsymbol{H}_{\boldsymbol{p}}^i \cdot \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} - \int_{\partial B_R(0)} (\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}_{\boldsymbol{p}}^i) \cdot \boldsymbol{\psi} \,\mathrm{d}s \qquad (4.9)$$

for any $\boldsymbol{\psi} \in H(\mathbf{curl}; B_R(0))$. Subtracting (4.9) with $\boldsymbol{\psi} = \overline{\boldsymbol{H}_{q,\boldsymbol{p}}^s}$ from the complex conjugate of (4.8) with $\boldsymbol{\psi} = \overline{\boldsymbol{H}_{\boldsymbol{p}}^i}$ shows that

$$0 \;=\; \int_{B_R(0)} q\, \overline{\mathbf{curl}\, \boldsymbol{H}_{q,\boldsymbol{p}}} \cdot \mathbf{curl}\, \boldsymbol{H}_{\boldsymbol{p}}^i \,\mathrm{d}\boldsymbol{x} \\ - \int_{\partial B_R(0)} \Bigl((\boldsymbol{\nu} \times \overline{\mathbf{curl}\, \boldsymbol{H}_{q,\boldsymbol{p}}^s}) \cdot \boldsymbol{H}_{\boldsymbol{p}}^i - (\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}_{\boldsymbol{p}}^i) \cdot \overline{\boldsymbol{H}_{q,\boldsymbol{p}}^s} \Bigr) \,\mathrm{d}s\,.$$

On the other hand, we obtain using (3.10) and (3.7) that

$$\int_{\partial B_R(0)} \Bigl((\boldsymbol{\nu} \times \overline{\mathbf{curl}\, \boldsymbol{H}_{q,\boldsymbol{p}}^s}) \cdot \boldsymbol{H}_{\boldsymbol{p}}^i - (\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}_{\boldsymbol{p}}^i) \cdot \overline{\boldsymbol{H}_{q,\boldsymbol{p}}^s} \Bigr) \,\mathrm{d}s \\ = \int_{S^2} \boldsymbol{p}(\boldsymbol{\theta}) \cdot \int_{\partial B_R(0)} \Bigl((\boldsymbol{\nu} \times \overline{\mathbf{curl}\, \boldsymbol{H}_{q,\boldsymbol{p}}^s})(\boldsymbol{x}) + \mathrm{i}k \Bigl((\boldsymbol{\nu} \times \overline{\boldsymbol{H}_{q,\boldsymbol{p}}^s})(\boldsymbol{x}) \times \boldsymbol{\theta} \Bigr) \Bigr) e^{\mathrm{i}k\boldsymbol{\theta}\cdot\boldsymbol{x}} \,\mathrm{d}s(\boldsymbol{x}) \,\mathrm{d}s(\boldsymbol{\theta}) \\ = \int_{S^2} \boldsymbol{p}(\boldsymbol{\theta}) \cdot \overline{\boldsymbol{H}_{q,\boldsymbol{p}}^\infty(\boldsymbol{\theta})} \,\mathrm{d}s(\boldsymbol{\theta}) \;=\; \int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s\,.$$

This shows (4.4).

Now let $q_1 \in \mathcal{Y}_{D_1}$ and $q_2 \in \mathcal{Y}_{D_2}$ for some $D_1, D_2 \subseteq B_R(0)$ that are open and of class C^0, and let $r > R$. Then, $\boldsymbol{H}_{q_j,\boldsymbol{p}}^s, \boldsymbol{H}_{q_l,\boldsymbol{p}}^s \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ fulfill

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{H}_{q,\boldsymbol{p}}^s - k^2 \boldsymbol{H}_{q,\boldsymbol{p}}^s \;=\; 0 \qquad \text{in } B_r(0) \setminus \overline{B_R(0)}$$

for $q \in \{q_j, q_l\}$. Thus, Green's formula gives

$$\int_{\partial B_R(0)} \Bigl((\boldsymbol{\nu} \times \overline{\mathbf{curl}\, \boldsymbol{H}_{q_l,\boldsymbol{p}}^s}) \cdot \boldsymbol{H}_{q_j,\boldsymbol{p}}^s - (\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}_{q_j,\boldsymbol{p}}^s) \cdot \overline{\boldsymbol{H}_{q_l,\boldsymbol{p}}^s} \Bigr) \,\mathrm{d}s \\ = \int_{\partial B_r(0)} \Bigl((\boldsymbol{\nu} \times \overline{\mathbf{curl}\, \boldsymbol{H}_{q_l,\boldsymbol{p}}^s}) \cdot \boldsymbol{H}_{q_j,\boldsymbol{p}}^s - (\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}_{q_j,\boldsymbol{p}}^s) \cdot \overline{\boldsymbol{H}_{q_l,\boldsymbol{p}}^s} \Bigr) \,\mathrm{d}s\,. \qquad (4.10)$$

Applying the Silver-Müller radiation condition (3.5d) and inserting the far field expansion (3.6) we obtain that

$$\int_{\partial B_r(0)} \Bigl((\boldsymbol{\nu} \times \overline{\mathbf{curl}\, \boldsymbol{H}_{q_l,\boldsymbol{p}}^s}) \cdot \boldsymbol{H}_{q_j,\boldsymbol{p}}^s - (\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}_{q_j,\boldsymbol{p}}^s) \cdot \overline{\boldsymbol{H}_{q_l,\boldsymbol{p}}^s} \Bigr) \,\mathrm{d}s \\ = 2\mathrm{i}k \int_{\partial B_r(0)} \boldsymbol{H}_{q_j,\boldsymbol{p}}^s \cdot \overline{\boldsymbol{H}_{q_l,\boldsymbol{p}}^s} \,\mathrm{d}s + o(1) \;=\; \frac{\mathrm{i}k}{8\pi^2} \int_{S^2} F_{q_j}\boldsymbol{p} \cdot \overline{F_{q_l}\boldsymbol{p}} \,\mathrm{d}s + o(1)$$

as $r \to \infty$. Together with (4.10) this shows (4.6). □

In the next lemma we establish an integral identity for the left hand side of (4.1) (see also Remark 4.7 below).

Lemma 4.6. *Let $D_1, D_2 \subseteq B_R(0)$ be open and of class C^0, and let $q_1 \in \mathcal{Y}_{D_1}$ and $q_2 \in \mathcal{Y}_{D_2}$. Then,*

$$\begin{aligned}
&\int_{S^2} \bigl(\boldsymbol{p} \cdot \overline{F_{q_2}\boldsymbol{p}} - \overline{\boldsymbol{p}} \cdot F_{q_1}\boldsymbol{p}\bigr) \,\mathrm{d}s + \frac{\mathrm{i}k}{8\pi^2} \int_{S^2} F_{q_1}\boldsymbol{p} \cdot \overline{F_{q_2}\boldsymbol{p}} \,\mathrm{d}s + \int_{B_R(0)} (q_1 - q_2) |\,\mathbf{curl}\, \boldsymbol{H}_{q_1,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \\
&= \int_{B_R(0)} \bigl(\varepsilon_{r,2}^{-1} |\,\mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})|^2 - k^2 |\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}}|^2\bigr) \,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\partial B_R(0)} \overline{(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})} \cdot \bigl(\boldsymbol{\nu} \times \mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})\bigr) \,\mathrm{d}s
\end{aligned} \tag{4.11}$$

for any $\boldsymbol{p} \in L^2_t(S^2, \mathbb{C}^3)$.

Proof. Let $\boldsymbol{p} \in L^2_t(S^2, \mathbb{C}^3)$. Using (4.6) with $j = 1$ and $l = 2$ we find that

$$\begin{aligned}
&2\,\mathrm{Re}\Bigl(\int_{\partial B_R(0)} \overline{\boldsymbol{H}^s_{q_1,\boldsymbol{p}}} \cdot \bigl(\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}^s_{q_2,\boldsymbol{p}}\bigr) \,\mathrm{d}s\Bigr) \\
&= \int_{\partial B_R(0)} \Bigl(\overline{\boldsymbol{H}^s_{q_1,\boldsymbol{p}}} \cdot \bigl(\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}^s_{q_2,\boldsymbol{p}}\bigr) + \overline{\boldsymbol{H}^s_{q_2,\boldsymbol{p}}} \cdot \bigl(\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}^s_{q_1,\boldsymbol{p}}\bigr)\Bigr) \,\mathrm{d}s + \frac{\mathrm{i}k}{8\pi^2} \int_{S^2} F_{q_1}\boldsymbol{p} \cdot \overline{F_{q_2}\boldsymbol{p}} \,\mathrm{d}s\,.
\end{aligned}$$

Therewith, we deduce that

$$\begin{aligned}
&\int_{B_R(0)} \bigl(\varepsilon_{r,2}^{-1} |\,\mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})|^2 - k^2 |\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}}|^2\bigr) \,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\partial B_R(0)} \overline{(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})} \cdot \bigl(\boldsymbol{\nu} \times \mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})\bigr) \,\mathrm{d}s \\
&= \int_{B_R(0)} \bigl(\varepsilon_{r,2}^{-1} |\,\mathbf{curl}\, \boldsymbol{H}^s_{q_2,\boldsymbol{p}}|^2 - k^2 |\boldsymbol{H}^s_{q_2,\boldsymbol{p}}|^2\bigr) \,\mathrm{d}\boldsymbol{x} + \int_{B_R(0)} \bigl(\varepsilon_{r,2}^{-1} |\,\mathbf{curl}\, \boldsymbol{H}^s_{q_1,\boldsymbol{p}}|^2 - k^2 |\boldsymbol{H}^s_{q_1,\boldsymbol{p}}|^2\bigr) \,\mathrm{d}\boldsymbol{x} \\
&\quad - 2\,\mathrm{Re}\Bigl(\int_{B_R(0)} \Bigl(\varepsilon_{r,2}^{-1}\, \mathbf{curl}\, \boldsymbol{H}^s_{q_2,\boldsymbol{p}} \cdot \overline{\mathbf{curl}\, \boldsymbol{H}^s_{q_1,\boldsymbol{p}}} - k^2 \boldsymbol{H}^s_{q_2,\boldsymbol{p}} \cdot \overline{\boldsymbol{H}^s_{q_1,\boldsymbol{p}}}\Bigr) \,\mathrm{d}\boldsymbol{x} \\
&\qquad\qquad + \int_{\partial B_R(0)} \overline{\boldsymbol{H}^s_{q_1,\boldsymbol{p}}} \cdot \bigl(\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}^s_{q_2,\boldsymbol{p}}\bigr) \,\mathrm{d}s\Bigr) \\
&\quad + \int_{\partial B_R(0)} \Bigl(\overline{\boldsymbol{H}^s_{q_2,\boldsymbol{p}}} \cdot \bigl(\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}^s_{q_2,\boldsymbol{p}}\bigr) + \overline{\boldsymbol{H}^s_{q_1,\boldsymbol{p}}} \cdot \bigl(\boldsymbol{\nu} \times \mathbf{curl}\, \boldsymbol{H}^s_{q_1,\boldsymbol{p}}\bigr)\Bigr) \,\mathrm{d}s \\
&\quad + \frac{\mathrm{i}k}{8\pi^2} \int_{S^2} F_{q_1}\boldsymbol{p} \cdot \overline{F_{q_2}\boldsymbol{p}} \,\mathrm{d}s\,.
\end{aligned}$$

Applying (4.5) gives

$$\begin{aligned}
&\int_{B_R(0)} \bigl(\varepsilon_{r,2}^{-1} |\,\mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})|^2 - k^2 |\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}}|^2\bigr) \,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\partial B_R(0)} \overline{(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})} \cdot \bigl(\boldsymbol{\nu} \times \mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})\bigr) \,\mathrm{d}s \\
&= \int_{B_R(0)} q_2\, \mathbf{curl}\, \boldsymbol{H}^i_{\boldsymbol{p}} \cdot \overline{\mathbf{curl}\, \boldsymbol{H}^s_{q_2,\boldsymbol{p}}} \,\mathrm{d}\boldsymbol{x} + \int_{B_R(0)} q_1\, \mathbf{curl}\, \boldsymbol{H}^i_{\boldsymbol{p}} \cdot \overline{\mathbf{curl}\, \boldsymbol{H}^s_{q_1,\boldsymbol{p}}} \,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{B_R(0)} (q_1 - q_2) |\,\mathbf{curl}\, \boldsymbol{H}^s_{q_1,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} - 2\,\mathrm{Re}\Bigl(\int_{B_R(0)} q_2\, \mathbf{curl}\, \boldsymbol{H}^i_{\boldsymbol{p}} \cdot \overline{\mathbf{curl}\, \boldsymbol{H}^s_{q_1,\boldsymbol{p}}} \,\mathrm{d}\boldsymbol{x}\Bigr) \\
&\quad + \frac{\mathrm{i}k}{8\pi^2} \int_{S^2} F_{q_1}\boldsymbol{p} \cdot \overline{F_{q_2}\boldsymbol{p}} \,\mathrm{d}s\,.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_{B_R(0)} \big(\varepsilon_{r,2}^{-1}|\,\mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})|^2 - k^2|\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}}|^2\big) \,\mathrm{d}\boldsymbol{x}\\
&\quad + \int_{\partial B_R(0)} \overline{(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})} \cdot \big(\boldsymbol{\nu} \times \mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})\big) \,\mathrm{d}s\\
&= \int_{B_R(0)} q_2\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}} \cdot \overline{\mathbf{curl}\,\boldsymbol{H}^s_{q_2,\boldsymbol{p}}} \,\mathrm{d}\boldsymbol{x} + 2\,\mathrm{Re}\Big(\int_{B_R(0)} (q_1 - q_2)\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}} \cdot \overline{\mathbf{curl}\,\boldsymbol{H}^s_{q_1,\boldsymbol{p}}} \,\mathrm{d}\boldsymbol{x}\Big)\\
&\quad - \int_{B_R(0)} q_1\overline{\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}} \cdot \mathbf{curl}\,\boldsymbol{H}^s_{q_1,\boldsymbol{p}} \,\mathrm{d}\boldsymbol{x} + \int_{B_R(0)} (q_1 - q_2)|\,\mathbf{curl}\,\boldsymbol{H}^s_{q_1,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x}\\
&\quad + \frac{\mathrm{i}k}{8\pi^2} \int_{S^2} F_{q_1}\boldsymbol{p} \cdot \overline{F_{q_2}\boldsymbol{p}} \,\mathrm{d}s\\
&= \int_{B_R(0)} q_2\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}} \cdot \overline{\mathbf{curl}\,\boldsymbol{H}_{q_2,\boldsymbol{p}}} \,\mathrm{d}\boldsymbol{x} - \int_{B_R(0)} q_1\overline{\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}} \cdot \mathbf{curl}\,\boldsymbol{H}_{q_1,\boldsymbol{p}} \,\mathrm{d}\boldsymbol{x}\\
&\quad + \int_{B_R(0)} (q_1 - q_2)|\,\mathbf{curl}\,\boldsymbol{H}_{q_1,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} + \frac{\mathrm{i}k}{8\pi^2} \int_{S^2} F_{q_1}\boldsymbol{p} \cdot \overline{F_{q_2}\boldsymbol{p}} \,\mathrm{d}s\,.
\end{aligned}$$

Finally, applying (4.4) gives

$$\begin{aligned}
&\int_{B_R(0)} \big(\varepsilon_{r,2}^{-1}|\,\mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})|^2 - k^2|\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}}|^2\big) \,\mathrm{d}\boldsymbol{x}\\
&\quad + \int_{\partial B_R(0)} \overline{(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})} \cdot \big(\boldsymbol{\nu} \times \mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})\big) \,\mathrm{d}s\\
&= \int_{S^2} \big(\boldsymbol{p} \cdot \overline{F_{q_2}\boldsymbol{p}} - \overline{\boldsymbol{p}} \cdot F_{q_1}\boldsymbol{p}\big) \,\mathrm{d}s + \int_{B_R(0)} (q_1 - q_2)|\,\mathbf{curl}\,\boldsymbol{H}_{q_1,\boldsymbol{p}}|^2 dx\\
&\quad + \frac{\mathrm{i}k}{8\pi^2} \int_{S^2} F_{q_1}\boldsymbol{p} \cdot \overline{F_{q_2}\boldsymbol{p}} \,\mathrm{d}s\,.
\end{aligned}$$

□

Remark 4.7. Using (3.9) we find that

$$\mathcal{S}^*_{q_1}(F_{q_2} - F_{q_1}) \;=\; F_{q_2} - F_{q_1} - \frac{\mathrm{i}k}{8\pi^2}(F^*_{q_1}F_{q_2} - F^*_{q_1}F_{q_1})\,,$$

and thus,

$$\mathrm{Re}\big(\mathcal{S}^*_{q_1}(F_{q_2} - F_{q_1})\big) \;=\; \mathrm{Re}\Big(F_{q_2} - F_{q_1} - \frac{\mathrm{i}k}{8\pi^2}F^*_{q_1}F_{q_2}\Big)\,.$$

Accordingly, the real part of the first two integrals on the left hand side of (4.11) satisfies

$$\begin{aligned}
&\mathrm{Re}\Big(\int_{S^2} \big(\boldsymbol{p} \cdot \overline{F_{q_2}\boldsymbol{p}} - \overline{\boldsymbol{p}} \cdot F_{q_1}\boldsymbol{p}\big) \,\mathrm{d}s + \frac{\mathrm{i}k}{8\pi^2} \int_{S^2} F_{q_1}\boldsymbol{p} \cdot \overline{F_{q_2}\boldsymbol{p}} \,\mathrm{d}s\Big)\\
&\quad = \mathrm{Re}\Big(\int_{S^2} \boldsymbol{p} \cdot \overline{\Big(F_{q_2} - F_{q_1} - \frac{\mathrm{i}k}{8\pi^2}F^*_{q_1}F_{q_2}\Big)\boldsymbol{p}} \,\mathrm{d}s\Big) \;=\; \mathrm{Re}\Big(\int_{S^2} \boldsymbol{p} \cdot \overline{\mathcal{S}^*_{q_1}(F_{q_2} - F_{q_1})\boldsymbol{p}} \,\mathrm{d}s\Big)\,. \qquad (4.12)
\end{aligned}$$

Since F_{q_1} and F_{q_2} are compact, the operator $\mathcal{S}^*_{q_1}(F_{q_2} - F_{q_1})$ is compact as well, and using (3.9) once more it is immediately seen that $\mathcal{S}^*_{q_1}(F_{q_2} - F_{q_1})$ is normal. ♢

Next we show that the right hand side of (4.11) is nonnegative if the density $\boldsymbol{p} \in L^2_t(S^2, \mathbb{C}^3)$ belongs to the complement of a certain finite dimensional subspace $V \subseteq L^2_t(S^2, \mathbb{C}^3)$. We consider the exterior Calderon operator $\Lambda : H^{-1/2}(\mathrm{Div};\partial B_R(0)) \to H^{-1/2}(\mathrm{Curl};\partial B_R(0))$, which maps boundary data $\boldsymbol{\psi} \in H^{-1/2}(\mathrm{Div};\partial B_R(0))$ to the tangential trace $(\boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{w}|_{\partial B_R(0)}) \times \boldsymbol{\nu}$ of the radiating solution $\boldsymbol{w} \in H(\mathbf{curl}; B_R(0))$ to the exterior boundary value problem

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{w} - k^2\boldsymbol{w} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B_R(0)}\,, \qquad \boldsymbol{\nu} \times \boldsymbol{w} = \boldsymbol{\psi} \quad \text{on } \partial B_R(0)\,,$$

(see, e.g., [44, pp. 248–250]). We note that this operator is invertible (see, e.g., [44, Lem. 9.20], and we define the space

$$\mathcal{X} := \Big\{ \boldsymbol{u} \in H(\mathbf{curl}; B_R(0)) \;\Big|\; \mathrm{div}\,\boldsymbol{u} = 0 \text{ in } B_R(0) \text{ and } \boldsymbol{\nu}\cdot\boldsymbol{u}|_{\partial B_R(0)} = k^{-2}\,\mathrm{Curl}\big(\Lambda(\boldsymbol{\nu} \times \boldsymbol{u}|_{\partial B_R(0)})\big) \Big\}$$

equipped with the norm $\|\cdot\|_{\mathcal{X}} := \|\cdot\|_{H(\mathbf{curl};B_R(0))}$. Then $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Hilbert space (see, e.g., [44, Lem. 10.3]) and the embedding operator $J : \mathcal{X} \to L^2(B_R(0), \mathbb{C}^3)$ is compact (see, e.g., [44, Lem. 10.4]).

From (3.5b) we see that $\mathrm{div}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}}) = 0$ in $\mathbb{R}^3$ and

$$\begin{aligned}
\boldsymbol{\nu}\cdot(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})\big|_{\partial B_R(0)} &= k^{-2}\boldsymbol{\nu}\cdot\mathbf{curl}\,\mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})\big|_{\partial B_R(0)}\\
&= k^{-2}\,\mathrm{Curl}\big(\big(\boldsymbol{\nu}\times\mathbf{curl}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})\big|_{\partial B_R(0)}\big)\times\boldsymbol{\nu}\big)\\
&= k^{-2}\,\mathrm{Curl}\big(\Lambda\big(\boldsymbol{\nu}\times(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})\big|_{\partial B_R(0)}\big)\big)\,.
\end{aligned}$$

This shows that $\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}} \in \mathcal{X}$.

Using the Lax-Milgram lemma, we define for any $q = 1 - \varepsilon_r^{-1} \in \mathcal{Y}_D$ with $D \subset\subset B_R(0)$ open and bounded of class C^0 a bounded linear self-adjoint operator $I_q : \mathcal{X} \to \mathcal{X}$ with bounded inverse by

$$\langle I_q\boldsymbol{u}, \boldsymbol{v}\rangle_{H(\mathbf{curl};B_R(0))} = \int_{B_R(0)} \varepsilon_r^{-1}\big(\overline{\mathbf{curl}\,\boldsymbol{u}}\cdot\mathbf{curl}\,\boldsymbol{v} + \overline{\boldsymbol{u}}\cdot\boldsymbol{v}\big)\,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \boldsymbol{u},\boldsymbol{v}\in\mathcal{X}\,.$$

Furthermore, let $K : \mathcal{X} \to \mathcal{X}$ and $K_q : \mathcal{X} \to \mathcal{X}$ be given by

$$K\boldsymbol{u} := J^*J\boldsymbol{u} \qquad \text{and} \qquad K_q\boldsymbol{v} := J^*\big(\varepsilon_r^{-1}J\boldsymbol{v}\big)\,,$$

respectively. Then K and K_q are compact self-adjoint linear operators, and for any $\boldsymbol{v} \in \mathcal{X}$,

$$\big\langle (I_q - K_q - k^2K)\boldsymbol{v}, \boldsymbol{v}\big\rangle_{\mathcal{X}} = \int_{B_R(0)} \big(\varepsilon_r^{-1}|\,\mathbf{curl}\,\boldsymbol{v}|^2 - k^2|\boldsymbol{v}|^2\big)\,\mathrm{d}\boldsymbol{x}\,.$$

For $0 < \varepsilon < R$ we denote by $N_\varepsilon : \mathcal{X} \to H^{-1/2}(\mathrm{Curl};\partial B_R(0))$ the compact linear operator that maps $\boldsymbol{v} \in \mathcal{X}$ to the tangential trace $(\boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{v}_\varepsilon|_{\partial B_R(0)})\times\boldsymbol{\nu}$ of the radiating solution to the exterior boundary value problem

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v}_\varepsilon - k^2\boldsymbol{v}_\varepsilon = 0 \quad \text{in } \mathbb{R}^3\setminus\overline{B_{R-\varepsilon}(0)}\,, \qquad \boldsymbol{\nu}\times\boldsymbol{v}_\varepsilon = \boldsymbol{\nu}\times\boldsymbol{v} \quad \text{on } \partial B_{R-\varepsilon}(0)\,.$$

Given any $\boldsymbol{v} \in \mathcal{X}$ that can be extended to a radiating solution of Maxwell's equations

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} - k^2\boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^3\setminus\overline{B_{R-\varepsilon}(0)}\,,$$

we find that

$$N_\varepsilon\boldsymbol{v} = \big(\boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{v}|_{\partial B_R(0)}\big)\times\boldsymbol{\nu} \qquad \text{and} \qquad \Lambda^{-1}N_\varepsilon\boldsymbol{v} = \boldsymbol{\nu}\times\boldsymbol{v}|_{\partial B_R(0)}\,.$$

Accordingly,

$$\big\langle N_\varepsilon^*\Lambda^{-1}N_\varepsilon\boldsymbol{v}, \boldsymbol{v}\big\rangle_{\mathcal{X}} = \big\langle \Lambda^{-1}N_\varepsilon\boldsymbol{v}, N_\varepsilon\boldsymbol{v}\big\rangle_{L^2(\partial B_R(0),\mathbb{C}^3)} = -\int_{\partial B_R(0)} (\boldsymbol{\nu}\times\mathbf{curl}\,\boldsymbol{v})\cdot\overline{\boldsymbol{v}}\,\mathrm{d}s\,,$$

and in particular this holds for $\boldsymbol{v} = \boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}}$ if the ball $B_{R-\varepsilon}(0)$ contains $\overline{D_1}\cup\overline{D_2}$.

Lemma 4.8. *Let $D_1, D_2 \subset\subset B_R(0)$ be open and of class C^0, and let $q_1 \in \mathcal{Y}_{D_1}$ and $q_2 \in \mathcal{Y}_{D_2}$. Then there exists a finite dimensional subspace $V \subset L^2_t(S^2, \mathbb{C}^3)$ such that*

$$\int_{B_R(0)} \Big(\varepsilon_{r,2}^{-1} |\operatorname{\mathbf{curl}}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})|^2 - k^2 |\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}}|^2 \Big) \,\mathrm{d}\boldsymbol{x}$$
$$+ \operatorname{Re}\Big(\int_{\partial B_R(0)} \overline{(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})} \cdot \big(\boldsymbol{\nu} \times \operatorname{\mathbf{curl}}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})\big) \times \boldsymbol{\nu} \,\mathrm{d}s \Big) \geq 0 \qquad \textit{for all } \boldsymbol{p} \in V^\perp .$$

Proof. Let $q_1 \in \mathcal{Y}_{D_1}$ and $q_2 \in \mathcal{Y}_{D_2}$ for some $D_1, D_2 \subset\subset B_R(0)$ that are open and of class C^0, and let $\varepsilon > 0$ be sufficiently small, so that $\overline{D_1} \cup \overline{D_2} \subset B_{R-\varepsilon}(0)$. Then

$$\int_{B_R(0)} \Big(\varepsilon_{r,2}^{-1} |\operatorname{\mathbf{curl}}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})|^2 - k^2 |\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}}|^2 \Big) \,\mathrm{d}\boldsymbol{x}$$
$$+ \operatorname{Re}\Big(\int_{\partial B_R(0)} \overline{(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}})} \cdot (\boldsymbol{\nu} \times \operatorname{\mathbf{curl}}(\boldsymbol{H}^s_{q_2,\boldsymbol{p}} - \boldsymbol{H}^s_{q_1,\boldsymbol{p}}) \,\mathrm{d}s \Big)$$
$$= \big\langle \big(I_{q_2} - K_{q_2} - k^2 K - \operatorname{Re}(N_\varepsilon^* \Lambda^{-1} N_\varepsilon)\big)(A_2 - A_1)\boldsymbol{p}, (A_2 - A_1)\boldsymbol{p} \big\rangle_{\mathcal{X}} ,$$

where, for $j = 1, 2$ we denote by $A_j : L^2_t(S^2, \mathbb{C}^3) \to \mathcal{X}$ the bounded linear operator that maps densities $\boldsymbol{p} \in L^2_t(S^2, \mathbb{C}^3)$ to the restriction of the corresponding scattered magnetic field $\boldsymbol{H}^s_{q_j,\boldsymbol{p}}$ to $B_R(0)$.

We denote by W the sum of eigenspaces of the compact self-adjoint operator $K_{q_2} + k^2 K + \operatorname{Re}(N_\varepsilon^* \Lambda^{-1} N_\varepsilon)$ associated to eigenvalues larger than

$$c_{\min} := \operatorname*{ess\,inf}_{x \in B_R(0)} \varepsilon_{r,2}^{-1}(x) > 0 .$$

The subspace W is finite dimensional and

$$\big\langle \big(I_{q_2} - K_{q_2} - k^2 K - \operatorname{Re}(N_\varepsilon^* \Lambda^{-1} N_\varepsilon)\big)\boldsymbol{w}, \boldsymbol{w} \big\rangle_{\mathcal{X}} \geq 0 \qquad \text{for all } \boldsymbol{w} \in W^\perp.$$

We observe that, for any $\boldsymbol{p} \in L^2_t(S^2, \mathbb{C}^3)$,

$$(A_2 - A_1)\boldsymbol{p} \in W^\perp \qquad \text{if and only if} \qquad \boldsymbol{p} \in \big((A_2 - A_1)^* W\big)^\perp .$$

Since $\dim\big((A_2 - A_1)^* W\big) \leq \dim(W) < \infty$, choosing $V := (A_2 - A_1)^* W$ ends the proof. □

Proof of Theorem 4.2. We take the real part of (4.11) and use (4.12). Then Lemma 4.8 yields the result. □

5 Localized vector wave functions

We establish the existence of *localized vector wave functions*, which are solutions to (3.5) that have arbitrarily large energy in some prescribed region and arbitrarily small energy in another prescribed region. This extends related results for solutions to Maxwell's equations on bounded domains from [27]. The localized vector wave functions will be used to justify the shape characterizations for sign definite scattering objects in Section 6 below.

Theorem 5.1. *Suppose that $D \subseteq \mathbb{R}^3$ is open and bounded of class C^0, let $q \in \mathcal{Y}_D$, and let $B, \Omega \subseteq \mathbb{R}^3$ be open and bounded such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected.*

If $B \not\subseteq \Omega$, then for any finite dimensional subspace $V \subseteq L^2_t(S^2, \mathbb{C}^3)$ there exists a sequence $(\boldsymbol{p}_m)_{m \in \mathbb{N}} \subseteq V^\perp$ such that

$$\int_B |\operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to \infty \quad \textit{and} \quad \int_\Omega |\operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to 0 \quad \textit{as } m \to \infty , \tag{5.1}$$

where $\boldsymbol{H}_{q,\boldsymbol{p}_m} \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ is given by (3.11) with $\boldsymbol{p} = \boldsymbol{p}_m$.

The proof of Theorem 5.1 combines the following three lemmas.

Lemma 5.2. *Suppose that $D \subseteq \mathbb{R}^3$ is open and of class C^0, let $q \in \mathcal{Y}_D$, and assume that $\Omega \subseteq \mathbb{R}^3$ is open and bounded. We define*

$$L_{q,\Omega} : L^2_t(S^2, \mathbb{C}^3) \to L^2(\Omega, \mathbb{C}^3)\,, \quad L_{q,\Omega}\, \boldsymbol{p} \,:=\, \mathbf{curl}\, \boldsymbol{H}_{q,\boldsymbol{p}}|_\Omega \,=\, -\mathrm{i}\omega\varepsilon \boldsymbol{E}_{q,\boldsymbol{p}}|_\Omega\,. \tag{5.2}$$

Then, $L_{q,\Omega}$ is a compact linear operator and its adjoint is given by

$$L^*_{q,\Omega} : L^2(\Omega, \mathbb{C}^3) \to L^2_t(S^2, \mathbb{C}^3)\,, \quad L^*_{q,\Omega} \boldsymbol{f} \,:=\, \sqrt{\frac{\mu_0}{\varepsilon_0}} \mathcal{S}^*_q(\boldsymbol{\nu} \times \boldsymbol{e}^\infty)\,,$$

where $\boldsymbol{e}^\infty \in L^2_t(S^2, \mathbb{C}^3)$ is the far field pattern of the radiating solution $\boldsymbol{e} \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ to

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e} - k^2 \varepsilon_r \boldsymbol{e} \,=\, \mathrm{i}\omega\varepsilon \boldsymbol{f} \qquad \text{in } \mathbb{R}^3\,. \tag{5.3}$$

Proof. The integral representation (3.11) shows that $L_{q,\Omega}$ is a Fredholm integral operator with square integrable kernel, which implies the compactness (see, e.g., [10, p. 354]).

The existence of a unique radiating solution $\boldsymbol{e} \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ to (5.3) follows again by combining the uniqueness result from [4] with Riesz–Fredholm theory (see, e.g., [38, pp. 113–118] or [44, pp. 262–272]). Let $R > 0$ sufficiently large such that $\overline{D} \cup \overline{\Omega} \subseteq B_R(0)$. Multiplying (5.3) by $\boldsymbol{\psi} \in H(\mathbf{curl}; B_R(0))$ and integrating by parts shows that

$$\begin{aligned}\int_{B_R(0)} \big(\mathbf{curl}\,\boldsymbol{e} \cdot \mathbf{curl}\,\boldsymbol{\psi} - k^2 \varepsilon_r \boldsymbol{e} \cdot \boldsymbol{\psi}\big) \,\mathrm{d}\boldsymbol{x}& \\ = \int_{B_R(0)} \mathrm{i}\omega\varepsilon \boldsymbol{f} \cdot \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x} &+ \int_{\partial B_R(0)} (\boldsymbol{\nu} \times \boldsymbol{\psi}) \cdot \mathbf{curl}\,\boldsymbol{e} \,\mathrm{d}s\,.\end{aligned} \tag{5.4}$$

Combining (5.2), the complex conjugate of (5.4), and integrating by parts we obtain from (3.5) that, for any $\boldsymbol{f} \in L^2(\Omega, \mathbb{C}^3)$ and $\boldsymbol{p} \in L^2_t(S^2, \mathbb{C}^3)$,

$$\begin{aligned}\int_\Omega (L_{q,\Omega}\boldsymbol{p}) \cdot \overline{\boldsymbol{f}} \,\mathrm{d}\boldsymbol{x} \,&=\, -\int_{B_R(0)} \mathrm{i}\omega\varepsilon \boldsymbol{E}_{q,\boldsymbol{p}} \cdot \overline{\boldsymbol{f}} \,\mathrm{d}\boldsymbol{x} \\ &= \int_{B_R(0)} \big(\mathbf{curl}\,\boldsymbol{E}_{q,\boldsymbol{p}} \cdot \overline{\mathbf{curl}\,\boldsymbol{e}} - k^2 \varepsilon_r \boldsymbol{E}_{q,\boldsymbol{p}} \cdot \overline{\boldsymbol{e}}\big) \,\mathrm{d}\boldsymbol{x} - \int_{\partial B_R(0)} (\boldsymbol{\nu} \times \boldsymbol{E}_{q,\boldsymbol{p}}) \cdot \overline{\mathbf{curl}\,\boldsymbol{e}} \,\mathrm{d}s \\ &= \int_{\partial B_R(0)} \big((\boldsymbol{\nu} \times \overline{\boldsymbol{e}}) \cdot \mathbf{curl}\,\boldsymbol{E}^i_{\boldsymbol{p}} - (\boldsymbol{\nu} \times \boldsymbol{E}^i_{\boldsymbol{p}}) \cdot \overline{\mathbf{curl}\,\boldsymbol{e}}\big) \,\mathrm{d}s \\ &\quad + \int_{\partial B_R(0)} \big((\boldsymbol{\nu} \times \overline{\boldsymbol{e}}) \cdot \mathbf{curl}\,\boldsymbol{E}^s_{q,\boldsymbol{p}} - (\boldsymbol{\nu} \times \boldsymbol{E}^s_{q,\boldsymbol{p}}) \cdot \overline{\mathbf{curl}\,\boldsymbol{e}}\big) \,\mathrm{d}s\,.\end{aligned} \tag{5.5}$$

We discuss the two integrals on the right hand side of (5.5) separately. Using (3.10) we find for the first integral that

$$\begin{aligned}&\int_{\partial B_R(0)} \big((\boldsymbol{\nu} \times \overline{\boldsymbol{e}}) \cdot \mathbf{curl}\,\boldsymbol{E}^i_{\boldsymbol{p}} - (\boldsymbol{\nu} \times \boldsymbol{E}^i_{\boldsymbol{p}}) \cdot \overline{\mathbf{curl}\,\boldsymbol{e}}\big) \,\mathrm{d}s \\ &= \int_{\partial B_R(0)} \Bigg(\big(\boldsymbol{\nu}(\boldsymbol{x}) \times \overline{\boldsymbol{e}(\boldsymbol{x})}\big) \cdot \Big(\mathrm{i}\omega\mu_0 \int_{S^2} \boldsymbol{p}(\boldsymbol{\theta}) e^{\mathrm{i}k\boldsymbol{x}\cdot\boldsymbol{\theta}} \,\mathrm{d}s(\boldsymbol{\theta})\Big) \\ &\qquad - \Big(\sqrt{\frac{\mu_0}{\varepsilon_0}} \int_{S^2} (\boldsymbol{\theta} \times \boldsymbol{p}(\boldsymbol{\theta})) e^{\mathrm{i}k\boldsymbol{x}\cdot\boldsymbol{\theta}} \,\mathrm{d}s(\boldsymbol{\theta})\Big) \cdot \big(\boldsymbol{\nu}(\boldsymbol{x}) \times \overline{\mathbf{curl}\,\boldsymbol{e}(\boldsymbol{x})}\big) \Bigg) \,\mathrm{d}s(\boldsymbol{x}) \\ &= \int_{S^2} \boldsymbol{p}(\boldsymbol{\theta}) \cdot \int_{\partial B_R(0)} \Big(\mathrm{i}\omega\mu_0 \big(\boldsymbol{\nu}(\boldsymbol{x}) \times \overline{\boldsymbol{e}(\boldsymbol{x})}\big) - \sqrt{\frac{\mu_0}{\varepsilon_0}} \big(\boldsymbol{\nu}(\boldsymbol{x}) \times \overline{\mathbf{curl}\,\boldsymbol{e}(\boldsymbol{x})}\big) \times \boldsymbol{\theta}\Big) e^{\mathrm{i}k\boldsymbol{x}\cdot\boldsymbol{\theta}} \,\mathrm{d}s(\boldsymbol{x}) \,\mathrm{d}s(\boldsymbol{\theta})\,.\end{aligned}$$

On the other hand, the representation formula for the far field pattern $\boldsymbol{e}^\infty$ of $\boldsymbol{e}$ analogous to (3.7) gives

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\boldsymbol{\theta}\times\overline{\boldsymbol{e}^\infty(\boldsymbol{\theta})} = \int_{\partial B_R(0)}\Big(\Big(\boldsymbol{\theta}\times\Big(\mathrm{i}\omega\mu_0\big(\boldsymbol{\nu}(\boldsymbol{x})\times\overline{\boldsymbol{e}(\boldsymbol{x})}\big)\Big)\Big)\times\boldsymbol{\theta} \\ -\sqrt{\frac{\mu_0}{\varepsilon_0}}\big(\boldsymbol{\nu}(\boldsymbol{x})\times\overline{\mathbf{curl}\,\boldsymbol{e}(\boldsymbol{x})}\big)\times\boldsymbol{\theta}\Big)e^{\mathrm{i}k\boldsymbol{x}\cdot\boldsymbol{\theta}}\,\mathrm{d}s(\boldsymbol{x})$$

for $\boldsymbol{\theta}\in S^2$, and thus

$$\int_{\partial B_R(0)}\Big((\boldsymbol{\nu}\times\overline{\boldsymbol{e}})\cdot\mathbf{curl}\,\boldsymbol{E}^i_{\boldsymbol{p}}-(\boldsymbol{\nu}\times\boldsymbol{E}^i_{\boldsymbol{p}})\cdot\overline{\mathbf{curl}\,\boldsymbol{e}}\Big)\,\mathrm{d}s = \sqrt{\frac{\mu_0}{\varepsilon_0}}\int_{S^2}\boldsymbol{p}(\boldsymbol{\theta})\cdot\big(\boldsymbol{\theta}\times\overline{\boldsymbol{e}^\infty(\boldsymbol{\theta})}\big)\,\mathrm{d}s(\boldsymbol{\theta})\,. \tag{5.6}$$

Next, we consider the second integral on the right hand side of (5.5) and apply the radiation condition (3.5d) as well as the far field expansion (3.6). This gives, as $R\to\infty$,

$$\begin{aligned}&\int_{\partial B_R(0)}\big((\boldsymbol{\nu}\times\overline{\boldsymbol{e}})\cdot\mathbf{curl}\,\boldsymbol{E}^s_{q,\boldsymbol{p}}-(\boldsymbol{\nu}\times\boldsymbol{E}^s_{q,\boldsymbol{p}})\cdot\overline{\mathbf{curl}\,\boldsymbol{e}}\big)\,\mathrm{d}s\\ &= \int_{S^2}\int_{\partial B_R(0)}\Big(-\Big(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\times\mathbf{curl}\,\boldsymbol{E}^s_q(\boldsymbol{x};\boldsymbol{\theta},\boldsymbol{p})\Big)\cdot\overline{\boldsymbol{e}(\boldsymbol{x})}\\ &\qquad\qquad +\boldsymbol{E}^s_q(\boldsymbol{x};\boldsymbol{\theta},\boldsymbol{p})\cdot\Big(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\times\overline{\mathbf{curl}\,\boldsymbol{e}(\boldsymbol{x})}\Big)\Big)\,\mathrm{d}s(\boldsymbol{x})\,\mathrm{d}s(\boldsymbol{\theta})\\ &= 2\mathrm{i}k\int_{S^2}\int_{\partial B_R(0)}\boldsymbol{E}^s_q(\boldsymbol{x};\boldsymbol{\theta},\boldsymbol{p})\cdot\overline{\boldsymbol{e}(\boldsymbol{x})}\,\mathrm{d}s(\boldsymbol{x})\,\mathrm{d}s(\boldsymbol{\theta})+o(1)\\ &= \frac{\mathrm{i}k}{8\pi^2}\int_{S^2}\int_{S^2}\boldsymbol{E}^\infty_q(\widehat{\boldsymbol{x}};\boldsymbol{\theta},\boldsymbol{p})\cdot\overline{\boldsymbol{e}^\infty(\widehat{\boldsymbol{x}})}\,\mathrm{d}s(\widehat{\boldsymbol{x}})\,\mathrm{d}s(\boldsymbol{\theta})+o(1)\,.\end{aligned}$$

Recalling that $\boldsymbol{H}^\infty_q(\widehat{\boldsymbol{x}}) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\widehat{\boldsymbol{x}}\times\boldsymbol{E}^\infty_q(\widehat{\boldsymbol{x}})$ for all $\widehat{\boldsymbol{x}}\in S^2$, we obtain

$$\int_{S^2}\int_{S^2}\boldsymbol{E}^\infty_q(\widehat{\boldsymbol{x}};\boldsymbol{\theta},\boldsymbol{p})\cdot\overline{\boldsymbol{e}^\infty(\widehat{\boldsymbol{x}})}\,\mathrm{d}s(\widehat{\boldsymbol{x}})\,\mathrm{d}s(\boldsymbol{\theta}) = \sqrt{\frac{\mu_0}{\varepsilon_0}}\int_{S^2}(F_q\boldsymbol{p})(\widehat{\boldsymbol{x}})\cdot\big(\widehat{\boldsymbol{x}}\times\overline{\boldsymbol{e}^\infty(\widehat{\boldsymbol{x}})}\big)\,\mathrm{d}s(\widehat{\boldsymbol{x}})\,,$$

and the second integral on the right hand side of (5.5) becomes

$$\int_{\partial B_R(0)}\big((\boldsymbol{\nu}\times\overline{\boldsymbol{e}})\cdot\mathbf{curl}\,\boldsymbol{E}^s_{q,\boldsymbol{p}}-(\boldsymbol{\nu}\times\boldsymbol{E}^s_{q,\boldsymbol{p}})\cdot\overline{\mathbf{curl}\,\boldsymbol{e}}\big)\,\mathrm{d}s \\ = \sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\mathrm{i}k}{8\pi^2}\int_{S^2}\boldsymbol{p}(\widehat{\boldsymbol{x}})\cdot\overline{F_q^*\big(\widehat{\boldsymbol{x}}\times\boldsymbol{e}^\infty(\widehat{\boldsymbol{x}})\big)}\,\mathrm{d}s(\widehat{\boldsymbol{x}})+o(1)\,. \tag{5.7}$$

Combining (5.5), (5.6), (5.7), and (3.9) we finally obtain that

$$\int_D(L_{q,\Omega}\boldsymbol{p})\cdot\overline{\boldsymbol{f}}\,\mathrm{d}\boldsymbol{x} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\int_{S^2}\boldsymbol{p}(\widehat{\boldsymbol{x}})\cdot\overline{S_q^*\big(\widehat{\boldsymbol{x}}\times\boldsymbol{e}^\infty(\widehat{\boldsymbol{x}})\big)}\,\mathrm{d}s(\widehat{\boldsymbol{x}})\,.$$

□

Lemma 5.3. *Suppose that $D\subseteq\mathbb{R}^3$ is open and of class C^0, and let $q\in\mathcal{Y}_D$. Let $B,\Omega\subseteq\mathbb{R}^3$ be open and bounded such that $\mathbb{R}^3\setminus(\overline{B}\cup\overline{\Omega})$ is connected and $\overline{B}\cap\overline{\Omega}=\emptyset$. Then,*

$$\mathcal{R}(L^*_{q,B})\cap\mathcal{R}(L^*_{q,\Omega}) = \{0\}\,,$$

and $\mathcal{R}(L^*_{q,B}),\mathcal{R}(L^*_{q,\Omega})\subseteq L^2_t(S^2,\mathbb{C}^3)$ are both dense.

Proof. We assume that $\boldsymbol{\phi} \in \mathcal{R}(L_{q,B}^*) \cap \mathcal{R}(L_{q,\Omega}^*)$. Then, we know from Lemma 5.2 that there exist sources $\boldsymbol{f}_B \in L^2(B, \mathbb{C}^3)$ and $\boldsymbol{f}_\Omega \in L^2(\Omega, \mathbb{C}^3)$ such that

$$\boldsymbol{\phi} \;=\; \sqrt{\frac{\mu_0}{\varepsilon_0}} \mathcal{S}_q^*(\boldsymbol{\nu} \times \boldsymbol{e}_B^\infty) \;=\; \sqrt{\frac{\mu_0}{\varepsilon_0}} \mathcal{S}_q^*(\boldsymbol{\nu} \times \boldsymbol{e}_\Omega^\infty)\,,$$

where $\boldsymbol{e}_B, \boldsymbol{e}_\Omega \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ are radiating solutions to

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e}_B - k^2\varepsilon_r \boldsymbol{e}_B \;=\; \mathrm{i}\omega\varepsilon \boldsymbol{f}_B \quad \text{and} \quad \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e}_\Omega - k^2\varepsilon_r \boldsymbol{e}_\Omega \;=\; \mathrm{i}\omega\varepsilon \boldsymbol{f}_\Omega \qquad \text{in } \mathbb{R}^3\,.$$

Since $\mathcal{S}_q$ is unitary, Rellich's lemma (see, e.g., [44, Cor. 9.29]) and the unique continuation principle (see [4]) imply that $\boldsymbol{e}_B = \boldsymbol{e}_\Omega$ in $\mathbb{R}^3 \setminus (\overline{B} \cup \overline{\Omega})$, and we may define $\boldsymbol{e} \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ by

$$\boldsymbol{e} := \begin{cases} \boldsymbol{e}_B = \boldsymbol{e}_\Omega & \text{in } \mathbb{R}^3 \setminus (\overline{B} \cup \overline{\Omega})\,,\\ \boldsymbol{e}_B & \text{in } \Omega\,,\\ \boldsymbol{e}_\Omega & \text{in } B\,. \end{cases}$$

Then $\boldsymbol{e}$ is a radiating solution to

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e} - k^2\varepsilon_r \boldsymbol{e} \;=\; 0 \qquad \text{in } \mathbb{R}^3\,.$$

The uniqueness result [4, Thm. 2] shows that $\boldsymbol{e}$ must vanish identically in $\mathbb{R}^3$. In particular $\boldsymbol{e}^\infty = 0$, and thus $\boldsymbol{\phi} = 0$.

To show that $\mathcal{R}(L_{q,B}^*) \subseteq L_t^2(S^2, \mathbb{C}^3)$ is dense, we prove the injectivity of the operator $L_{q,B}$. Suppose that $L_{q,B}\boldsymbol{p} = -\mathrm{i}k\varepsilon \boldsymbol{E}_{q,\boldsymbol{p}}|_B = 0$. Then $\boldsymbol{E}_{q,\boldsymbol{p}}|_B = 0$, and unique continuation (see [4]) implies that $\boldsymbol{E}_{q,\boldsymbol{p}} = 0$ in $\mathbb{R}^3$. In particular, $\boldsymbol{E}_{\boldsymbol{p}}^i = \boldsymbol{E}_{q,\boldsymbol{p}}^s$ is an entire radiating solution to Maxwell's equations (3.5a), and therefore $\boldsymbol{E}_{\boldsymbol{p}}^i = \boldsymbol{H}_{\boldsymbol{p}}^i = 0$ in $\mathbb{R}^3$. Thus, [10, Thm. 3.27] gives $\boldsymbol{p} = 0$. The denseness of $\mathcal{R}(L_{q,\Omega}^*) \subseteq L_t^2(S^2, \mathbb{C}^3)$ follows analogously. □

In the next lemma we quote a special case of Lemma 2.5 in [31].

Lemma 5.4. *Let X, Y and Z be Hilbert spaces, and let $A : X \to Y$ and $B : X \to Z$ be bounded linear operators. Then,*

$$\exists C > 0 :\ \|Ax\| \leq C\|Bx\| \quad \forall x \in X \qquad \textit{if and only if} \qquad \mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)\,.$$

Now we establish the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $V \subseteq L_t^2(S^2, \mathbb{C}^3)$ be a finite dimensional subspace. Without loss of generality we assume that $\overline{B} \cap \overline{\Omega} = \emptyset$ and that $\mathbb{R}^3 \setminus (\overline{B} \cup \overline{\Omega})$ is connected (otherwise we replace B by a sufficiently small ball $\widetilde{B} \subseteq B \setminus \Omega_\rho$, where Ω_ρ denotes a sufficiently small neighborhood of Ω). We introduce the orthogonal projection $P_V : L_t^2(S^2, \mathbb{C}^3) \to L_t^2(S^2, \mathbb{C}^3)$ onto V. From Lemma 5.3 we know that $\mathcal{R}(L_{q,B}^*) \subseteq L_t^2(S^2, \mathbb{C}^3)$ is dense and therefore $\mathcal{R}(L_{q,B}^*)$ is infinite dimensional. Together with the fact that $\mathcal{R}(L_{q,B}^*) \cap \mathcal{R}(L_{q,\Omega}^*) = \{0\}$, a dimensionality argument (cf. [31, Lem. 4.7]) shows that

$$\mathcal{R}(L_{q,B}^*) \not\subseteq \mathcal{R}(L_{q,\Omega}^*) + V \;=\; \mathcal{R}\big([L_{q,\Omega}^* \,|\, P_V^*]\big) \;=\; \mathcal{R}\left(\begin{bmatrix} L_{q,\Omega} \\ P_V \end{bmatrix}^*\right).$$

From Lemma 5.4 it follows that there does not exist a constant $C > 0$ such that

$$\|L_{q,B}\boldsymbol{p}\|_{L^2(B)}^2 \;\leq\; C^2 \left\| \begin{bmatrix} L_{q,\Omega} \\ P_V \end{bmatrix} \boldsymbol{p} \right\|_{L^2(\Omega)\times L_t^2(S^2)}^2 \;=\; C^2\big(\|L_{q,\Omega}\boldsymbol{p}\|_{L^2(\Omega)}^2 + \|P_V\boldsymbol{p}\|_{L_t^2(S^2)}^2\big)$$

holds for all $\boldsymbol{p} \in L^2_t(S^2,\mathbb{C}^3)$. This means that one can find a sequence $(\widetilde{\boldsymbol{p}}_m)_{m\in\mathbb{N}} \subseteq L^2_t(S^2,\mathbb{C}^3)$ such that

$$\|L_{q,B}\widetilde{\boldsymbol{p}}_m\|^2_{L^2(B)} \to \infty \qquad \text{and} \qquad \|L_{q,\Omega}\widetilde{\boldsymbol{p}}_m\|^2_{L^2(\Omega)} + \|P_V\widetilde{\boldsymbol{p}}_m\|^2_{L^2_t(S^2)} \to 0$$

as $m \to \infty$. Setting $\boldsymbol{p}_m := \widetilde{\boldsymbol{p}}_m - P_V\widetilde{\boldsymbol{p}}_m \in V^\perp$ for all $m \in \mathbb{N}$ yields

$$\begin{aligned} \|L_{q,B}\boldsymbol{p}_m\|_{L^2(B)} &\geq \|L_{q,B}\widetilde{\boldsymbol{p}}_m\|_{L_2(B)} - \|L_{q,B}\|\|P_V\widetilde{\boldsymbol{p}}_m\|_{L^2_t(S^2)} \to \infty && \text{as } m\to\infty\,,\\ \|L_{q,\Omega}\boldsymbol{p}_m\|_{L^2(\Omega)} &\leq \|L_{q,\Omega}\widetilde{\boldsymbol{p}}_m\|_{L^2(\Omega)} + \|L_{q,\Omega}\|\|P_V\widetilde{\boldsymbol{p}}_m\|_{L^2_t(S^2)} \to 0 && \text{as } m\to\infty\,. \end{aligned}$$

Recalling that $L_{q,B}\boldsymbol{p}_m = \operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}_m}|_B$ and $L_{q,\Omega}\boldsymbol{p}_m = \operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}_m}|_\Omega$, this ends the proof. □

Theorem 5.5. *Suppose that $D_1, D_2 \subseteq \mathbb{R}^3$ are open and bounded of class C^0, let $q_1 \in \mathcal{Y}_{D_1}$ and $q_2 \in \mathcal{Y}_{D_2}$, and assume that $\Omega \subseteq \mathbb{R}^3$ is open and bounded. If $q_1(x) = q_2(x)$ for a.e. $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, then there exist constants $c, C > 0$ such that*

$$c\int_\Omega |\operatorname{\mathbf{curl}} \boldsymbol{H}_{q_1,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \leq \int_\Omega |\operatorname{\mathbf{curl}} \boldsymbol{H}_{q_2,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \leq C\int_\Omega |\operatorname{\mathbf{curl}} \boldsymbol{H}_{q_1,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x}$$

for all $\boldsymbol{p} \in L^2_t(S^2,\mathbb{C}^3)$.

Proof. Lemma 5.2 shows that, for any $\boldsymbol{f} \in L^2(\Omega,\mathbb{C}^3)$,

$$L^*_{q_1,\Omega}\boldsymbol{f} \;=\; \sqrt{\frac{\mu_0}{\varepsilon_0}}\mathcal{S}^*_{q_1}(\boldsymbol{\nu}\times\boldsymbol{e}^\infty_1) \qquad \text{and} \qquad L^*_{q_2,\Omega}\boldsymbol{f} \;=\; \sqrt{\frac{\mu_0}{\varepsilon_0}}\mathcal{S}^*_{q_2}(\boldsymbol{\nu}\times\boldsymbol{e}^\infty_2)\,, \tag{5.8}$$

where $\boldsymbol{e}^\infty_j$, $j = 1, 2$, are the far field patterns of radiating solutions to

$$\operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}} \boldsymbol{e}_j - k^2\varepsilon_{r,j}\boldsymbol{e}_j \;=\; \mathrm{i}\omega\varepsilon\boldsymbol{f} \qquad \text{in } \mathbb{R}^3\,.$$

Moreover, we observe that

$$\begin{aligned} \operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}} \boldsymbol{e}_1 - k^2\varepsilon_{r,2}\boldsymbol{e}_1 &= \mathrm{i}\omega\varepsilon_2\Big(\frac{\varepsilon_1}{\varepsilon_2}\boldsymbol{f} - \frac{k^2}{\mathrm{i}\omega\varepsilon_2}(\varepsilon_{r,2}-\varepsilon_{r,1})\boldsymbol{e}_1\Big) && \text{in } \mathbb{R}^3\,,\\ \operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}} \boldsymbol{e}_2 - k^2\varepsilon_{r,1}\boldsymbol{e}_1 &= \mathrm{i}\omega\varepsilon_1\Big(\frac{\varepsilon_2}{\varepsilon_1}\boldsymbol{f} - \frac{k^2}{\mathrm{i}\omega\varepsilon_1}(\varepsilon_{r,1}-\varepsilon_{r,2})\boldsymbol{e}_2\Big) && \text{in } \mathbb{R}^3\,. \end{aligned}$$

Since by assumption $\varepsilon_{r,1} - \varepsilon_{r,2}$ vanishes a.e. outside of Ω, this implies that

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\mathcal{S}^*_{q_2}(\boldsymbol{\nu}\times\boldsymbol{e}^\infty_1) \;=\; L^*_{q_2,\Omega}\Big(\frac{\varepsilon_1}{\varepsilon_2}\boldsymbol{f} + \mathrm{i}k^2\frac{(\varepsilon_{r,2}-\varepsilon_{r,1})}{\omega\varepsilon_2}\boldsymbol{e}_1\Big)\,, \tag{5.9a}$$

$$\sqrt{\frac{\mu_0}{\varepsilon_0}}\mathcal{S}^*_{q_1}(\boldsymbol{\nu}\times\boldsymbol{e}^\infty_2) \;=\; L^*_{q_1,\Omega}\Big(\frac{\varepsilon_2}{\varepsilon_1}\boldsymbol{f} + \mathrm{i}k^2\frac{(\varepsilon_{r,1}-\varepsilon_{r,2})}{\omega\varepsilon_1}\boldsymbol{e}_2\Big)\,. \tag{5.9b}$$

Combining (5.8) and (5.9), we obtain that $\mathcal{R}(\mathcal{S}_{q_1}L^*_{q_1,\Omega}) = \mathcal{R}(\mathcal{S}_{q_2}L^*_{q_2,\Omega})$. It remains to show that $\mathcal{R}(\mathcal{S}_{q_j}L^*_{q_j,\Omega}) = \mathcal{R}(L^*_{q_j,\Omega})$ for $j = 1, 2$. Then the assertion follows from Lemma 5.4.

Using (3.9) we find that for any $\boldsymbol{f} \in L^2(\Omega,\mathbb{C}^3)$,

$$\mathcal{S}_{q_j}L^*_{q_j,\Omega}\boldsymbol{f} \;=\; L^*_{q_j,\Omega}\boldsymbol{f} + \frac{\mathrm{i}k}{8\pi^2}F_{q_j}L^*_{q_j,\Omega}\boldsymbol{f}\,. \tag{5.10}$$

The definition of the far field operator F_{q_j} in (3.8) together with our notation from (3.10)–(3.11) shows that $F_{q_j}L^*_{q_j,\Omega}\boldsymbol{f} = \boldsymbol{H}^\infty_{q_j,\boldsymbol{p}}$ with $\boldsymbol{p}_{j,\boldsymbol{f}} := L^*_{q_j,\Omega}\boldsymbol{f}$. Since (3.5) implies that the corresponding scattered electric field $\boldsymbol{E}^s_{q,\boldsymbol{p}}$ is a radiating solution to

$$\operatorname{\mathbf{curl}}\operatorname{\mathbf{curl}} \boldsymbol{E}^s_{q_j,\boldsymbol{p}_{j,\boldsymbol{f}}} - k^2\varepsilon_{r,j}\boldsymbol{E}^s_{q_j,\boldsymbol{p}_{j,\boldsymbol{f}}} \;=\; -k^2(1-\varepsilon_{r,j})\boldsymbol{E}^i_{\boldsymbol{p}_{j,\boldsymbol{f}}} \qquad \text{in } \mathbb{R}^3\,,$$

we find using Lemma 5.2 that

$$\boldsymbol{H}^{\infty}_{q_j,\boldsymbol{p}_{j,\boldsymbol{f}}} \;=\; \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\boldsymbol{\nu}\times\boldsymbol{E}^{\infty}_{q_j,\boldsymbol{p}_{j,\boldsymbol{f}}} \;=\; \mathcal{S}_{q_j}L^*_{q_j,\Omega}\Big(\mathrm{i}\omega\varepsilon_0\frac{1-\varepsilon_{r,j}}{\varepsilon_{r,j}}\,\boldsymbol{E}^i_{\boldsymbol{p}_{j,\boldsymbol{f}}}\Big)\,.$$

Substituting this into (5.10) and rearranging terms shows that, for any $\boldsymbol{f}\in L^2(\Omega,\mathbb{C}^3)$,

$$L^*_{q_j,\Omega}\boldsymbol{f} \;=\; \mathcal{S}_{q_j}L^*_{q_j,\Omega}\Big(\boldsymbol{f}+\frac{k}{8\pi^2}\omega\varepsilon_0\frac{1-\varepsilon_{r,j}}{\varepsilon_{r,j}}\,\boldsymbol{E}^i_{\boldsymbol{p}_{j,\boldsymbol{f}}}\Big)\,,$$

i.e., $\mathcal{R}(L^*_{q_j,\Omega})\subseteq\mathcal{R}(\mathcal{S}_{q_j}L^*_{q_j,\Omega})$ for $j=1,2$.

Similarly, using (5.8) we have that, for any $\boldsymbol{f}\in L^2(\Omega,\mathbb{C}^3)$,

$$\mathcal{S}_{q_j}L^*_{q_j,\Omega}\boldsymbol{f} \;=\; \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\mathcal{S}^*_{q_j}\big(\mathcal{S}_{q_j}(\boldsymbol{\nu}\times\boldsymbol{e}^{\infty}_j)\big) \;=\; \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\mathcal{S}^*_{q_j}\Big(\boldsymbol{\nu}\times\boldsymbol{e}^{\infty}_j+\frac{\mathrm{i}k}{8\pi^2}F_{q_j}(\boldsymbol{\nu}\times\boldsymbol{e}^{\infty}_j)\Big) \qquad (5.11)$$

Writing $\boldsymbol{p}_{j,\boldsymbol{f}}:=\boldsymbol{\nu}\times\boldsymbol{e}^{\infty}_j$ we obtain as before that

$$F_{q_j}(\boldsymbol{\nu}\times\boldsymbol{e}^{\infty}_j) \;=\; \boldsymbol{H}^{\infty}_{q_j,\boldsymbol{p}_{j,\boldsymbol{f}}} \;=\; \mathcal{S}_{q_j}L^*_{q_j,\Omega}\Big(\mathrm{i}\omega\varepsilon_0\frac{1-\varepsilon_{r,j}}{\varepsilon_{r,j}}\,\boldsymbol{E}^i_{\boldsymbol{p}_{j,\boldsymbol{f}}}\Big)\,.$$

Accordingly, substituting this into (5.11) and applying (5.8) we find that, for any $\boldsymbol{f}\in L^2(\Omega,\mathbb{C}^3)$,

$$\mathcal{S}_{q_j}L^*_{q_j,\Omega}\boldsymbol{f} \;=\; L^*_{q_j,\Omega}\Big(\boldsymbol{f}-\frac{k}{8\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\omega\varepsilon_0\frac{1-\varepsilon_{r,j}}{\varepsilon_{r,j}}\,\boldsymbol{E}^i_{\boldsymbol{p}_{j,\boldsymbol{f}}}\Big)\,,$$

i.e., $\mathcal{R}(\mathcal{S}_{q_j}L^*_{q_j,\Omega})\subseteq\mathcal{R}(L^*_{q_j,\Omega})$ for $j=1,2$. □

Our first application of Theorem 5.1 is the following simple uniqueness result for the inverse scattering problem. This should be compared to (4.2) in Theorem 4.2.

Theorem 5.6. *Suppose that $D_1,D_2\subseteq\mathbb{R}^3$ are open and bounded of class C^0, let $q_1\in\mathcal{Y}_{D_1}$ and $q_2\in\mathcal{Y}_{D_2}$. If $O\subseteq\mathbb{R}^3$ is an unbounded domain such that*

$$q_1 \;\le\; q_2 \qquad \text{a.e. in } O\,, \qquad (5.12)$$

and if $B\subseteq O$ is open with

$$q_1 \;\le\; q_2-c \qquad \text{a.e. in } B \text{ for some } c>0\,,$$

then

$$\mathrm{Re}(\mathcal{S}^*_{q_1}F_{q_1}) \not\ge_{\mathrm{fin}} \mathrm{Re}(\mathcal{S}^*_{q_1}F_{q_2})\,.$$

In particular, $F_{q_1}\neq F_{q_2}$.

Proof. Suppose that there is a finite dimensional subspace $V_1\subseteq L^2_t(S^2,\mathbb{C}^3)$ such that

$$\mathrm{Re}\Big(\int_{S^2}\boldsymbol{p}\cdot\overline{\mathcal{S}^*_{q_1}(F_{q_2}-F_{q_1})\boldsymbol{p}}\,\mathrm{d}s\Big) \;\le\; 0 \qquad \text{for all } \boldsymbol{p}\in V_1^{\perp}\,.$$

Then, Theorem 4.2 shows that there exists another finite dimensional subspace $V_2\subseteq L^2_t(S^2,\mathbb{C}^3)$ such that

$$\mathrm{Re}\Big(\int_{S^2}\boldsymbol{p}\cdot\overline{\mathcal{S}^*_{q_1}(F_{q_2}-F_{q_1})\boldsymbol{p}}\,\mathrm{d}s\Big) \;\ge\; \int_{\mathbb{R}^3}(q_2-q_1)\,|\,\mathbf{curl}\,\boldsymbol{H}_{q_1,\boldsymbol{p}}|^2\,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \boldsymbol{p}\in V_2^{\perp}\,.$$

Defining $V := V_1 + V_2$, we obtain from (5.12) that, for any $\boldsymbol{p} \in V^\perp$,

$$\begin{aligned} 0 \,\geq\, \mathrm{Re}\Big(\int_{S^2} \boldsymbol{p} \cdot \overline{\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})\boldsymbol{p}} \,\mathrm{d}s \Big) \,&\geq\, \int_{\mathbb{R}^3} (q_2 - q_1)\, |\,\mathbf{curl}\, \boldsymbol{H}_{q_1,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \\ &= \int_O (q_2 - q_1)\, |\,\mathbf{curl}\, \boldsymbol{H}_{q_1,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^3 \setminus \overline{O}} (q_2 - q_1)\, |\,\mathbf{curl}\, \boldsymbol{H}_{q_1,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \\ &\geq c \int_B |\,\mathbf{curl}\, \boldsymbol{H}_{q_1,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} - \big(\|q_1\|_{L^\infty(\mathbb{R}^3)} + \|q_2\|_{L^\infty(\mathbb{R}^3)} \big) \int_{\mathbb{R}^3 \setminus \overline{O}} |\,\mathbf{curl}\, \boldsymbol{H}_{q_1,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \,. \end{aligned}$$

However, this contradicts Theorem 5.1 with $D = D_1$, $q = q_1$, and $\Omega = \mathbb{R}^3 \setminus \overline{O}$, which guarantees the existence of $(\boldsymbol{p}_m)_{m\in\mathbb{N}} \subseteq V^\perp$ with

$$\int_B |\,\mathbf{curl}\, \boldsymbol{H}_{q_1,\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to \infty \quad \text{and} \quad \int_{\mathbb{R}^3 \setminus \overline{O}} |\,\mathbf{curl}\, \boldsymbol{H}_{q_1,\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to 0 \quad \text{as } m \to \infty \,.$$

Thus, $\mathrm{Re}(\mathcal{S}_{q_1}^*(F_{q_2} - F_{q_1})) \not\leq_{\mathrm{fin}} 0$. □

6 Shape reconstruction for sign definite scatterers

We discuss criteria to determine the shape of a scattering object D with permittivity contrast $q \in \mathcal{Y}_D$ from observations of the corresponding far field operator F_q. In this section we consider the special case when the contrast function q is either strictly positive or strictly negative a.e. on D. The general case will be treated in Section 8 below.

Let $B \subseteq \mathbb{R}^3$ be open and bounded. The *Herglotz operator* $H_B : L^2_t(S^2, \mathbb{C}^3) \to L^2(B, \mathbb{C}^3)$ is defined by

$$(H_B \boldsymbol{p})(\boldsymbol{y}) \,:=\, \int_{S^2} \mathbf{curl}_{\boldsymbol{y}} \big(e^{\mathrm{i}k\boldsymbol{y}\cdot\boldsymbol{\theta}} \boldsymbol{p}(\boldsymbol{\theta}) \big) \,\mathrm{d}s(\boldsymbol{\theta}) \,=\, \mathrm{i}k \int_{S^2} e^{\mathrm{i}k\boldsymbol{y}\cdot\boldsymbol{\theta}} \big(\boldsymbol{\theta} \times \boldsymbol{p}(\boldsymbol{\theta}) \big) \,\mathrm{d}s(\boldsymbol{\theta}) \,, \qquad \boldsymbol{y} \in B \,.$$

Accordingly, the adjoint operator $H_B^* : L^2(B, \mathbb{C}^3) \to L^2_t(S^2, \mathbb{C}^3)$ satisfies

$$(H_B^* \boldsymbol{f})(\widehat{\boldsymbol{x}}) \,=\, \mathrm{i}k\, \widehat{\boldsymbol{x}} \times \int_B e^{-\mathrm{i}k\boldsymbol{y}\cdot\widehat{\boldsymbol{x}}} \boldsymbol{f}(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \,, \qquad \widehat{\boldsymbol{x}} \in S^2 \,,$$

and

$$(H_B^* H_B \boldsymbol{p})(\widehat{\boldsymbol{x}}) \,=\, -k^2\, \widehat{\boldsymbol{x}} \times \Big(\int_{S^2} \Big(\int_B e^{\mathrm{i}k\boldsymbol{y}\cdot(\boldsymbol{\theta} - \widehat{\boldsymbol{x}})} \,\mathrm{d}\boldsymbol{y} \Big) \big(\boldsymbol{\theta} \times \boldsymbol{p}(\boldsymbol{\theta}) \big) \,\mathrm{d}s(\boldsymbol{\theta}) \Big) \,, \qquad \widehat{\boldsymbol{x}} \in S^2 \,.$$

In the following, we consider the *probing operator* $T_B : L^2_t(S^2, \mathbb{C}^3) \to L^2_t(S^2, \mathbb{C}^3)$ corresponding to the *probing domain* B, which is defined by

$$T_B \boldsymbol{p} \,:=\, H_B^* H_B \boldsymbol{p} \,. \tag{6.1}$$

This operator is compact and self-adjoint, and for all $\boldsymbol{p} \in L^2_t(S^2, \mathbb{C}^3)$ we have that

$$\begin{aligned} \int_{S^2} \boldsymbol{p} \cdot \overline{T_B \boldsymbol{p}} \,\mathrm{d}s \,&=\, \int_B \Big(\mathrm{i}k \int_{S^2} e^{\mathrm{i}k\boldsymbol{y}\cdot\widehat{\boldsymbol{x}}} \big(\widehat{\boldsymbol{x}} \times \boldsymbol{p}(\widehat{\boldsymbol{x}}) \big) \,\mathrm{d}s(\widehat{\boldsymbol{x}}) \Big) \cdot \overline{\Big(\mathrm{i}k \int_{S^2} e^{\mathrm{i}k\boldsymbol{y}\cdot\boldsymbol{\theta}} \big(\boldsymbol{\theta} \times \boldsymbol{p}(\boldsymbol{\theta}) \big) \,\mathrm{d}s(\boldsymbol{\theta}) \Big)} \,\mathrm{d}\boldsymbol{y} \\ &=\, \int_B |\,\mathbf{curl}\, \boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \,, \end{aligned} \tag{6.2}$$

where $\boldsymbol{H}^i_{\boldsymbol{p}}$ is the incident magnetic field from (3.10). This should be compared to (4.4).

The theorem below considers the case when the contrast function q is strictly positive a.e. on D.

Theorem 6.1. *Let $D \subseteq \mathbb{R}^3$ be open and bounded of class C^0 such that $\mathbb{R}^3 \setminus \overline{D}$ is connected, and let $q \in \mathcal{Y}_D$. Suppose that $0 < q_{\min} \le q \le q_{\max} < 1$ for some constants $q_{\min}, q_{\max} \in \mathbb{R}$, and let $B \subseteq B_R(0)$ be open and bounded.*

(a) If $B \subseteq D$, then

$$\alpha T_B \le_{\text{fin}} \operatorname{Re}(F_q) \qquad \textit{for all } \alpha \le q_{\min}\,.$$

(b) If $B \not\subseteq D$, then

$$\alpha T_B \not\le_{\text{fin}} \operatorname{Re}(F_q) \qquad \textit{for any } \alpha > 0\,.$$

Proof. Let $B \subseteq D$ and $\alpha \le q_{\min}$. Theorem 4.2 with $q_1 = 0$ and $q_2 = q$ guarantees the existence of a finite dimensional subspace $V \subseteq L^2_t(S^2, \mathbb{C}^3)$ such that

$$\operatorname{Re}\Big(\int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s\Big) \ge \int_D q\,|\operatorname{\mathbf{curl}} \boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \boldsymbol{p} \in V^\perp\,.$$

Since $B \subseteq D$ and $q_{\min} \ge \alpha$, (6.2) yields

$$\operatorname{Re}\Big(\int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s\Big) \ge \alpha \int_B |\operatorname{\mathbf{curl}} \boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} = \alpha \int_{S^2} \boldsymbol{p} \cdot \overline{T_B \boldsymbol{p}} \,\mathrm{d}s \qquad \text{for all } \boldsymbol{p} \in V^\perp\,.$$

Now applying Lemma 4.1 shows part (a).

Next we assume that $B \not\subseteq D$ and that there exists $\alpha > 0$ with $\alpha T_B \le_{\text{fin}} \operatorname{Re}(F_q)$. The latter implies the existence of a finite dimensional subspace $V_1 \subseteq L^2_t(S^2, \mathbb{C}^3)$ such that

$$\alpha \int_{S^2} \boldsymbol{p} \cdot \overline{T_B \boldsymbol{p}} \,\mathrm{d}s \le \operatorname{Re}\Big(\int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s\Big) \qquad \text{for all } \boldsymbol{p} \in V_1^\perp. \tag{6.3}$$

Moreover, Corollary 4.4 with $q_1 = 0$ and $q_2 = q$ shows that there is a finite dimensional subspace $V_2 \subseteq L^2_t(S^2, \mathbb{C}^3)$ such that

$$\operatorname{Re}\Big(\int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s\Big) \le \int_D q\,|\operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \le q_{\max} \int_D |\operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \boldsymbol{p} \in V_2^\perp. \tag{6.4}$$

We set $V := V_1 + V_2$. Combining (6.3) and (6.4) we obtain that

$$\alpha \int_B |\operatorname{\mathbf{curl}} \boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \le q_{\max} \int_D |\operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \boldsymbol{p} \in V^\perp\,.$$

To further estimate the right hand side we use Theorem 5.5 with $q_1 = 0$, $q_2 = q$, and $\Omega = D$, and we find that

$$\alpha \int_B |\operatorname{\mathbf{curl}} \boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \le C q_{\max} \int_D |\operatorname{\mathbf{curl}} \boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \boldsymbol{p} \in V^\perp$$

with some $C > 0$. However, this contradicts Theorem 5.1 with $q = 0$ and $\Omega = D$, which implies the existence of a sequence $(\boldsymbol{p}_m)_{m\in\mathbb{N}} \subseteq V^\perp$ such that

$$\int_B |\operatorname{\mathbf{curl}} \boldsymbol{H}^i_{\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to \infty \qquad \text{and} \qquad \int_D |\operatorname{\mathbf{curl}} \boldsymbol{H}^i_{\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to 0 \qquad \text{as } m \to \infty\,.$$

□

The next result is analogous to Theorem 6.1, but with contrast functions that are strictly negative a.e. on D.

Theorem 6.2. *Let $D \subseteq \mathbb{R}^3$ be open and bounded of class C^0 such that $\mathbb{R}^3 \setminus \overline{D}$ is connected, and let $q \in \mathcal{Y}_D$. Suppose that $-\infty < q_{\min} \le q \le q_{\max} < 0$ for some constants $q_{\min}, q_{\max} \in \mathbb{R}$, and let $B \subseteq B_R(0)$ be open and bounded.*

(a) If $B \subseteq D$, then there exists a constant $C > 0$ such that

$$\alpha T_B \ge_{\text{fin}} \operatorname{Re}(F_q) \qquad \textit{for all } \alpha \ge C q_{\max}\,.$$

(b) If $B \not\subseteq D$, then

$$\alpha T_B \not\ge_{\text{fin}} \operatorname{Re}(F_q) \qquad \textit{for any } \alpha < 0\,.$$

Proof. Suppose that $B \subseteq D$. Applying Corollary 4.4 with $q_1 = 0$ and $q_2 = q$ we obtain a finite dimensional subspace $V \subseteq L^2_t(S^2, \mathbb{C}^3)$ such that

$$\operatorname{Re}\Bigl(\int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s\Bigr) \le \int_D q\,|\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \le q_{\max} \int_D |\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \qquad \text{for all } p \in V^\perp\,.$$

Furthermore, Theorem 5.5 with $q_1 = 0$, $q_2 = q$, and $\Omega = D$ shows that there exists a constant $C > 0$ such that

$$\operatorname{Re}\Bigl(\int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s\Bigr) \le C q_{\max} \int_D |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \qquad \text{for all } p \in V^\perp\,.$$

In particular,

$$\operatorname{Re}(F_q) \le_{\text{fin}} \alpha T_B \qquad \text{for all } \alpha \ge C q_{\max}\,,$$

and part (a) is proven.

For part (b) we assume that $B \not\subseteq D$, and that there exists $\alpha < 0$ with $\alpha T_B \ge_{\text{fin}} \operatorname{Re}(F_q)$. This means that there exists a finite dimensional subspace $V_1 \subseteq L^2_t(S^2, \mathbb{C}^3)$ such that

$$\alpha \int_{S^2} \boldsymbol{p} \cdot \overline{T_B \boldsymbol{p}} \,\mathrm{d}s \ge \operatorname{Re}\Bigl(\int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s\Bigr) \qquad \text{for all } \boldsymbol{p} \in V_1^\perp\,. \tag{6.5}$$

On the other hand, Theorem 4.2 with $q_1 = 0$ and $q_2 = q$ gives a finite dimensional subspace $V_2 \subseteq L^2_t(S^2, \mathbb{C}^3)$ such that

$$\operatorname{Re}\Bigl(\int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s\Bigr) \ge \int_D q\,|\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \ge q_{\min} \int_D |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x}\,. \tag{6.6}$$

Let $V := V_1 + V_2$. Combining (6.5) and (6.6) we deduce that

$$\alpha \int_B |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \ge q_{\min} \int_D |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \boldsymbol{p} \in V^\perp\,.$$

Applying Theorem 5.1 with $q = 0$ and $\Omega = D$ gives a sequence $(\boldsymbol{p}_m)_{m\in\mathbb{N}} \subseteq V^\perp$ satisfying

$$\int_B |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to \infty \quad \text{and} \quad \int_D |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to 0 \quad \text{as } m \to \infty\,.$$

Since $\alpha < 0$, this yields a contradiction. □

7 Simultaneously localized vector wave functions

To justify a shape characterization similar to Theorems 6.1 and 6.2 for indefinite scattering objects, i.e., for the general case when the constrast function q is neither strictly positive nor strictly negative a.e. on D, we require a refined version of Theorem 5.1. In Theorem 7.1 we not only control the energy of the total field $\boldsymbol{H}_{q,\boldsymbol{p}}$, as was done in Theorem 5.1, but also the energy of the incident field $\boldsymbol{H}^i_{\boldsymbol{p}}$. Similar results have been established for the Schrödinger equation in [26], for the Helmholtz obstacle scattering problem in [1], and for the Helmholtz medium scattering problem in [19].

Theorem 7.1. *Let $D \subseteq \mathbb{R}^3$ be open and bounded of class C^0, and let $q \in \mathcal{Y}_D$ with $q|_D \in C^1(\overline{D})$. Let $E, M \subseteq \mathbb{R}^3$ be open and Lipschitz bounded such that $\operatorname{supp}(q) \subseteq \overline{E} \cup \overline{M}$, $\mathbb{R}^3 \setminus (\overline{E} \cup \overline{M})$ is connected, and $E \cap M = \emptyset$. Assume furthermore that there is a connected subset $\Gamma \subseteq \partial E \setminus \overline{M}$ that is relatively open and $C^{2,1}$-smooth.*

Then for any finite dimensional subspace $V \subseteq L^2_t(S^2, \mathbb{C}^3)$ there exists a sequence $(\boldsymbol{p}_m)_{m\in\mathbb{N}} \subseteq V^\perp$ such that

$$\int_E |\operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to \infty \qquad \textit{and} \qquad \int_M \big(|\operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}_m}|^2 + |\operatorname{\mathbf{curl}} \boldsymbol{H}^i_{\boldsymbol{p}_m}|^2\big) \,\mathrm{d}\boldsymbol{x} \to 0$$

as $m \to \infty$, where $\boldsymbol{H}^i_{\boldsymbol{p}_m}, \boldsymbol{H}_{q,\boldsymbol{p}_m} \in H_{\mathrm{loc}}(\operatorname{\mathbf{curl}}; \mathbb{R}^3)$ are given by (3.10) and (3.11) with $\boldsymbol{p} = \boldsymbol{p}_m$.

The proof of Theorem 7.1 relies on the following two lemmas. Lemma 7.2 extends the result of Lemma 5.2. The goal is to allow for more general arguments for the adjoint $L^*_{q,\Omega}$.

Lemma 7.2. *Suppose that $D \subseteq \mathbb{R}^3$ is open and of class C^0, let $q \in \mathcal{Y}_D$, and assume that $\Omega \subseteq \mathbb{R}^3$ is open and bounded. We define*

$$L_{q,\Omega} : L^2_t(S^2, \mathbb{C}^3) \to H(\operatorname{\mathbf{curl}}; \Omega)\,, \quad L_{q,\Omega}\, \boldsymbol{p} \,:=\, \operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}}|_\Omega \,=\, -\mathrm{i}\omega\varepsilon \boldsymbol{E}_{q,\boldsymbol{p}}|_\Omega\,.$$

Then, $L_{q,\Omega}$ is a linear operator and its adjoint is given by

$$L^*_{q,\Omega} : H(\operatorname{\mathbf{curl}}; \Omega)^* \to L^2_t(S^2, \mathbb{C}^3)\,, \quad L^*_{q,\Omega} \boldsymbol{f} \,:=\, \sqrt{\frac{\mu_0}{\varepsilon_0}} \mathcal{S}^*_q(\boldsymbol{\nu} \times \boldsymbol{e}^\infty)\,,$$

where $H(\operatorname{\mathbf{curl}}; \Omega)^*$ is the dual of $H(\operatorname{\mathbf{curl}}; \Omega)$, and $\boldsymbol{e}^\infty \in L^2_t(S^2, \mathbb{C}^3)$ is the far field pattern of the radiating solution $\boldsymbol{e} \in H_{\mathrm{loc}}(\operatorname{\mathbf{curl}}; \mathbb{R}^3)$ to

$$\operatorname{\mathbf{curl}} \operatorname{\mathbf{curl}} \boldsymbol{e} - k^2 \varepsilon_r \boldsymbol{e} \,=\, \mathrm{i}\omega\varepsilon \boldsymbol{f} \qquad \textit{in } \mathbb{R}^3\,.$$

Proof. This follows from the same arguments that have been used in the proof of Lemma 5.2. □

Lemma 7.3. *Let $D \subseteq \mathbb{R}^3$ be open and bounded of class C^0, and let $q \in \mathcal{Y}_D$ with $q|_D \in C^1(\overline{D})$. Let $E, M \subseteq \mathbb{R}^3$ be open and Lipschitz bounded such that $\operatorname{supp}(q) \subseteq \overline{E} \cup \overline{M}$, $\mathbb{R}^3 \setminus (\overline{E} \cup \overline{M})$ is connected, and $E \cap M = \emptyset$. Assume furthermore that there is a connected subset $\Gamma \subseteq \partial E \setminus \overline{M}$ that is relatively open and $C^{2,1}$-smooth. Then,*

$$\mathcal{R}(L^*_{q,E}) \not\subseteq \mathcal{R}\big([L^*_{q,M} \,|\, L^*_{0,M}]\big)$$

and there exists an infinite dimensional subspace $Z \subseteq \mathcal{R}(L^*_{q,E})$ such that

$$Z \cap \mathcal{R}\big([L^*_{q,M} \,|\, L^*_{0,M}]\big) \,=\, \{0\}\,.$$

Proof. Let $\boldsymbol{h} \in \mathcal{R}(L_{q,E}^*) \cap \mathcal{R}\big(\big[L_{q,M}^* \,\big|\, L_{0,M}^*\big]\big)$. Lemma 7.2 shows that there are $\boldsymbol{f}_{q,E} \in H(\mathbf{curl}; E)^*$ and $\boldsymbol{f}_{q,M}, \boldsymbol{f}_{0,M} \in H(\mathbf{curl}; M)^*$ such that the far field patterns $\boldsymbol{e}_{q,E}^\infty, \boldsymbol{e}_{q,M}^\infty, \boldsymbol{e}_{0,M}^\infty$ of the radiating solutions $\boldsymbol{e}_{q,E}, \boldsymbol{e}_{q,M}, \boldsymbol{e}_{0,M} \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ to

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e}_{q,E} - k^2\varepsilon_r\boldsymbol{e}_{q,E} &= \mathrm{i}\omega\varepsilon\boldsymbol{f}_{q,E} && \text{in } \mathbb{R}^3\,,\\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e}_{q,M} - k^2\varepsilon_r\boldsymbol{e}_{q,M} &= \mathrm{i}\omega\varepsilon\boldsymbol{f}_{q,M} && \text{in } \mathbb{R}^3\,,\\ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e}_{0,M} - k^2\boldsymbol{e}_{0,M} &= \mathrm{i}\omega\varepsilon\boldsymbol{f}_{0,M} && \text{in } \mathbb{R}^3\,, \end{aligned}$$

satisfy

$$\sqrt{\frac{\varepsilon_0}{\mu_0}}\boldsymbol{h} = \mathcal{S}_q^*\big(\boldsymbol{\nu}\times\boldsymbol{e}_{q,E}^\infty\big) = \boldsymbol{\nu}\times\boldsymbol{e}_{0,M}^\infty + \mathcal{S}_q^*\big(\boldsymbol{\nu}\times\boldsymbol{e}_{q,M}^\infty\big)\,.$$

Here we used that $\mathcal{S}_0$ is the identity operator. Accordingly, recalling the definition of the scattering operator in (3.9), we find that

$$\begin{aligned} 0 &= \boldsymbol{\nu}\times\boldsymbol{e}_{q,E}^\infty - \boldsymbol{\nu}\times\boldsymbol{e}_{q,M}^\infty - \mathcal{S}_q\big(\boldsymbol{\nu}\times\boldsymbol{e}_{0,M}^\infty\big)\\ &= \boldsymbol{\nu}\times\boldsymbol{e}_{q,E}^\infty - \boldsymbol{\nu}\times\boldsymbol{e}_{q,M}^\infty - \boldsymbol{\nu}\times\boldsymbol{e}_{0,M}^\infty - \frac{\mathrm{i}k}{8\pi^2}F_q\boldsymbol{e}_{0,M}^\infty\\ &= \boldsymbol{\nu}\times\boldsymbol{e}_{q,E}^\infty - \big(\boldsymbol{\nu}\times\boldsymbol{e}_{q,M}^\infty + \boldsymbol{\nu}\times\boldsymbol{e}_{0,M}^\infty + \boldsymbol{\nu}\times\boldsymbol{e}_q^\infty\big)\,, \end{aligned}$$

where $\boldsymbol{e}_q^\infty$ is the far field of a radiating solution $\boldsymbol{e}_q^s \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ to

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e}_q^s - k^2\varepsilon_r\boldsymbol{e}_q^s = k^2(1-\varepsilon_r)\boldsymbol{e}^i \qquad \text{in } \mathbb{R}^3$$

for some entire solution $\boldsymbol{e}^i \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ of

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e}^i - k^2\boldsymbol{e}^i = 0 \qquad \text{in } \mathbb{R}^3\,.$$

Since $\mathrm{supp}(q) \subseteq \overline{E}\cup\overline{M}$ and $\mathbb{R}^3 \setminus (\overline{E}\cup\overline{M})$ is connected, Rellich's lemma and unique continuation guarantee that

$$\boldsymbol{e}_{q,E} - (\boldsymbol{e}_{q,M} + \boldsymbol{e}_{0,M} + \boldsymbol{e}_q^s) = 0 \qquad \text{in } \mathbb{R}^3 \setminus (\overline{E}\cup\overline{M})$$

(cf., e.g., [10, Thm. 6.10]).

Next we discuss the regularity of the traces of $\boldsymbol{\nu}\times\boldsymbol{e}_{q,E}|_\Gamma = \boldsymbol{\nu}\times(\boldsymbol{e}_{q,M}+\boldsymbol{e}_{0,M}+\boldsymbol{e}_q^s)|_\Gamma$ at the boundary segment $\Gamma \subseteq \partial E\setminus\overline{M}$. W.l.o.g. we may assume that Γ is bounded away from $\overline{M}$. Since $\mathrm{supp}(\boldsymbol{f}_{q,M}+\boldsymbol{f}_{0,M}) \subseteq \overline{M}$, regularity results for time-harmonic Maxwell's equations from [50] show that any point $\boldsymbol{x}\in\Gamma$ has an open neighborhood $U\subseteq\mathbb{R}^3$ such that $(\boldsymbol{e}_{q,M}+\boldsymbol{e}_{0,M}+\boldsymbol{e}_q)|_{E\cap U} \in H^2(E\cap U, \mathbb{C}^3)$ and $(\boldsymbol{e}_{q,M}+\boldsymbol{e}_{0,M}+\boldsymbol{e}_q)|_{U\setminus\overline{E}} \in H^2(U\setminus\overline{E}, \mathbb{C}^3)$, where $\boldsymbol{e}_q = \boldsymbol{e}^i + \boldsymbol{e}_q^s$. Accordingly, applying the trace operator on $H^2(U\setminus\overline{E}, \mathbb{C}^3)$ and taking the cross product with $\boldsymbol{\nu}\in C^{1,1}(\Gamma,\mathbb{R}^3)$, we find that

$$\boldsymbol{\nu}\times(\boldsymbol{e}_{q,M}+\boldsymbol{e}_{0,M}+\boldsymbol{e}_q)|_\Gamma \in H_t^{\frac{3}{2}}(\Gamma\cap U, \mathbb{C}^3)$$

(see [23, p. 21])

Since $\boldsymbol{x}\in\Gamma$ was arbitrary and $\boldsymbol{e}^i$ is smooth this implies that

$$\boldsymbol{\nu}\times\boldsymbol{e}_{q,E}|_\Gamma = \boldsymbol{\nu}\times(\boldsymbol{e}_{q,M}+\boldsymbol{e}_{0,M}+\boldsymbol{e}_q^s)|_\Gamma \in H_t^{\frac{3}{2}}(\Gamma, \mathbb{C}^3)$$

To prove the lemma, we will construct a sufficiently large class of sources $\boldsymbol{f}\in H(\mathbf{curl}; E)^*$ such that $L_{q,E}^*\boldsymbol{f} \notin \mathcal{R}\big(\big[L_{q,M}^* \,\big|\, L_{0,M}^*\big]\big)$. Let $\boldsymbol{g}\in H^{-\frac{1}{2}}(\mathrm{Div}; \partial E)$ such that $\mathrm{supp}(\boldsymbol{g})\subseteq\Gamma$. Accordingly, let $\boldsymbol{U}^+ \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3\setminus\overline{E})$ be the radiating solution to the exterior boundary problem

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{U}^+ - k^2\varepsilon_r\boldsymbol{U}^+ = 0 \quad \text{in } \mathbb{R}^3\setminus\overline{E}\,, \qquad \boldsymbol{\nu}\times\boldsymbol{U}^+ = \boldsymbol{g} \quad \text{on } \partial E\,, \tag{7.1}$$

(see, e.g., [39, Thm. 5.64]). Similarly, we define $\boldsymbol{U}^- \in H(\mathbf{curl}; E)$ as the solution to the interior boundary value problem

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{U}^- - k^2(\varepsilon_r + \mathrm{i})\boldsymbol{U}^- \;=\; 0 \quad \text{in } E\,, \qquad \boldsymbol{\nu} \times \boldsymbol{U}^- \;=\; \boldsymbol{g} \quad \text{on } \partial E\,, \tag{7.2}$$

(see, e.g., [39, Thm. 4.41]). Therewith we define $\boldsymbol{U} \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ by

$$\boldsymbol{U} \;:=\; \begin{cases} \boldsymbol{U}^- & \text{in } E\,, \\ \boldsymbol{U}^+ & \text{in } \mathbb{R}^3 \setminus \overline{E}\,, \end{cases}$$

and $\boldsymbol{f} \in H(\mathbf{curl}; E)^*$ by

$$\boldsymbol{f} \;:=\; \frac{1}{\mathrm{i}\omega\varepsilon}\Big(\mathrm{i}k^2\boldsymbol{U}^- - \pi_t^*\big(\boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{U}\big|^+_{\partial E} - \boldsymbol{\nu} \times \mathbf{curl}\,\boldsymbol{U}\big|^-_{\partial E}\big)\Big)\,,$$

where $\pi_t^* : H^{-1/2}(\mathrm{Div}; \partial E) \to H(\mathbf{curl}; E)^*$ denotes the adjoint of the interior tangential trace operator $\pi_t : H(\mathbf{curl}; E) \to H^{-1/2}(\mathrm{Curl}; \partial E)$ with $\pi_t(\boldsymbol{V}) = (\boldsymbol{\nu} \times \boldsymbol{V}|_{\partial E}) \times \boldsymbol{\nu}$. Then $\boldsymbol{U} \in H_{\mathrm{loc}}(\mathbf{curl}; \mathbb{R}^3)$ (see, e.g., [44, Lem. 5.3]), and the weak formulations of (7.1) and (7.2) show that

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{U} - k^2\varepsilon_r\boldsymbol{U} \;=\; \mathrm{i}\omega\varepsilon\boldsymbol{f} \quad \text{in } \mathbb{R}^3\,.$$

Accordingly, $L^*_{q,E}\boldsymbol{f} = \sqrt{\mu_0/\varepsilon_0}\,\mathcal{S}_q^*(\boldsymbol{\nu} \times \boldsymbol{U}^\infty)$, where $\boldsymbol{U}^\infty \in L^2_t(S^2, \mathbb{C}^3)$ coincides with the far field of the radiating solution $\boldsymbol{U}^+$ to the exterior boundary value problem (7.1). If $\boldsymbol{g} \notin H_t^{\frac{3}{2}}(\partial E, \mathbb{C}^3)$, then our regularity considerations from above show that $L^*_{q,E} f \notin \mathcal{R}\big([L^*_{q,M} \,|\, L^*_{0,M}]\big)$.

Now let

$$X \subseteq \big\{\boldsymbol{g} \in H^{-\frac{1}{2}}(\mathrm{Div}; \partial E) \;\big|\; \mathrm{supp}(\boldsymbol{g}) \subseteq \Gamma\big\}$$

be an infinite dimensional subspace of $H^{-1/2}(\mathrm{Div}; \partial E)$ such that $X \cap H_t^{\frac{3}{2}}(\partial E, \mathbb{C}^3) = \{0\}$. Let $G_E : H^{-\frac{1}{2}}(\mathrm{Div}; \partial E) \to L^2_t(S^2, \mathbb{C}^3)$ be the operator that maps $\boldsymbol{g} \in H^{-\frac{1}{2}}(\mathrm{Div}; \partial E)$ to the far field pattern of the radiating solution $\boldsymbol{U}^+$ of the exterior boundary value problem (7.1). Then Rellich's lemma and unique continuation show that G_E is one-to-one, and thus

$$Z \;:=\; \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\mathcal{S}_q^* G_E(X) \subseteq L^2_t(S^2, \mathbb{C}^3)$$

is an infinite dimensional subspace as well. Furthermore, we have just shown that

$$Z \subseteq \mathcal{R}(L^*_{q,E}) \qquad \text{and} \qquad Z \cap \mathcal{R}\big([L^*_{q,M} \,|\, L^*_{0,M}]\big) \;=\; \{0\}\,.$$

□

Now we give the proof of Theorem 7.1.

Proof of Theorem 7.1. Let $V \subseteq L^2_t(S^2, \mathbb{C}^3)$ be a finite dimensional subspace. We denote by $P_V : L^2_t(S^2, \mathbb{C}^3) \to L^2_t(S^2, \mathbb{C}^3)$ the orthogonal projection on V. Combining Lemma 7.3 with a simple dimensionality argument (see [31, Lem. 4.7]) shows that

$$Z \not\subseteq \mathcal{R}\big([L^*_{q,M} \,|\, L^*_{0,M}]\big) + V \;=\; \mathcal{R}([L^*_{q,M} \,|\, L^*_{0,M} | P_V])\,,$$

where $Z \subseteq \mathcal{R}(L^*_{q,E})$ denotes the subspace in Lemma 7.3. Thus,

$$\mathcal{R}(L^*_{q,E}) \not\subseteq \mathcal{R}([L^*_{q,M} \,|\, L^*_{0,M} | P_V])\,,$$

and accordingly Lemma 5.4 implies that there is no constant $C > 0$ such that

$$\begin{aligned}\|L_{q,E}\boldsymbol{p}\|^2_{L^2(E)} &\leq C^2 \left\| \begin{bmatrix} L_{q,M} \\ L_{0,M} \\ P_V \end{bmatrix} \boldsymbol{p} \right\|^2_{L^2(M)\times L^2(M)\times L^2_t(S^2,\mathbb{C}^3)} \\ &= C^2 \big(\|L_{q,M}\boldsymbol{p}\|^2_{L^2(M)} + \|L_{0,M}\boldsymbol{p}\|^2_{L^2(M)} + \|P_V\boldsymbol{p}\|^2_{L^2_t(S^2,\mathbb{C}^3)} \big)\end{aligned}$$

for all $\boldsymbol{p} \in L^2_t(S^2,\mathbb{C}^3)$. Hence, there exists as sequence $(\widetilde{\boldsymbol{p}}_m)_{m\in\mathbb{N}} \subseteq L^2_t(S^2,\mathbb{C}^3)$ such that

$$\begin{aligned} \|L_{q,E}\widetilde{\boldsymbol{p}}_m\|_{L^2(E)} &\to \infty && \text{as } m \to \infty\,, \\ \|L_{q,M}\widetilde{\boldsymbol{p}}_m\|_{L^2(M)} + \|L_{0,M}\widetilde{\boldsymbol{p}}_m\|_{L^2(M)} + \|P_V\widetilde{\boldsymbol{p}}_m\|_{L^2_t(S^2,\mathbb{C}^3)} &\to 0 && \text{as } m \to \infty\,. \end{aligned}$$

Setting $\boldsymbol{p}_m := \widetilde{\boldsymbol{p}}_m - P_V\widetilde{\boldsymbol{p}}_m \in V^\perp \subseteq L^2_t(S^2,\mathbb{C}^3)$ for any $m \in \mathbb{N}$, we finally obtain

$$\|L_{q,E}\boldsymbol{p}_m\|_{L^2(E)} \geq \|L_{q,E}\widetilde{\boldsymbol{p}}_m\|_{L^2(E)} - \|L_{q,E}\|\|P_V\widetilde{\boldsymbol{p}}_m\|_{L^2_t(S^2,\mathbb{C}^3)} \to \infty \qquad \text{as } m \to \infty\,,$$

and

$$\begin{aligned}\|L_{q,M}\boldsymbol{p}_m\|_{L^2(M)} + \|L_{0,M}\boldsymbol{p}_m\|_{L^2(M)} &\leq \|L_{q,M}\widetilde{\boldsymbol{p}}_m\|_{L^2(M)} + \|L_{0,M}\widetilde{\boldsymbol{p}}_m\|_{L^2(M)} \\ &\quad + (\|L_{q,M}\| + \|L_{0,M}\|)\|P_V\widetilde{\boldsymbol{p}}_m\|_{L^2_t(S^2,\mathbb{C}^3)} \to 0 \qquad \text{as } m \to \infty\,.\end{aligned}$$

Since $L_{q,E}\boldsymbol{p}_m = \mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}_m}|_E$, $L_{q,M}\boldsymbol{p}_m = \mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}_m}|_M$, and $L_{0,M}\boldsymbol{p}_m = \mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}_m}|_M$, this ends the proof. □

8 Shape reconstruction for indefinite scatterers

We consider the general case when the constrast function q is neither strictly positive nor strictly negative a.e. on the support D of the scatterer. While the criteria developed in Theorems 6.1 and 6.2 determine whether a certain probing domain B is contained in the support D of the scattering object or not, the criterion in Theorem 8.1 characterizes whether a certain probing domain B contains the support D of the scatterer or not.

Theorem 8.1. *Let $D \subseteq \mathbb{R}^3$ be open and bounded such that ∂D is piecewise $C^{2,1}$, and $\mathbb{R}^3 \setminus \overline{D}$ is connected. Let $q \in \mathcal{Y}_D$ with $q|_D \in C^1(\overline{D})$, and suppose that $-\infty < q_{\min} \leq q \leq q_{\max} < 1$ a.e. on D for some constants $q_{\min}, q_{\max} \in \mathbb{R}$. Furthermore, we assume that for any point $\boldsymbol{x} \in \partial D$ on the boundary of D, and for any neighborhood $U \subseteq D$ of $\boldsymbol{x}$ in D, there exists a connected unbounded domain $O \subseteq \mathbb{R}^3$ with $\emptyset \neq E := O \cap D \subseteq U$ such that*

$$q|_E \geq q_{E,\min} > 0 \qquad \text{or} \qquad q|_E \leq q_{E,\max} < 0 \tag{8.1}$$

for some constants $q_{E,\min}, q_{E,\max} \in \mathbb{R}$.

Let $B \subseteq \mathbb{R}^3$ be open such that $\mathbb{R}^3 \setminus \overline{B}$ is connected.

(a) If $D \subseteq B$, then there exists a constant $C > 0$ such that

$$\alpha T_B \leq_{\text{fin}} \mathrm{Re}(F_q) \leq_{\text{fin}} \beta T_B \quad \text{for all } \alpha \leq \min\{0, q_{\min}\}\,,\ \beta \geq \max\{0, Cq_{\max}\}\,. \tag{8.2}$$

(b) If $D \not\subseteq B$, then

$$\alpha T_B \not\leq_{\text{fin}} \mathrm{Re}(F_q) \quad \text{for any } \alpha \in \mathbb{R} \quad \text{or} \quad \mathrm{Re}(F_q) \not\leq_{\text{fin}} \beta T_B \quad \text{for any } \beta \in \mathbb{R}\,. \tag{8.3}$$

Proof of Theorem 8.1. Let $D \subseteq B$. Using Corollary 4.4 and Theorem 5.5 with $q_1 = 0$ and $q_2 = q$ we find that there exists a constant $C > 0$ and a finite dimensional subspace $V_1 \subseteq L^2_t(S^2, \mathbb{C}^3)$ such that, for all $\boldsymbol{p} \in V_1^\perp$ and any $\beta \geq \max\{0, Cq_{\max}\}$,

$$\begin{aligned}\operatorname{Re}\Big(\int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s\Big) \,&\leq\, \int_D q|\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \,\leq\, q_{\max}\int_D |\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x}\\ &\leq\, Cq_{\max}\int_D |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \,\leq\, \beta\int_B |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x}\,.\end{aligned}$$

On the other hand, Theorem 4.2 with $q_1 = 0$ and $q_2 = q$ gives a finite dimensional subspace $V_2 \subseteq L^2_t(S^2, \mathbb{C}^3)$ such that, for all $\boldsymbol{p} \in V_2^\perp$ and any $\alpha \leq \min\{0, q_{\min}\}$,

$$\operatorname{Re}\Big(\int_{S^2} \boldsymbol{p} \cdot \overline{F_q \boldsymbol{p}} \,\mathrm{d}s\Big) \,\geq\, \int_D q|\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \,\geq\, q_{\min}\int_D |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} \,\geq\, \alpha\int_B |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x}\,.$$

Thus, part (a) is proven.

Part (b) is shown by contradiction. Let $D \not\subseteq B$, then $U := D \backslash B$ is not empty. By assumption there exists a point $\boldsymbol{x} \in \overline{U} \cap \partial D$ and a connected unbounded open neighborhood $O \subseteq \mathbb{R}^3$ of $\boldsymbol{x}$ with $O \cap D \subseteq U$ and $O \cap B = \emptyset$, such that (8.1) is satisfied with $E := O \cap D$. Let $R > 0$ be large enough such that $B, D \subseteq B_R(0)$. Without loss of generality we suppose that $O \cap B_R(0)$, and $B_R(0) \setminus \overline{O}$ are connected.

If $q|_E \geq q_{E,\min} > 0$ we assume that $\operatorname{Re}(F_q) \leq_{\text{fin}} \beta T_B$ for some $\beta \in \mathbb{R}$. Applying the monotonicity relation (4.1) in Theorem 4.2 with $q_1 = 0$ and $q_2 = q$, we find that there exists a finite dimensional subspace $V_3 \subseteq L^2_t(S^2, \mathbb{C}^3)$ such that, for any $\boldsymbol{p} \in V_3^\perp$,

$$\begin{aligned}0 \,&\geq\, \int_{S^2} \boldsymbol{p} \cdot \big(\overline{\operatorname{Re}(F_q)\boldsymbol{p} - \beta T_B \boldsymbol{p}}\big) \,\mathrm{d}s \,\geq\, \int_{B_R(0)} (q - \beta\chi_B)|\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x}\\ &=\, \int_{B_R(0)\setminus\overline{O}} (q - \beta\chi_B)|\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} + \int_{B_R(0)\cap O} (q - \beta\chi_B)|\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x}\\ &\geq\, -(\|q\|_{L^\infty(\mathbb{R}^3)} + |\beta|)\int_{B_R(0)\setminus\overline{O}} |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} + q_{E,\min}\int_E |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x}\,.\end{aligned}$$

However, this contradicts Theorem 5.1 with $B = E$, $\Omega = B_R(0) \setminus \overline{O}$, and $q = 0$, which yields a sequence $(\boldsymbol{p}_m)_{m\in\mathbb{N}} \subseteq V_3^\perp$ with

$$\int_E |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to \infty \qquad \text{and} \qquad \int_{B_R(0)\setminus\overline{O}} |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to 0 \qquad \text{as } m \to \infty\,.$$

Thus, $\operatorname{Re}(F_q) \not\leq_{\text{fin}} \beta T_B$ for all $\beta \in \mathbb{R}$.

Now assume that $q|_E \leq q_{E,\max} < 0$, and that $\alpha T_B \leq_{\text{fin}} \operatorname{Re}(F_q)$ for some $\alpha \in \mathbb{R}$. Then the monotonicity relation (4.3) in Corollary 4.4 with $q_1 = 0$ and $q_2 = q$ shows that there exists a finite dimensional subspace $V_4 \subseteq L^2_t(S^2, \mathbb{C}^3)$ such that, for any $\boldsymbol{p} \in V_4^\perp$,

$$\begin{aligned}0 \,&\leq\, \int_{S^2} \boldsymbol{p} \cdot \big(\overline{\operatorname{Re}(F_q)\boldsymbol{p} - \alpha T_B \boldsymbol{p}}\big) \,\mathrm{d}s \,\leq\, \int_{B_R(0)} \big(q|\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}|^2 - \alpha\chi_B|\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2\big) \,\mathrm{d}\boldsymbol{x}\\ &=\, \int_{B_R(0)\setminus\overline{O}} \big(q|\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}|^2 - \alpha\chi_B|\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2\big) \,\mathrm{d}\boldsymbol{x}\\ &\quad+ \int_{B_R(0)\cap O} \big(q|\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}|^2 - \alpha\chi_B|\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2\big) \,\mathrm{d}\boldsymbol{x}\\ &\leq\, q_{\max}\int_{B_R(0)\setminus\overline{O}} |\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} + |\alpha|\int_{B_R(0)\setminus\overline{O}} |\,\mathbf{curl}\,\boldsymbol{H}^i_{\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x} + q_{E,\max}\int_E |\,\mathbf{curl}\,\boldsymbol{H}_{q,\boldsymbol{p}}|^2 \,\mathrm{d}\boldsymbol{x}\,.\end{aligned}$$

Let $M := B_R(0)\setminus\overline{O}$. Since ∂D is piecewise $C^{2,1}$ smooth, there is a connected subset $\Gamma \subseteq \partial E\setminus\overline{M}$ that is relatively open and $C^{2,1}$ smooth. Using Theorem 7.1 we obtain a sequence $(\boldsymbol{p}_m)_{m\in\mathbb{N}} \subseteq V_4^\perp$ such that

$$\int_E |\operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}_m}|^2 \,\mathrm{d}\boldsymbol{x} \to \infty \qquad \text{and} \qquad \int_{B_R(0)\setminus\overline{O}} \big(|\operatorname{\mathbf{curl}} \boldsymbol{H}_{q,\boldsymbol{p}_m}|^2 + |\operatorname{\mathbf{curl}} \boldsymbol{H}^i_{\boldsymbol{p}_m}|^2\big) \,\mathrm{d}\boldsymbol{x} \to 0$$

as $m \to \infty$. However, since $q_{E,\max} < 0$ this gives a contradiction. Therefore, $\alpha T_B \not\leq_{\text{fin}} \operatorname{Re}(F_q)$ for all $\alpha \in \mathbb{R}$, and this ends the proof of part (b). $\square$

The following corollary is an immediate consequence of the proof of Theorem 8.1. We consider the special case of an indefinite scattering object $D = D_1 \cup D_2$ with contrast function q such that $q_1 := q|_{D_1}$ is strictly positive on D_1 while $q_2 := q|_{D_2}$ is strictly negative on D_2. The result characterizes whether a certain probing domain B contains the postive part D_1 of the scatterer or its negative part D_2 or none of them.

Corollary 8.2. *Let $D = D_1 \cup D_2 \subseteq \mathbb{R}^3$ be open and bounded such that $\overline{D_1} \cap \overline{D_2} = \emptyset$, ∂D is piecewise $C^{2,1}$, and $\mathbb{R}^3 \setminus \overline{D}$ is connected. Let $q \in \mathcal{Y}_D$ with $q_j := q|_{D_j} \in C^1(\overline{D_j})$, $j = 1, 2$, and suppose that*

$$\begin{aligned} 0 \,<\, q_{1,\min} \,\leq\, q_1 \,\leq\, q_{1,\max} < 1 \qquad\qquad & \text{a.e. on } D_1\,,\\ -\infty \,<\, q_{2,\min} \,\leq\, q_2 \,\leq\, q_{2,\max} \,<\, 0 \qquad\qquad & \text{a.e. on } D_2\,, \end{aligned}$$

for some constants $q_{1,\min}, q_{1,\max}, q_{2,\min}, q_{2,\max} \in \mathbb{R}$.

Let $B \subseteq \mathbb{R}^3$ be open such that $\mathbb{R}^3 \setminus \overline{B}$ is connected.

(a) If $D_1 \subseteq B$, then there exists a constant $C > 0$ such that

$$\operatorname{Re}(F_q) \,\leq_{\text{fin}}\, \alpha T_B \quad \textit{for all } \alpha \geq C q_{1,\max}\,.$$

(b) If $D_1 \not\subseteq B$, then

$$\operatorname{Re}(F_q) \,\not\leq_{\text{fin}}\, \alpha T_B \quad \textit{for any } \alpha \in \mathbb{R}\,.$$

(c) If $D_2 \subseteq B$, then

$$\operatorname{Re}(F_q) \,\geq_{\text{fin}}\, \alpha T_B \quad \textit{for all } \alpha \leq q_{2,\min}\,.$$

(d) If $D_2 \not\subseteq B$, then

$$\operatorname{Re}(F_q) \,\not\geq_{\text{fin}}\, \alpha T_B \quad \textit{for any } \alpha \in \mathbb{R}\,.$$

At the end of the next section we will comment on a sampling strategy that implements the criteria in Corollary 8.2 to geometrically separate positive and negative components of mixed scattering configurations. Using techniques from [21, 22, 45] this information could be used to obtain a full shape reconstruction of the unknown scatterers. A stable numerical implementation of the monotonicity based shape characterization for the general indefinite case from Theorem 8.1 seems to require a better understanding of the dimensions of the finite dimensional subspaces that are excluded in the monotonicity relations in (8.2)–(8.3).

9 Numerical examples

We discuss numerical examples for the shape characterizations developed in Sections 6 and 8. The main issue here is that numerical approximations of the operators F_q and T_B are necessarily finite dimensional. Accordingly, the question, whether suitable combinations of these operators are positive or negative definite up to some finite dimensional subspace (see Theorems 6.1, 6.2, and 8.1) needs to be carefully relaxed.

9.1 An explicit radially symmetric example

To illustrate the results from Theorems 6.1, 6.2, and 8.1 we consider the special case when the scatterer D and the probing domain B are concentric balls.

Let $D = B_{r_D}(0)$ be a ball of radius $r_D > 0$ centered at the origin with constant electric permittivity contrast $q < 1$, i.e., the relative electric permittivity is $\varepsilon_r^{-1} = 1 - q > 0$. We derive series expansions for the incident magnetic field and for the corresponding magnetic far field pattern to obtain explicit formulas for the eigenvalue decomposition of the magnetic far field operator F_q from (3.8).

Let Y_n^m, $m = -n, \dots, n$, $n \in \mathbb{N}$, denote a complete orthonormal system of spherical harmonics of order n in $L^2(S^2)$. Then, the *vector spherical harmonics*

$$\boldsymbol{U}_n^m(\boldsymbol{\theta}) := \frac{1}{\sqrt{n(n+1)}} \,\mathbf{Grad}_{S^2} Y_n^m(\boldsymbol{\theta}), \qquad \boldsymbol{V}_n^m(\boldsymbol{\theta}) := \boldsymbol{\theta} \times \boldsymbol{U}_n^m(\boldsymbol{\theta}), \qquad \boldsymbol{\theta} \in S^2,$$

for $m = -n, \dots, n$, $n = 1, 2, \dots$, form a complete orthonormal system in $L_t^2(S^2, \mathbb{C}^3)$. Accordingly, we define the *spherical vector wave functions*

$$\boldsymbol{M}_n^m(\boldsymbol{x}) := -j_n(k|\boldsymbol{x}|)\boldsymbol{V}_n^m(\widehat{\boldsymbol{x}}), \qquad \boldsymbol{N}_n^m(\boldsymbol{x}) := -h_n^{(1)}(k|\boldsymbol{x}|)\boldsymbol{V}_n^m(\widehat{\boldsymbol{x}}), \qquad \boldsymbol{x} \in \mathbb{R}^3, \tag{9.1}$$

for $m = -n, \dots, n$, $n = 1, 2, \dots$, where j_n and $h_n^{(1)}$ denote the spherical Bessel and Hankel function of degree n. We note that the normalization factors used in (9.1) differ from what is used elsewhere in the literature (see, e.g., [10, Sec. 6.5]).

Given a tangential vector field

$$\boldsymbol{p} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \big(a_n^m \boldsymbol{U}_n^m + b_n^m \boldsymbol{V}_n^m\big) \in L_t^2(S^2, \mathbb{C}^3), \tag{9.2}$$

we obtain from (3.10) and [10, Thm. 6.29] that

$$\boldsymbol{H}_{\boldsymbol{p}}^i(\boldsymbol{x}) = \frac{4\pi\, \mathrm{i}^{n-1}}{k} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \big(a_n^m \,\mathbf{curl}\, \boldsymbol{M}_n^m(\boldsymbol{x}) - \mathrm{i}k\, b_n^m \boldsymbol{M}_n^m(\boldsymbol{x})\big), \qquad \boldsymbol{x} \in \mathbb{R}^3.$$

Applying separation of variables a short computation shows that the corresponding scattered magnetic field outside the support of the scatterer is given by

$$\boldsymbol{H}_{q,\boldsymbol{p}}^s(\boldsymbol{x}) = \frac{4\pi\, \mathrm{i}^{n-1}}{k} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \big(c_n^m \,\mathbf{curl}\, \boldsymbol{N}_n^m(\boldsymbol{x}) - \mathrm{i}k\, d_n^m \boldsymbol{N}_n^m(\boldsymbol{x})\big), \qquad \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{D},$$

with

$$c_n^m = a_n^m \frac{(\kappa r_D) j_n(k r_D) j_n'(\kappa r_D) - (k r_D) j_n(\kappa r_D) j_n'(k r_D)}{(k r_D) j_n(\kappa r_D)(h_n^{(1)})'(k r_D) - (\kappa r_D) h_n^{(1)}(k r_D) j_n'(\kappa r_D)},$$

$$d_n^m = b_n^m \frac{\varepsilon_r^{-1} j_n(k r_D)\big(j_n(\kappa r_D) + (\kappa r_D) j_n'(\kappa r_D)\big) - j_n(\kappa r_D)\big(j_n(k r_D) + (k r_D) j_n'(k r_D)\big)}{(k r_D) j_n(\kappa r_D)(h_n^{(1)})'(k r_D) - (\kappa r_D)\varepsilon_r^{-1} j_n'(\kappa r_D) h_n^{(1)}(k r_D) + q h_n^{(1)}(k r_D) j_n(\kappa r_D)},$$

and $\kappa := k\sqrt{\varepsilon_r}$. Recalling that the far field patterns of the spherical vector wave functions $\boldsymbol{N}_n^m$ and $\mathbf{curl}\, \boldsymbol{N}_n^m$ are given by

$$(\boldsymbol{N}_n^m)^\infty(\widehat{\boldsymbol{x}}) = -\frac{4\pi\, (-\mathrm{i})^{n+1}}{k} \boldsymbol{V}_n^m(\widehat{\boldsymbol{x}}), \qquad (\mathbf{curl}\, \boldsymbol{N}_n^m)^\infty(\widehat{\boldsymbol{x}}) = 4\pi\, (-\mathrm{i})^n \boldsymbol{U}_n^m(\widehat{\boldsymbol{x}}), \qquad \widehat{\boldsymbol{x}} \in S^2,$$

for $m=-n,\dots,n$, $n=1,2,\dots$ (see, e.g., [10, Thm. 6.28]), we find that

$$\boldsymbol{H}_{q,\boldsymbol{p}}^\infty(\widehat{\boldsymbol{x}}) \;=\; \frac{(4\pi)^2}{\mathrm{i}k}\sum_{n=1}^\infty\sum_{m=-n}^n \big(c_n^m \boldsymbol{U}_n^m(\widehat{\boldsymbol{x}}) + d_n^m \boldsymbol{V}_n^m(\widehat{\boldsymbol{x}})\big)\,, \qquad \widehat{\boldsymbol{x}}\in S^2\,.$$

Accordingly, the eigenvalues and eigenvectors of the magnetic far field operator F_q are given by $(\lambda_n^{(j)}, \boldsymbol{v}_{m,n}^{(j)})$, $n\geq 1$, $-n\leq m\leq n$, $j=s,t$ with

$$\lambda_n^{(s)} = \frac{(4\pi)^2}{\mathrm{i}k}\,\frac{(\kappa r_D)j_n(kr_D)j_n'(\kappa r_D)-(kr_D)j_n(\kappa r_D)j_n'(kr_D)}{(kr_D)j_n(\kappa r_D)(h_n^{(1)})'(kr_D)-(\kappa r_D)h_n^{(1)}(kr_D)j_n'(\kappa r_D)}, \tag{9.3a}$$

$$\lambda_n^{(t)} = \frac{(4\pi)^2}{\mathrm{i}k}\,\frac{\varepsilon_r^{-1}j_n(kr_D)\big(j_n(\kappa r_D)+(\kappa r_D)j_n'(\kappa r_D)\big)-j_n(\kappa r_D)\big(j_n(kr_D)+(kr_D)j_n'(kr_D)\big)}{(kr_D)j_n(\kappa r_D)(h_n^{(1)})'(kr_D)-(\kappa r_D)\varepsilon_r^{-1}j_n'(\kappa r_D)h_n^{(1)}(kr_D)+qh_n^{(1)}(kr_D)j_n(\kappa r_D)}, \tag{9.3b}$$

and

$$\boldsymbol{v}_{m,n}^{(s)}(\widehat{\boldsymbol{x}}) \;=\; \boldsymbol{U}_n^m(\widehat{\boldsymbol{x}})\,, \qquad \boldsymbol{v}_{m,n}^{(t)}(\widehat{\boldsymbol{x}}) \;=\; \boldsymbol{V}_n^m(\widehat{\boldsymbol{x}})\,, \qquad \widehat{\boldsymbol{x}}\in S^2\,. \tag{9.3c}$$

Similarly, we consider for the test domain $B=B_{r_B}(0)$ a ball of radius $r_B>0$ centered at the origin. Then the probing operator $T_B: L_t^2(S^2,\mathbb{C}^3)\to L_t^2(S^2,\mathbb{C}^3)$ from (6.1) satisfies

$$\begin{aligned}(T_B\boldsymbol{p})(\widehat{\boldsymbol{x}}) \;&=\; k^2\Big(\int_{S^2}\Big(\int_{B_{r_B}(0)} e^{\mathrm{i}k\boldsymbol{y}\cdot(\boldsymbol{\theta}-\widehat{\boldsymbol{x}})}\,\mathrm{d}\boldsymbol{y}\Big)\big(\boldsymbol{\theta}\times\boldsymbol{p}(\boldsymbol{\theta})\big)\,\mathrm{d}s(\boldsymbol{\theta})\Big)\times\widehat{\boldsymbol{x}}\\ &=\; k^2\Big(\int_{S^2}\Big(\int_0^{r_B} 4\pi\rho^2 j_0(k\rho|\boldsymbol{\theta}-\widehat{\boldsymbol{x}}|)\,\mathrm{d}\rho\Big)\big(\boldsymbol{\theta}\times\boldsymbol{p}(\boldsymbol{\theta})\big)\,\mathrm{d}s(\boldsymbol{\theta})\Big)\times\widehat{\boldsymbol{x}}\,, \qquad \widehat{\boldsymbol{x}}\in S^2\,.\end{aligned} \tag{9.4}$$

Here we used the integral representation of j_0 (see, e.g., [10, (2.45)]. Substituting the vector spherical harmonics expansion (9.2) into (9.4) we find that

$$\begin{aligned}(T_B\boldsymbol{p})(\widehat{\boldsymbol{x}}) \;&=\; 4\pi k^2\sum_{n=1}^\infty\sum_{m=-n}^n\Big(a_n^m\int_0^{r_B}\int_{S^2} j_0(k\rho|\boldsymbol{\theta}-\widehat{\boldsymbol{x}}|)\boldsymbol{V}_n^m(\boldsymbol{\theta})\,\mathrm{d}s(\boldsymbol{\theta})\,\rho^2\,\mathrm{d}\rho\\ &\qquad\qquad - b_n^m\int_0^{r_B}\int_{S^2} j_0(k\rho|\boldsymbol{\theta}-\widehat{\boldsymbol{x}}|)\boldsymbol{U}_n^m(\boldsymbol{\theta})\,\mathrm{d}s(\boldsymbol{\theta})\,\rho^2\,\mathrm{d}\rho\Big)\times\widehat{\boldsymbol{x}}\\ &= \sum_{n=1}^\infty\sum_{m=-n}^n\Big(a_n^m\Big((4\pi k)^2\int_0^{r_B} j_n^2(k\rho)\rho^2\,\mathrm{d}\rho\Big)\boldsymbol{U}_n^m(\widehat{\boldsymbol{x}})\\ &\qquad + b_n^m\Big((4\pi)^2\int_0^{r_B}\Big(\big(j_n(k\rho)+k\rho j_n'(k\rho)\big)^2+n(n+1)j_n^2(k\rho)\Big)\,\mathrm{d}\rho\Big)\boldsymbol{V}_n^m(\widehat{\boldsymbol{x}})\Big)\end{aligned} \tag{9.5}$$

(see, e.g., [10, Thm. 6.29]). Accordingly, the eigenvalues and eigenvectors of the probing operator T_B are given by $(\mu_n^{(j)}, \boldsymbol{v}_{m,n}^{(j)})$, $n\geq 1$, $-n\leq m\leq n$, $j=s,t$ with

$$\mu_n^{(s)} \;=\; \frac{(4\pi)^2}{k}\int_0^{kr_B} j_n^2(\rho)\rho^2\,\mathrm{d}\rho\,, \tag{9.6a}$$

$$\mu_n^{(t)} \;=\; \frac{(4\pi)^2}{k}\int_0^{kr_B}\big(n(n+1)j_n^2(\rho)+(j_n(\rho)+\rho j_n'(\rho))^2\big)\,\mathrm{d}\rho\,, \tag{9.6b}$$

and

$$\boldsymbol{v}_{m,n}^{(s)}(\widehat{\boldsymbol{x}}) \;=\; \boldsymbol{U}_n^m(\widehat{\boldsymbol{x}})\,, \qquad \boldsymbol{v}_{m,n}^{(t)}(\widehat{\boldsymbol{x}}) \;=\; \boldsymbol{V}_n^m(\widehat{\boldsymbol{x}})\,, \qquad \widehat{\boldsymbol{x}}\in S^2\,. \tag{9.6c}$$

Assuming that $0<q<1$, the criteria established in Theorem 6.1 and Theorem 8.1 show that

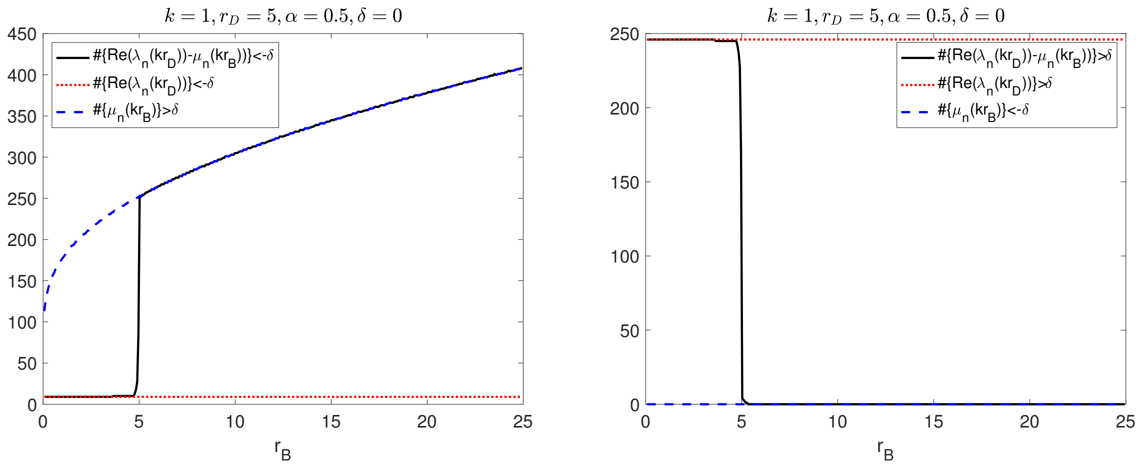

Figure 9.1: Number of positive eigenvalues (left) and number of negative eigenvalues (right) $\mathrm{Re}\big(\lambda_n(r_D)\big)$ (dotted), $\mu_n(r_B)$ (dashed), and $\mathrm{Re}\big(\lambda_n(r_D)\big)+\mu_n(r_B)$ (solid) within the range $n = 0, \ldots, 1000$ as function of r_B.

(a) if $r_B < r_D$, i.e., when $B \subseteq D$, then $0 \leq_{\mathrm{fin}} \mathrm{Re}(F_q) - \alpha T_B$ when $\alpha \leq q$ but $0 \not\geq_{\mathrm{fin}} \mathrm{Re}(F_q) - \alpha T_B$ for any $\alpha \in \mathbb{R}$. This means that $\mathrm{Re}(F_q) - \alpha T_B$ has infinitely many positive eigenvalues for any $\alpha \in \mathbb{R}$ but only finitely many negative eigenvalues when $\alpha \leq q$.

(b) if $r_B > r_D$, i.e., when $B \not\subseteq D$, then $\mathrm{Re}(F_q) - \alpha T_B \leq_{\mathrm{fin}} 0$ when $\alpha \geq Cq$ with $C > 0$ as in Theorem 5.5, but $0 \not\leq_{\mathrm{fin}} \mathrm{Re}(F_q) - \alpha T_B$ for any $\alpha \in \mathbb{R}$. This means that $\mathrm{Re}(F_q) - \alpha T_B$ has infinitely many negative eigenvalues for any $\alpha \in \mathbb{R}$ but only finitely many positive eigenvalues when $\alpha \geq Cq$.

A similar characterization for negative contrasts $-\infty < q < 0$ can be obtained from Theorems 6.2 and 8.1.

To illustrate this characterization, we choose $q = 0.5$, i.e., $\varepsilon_r = 2$, and we evaluate the eigenvalues $\mathrm{Re}\big(\lambda_n^{(j)}(r_D)\big)$, $\mu_n^{(j)}(r_B)$, and $\mathrm{Re}\big(\lambda_n^{(j)}(r_D)\big) - \alpha\mu_n^{(j)}(r_B)$, $j = s, t$, with wave number $k = 1$, radius of the obstacle $r = 5$, $\alpha = 0.5$, and $n = 1, \ldots, 1000$ for different values of the radius $r_B \in [0, 25]$ of the test domain B using (9.3) and (9.6). In Figure 9.1 we show plots of the number of negative eigenvalues (left plot) and of the number of postive eigenvalues (right plot) $\mathrm{Re}\big(\lambda_n^{(j)}(r_D)\big)$ (dotted), $\mu_n^{(j)}(r_B)$ (dashed), and $\mathrm{Re}\big(\lambda_n^{(j)}(r_D)\big) - \alpha\mu_n^{(j)}(r_D)$ (solid), $j = s, t$, within the range $n = 0, \ldots, 1000$ as a function of r_B.

As suggested by Theorems 6.1 and 8.1 there is a sharp transition in the behavior of the eigenvalues of $\mathrm{Re}(F_q) - \alpha T_B$ at $r_B = r_D = 5$, which could be used to estimate the value of r_D. In these plots the contribution of the operator $\mathrm{Re}(F_q)$ dominates in the superposition $\mathrm{Re}(F_q) - \alpha T_B$ as long as $r_B < r_D$ (i.e., when $B \subseteq D$), while the contribution of the operator αT_B dominates when $r_B > r_D$ (i.e., when $D \subseteq B$).

9.2 A sampling strategy for sign-definite scatterers

We discuss a numerical realization of the criteria established in Theorems 6.1 and 6.2. To discretize the magnetic far field operator F_q from (3.8) we use a truncated vector spherical harmonics expansion. Let $\boldsymbol{p} \in L^2_t(S^2, \mathbb{C}^3)$ as in (9.2), then applying F_q gives

$$F_q\boldsymbol{p} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \big(a_n^m F_q \boldsymbol{U}_n^m + b_n^m F_q \boldsymbol{V}_n^m\big) \in L^2_t(S^2, \mathbb{C}^3)\,. \tag{9.7}$$

Studying the singular value decomposition of the linear operator that maps current densities supported in the ball $B_R(0)$ of radius R around the origin to their radiated far field patterns, it has been observed in [20] that for a large class of practically relevant source distributions the radiated far field pattern is well approximated by a vector spherical harmonics expansion of order $N \gtrsim kR$. This study suggests to truncate the series in (9.7) at an index N that is at least slightly larger than the radius of the smallest ball around the origin that contains the scattering object. Accordingly, we use the matrix

$$\boldsymbol{F}_q := \begin{bmatrix} \langle F_q \boldsymbol{U}_n^m, \boldsymbol{U}_{n'}^{m'} \rangle_{L_t^2(S^2,\mathbb{C}^3)} & \langle F_q \boldsymbol{V}_n^m, \boldsymbol{U}_{n'}^{m'} \rangle_{L_t^2(S^2,\mathbb{C}^3)} \\ \langle F_q \boldsymbol{U}_n^m, \boldsymbol{V}_{n'}^{m'} \rangle_{L_t^2(S^2,\mathbb{C}^3)} & \langle F_q \boldsymbol{V}_n^m, \boldsymbol{V}_{n'}^{m'} \rangle_{L_t^2(S^2,\mathbb{C}^3)} \end{bmatrix} \in \mathbb{C}^{Q\times Q} \tag{9.8}$$

with $Q = 2N(N+2)$ as a discrete approximation of F_q.

Next, we consider an equidistant grid of sampling points

$$\triangle = \{\boldsymbol{z}_{ij\ell} = (ih, jh, \ell h) \mid -J \le i,j,\ell \le J\} \subseteq [-R,R]^3 \tag{9.9}$$

with step size $h = R/J$ in the *region of interest* $[-R,R]^3$. For each $\boldsymbol{z}_{ij\ell} \in \triangle$ we consider a probing operator $T_{B_{ij\ell}}$ as in (6.1), where the probing domain $B_{ij\ell} = B_h(\boldsymbol{z}_{ij\ell})$ is a ball of radius h centered at $\boldsymbol{z}_{ij\ell}$. This probing operator satisfies, for any $\boldsymbol{p} \in L_t^2(S^2,\mathbb{C}^3)$ and $\widehat{\boldsymbol{x}} \in S^2$,

$$\begin{aligned}(T_{B_{ij\ell}}\boldsymbol{p})(\widehat{\boldsymbol{x}}) &= k^2\Big(\int_{S^2} e^{\mathrm{i}k\boldsymbol{z}\cdot(\boldsymbol{\theta}-\widehat{\boldsymbol{x}})}\Big(\int_{B_h(0)} e^{\mathrm{i}k\boldsymbol{y}\cdot(\boldsymbol{\theta}-\widehat{\boldsymbol{x}})}\,\mathrm{d}\boldsymbol{y}\Big)\big(\boldsymbol{\theta}\times\boldsymbol{p}(\boldsymbol{\theta})\big)\,\mathrm{d}s(\boldsymbol{\theta})\Big)\times\widehat{\boldsymbol{x}}\\ &= e^{-\mathrm{i}k\boldsymbol{z}\cdot\widehat{\boldsymbol{x}}}\Big(T_{B_h(0)}\big(e^{\mathrm{i}k\boldsymbol{z}\cdot(\cdot)}\boldsymbol{p}\big)\Big)(\widehat{\boldsymbol{x}})\,.\end{aligned}$$

Combining this representation with the eigenvalue expansion of $T_{B_h(0)}$ that we have derived in the previous subsection (see (9.5) and (9.6)), we find that $T_{B_{ij\ell}}$ has the same eigenvalues $\mu_n^{(s)}, \mu_n^{(t)}$ as $T_{B_h(0)}$, but the corresponding eigenvectors for $T_{B_{ij\ell}}$ are

$$\widetilde{\boldsymbol{v}}_{m,n}^{(s)}(\widehat{\boldsymbol{x}}) = e^{-\mathrm{i}k\boldsymbol{z}\cdot\widehat{\boldsymbol{x}}}\boldsymbol{U}_n^m(\widehat{\boldsymbol{x}}) \qquad \text{and} \qquad \widetilde{\boldsymbol{v}}_{m,n}^{(t)}(\widehat{\boldsymbol{x}}) = e^{-\mathrm{i}k\boldsymbol{z}\cdot\widehat{\boldsymbol{x}}}\boldsymbol{V}_n^m(\widehat{\boldsymbol{x}})\,, \qquad \widehat{\boldsymbol{x}} \in S^2\,.$$

Accordingly we find for $\boldsymbol{A}_n^m \in \{\boldsymbol{U}_n^m, \boldsymbol{V}_n^m\}$ and $\boldsymbol{B}_{n'}^{m'} \in \{\boldsymbol{U}_{n'}^{m'}, \boldsymbol{V}_{n'}^{m'}\}$ with $n, n' \ge 1$, $-n \le m \le n$, and $-n' \le m' \le n'$ that

$$\begin{aligned}&\langle T_{B_{ij\ell}}\boldsymbol{A}_n^m, \boldsymbol{B}_{n'}^{m'}\rangle_{L_t^2(S^2,\mathbb{C}^3)}\\ &\quad= \sum_{b=1}^{\infty}\sum_{a=-b}^{b}\Big(\mu_a^{(s)}\langle \boldsymbol{A}_n^m, e^{-\mathrm{i}k\boldsymbol{z}\cdot(\cdot)}\boldsymbol{U}_b^a\rangle_{L_t^2(S^2,\mathbb{C}^3)}\langle e^{-\mathrm{i}k\boldsymbol{z}\cdot(\cdot)}\boldsymbol{U}_b^a, \boldsymbol{B}_{n'}^{m'}\rangle_{L_t^2(S^2,\mathbb{C}^3)}\\ &\qquad\qquad + \mu_a^{(t)}\langle \boldsymbol{A}_n^m, e^{-\mathrm{i}k\boldsymbol{z}\cdot(\cdot)}\boldsymbol{V}_b^a\rangle_{L_t^2(S^2,\mathbb{C}^3)}\langle e^{-\mathrm{i}k\boldsymbol{z}\cdot(\cdot)}\boldsymbol{V}_b^a, \boldsymbol{B}_{n'}^{m'}\rangle_{L_t^2(S^2,\mathbb{C}^3)}\Big)\,.\end{aligned} \tag{9.10}$$

Truncating the series in (9.10) and applying a quadrature rule on S^2 to evaluate the inner products (see, e.g., [2, Sec. 5.1]), we obtain a discrete approximation

$$\boldsymbol{T}_{B_{ij\ell}} := \begin{bmatrix} \langle T_{B_{ij\ell}} \boldsymbol{U}_n^m, \boldsymbol{U}_{n'}^{m'} \rangle_{L_t^2(S^2,\mathbb{C}^3)} & \langle T_{B_{ij\ell}} \boldsymbol{V}_n^m, \boldsymbol{U}_{n'}^{m'} \rangle_{L_t^2(S^2,\mathbb{C}^3)} \\ \langle T_{B_{ij\ell}} \boldsymbol{U}_n^m, \boldsymbol{V}_{n'}^{m'} \rangle_{L_t^2(S^2,\mathbb{C}^3)} & \langle T_{B_{ij\ell}} \boldsymbol{V}_n^m, \boldsymbol{V}_{n'}^{m'} \rangle_{L_t^2(S^2,\mathbb{C}^3)} \end{bmatrix} \in \mathbb{C}^{Q\times Q} \tag{9.11}$$

of $T_{B_{ij\ell}}$ for any $-J \le i,j,\ell \le J$. The results from [20] suggest to truncate the series in (9.10) at an index larger than $k|\boldsymbol{z}_{ij\ell}|$. In the following we use the same truncation index $N \gtrsim \sqrt{3}kR$ for $\boldsymbol{F}_q$ and $T_{B_{ij\ell}}$ for any $-J \le i,j,\ell \le J$, and thus also the same $Q = 2N(N+2)$.

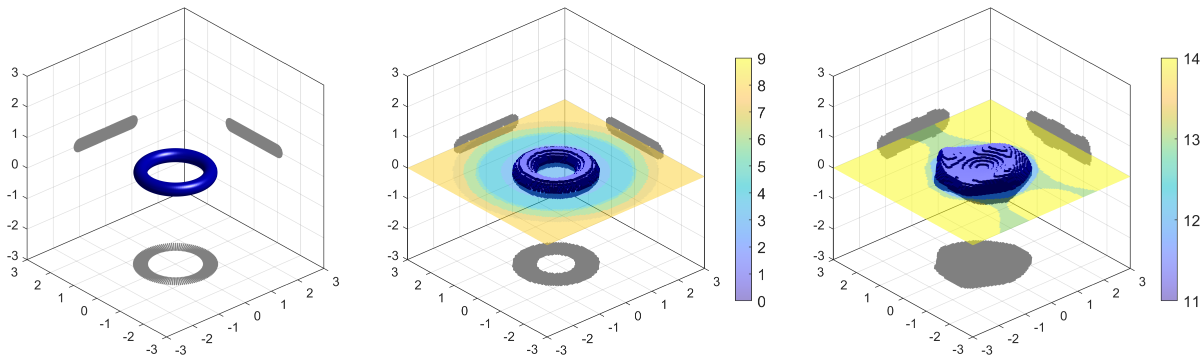

Figure 9.2: Visualization of exact shape of scattering object in Example 9.1 (left), visualization of isosurface $I_{20} = 2$ of indicator function from (9.13) using simulated far field data without additional noise (center), and visualization of isosurface $I_{20} = 11$ using simulated far field data with 0.1% noise (right).

To implement the criteria from Theorems 6.1 and 6.2 we compute for each grid point $\boldsymbol{z}_{ij\ell} \in \triangle$ the eigenvalues $\lambda_1^{(ij\ell)}, \dots, \lambda_Q^{(ij\ell)} \in \mathbb{R}$ of the self-adjoint matrix

$$\boldsymbol{A}_{B_{ij\ell}} := \operatorname{sign}(q)(\operatorname{Re}(\boldsymbol{F}_q) - \alpha \boldsymbol{T}_{B_{ij\ell}}) \in \mathbb{C}^{Q\times Q}\,, \qquad 1 \le i,j,\ell \le J\,. \tag{9.12}$$

For numerical stabilization, we discard those eigenvalues whose absolute values are smaller than some threshold. This number depends on the quality of the data. If there are good reasons to believe that $\boldsymbol{A}_{B_{ij\ell}}$ is known up to a perturbation of size $\delta > 0$ (with respect to the spectral norm), then we can only trust in those eigenvalues with magnitude larger than δ (see, e.g., [17, Thm. 7.2.2]). To obtain a reasonable estimate for δ, we use the magnitude of the non-unitary part of $\boldsymbol{S}_q := (\boldsymbol{I}_Q + (\mathrm{i}k/(8\pi^2)\boldsymbol{F}_q)$, i.e. we take $\delta = \|\boldsymbol{S}_q^*\boldsymbol{S}_q - \boldsymbol{I}_Q\|_2$, since this quantity should be zero for exact data and be of the order of the data error, otherwise.

Assuming that the electric permittivity contrast q is either larger or smaller than zero a.e. in $\operatorname{supp}(q)$, and that the parameter $\alpha \in \mathbb{R}$ satisfies the conditions in part (a) of Theorems 6.1 or 6.2, respectively, we then simply count for each test ball $B_{ij\ell}$ the number of negative eigenvalues of $\boldsymbol{A}_{B_{ij\ell}}$, and we define the *indicator function* $I_\alpha : \triangle \to \mathbb{N}$,

$$I_\alpha(\boldsymbol{z}_{ij\ell}) := \#\{\lambda_n^{(ij\ell)} \mid \lambda_n^{(ij\ell)} < -\delta\,,\ 1 \le n \le N\}\,, \qquad 1 \le i,j,\ell \le J\,. \tag{9.13}$$

Theorems 6.1–6.2 suggest that I_α is larger on sampling points $\boldsymbol{z}_{ij\ell} \in \triangle$ that are not contained in the support $\operatorname{supp}(q)$ of the scattering object than on sampling points $\boldsymbol{z}_{ij\ell} \in \triangle$ that are contained in $\operatorname{supp}(q)$.

Example 9.1. We consider a scattering object D that has the shape of a torus as shown in Figure 9.2 (left). We use $q = 0.5$ for the contrast function (i.e., the relative electric permittivity is $\varepsilon_r = 2$), $k = 1$ for the wave number, and $N = 5$ for the truncation index in the vector spherical harmonics expansions (9.7) and (9.10) (i.e., $Q = 70$ in (9.8), (9.11) and (9.12)). We simulate the far field matrix $\boldsymbol{F}_q \in \mathbb{C}^{Q\times Q}$ using the C++ boundary element library Bempp [46].

For the reconstructions we use the sampling grid $\triangle$ from (9.9) with step size $h = 0.05$ in the region of interest $[-3,3]^3$, i.e., we have 161 grid points in each direction. In Figure 9.3 we show color coded plots of the indicator function I_α from (9.13) in the $\boldsymbol{x}_1, \boldsymbol{x}_2$-plane, i.e., we plot the number of those eigenvalues of $\boldsymbol{A}_{B_{ij\ell}}$ from (9.12) that are smaller than $-\delta$ for all grid points with vanishing third component. We use $\delta = 10^{-14}$ for the threshold parameter, and we examine six different values for α, namely $\alpha \in \{0.01, 0.1, 0.5, 1, 10, 20\}$. We observe that the values of I_α are smaller for grid points inside the scattering object than outside, and that this number increases the farther away a grid point is from the scattering object, as we would expect

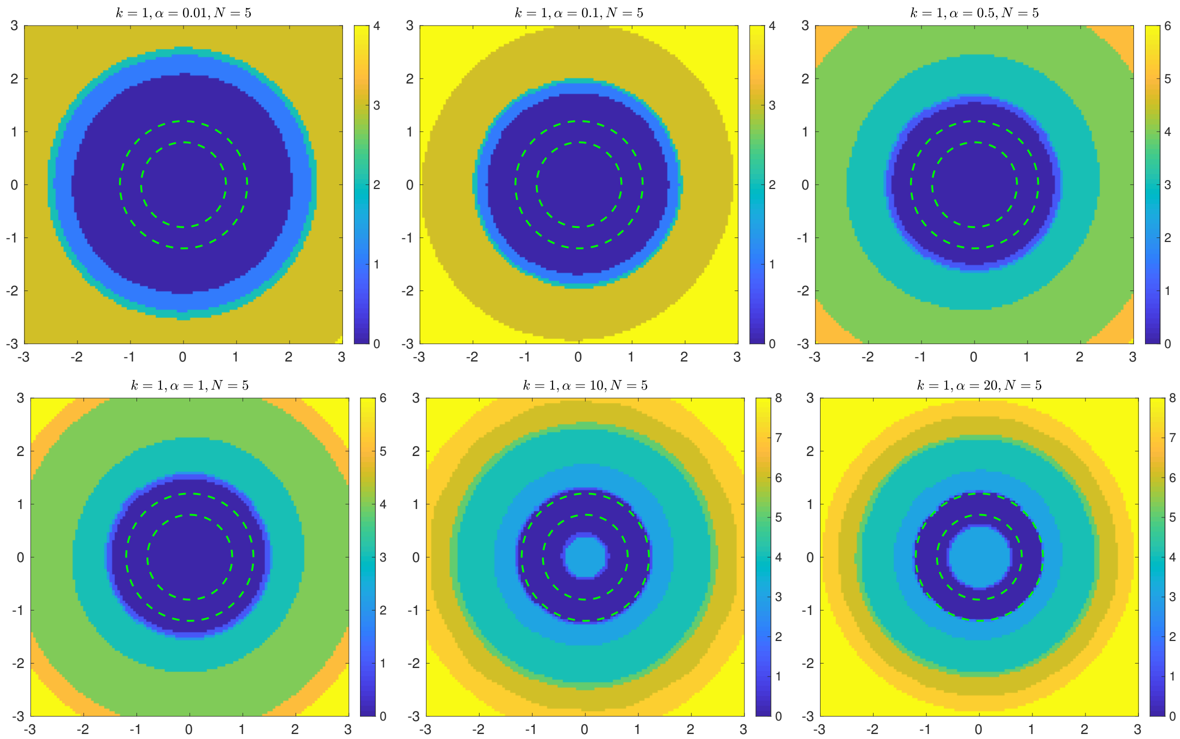

Figure 9.3: Visualization of the indicator function I_α for $\alpha \in \{0.01, 0.1, 0.5, 1, 10, 20\}$ in the $\boldsymbol{x}_1, \boldsymbol{x}_2$-plane using simulated far field data without additional noise. The dashed lines show the exact boundaries of the cross-section of the scatterer.

from Theorem 6.1. The condition $\alpha \leq q_{\min}$ in the second part of Theorem 6.1 is satisfied only for $\alpha \in \{0.01, 0.1, 0.5\}$. On the other hand the hole inside the cross-section of the torus becomes visible in these reconstruction when α is chosen sufficiently large. For $\alpha = 20$, we provide a three dimensional reconstruction in Figure 9.2. Inspecting the middle picture in the bottom row of Figure 9.3 suggests to plot the isosurface $I_{20} = 2$, which is shown in Figure 9.3 (center). The position and the shape of the torus are nicely reconstructed. We note that it was observed in [18] for the corresponding scalar scattering problem governed by the Helmholtz equation that the quality of the reconstructions of this monotonicity based scheme increases with increasing wave number also for smaller values of α.

To get an idea about the sensitivity of the reconstruction algorithm with respect to noise in the data, we redo this computation but add 0.1% complex-valued uniformly distributed additive error to the simulated far field data before starting the reconstruction procedure. The resulting reconstruction is shown in Figure 9.2 (right). In these reconstructions the noise is only accounted for via the threshold parameter δ in (9.13): We use $\delta = 0.001$. The result clearly gets worse, but it still contains useful information on the location and the shape of the scatterer.

9.3 Separating mixed scatterers

We discuss a numerical realization of the criteria established in Corollary 8.2. Suppose that $D = D_1 \cup D_2$ is an indefinite scattering object with contrast function q such that $q_1 := q|_{D_1}$ is strictly positive on D_1 and $q_2 := q|_{D_2}$ is strictly negative on D_2. While the algorithm for sign definite scattering objects in the previous subsection determines whether a sufficiently small probing domain B is contained inside the support D of the unknown scattering object or not, the criteria from Corollary 8.2 describe whether a sufficiently large probing domain B contains the support D_1 or D_2 of the component of the scattering object with stricly positive or stricly negative contrast function, respectively. We develop an algorithm to determine upper bounds $B_1, B_2 \subseteq \mathbb{R}^3$ such that $D_1 \subseteq B_1$ and $D_2 \subseteq B_2$. This is clearly less than full shape reconstruction but our numerical results below confirm that we can separate the two components of the scatterer with positive and negative scattering contrasts from far field data, at least when their supports are sufficiently far apart from each other.

We work on an equidistant sampling grid $\triangle$ as in (9.9). For each $\boldsymbol{z}_{ij\ell} \in \triangle$ we consider a probing operator $T_{B_{ij\ell}}$ as in (6.1), where the probing domain $B_{ij\ell} = B_\rho(\boldsymbol{z}_{ij\ell})$ is a ball of radius ρ centered at $\boldsymbol{z}_{ij\ell}$. Here we assume that $\rho \geq \rho_0 > 0$, where $2\rho_0$ is an upper bound for the diameters of D_1 and D_2.

To implement the criteria from Corollary 8.2 we compute for each grid point $\boldsymbol{z}_{ij\ell} \in \triangle$ the eigenvalues $\lambda_1^{(ij\ell)}, \dots, \lambda_Q^{(ij\ell)} \in \mathbb{R}$ of the self-adjoint matrix

$$\boldsymbol{A}^{\pm}_{B_{ij\ell}} := \pm(\mathrm{Re}(\boldsymbol{F}_q) \mp \alpha \boldsymbol{T}_{B_{ij\ell}}) \in \mathbb{C}^{Q\times Q}\,, \qquad 1 \leq i,j,\ell \leq J\,.$$

Then Corollary 8.2 says the following.

(a) If $D_1 \subseteq B$, then $\boldsymbol{A}^{+}_{B_{ij\ell}}$ has only finitely many positive eigenvalues for all $\alpha \geq C q_{1,\max}$.

(b) If $D_1 \not\subseteq B$, then $\boldsymbol{A}^{+}_{B_{ij\ell}}$ has infinitely many positive eigenvalues for all $\alpha \in \mathbb{R}$.

(c) If $D_2 \subseteq B$, then $\boldsymbol{A}^{-}_{B_{ij\ell}}$ has only finitely many positive eigenvalues for all $\alpha \leq q_{2,\min}$.

(d) If $D_2 \not\subseteq B$, then $\boldsymbol{A}^{-}_{B_{ij\ell}}$ has infinitely many positive eigenvalues for all $\alpha \in \mathbb{R}$.

Accordingly, assuming that the parameter $\alpha > 0$ is sufficiently large, we count for each test ball $B_{ij\ell}$ the number of positive eigenvalues of $\boldsymbol{A}^{\pm}_{B_{ij\ell}}$, and we define for any $\boldsymbol{z} \in \triangle \cap B_{ij\ell}$,

$$I^{\pm}_{\alpha,ij\ell}(\boldsymbol{z}) := \#\{\lambda_n^{(ij\ell)} \mid \lambda_n^{(ij\ell)} > \delta\,,\ 1 \leq n \leq N\}\,,$$

where $\delta > 0$ is a theshold parameter that depends on the quality of the data as in Section 9.2. Therewith we define the *indicator function* $I^{\pm}_{\alpha} : \triangle \to \mathbb{N}$,

$$I^{\pm}_{\alpha}(\boldsymbol{z}) := \min\{I^{\pm}_{\alpha,ij\ell}(\boldsymbol{z}) \mid 1 \leq i,j,\ell \leq J\}\,, \qquad \boldsymbol{z} \in \triangle\,. \tag{9.14}$$

Corollary 8.2 suggests that I^{+}_{α} and I^{-}_{α} are smaller on sampling points $\boldsymbol{z}_{ij\ell} \in \triangle$ that are close to D_1 and D_2 than on sampling points away from D_1 and D_2, respectively.

Example 9.2. We consider an indefinite scattering configuration with two scattering objects that are supported on cubes as shown in Figure 9.4 (left). The contrast function of the scatterer supported on the lower cube D_1 is $q_1 = 0.5$ (i.e., the relative electric permittivity is $\varepsilon_r = 2$), and the contrast function of the scatterer supported on the upper cube D_2 is $q_2 = -1$ (i.e., $\varepsilon_r = 0.5$). We use $k = 1$ for the wave number and $N = 5$ for the truncation index in the vector spherical harmonics expansions (9.7) and (9.10) (i.e., $Q = 70$ in (9.8), (9.11) and (9.12)). We simulate the far field matrix $\boldsymbol{F}_q \in \mathbb{C}^{Q\times Q}$ using the C++ boundary element library Bempp [46].

For the reconstructions we use the sampling grid $\triangle$ from (9.9) with step size $h = 0.05$ in the region of interest $[-3,3]^3$. In Figure 9.4 (center and right) we show horizontal cross sections of color coded plots of the indicator function $I^{\pm}_{\alpha}$ from (9.14), where the radius of the test balls is $\rho = 0.75$. We use $\alpha = 0.5$ (center) to recover the approximate position and size of the component D_1, where the constrast function q is stricly positive, and $\alpha = -1$ (right) to recover the approximate position and size of the component D_2 where the constrast function q is stricly negative. For the threshold parameter we use $\delta = 10^{-14}$. We observe that the values of $I^{+}_{0.5}$ are

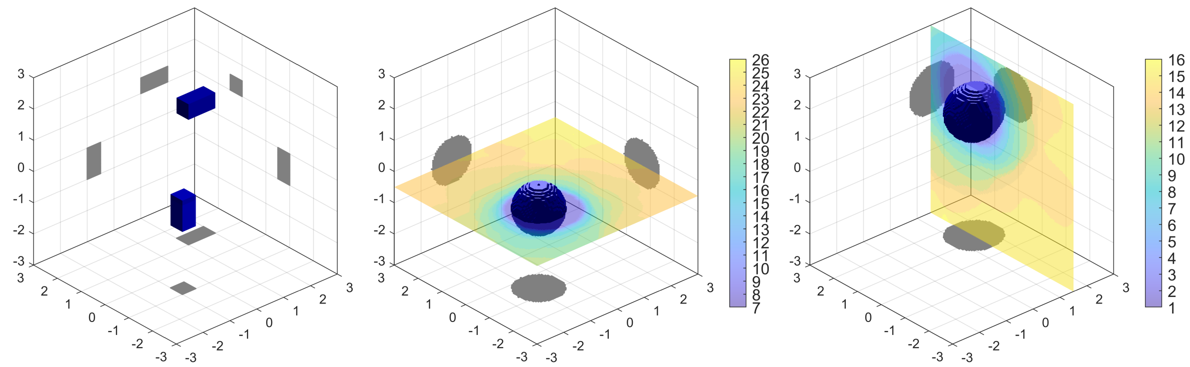

Figure 9.4: Visualization of the exact shape of the mixed scattering object in Example 9.2 (left), and of the isosurfaces $I^+_{0.5} = 7$ (center) and $I^-_{-1} = 1$ (right).

smaller for grid points inside the component D_1 of the scattering object than outside. Similarly, the values of I^-_{-1} are smaller for grid points inside the component D_2 of the scattering object than outside. These numbers increase the farther away a grid point is from the corresponding component of the scattering object, as we would expect from Corollary 8.2. The two isosurface plots $I^+_{0.5} = 7$ (center) and $I^-_{-1} = 1$ (right) in Figure 9.4 show that the components D_1 and D_2 of the indefinite scattering configuration can be nicely separated by the algorithm.

Conclusions

In this work we have considered the inverse scattering problem to reconstruct the shape of a scattering object from far field observations of scattered electromagnetic waves. We have established new rigorous characterizations of the support of inhomogeneous non-magnetic scattering objects in terms of the corresponding far field operator. These characterizations are based on novel monotonicity relations for the difference of two far field operators corresponding to two different permittivity contrasts. We have also established the existence of solutions to the direct scattering problem that have arbitrarily large energy in some prescribed region, while at the same time having arbitrarily small energy in a different prescribed region. This has been an important theoretical tool in our analysis. We have provided some simple numerical demonstrations of our theoretical results. A stable numerical implementation of the new monotonicity based shape characterizations still requires a better understanding of the dimensions of the finite dimensional subspaces that are excluded in the monotonicity relations. Corresponding results have been established for a related inverse boundary value problem for the Helmholtz equation in [30, 31]. Similarly, a combination of the new monotonicity relations with traditional regularization methods, as done for electrical impedance tomography in [28], is an interesting open problem.

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