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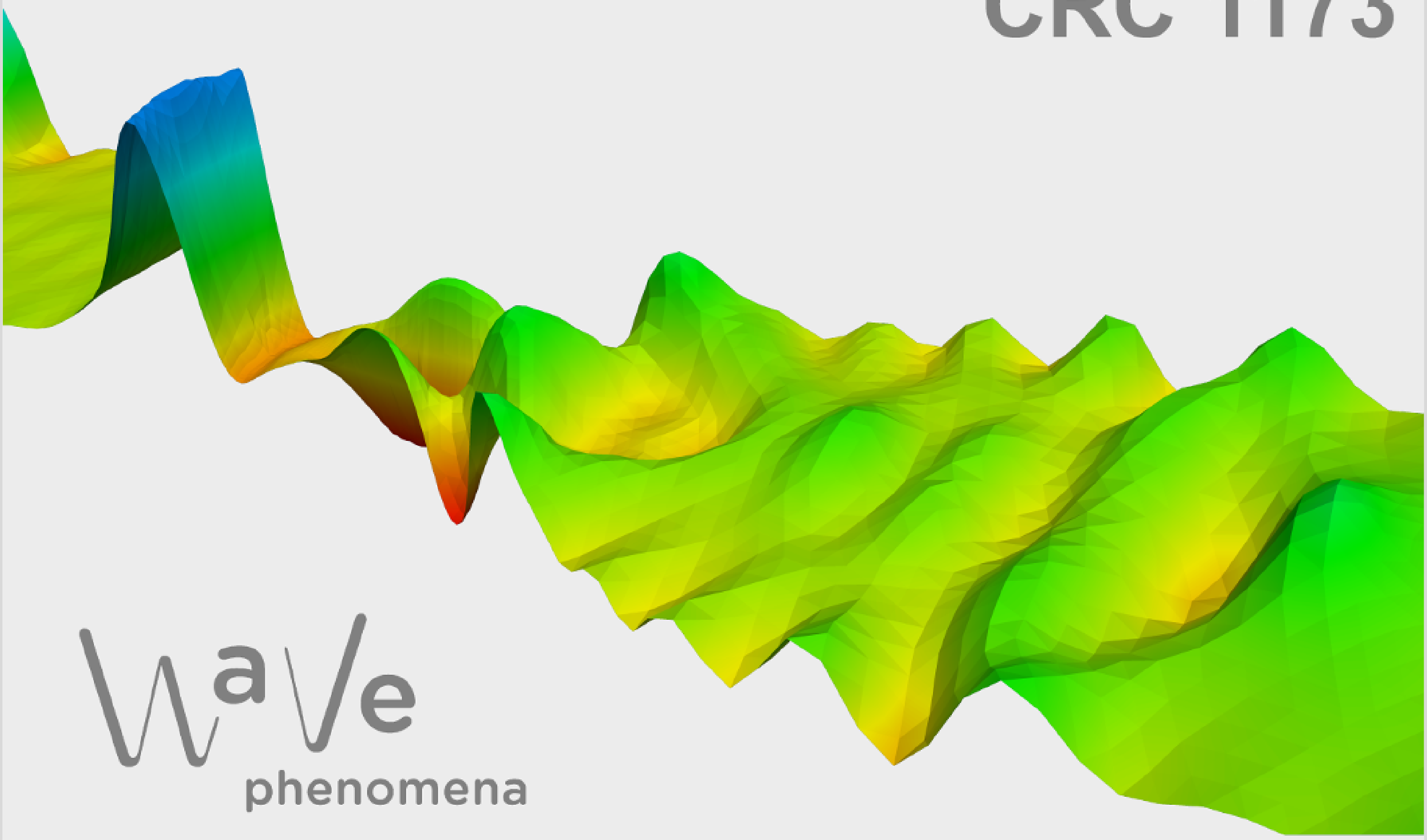

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IMPROVED RESOLVENT ESTIMATES FOR CONSTANT-COEFFICIENT ELLIPTIC OPERATORS IN THREE DIMENSIONS

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Abstract. We prove new L^p-L^q-estimates for solutions to elliptic differential operators with constant coefficients in $\mathbb{R}^3$. We use the estimates for the decay of the Fourier transform of particular surfaces in $\mathbb{R}^3$ with vanishing Gaussian curvature due to Erdős–Salmhofer to derive new Fourier restriction–extension estimates. These allow for constructing distributional solutions in $L^q(\mathbb{R}^3)$ for L^p-data via limiting absorption by well-known means.

1. Introduction

The purpose of this note is to show new L^p-L^q-estimates for solutions to elliptic differential equations in $\mathbb{R}^3$. Let

$$p(\xi) = \sum_{\substack{\alpha \in \mathbb{N}_0^3, \\ |\alpha| \leq N}} a_\alpha \xi^\alpha$$

be a multi-variate polynomial in $\mathbb{R}^3$ with real coefficients and suppose that $a_\alpha \neq 0$ for some $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = N$. We consider partial differential operators

$$P(D) = p(-i\nabla_x) = \sum_{|\alpha| \leq N} a_\alpha (-i)^{|\alpha|} \partial^\alpha \tag{1}$$

such that for $u \in \mathcal{S}'(\mathbb{R}^3)$ we have

$$\mathcal{F}(P(D)u)(\xi) = p(\xi)\hat{u}(\xi).$$

By ellipticity we mean that

$$p_N(\xi) = \sum_{|\alpha|=N} a_\alpha \xi^\alpha \neq 0$$

for $\xi \neq 0$. We assume $p_N(\xi) > 0$ for the sake of definiteness. In the following we prove existence of solutions $u \in L^q(\mathbb{R}^3)$ such that

$$P(D)u = f$$

for $f \in L^p(\mathbb{R}^3)$ in a certain range of p and q, which satisfy the estimate

$$\|u\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}.$$

The properties of the vanishing set of $p(\xi)$ play a key role for constructing solutions: Gutiérrez [8] constructed solutions for $p(\xi) = |\xi|^2 - 1$. In most previous works on elliptic operators was assumed that $\Sigma_0 = \{p(\xi) = 0\}$ is a smooth manifold with non-vanishing Gaussian curvature $K \neq 0$. In this case the analysis of Gutiérrez applies. Recently, Castéras–Földes [3] analyzed fourth-order Schrödinger operators (in dimensions $d \geq 2$) with smooth characteristic surface, and estimates depending on the number of non-vanishing principal curvatures were proved. A wider range was covered in [14], where also surfaces with conic singularities were treated. Presently, we consider the effect of vanishing Gaussian curvature

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in a generic case, which was described by Erdős–Salmhofer [6]. The idea of constructing solutions is to consider approximates

$$\hat{u}_\delta(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{ix.\xi} \hat{f}(\xi)}{p(\xi) + i\delta} d\xi$$

for $\delta \neq 0$ and show uniform bounds

$$\|u_\delta\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)} \tag{2}$$

for fixed $P(D)$.

Then we shall find distributional limits $u \in L^q(\mathbb{R}^3)$, which satisfy

$$P(D)u = f \text{ in } \mathcal{S}'(\mathbb{R}^3)$$

and

$$\|u\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}.$$

This is referred to as limiting absorption principle. We shall still assume that $\nabla p(\xi) \neq 0$ for $\xi \in \Sigma_0$. This is a generic assumption for polynomials. In this case Sokhotsky's formula yields for solutions as described above

$$\begin{aligned} u(x) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{ix.\xi} \hat{f}(\xi)}{p(\xi) \pm i0} d\xi \\ &= \mp \frac{i\pi}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix.\xi} \hat{f}(\xi) \delta_{\Sigma_0}(\xi) d\xi + \frac{1}{(2\pi)^3} v.p. \int_{\mathbb{R}^3} \frac{e^{ix.\xi} \hat{f}(\xi)}{p(\xi)} d\xi. \end{aligned}$$

This points out a close connection to Fourier restriction. The most basic L^p-L^q-results rely on the decay of the Fourier transform of the surface measure. This in term is caused by the curvature of the surface. If $K \neq 0$, the estimate

$$|\hat{\mu}_S(\xi)| = \Big| \int_S e^{ix.\xi} dx \Big| \lesssim \langle \xi \rangle^{-1}$$

is classical (cf. [13, 15]). Corresponding L^p-L^q-estimates for solutions were proved in [14].

In this note we consider vanishing total curvature in a generic sense. For constructing solutions as laid out above, we also have to consider level sets $\Sigma_a = \{p(\xi) = a\}$ for $|a| \leq \delta_0$. We recall the assumptions in Erdős–Salmhofer:

Let I be a compact interval and let $\mathcal{D} = e^{-1}(I)$. Suppose that Σ_a is a two-dimensional submanifold for each $a \in I$. Let $f \in C^\infty_c(\mathcal{D})$ and define

$$\hat{\mu}_a(x) = \int_{\Sigma_a} e^{ix.\xi} f(\xi) d\sigma_a(\xi) \tag{3}$$

the Fourier transform of the surface carried measure $f d\sigma_a$.

Let $C_0 = \text{diam}(\mathcal{D})$, $C_1 = \|p\|_{C^5(\mathcal{D})}$. The following assumptions have to be met:

Assumption 1:

$$C_2 = \min_{\xi \in \mathcal{D}} |\nabla p(\xi)| > 0, \tag{4}$$

which means that $(\Sigma_a)_{a \in I}$ is a regular foliation of $\mathcal{D}$.

Let $K : \mathcal{D} \to \mathbb{R}$ be the Gaussian curvature of the foliation, i.e., for $\xi \in \Sigma_a \subseteq \mathcal{D}$, $K(\xi)$ denotes the curvature of Σ_a at ξ.

The crucial assumption is that the vanishing set of the Gaussian curvature is a submanifold, which intersects $(\Sigma_a)_{a \in I}$ transversally:

Assumtion 2: Let $\mathcal{C} = \{\xi \in \mathcal{D} : K(\xi) = 0\}$. Then

$$C_3 = \min_{\xi \in \mathcal{D}} (\{ |\nabla p(\xi) \times \nabla K(\xi)| \, : \xi \in \mathcal{C}\}) > 0.$$

With ∇K non-vanishing on $\mathcal{C}$, it is a two-dimensional submanifold by the regular value theorem. Since p and K are smooth, we find that

$$\Gamma_a = \mathcal{C} \cap \Sigma_a$$

is a finite union of disjoint regular curves on Σ_a for each $a \in I$.
Let

$$\xi \mapsto w(\xi) = \frac{\nabla p(\xi) \times \nabla K(\xi)}{|\nabla p(\xi) \times \nabla K(\xi)|}$$

be the unit vectorfield tangent to Γ_a. Denote the normal map $\nu : \mathcal{D} \to \mathbb{S}^2$ by

$$\nu(\xi) = \frac{\nabla p(\xi)}{|\nabla p(\xi)|}.$$

Recall that the Gaussian curvature is given by the Jacobian of the normal map restricted to each surface, $\nu : \Sigma_a \to \mathbb{S}^2$: $K(\xi) = \det \nu'(\xi)$.

We further require the following regularity assumption on the Gauss map.

Assumption 3: The number of preimages of $\nu : \Sigma_a \to \mathbb{S}^2$ is finite, i.e.,

$$C_4 = \sup_{a \in I} \sup_{\omega \in \mathbb{S}^2} \text{card}\{p \in \Sigma_a \,:\, \nu(p) = \omega\} < \infty.$$

On the curves Γ_a, exactly one of the principal curvatures vanish. We define a (local) unit vectorfield $Z \in T\Sigma_a$ along Γ_a in the tangent plane of Σ_a. Z can be extended to a neighbourhood of Γ_a as the direction of the principal curvature that is small and vanishes on Γ_a. We assume that Z is transversal to Γ_a up to finitely many points (called *tangential points*) and the angle between Z and Γ_a increases linearly:

Assumption 4: There exist positive constants C_5, C_6 such that for any $a \in I$ the set of tangential points

$$\mathcal{T}_a = \{\xi \in \Gamma_a \,:\, Z(\xi) \times w(\xi) = 0\},$$

is finite with cardinality $N_a = |\mathcal{T}_a| \leq C_5$. For all $\xi \in \Gamma_a$

$$|Z(\xi) \times w(\xi)| \geq C_6 \cdot d_a(\xi),$$

where $d_a(\xi)$ is defined as follows:
If $N_a = 0$, then $d_a(\xi) = 1$. If $N_a \neq 0$, and $\mathcal{T}_a = \{\xi_a^{(1)}, \dots, \xi_a^{(N_a)}\}$, then

$$d_a(\xi) = \min(\{|\xi - \xi_a^{(j)}| \,:\, j = 1, \dots, N_a\}), \quad a \in I,\ p \in \Sigma_a.$$

Define

$$D_a(\omega) = \min\{|\nu(\xi_a^{(j)} \times \omega| \,:\, 1 \leq j \leq N_a\},\ \omega \in \mathbb{S}^2.$$

if $N_a \neq 0$ and $D_a(\omega) = 1$ if $N_a = 0$.

Under the above assumptions, Erdős–Salmhofer [6, Theorem 2.1] proved the following dispersive estimate for the Fourier transform of the surface measure μ_a:

$$|\hat{\mu}_a(\xi)| \leq C \langle \xi \rangle^{-\frac{3}{4}} \tag{5}$$

with $C = C(C_0, \dots, C_6, \|f\|_{C^2(\mathcal{D})})$. This morally corresponds to a decay from $\frac{3}{2}$ principal curvatures bounded from below in modulus and thus improves the previous result for one non-vanishing principal curvature (cf. [14, Theorem 1.3]). In this article we record its consequence for solutions to elliptic differential operators. As argued in [6, Remark 1, p. 268], the above assumptions are generic for surfaces in $\mathbb{R}^3$. Thus, we say that the results apply to generic elliptic operators in $\mathbb{R}^3$.

In the first step, we derive a Fourier restriction–extension theorem for surfaces Σ_a by following along the lines of the preceding work [14]. We prove strong bounds

$$\| \int_{\mathbb{R}^3} e^{ix.\xi} \delta_{\Sigma_a}(\xi) \beta(\xi) \hat{f}(\xi) d\xi \|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)} \tag{6}$$

within a pentagonal region. Here $\beta \in C_c^\infty$ localizes to a suitable neighbourhood of $\{K = 0\}$ in $\big(\Sigma_a\big)_{a \in [-\delta_0, \delta_0]}$. Away from $\{K = 0\}$, [14, Theorem 1.3] provides better estimates for $d = 3$, $k = 2$.

On part of the boundary of the pentagonal region, we show weak bounds

$$\| \int_{\mathbb{R}^3} e^{ix.\xi} \delta_{\Sigma_a}(\xi) \beta(\xi) \hat{f}(\xi) d\xi \|_{L^{q,\infty}(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)} \tag{7}$$

$$\| \int_{\mathbb{R}^3} e^{ix.\xi} \delta_{\Sigma_a}(\xi) \beta(\xi) \hat{f}(\xi) d\xi \|_{L^{q}(\mathbb{R}^3)} \lesssim \|f\|_{L^{p,1}(\mathbb{R}^3)}, \tag{8}$$

and lastly, restricted weak bounds

$$\| \int_{\mathbb{R}^3} e^{ix.\xi} \delta_{\Sigma_a}(\xi) \beta(\xi) \hat{f}(\xi) d\xi \|_{L^{q,\infty}(\mathbb{R}^3)} \lesssim \|f\|_{L^{p,1}(\mathbb{R}^3)} \tag{9}$$

at its inner endpoints. We refer to Figure 2 for a diagram. For $X, Y \in [0,1]^2$ we write $[X,Y] = \{Z : \exists \lambda \in [0,1] : Z = \lambda X + (1-\lambda) Y\}$ and correspondingly (X,Y), $(X,Y]$, etc.

Proposition 1.1. *Let $p : \mathbb{R}^3 \to \mathbb{R}$ be an elliptic polynomial with $\delta_0 > 0$ such that for $\Sigma_a = \{p(\xi) = a\}$, $-\delta_0 \leq a \leq \delta_0$ Assumptions 1-4 are satisfied in a neighbourhood of $K = 0$ in Σ_a. Then, we find* (6) *to hold for $(\frac{1}{p}, \frac{1}{q}) \in [0,1]^2$ provided that*

$$\frac{1}{p} > \frac{7}{10}, \quad \frac{1}{q} < \frac{3}{10}, \quad \frac{1}{p} - \frac{1}{q} \geq \frac{4}{7}.$$

Let

$$B = \big(\frac{7}{10}, \frac{9}{70}\big), \; C = \big(\frac{7}{10}, 0\big), \quad B' = \big(\frac{61}{70}, \frac{3}{10}\big), \; C' = \big(1, \frac{3}{10}\big) :$$

Furthermore, we find (7) *to hold for $(1/p, 1/q) \in (B', C']$,* (8) *for $(1/p, 1/q) \in (B, C]$, and* (9) *for $(1/p, 1/q) \in \{B, B'\}$.*

In the second step we foliate a neighbourhood U of Σ_0 with level sets of p to show bounds $\|A_\delta f\|_{L^q} \lesssim \|f\|_{L^p(\mathbb{R}^3)}$ for

$$A_\delta f(x) = \int_{\mathbb{R}^3} \frac{e^{ix.\xi} \beta_1(\xi)}{p(\xi) + i\delta} \hat{f}(\xi) d\xi \tag{10}$$

independent of δ. Here, p, q are as in Proposition 1.1 and $|p(\xi)| \leq \delta_0$ for $\xi \in \text{supp}\,(\beta_1)$ with $\Sigma_0 \subseteq \text{supp}\,(\beta_1)$. Away from the singular set, estimates for

$$B_\delta f(x) = \int_{\mathbb{R}^3} \frac{e^{ix.\xi} \beta_2(\xi)}{p(\xi) + i\delta} \hat{f}(\xi) d\xi \tag{11}$$

with $\beta_1 + \beta_2 \equiv 1$ follow from Young's inequality and properties of the Bessel potential. The estimate of $\|B_\delta\|_{L^p \to L^q}$ depends on the order of the elliptic operator.

The method of proof is well-known and detailed in [14]; see also [11, 9] and references therein. We shall be brief. It turns out that one can follow along the lines of [14] very closely, substituting $k = \frac{3}{2}$ non-vanishing principal curvatures. We prove the following:

Theorem 1.2. *Let $p : \mathbb{R}^3 \to \mathbb{R}$ be an elliptic polynomial of degree $N \geq 2$. Let $1 < p_1, p_2, q < \infty$ and $f \in L^{p_1}(\mathbb{R}^3) \cap L^{p_2}(\mathbb{R}^3)$. Suppose that there is $\delta_0 > 0$ such that Assumptions 1-4 are satisfied for $(\Sigma_a)_{a \in [-\delta_0, \delta_0]}$. Then, there is $u \in L^q(\mathbb{R}^3)$ satisfying*

$$P(D)u = f$$

in the distributional sense and the estimate

$$\|u\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^{p_1} \cap L^{p_2}(\mathbb{R}^3)}$$

provided that

$$\frac{1}{p_1} > \frac{7}{10}, \quad \frac{1}{q} < \frac{3}{10}, \quad \frac{1}{p_1} - \frac{1}{q} \geq \frac{4}{7}$$

and for $N \leq 3$

$$0 \leq \frac{1}{p_2} - \frac{1}{q} \leq \frac{N}{3}, \quad \big(\frac{1}{q}, \frac{1}{p_2}\big) \notin \begin{cases} \{(0, \frac{2}{3}), \; (\frac{1}{3}, 1)\} \text{ for } N = 2, \\ \{(0,1)\} \text{ for } N = 3. \end{cases}$$

2. The Fourier restriction-extension estimate

The purpose of this section is to prove Proposition 1.1. We shall follow the argument of [14, Section 4]. In the first step, we localize to a small neighbourhood of the vanishing set $\{K = 0\}$, which by assumptions is a two-dimensional manifold in $\mathcal{D}$. In the complementary set, by compactness, we can apply [14, Theorem 1.3], which gives uniform L^p-L^q-estimates in a broader range. Thus, it is enough to suppose that Assumptions 1-4 are valid in a neighbourhood of $\{K = 0\}$. The proof follows [14, Section 4] closely. In the first step, by finite decomposition and rotations, we change to parametric representation of $\Sigma_a = \{(\xi', \psi(\xi')) : \xi' \in B(0,c)\}$. We show bounds $T : L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)$ for

$$Tf(x) = \int_{\mathbb{R}^3} \delta(\xi_3 - \psi(\xi')) e^{ix.\xi} \chi(\xi') \hat{f}(\xi) d\xi.$$

The following decay estimate, which is (5), is central.

$$\left| \int e^{i(x'.\xi' + x_3 \psi(\xi'))} \beta(\xi') d\xi' \right| \lesssim (1 + |x_3|)^{-\frac{3}{4}}.$$

Applying the TT^* argument (cf. [16, 7, 10]), we find the following Strichartz estimate:

$$\left\| \int e^{i(x'.\xi' + x_3 \psi(\xi'))} \beta(\xi') \hat{f}(\xi') d\xi' \right\|_{L_x^{\frac{14}{3}}(\mathbb{R}^3)} \lesssim \|f\|_{L^2_{\xi'}(B(0,c))}. \tag{12}$$

We recall the following lemma to decompose the delta distribution:

Lemma 2.1 ([4, Lemma 2.1])**.** *There is a smooth function ϕ satisfying supp$(\hat{\phi}) \subseteq \{t : |t| \sim 1\}$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\langle \delta(\xi_3 - \psi(\xi')), f \rangle = \sum_{j \in \mathbb{Z}} 2^j \int_{\mathbb{R}^3} \phi(2^j(\xi_3 - \psi(\xi'))) \chi(\xi') f(\xi) d\xi.$$

By this, we can write

$$Tf(x) = \sum_{j \in \mathbb{Z}} 2^j \int_{\mathbb{R}^3} \phi(2^j(\xi_3 - \psi(\xi'))) e^{ix.\xi} \chi(\xi') \hat{f}(\xi) d\xi := \sum_{j \in \mathbb{Z}} 2^j T_{2^{-j}} f.$$

As pointed out in [4], the contribution of $j \leq 0$ is easier to estimate.

The contribution of $j \geq 0$, i.e., close to the singularity, is estimated by Strichartz and kernel estimates:

Lemma 2.2 (cf. [14, Lemma 4.3])**.** *Let $q \geq \frac{14}{3}$. Then, we find the following estimate to hold:*

$$\|T_{2^j} f\|_{L^q(\mathbb{R}^3)} \lesssim 2^{\frac{-j}{2}} \|f\|_{L^2(\mathbb{R}^3)}.$$

This estimate does not admit summation. For this purpose, we interpolate with the kernel estimate:

Lemma 2.3 (cf. [14, Lemma 4.4])**.** *Let*

$$K_\delta(x) = \int_{\mathbb{R}^3} e^{ix.\xi} \beta(\xi') \phi\Big(\frac{\xi_3 - \psi(\xi')}{\delta}\Big) d\xi.$$

Then K_δ is supported in $\{(x', x_3) : |x_3| \sim \delta^{-1}\}$, and we find the following estimates to hold:

$$|K_\delta(x)| \lesssim_N \delta^N (1 + \delta|x|)^{-N}, \text{ if } |x'| \geq c|x_3|,$$
$$|K_\delta(x)| \lesssim \delta^{\frac{7}{4}}, \text{ if } |x'| \leq c|x_3|.$$

The last ingredient to show (restricted) weak endpoint estimates is Bourgain's summation argument (cf. [1, 2] and [12, Lemma 2.3] for an elementary proof):

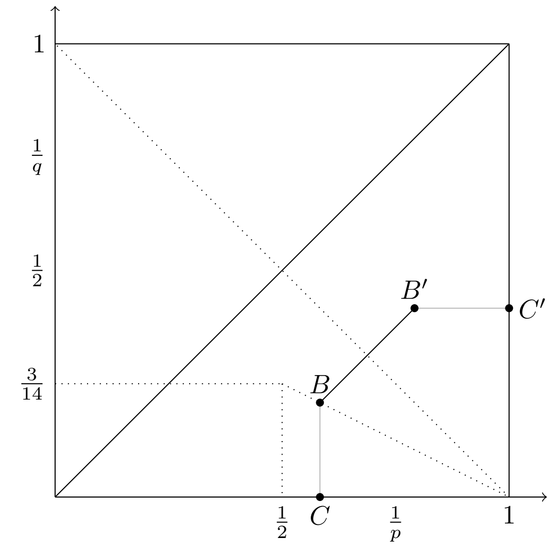

FIGURE 1. Pentagonal region, within which strong L^p-L^q-Fourier restriction extension estimates hold.

Lemma 2.4. *Let $\varepsilon_1, \varepsilon_2 > 0$, $1 \leq p_1, p_2 \leq \infty$, $1 \leq q_1, q_2 < \infty$. For every $j \in \mathbb{Z}$ let T_j be a linear operator, which satisfies*

$$\|T_j(f)\|_{q_1} \leq M_1 2^{\varepsilon_1 j} \|f\|_{p_1}$$
$$\|T_j(f)\|_{q_2} \leq M_2 2^{-\varepsilon_2 j} \|f\|_{p_2}.$$

Then, for θ, q and p_i defined by $\theta = \frac{\varepsilon_2}{\varepsilon_1 \varepsilon_2}$, $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$ and $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, the following hold:

$$\| \sum_j T_j(f) \|_{q,\infty} \leq C M_1^\theta M_2^{1-\theta} \|f\|_{p,1}, \tag{13}$$

$$\| \sum_j T_j(f) \|_{q} \leq C M_1^\theta M_2^{1-\theta} \|f\|_{p,1} \text{ if } q_1 = q_2 = q, \tag{14}$$

$$\| \sum_j T_j(f) \|_{q,\infty} \leq C M_1^\theta M_2^{1-\theta} \|f\|_{p} \text{ if } p_1 = p_2. \tag{15}$$

We interpolate the bounds

$$2^j \|T_{2^{-j}} f\|_{L^q(\mathbb{R}^3)} \lesssim 2^{\frac{j}{2}} \|f\|_{L^2(\mathbb{R}^3)}, \quad \frac{14}{3} \leq q \leq \infty,$$

and

$$2^j \|T_{2^{-j}} f\|_{L^\infty(\mathbb{R}^3)} \lesssim 2^{-\frac{3j}{4}} \|f\|_{L^1(\mathbb{R}^3)}$$

as above together with duality to find restricted weak endpoint bounds

$$\|Tf\|_{L^{q,\infty}(\mathbb{R}^3)} \lesssim \|f\|_{L^{p,1}(\mathbb{R}^3)}$$

for $(1/p, 1/q) \in \{B, B'\}$, weak bounds

$$\|Tf\|_{L^{q,\infty}} \lesssim \|f\|_{L^p}, \quad \|Tf\|_{L^q} \lesssim \|f\|_{L^{p,1}}$$

for $(1/p, 1/q) \in (B', C']$, respectively, $(1/p, 1/q) \in (B, C]$, and strong bounds in the interior of the pentagon $\mathrm{conv}(A, B, C, C', B')$ with $A = (1, 0)$,

$$B = (\frac{7}{10}, \frac{9}{70}), \ C = (\frac{7}{10}, 0), \quad B' = (\frac{61}{70}, \frac{3}{10}), \ C' = (1, \frac{3}{10}) :$$

Real interpolation of the weak bounds at B and B' gives strong bounds on (B, B'). This finishes the proof of Proposition 1.1. □

3. L^p-L^q-ESTIMATES FOR SOLUTIONS TO ELLIPTIC DIFFERENTIAL OPERATORS

In this section we prove Theorem 1.2 relying on Proposition 1.1. The argument parallels [14, Section 5.2] very closely, to avoid repitition we shall be brief. Let A_δ and B_δ be as in (10) and (11). We start with the more difficult estimate of A_δ. We show boundedness of $A_\delta : L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)$ independently of δ with p, q as in Proposition 1.1. For this it is enough to show restricted weak type bounds

$$\|A_\delta\|_{L^{q_0,\infty}} \lesssim \|f\|_{L^{p_0,1}}$$

for $(1/p_0, 1/q_0) = (61/70, 3/10)$ and the bounds

$$\|A_\delta f\|_{L^q} \lesssim \|f\|_{L^{p,1}}$$

for $(1/p, 1/q) \in ((61/70, 3/10), (1, 3/10)]$ as strong bounds for A_δ with p, q as in Proposition 1.1 are recovered by interpolation and duality. As $\nabla p(\xi) \neq 0$ for $\xi \in \text{supp}(\beta_1)$ by construction, we can change to generalized polar coordinates. Let $\xi = \xi(p, q)$, where p and q are complementary coordinates.
Write

$$A_\delta f(x) = \int \frac{e^{ix.\xi}\beta_1(\xi)}{p(\xi) + i\delta}\hat{f}(\xi)d\xi = \int dp \int dq \frac{e^{ix.\xi(p,q)}\beta(\xi(p,q))h(p,q)\hat{f}(\xi(p,q))}{p + i\delta},$$

where h denotes the Jacobian. We can suppose that $|\partial^\alpha h| \lesssim_\alpha 1$ choosing supp(β) small enough. The expression is estimated as in [14, Subsection 5.2] by suitable decompositions in Fourier space and crucially depending on the Fourier restriction estimates for Proposition 1.1; see [11] for $p(\xi) = |\xi|^\alpha$. We write

$$\frac{1}{p(\xi) + i\delta} = \frac{p(\xi)}{p^2(\xi) + \delta^2} - i\frac{\delta}{p^2(\xi) + \delta^2} = \mathfrak{R}(\xi) - i\mathfrak{I}(\xi).$$

As in [14], $\mathfrak{I}(D)$ is estimated by Minkowski's inequality and Fourier restriction–extension estimates, in the present context from Proposition 1.1. The only difference in the estimate of $\mathfrak{R}(D)$ is that [14, Lemma 5.1] is applied for $k = \frac{3}{2}$ according to the dispersive estimate (5). For details we refer to [14, Section 4]. This finishes the proof of the estimate for A_δ.

For the estimate of B_δ, we carry out a further decomposition in Fourier space: By ellipticity, there is $R \geq 1$ such that

$$|p(\xi)| \gtrsim |\xi|^N$$

provided that $|\xi| \geq R$. Let $\beta_2(\xi) = \beta_{21}(\xi) + \beta_{22}(\xi)$ with $\beta_{21}, \beta_{22} \in C^\infty$ and $\beta_{22}(\xi) = 0$ for $|\xi| \leq R$, $\beta_{22}(\xi) = 1$ for $|\xi| \geq 2R$.
We can estimate

$$\|B_\delta(\beta_{21}(D)f)\|_{L^q} \lesssim \|f\|_{L^p}$$

for any $1 \leq p \leq q \leq \infty$ by Young's inequality uniform in δ. This gives no additional assumptions on p and q. We estimate the contribution of β_{22} by properties of the Bessel kernel (cf. [5, Theorem 30])

$$\|B_\delta(\beta_{22}(D)f)\|_{L^q(\mathbb{R}^3)} \lesssim \|\beta_{22}(D)f\|_{L^p(\mathbb{R}^3)}$$

for $1 \leq p, q \leq \infty$ and $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{N}{3}$ with the endpoints excluded for $N \leq 3$. For $N \geq 4$ this estimate holds true for $1 \leq p \leq q \leq \infty$. This corresponds to the second assumption on p and q in Theorem 1.2. Lastly, we give the standard argument for constructing solutions: For $\delta > 0$, consider the approximate solutions $u_\delta \in L^q(\mathbb{R}^3)$

$$\hat{u}_\delta(\xi) = \frac{\hat{f}(\xi)}{p(\xi) + i\delta}.$$

By the above, we have uniform bounds

$$\|u_\delta\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^3) \cap L^{p_2}(\mathbb{R}^3)}.$$

By the Banach–Alaoglu–Bourbaki theorem, we find a weak limit $u_\delta \to u$, which satisfies the same bound. We observe that

$$P(D)u_\delta = f - i\frac{\delta}{P(D)+i\delta} f.$$

Since

$$\| \frac{\delta}{P(D)+i\delta} f \|_{L^q} \lesssim \delta \|f\|_{L^{p_1} \cap L^{p_2}},$$

we find that $P(D)u_\delta \to f$ in $L^q(\mathbb{R}^3)$. Since $P(D)u_\delta \to P(D)u$ in $\mathcal{S}'(\mathbb{R}^3)$, this shows that

$$P(D)u = f$$

in $\mathcal{S}'(\mathbb{R}^3)$. The proof is complete. □

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