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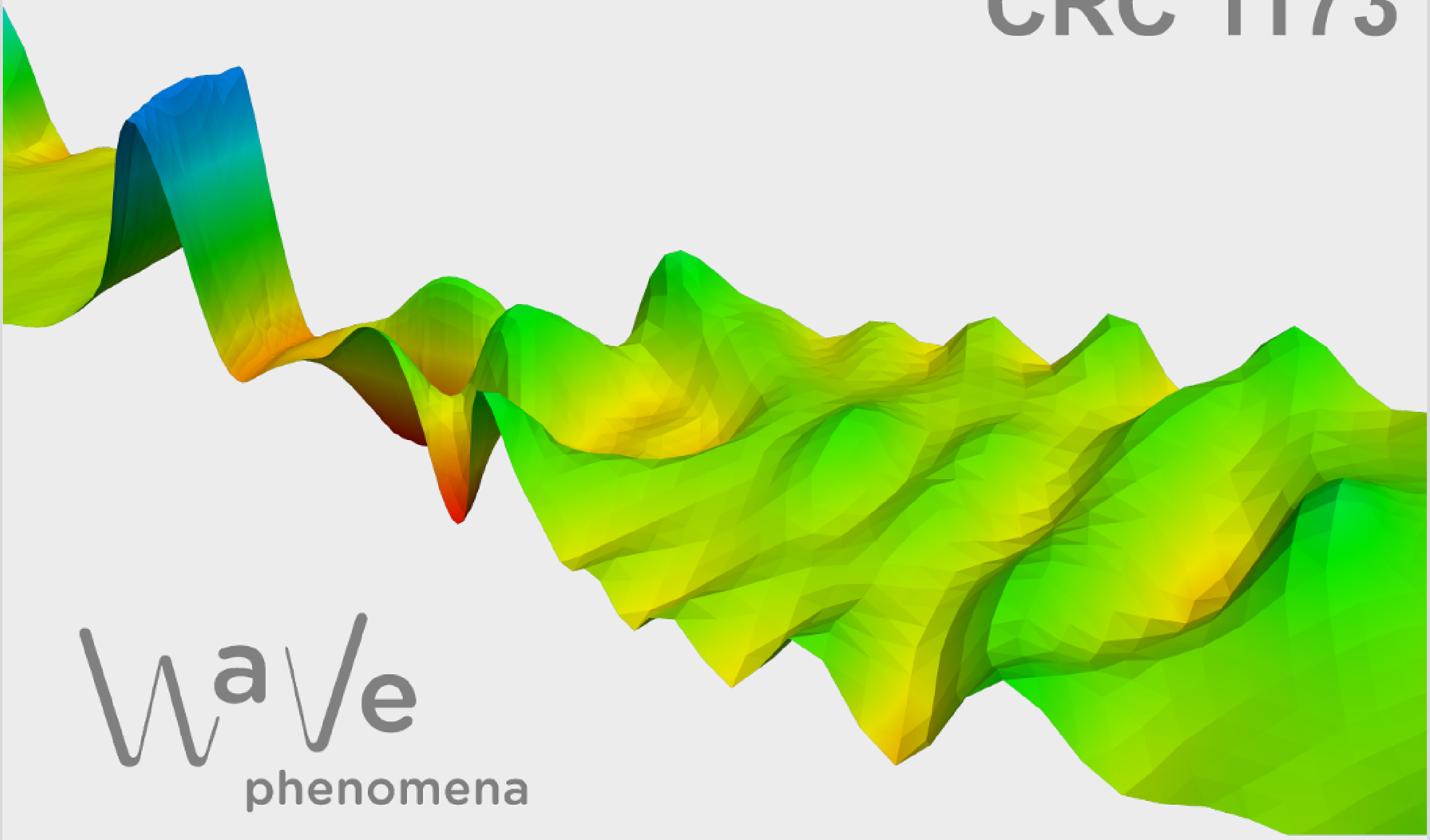

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INTERPOLATION OF A REGULAR SUBSPACE COMPLEMENTING THE SPAN OF A RADIALLY SINGULAR FUNCTION

KONSTANTIN ZERULLA

Abstract. We analyze the interpolation of the sum of a subspace, consisting of regular functions, with the span of a function with r^α-type singularity. In particular, we determine all interpolation parameters, for which the interpolation space of the subspace of regular functions is still a closed subspace. The main tool is here a result by Ivanov and Kalton on interpolation of subspaces. To apply it, we study the K-functional of the r^α-singular function. It turns out that the K-functional possesses upper and lower bounds that have a common decay rate at zero.

1. Introduction

Many relevant problems in mathematics and physics demand for a thorough study of functions that have a radial singularity of r^α-type. Important examples are elliptic boundary value problems on domains with irregular boundary, or interface problems for an elliptic operator, see [12, 10, 21, 3, 16, 8, 9] for instance. These functions furthermore play an important role in the analysis and numerical solution of Maxwell equations on homogeneous and heterogeneous domains with irregular boundary, see [8, 9, 7, 2] among others. The error in numerical approximations for problems involving singular functions of r^α-type is also investigated in [4, 5, 6] for instance.

This paper is motivated by the regularity analysis of Maxwell equations in heterogeneous cuboids, which is in preparation. Indeed, the below Theorem 1.1 is essential to study the behavior of the electric field near interior edges of the heterogeneous material, as the regularity of the electric field can be expressed by means of the first interpolation space in Theorem 1.1.

We consider here the singular function

$$\omega(r\cos\varphi, r\sin\varphi) = \chi(r) r^\alpha \psi(\varphi), \qquad r \in [0,1],\ \varphi \in [0,2\pi), \tag{1.1}$$

on the open unit disc D. Definition (1.1) involves a smooth cut-off function $\chi : [0,\infty) \to [0,1]$ with $\chi = 1$ on $[0,1/2]$ and support in $[0,3/4]$, a number $\alpha \in (0,1)$, and a piecewise C^2-function $\psi : [0,2\pi] \to \mathbb{R}$ with $\psi(0) = 1$. For simplicity, we assume that ψ is C^2-regular on $[0,\pi/2]$ and on $[\pi/2,2\pi]$. Other partitions of $[0,2\pi]$ or restrictions to subintervals can be handled with

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similar arguments. To have a short notation, we always write $\omega(r,\varphi)$ instead of $\omega(r\cos\varphi, r\sin\varphi)$. The span of ω is denoted by

$$V := \operatorname{span}\{\omega\}. \tag{1.2}$$

We next present the main result of this paper. The relevant notation of the statement is introduced in Section 2. In particular, $\mathrm{PH}^s(D)$ denotes the space of piecewise H^s-regular functions on D for $s > 0$, see (2.1).

Theorem 1.1. *The identities*

$$\begin{aligned} \big(\mathrm{PH}^1(D), \mathrm{PH}^2(D) \oplus V\big)_{\theta_1,2} &= \mathrm{PH}^{1+\theta_1}(D) \oplus V, \qquad && \theta_1 \in (\alpha, 1], \\ \big(\mathrm{PH}^1(D), \mathrm{PH}^2(D) \oplus V\big)_{\theta_2,2} &= \mathrm{PH}^{1+\theta_2}(D), && \theta_2 \in [0, \alpha), \end{aligned}$$

are valid. For the critical value α, the space $\mathrm{PH}^{1+\alpha}(D)$ is not closed in $(\mathrm{PH}^1(D), \mathrm{PH}^2(D) \oplus V)_{\alpha,2}$.

We point out that the statements in Theorem 1.1 are of a sharp nature. Note also that the space $\mathrm{PH}^{1+\theta_1}(D) \oplus V$ from the first line in Theorem 1.1 is equipped with the sum of the norms in $\mathrm{PH}^{1+\theta_1}(D)$ and V.

To the best of our knowledge, Theorem 1.1 is the first one to answer the question, for which interpolation parameters θ the interpolation of the closed subspace $\mathrm{PH}^2(D)$ is still a closed subspace of the interpolation space $(\mathrm{PH}^1(D), \mathrm{PH}^2(D) \oplus V)_{\theta,2}$. This question has been investigated for other spaces several times in the literature:

> Consider an interpolation couple (X_0, X_1), and closed subspaces Y_0 in X_0 and Y_1 in X_1. *Is the interpolation space $(Y_0, Y_1)_{\theta,p}$ still a closed subspace of $(X_0, X_1)_{\theta,p}$?*

The issue is addressed in Problem 18.5 in Chapter 1 of [17]. Remark 11.4 in Chapter 1 of [17] yields that the answer is no, in general. In Satz 5 of [22], Triebel gives an example for Hilbert spaces H_0, H_1, H_2 with $H_1 \hookrightarrow H_0$, $H_2 \subseteq H_1$ being closed with (arbitrary) finite codimension, and $(H_0, H_2)_{1/2,2}$ not being closed in $(H_0, H_1)_{1/2,2}$. Wallstén analyzes this issue for a codimension one subspace M in L^1, and interpolates to L^∞. Depending on the choice of M and the interpolation parameter, the interpolation space between M and L^∞ is a closed subspace of the interpolation space between L^1 and L^∞. Note that it can also happen that the above statement is not fulfilled for any interpolation parameter in $(0, 1)$, see [23].

Ivanov and Kalton study Banach spaces X_0, X_1 and Y_0, with Y_0 being a closed subspace of codimension one in X_0, and (X_0, X_1) being an interpolation couple. They derive formulas for numbers $\sigma_0 \le \sigma_1$ with $(Y_0, X_1)_{\theta,p}$ being a closed subspace of $(X_0, X_1)_{\theta,p}$ for $\theta \in (0, \sigma_0) \cup (\sigma_1, 1)$, $p \in [1, \infty)$, see Theorem 2.1 in [13]. Note that some of the statements in [13] have earlier been obtained in [18]. Theorem 2.1 from [13] is the essential tool in the proof of Theorem 1.1 in this paper. In [1], the findings of [13] are generalized. In particular, the closed subspace Y_0 is allowed to have arbitrary finite codimension in X_0.

The major difficulty in the proof of Theorem 1.1 is to obtain a sharp lower bound for the K-functional of the singular function ω. To that end, we study the modulus of smoothness of ω. It then turns out that a subtle analysis of the singular function ω near zero is needed to provide estimates for the modulus of smoothness of ω, see the proof of Lemma 3.1.

The paper is organized in the following way. In the next section, we fix a common notation for the occurring objects from interpolation theory, and we introduce the relevant (broken) Sobolev spaces of fractional order. In Section 3, we derive the crucial upper and lower estimates for the K-functional of the singular function ω. Finally, we conclude Theorem 1.1 in Section 4 by combining our results from Section 3 with Theorem 2.1 from [13].

As a byproduct of our preparations in Section 3, we also obtain a precise regularity statement for the singular function ω in terms of interpolation spaces, see Corollary 3.3. To the best of our knowledge, only parts of the statement are available in the literature, see [4] for instance.

2. Analytical preliminaries

We first recall basic constructions from real interpolation theory via the K-method. Our presentation follows Section 1.1 in [19]. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two real Banach spaces, which both embed into a common Hausdorff space. The K-functional is given by the formula

$$K(t, z, X, Y) := \inf_{\substack{z=x+y,\\ x\in X,\ y\in Y}} \left(\|x\|_X + t\|y\|_Y\right)$$

for $z \in X + Y$ and $t > 0$. It is used to define the real interpolation spaces

$$(X, Y)_{\theta,p} := \{z \in X + Y \mid \|z\|^p_{(X,Y)_{\theta,p}} := \int_0^\infty t^{-1-\theta p} K(t, z, X, Y)^p \,\mathrm{d}t < \infty\},$$

for $\theta \in (0, 1)$, and $p \in [1, \infty)$. The spaces

$$(X, Y)_\theta := \{z \in X + Y \mid \lim_{t\to 0} t^{-\theta} K(t, z, X, Y) = \lim_{t\to\infty} t^{-\theta} K(t, z, X, Y) = 0\},$$

$$(X, Y)_{\theta,\infty} := \{z \in X + Y \mid t \mapsto t^{-\theta} K(t, z, X, Y) \in \mathrm{L}^\infty(0, \infty)\},$$

also arise in this paper. Both are complete with respect to the norm

$$\|z\|_{(X,Y)_{\theta,\infty}} := \|t^{-\theta} K(t, z, X, Y)\|_{\mathrm{L}^\infty(0,\infty)}, \qquad z \in (X, Y)_{\theta,\infty}.$$

We next recall the definition of fractional order Sobolev spaces. Let $O \subseteq \mathbb{R}^2$ be a bounded domain with compact Lipschitz boundary. We denote the standard Sobolev space of order $k \in \mathbb{N}$ by $\mathrm{H}^k(O)$, and set

$$\mathrm{H}^s(O) := \left(\mathrm{L}^2(O), \mathrm{H}^2(O)\right)_{s/2,2}, \qquad s \in [0, 2].$$

Fractional Sobolev spaces can also be defined by means of weighted difference quotients, see Section 2.3.8 in [20] for instance. Note that both definitions are equivalent in the current setting. (This well known fact is for instance verified by combining the main theorem in Section 1 of [15] with Corollary 6.8 in [11].)

Having the choice of the function ψ in (1.1) in mind, we introduce the open disc segments

$$D_1 := \{(r \cos \varphi, r \sin \varphi) \mid r \in (0, 1),\ \varphi \in (0, \tfrac{\pi}{2})\},$$
$$D_2 := \{(r \cos \varphi, r \sin \varphi) \mid r \in (0, 1),\ \varphi \in (\tfrac{\pi}{2}, 2\pi)\},$$

using polar coordinates (r,φ) on D. To study functions that are only regular on D_1 and D_2, but not on D, we use the broken fractional order Sobolev spaces

$$\mathrm{PH}^s(D) := \{f \in \mathrm{L}^2(D) \mid f|_{D_i} \in \mathrm{H}^s(D_i),\ i \in \{1,2\}\}, \qquad s \in [0,2], \quad (2.1)$$

being complete with respect to the norm

$$\|f\|_{\mathrm{PH}^s(D)} := \Big(\sum_{i=1}^{2} \|f|_{D_i}\|^2_{\mathrm{H}^s(D_i)}\Big)^{1/2}, \qquad f \in \mathrm{PH}^s(D).$$

We also note the interpolation property

$$\big(\mathrm{L}^2(D), \mathrm{PH}^2(D)\big)_{s/2,2} = \mathrm{PH}^s(D), \qquad s \in [0,2].$$

An essential part of the proof for Theorem 1.1 consists in the derivation of sharp upper and lower estimates for the functional $K(\cdot,\omega,\mathrm{L}^2(D),\mathrm{PH}^2(D))$, see Lemmas 3.1 and 3.2. By *sharp* we mean that the upper and lower bounds have the same decay rate near zero. To obtain the inequalities, it is useful to analyze the second modulus of smoothness for the singular function ω on an appropriately chosen open subset D_0 of D_1. To define the second modulus of smoothness, we use the set

$$D_0(h) = \{v \in D_0 \mid v + th \in D_0 \text{ for all } 0 \le t \le 1\}$$

for $h \in \mathbb{R}^2$. Denoting the characteristic function of a set $O \subseteq \mathbb{R}^2$ by $\mathbb{1}_O$, the second modulus of smoothness of ω on D_0 is defined as

$$m_2(t,\omega) := \sup_{0<|h|\le t} \|\mathbb{1}_{D_0(2h)}(\omega - 2\omega(\cdot+h) + \omega(\cdot+2h))\|_{L^2(D_0)}, \qquad (2.2)$$

for $t > 0$, see Section 1 in [14] for instance. Lemma 1 in [14] and the definition of the K-functional then provide the inequality

$$\begin{aligned}
K(t^2,\omega,\mathrm{L}^2(D_0),&\mathrm{H}^2(D_0)) \\
&\ge \inf_{g\in \mathrm{H}^2(D_0)} \big(\|\omega - g\|_{\mathrm{L}^2(D_0)} + t^2 \sup_{k_1+k_2=2} \|\partial_x^{k_1}\partial_y^{k_2} g\|_{\mathrm{L}^2(D_0)}\big) \\
&\ge C m_2(t,\omega), \qquad t > 0,
\end{aligned}$$

with a uniform constant $C > 0$. We then infer the useful estimate

$$\begin{aligned}
K(t^2,\omega,\mathrm{L}^2(D),\mathrm{PH}^2(D)) &= \inf_{\substack{\omega=f+g,\\ f\in \mathrm{L}^2(D),\ g\in \mathrm{PH}^2(D)}} \big(\|f\|_{\mathrm{L}^2(D)} + t^2\|g\|_{\mathrm{PH}^2(D)}\big) \\
&\ge \inf_{\substack{\omega=\tilde f+\tilde g,\\ \tilde f\in \mathrm{L}^2(D_0),\ \tilde g\in \mathrm{H}^2(D_0)}} \big(\|\tilde f\|_{\mathrm{L}^2(D_0)} + t^2\|\tilde g\|_{\mathrm{H}^2(D_0)}\big) \\
&= K(t^2,\omega,\mathrm{L}^2(D_0),\mathrm{H}^2(D_0)) \ge C m_2(t,\omega) \qquad (2.3)
\end{aligned}$$

for $t > 0$, that comes into play in the proof of Lemma 3.1.

3. Estimates for the K-functional

In this section, we derive upper and lower estimates for the K-functional of the singular function ω from (1.1). The inequalities are crucial for the proof of Theorem 1.1 in Section 4. In particular, it turns out that it is important to have upper and lower bounds for the K-functional that have the same decay rate near zero.

In the next lemma, we start with the desired lower estimate.

Lemma 3.1. *The inequality*

$$K(t^2, \omega, \mathrm{L}^2(D), \mathrm{PH}^2(D)) \geq C_l t^{\alpha+1}, \qquad t \in (0,1],$$

is valid with a uniform constant $C_l = C_l(\omega) > 0$.

Proof. 1) We consider the problem in cartesian coordinates (x, y) on the open subset

$$D_0 := \{(x,y) \mid 0 < x^2 + y^2 < \tfrac{1}{16},\ 0 < y < x\}$$

of D_1. Note that the cut-off function χ from (1.1) is then equal to one on D_0. As a result, ω has the representation

$$\omega(x,y) = \psi(\arctan(\tfrac{y}{x}))|(x,y)|^\alpha, \qquad (x,y) \in D_0.$$

On D_0, we then calculate

$$\begin{aligned}
\partial_x \omega(x,y) &= -y\psi'(\arctan(\tfrac{y}{x}))|(x,y)|^{\alpha-2} + \alpha x \psi(\arctan(\tfrac{y}{x}))|(x,y)|^{\alpha-2},\\
\partial_x^2 \omega(x,y) &= \Big(y^2\psi''(\arctan(\tfrac{y}{x})) - 2(\alpha-1)yx\psi'(\arctan(\tfrac{y}{x}))\\
&\qquad + \alpha(\alpha-2)x^2\psi(\arctan(\tfrac{y}{x}))\Big)|(x,y)|^{\alpha-4}\\
&\quad + \alpha\psi(\arctan(\tfrac{y}{x}))|(x,y)|^{\alpha-2}.
\end{aligned}$$

We next derive a lower estimate for the function $-\partial_x^2\omega$ on an appropriate part of D_0. For convenience, we denote the piecewise C^2-norm of ψ by $\|\psi\|_{\mathrm{C}^2}$ (meaning the supremum of the C^2-norms on D_1 and D_2). Recall that ψ satisfies the condition $\psi(0) = 1$, see Section 1. By continuity, there hence is a number $\delta \in (0, \pi/2)$ with $\psi(\varphi) \geq 1/2$ for $\varphi \in [0, \delta]$. Let furthermore $x \in (0, \frac{1}{4\sqrt{10}})$ and $0 < y \leq \gamma x$ with $\gamma := \min\{\tan\delta, \frac{(1-\alpha)\alpha}{12\|\psi\|_{\mathrm{C}^2}}\}$. Note that $\gamma < 1$, as $\psi(0) = 1$ and $\alpha < 1$. The choice of γ then in particular implies the fact $\arctan(\frac{y}{x}) \in (0, \delta]$. In view of the assumption $\alpha \in (0,1)$, the relations

$$\begin{aligned}
&\Big(\alpha(2-\alpha)x^2\psi(\arctan(\tfrac{y}{x})) - y^2\psi''(\arctan(\tfrac{y}{x}))\\
&\quad - 2(1-\alpha)yx\psi'(\arctan(\tfrac{y}{x}))\Big)|(x,y)|^{\alpha-4}\\
&\geq \Big(\tfrac{1-\alpha-2\gamma}{2-\alpha}\alpha(2-\alpha)x^2\psi(\arctan(\tfrac{y}{x})) + \tfrac{1+2\gamma}{2-\alpha}\alpha(2-\alpha)x^2\psi(\arctan(\tfrac{y}{x}))\\
&\qquad - y^2\|\psi\|_{\mathrm{C}^2} - 2yx\|\psi\|_{\mathrm{C}^2}\Big)|(x,y)|^{\alpha-4}\\
&\geq \Big(\tfrac{1}{2}(1-\alpha-2\gamma)\alpha x^2 + (1+2\gamma)\alpha x^2\psi(\arctan(\tfrac{y}{x}))\\
&\qquad - 3\gamma x^2\|\psi\|_{\mathrm{C}^2}\Big)|(x,y)|^{\alpha-4}
\end{aligned} \tag{3.1}$$

then follow. We next use the inequalities $\gamma \leq \frac{(1-\alpha)\alpha}{12\|\psi\|_{\mathrm{C}^2}} < \frac{1-\alpha}{4}$ to conclude the estimates

$$\begin{aligned}
&\Big(\tfrac{1}{2}(1-\alpha-2\gamma)\alpha x^2 + (1+2\gamma)\alpha x^2\psi(\arctan(\tfrac{y}{x})) - 3\gamma x^2\|\psi\|_{\mathrm{C}^2}\Big)|(x,y)|^{\alpha-4}\\
&\geq \Big(\tfrac{1}{4}(1-\alpha)\alpha x^2 + (1+2\gamma)\alpha x^2\psi(\arctan(\tfrac{y}{x})) - 3\gamma x^2\|\psi\|_{\mathrm{C}^2}\Big)|(x,y)|^{\alpha-4}\\
&\geq (1+2\gamma)\alpha x^2\psi(\arctan(\tfrac{y}{x}))|(x,y)|^{\alpha-4}.
\end{aligned} \tag{3.2}$$

Combining (3.1)–(3.2) and using again the relation $y \le \gamma x$, we then infer the useful inequalities

$$\begin{aligned}
&-\partial_x^2\omega(x,y)\\
&\quad\ge -\alpha\psi(\arctan(\tfrac{y}{x}))|(x,y)|^{\alpha-2} + (1+2\gamma)\alpha x^2\psi(\arctan(\tfrac{y}{x}))|(x,y)|^{\alpha-4}\\
&\quad= \alpha\big(-(x^2+y^2)+(1+2\gamma)x^2\big)\psi(\arctan(\tfrac{y}{x}))|(x,y)|^{\alpha-4}\\
&\quad\ge \alpha\big(-(1+\gamma)+1+2\gamma\big)x^2\psi(\arctan(\tfrac{y}{x}))|(x,y)|^{\alpha-4}\\
&\quad= \alpha\gamma x^2\psi(\arctan(\tfrac{y}{x}))|(x,y)|^{\alpha-4}. \qquad (3.3)
\end{aligned}$$

2) We now bound the second modulus of smoothness $m_2(\cdot,\omega)$ for ω on D_0 from below, see (2.2). To that end, we choose $t < \frac{1}{12\sqrt{10}}$ and $h = (h_1, 0)$ with $h_1 = t$ in (2.2). Combining the choice of h_1 and $\gamma < 1$, the inequalities

$$\begin{aligned}
m_2(t,\omega)^2 &\ge \int_0^{h_1}\int_0^{\gamma x} \big(\omega(x,y) - 2\omega(x+h_1,y) + \omega(x+2h_1,y)\big)^2 \,\mathrm{d}y\,\mathrm{d}x\\
&= \int_0^{h_1}\int_0^{\gamma x} \Big(\int_0^{h_1} -\partial_x\omega(x+s,y) + \partial_x\omega(x+h_1+s,y)\,\mathrm{d}s\Big)^2 \,\mathrm{d}y\,\mathrm{d}x\\
&= \int_0^{h_1}\int_0^{\gamma x} \Big(\int_0^{h_1}\int_0^{h_1} \partial_x^2\omega(x+s+\tau,y)\,\mathrm{d}\tau\,\mathrm{d}s\Big)^2 \,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

are obtained. Inserting now also (3.3), we infer the relation

$$\begin{aligned}
m_2(t,\omega)^2 \ge \int_0^{h_1}\int_0^{\gamma x} \Big(\int_0^{h_1}\int_0^{h_1} &\alpha\gamma(x+s+\tau)^2\psi(\arctan(\tfrac{y}{x+s+\tau}))\\
&\cdot|(x+s+\tau,y)|^{\alpha-4}\,\mathrm{d}\tau\,\mathrm{d}s\Big)^2 \,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Taking also the fact $\psi(\arctan(\frac{y}{x+s+\tau})) \ge \frac{1}{2}$ for $x, s, \tau \in (0, h_1)$ and $y < \gamma x$ into account, we arrive at the estimates

$$\begin{aligned}
&m_2(t,\omega)^2\\
&\ge \int_0^{h_1}\int_0^{\gamma x} \Big(\int_0^{h_1}\int_0^{h_1} \tfrac{\alpha}{2}\gamma(x+s+\tau)^2(1+\gamma)^{\alpha-4}(x+s+\tau)^{\alpha-4}\,\mathrm{d}\tau\,\mathrm{d}s\Big)^2 \,\mathrm{d}y\,\mathrm{d}x\\
&= \int_0^{h_1}\int_0^{\gamma x} \Big(\int_0^{h_1} \tfrac{\alpha}{2}\gamma\tfrac{(1+\gamma)^{\alpha-4}}{1-\alpha}\big((x+s)^{\alpha-1} - (x+s+h_1)^{\alpha-1}\big)\,\mathrm{d}s\Big)^2 \,\mathrm{d}y\,\mathrm{d}x\\
&\ge \int_0^{h_1}\int_0^{\gamma x} \Big(\int_0^{h_1} \tfrac{\alpha}{2}\gamma\tfrac{(1+\gamma)^{\alpha-4}}{1-\alpha}(1-(\tfrac{3}{2})^{\alpha-1})(x+s)^{\alpha-1}\,\mathrm{d}s\Big)^2 \,\mathrm{d}y\,\mathrm{d}x\\
&\ge \int_0^{h_1}\int_0^{\gamma x} \Big(\int_0^{h_1} \tfrac{\alpha}{2}\gamma\tfrac{(1+\gamma)^{\alpha-4}}{1-\alpha}(1-(\tfrac{3}{2})^{\alpha-1})2^{\alpha-1}h_1^{\alpha-1}\,\mathrm{d}s\Big)^2 \,\mathrm{d}y\,\mathrm{d}x\\
&= \tfrac{\alpha^2}{4^{1-\alpha}}\gamma^3\tfrac{(1+\gamma)^{2\alpha-8}}{8(1-\alpha)^2}\big(1-(\tfrac{3}{2})^{\alpha-1}\big)^2 h_1^{2\alpha+2}\\
&=: C_1 h_1^{2\alpha+2}.
\end{aligned}$$

In view of (2.3), we hence conclude the result

$$K(t^2,\omega,\mathrm{L}^2(D),\mathrm{PH}^2(D)) \ge C\sqrt{C_1}t^{\alpha+1}, \qquad 0 < t < \tfrac{1}{12\sqrt{10}} =: t_0.$$

The monotonicity of the K-functional furthermore implies the inequality

$$K(t^2,\omega,\mathrm{L}^2(D),\mathrm{PH}^2(D)) \ge K(t_0^2,\omega,\mathrm{L}^2(D),\mathrm{PH}^2(D)) =: C_2 \ge C_2 t^{\alpha+1}$$

for $t \in [t_0, 1]$. Altogether, we arrive at the desired statement. □

The remaining upper estimate for the K-functional essentially follows from a modification of the arguments in the proofs of Theorem 2.3 in [5] and Theorem 2.5 in [6]. For the sake of a clear presentation, however, we elaborate the proof.

Lemma 3.2. *There is a uniform constant* $C_u = C_u(\omega) > 0$ *with*

$$K(t^2, \omega, \mathrm{L}^2(D), \mathrm{PH}^2(D)) \leq C_u t^{\alpha+1}, \qquad t \in (0,1].$$

Proof. 1) Throughout the proof, $C = C(\omega) > 0$ is a constant that is allowed to change from line to line. Let $\delta \in (0,1)$ be a fixed number that will be determined later. Let furthermore $\chi_\delta : [0,1] \to [0,1]$ be a smooth cut-off function with $\chi_\delta = 1$ on $[0, \delta/2]$, support in $[0,\delta]$, $\|\chi_\delta'\|_\infty \leq C/\delta$, and $\|\chi_\delta''\|_\infty \leq C/\delta^2$. We then write ω as the sum

$$\omega(r,\varphi) = \chi_\delta(r)\omega(r,\varphi) + (1-\chi_\delta(r))\omega(r,\varphi) =: v_1(r,\varphi) + v_2(r,\varphi)$$

for $r \in [0,1]$ and $\varphi \in [0, 2\pi)$. By construction, v_2 is piecewise C^2-regular on the partition $\cup_{i=1}^2 \overline{D_i}$. As a result, the estimate

$$\begin{aligned} K(t^2, \omega, \mathrm{L}^2(D), \mathrm{PH}^2(D)) &= \inf_{\substack{\omega = \tilde{v}_1 + \tilde{v}_2, \\ \tilde{v}_1 \in \mathrm{L}^2(D),\ \tilde{v}_2 \in \mathrm{PH}^2(D)}} \big(\|\tilde{v}_1\|_{\mathrm{L}^2(D)} + t^2 \|\tilde{v}_2\|_{\mathrm{PH}^2(D)}\big) \\ &\leq \|v_1\|_{\mathrm{L}^2(D)} + t^2 \|v_2\|_{\mathrm{PH}^2(D)} \end{aligned} \tag{3.4}$$

is true. We next bound the quantities on the right hand side of (3.4) separately. Note that we only focus on the disc segment D_1. The remaining one can be handled in the same way, due to symmetry.

2) Recall the definition

$$\omega(r,\varphi) = \chi(r) r^\alpha \psi(\varphi)$$

of ω in (1.1). Since the cut-off functions χ and χ_δ are bounded by one, the relation

$$\|v_1\|^2_{\mathrm{L}^2(D_1)} \leq C \|\psi\|^2_\infty \int_0^\delta r^{2\alpha+1} \,\mathrm{d}r \leq C \delta^{2\alpha+2} \tag{3.5}$$

is valid.

3) Similar to 2), we first bound the L^2-norm of v_2 by

$$\|v_2\|^2_{\mathrm{L}^2(D_1)} \leq C \delta^{2\alpha-2}. \tag{3.6}$$

To estimate the H^2-norm of v_2 on D_1, we note the fact

$$\begin{aligned} \|v_2\|^2_{\mathrm{H}^2(D_1)} \leq C\Big(\|v_2\|^2_{\mathrm{L}^2(D_1)} + \int_{\delta/2}^1 \int_0^{\pi/2} & \big(r|\partial_r^2 v_2|^2 + \tfrac{1}{r}|\partial_r \partial_\varphi v_2|^2 + \tfrac{1}{r^3}|\partial_\varphi v_2|^2 \\ & + \tfrac{1}{r}|\partial_r v_2|^2 + \tfrac{1}{r^3}|\partial_\varphi^2 v_2|^2\big) \,\mathrm{d}\varphi \,\mathrm{d}r\Big). \end{aligned} \tag{3.7}$$

The inequality follows from the representation of all first and second order derivatives in polar coordinates, and the location of the support of χ_δ. The expressions on the right hand side of (3.7) are given by the formulas

$$\begin{aligned} \partial_r v_2 &= \big(-\chi_\delta'\chi r^\alpha + (1-\chi_\delta)(\chi' r^\alpha + \alpha\chi r^{\alpha-1})\big)\psi, \\ \partial_r^2 v_2 &= \Big(\big(-\chi_\delta''\chi - 2\chi_\delta'\chi' + (1-\chi_\delta)\chi''\big)r^\alpha + 2\alpha\big((1-\chi_\delta)\chi' - \chi_\delta'\chi\big)r^{\alpha-1} \end{aligned}$$

$$
\begin{aligned}
&\quad + \alpha(\alpha-1)(1-\chi_\delta)\chi r^{\alpha-2}\Big)\psi,\\
\partial_\varphi v_2 &= (1-\chi_\delta)\chi r^\alpha \psi',\\
\partial_\varphi^2 v_2 &= (1-\chi_\delta)\chi r^\alpha \psi'',\\
\partial_\varphi \partial_r v_2 &= \big(-\chi_\delta' \chi r^\alpha + (1-\chi_\delta)(\chi' r^\alpha + \alpha\chi r^{\alpha-1})\big)\psi'.
\end{aligned}
$$

Combining the choice of χ_δ, (3.6) and (3.7), we then obtain the estimates

$$
\begin{aligned}
\|v_2\|^2_{\mathrm{H}^2(D_1)} &\le C\Big(\|v_2\|^2_{\mathrm{L}^2(D_1)} + \int_{\delta/2}^{\delta} (|\chi_\delta''|^2 r^{2\alpha+1} + |\chi_\delta'|^2 r^{2\alpha-1})\|\psi\|_{\mathrm{C}^2}\,\mathrm{d}r\\
&\qquad + \int_{\delta/2}^{1} r^{2\alpha-3}\,\mathrm{d}r \|\psi\|_{\mathrm{C}^2}\Big)\\
&\le C\Big(\delta^{2\alpha-2} + \int_{\delta/2}^{\delta} (\tfrac{1}{\delta^4} r^{2\alpha+1} + \tfrac{1}{\delta^2} r^{2\alpha-1})\,\mathrm{d}r\Big)\\
&\le C\delta^{2\alpha-2}.
\end{aligned}
$$

Due to symmetry, an analogous inequality is true on D_2. As a result, we infer the relation

$$
\|v_2\|_{\mathrm{PH}^2(D)} \le C\delta^{\alpha-1}. \tag{3.8}
$$

4) In view of (3.4), (3.5) and (3.8), we arrive at the result

$$
K(t^2, \omega, \mathrm{L}^2(D), \mathrm{PH}^2(D)) \le C(\delta^{\alpha+1} + t^2\delta^{\alpha-1}).
$$

The asserted statement follows by choosing $\delta = t$. □

Combining Lemmas 3.1 and 3.2, we can directly derive the following regularity statement for ω in terms of interpolation spaces. The first part of the statement is well known, see [4] for instance.

Corollary 3.3. *Let $p \in [1,\infty)$, and $\theta \in (0, \frac{1+\alpha}{2})$. The function ω is an element of the space*

$$
\big(\mathrm{L}^2(D), \mathrm{PH}^2(D)\big)_{\theta,p} \cap \big(\mathrm{L}^2(D), \mathrm{PH}^2(D)\big)_{(1+\alpha)/2,\infty}.
$$

The mapping is, however, not contained in the (continuous) interpolation space $(\mathrm{L}^2(D), \mathrm{PH}^2(D))_{(1+\alpha)/2}$.

Proof. The first statement is a direct consequence of Lemma 3.2 and the embeddings

$$
\big(\mathrm{L}^2(D), \mathrm{PH}^2(D)\big)_{(1+\alpha)/2,\infty} \subset \big(\mathrm{L}^2(D), \mathrm{PH}^2(D)\big)_{\theta,1} \subset \big(\mathrm{L}^2(D), \mathrm{PH}^2(D)\big)_{\theta,p},
$$

see for instance Propositions 1.3 and 1.4 in [19]. The last claim follows from Lemma 3.1. □

4. Proof of Theorem 1.1

This section is devoted to the proof of the main result Theorem 1.1. The essential ingredients of the proof are an application of Theorem 2.1 in [13], and the estimates for the K-functional of ω from Lemmas 3.1 and 3.2.

To transform our problem into the setting of Ivanov and Kalton, we introduce the linear functional

$$
\Phi(v + \lambda\omega) := \lambda, \qquad v \in \mathrm{PH}^2(D),\ \lambda \in \mathbb{R},
$$

on the space $\mathrm{PH}^2(D) \oplus V$. The latter is equipped with the norm

$$\|v + \lambda\omega\|' := |\lambda| + \|v\|_{\mathrm{PH}^2(D)}, \qquad v \in \mathrm{PH}^2(D),\ \lambda \in \mathbb{R}.$$

The kernel of Φ then coincides with $\mathrm{PH}^2(D)$, and Φ is bounded. Following Section 2 in [13], we now also define the quantities

$$\begin{aligned} \|\Phi\|_t &:= \sup\{|\Phi(f)| \mid f \in \mathrm{PH}^2(D) \oplus V \text{ with } t\|f\|_{\mathrm{L}^2(D)} + \|f\|' \leq 1\}, \\ \sigma_0 &:= \lim_{\tau\to\infty} \inf_{0<\tau t\leq 1} \tfrac{1}{\log\tau} \log \tfrac{K(\tau t,\Phi)}{K(t,\Phi)}, \\ \sigma_1 &:= \lim_{\tau\to\infty} \sup_{0<\tau t\leq 1} \tfrac{1}{\log\tau} \log \tfrac{K(\tau t,\Phi)}{K(t,\Phi)}, \end{aligned}$$

involving the function

$$K(t,\Phi) := K(t,\Phi,(\mathrm{PH}^2(D)\oplus V)^*, \mathrm{L}^2(D)^*), \qquad t > 0.$$

(The symbol W^* denotes the dual space of $W \in \{\mathrm{PH}^2(D) \oplus V, \mathrm{L}^2(D)\}$.) The reasoning in the proof of Proposition 3.2 in [13] then gives rise to the useful formulas

$$\sigma_0 = \lim_{\tau\to\infty} \inf_{s\geq 1} \tfrac{1}{\log\tau} \log \tfrac{\|\Phi\|_s}{\|\Phi\|_{s\tau}}, \qquad \sigma_1 = \lim_{\tau\to\infty} \sup_{s\geq 1} \tfrac{1}{\log\tau} \log \tfrac{\|\Phi\|_s}{\|\Phi\|_{s\tau}}. \tag{4.1}$$

We next determine σ_0 and σ_1 in terms of the exponent α.

Lemma 4.1. *The identity $\sigma_0 = \sigma_1 = \frac{1-\alpha}{2}$ is valid.*

Proof. 1) For convenience, we write $K(\cdot,\omega)$ for $K(\cdot,\omega,\mathrm{L}^2(D),\mathrm{PH}^2(D))$. Let $t > 0$, and $f = v + \lambda\omega$ in $\mathrm{PH}^2(D) \oplus V$ with $t\|f\|_{\mathrm{L}^2(D)} + \|f\|' \leq 1$. In case λ is zero, the relation

$$|\Phi(f)| = 0 \leq \tfrac{1}{1+tK(\frac{1}{t},\omega)}$$

is clearly true. The next goal is to establish the same estimate for nonzero real λ. By definition of the norm $\|\cdot\|'$, we infer the estimates

$$\begin{aligned} 1 &\geq t|\lambda|\big(\|\tfrac{1}{\lambda}v + \omega\|_{\mathrm{L}^2(D)} + \tfrac{1}{t}\|\tfrac{1}{\lambda}v\|_{\mathrm{PH}^2(D)}\big) + |\lambda| \geq t|\lambda|K(\tfrac{1}{t},\omega) + |\lambda| \\ &= (1 + tK(\tfrac{1}{t},\omega))|\Phi(f)|. \end{aligned}$$

Taking now the supremum with respect to all functions f in $\mathrm{PH}^2(D) \oplus V$ with $t\|f\|_{\mathrm{L}^2(D)} + \|f\|' \leq 1$, we conclude the result

$$\|\Phi\|_t \leq \tfrac{1}{1+tK(\frac{1}{t},\omega)}. \tag{4.2}$$

Next, we derive a similar lower inequality for $\|\Phi\|_t$. To that end, let $w \in \mathrm{PH}^2(D)$ with

$$K(\tfrac{1}{t},\omega) + \tfrac{1}{t} \geq \|w + \omega\|_{\mathrm{L}^2(D)} + \tfrac{1}{t}\|w\|_{\mathrm{PH}^2(D)}. \tag{4.3}$$

We then put $v := \|\Phi\|_t w \in \mathrm{PH}^2(D)$. Note here the relation

$$t\|v + \|\Phi\|_t\omega\|_{\mathrm{L}^2(D)} + \|\Phi\|_t + \|v\|_{\mathrm{PH}^2(D)} \geq 1, \tag{4.4}$$

being a consequence of the definition of $\|\Phi\|_t$. Combining (4.3) and (4.4), we conclude the estimates

$$1 \geq \frac{\|\frac{1}{\|\Phi\|_t}v + \omega\|_{\mathrm{L}^2(D)} + \frac{1}{t}\|\frac{1}{\|\Phi\|_t}v\|_{\mathrm{PH}^2(D)}}{K(\frac{1}{t},\omega) + \frac{1}{t}}$$

$$= \frac{\frac{1}{t\|\Phi\|_t}\big(t\|v + \|\Phi\|_t \omega\|_{\mathrm{L}^2(D)} + \|v\|_{\mathrm{PH}^2(D)}\big)}{K(\frac{1}{t},\omega) + \frac{1}{t}}$$
$$= \frac{\frac{1}{t\|\Phi\|_t}\big(t\|v + \|\Phi\|_t \omega\|_{\mathrm{L}^2(D)} + \|\Phi\|_t + \|v\|_{\mathrm{PH}^2(D)} - \|\Phi\|_t\big)}{K(\frac{1}{t},\omega) + \frac{1}{t}}$$
$$\geq \frac{\frac{1}{t\|\Phi\|_t}(1 - \|\Phi\|_t)}{K(\frac{1}{t},\omega) + \frac{1}{t}}.$$

Simple algebraic manipulations then lead to the desired lower inequality

$$\|\Phi\|_t \geq \tfrac{1}{tK(\frac{1}{t},\omega)+2}. \tag{4.5}$$

2) To obtain a lower bound for σ_0, we plug (4.2) and (4.5) into formula (4.1). In this way, the relations

$$\sigma_0 = \lim_{\tau\to\infty} \inf_{s\geq 1} \tfrac{1}{\log\tau} \log \tfrac{\|\Phi\|_s}{\|\Phi\|_{s\tau}} \geq \lim_{\tau\to\infty} \inf_{s\geq 1} \tfrac{1}{\log\tau} \log\Big(\tfrac{s\tau K(\frac{1}{s\tau},\omega)+1}{sK(\frac{1}{s},\omega)+2}\Big)$$

follow. Taking now also Lemmas 3.1 and 3.2 into account, we arrive at the crucial result

$$\sigma_0 \geq \lim_{\tau\to\infty} \inf_{s\geq 1} \tfrac{1}{\log\tau} \log\Big(\tfrac{C_l s^{\frac{1-\alpha}{2}} \tau^{\frac{1-\alpha}{2}}}{C_u s^{\frac{1-\alpha}{2}}+2}\Big) \geq \tfrac{1-\alpha}{2}.$$

Similar reasoning leads to the analogous statement

$$\sigma_1 \leq \tfrac{1-\alpha}{2}.$$

Altogether, we conclude the asserted identity $\sigma_0 = \sigma_1 = \frac{1-\alpha}{2}$. □

Combining the above Lemma 4.1 with Theorem 2.1 in [13], we are now in the position to establish Theorem 1.1.

Proof of Theorem 1.1. 1) We first derive the second asserted identity

$$\big(\mathrm{PH}^1(D), \mathrm{PH}^2(D) \oplus V\big)_{\theta_2,2} = \mathrm{PH}^{1+\theta_2}(D), \qquad \theta_2 \in [0,\alpha).$$

Applying Theorem 2.1 in [13] together with Lemma 4.1, we obtain the identities

$$\begin{aligned}\big(\mathrm{PH}^2(D) \oplus V, \mathrm{L}^2(D)\big)_{\frac{1-\theta_2}{2},2} &= \big(\mathrm{PH}^2(D), \mathrm{L}^2(D)\big)_{\frac{1-\theta_2}{2},2} \\ &= \big(\mathrm{L}^2(D), \mathrm{PH}^2(D)\big)_{\frac{1+\theta_2}{2},2} = \mathrm{PH}^{1+\theta_2}(D).\end{aligned}$$

In particular, the statement

$$\big(\mathrm{PH}^2(D) \oplus V, \mathrm{L}^2(D)\big)_{1/2,2} = \mathrm{PH}^1(D) \tag{4.6}$$

is obtained. Using now additionally reiteration interpolation, see Corollary 1.24 in [19] for instance, we infer the second desired formula

$$\begin{aligned}\big(\mathrm{PH}^1(D), \mathrm{PH}^2(D) \oplus V\big)_{\theta_2,2} &= \big(\mathrm{PH}^2(D) \oplus V, \mathrm{PH}^1(D)\big)_{1-\theta_2,2} \\ &= \big(\mathrm{PH}^2(D) \oplus V, (\mathrm{PH}^2(D) \oplus V, \mathrm{L}^2(D))_{1/2,2}\big)_{1-\theta_2,2} \\ &= \big(\mathrm{PH}^2(D) \oplus V, \mathrm{L}^2(D)\big)_{\frac{1-\theta_2}{2},2} = \mathrm{PH}^{1+\theta_2}(D).\end{aligned}$$

2) Let $\theta_1 \in (\alpha, 1)$. By Theorem 2.1 in [13] and Lemma 4.1, the interpolation space

$$\big(\mathrm{PH}^2(D), \mathrm{L}^2(D)\big)_{\frac{1-\theta_1}{2},2} = \big(\mathrm{L}^2(D), \mathrm{PH}^2(D)\big)_{\frac{1+\theta_1}{2},2} = \mathrm{PH}^{1+\theta_1}(D)$$

is a closed subspace of codimension one in the space

$$\begin{aligned}
&\big(\mathrm{PH}^2(D) \oplus V, \mathrm{L}^2(D)\big)_{\frac{1-\theta_1}{2},2} \\
&\qquad = \big(\mathrm{PH}^2(D) \oplus V, (\mathrm{PH}^2(D) \oplus V, \mathrm{L}^2(D))_{1/2,2}\big)_{1-\theta_1,2} \\
&\qquad = \big(\mathrm{PH}^2(D) \oplus V, \mathrm{PH}^1(D)\big)_{1-\theta_1,2} \\
&\qquad = \big(\mathrm{PH}^1(D), \mathrm{PH}^2(D) \oplus V\big)_{\theta_1,2}. \qquad (4.7)
\end{aligned}$$

Note here that we employ formula (4.6) in the second identity. Furthermore, the one-dimensional space V from (1.2) is also a closed subspace of $\big(\mathrm{PH}^1(D), \mathrm{PH}^2(D) \oplus V\big)_{\theta_1,2}$ with $V \cap \mathrm{PH}^{1+\theta_1}(D) = \{0\}$, see Corollary 3.3 or [4] for instance. Altogether, we arrive at the first asserted identity

$$\big(\mathrm{PH}^1(D), \mathrm{PH}^2(D) \oplus V\big)_{\theta_1,2} = \mathrm{PH}^{1+\theta_1}(D) \oplus V.$$

3) It remains to conclude that the space $\mathrm{PH}^{1+\alpha}(D)$ is not closed in $\big(\mathrm{PH}^1(D), \mathrm{PH}^2(D) \oplus V\big)_{\alpha,2}$. This statement is a consequence of the formula

$$\big(\mathrm{PH}^1(D), \mathrm{PH}^2(D) \oplus V\big)_{\alpha,2} = \big(\mathrm{PH}^2(D) \oplus V, \mathrm{L}^2(D)\big)_{\frac{1-\alpha}{2},2}, \qquad (4.8)$$

Theorem 2.1 in [13], Lemma 4.1, and the relation

$$\mathrm{PH}^{1+\alpha}(D) = \big(\mathrm{PH}^2(D), \mathrm{L}^2(D)\big)_{\frac{1-\alpha}{2},2}.$$

(Note that (4.8) is verified in the same way as (4.7).) □

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