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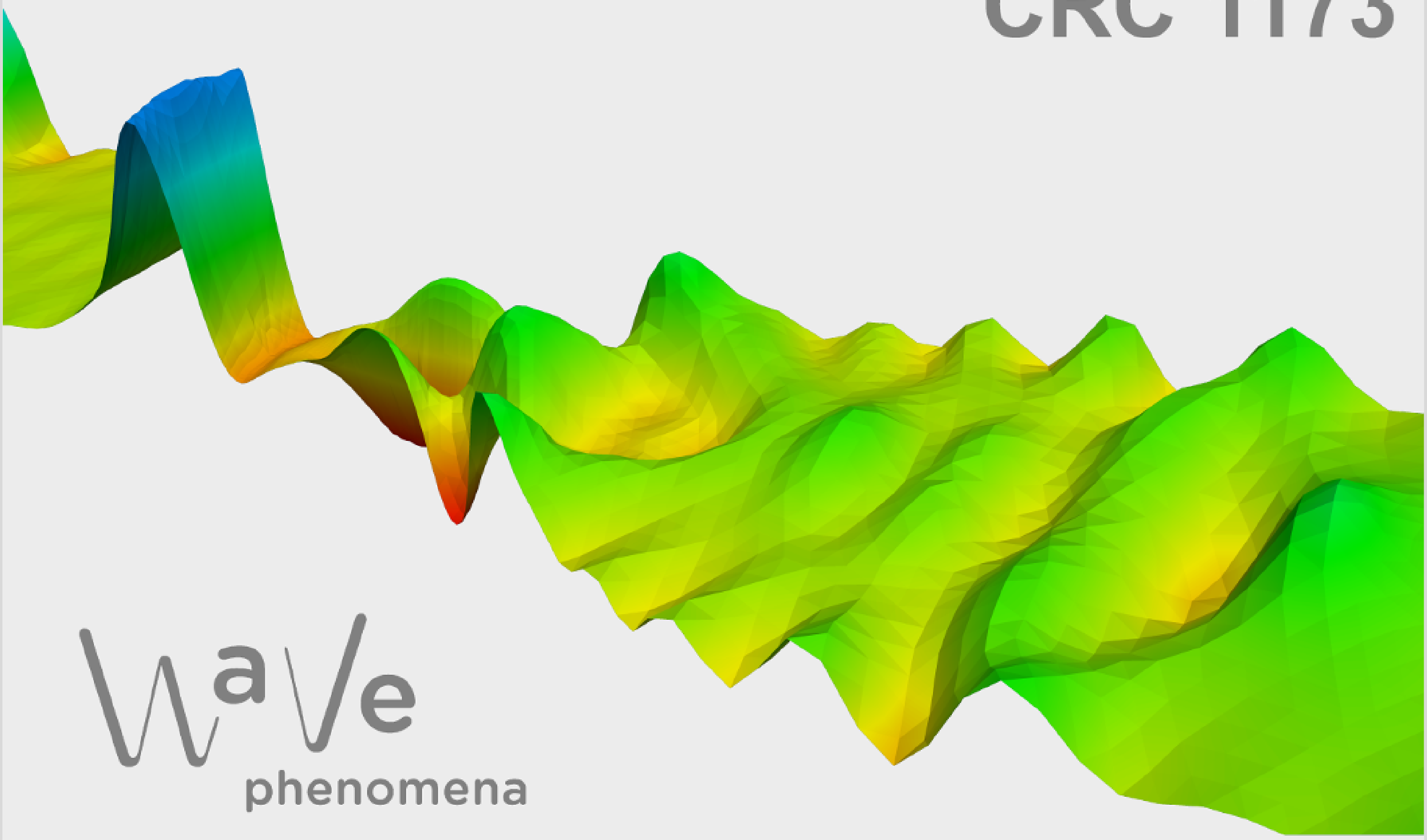

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Time-dependent electromagnetic scattering from thin layers

Jörg Nick · Balázs Kovács · Christian Lubich

Abstract The scattering of electromagnetic waves from obstacles with wave-material interaction in thin layers on the surface is described by generalized impedance boundary conditions, which provide effective approximate models. In particular, this includes a thin coating around a perfect conductor and the skin effect of a highly conducting material. The approach taken in this work is to derive, analyse and discretize a system of time-dependent boundary integral equations that determines the tangential traces of the scattered electric and magnetic fields. In a second step the fields are evaluated in the exterior domain by a representation formula, which uses the time-dependent potential operators of Maxwell's equations. A key role in the well-posedness of the time-dependent boundary integral equations and the stability of the numerical discretization is taken by the coercivity of the Calderón operator for the time-harmonic Maxwell's equations with frequencies in a complex half-plane. This entails the coercivity of the full boundary operator that includes the impedance operator. The system of time-dependent boundary integral equations is discretized with Runge–Kutta based convolution quadrature in time and Raviart–Thomas boundary elements in space. The full discretization is proved to be stable and convergent, with explicitly given rates in the case of sufficient regularity. The theoretical results are illustrated by numerical experiments.

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1 Introduction

This work studies a numerical approach to computing time-domain electromagnetic scattering from obstacles that, due to their material properties, involve multiple scales and yield effective boundary conditions known as generalized impedance boundary conditions.

On an exterior domain Ω, which is the complement of one or multiple bounded domains, we consider the *time-dependent Maxwell's equations* for the total electric field $E^{\mathrm{tot}}(x,t)$ and the total magnetic field $H^{\mathrm{tot}}(x,t)$,

$$\begin{aligned} \varepsilon\, \partial_t E^{\mathrm{tot}} - \mathrm{curl}\, H^{\mathrm{tot}} &= 0 \\ \mu\, \partial_t H^{\mathrm{tot}} + \mathrm{curl}\, E^{\mathrm{tot}} &= 0 \end{aligned} \qquad \text{in the exterior domain } \Omega. \tag{1.1}$$

The permittivity ε and the permeability μ are taken here as positive constants in Ω.

We assume to be given incident electric and magnetic fields $(E^{\mathrm{inc}}, H^{\mathrm{inc}})$, which are a solution to Maxwell's equations in $\mathbb{R}^3$, and which initially, at time $t = 0$, have their support in Ω and are thus bounded away from the boundary $\Gamma = \partial\Omega$. The objective is to compute the scattered fields $E^{\mathrm{scat}} = E^{\mathrm{tot}} - E^{\mathrm{inc}}$ and $H^{\mathrm{scat}} = H^{\mathrm{tot}} - H^{\mathrm{inc}}$ on a time interval $0 \le t \le T$, possibly only at selected space points $x \in \Omega$, such that the total fields $(E^{\mathrm{tot}}, H^{\mathrm{tot}})$ are a solution to Maxwell's equations (1.1) that satisfies the specified boundary conditions on the boundary Γ. As we will construct a numerical method for the computation of the scattered fields, we simply write them as $(E, H) = (E^{\mathrm{scat}}, H^{\mathrm{scat}})$.

The *generalized impedance boundary conditions* studied here are of the form

$$E_T^{\mathrm{tot}} + Z(\partial_t)\left(H^{\mathrm{tot}} \times \nu\right) = 0 \quad \text{ on } \Gamma = \partial\Omega, \tag{1.2}$$

where ν denotes the unit outward surface normal, E_T^{tot} denotes the tangential component of the total electric field E^{tot} on the scattering surface Γ, and $Z(\partial_t)$ is a combined surface differential operator and temporal convolution operator, which in the following is called the *time-dependent impedance operator*.

We now list some examples of operators $Z(\partial_t)$ that appear in the literature. These boundary operators often contain small quantities, each corresponding to a different physical value. To unify our notation, we will make use of a *small parameter* $\delta > 0$.

The first boundary condition we are interested in is an approximate model for a material with a thin coating, as introduced by Engquist & Nédélec [21, equation (4.9)] in the time-harmonic setting. Transferred to the time domain, it is given by (1.2) with

$$Z(\partial_t) = \delta \left(\mu^\delta \partial_t - \frac{1}{\varepsilon^\delta} \partial_t^{-1} \nabla_\Gamma \operatorname{div}_\Gamma \right), \tag{1.3}$$

where $\delta \ll 1$ is the layer depth and $\varepsilon^\delta, \mu^\delta$ describe the permittivity and permeability inside the thin layer. Here, ∂_t^{-1} denotes integration in time. This boundary condition is of first-order accuracy in δ. The problem in the time-harmonic setting with a fixed frequency was analysed by Ammari & Nédélec [3,4] using boundary integral equations. In the time-dependent case we are not aware of an analysis of well-posedness or of numerical analysis. Both will be given here.

The boundary condition (1.2) with (1.3) has been extended in several ways, of which we present a small selection in the following. The second-order boundary condition for thin layers was derived by Haddar & Joly [23, Eq. (95)]. Its time domain formulation reads

$$Z(\partial_t) = \delta \left(\mu^\delta \partial_t + \delta \mu^\delta \left(\mathcal{H} - \mathcal{C} \right) \partial_t - \frac{1}{\varepsilon^\delta} \partial_t^{-1} \nabla_\Gamma \left[(1 - \delta \mathcal{H}) \operatorname{div}_\Gamma \right] \right), \tag{1.4}$$

where $\mathcal{H}$ is the mean curvature and $\mathcal{C}$ is the curvature tensor.

In [2], the first-order boundary condition is generalized to a model where the permittivity of the thin coating is not homogenous but depends on the location on Γ.

Multiple layers on top of each other is another case of interest, for which effective boundary conditions were recently given in [22]. The corresponding impedance operator is a linear combination of operators (1.3) with different permittivities.

A boundary condition for the approximation of scattering from highly conductive obstacles was developed by Haddar, Joly & Nguyen [24]. The skin effect limits the penetration of the wave to a thin layer near the surface, which then can be asymptotically approximated to create a reduced model. The authors deduce absorbing impedance boundary conditions for time-harmonic Maxwell's equations of multiple orders. Here we restrict our attention to the first- and second-order boundary conditions. The impedance operator corresponding to the first-order boundary condition reads in the time domain

$$Z(\partial_t) = \delta\, \partial_t^{1/2}, \tag{1.5}$$

where δ is inversely proportional to the high conductivity, and the fractional derivative $\partial_t^{1/2}$ is the time derivative of convolution with the kernel $(\pi t)^{-1/2}$. The second-order impedance operator reads

$$Z(\partial_t) = \delta\, \partial_t^{1/2} - \delta^2 (\mathcal{H} - \mathcal{C}), \tag{1.6}$$

where again $\mathcal{H}$ is the mean curvature and $\mathcal{C}$ is the curvature tensor.

The above generalized impedance boundary conditions have been analysed in the time-harmonic setting (for a fixed frequency) in the stated references; see also Chaulet [19] for well-posedness results in a general framework that partly inspired ours. While there is some numerical analysis in the time-harmonic case by Schmidt & Hiptmair [36], we are not aware of any existing numerical analysis of the time-dependent problem as studied here, which requires estimates for the corresponding time-harmonic problem for all frequencies in a complex half-plane in combination with Laplace transform techniques on the analytical side, and stable time discretization on the numerical side.

This paper transfers the approach to numerical discretization and its analysis from time-dependent acoustic scattering with generalized impedance boundary conditions, as studied in [9], to the time-dependent electromagnetic case. While the basic numerical approach via the discretization of time-dependent boundary integral equations by convolution quadrature and boundary elements is conceptually similar, the functional-analytic framework is very different. A basic tool in [9] was a coercivity property of the Calderón operator of the Helmholtz equation for frequencies in a half-plane, which was previously proved in [10]. In this paper we use an analogous result for the Calderón operator of the time-harmonic Maxwell's equations, which was proved in [27]; see also [34]. Using this result together with a positivity property of the impedance operator is the key to proving well-posedness of the time-dependent problem and the stability of the numerical discretization.

For the time discretization of the system of time-dependent boundary integral equations for the tangential traces and the time-dependent representation formulas for the electric and magnetic fields, we use Runge–Kutta based convolution quadrature, which was first introduced in [30] in the context of parabolic problems and was later studied for wave propagation problems in [8]. Convolution quadrature was used and analysed for the numerical solution of various exterior Maxwell problems in [5, 18, 20] and of an eddy current problem with an impedance boundary condition in [26].

For space discretization we use boundary elements as described for boundary integral equations related to Maxwell's equations in the monographs by Nédélec [32] and Monk [31]. For our numerical experiments we have chosen Raviart–Thomas elements.

The paper is organized as follows:

In Section 2 we introduce the functional-analytic setting of the paper, prove that the above impedance operators fit into this framework, and give an appropriate weak formulation of the generalized impedance boundary condition. Moreover, we introduce basic notation used throughout the paper.

Section 3 studies the time-harmonic Maxwell's equations with generalized impedance boundary conditions for frequencies in a complex half-plane. The main result here is a well-posedness result for the time-harmonic scattering problem with a bound that gives an explicit dependence on the complex frequency (Theorem 3.1) and behaves well with respect to the small parameter δ that appears in the time-harmonic impedance operators described above. On the way to proving this result we prove and use boundedness and coercivity results for time-harmonic boundary integral operators, in particular the Calderón operator and a related operator that adds the impedance operator.

We formulate the system of boundary integral equations for the tangential traces of the electric and magnetic fields under generalized impedance boundary conditions and prove its well-posedness, showing a bound that is proportional to the square of the absolute value of the complex frequency divided by its real part. With the tangential traces, the scattered electric and magnetic fields are obtained from the representation formula that involves the single and double layer electromagnetic potential operators.

Section 4 transfers the results of Section 3 from the Laplace domain to the time domain, using the polynomial bounds in the frequency together with Laplace transform techniques. We thus obtain well-posedness of the time-dependent electromagnetic scattering problem with generalized impedance boundary conditions (Theorem 4.1) via a system of time-dependent boundary integral equations for the tangential traces of the electric and magnetic fields, which is discretized numerically in the following sections.

Section 5 briefly recapitulates Runge–Kutta based convolution quadratures and their error bounds as proved in [8]. Combining these quadrature error bounds with the time-harmonic well-posedness results of Section 3, we obtain error bounds for the semi-discretization in time of the system of time-dependent boundary integral equations of Section 4 and of the scattered time-dependent electric and magnetic fields obtained from the convolution quadrature time discretization of the time-dependent representation formulas.

In Section 6 we consider the full discretization of the time-dependent boundary integral equation by Runge–Kutta convolution quadrature in time and Raviart–Thomas boundary elements in space. We obtain error bounds for the approximate scattered electric and magnetic fields (Theorem 6.1). We prove full-order error bounds in time and space in exterior subdomains $\Omega_d \subset \Omega$ with a fixed positive distance d to the boundary Γ, both in the $H(\mathrm{curl}, \Omega_d)$ norm and in the maximum norm on Ω_d, and we prove error bounds of reduced (actually halved) temporal order on the whole exterior domain Ω in the $H(\mathrm{curl}, \Omega)$ norm, uniformly over bounded time intervals. The error bounds are uniform in the small parameter δ of the impedance operators described above.

In Section 7 we present numerical experiments to illustrate our theoretical results and computational aspects.

2 Framework and analytical background

We are interested in the solution of the time-dependent Maxwell's equations with generalized impedance boundary conditions in the context of wave scattering. Given an incident wave $\left(E^{\mathrm{inc}}, H^{\mathrm{inc}}\right)$, which is a solution to the time-dependent Maxwell's equations on $\mathbb{R}^3$ with initial support in the exterior domain Ω away from the boundary Γ, we are interested in computing (possibly in a few selected points x only) the scattered fields $E = E^{\mathrm{tot}} - E^{\mathrm{inc}}$ and $H = H^{\mathrm{tot}} - H^{\mathrm{inc}}$, which are an outgoing solution to the following initial–

boundary value problem of Maxwell's equations:

$$\varepsilon\,\partial_t E - \operatorname{curl} H = 0 \qquad \text{in} \quad \Omega, \tag{2.1}$$

$$\mu\,\partial_t H + \operatorname{curl} E = 0 \qquad \text{in} \quad \Omega, \tag{2.2}$$

$$E_T + Z(\partial_t)\,(H \times \nu) = g^{\mathrm{inc}} \quad \text{on} \quad \Gamma, \tag{2.3}$$

where $E_T = (I - \nu\nu^\top)E = -(E \times \nu) \times \nu$ is the tangential component of E and

$$g^{\mathrm{inc}} = -\big(E_T^{\mathrm{inc}} + Z(\partial_t)(H^{\mathrm{inc}} \times \nu)\big) \quad \text{ on } \Gamma. \tag{2.4}$$

The initial values at $t = 0$ are zero in Ω for both E and H.

Throughout the paper, we assume that the physical units are chosen such that

$$\varepsilon\mu = c^{-2} = 1, \tag{2.5}$$

which can always be achieved by rescaling time $t \to ct$ or frequency $s \to s/c$.

As the problem has finite wave speed $c = 1$, the fields (E, H) have bounded support at any time, vanishing beyond a distance ct from the boundary at time t. (In contrast to the time-harmonic problem we therefore need not care about asymptotic conditions as $|x| \to \infty$.)

In this section we describe the functional-analytic framework and show that the above-mentioned examples for $Z(\partial_t)$ fit into this general setting. We then give a weak formulation of the boundary condition (2.3) that is appropriate for our analysis.

2.1 Tangential trace, trace space X_Γ and a further Hilbert space $\mathrm{V}_\Gamma \subset \mathrm{X}_\Gamma$

Throughout this paper, we assume that Ω is the complement of one or several bounded domains in $\mathbb{R}^3$ with a piecewise smooth boundary surface $\Gamma = \partial\Omega$. For a continuous vector field in the domain, $v : \overline{\Omega} \to \mathbb{C}^3$, we define the *tangential trace*

$$\gamma_T v = v|_\Gamma \times \nu \qquad \text{on } \Gamma,$$

where ν denotes the unit surface normal pointing into the exterior domain. We note that the tangential component of $v|_\Gamma$ is $v_T = (I - \nu\nu^\top)v|_\Gamma = -(\gamma_T v) \times \nu$.

By the version of Green's formula for the curl operator, we have for sufficiently regular vector fields $u, v : \overline{\Omega} \to \mathbb{C}^3$ that

$$\int_\Omega \big(u \cdot \operatorname{curl} v - \operatorname{curl} u \cdot v\big)\mathrm{d}x = \int_\Gamma (\gamma_T u \times \nu) \cdot \gamma_T v \,\mathrm{d}\sigma, \tag{2.6}$$

where the dot $\cdot$ stands for the Euclidean inner product on $\mathbb{C}^3$, i.e., $a \cdot b = \overline{a}^\top b$ for $a, b \in \mathbb{C}^3$. The right-hand side in this formula defines a *skew-hermitian sesquilinear form* on continuous tangential vector fields on the boundary, say $\phi, \psi : \Gamma \to \mathbb{C}^3$, which we write as

$$[\phi, \psi]_\Gamma = \int_\Gamma (\phi \times \nu) \cdot \psi \,\mathrm{d}\sigma. \tag{2.7}$$

As it was shown by Alonso & Valli [1] for smooth domains and by Buffa, Costabel & Sheen [16] for Lipschitz domains (see also the surveys in [17, Sect. 2.2] and [32, Sect. 5.4]), the trace operator γ_T can be extended to a surjective bounded linear operator from the space that appears naturally for Maxwell's equations, $H(\mathrm{curl}, \Omega) = \{v \in L^2(\Omega)^3 \,:\, \mathrm{curl}\, v \in L^2(\Omega)^3\}$, to the

proper trace space: a Hilbert space denoted X_Γ, with norm $\|\cdot\|_{\mathrm{X}_\Gamma}$.

This space is characterized as the tangential subspace of the Sobolev space $H^{-1/2}(\Gamma)^3$ with surface divergence in $H^{-1/2}(\Gamma)$ (see the papers cited above for the precise formulation, e.g. [17, Section 2.2]). It has the property that the pairing $[\cdot,\cdot]_\Gamma$ can be extended to a non-degenerate continuous sesquilinear form on $\mathrm{X}_\Gamma \times \mathrm{X}_\Gamma$. With this pairing the space X_Γ becomes its own dual.

For the treatment of generalized impedance boundary conditions we need a further Hilbert space, which is chosen as a dense subspace $\mathrm{V}_\Gamma \subset \mathrm{X}_\Gamma$ equipped with a (semi-)norm $|\cdot|_{\mathrm{V}_\Gamma}$ and the full norm

$$\|\phi\|^2_{\mathrm{V}_\Gamma} = \|\phi\|^2_{\mathrm{X}_\Gamma} + |\phi|^2_{\mathrm{V}_\Gamma}\,. \tag{2.8}$$

We will choose $\mathrm{V}_\Gamma = \mathrm{X}_\Gamma \cap H(\mathrm{div}_\Gamma, \Gamma)$ with $H(\mathrm{div}_\Gamma, \Gamma) = \{\phi \in L^2(\Gamma)^3 \,:\, \mathrm{div}_\Gamma\, \phi \in L^2(\Gamma)\}$ for the impedance operators (1.3) and (1.4), and we choose $\mathrm{V}_\Gamma = \mathrm{X}_\Gamma \cap L^2(\Gamma)^3$ for (1.5) and (1.6), in all cases with $|\cdot|_{\mathrm{V}_\Gamma}$ depending on the small parameter δ.

2.2 Impedance operator and temporal convolution

Let $Z(s)\colon \mathrm{V}_\Gamma \to \mathrm{V}_\Gamma{}'$, for $\mathrm{Re}\, s > 0$, be an analytic family of bounded linear operators. We assume that Z is *polynomially bounded*: there exists a real κ, and for every $\sigma > 0$ there exists $M_\sigma < \infty$, such that

$$\|Z(s)\|_{\mathrm{V}_\Gamma{}' \leftarrow \mathrm{V}_\Gamma} \le M_\sigma\, |s|^\kappa\,, \qquad \mathrm{Re}\ s \ge \sigma > 0. \tag{2.9}$$

As a key property, we further assume that Z is of *positive type*: for every $\sigma > \sigma_0 \ge 0$, there exists $c_\sigma > 0$ such that

$$\mathrm{Re}\langle \phi, Z(s)\phi\rangle \ge c_\sigma\, \mathrm{Re}\, s\, \left|s^{-1}\phi\right|^2_{\mathrm{V}_\Gamma} \quad \text{for all } \phi \in \mathrm{V}_\Gamma \text{ and } \mathrm{Re}\ s \ge \sigma, \tag{2.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the anti-duality between V_Γ and $\mathrm{V}_\Gamma{}'$, taken anti-linear in the first argument.

The bound (2.9) ensures that Z is the Laplace transform of a distribution of finite order of differentiation with support on the non-negative real half-line $t \ge 0$. For a function $g : [0,T] \to \mathrm{V}_\Gamma$, which together with its extension by 0 to the negative real half-line is sufficiently regular, we use the operational calculus notation

$$Z(\partial_t)g = (\mathcal{L}^{-1}Z) * g \tag{2.11}$$

for the temporal convolution of the inverse Laplace transform of Z with g. For the identity operator $\mathrm{Id}(s) = s$, we have $\mathrm{Id}(\partial_t)g = \partial_t g$, the time derivative of g. For two such families of operators $K(s)$ and $L(s)$ mapping into compatible spaces, the associativity of convolution and the product rule of Laplace transforms yield the composition rule

$$K(\partial_t)L(\partial_t)g = (KL)(\partial_t)g. \tag{2.12}$$

For a Hilbert space V, we let $H^r(\mathbb{R}, V)$ be the Sobolev space of order r of V-valued functions on $\mathbb{R}$, and on finite intervals $(0, T)$ we denote[1]

$$H_0^r(0, T; V) = \{g|_{(0,T)} \ : \ g \in H^r(\mathbb{R}, V) \ \text{ with } \ g = 0 \ \text{ on } \ (-\infty, 0)\}.$$

For integer $r \geq 0$, the norm $\|\partial_t^r g\|_{L^2(0,T;V)}$ is equivalent to the natural norm on $H_0^r(0, T; V)$. The Plancherel formula yields the following [29, Lemma 2.1]: If $Z(s)$ is bounded by (2.9) in the half-plane Re $s > 0$, then $Z(\partial_t)$ extends by density to a bounded linear operator $Z(\partial_t)$ from $H_0^{r+\kappa}(0, T; \mathrm{V}_\Gamma)$ to $H_0^r(0, T; \mathrm{V}_\Gamma{}')$ with the bound

$$\|Z(\partial_t)\|_{H_0^r(0,T;\mathrm{V}_\Gamma{}') \leftarrow H_0^{r+\kappa}(0,T;\mathrm{V}_\Gamma)} \leq e M_{1/T} \tag{2.13}$$

for arbitrary real r. (The bound on the right-hand side arises from the bound $e^{\sigma T} M_\sigma$ on choosing $\sigma = 1/T$.) We note that for any integer k and $\alpha > 1/2$, we have the continuous embedding $H_0^{k+\alpha}(0, T; \mathrm{V}_\Gamma{}') \subset C^k([0, T]; \mathrm{V}_\Gamma{}')$.

The passage from the operators $Z(s)$, satisfying a polynomial bound (2.9), to the convolution operators $Z(\partial_t)$ and their bound (2.13) will be used in the same way also for other operators between different Hilbert spaces in the course of this paper.

2.3 The impedance operators (1.3)–(1.6)

As the following two lemmas show, the impedance operators listed in the introduction fit into the abstract framework given above.

Lemma 2.1 (Thin coating) *With the space* $\mathrm{V}_\Gamma = \mathrm{X}_\Gamma \cap H(\mathrm{div}_\Gamma, \Gamma)$ *and, in* (2.8)*, the norm* $|\phi|_{\mathrm{V}_\Gamma}^2 = \delta\big(\|\phi\|_{L^2(\Gamma)^3}^2 + \|\operatorname{div}_\Gamma \phi\|_{L^2(\Gamma)}^2\big)$*, the transfer operators* $Z(s) : \mathrm{V}_\Gamma \to \mathrm{V}_\Gamma{}'$ *for* Re $s > 0$ *corresponding to the impedance operators* (1.3) *and* (1.4) *satisfy the bound* (2.9) *with* $\kappa = 1$ *and the positivity condition* (2.10)*, with* M_σ *and* $c_\sigma > 0$ *independent of the small parameter* δ*. In the case of* (1.3)*,* $\sigma_0 = 0$ *for* (2.10)*.*

Proof We prove the result only for (1.3), as the proof for (1.4) is a straightforward extension. Moreover, we assume ε^δ and μ^δ to be positive and restrict our attention, for the ease of presentation, to the transfer operator

$$Z(s) = \delta\big(s - s^{-1}\nabla_\Gamma \operatorname{div}_\Gamma\big),$$

[1] We note that the subscript 0 in H_0^r only refers to the left end-point of the interval.

for which the anti-duality between V_Γ and ${\mathrm{V}_\Gamma}'$ is to be understood as

$$\langle \phi, Z(s)\psi\rangle = \delta s\big(\phi,\psi\big)_{L^2(\Gamma)^3} + \delta s^{-1}\big(\mathrm{div}_\Gamma\,\phi, \mathrm{div}_\Gamma\,\psi\big)_{L^2(\Gamma)}, \tag{2.14}$$

where the round brackets denote the L^2 inner product, taken anti-linear in the first argument. This is bounded as follows, abbreviating $m(|s|) = \max(|s|, |s|^{-1})$:

$$\begin{aligned}
|\langle \phi, Z(s)\psi\rangle| &\le m(|s|)\,\delta\big(\|\phi\|_{L^2(\Gamma)^3}\,\|\psi\|_{L^2(\Gamma)^3} + \|\,\mathrm{div}_\Gamma\,\phi\|_{L^2(\Gamma)}\,\|\,\mathrm{div}_\Gamma\,\psi\|_{L^2(\Gamma)}\big)\\
&\le m(|s|)\,\delta\big(\|\phi\|_{L^2(\Gamma)^3} + \|\,\mathrm{div}_\Gamma\,\phi\|_{L^2(\Gamma)}\big)\big(\|\psi\|_{L^2(\Gamma)^3} + \|\,\mathrm{div}_\Gamma\,\psi\|_{L^2(\Gamma)}\big)\\
&\le 2\,m(|s|)\,|\phi|_{\mathrm{V}_\Gamma}\,|\psi|_{\mathrm{V}_\Gamma}\\
&\le 2\,m(|s|)\,\|\phi\|_{\mathrm{V}_\Gamma}\,\|\psi\|_{\mathrm{V}_\Gamma}.
\end{aligned}$$

This yields (2.9) with $\kappa = 1$. On the other hand, taking $\phi = \psi$, we have for $\mathrm{Re}\, s \ge \sigma > 0$

$$\begin{aligned}
\mathrm{Re}\langle \phi, Z(s)\phi\rangle &= \delta\,(\mathrm{Re}\, s)\,\|\phi\|^2_{L^2(\Gamma)^3} + \delta\,\frac{\mathrm{Re}\, s}{|s|^2}\,\|\,\mathrm{div}_\Gamma\,\phi\|^2_{L^2(\Gamma)}\\
&\ge \delta(\mathrm{Re}\, s)\sigma^2\|s^{-1}\phi\|^2_{L^2(\Gamma)^3} + \delta(\mathrm{Re}\, s)\|s^{-1}\,\mathrm{div}_\Gamma\,\phi\|^2_{L^2(\Gamma)}\\
&\ge \min(\sigma^2, 1)\,(\mathrm{Re}\, s)\,|s^{-1}\phi|^2_{\mathrm{V}_\Gamma},
\end{aligned}$$

which yields (2.10). □

Lemma 2.2 (Strong absorption) *With the space* $\mathrm{V}_\Gamma = \mathrm{X}_\Gamma \cap L^2(\Gamma)^3$ *and* $|\phi|^2_{\mathrm{V}_\Gamma} = \delta\|\phi\|^2_{L^2(\Gamma)^3}$*, the transfer operators* $Z(s) : \mathrm{V}_\Gamma \to {\mathrm{V}_\Gamma}'$ *for* $\mathrm{Re}\ s > 0$ *corresponding to the impedance operators* (1.5) *and* (1.6) *satisfy the bound* (2.9) *with* $\kappa = 1/2$ *and the positivity condition* (2.10)*, with* M_σ *and* $c_\sigma > 0$ *independent of the small parameter* δ*. In the case of* (1.5)*,* $\sigma_0 = 0$ *for* (2.10)*.*

Proof We prove the result only for (1.5), as the proof for (1.6) is a straightforward extension. Here, the transfer operator is

$$Z(s) = \delta\, s^{1/2},$$

for which the anti-duality between V_Γ and ${\mathrm{V}_\Gamma}'$ is to be understood as

$$\langle \phi, Z(s)\psi\rangle = \delta s^{1/2}\big(\phi,\psi\big)_{L^2(\Gamma)^3}. \tag{2.15}$$

Here we obtain without ado

$$|\langle \phi, Z(s)\phi\rangle| \le |s|^{1/2}\|\phi\|_{\mathrm{V}_\Gamma}\,\|\phi\|_{\mathrm{V}_\Gamma},$$

which yields (2.9) with $\kappa = 1/2$, and for $\mathrm{Re}\, s \ge \sigma > 0$ we have

$$\mathrm{Re}\langle \phi, Z(s)\phi\rangle \ge \delta(\mathrm{Re}\, s^{1/2})\,\|\phi\|^2_{L^2(\Gamma)^3} \ge \sigma^{3/2}(\mathrm{Re}\, s)|s^{-1}\phi|^2_{\mathrm{V}_\Gamma},$$

which yields (2.10). □

2.4 Weak formulation of the generalized impedance boundary condition

Formally taking the $L^2(\varGamma)^3$ inner product $(\cdot,\cdot)_\varGamma$ of the boundary condition (2.3) with an arbitrary continuous tangential vector field ϕ on $\varGamma$, we obtain the equation

$$(\phi, E_T)_\varGamma + (\phi, Z(\partial_t)\gamma_T H)_\varGamma = (\phi, g^{\mathrm{inc}})_\varGamma, \tag{2.16}$$

which is the starting point for motivating the weak formulation given below.

Noting that for continuous E we have $E_T \times \nu = E \times \nu = \gamma_T E$, we find

$$(\phi, E_T)_\varGamma = (\phi \times \nu, E_T \times \nu)_\varGamma = (\phi \times \nu, \gamma_T E)_\varGamma = [\phi, \gamma_T E]_\varGamma,$$

with the skew-hermitian sesquilinear form (2.7). Starting from a combined surface differential and temporal convolution operator $Z(\partial_t)$ in the strong formulation (2.3), we construct the transfer operator $Z(s) : \mathrm{V}_\varGamma \to \mathrm{V}_\varGamma{}'$ such that for sufficiently regular $\gamma_T H$, the duality coincides with the $L^2(\varGamma)^3$ inner product:

$$\langle \phi, Z(\partial_t)\gamma_T H\rangle_\varGamma = (\phi, Z(\partial_t)\gamma_T H)_\varGamma, \qquad \phi \in \mathrm{V}_\varGamma,$$

as we did for (1.3)–(1.6) in (2.14) and (2.15). Similarly, a regular tangential vector field g^{inc} defines a functional on $\mathrm{V}_\varGamma$ by

$$\langle \phi, g^{\mathrm{inc}}\rangle_\varGamma = (\phi, g^{\mathrm{inc}})_\varGamma, \qquad \phi \in \mathrm{V}_\varGamma.$$

Inserting the identities above into (2.16) motivates us to study the following *weak formulation of the boundary condition* (2.3): the tangential traces of solutions $E, H \in L^2(0,T; H(\mathrm{curl}, \varOmega)) \cap H^1(0,T; L^2(\varOmega)^3)$ to the Maxwell's equations in $\varOmega$ with zero initial conditions are to be determined as $\gamma_T E \in L^2(0,T; \mathrm{X}_\varGamma)$ and $\gamma_T H \in H_0^\kappa(0,T; \mathrm{V}_\varGamma)$, for κ of (2.9), such that for almost every $t \in (0,T)$,

$$[\phi, \gamma_T E]_\varGamma + \langle \phi, Z(\partial_t)\gamma_T H\rangle_\varGamma = \langle \phi, g^{\mathrm{inc}}\rangle_\varGamma \qquad \text{for all } \phi \in \mathrm{V}_\varGamma. \tag{2.17}$$

This boundary condition relates the tangential traces of E and H. The terms on the left-hand side are well-defined under the stated regularity requirements on $\gamma_T E$ and $\gamma_T H$.

In the following two sections we will prove that this initial and boundary value problem is well-posed in the stated Hilbert spaces if g^{inc} has sufficient temporal regularity: $g^{\mathrm{inc}} \in H_0^3(0,T; \mathrm{V}_\varGamma{}')$ (provided that $\kappa \le 1$, else $H_0^{2+\kappa}$). The arguments and intermediate results in the proof of well-posedness will again be used in the stability and error analysis of the numerical methods.

3 Time-harmonic Maxwell's equations

Although the main interest of this work lies on time-domain scattering, it will turn out useful to start with the analysis of the corresponding problem in the Laplace domain, the *time-harmonic Maxwell's equations*. These equations read, for $s \in \mathbb{C}$ considered here with $\operatorname{Re} s > 0$ (and with $\varepsilon\mu = 1$; see (2.5))

$$s\varepsilon\widehat{E} - \operatorname{curl}\widehat{H} = 0 \quad \text{in } \Omega, \tag{3.1}$$

$$s\mu\widehat{H} + \operatorname{curl}\widehat{E} = 0 \quad \text{in } \Omega. \tag{3.2}$$

This is complemented with the asymptotic conditions as $|x| \to \infty$ for an outgoing wave, which are automatically satisfied by the solutions constructed via the representation formula from the tangential traces on Γ, as we will do in the following. We will then obtain $\widehat{E}, \widehat{H} \in H(\operatorname{curl}, \Omega)$.

For solutions of the time-harmonic Maxwell's equations, Green's formula (2.6) reduces to

$$\begin{aligned}\left[\gamma_T\widehat{E}, \gamma_T\widehat{H}\right]_\Gamma &= \int_\Omega \widehat{E} \cdot \operatorname{curl}\widehat{H} - \operatorname{curl}\widehat{E} \cdot \widehat{H} \mathrm{d}x \\ &= \int_\Omega s\varepsilon\big|\widehat{E}\big|^2 + \bar{s}\mu\big|\widehat{H}\big|^2 \mathrm{d}x.\end{aligned} \tag{3.3}$$

The conjugation of s in the second summand stems from the convention that $\cdot$ denotes the inner product $a \cdot b = \overline{a}^\top b$ on $\mathbb{C}^3$.

3.1 Potential operators and representation formulas

We recall the usual potential operators for the time-harmonic Maxwell's equations; cf. [17,32]. The *fundamental solution* is given by

$$G(s,x) = \frac{e^{-s|x|}}{4\pi\,|x|}, \qquad \operatorname{Re} s > 0,\ x \in \mathbb{R}^3 \setminus \{0\}.$$

The electromagnetic *single layer potential* operator $\mathcal{S}(s)$, applied to a regular complex-valued function φ and evaluated at $x \in \mathbb{R}^3 \setminus \Gamma$, is given by

$$\mathcal{S}(s)\varphi(x) = -s\int_\Gamma G(s, x-y)\varphi(y)\mathrm{d}y + s^{-1}\nabla\int_\Gamma G(s,x-y)\operatorname{div}_\Gamma\varphi(y)\mathrm{d}y,$$

and the electromagnetic *double layer potential* operator $\mathcal{D}(s)$ is given by

$$\mathcal{D}(s)\varphi(x) = \operatorname{curl}\int_\Gamma G(s,x-y)\varphi(y)\mathrm{d}y.$$

The potential operators satisfy the relations

$$s\mathcal{S}(s) - \operatorname{curl}\circ\mathcal{D}(s) = 0, \qquad s\mathcal{D}(s) + \operatorname{curl}\circ\mathcal{S}(s) = 0. \tag{3.4}$$

This implies that for any regular function φ, the fields $\widehat{E} = \mathcal{S}(s)\varphi$ and $\mu\widehat{H} = \mathcal{D}(s)\varphi$ are a solution to the time-harmonic Maxwell's equations (3.1)–(3.2) on $\mathbb{R}^3 \setminus \Gamma$ (recall $\varepsilon\mu = 1$). Likewise, this also holds true for the fields $\widehat{E} = \mathcal{D}(s)\varphi$ and $\mu\widehat{H} = -\mathcal{S}(s)\varphi$.

In our problem setting only the exterior domain Ω occurs. As a theoretical tool, however, it will be useful to analyse transmission problems on $\mathbb{R}^3 \setminus \Gamma$. We introduce some standard notation designed to simplify the description of such problems.

In the context of transmission problems, we denote the interior of the bounded scatterer by Ω^- and the exterior domain by Ω^+ (elsewhere in this paper denoted by Ω), such that $\mathbb{R}^3$ is decomposed into $\mathbb{R}^3 = \Omega^- \,\dot\cup\, \Gamma \,\dot\cup\, \Omega^+$. Furthermore, γ_T^- and γ_T^+ denote the tangential traces on Ω^- and Ω^+, respectively. We denote *jumps* and *averages* by

$$[\gamma_T] = \gamma_T^+ - \gamma_T^-, \qquad \{\gamma_T\} = \tfrac{1}{2}\left(\gamma_T^+ + \gamma_T^-\right).$$

The sign convention for the jumps has been chosen to coincide with that of [17]. A fundamental role is played by the *jump relations* of the potential operators:

$$[\gamma_T] \circ \mathcal{S}(s) = 0, \qquad [\gamma_T] \circ \mathcal{D}(s) = -\mathrm{Id}. \tag{3.5}$$

As a direct consequence of (3.4) and (3.5), for any given boundary densities $(\widehat{\varphi}, \widehat{\psi})$ (regular in a dense subspace of $\mathrm{X}_\Gamma \times \mathrm{X}_\Gamma$), the electric and magnetic fields defined by[2]

$$\widehat{E} = -\mathcal{S}(s)\widehat{\varphi} + \mathcal{D}(s)\widehat{\psi}, \tag{3.6}$$

$$\mu\widehat{H} = -\mathcal{D}(s)\widehat{\varphi} - \mathcal{S}(s)\,\widehat{\psi}, \tag{3.7}$$

are a solution to the transmission problem (assuming $\varepsilon\mu = 1$)

$$s\varepsilon\widehat{E} - \operatorname{curl}\widehat{H} = 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \Gamma, \tag{3.8}$$

$$s\mu\widehat{H} + \operatorname{curl}\widehat{E} = 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \Gamma, \tag{3.9}$$

$$\mu[\gamma_T]\widehat{H} = \widehat{\varphi}, \tag{3.10}$$

$$-[\gamma_T]\widehat{E} = \widehat{\psi}. \tag{3.11}$$

So far, our presentation was restricted to regular boundary densities. The following lemma shows that the linear map $(\widehat{\varphi}, \widehat{\psi}) \mapsto (\widehat{E}, \mu\widehat{H})$ extends by density to a bounded linear operator from $\mathrm{X}_\Gamma \times \mathrm{X}_\Gamma$ to $H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$, and it gives an s-explicit bound; cf. [18, Lemma 6.4] for a related, yet more complicated result.

Lemma 3.1 *For* $\operatorname{Re} s > 0$*, the solution* $(\widehat{E}, \widehat{H})$ *of the transmission problem* (3.8)–(3.11) *defined by* (3.6)–(3.7) *is bounded by*

$$\left\| \begin{pmatrix} \widehat{E} \\ \mu\widehat{H} \end{pmatrix} \right\|_{H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)^2} \leq C_\Gamma \,\frac{|s|^2 + 1}{\operatorname{Re} s}\, \left\| \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right\|_{\mathrm{X}_\Gamma{}^2},$$

where $C_\Gamma = \|\{\gamma_T\}\|_{\mathrm{X}_\Gamma \leftarrow H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)}$.

[2] We write $(\widehat{\varphi}, \widehat{\psi})$ when these functions appear as boundary densities defining fields $(\widehat{E}, \widehat{H})$ as in (3.6)–(3.7), where the hats recall that these variables correspond to Laplace transforms of time-dependent functions, which will be studied in the next section. On the other hand, we omit the hats for generic functions to which potential operators or boundary operators are applied.

Proof We start from Green's formula (3.3) on the exterior and interior domain, which after multiplying with μ and using $\varepsilon\mu = 1$ gives

$$I := \int_{\mathbb{R}^3\setminus\Gamma} s\big|\widehat{E}\big|^2 + \bar{s}\big|\mu\widehat{H}\big|^2 \mathrm{d}x = \Big[\gamma_T^+\widehat{E}, \mu\gamma_T^+\widehat{H}\Big]_\Gamma - \Big[\gamma_T^-\widehat{E}, \mu\gamma_T^-\widehat{H}\Big]_\Gamma . \tag{3.12}$$

On inserting (3.8) and (3.9) for $\widehat{E}$ and $\mu\widehat{H}$ into θ times the integrand, where $0 < \theta < 1$ is arbitrary, the left-hand side is rewritten as

$$\begin{aligned} I = \int_{\mathbb{R}^3\setminus\Gamma} \Big(&(1-\theta)s\big|\widehat{E}\big|^2 + \theta\bar{s}\big|s^{-1}\operatorname{curl}\widehat{E}\big|^2 \\ &+ (1-\theta)\bar{s}\big|\mu\widehat{H}\big|^2 + \theta s\big|s^{-1}\operatorname{curl}(\mu\widehat{H})\big|^2\Big)\mathrm{d}x. \end{aligned}$$

Choosing θ such that $1-\theta = \theta|s|^{-2}$, i.e. $\theta = 1/(1+|s|^{-2})$, and taking the real part then gives

$$\operatorname{Re} I = \frac{\operatorname{Re} s}{|s^2|+1}\Big(\|\widehat{E}\|^2_{H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)} + \|\mu\widehat{H}\|^2_{H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)}\Big). \tag{3.13}$$

On the other hand, by (3.12) we also have

$$\operatorname{Re} I = \operatorname{Re}\Big(\Big[\gamma_T^+\widehat{E}, \mu\gamma_T^+\widehat{H}\Big]_\Gamma - \Big[\gamma_T^-\widehat{E}, \mu\gamma_T^-\widehat{H}\Big]_\Gamma\Big).$$

Rewriting the right-hand side in terms of jumps and averages and using the transmission conditions (3.10)–(3.11), we obtain

$$\begin{aligned} \operatorname{Re} I &= \operatorname{Re}\Big(\Big[\mu[\gamma_T]\widehat{H}, \{\gamma_T\}\widehat{E}\Big]_\Gamma + \Big[-[\gamma_T]\widehat{E}, \mu\{\gamma_T\}\widehat{H}\Big]_\Gamma\Big) \\ &= \operatorname{Re}\Big(\Big[\widehat{\varphi}, \{\gamma_T\}\widehat{E}\Big]_\Gamma + \Big[\widehat{\psi}, \mu\{\gamma_T\}\widehat{H}\Big]_\Gamma\Big). \end{aligned} \tag{3.14}$$

We now recall that X_Γ is its own dual with the duality pairing $[\cdot,\cdot]_\Gamma$ and we use the Cauchy–Schwarz inequality on $\mathbb{R}^2$ to estimate

$$\begin{aligned} \operatorname{Re} I &\le \|\widehat{\varphi}\|_{\mathrm{X}_\Gamma}\, \|\{\gamma_T\}\widehat{E}\|_{\mathrm{X}_\Gamma} + \|\widehat{\psi}\|_{\mathrm{X}_\Gamma}\, \|\{\gamma_T\}\mu\widehat{H}\|_{\mathrm{X}_\Gamma} \\ &\le \Big(\|\widehat{\varphi}\|^2_{\mathrm{X}_\Gamma} + \|\widehat{\psi}\|^2_{\mathrm{X}_\Gamma}\Big)^{1/2}\Big(\|\{\gamma_T\}\widehat{E}\|^2_{\mathrm{X}_\Gamma} + \|\{\gamma_T\}\mu\widehat{H}\|^2_{\mathrm{X}_\Gamma}\Big)^{1/2}. \end{aligned}$$

The right-hand side is finite because it is known from [17] that $\widehat{E}$ and $\widehat{H}$ are in the local Sobolev space $H_{\mathrm{loc}}(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)$ and moreover, $\{\gamma_T\}$ is a bounded operator from $H(\mathrm{curl},\Omega_R)$ onto X_Γ, where Ω_R is a ball of sufficiently large radius R that contains Γ. So we find that $\operatorname{Re} I$ has a finite bound, and by (3.13), $\widehat{E}$ and $\widehat{H}$ are therefore in $H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)$. We then use the bound C_Γ of $\{\gamma_T\} : H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma) \to \mathrm{X}_\Gamma$ to conclude

$$\operatorname{Re} I \le C_\Gamma\Big(\|\widehat{\varphi}\|^2_{\mathrm{X}_\Gamma} + \|\widehat{\psi}\|^2_{\mathrm{X}_\Gamma}\Big)^{1/2}\Big(\|\widehat{E}\|^2_{H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)} + \|\mu\widehat{H}\|^2_{H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)}\Big)^{1/2}.$$

In view of (3.13), this yields the stated result. □

On setting $\widehat{\psi} = 0$ in Lemma 3.1, we immediately obtain the following corollary.

Lemma 3.2 *For* $\operatorname{Re} s > 0$*, the single and double layer potential operators* $\mathcal{S}(s)$ *and* $\mathcal{D}(s)$ *extend by density to bounded linear operators from* X_Γ *to* $H(\mathrm{curl}, \mathbb{R}^3 \setminus \Gamma)$*, which are bounded by*

$$\|\mathcal{S}(s)\|_{H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)\leftarrow \mathrm{X}_\Gamma} \le C_\Gamma \frac{|s|^2+1}{\operatorname{Re} s}, \qquad \|\mathcal{D}(s)\|_{H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)\leftarrow \mathrm{X}_\Gamma} \le C_\Gamma \frac{|s|^2+1}{\operatorname{Re} s},$$

where again $C_\Gamma = \|\{\gamma_T\}\|_{\mathrm{X}_\Gamma \leftarrow H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)}$.

We return to the transmission problem (3.8)–(3.11). Electromagnetic scattered fields $\widehat{E}, \widehat{H}$ that solve (3.1)–(3.2) in the exterior domain $\Omega = \Omega^+$ are extended by zero into the interior, so that the jumps are just the exterior tangential traces in (3.10)–(3.11), as are the averages up to the factor $1/2$. The scattered fields are then recovered from their tangential traces by the representation formulas (recall that $\varepsilon\mu = 1$)

$$\widehat{E} = -\,\mathcal{S}(s)\big(\mu\gamma_T \widehat{H}\big) + \mathcal{D}(s)\big(-\gamma_T \widehat{E}\big) \quad \text{in } \Omega, \tag{3.15}$$

$$\mu\widehat{H} = -\mathcal{D}(s)\big(\mu\gamma_T \widehat{H}\big) \,-\, \mathcal{S}(s)\big(-\gamma_T \widehat{E}\big) \quad \text{in } \Omega. \tag{3.16}$$

Our analytical as well as numerical approach will consist in determining the tangential traces from boundary integral equations that incorporate the generalized impedance boundary conditions, and then obtain the electromagnetic fields from the above representation formulas (or their time-domain analogues).

In this situation the bound of Lemma 3.1 improves as follows.

Lemma 3.3 *In the situation of Lemma 3.1, assume further that the interior tangential traces of* $\widehat{E}$ *and* $\widehat{H}$ *are identically* 0*, which implies* $\mu\gamma_T\widehat{H} = \widehat{\varphi}$ *and* $-\gamma_T\widehat{E} = \widehat{\psi}$*. Then, the bound of Lemma 3.1 improves to*

$$\left\| \begin{pmatrix} \widehat{E} \\ \mu\widehat{H} \end{pmatrix} \right\|_{H(\mathrm{curl},\Omega)^2} \le \left(\frac{|s|^2+1}{2\operatorname{Re} s} \right)^{1/2} \left\| \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right\|_{\mathrm{X}_\Gamma{}^2}.$$

Furthermore, we have the L^2 *bound*

$$\left\| \begin{pmatrix} \widehat{E} \\ \mu\widehat{H} \end{pmatrix} \right\|_{(L^2(\Omega)^3)^2} \le \left(\frac{1}{2\operatorname{Re} s} \right)^{1/2} \left\| \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right\|_{\mathrm{X}_\Gamma{}^2}.$$

Proof The proof of the $H(\mathrm{curl}, \Omega)$ bound is identical to that of Lemma 3.1 down to (3.14), which now implies the bound $\operatorname{Re} I \le \frac{1}{2}\big(\|\widehat{\varphi}\|_{\mathrm{X}_\Gamma}^2 + \|\widehat{\psi}\|_{\mathrm{X}_\Gamma}^2\big)$ and yields the stated result. The proof of the L^2 bound is even simpler, working directly with (3.12) instead of (3.13). $\square$

3.2 Time-harmonic boundary operators and the Calderón operator

The electromagnetic *single and double layer boundary operators* are the operators from X_Γ to X_Γ defined as

$$V(s) = \{\gamma_T\} \circ \mathcal{S}(s), \qquad K(s) = \{\gamma_T\} \circ \mathcal{D}(s).$$

We define the *Calderón operator* as introduced in [27] (with a sign corrected in [34]):

$$B(s) = \begin{pmatrix} -V(s) & K(s) \\ -K(s) & -V(s) \end{pmatrix} = \{\gamma_T\} \circ \begin{pmatrix} -\mathcal{S}(s) & \mathcal{D}(s) \\ -\mathcal{D}(s) & -\mathcal{S}(s) \end{pmatrix}, \tag{3.17}$$

where we note that the right-most block operator is the one appearing in the representation formula (3.6)–(3.7). Let $\widehat{E}, \widehat{H}$ be Maxwell solutions that are given by this representation formula. Then, the Calderón operator satisfies by construction (see (3.8)–(3.11))

$$B(s) \begin{pmatrix} \mu[\gamma_T]\widehat{H} \\ -[\gamma_T]\widehat{E} \end{pmatrix} = \begin{pmatrix} \{\gamma_T\}\widehat{E} \\ \mu\{\gamma_T\}\widehat{H} \end{pmatrix}. \tag{3.18}$$

The following bound of $B(s)$ follows immediately from (3.17) and Lemma 3.1. This bound improves on existing time-harmonic s-explicit bounds of the boundary operators; see [5, Theorem 4.4] and [27, Lemma 2.3].

Lemma 3.4 *For* $\operatorname{Re} s > 0$, *the Calderón operator* $B(s) : {\mathrm{X}_\Gamma}^2 \to {\mathrm{X}_\Gamma}^2$ *is bounded by*

$$\|B(s)\|_{{\mathrm{X}_\Gamma}^2 \leftarrow {\mathrm{X}_\Gamma}^2} \le C_\Gamma^2 \, \frac{|s|^2 + 1}{\operatorname{Re} s}, \tag{3.19}$$

where again $C_\Gamma = \|\{\gamma_T\}\|_{\mathrm{X}_\Gamma \leftarrow H(\mathrm{curl}, \mathbb{R}^3 \setminus \Gamma)}$. *The same bound also holds for* $\|V(s)\|_{\mathrm{X}_\Gamma \leftarrow \mathrm{X}_\Gamma} + \|K(s)\|_{\mathrm{X}_\Gamma \leftarrow \mathrm{X}_\Gamma}$.

We extend the skew-hermitian pairing $[\cdot,\cdot]_\Gamma$ from $\mathrm{X}_\Gamma \times \mathrm{X}_\Gamma$ to ${\mathrm{X}_\Gamma}^2 \times {\mathrm{X}_\Gamma}^2$ in the obvious way:

$$\left[\begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} \upsilon \\ \xi \end{pmatrix} \right]_\Gamma = [\varphi, \upsilon]_\Gamma + [\psi, \xi]_\Gamma.$$

The Calderón operator $B(s)$ is coercive with respect to the pairing $[\cdot,\cdot]_\Gamma$, as was shown in [27, Lemma 3.1]. Here we give a formulation of this key lemma with an explicit bound, and we include the short proof that reframes the arguments of the proof in [27] within the present setting.

Lemma 3.5 (essentially [27, Lemma 3.1]) *For* $\operatorname{Re} s > 0$, *we have the coercivity*

$$\operatorname{Re} \left[\begin{pmatrix} \varphi \\ \psi \end{pmatrix}, B(s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right]_\Gamma \ge \frac{1}{c_\Gamma^2} \, \frac{\operatorname{Re} s}{|s|^2 + 1} \left(\|\varphi\|_{\mathrm{X}_\Gamma}^2 + \|\psi\|_{\mathrm{X}_\Gamma}^2 \right) \tag{3.20}$$

for all $(\varphi, \psi) \in {\mathrm{X}_\Gamma}^2$. *Here,* $c_\Gamma = \|[\gamma_T]\|_{\mathrm{X}_\Gamma \leftarrow H(\mathrm{curl}, \mathbb{R}^3 \setminus \Gamma)}$.

Proof Let $(\widehat{\varphi}, \widehat{\psi}) \in \mathrm{X}_\Gamma{}^2$ be arbitrary and $\widehat{E}, \widehat{H} \in H(\mathrm{curl}, \mathbb{R}^3 \setminus \Gamma)$ be the solutions to the associated transmission problem of Lemma 3.1. We then have consecutively by (3.10)–(3.11), by the bound c_Γ of the jump operator $[\gamma_T]$, and by (3.13)–(3.14),

$$\begin{aligned}
\left\| \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right\|^2_{\mathrm{X}_\Gamma \times \mathrm{X}_\Gamma} &= \left\| \begin{pmatrix} \mu[\gamma_T]\widehat{H} \\ -[\gamma_T]\widehat{E} \end{pmatrix} \right\|^2_{\mathrm{X}_\Gamma \times \mathrm{X}_\Gamma} \\
&\le c_\Gamma^2 \left(\|\mu \widehat{H}\|^2_{H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)} + \|\widehat{E}\|^2_{H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)} \right) \\
&= c_\Gamma^2 \, \frac{|s|^2+1}{\operatorname{Re} s} \operatorname{Re} \left[\begin{pmatrix} \mu[\gamma_T]\widehat{H} \\ -[\gamma_T]\widehat{E} \end{pmatrix}, \begin{pmatrix} \{\gamma_T\}\widehat{E} \\ \mu\{\gamma_T\}\widehat{H} \end{pmatrix} \right]_\Gamma \\
&= c_\Gamma^2 \, \frac{|s|^2+1}{\operatorname{Re} s} \operatorname{Re} \left[\begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix}, B(s) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right]_\Gamma ,
\end{aligned}$$

where the last equality follows from (3.18) on inserting (3.10)–(3.11). This yields the result. □

3.3 Boundary integral equation for tangential traces under time-harmonic generalized impedance boundary conditions

We now derive a well-posed boundary integral equation of the time-harmonic Maxwell's equations (3.1)–(3.2) for $\operatorname{Re} s > 0$ with the weak formulation of the generalized impedance boundary condition (2.17),

$$[\phi, \gamma_T \widehat{E}]_\Gamma + \langle \phi, Z(s)\gamma_T \widehat{H} \rangle = \langle \phi, \widehat{g}^{\mathrm{inc}} \rangle_\Gamma \qquad \text{for all } \phi \in \mathrm{V}_\Gamma, \tag{3.21}$$

where the transfer operator $Z(s)$ satisfies (2.9)–(2.10), and $\widehat{g}^{\mathrm{inc}} \in \mathrm{V}_\Gamma{}'$ is arbitrary.

We start with the observation that any solution of the time-harmonic Maxwell's equations on the exterior domain $\Omega = \Omega^+$, trivially extended by zero into the bounded interior Ω^-, solves an associated transmission problem as in Lemma 3.1. As the inner traces of the extended fields vanish by construction, the jumps and the averages reduce to the outer traces and the representation formulas can be evaluated by the boundary data, as in (3.15)–(3.16).

Therefore, the relation (3.18) of the Calderón operator then reads

$$B(s) \begin{pmatrix} \mu\gamma_T \widehat{H} \\ -\gamma_T \widehat{E} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \gamma_T \widehat{E} \\ \mu\gamma_T \widehat{H} \end{pmatrix}. \tag{3.22}$$

Following [12,7,9] in the acoustic case, we rewrite this identity by adding a symmetric block operator and arrive at

$$B_{\mathrm{imp}}(s) \begin{pmatrix} \mu\gamma_T \widehat{H} \\ -\gamma_T \widehat{E} \end{pmatrix} = \begin{pmatrix} \gamma_T \widehat{E} \\ 0 \end{pmatrix}, \qquad B_{\mathrm{imp}}(s) = B(s) + \begin{pmatrix} 0 & -\frac{1}{2}I \\ -\frac{1}{2}I & 0 \end{pmatrix}. \tag{3.23}$$

Introducing the boundary densities

$$\widehat{\varphi} = \mu\gamma_T \widehat{H}, \qquad \widehat{\psi} = -\gamma_T \widehat{E}, \tag{3.24}$$

and testing both sides with $(\upsilon, \xi) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$ yields

$$\left[\begin{pmatrix} \upsilon \\ \xi \end{pmatrix}, B_{\mathrm{imp}}(s) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right]_\Gamma = \left[\upsilon, \gamma_T \widehat{E} \right]_\Gamma .$$

Inserting the boundary condition (3.21) on the right-hand side leads, upon rearranging the impedance operator to the left-hand side, to the weak formulation of the boundary integral equation that will be studied here: Find $(\widehat{\varphi}, \widehat{\psi}) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$ such that, for all $(\upsilon, \xi) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$,

$$\left[\begin{pmatrix} \upsilon \\ \xi \end{pmatrix}, B_{\mathrm{imp}}(s) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right]_\Gamma + \mu^{-1} \langle \upsilon, Z(s)\widehat{\varphi} \rangle = \langle \upsilon, \widehat{g}^{\mathrm{inc}} \rangle. \tag{3.25}$$

We introduce the family of operators $A(s) : \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma \to \mathrm{V}_\Gamma{}' \times \mathrm{X}_\Gamma{}'$ that is defined by the left-hand side above, i.e., for all (φ, ψ) and $(\upsilon, \xi) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$,

$$\left\langle \begin{pmatrix} \upsilon \\ \xi \end{pmatrix}, A(s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle = \left[\begin{pmatrix} \upsilon \\ \xi \end{pmatrix}, B_{\mathrm{imp}}(s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right]_\Gamma + \mu^{-1} \langle \upsilon, Z(s)\varphi \rangle, \tag{3.26}$$

where $\langle \cdot, \cdot \rangle$ denotes the anti-duality between $\mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$ and $\mathrm{V}_\Gamma{}' \times \mathrm{X}_\Gamma{}'$ on the left-hand side, and between V_Γ and $\mathrm{V}_\Gamma{}'$ on the right-hand side. The boundary integral equation (3.25) then reads more compactly as follows: find $(\widehat{\varphi}, \widehat{\psi}) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$ such that

$$\left\langle \begin{pmatrix} \upsilon \\ \xi \end{pmatrix}, A(s) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right\rangle = \langle \upsilon, \widehat{g}^{\mathrm{inc}} \rangle \qquad \text{for all} \quad (\upsilon, \xi) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma. \tag{3.27}$$

We write (3.27) even more compactly as

$$A(s) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} = \begin{pmatrix} \widehat{g}^{\mathrm{inc}} \\ 0 \end{pmatrix}. \tag{3.28}$$

The boundary integral operator $A(s)$ defined by (3.26) inherits the bounds and positivity properties of the Calderón operator $B(s)$ and the impedance operator $Z(s)$, respectively, from Lemma 3.4–3.5 and (2.9)–(2.10), as the following two lemmas state.

Lemma 3.6 *The operators* $A(s) : \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma \to \mathrm{V}_\Gamma{}' \times \mathrm{X}_\Gamma{}'$ *defined by* (3.26) *form an analytic family of bounded linear operators that satisfy the bound, for* $\mathrm{Re}\, s \geq \sigma > 0$,

$$\| A(s) \|_{\mathrm{V}_\Gamma{}' \times \mathrm{X}_\Gamma{}' \leftarrow \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma} \leq C_\sigma \frac{|s|^2}{\mathrm{Re}\, s}.$$

The constant C_σ *only depends polynomially on* σ^{-1} *and on the boundary* Γ *via the norm of the tangential trace operator.*

Proof The bound (3.19) of the Calderón operator $B(s)$ given in Lemma 3.5 and the polynomial bound (2.9) of the impedance operator $Z(s)$ yield estimates on all the terms appearing on the right-hand side of (3.26) with the exceptions of the identity operators $I_{\mathrm{V}_\Gamma' \leftarrow \mathrm{X}_\Gamma}$ and $I_{\mathrm{X}_\Gamma' \leftarrow \mathrm{V}_\Gamma}$ occurring in B_{imp}, which are bounded in view of the continuous embeddings $\mathrm{V}_\Gamma \subset \mathrm{X}_\Gamma = \mathrm{X}_\Gamma' \subset \mathrm{V}_\Gamma'$. □

Lemma 3.7 *The operator family $A(s)$ has the following coercivity property: For every $\sigma > \sigma_0$ (with $\sigma_0 \geq 0$ of (2.10)), there exists a constant $c_\sigma > 0$ such that for* $\operatorname{Re} s \geq \sigma$,

$$\operatorname{Re} \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, A(s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle \geq c_\sigma \frac{\operatorname{Re} s}{|s|^2} \left(\|\varphi\|_{\mathrm{V}_\Gamma}^2 + \|\psi\|_{\mathrm{X}_\Gamma}^2 \right),$$

for all $(\varphi, \psi) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$. The constant c_σ only depends polynomially on σ^{-1} and on the boundary Γ via the norm of the tangential trace operator.

Proof The operator $B_{\mathrm{imp}}(s)$ has the same coercivity property as $B(s)$ because the additional sum is skew-symmetric with respect to the skew-hermitian pairing, since

$$\left[\begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right]_\Gamma = [\varphi, \psi]_\Gamma + [\psi, \varphi]_\Gamma = 0.$$

Combining the coercivity of the Calderón operator, as stated in Lemma 3.5, and the positivity condition (2.10) on $Z(s)$ then yield

$$\begin{aligned} \operatorname{Re} \left\langle \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, A(s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle &= \operatorname{Re} \left[\begin{pmatrix} \varphi \\ \psi \end{pmatrix}, B(s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right]_\Gamma + \mu^{-1} \operatorname{Re} \langle \upsilon, Z(s) \varphi \rangle \\ &\geq c_\sigma^{(B)} \frac{\operatorname{Re} s}{|s|^2} \left(\|\varphi\|_{\mathrm{X}_\Gamma}^2 + \|\psi\|_{\mathrm{X}_\Gamma}^2 \right) + c_\sigma^{(Z)} \frac{\operatorname{Re} s}{|s|^2} |\varphi|_{\mathrm{V}_\Gamma}^2 \\ &\geq c_\sigma \frac{\operatorname{Re} s}{|s|^2} \left(\|\varphi\|_{\mathrm{V}_\Gamma}^2 + \|\psi\|_{\mathrm{X}_\Gamma}^2 \right), \end{aligned}$$

which is the stated result. □

From the previous two lemmas we obtain the following result.

Proposition 3.1 (Well-posedness of the time-harmonic boundary integral equation) *For $\operatorname{Re} s > \sigma_0 \geq 0$, the boundary integral equation* (3.27), *with the boundary operator $A(s) : \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma \to \mathrm{V}_\Gamma' \times \mathrm{X}_\Gamma'$ defined by* (3.26), *has a unique solution $(\widehat{\varphi}, \widehat{\psi}) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$, and*

$$\left\| \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right\|_{\mathrm{V}_\Gamma \times \mathrm{X}_\Gamma} \leq C_\sigma \frac{|s|^2}{\operatorname{Re} s} \left\| \widehat{g}^{\mathrm{inc}} \right\|_{\mathrm{V}_\Gamma'}. \tag{3.29}$$

The constant C_σ only depends polynomially on σ^{-1} and on the boundary Γ via the norm of the tangential trace operator.

Proof By the Lax–Milgram theorem, Lemmas 3.6 and 3.7 yield that $A(s)$ is invertible and its inverse is bounded, for $\mathrm{Re}\, s \geq \sigma > \sigma_0$, by

$$\left\| A(s)^{-1} \right\|_{\mathrm{V}_\Gamma \times \mathrm{X}_\Gamma \leftarrow \mathrm{V}_\Gamma{}' \times \mathrm{X}_\Gamma{}'} \leq C_\sigma \frac{|s|^2}{\mathrm{Re}\, s}. \tag{3.30}$$

This gives the result. $\square$

Remark 3.1 In Lemmas 2.1 and 2.2 we have $\delta^{1/2}\|\phi\|_{L^2(\Gamma)^3} \leq \|\phi\|_{\mathrm{V}_\Gamma}$ for all $\phi \in \mathrm{V}_\Gamma$. This implies that for a tangential vector field $\widehat{g}^{\mathrm{inc}} \in L^2(\Gamma)^3$,

$$\begin{aligned} \left\| \widehat{g}^{\mathrm{inc}} \right\|_{\mathrm{V}_\Gamma{}'} &= \sup_{\|\phi\|_{\mathrm{V}_\Gamma}=1} \langle \phi, \widehat{g}^{\mathrm{inc}} \rangle = \sup_{\|\phi\|_{\mathrm{V}_\Gamma}=1} (\phi, \widehat{g}^{\mathrm{inc}})_\Gamma \\ &\leq \sup_{\|\phi\|_{L^2(\Gamma)^3} \leq \delta^{-1/2}} (\phi, \widehat{g}^{\mathrm{inc}})_\Gamma = \delta^{-1/2} \, \|\widehat{g}^{\mathrm{inc}}\|_{L^2(\Gamma)^3}. \end{aligned}$$

On the other hand, we have $\|\phi\|_{\mathrm{X}_\Gamma} \leq \|\phi\|_{\mathrm{V}_\Gamma}$ for all $\phi \in \mathrm{V}_\Gamma$. If $\widehat{g}^{\mathrm{inc}}$ is in X_Γ, we therefore obtain

$$\left\| \widehat{g}^{\mathrm{inc}} \right\|_{\mathrm{V}_\Gamma{}'} \leq \left\| \widehat{g}^{\mathrm{inc}} \right\|_{\mathrm{X}_\Gamma}$$

without any dependence on the small parameter δ. We do have $\widehat{g}^{\mathrm{inc}} \in \mathrm{X}_\Gamma$ in the case where $\widehat{g}^{\mathrm{inc}} = -\widehat{E}_T^{\mathrm{inc}} - Z(s)\gamma_T \widehat{H}^{\mathrm{inc}}$, cf. (2.4), for a sufficiently regular boundary Γ and sufficiently regular fields $\widehat{E}^{\mathrm{inc}}$ and $\widehat{H}^{\mathrm{inc}}$ for $Z(s)$ in the situations of Lemmas 2.1 and 2.2.

3.4 Well-posedness of time-harmonic scattering from generalized impedance boundary conditions

Using the above properties, we prove the following result.

Theorem 3.1 (Well-posedness of the time-harmonic scattering problem) *For* $\mathrm{Re}\, s > \sigma_0 \geq 0$, *consider the time-harmonic scattering problem* (3.1)–(3.2) *(with the normalization* $\varepsilon\mu = 1$*) under the generalized impedance boundary condition* (3.21), *with* $Z(s)$ *satisfying conditions* (2.9)–(2.10) *and with* $\widehat{g}^{\mathrm{inc}} \in \mathrm{V}_\Gamma{}'$.

(a) This problem has a solution $(\widehat{E}, \widehat{H}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ *given by the representation formulas* (3.15)–(3.16). *The tangential traces are uniquely determined by the solution* $(\widehat{\varphi}, \widehat{\psi}) = (\mu\gamma_T \widehat{H}, -\gamma_T \widehat{E}) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$ *of the system of boundary integral equations of Proposition 3.1.*

(b) The electromagnetic fields are bounded by

$$\|\widehat{E}\|_{H(\mathrm{curl},\Omega)} + \|\mu \widehat{H}\|_{H(\mathrm{curl},\Omega)} \leq C_\sigma \frac{|s|^3}{(\mathrm{Re}\, s)^{3/2}} \left\| \widehat{g}^{\mathrm{inc}} \right\|_{\mathrm{V}_\Gamma{}'},$$

where C_σ *depends on* σ, *on* c_σ *of* (2.10), *and on* Γ *through norms of tangential trace operators, but is independent of* ε *and* μ *with* $\varepsilon\mu = 1$ *and, in the case of the impedance operators* (1.3)–(1.6), *independent of the small parameter* δ *(but see Remark 3.1).*

Proof By Proposition 3.1, the boundary integral equation (3.28) has a unique solution $(\widehat{\varphi}, \widehat{\psi}) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$, which is bounded by (4.4).

We are now in the situation of Lemma 3.1: The representation formulas (3.6)–(3.7) define $\widehat{E}, \widehat{H} \in H(\mathrm{curl}, \mathbb{R}^3 \setminus \Gamma)$, which solve the transmission problem (3.8)–(3.11). Furthermore, upon expressing $(\widehat{\varphi}, \widehat{\psi})$ in terms of $(\widehat{E}, \widehat{H})$ by means of (3.10)–(3.11), the fundamental identity (3.18) of the Calderón operator implies the identity

$$\begin{aligned} B_{\mathrm{imp}}(s) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} &= B(s) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \widehat{\psi} \\ \widehat{\varphi} \end{pmatrix} \\ &= \begin{pmatrix} \{\gamma_T \widehat{E}\} \\ \mu\{\gamma_T \widehat{H}\} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -[\gamma_T \widehat{E}] \\ \mu[\gamma_T \widehat{H}] \end{pmatrix} = \begin{pmatrix} \gamma_T^+ \widehat{E} \\ \mu\gamma_T^- \widehat{H} \end{pmatrix}. \end{aligned} \tag{3.31}$$

By definition, $(\widehat{\varphi}, \widehat{\psi})$ solve the weak formulation (3.25) of the boundary integral equation. Using the identity above reduces the weak formulation to

$$\big[\upsilon, \gamma_T^+ \widehat{E}\big]_\Gamma + \mu^{-1}\big\langle \upsilon, Z(s)\widehat{\varphi}\big\rangle = \big(\upsilon, \widehat{g}^{\mathrm{inc}}\big)_\Gamma \qquad \text{for all } \upsilon \in \mathrm{V}_\Gamma, \tag{3.32}$$

$$\big[\xi, \mu\gamma_T^- \widehat{H}\big]_\Gamma = 0 \qquad \text{for all } \xi \in \mathrm{X}_\Gamma. \tag{3.33}$$

As X_Γ coincides with its own dual, we deduce $\gamma_T^- \widehat{H} = 0$ and hence $\widehat{\varphi} = \mu\gamma_T^+ \widehat{H}$. Therefore, (3.32) implies that $(\widehat{E}, \widehat{H})|_{\Omega^+}$ are indeed solutions to the time-harmonic Maxwell's equations which satisfy the generalized impedance boundary condition (3.21).

Green's formula (2.6) for the solution $\widehat{E}|_{\Omega^-}$ in the interior domain Ω^- of (3.1)–(3.2) yields

$$\int_{\Omega^-} s\varepsilon\big|\widehat{E}\big|^2 + \bar{s}\mu\big|\widehat{H}\big|^2 \mathrm{d}x = -\int_\Gamma (\gamma_T^- \widehat{E} \times \nu) \cdot \gamma_T^- \widehat{H} \, \mathrm{d}\sigma = 0,$$

which, after taking the real part, gives $\widehat{E}|_{\Omega^-} = \widehat{H}|_{\Omega^-} = 0$ and therefore $\gamma_T^- \widehat{E} = 0$ and $\widehat{\psi} = -\gamma_T^+ \widehat{E}$. This completes the proof of part (a).

With $\gamma_T^- \widehat{E} = 0$ and $\gamma_T^- \widehat{H} = 0$ as shown, we are now in the situation of Lemma 3.3, which together with the bound of Proposition 3.1 yields the bound of part (b) of the theorem. □

Remark 3.2 In view of the L^2 bound of Lemma 3.3, we further have the L^2 bound

$$\|\widehat{E}\|_{L^2(\Omega)^3} + \|\mu\widehat{H}\|_{L^2(\Omega)^3} \le C_\sigma \frac{|s|^2}{(\mathrm{Re}\, s)^{3/2}} \left\|\widehat{g}^{\mathrm{inc}}\right\|_{\mathrm{V}_\Gamma'}. \tag{3.34}$$

3.5 Bounds for the time-harmonic potential operators away from the boundary

Point evaluations of the potential operators are bounded by means of the following lemma, which yields a more favourable dependence on s for large

$\operatorname{Re} s$ than the $H(\operatorname{curl}, \mathbb{R}^3 \setminus \Gamma)$-norm bound of Lemma 3.2. On smooth domains similar pointwise bounds already exist, obtained with more straightforward techniques; see [5, Theorem 4.4 (c)] for the single layer operator. The proof of the following lemma generalizes the idea given there to the more technical situation of non-smooth boundaries.

Lemma 3.8 *The single and double layer potential operators $\mathcal{S}(s), \mathcal{D}(s)$ evaluated at a point $x \in \mathbb{R}^3 \setminus \Gamma$ with $d = \operatorname{dist}(x, \Gamma) > 0$ satisfy the following bounds:*

$$|\mathcal{S}(s)\varphi(x)| \leq C\, |s|^2\, e^{-d \operatorname{Re} s}\, \|\varphi\|_{\mathrm{X}_\Gamma}\,,$$
$$|\mathcal{D}(s)\varphi(x)| \leq C\, |s|^2\, e^{-d \operatorname{Re} s}\, \|\varphi\|_{\mathrm{X}_\Gamma}\,,$$

for $\operatorname{Re} s \geq \sigma > 0$, and for any $\varphi \in \mathrm{X}_\Gamma$. The constant C depends only on x, Γ and σ.

Proof Let e_j denote the j-th unit vector in $\mathbb{R}^3$, and let $x \in \Omega$ with $d = \operatorname{dist}(x, \Gamma) > 0$. We then start by analysing the corresponding component of the integral

$$\begin{aligned}
\left| e_j \cdot \int_\Gamma G(s, x-y)\varphi(y)\mathrm{d}y \right| &= \left| \int_\Gamma G(s, x-y) e_j \cdot \varphi(y)\mathrm{d}y \right| \\
&= \left| \int_\Gamma G(s, x-y)\,(e_j \times \nu) \cdot (\varphi(y) \times \nu)\,\mathrm{d}y \right| \\
&\leq C\, \|\gamma_T\,(G(s, x-\cdot)e_j)\|_{\mathrm{X}_\Gamma}\, \|\varphi\|_{\mathrm{X}_\Gamma} \\
&\leq C\, \|G(s, x-\cdot)e_j\|_{H(\mathrm{curl},\Omega)}\, \|\varphi\|_{\mathrm{X}_\Gamma} \\
&\leq C\, \|G(s, x-\cdot)\|_{H^1(\Omega)}\, \|\varphi\|_{\mathrm{X}_\Gamma}\,,
\end{aligned}$$

where the estimate on the trace holds due to [16, Theorem 4.1]. The second summand of the single layer operator is estimated more straightforwardly, as

$$\begin{aligned}
\left| \nabla \int_\Gamma G(s, x-y) \operatorname{div}_\Gamma \varphi(y)\mathrm{d}y \right| &\leq \|\nabla G(s, x-\cdot)\|_{H^{1/2}(\Gamma)}\, \|\operatorname{div}_\Gamma \varphi(y)\|_{H^{-1/2}(\Gamma)} \\
&\leq \|G(s, x-\cdot)\|_{H^2(\Omega)}\, \|\varphi\|_{\mathrm{X}_\Gamma}\,.
\end{aligned}$$

We estimate the double layer potential similarly to the first summand of the single layer, by taking a partial derivative with respect to a coordinate x_i and obtain

$$\begin{aligned}
\left| \partial_{x_i} e_j \cdot \int_\Gamma G(s, x-y)\varphi(y)\mathrm{d}y \right| &= \left| e_j \cdot \int_\Gamma \partial_{x_i} G(s, x-y)\varphi(y)\mathrm{d}y \right| \\
&\leq \|\partial_{x_i} G(s, x-\cdot)\|_{H^1(\Omega)}\, \|\varphi\|_{\mathrm{X}_\Gamma}\,.
\end{aligned}$$

Since the curl operator is a linear combination of partial derivatives, this estimate implies the stated bound for the double layer potential operator. □

The proof of the above result immediately implies the following extension for any spatial differential operator. In particular it implies that, given traces $\widehat{\varphi}, \widehat{\psi} \in \mathrm{X}_\Gamma$, the corresponding (time-harmonic) solution field $\widehat{E} = -\mathcal{S}(s)\widehat{\varphi} + \mathcal{D}(s)\widehat{\psi}$ is smooth in every point $x \in \Omega \setminus \Gamma$.

Lemma 3.9 *For every positive integer k and for $j = 1, 2, 3$, we have the following bounds at $x \in \mathbb{R}^3 \setminus \Gamma$ with $d = \mathrm{dist}(x, \Gamma) > 0$ for $\mathrm{Re}\, s \geq \sigma > 0$:*

$$\begin{aligned} \left|\partial_{x_j}^k \mathcal{S}(s)\varphi(x)\right| &\leq C\, |s|^{2+k}\, e^{-\sigma d}\, \|\varphi\|_{\mathrm{X}_\Gamma}, \\ \left|\partial_{x_j}^k \mathcal{D}(s)\varphi(x)\right| &\leq C\, |s|^{2+k}\, e^{-\sigma d}\, \|\varphi\|_{\mathrm{X}_\Gamma}, \end{aligned} \qquad \text{for all } \varphi \in \mathrm{X}_\Gamma.$$

The following lemma gives a bound of the potential operators in the operator norm from X_Γ to $H(\mathrm{curl}, \Omega_d)$, where the boundary of $\Omega_d \subset \Omega$ has distance d to Γ.

Lemma 3.10 *Let $\Omega_d = \{x \in \Omega \mid \mathrm{dist}(x, \Gamma) > d\}$ be the domain away from the boundary by at least some fixed distance $d > 0$. Then, the single and double layer potential operators $\mathcal{S}(s), \mathcal{D}(s)$ satisfy the following bounds:*

$$\begin{aligned} \|\mathcal{S}(s)\|_{H(\mathrm{curl},\Omega_d)\leftarrow \mathrm{X}_\Gamma} &\leq C e^{-d\,\mathrm{Re}\, s} \max\{\sigma^{-1}, \sigma^{-3}\}\, |s|^3, \\ \|\mathcal{D}(s)\|_{H(\mathrm{curl},\Omega_d)\leftarrow \mathrm{X}_\Gamma} &\leq C e^{-d\,\mathrm{Re}\, s} \max\{1, \sigma^{-3/2}\}\, |s|^3, \end{aligned}$$

for $\mathrm{Re}\, s \geq \sigma > 0$.

Proof To show the bound for the double layer potential, we start with the square of the $H(\mathrm{curl}, \Omega)$-norm of an image of the double layer potential and employ the bounds from Lemma 3.9:

$$\begin{aligned} \|\mathcal{S}(s)\varphi(x)\|^2_{H(\mathrm{curl},\Omega_d)} &= \int_{\Omega_d} |\mathcal{S}(s)\varphi(x)|^2 + |\mathrm{curl}\, \mathcal{S}(s)\varphi(x)|^2 \,\mathrm{d}x \\ &\leq C_\sigma\, \|\varphi\|^2_{\mathrm{X}_\Gamma}\, |s|^6 \int_{\Omega_d} e^{-2\,\mathrm{dist}(x,\Gamma)\,\mathrm{Re}\, s} \mathrm{d}x. \end{aligned}$$

Estimating the last integral then yields the stated result for $\mathcal{S}(s)$. The result for $\mathcal{D}(s)$ is obtained by the same argument. □

4 Time-dependent Maxwell's equations with generalized impedance boundary conditions

The time-harmonic treatment of the previous section extends to the time domain in a direct way, using the passage from the Laplace domain to the time domain described in Section 2.2. We start from the time-dependent version of the boundary integral equation (3.23), obtained by formally replacing the Laplace transform variable s by the time differentiation operator ∂_t:

Find time-dependent boundary densities $(\varphi, \psi) : [0,T] \to \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$ (of temporal regularity to be specified later) such that for almost every $t \in [0,T]$ we have for all $(\upsilon, \xi) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$,

$$\left[\begin{pmatrix}\upsilon \\ \xi\end{pmatrix}, B_{\mathrm{imp}}(\partial_t)\begin{pmatrix}\varphi \\ \psi\end{pmatrix}\right]_\Gamma + \mu^{-1}\langle \upsilon, Z(\partial_t)\gamma_T \varphi\rangle = \langle \upsilon, g^{\mathrm{inc}}\rangle, \tag{4.1}$$

where $g^{\mathrm{inc}} : [0,T] \to \mathrm{V}_\Gamma{}'$ is given by (2.4), assuming that $g^{\mathrm{inc}} \in H_0^m(0,T;\mathrm{V}_\Gamma{}')$ with sufficiently large m (to be specified later). We refer to Section 2.2 for the definition of this spatio-temporal Hilbert space.

With the operators $A(s) : \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma \to \mathrm{V}_\Gamma{}' \times \mathrm{X}_\Gamma{}'$ defined by (3.26), this boundary integral equation is rewritten more compactly as in (3.28),

$$A(\partial_t)\begin{pmatrix}\varphi \\ \psi\end{pmatrix} = \begin{pmatrix}g^{\mathrm{inc}} \\ 0\end{pmatrix}. \tag{4.2}$$

In view of the bound (3.30) on the operator family $A(s)^{-1}$ for $\operatorname{Re} s > \sigma_0$, the temporal convolution operator $A^{-1}(\partial_t)$ is well-defined by (2.11), and by the composition rule we have $A^{-1}(\partial_t)A(\partial_t) = \mathrm{Id}$ and $A(\partial_t)A^{-1}(\partial_t) = \mathrm{Id}$. So we have the temporal convolution

$$\begin{pmatrix}\varphi \\ \psi\end{pmatrix} = A^{-1}(\partial_t)\begin{pmatrix}g^{\mathrm{inc}} \\ 0\end{pmatrix} \tag{4.3}$$

as the unique solution of (4.2). More precisely, with the argument given above and the bound of [29, Lemma 2.1], i.e. (2.13) used for $A^{-1}(\partial_t)$ instead of $Z(\partial_t)$ and with the exponent $\kappa = 2$ by (3.30), we obtain the following result.

Proposition 4.1 (Well-posedness of the time-dependent boundary integral equation) *Let $r \geq 0$. For $g^{\mathrm{inc}} \in H_0^{r+3}(0,T;\mathrm{V}_\Gamma{}')$, the boundary integral equation* (4.2)*, with the boundary operator $A(s) : \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma \to \mathrm{V}_\Gamma{}' \times \mathrm{X}_\Gamma{}'$ defined by* (3.26)*, has a unique solution $(\varphi, \psi) \in H_0^{r+1}(0,T;\mathrm{V}_\Gamma \times \mathrm{X}_\Gamma)$, and*

$$\left\| \begin{pmatrix}\varphi \\ \psi\end{pmatrix} \right\|_{H_0^{r+1}(0,T;\mathrm{V}_\Gamma \times \mathrm{X}_\Gamma)} \leq C_T \left\| g^{\mathrm{inc}} \right\|_{H_0^{r+3}(0,T;\mathrm{V}_\Gamma{}')}. \tag{4.4}$$

Here, C_T depends on T (polynomially if $\sigma_0 = 0$ in (2.9)–(2.10)*) and on the boundary Γ via norms of tangential trace operators.*

With the time-dependent boundary densities φ, ψ of Proposition 4.1, the scattered wave is obtained by the time-dependent representation formula (assuming here again $\varepsilon\mu = 1$)

$$E = -\mathcal{S}(\partial_t)\varphi + \mathcal{D}(\partial_t)\psi, \tag{4.5}$$

$$\mu H = -\mathcal{D}(\partial_t)\varphi - \mathcal{S}(\partial_t)\psi. \tag{4.6}$$

We now give the well-posedness result for the time-dependent scattering problem under the generalized impedance boundary condition, which follows from the time-harmonic well-posedness result Theorem 3.1.

Theorem 4.1 (Well-posedness of the time-dependent scattering problem) *Consider the time-dependent scattering problem* (2.1)–(2.2) *(with the normalization $\varepsilon\mu = 1$) under the generalized impedance boundary condition* (2.17)*, with $Z(s)$ satisfying conditions* (2.9)–(2.10) *with $\kappa \le 1$ and with $\widehat{g}^{\mathrm{inc}} \in H_0^{r+3}(0,T;\mathrm{V}_\Gamma{}')$ for some arbitrary $r \ge 0$.*

(a) This problem has a unique solution

$$(E,H) \in H_0^r(0,T;H(\mathrm{curl},\Omega)^2) \cap H_0^{r+1}(0,T;(L^2(\Omega)^3)^2),$$

which is given by the representation formulas (4.5)–(4.6)*. The tangential traces are uniquely determined by the solution of the system of boundary integral equations of Proposition 4.1,*

$$(\varphi,\psi) = (\mu\gamma_T H, -\gamma_T E) \in H_0^{r+1}(0,T;\mathrm{V}_\Gamma \times \mathrm{X}_\Gamma).$$

(b) The electromagnetic fields are bounded by

$$\|E\|_{H_0^r(0,T;H(\mathrm{curl},\Omega))} + \|\mu H\|_{H_0^r(0,T;H(\mathrm{curl},\Omega))} \le C_T \|g^{\mathrm{inc}}\|_{H_0^{r+3}(0,T;\mathrm{V}_\Gamma{}')},$$

and the same bound is valid for the $H_0^{r+1}(0,T;(L^2(\Omega)^3)^2)$ norms. Here, C_T depends on T (polynomially if $\sigma_0 = 0$ in (2.9)–(2.10)*) and on the boundary Γ via norms of tangential trace operators, but is independent of ε and μ with $\varepsilon\mu = 1$ and, in the case of the impedance operators* (1.3)–(1.6)*, independent of the small parameter δ.*

Proof We extend $g^{\mathrm{inc}} \in H_0^r(0,T;\mathrm{V}_\Gamma{}')$ from the interval $(0,T)$ to a function in $H^r(\mathbb{R};\mathrm{V}_\Gamma{}')$ on the whole real line, with support in $[0,2T]$. The fields (E,H) defined by the time-dependent boundary integral equation (4.2) and the time-dependent representation formulas (4.5)–(4.6) have the regularity as stated because of (2.13) used for the time-harmonic solution operator with the bounds given in Theorem 3.1, and they satisfy the stated bounds on every finite interval $(0,\bar{T})$, with at most exponential growth in $\bar{T}$ of the norm with an arbitrary exponent $\sigma_1 > \sigma_0$. The Laplace transform $(\widehat{E}(s),\widehat{H}(s))$ then exists for $\mathrm{Re}\, s > \sigma_0$, and it is obtained by the solution of the time-harmonic boundary integral equation (3.28) and the time-harmonic representation formulas (3.15)–(3.16). By Theorem 3.1, $(\widehat{E}(s),\widehat{H}(s))$ is the solution to the time-harmonic scattering problem with the time-harmonic generalized impedance boundary conditions. Taking the inverse Laplace transform then shows that (E,H) solve the time-dependent scattering problem (2.1)–(2.2) under the generalized impedance boundary condition (2.17). Finally, the uniqueness of the time-dependent solution (E,H) follows from the uniqueness of the tangential traces and the well-posedness of the time-dependent exterior Maxwell problem with a given tangential trace. □

5 Semi-discretization in time by Runge–Kutta convolution quadrature

5.1 Recap: Runge–Kutta convolution quadrature

Runge–Kutta convolution quadratures will be used here to approximate temporal convolutions $K(\partial_t)g$; cf. (2.11). Let us first recall an m-stage implicit Runge–Kutta discretization of the initial value problem $y' = f(t, y)$, $y(0) = y_0$; see [25]. For a time step $\tau > 0$, the approximations y^n to $y(t_n)$ at time $t_n = n\tau$, and the internal stages Y^{ni} approximating $y(t_n + c_i\tau)$, are obtained from

$$\begin{aligned} Y^{ni} &= y^n + \tau \sum_{j=1}^{m} a_{ij} f(t_n + c_j h, Y^{nj}), \qquad i = 1, \dots, m, \\ y^{n+1} &= y^n + \tau \sum_{j=1}^{m} b_j f(t_n + c_j h, Y^{nj}). \end{aligned}$$

The method is given by its coefficients

$$\mathscr{A} = (a_{ij})_{i,j=1}^m, \quad b = (b_1, \dots, b_m)^T, \quad \text{and} \quad c = (c_1, \dots, c_m)^T.$$

The stability function of the Runge–Kutta method is given by $R(z) = 1 + zb^T(I - z\mathscr{A})^{-1}\mathbb{1}$, where $\mathbb{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^m$. We always assume that $\mathscr{A}$ is invertible.

Runge–Kutta methods can be used to construct convolution quadrature methods. Such methods were first introduced in [30] in the context of parabolic problems and were studied for wave propagation problems in [8] and subsequently, e.g., in [6, 7, 11, 12]. Runge–Kutta convolution quadrature was studied for the numerical solution of some exterior Maxwell problems in [5, 20] and of an eddy current problem with an impedance boundary condition in [26]. For wave problems, Runge–Kutta convolution quadrature methods such as those based on the Radau IIA methods (see [25, Section IV.5]), often enjoy more favourable properties than their BDF-based counterparts, which are more dissipative and cannot exceed order 2 but are easier to understand and slightly easier to implement.

Let $K(s) : X \to Y$, $\operatorname{Re} s \geq \sigma_0 > 0$, be an analytic family of linear operators between Banach spaces X and Y, satisfying the bound, for some exponents $\kappa \in \mathbb{R}$ and $\nu \geq 0$,

$$\|K(s)\|_{Y \leftarrow X} \leq M_\sigma \frac{|s|^\kappa}{(\operatorname{Re} s)^\nu}, \qquad \operatorname{Re} s \geq \sigma > \sigma_0. \tag{5.1}$$

As in Section 2.2, this yields a convolution operator $K(\partial_t) : H_0^{r+\kappa}(0, T; X) \to H_0^r(0, T; X)$ for arbitrary real r. For functions $g : [0, T] \to X$ that are sufficiently regular (together with their extension by 0 to the negative real half-axis $t < 0$), we wish to approximate the convolution $(K(\partial_t)g)(t)$ at discrete times $t_n = n\tau$ with a stepsize $\tau > 0$, using a discrete convolution.

To construct the convolution quadrature weights, we use the *Runge–Kutta differentiation symbol*

$$\Delta(\zeta) = \Big(\mathscr{A} + \frac{\zeta}{1-\zeta}\mathbb{1}b^T\Big)^{-1} \in \mathbb{C}^{m\times m}, \qquad \zeta \in \mathbb{C} \text{ with } |\zeta| < 1. \tag{5.2}$$

This is well-defined for $|\zeta| < 1$ if $R(\infty) = 1 - b^T\mathscr{A}^{-1}\mathbb{1}$ satisfies $|R(\infty)| \le 1$, as is seen from the Sherman–Woodbury formula. Moreover, for A-stable Runge–Kutta methods (e.g. the Radau IIA methods), the eigenvalues of the matrices $\Delta(\zeta)$ have positive real part for $|\zeta| < 1$ [8, Lemma 3].

To formulate the Runge–Kutta convolution quadrature for $K(\partial_t)g$, we replace the complex argument s in $K(s)$ by the matrix $\Delta(\zeta)/\tau$ and expand

$$K\Big(\frac{\Delta(\zeta)}{\tau}\Big) = \sum_{n=0}^{\infty} W_n(K)\zeta^n. \tag{5.3}$$

The operators $W_n(K) : X^m \to Y^m$ are used as the convolution quadrature "weights". For the discrete convolution of these operators with a sequence $g = (g^n)$ with $g^n = (g_i^n)_{i=1}^m \in X^m$ we use the notation

$$\big(K(\partial_t^\tau)g\big)^n = \sum_{j=0}^{n} W_{n-j}(K)g^j \in Y^m. \tag{5.4}$$

Given a function $g : [0,T] \to X$, we use this notation for the vectors $g^n = \big(g(t_n + c_i\tau)\big)_{i=1}^m$ of values of g. The i-th component of the vector $\big(K(\partial_t^\tau)g\big)^n$ is then an approximation to $\big(K(\partial_t)g\big)(t_n + c_i\tau)$; see [7, Theorem 4.2].

In particular, if $c_m = 1$, as is the case with Radau IIA methods, the continuous convolution at t_n is approximated by the m-th, i.e. last component of the m-vector (5.4) for $n-1$:

$$\big(K(\partial_t)g\big)(t_n) \approx \big(K(\partial_t^\tau)g\big)_m^{n-1}.$$

An essential property is that the composition rule (2.12) is preserved under this discretization: for two such operator families $K(s)$ and $L(s)$ that map to compatible spaces, we have

$$K(\partial_t^\tau)L(\partial_t^\tau)g = (KL)(\partial_t^\tau)g. \tag{5.5}$$

The following error bound for Runge–Kutta convolution quadrature from [8], here directly stated for the Radau IIA methods [25, Section IV.5] and transferred to a Banach space setting, will be the basis for our error bounds of the time discretization.

Lemma 5.1 ([8, Theorem 3]) *Let $K(s) : X \to Y$, $\mathrm{Re}\, s > \sigma_0 \ge 0$, be an analytic family of linear operators between Banach spaces X and Y satisfying the bound* (5.1) *with exponents κ and ν. Consider the Runge–Kutta convolution quadrature based on the Radau IIA method with m stages. Let $1 \le q \le m$ (the most interesting case is $q = m$) and $r > \max(2q - 1 + \kappa, 2q - 1, q + 1)$.*

Let $g \in C^r([0,T],X)$ satisfy $g(0) = g'(0) = ... = g^{(r-1)}(0) = 0$. Then, the following error bound holds at $t_n = n\tau \in [0,T]$:

$$\begin{aligned}\|(K(\partial_t^\tau)g)_m^{n-1} - (K(\partial_t)g)(t_n)\|_Y \\ \leq C\, M_{1/T}\, \tau^{\min(2q-1,q+1-\kappa+\nu)} \left(\|g^{(r)}(0)\|_X + \int_0^t \|g^{(r+1)}(t')\|_X \,\mathrm{d}t'\right).\end{aligned}$$

The constant C is independent of τ and g and M_σ of (5.1)*, but depends on the exponents κ and ν in* (5.1) *and on the final time T.*

5.2 Convolution quadrature for the scattering problem

Using a Runge–Kutta based convolution quadrature for the semi-discretization in time of the time-dependent boundary integral equation (4.2) yields the discrete convolution equation

$$A(\partial_t^\tau)\begin{pmatrix}\varphi^\tau\\ \psi^\tau\end{pmatrix} = \begin{pmatrix} g^{\mathrm{inc}} \\ 0\end{pmatrix}. \tag{5.6}$$

By the discrete composition rule (5.5), the solution to this equation is given by the convolution quadrature semi-discretization of the convolution (4.3),

$$\begin{pmatrix}\varphi^\tau\\ \psi^\tau\end{pmatrix} = A^{-1}(\partial_t^\tau)\begin{pmatrix} g^{\mathrm{inc}} \\ 0\end{pmatrix}.$$

While this formula is of no computational use, as the inverse boundary integral operator $A(s)^{-1}$ is not computationally available, it is extremely helpful for the convergence analysis, since it interprets the solution of the discretized boundary integral equation as a mere convolution quadrature, to which we can apply the error bound of Lemma 5.1 using the bound (3.30) of $A(s)^{-1}$. (Such an argument was first used in [29] for a time-dependent boundary integral equation in the acoustic case.) In particular, no stability issues arise for this time discretization.

The time discretizations of the electromagnetic fields are then obtained by applying the convolution quadrature to the representation formulas (4.5)–(4.6):

$$E^\tau = -\mathcal{S}(\partial_t^\tau)\varphi^\tau + \mathcal{D}(\partial_t^\tau)\psi^\tau, \tag{5.7}$$

$$\mu H^\tau = -\mathcal{D}(\partial_t^\tau)\varphi^\tau - \mathcal{S}(\partial_t^\tau)\psi^\tau. \tag{5.8}$$

Again by the composition rule, this is the convolution quadrature discretization

$$\begin{pmatrix}E^\tau\\ \mu H^\tau\end{pmatrix} = U(\partial_t^\tau)g^{\mathrm{inc}} \qquad \text{of} \qquad \begin{pmatrix}E\\ \mu H\end{pmatrix} = U(\partial_t)g^{\mathrm{inc}}, \tag{5.9}$$

where we have by Theorem 4.1 that

$$U(s) = \begin{pmatrix}-\mathcal{S}(s) & \mathcal{D}(s)\\ -\mathcal{D}(s) & -\mathcal{S}(s)\end{pmatrix} A(s)^{-1}\begin{pmatrix}\mathrm{Id}\\ 0\end{pmatrix} : \mathrm{V}_\Gamma{}' \to H(\mathrm{curl},\Omega)^2,$$

for which the bound

$$\|U(s)\|_{H(\mathrm{curl},\Omega)^2 \leftarrow \mathrm{V}_\Gamma{}'} \leq C_\sigma \frac{|s|^3}{(\operatorname{Re} s)^{3/2}}, \quad \text{for} \quad \operatorname{Re} s \geq \sigma > \sigma_0 \geq 0,$$

is given in Theorem 3.1. Moreover, away from the boundary we obtain by concatenating Lemmas 3.8–3.10 and Proposition 3.1 that on $\Omega_d = \{x \in \Omega : \operatorname{dist}(x, \Gamma) > d\}$ with $d > 0$, we have for $\operatorname{Re} s \geq \sigma > \sigma_0 \geq 0$ that

$$\|U(s)\|_{(C^1(\overline{\Omega}_d)^3)^2 \leftarrow \mathrm{V}_\Gamma{}'} + \|U(s)\|_{H(\mathrm{curl},\Omega_d)^2 \leftarrow \mathrm{V}_\Gamma{}'} \leq C_\sigma |s|^5 e^{-d \operatorname{Re} s},$$

where the $C^1(\overline{\Omega}_d)$-norm is the maximum norm on continuously differentiable functions and their derivatives on the closure of Ω_d.

Using $U(s)$ in the role of $K(s)$ and these bounds as (5.1) in Lemma 5.1 then directly yields the following result.

Proposition 5.1 (Error bound of the semi-discretization in time) *In the situation of Theorem 4.1, consider the Runge–Kutta convolution quadrature based on the Radau IIA method with m stages used for the semi-discretization in time* (5.6) *and* (5.7)–(5.8) *of the boundary integral equation* (4.2) *and the representation formulas* (4.5)–(4.6)*, respectively. For $r > 2m + 3$, assume that $g^{\mathrm{inc}} \in C^r([0,T], \mathrm{V}_\Gamma{}')$, vanishing at $t = 0$ together with its first $r - 1$ time derivatives. Then, the approximations to the electromagnetic fields $E^n = (E^\tau)_m^{n-1}$ and $H^n = (H^\tau)_m^{n-1}$ satisfy the following error bound of order $m - 1/2$ at $t_n = n\tau \in [0,T]$:*

$$\left\| \begin{pmatrix} E^n - E(t_n) \\ H^n - H(t_n) \end{pmatrix} \right\|_{H(\mathrm{curl},\Omega)^2} \leq C\, \tau^{m-1/2}\, M(g^{\mathrm{inc}}, t_n).$$

On $\Omega_d = \{x \in \Omega : \operatorname{dist}(x, \Gamma) > d\}$ with $d > 0$, there is the full order $2m - 1$:

$$\left\| \begin{pmatrix} E^n - E(t_n) \\ H^n - H(t_n) \end{pmatrix} \right\|_{\left(H(\mathrm{curl},\Omega_d) \cap C^1(\overline{\Omega}_d)^3\right)^2} \leq C_d\, \tau^{2m-1}\, M(g^{\mathrm{inc}}, t_n).$$

Here, $M(g,t) = \|g^{(r)}(0)\|_{\mathrm{V}_\Gamma{}'} + \int_0^t \|g^{(r+1)}(t')\|_{\mathrm{V}_\Gamma{}'}\, \mathrm{d}t'$. The constants C and C_d are independent of n, τ and g, but depend on the final time T. C_d additionally depends on the distance d. In the case of the impedance operators (1.3)–(1.6)*, both C and C_d are independent of the small parameter δ.*

We remark that for acoustic scattering from a sound-soft obstacle, full-order convergence away from the boundary for the Runge-Kutta convolution quadrature time discretization was previously proved in [8]. Proposition 5.1 shows that this favourable error behaviour extends to the electromagnetic scattering from generalized impedance boundary conditions.

6 Full discretization

We use a Galerkin approximation of the boundary integral equation (5.6) with boundary element spaces $V_h \subset \mathrm{V}_\Gamma$ and $X_h \subset \mathrm{X}_\Gamma$ corresponding to a family of triangulations with mesh width h. We choose both V_h and X_h to be the Raviart–Thomas boundary element space of order $k \geq 0$ [35], which is defined on the unit triangle $\widehat{K}$ as reference element by

$$\mathrm{RT}_k(\widehat{K}) = \left\{ x \mapsto p_1(x) + p_2(x)x \ : \ p_1 \in P_k(\widehat{K})^2, \ p_2 \in P_k(\widehat{K}) \right\},$$

where $P_k(\widehat{K})$ is the polynomial space of degree k on $\widehat{K}$. Raviart–Thomas elements on an arbitrary grid are then obtained in the standard way by piecewise pull-back to the reference element.

We will use the following approximation results, which are obtained from the results collected in Lemma 15 and Theorem 14 of [17]; see also the original references [14, Section III.3.3] and [15]. Here we use the same notation $H^p_\times(\Gamma) = \gamma_T H^{p+1/2}(\Omega)^3$ as in [17].

Lemma 6.1 *Let $X_h = V_h$ be the k-th order Raviart–Thomas boundary element space on Γ. For every $\xi \in \mathrm{X}_\Gamma \cap H^{k+1}_\times(\Gamma)$ and $\upsilon \in \mathrm{V}_\Gamma \cap H^{k+1}_\times(\Gamma)$, with the space V_Γ of Lemma 2.1 or Lemma 2.2, the best-approximation error is bounded by*

$$\inf_{\xi_h \in X_h} \|\xi_h - \xi\|_{\mathrm{X}_\Gamma} \leq C h^{k+3/2} \|\xi\|_{H^{k+1}_\times(\Gamma)},$$
$$\inf_{\upsilon_h \in V_h} \|\upsilon_h - \upsilon\|_{\mathrm{V}_\Gamma} \leq C h^{k+1} \|\upsilon\|_{H^{k+1}_\times(\Gamma)}.$$

Remark 6.1 We would have expected that the best-approximation error bound in the V_Γ-norm is $O(h^{k+3/2}+\delta^{1/2}h^{k+1})$, in analogy to the situation for acoustic generalized impedance boundary conditions [9]. This would, however, require proving the V_Γ-norm stability of the projection of [15] from X_Γ to X_h that was used to show the best-approximation estimate in X_Γ. If at all possible, this is in any case beyond the scope of this paper.

The Galerkin approximation of the time-discretized boundary integral equation (5.6) on $V_h \times X_h$ then reads

$$\left\langle \begin{pmatrix} \upsilon_h \\ \xi_h \end{pmatrix}, A(\partial_t^\tau) \begin{pmatrix} \varphi_h^\tau \\ \psi_h^\tau \end{pmatrix} \right\rangle = \langle \upsilon_h, g^{\mathrm{inc}} \rangle \qquad \forall\, (\upsilon_h, \xi_h) \in (V_h \times X_h)^m. \tag{6.1}$$

This determines the approximate boundary densities $\varphi_h^\tau = \big((\varphi_h^\tau)^n\big)$ with $(\varphi_h^\tau)^n = \big((\varphi_h^\tau)_i^n\big)_{i=1}^m \in V_h^m$ and $\psi_h^\tau = \big((\psi_h^\tau)^n\big)$ with $(\psi_h^\tau)^n = \big((\psi_h^\tau)_i^n\big)_{i=1}^m \in X_h^m$, which are used to define the approximations to the electromagnetic fields via the time-discrete representation formulas

$$E_h^\tau = -\mathcal{S}(\partial_t^\tau)\varphi_h^\tau + \mathcal{D}(\partial_t^\tau)\psi_h^\tau, \tag{6.2}$$
$$\mu H_h^\tau = -\mathcal{D}(\partial_t^\tau)\varphi_h^\tau - \mathcal{S}(\partial_t^\tau)\psi_h^\tau. \tag{6.3}$$

We then have the following error bounds for the full discretization, obtained under regularity assumptions that are presumably stronger than necessary.

Theorem 6.1 (Error bound of the full discretization) *In the situation of Theorem 4.1, consider*
— Runge–Kutta convolution quadrature based on the Radau IIA method with $m \geq 2$ *stages used for the time discretization* (5.6) *and* (5.7)–(5.8) *of the boundary integral equation* (4.2) *and the representation formulas* (4.5)–(4.6)*, respectively; and*
— Raviart–Thomas boundary elements of order k *for the space discretization of the boundary integral equation* (4.2).
For $r > 2m+3$*, let* $g^{\mathrm{inc}} \in C^r([0,T], \mathrm{V}_\Gamma{}')$ *vanish at* $t = 0$ *together with its first* $r-1$ *time derivatives. Furthermore, it is assumed that the solution* (φ, ψ) *of the boundary integral equation* (4.2) *is in* $C^{10}([0,T], H_\times^{k+1}(\Gamma)^2)$*, vanishing at* $t = 0$ *together with its time derivatives.*
Then, the approximations to the electromagnetic fields $E_h^n = (E_h^\tau)_m^{n-1}$ *and* $H_h^n = (H_h^\tau)_m^{n-1}$ *satisfy the following error bound of order* $m - 1/2$ *in time and order* $k+1$ *in space at* $t_n = n\tau \in [0,T]$*:*

$$\left\| \begin{pmatrix} E_h^n - E(t_n) \\ H_h^n - H(t_n) \end{pmatrix} \right\|_{H(\mathrm{curl},\Omega)^2} \leq C\big(\tau^{m-1/2} + h^{k+1}\big).$$

On $\Omega_d = \{x \in \Omega : \mathrm{dist}(x, \Gamma) > d\}$ *with* $d > 0$*, there is the full order* $2m-1$ *in time:*

$$\left\| \begin{pmatrix} E_h^n - E(t_n) \\ H_h^n - H(t_n) \end{pmatrix} \right\|_{\big(H(\mathrm{curl},\Omega_d)\cap C^1(\overline{\Omega}_d)^3\big)^2} \leq C_d\big(\tau^{2m-1} + h^{k+1}\big).$$

The constants C *and* C_d *are independent of* n*,* τ *and* h*, but depend on the final time* T *and on the regularity of* g^{inc} *and* (φ, ψ) *as stated.* C_d *additionally depends on the distance* d*. In the case of the impedance operators* (1.3)–(1.6)*, both* C *and* C_d *are independent of the small parameter* δ*.*

Proof We structure the proof into three parts (a)–(c).

(a) *(Discretized time-harmonic boundary integral equation).* We first consider the time-harmonic boundary integral equation (3.27), for $\mathrm{Re}\, s \geq \sigma > \sigma_0 \geq 0$. We denote by $L_h(s) : \mathrm{V}_\Gamma{}' \to V_h \times X_h$ the solution operator $\widehat{g} \mapsto (\widehat{\varphi}_h, \widehat{\psi}_h)$ of the Galerkin approximation in $V_h \times X_h$,

$$\left\langle \begin{pmatrix} \upsilon_h \\ \xi_h \end{pmatrix}, A(s) \begin{pmatrix} \widehat{\varphi}_h \\ \widehat{\psi}_h \end{pmatrix} \right\rangle = \langle \upsilon_h, \widehat{g} \rangle \qquad \forall\, (\upsilon_h, \xi_h) \in V_h \times X_h, \tag{6.4}$$

which by the bound of $A(s)$ in Lemma 3.6, the coercivity estimate of Lemma 3.7 and the Lax–Milgram lemma is bounded by

$$\|L_h(s)\|_{V_h \times X_h \leftarrow \mathrm{V}_\Gamma{}'} \leq \frac{1}{c_\sigma} \frac{|s|^2}{\mathrm{Re}\, s}. \tag{6.5}$$

Next we consider the associated Ritz projection $R_h(s) : \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma \to V_h \times X_h$, which maps $(\widehat{\varphi}, \widehat{\psi}) \in \mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$ to $(\widehat{\varphi}_h, \widehat{\psi}_h) \in V_h \times X_h$ determined by

$$\left\langle \begin{pmatrix} \upsilon_h \\ \xi_h \end{pmatrix}, A(s) \begin{pmatrix} \widehat{\varphi}_h \\ \widehat{\psi}_h \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \upsilon_h \\ \xi_h \end{pmatrix}, A(s) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right\rangle \qquad \forall\, (\upsilon_h, \xi_h) \in V_h \times X_h.$$

Again by Lemmas 3.6 and 3.7 and the Lax–Milgram lemma, this problem has a unique solution $(\widehat{\varphi}_h, \widehat{\psi}_h) \in V_h \times X_h$, and by Céa's lemma,

$$\left\| \begin{pmatrix} \widehat{\varphi}_h \\ \widehat{\psi}_h \end{pmatrix} - \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right\|_{\mathrm{V}_\Gamma \times \mathrm{X}_\Gamma} \le \frac{C_\sigma}{c_\sigma} \left(\frac{|s|^2}{\operatorname{Re} s} \right)^2 \inf_{(\upsilon_h, \xi_h) \in V_h \times X_h} \left\| \begin{pmatrix} \upsilon_h \\ \xi_h \end{pmatrix} - \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right\|_{\mathrm{V}_\Gamma \times \mathrm{X}_\Gamma},$$

where the right-hand side is further bounded by Lemma 6.1. We can thus view the associated error operator $\mathcal{E}_h(s) = R_h(s) - \mathrm{Id}$ as a bounded operator from $H_\times^{k+1}(\Gamma)^2$ to $\mathrm{V}_\Gamma \times \mathrm{X}_\Gamma$ with the bound, for $\operatorname{Re} s \ge \sigma > \sigma_0 \ge 0$,

$$\|\mathcal{E}_h(s)\|_{\mathrm{V}_\Gamma \times \mathrm{X}_\Gamma \leftarrow H_\times^{k+1}(\Gamma)^2} \le \widetilde{C}_\sigma \frac{|s|^4}{(\operatorname{Re} s)^2}\, h^{k+1}. \tag{6.6}$$

(b) *(Error of the spatial semi-discretization).* The spatial semi-discretization of the time-dependent boundary integral equation (4.2),

$$\left\langle \begin{pmatrix} \upsilon_h \\ \xi_h \end{pmatrix}, A(\partial_t) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} \right\rangle = \langle \upsilon_h, g^{\mathrm{inc}} \rangle \qquad \forall\, (\upsilon_h, \xi_h) \in (V_h \times X_h)^m, \tag{6.7}$$

then has the unique solution

$$\begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} = L_h(\partial_t) g^{\mathrm{inc}} = R_h(\partial_t) \begin{pmatrix} \varphi \\ \psi \end{pmatrix},$$

where $(\varphi, \psi)^\top = A^{-1}(\partial_t)(g^{\mathrm{inc}}, 0)^\top$ is the solution of (4.2). We abbreviate

$$W(s) = \begin{pmatrix} -\mathcal{S}(s) & \mathcal{D}(s) \\ -\mathcal{D}(s) & -\mathcal{S}(s), \end{pmatrix}$$

and set

$$U_h(s) = W(s) L_h(s) : {\mathrm{V}_\Gamma}' \to H(\mathrm{curl}, \Omega)^2. \tag{6.8}$$

By (6.5) and Lemma 3.1, this is bounded by

$$\|U_h(s)\|_{H(\mathrm{curl},\Omega)^2 \leftarrow {\mathrm{V}_\Gamma}'} \le \bar{C}_\sigma \frac{|s|^4}{(\operatorname{Re} s)^2}. \tag{6.9}$$

The spatial semi-discretization of the scattering problem is then obtained as

$$\begin{pmatrix} E_h \\ \mu H_h \end{pmatrix} = U_h(\partial_t) g^{\mathrm{inc}}.$$

In view of (5.9), its error is

$$\begin{aligned} \begin{pmatrix} E_h \\ \mu H_h \end{pmatrix} - \begin{pmatrix} E \\ \mu H \end{pmatrix} &= U_h(\partial_t) g^{\mathrm{inc}} - U(\partial_t) g^{\mathrm{inc}} = W(\partial_t) \begin{pmatrix} \varphi_h \\ \psi_h \end{pmatrix} - W(\partial_t) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \\ &= W(\partial_t)(R_h - \mathrm{Id}) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} = W(\partial_t)\, \mathcal{E}_h(\partial_t) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix}. \end{aligned}$$

Using the bound of Lemma 3.1 for the potential operator $W(s)$, the bound (6.6) for the error operator $\mathcal{E}_h(s)$, and the bound (2.13) (with $\kappa = 6$) for their composition, and finally the Sobolev embedding $H^1(0,T;H) \subset C([0,T],H)$ for any Hilbert space H, we obtain for the error of the spatial semi-discretization

$$\max_{0\le t\le T} \left\| \begin{pmatrix} E_h(t) \\ \mu H_h(t) \end{pmatrix} - \begin{pmatrix} E(t) \\ \mu H(t) \end{pmatrix} \right\|_{H(\mathrm{curl},\Omega)^2} \tag{6.10}$$
$$\le C \left\| \begin{pmatrix} E_h \\ \mu H_h \end{pmatrix} - \begin{pmatrix} E \\ \mu H \end{pmatrix} \right\|_{H_0^1(0,T;H(\mathrm{curl},\Omega)^2)} \le C_T\, h^{k+1} \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|_{H_0^7(0,T;H_\times^{k+1}(\Gamma)^2)}.$$

Using the same argument with the pointwise bounds away from the boundary given by Lemmas 3.8 and 3.9, we further obtain

$$\max_{0\le t\le T} \left\| \begin{pmatrix} E_h(t) \\ \mu H_h(t) \end{pmatrix} - \begin{pmatrix} E(t) \\ \mu H(t) \end{pmatrix} \right\|_{C^1(\overline{\Omega}_d)^2} \le C_T\, h^{k+1} \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|_{H_0^8(0,T;H_\times^{k+1}(\Gamma)^2)}. \tag{6.11}$$

(c) *(Error of the full discretization).* The total error is

$$\begin{pmatrix} E_h^n \\ \mu H_h^n \end{pmatrix} - \begin{pmatrix} E^n \\ \mu H^n \end{pmatrix} \quad + \quad \begin{pmatrix} E^n \\ \mu H^n \end{pmatrix} - \begin{pmatrix} E(t_n) \\ \mu H(t_n) \end{pmatrix}.$$

The second difference is the error of the temporal semi-discretization, which is bounded by $O(\tau^{m-1/2})$ in the $H(\mathrm{curl},\Omega)^2$ norm in Proposition 5.1. The first difference is written as (omitting here the superscript $n-1$ and subscript m)

$$W(\partial_t^\tau)\mathcal{E}_h(\partial_t^\tau) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} = \left(W(\partial_t^\tau)\mathcal{E}_h(\partial_t^\tau) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} - W(\partial_t)\mathcal{E}_h(\partial_t) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix} \right)$$
$$+ \; W(\partial_t)\mathcal{E}_h(\partial_t) \begin{pmatrix} \widehat{\varphi} \\ \widehat{\psi} \end{pmatrix}.$$

The last term is the error of the spatial semi-discretization studied in part (b), which is bounded by (6.10). The difference written in brackets on the right-hand side is a convolution quadrature error, which can be bounded by Lemma 5.1. This gives an $O(h^{k+1})$ error in the $H(\mathrm{curl},\Omega)^2$ norm, using that by Lemma 3.1 and (6.6) we have here $M_\sigma \le C_\sigma h^{k+1}$, $\kappa = 6$, $\nu = 3$ in (5.1) with $W(s)\mathcal{E}_h(s)$ in the role of $K(s)$, and choosing $q = 2$ and $r = 10 > 2q - 1 + \kappa$. Note that here $\min(2q-1, q+1-\kappa+\nu) = q-2 = 0$. Altogether, this yields the stated $O(\tau^{m-1/2} + h^{k+1})$ error bound in the $H(\mathrm{curl},\Omega)^2$ norm.

To prove the full-order error bound away from the boundary, we rewrite the error as

$$\begin{pmatrix} E_h^n \\ \mu H_h^n \end{pmatrix} - \begin{pmatrix} E_h(t_n) \\ \mu H_h(t_n) \end{pmatrix} \quad + \quad \begin{pmatrix} E_h(t_n) \\ \mu H_h(t_n) \end{pmatrix} - \begin{pmatrix} E(t_n) \\ \mu H(t_n) \end{pmatrix}.$$

The second difference is the error of the spatial semi-discretization studied in part (b). The first difference is a convolution quadrature error for the transfer operator $U_h(s)$ of (6.8):

$$\begin{pmatrix} E_h^n \\ \mu H_h^n \end{pmatrix} - \begin{pmatrix} E_h(t_n) \\ \mu H_h(t_n) \end{pmatrix} = \big(U_h(\partial_t^\tau) g^{\mathrm{inc}} \big)_m^{n-1} - U_h(\partial_t) g^{\mathrm{inc}}(t_n).$$

(Estimating this error in the $H(\mathrm{curl},\Omega)^2$ norm by Lemma 5.1 would only give an $O(\tau^{m-1})$ bound instead of the stated $O(\tau^{m-1/2})$ bound, which is why we chose a different path before.)

The full-order error bound away from the boundary in the $H(\mathrm{curl},\Omega_d)$ norm and the $C^1(\overline{\Omega}_d)$ norm then follows from Lemma 5.1, using the bounds of Lemmas 3.8–3.10 that decay exponentially with $d\,\mathrm{Re}\,s$, concatenated with the bound (6.5). This completes the proof of the error bounds. □

7 Implementation and Numerical Experiments

We start this final section with a few words on the implementation and then present the results of numerical experiments. The codes which were used to generate the figures in this section are distributed via [33].

7.1 Implementation

The convolution quadrature weights are approximated by discretizing their Cauchy-integral representation with the trapezoidal rule, as already described in [28]. This gives the approximation to the weights

$$W_n(K) \approx \frac{\rho^{-n}}{L}\sum_{l=0}^{L-1} K\left(\frac{\Delta(\rho\,\zeta_L^{-l})}{\tau}\right)\zeta_L^{nl}, \quad \text{for } 0 \le n \le N, \tag{7.1}$$

where $\zeta_L = e^{2\pi i/L}$. The parameters are chosen such that $L = N+1$ and $\rho^N = \sqrt{\epsilon}$, where ϵ denotes the machine precision.

To evaluate the analytic operator family $K(\Delta(\zeta)/\tau)$, for the matrix valued characteristic function $\Delta(\zeta) \in \mathbb{C}^{m\times m}$ at a point $\zeta \in \mathbb{C}$ inside of the unit circle, it is convenient to diagonalize the characteristic function by

$$T^{-1}K\left(\Delta(\zeta)\right)T = K\left(T^{-1}\Delta(\zeta)T\right), \quad \text{for invertible} \quad T \in \mathbb{C}^{m\times m},$$

which reduces the evaluation $K(\Delta(\zeta)/\tau)$ to evaluating $K(\cdot)$ at the eigenvalues of $\Delta(\zeta)$. Plugging the approximations to the quadrature weights into (5.4) then gives the scheme

$$(K(\partial_t^\tau)g)^n \approx \frac{\rho^{-n}}{L}\sum_{l=0}^{L-1}\zeta_L^{ln}K\left(\frac{\Delta(\rho\,\zeta_L^{-l})}{\tau}\right)\left[\sum_{j=0}^{N}\rho^j g^j \zeta_L^{-jl}\right].$$

The sums above are realized effectively by the application of FFTs, which leaves the main computational obstacle at the evaluations of the Laplace domain operators $K(\cdot)$ at mL scalar frequencies $s_k \in \mathbb{C}$ for $k = 1,\dots,mL$ (i.e. the collection of eigenvalues of $K(\Delta(\rho\,\zeta_L^{-l})/\tau))$ with positive real part. Setting either $K(s) = A_h(s)^{-1}$ or $K(s) = U_h(s)$ then gives schemes to approximate the boundary densities (φ,ψ) or the electromagnetic fields E, H, respectively.

We note that due to symmetric properties of the time-harmonic operators, only half of the Laplace domain evaluations have to be computed [13].

Our numerical experiments were conducted in Python, where the appearing potential and boundary operators were discretized with the library Bempp [37]. As space discretization we choose Raviart–Thomas elements of order 0 and the arising linear systems were iteratively solved with GMRES. The anti-symmetric pairing appearing in the weak formulation (3.26) was realized by choosing corresponding Nédélec boundary elements as the test space.

7.2 Numerical Experiments

We present two types of numerical experiments.

- Convergence experiments, where the errors between the numerical solution and a reference solution are presented, for various mesh sizes and time step sizes, and for different values of δ.
- We present the computed numerical solution of a three-dimensional scattering problem with a torus as the obstacle.

We test the proposed numerical method with an incidental electric planar wave that solves Maxwell's equations on $\mathbb{R}^3$, which we set to be

$$E^{\mathrm{inc}}(t,x) = e^{-50(t-x_3-t_0)^2} e_1, \tag{7.2}$$

where $e_1 = (1,0,0)^T$ and $t_0 = -2$. This incidental wave is scattered from a unit sphere centered around the origin, where we applied the generalized impedance boundary condition corresponding to $Z(\partial_t) = \delta \partial_t^{1/2}$, with $\delta = 10^{-1}, 10$. The reference solution is computed using a 0-th order Raviart–Thomas boundary element space discretization with 47242 degrees of freedoms and the 3-stage Radau IIA time discretization of order 5 with $N = 2^{10}$ time steps.

In Figure 7.1 and 7.2 we report on a numerical experiment illustrating the error estimate of Theorem 6.1. We plot the error of the point evaluation at $P = (2,0,0)$ between numerical approximation $E_h^\tau(P,t_n)$ and the reference solution $E_{ref}(P,t_n)$.

The logarithmic plots in Figure 7.1 show the errors against the time step size τ, the lines marked with different symbols correspond to different mesh widths h given in the plot. Figure 7.2 contains the same plots for $\delta = 10^{-1}$ (left) and $\delta = 10$ (right), but reversing the roles of τ and h.

In Figure 7.1 we can observe a region where the temporal discretization error dominates, and a region where the spatial discretization error dominates (the curves are flattening out). In the region with small spatial error, we can observe that the error curves match the order of convergence of our theoretical results (note the reference lines), of full classical order $O(\tau^{2m-1})$.

Similarly, for Figure 7.2 an analogous description applies but with reversed roles. Although the error estimates of Theorem 6.1 are δ-independent, in view of Remark 6.1, we expect a δ-explicit error bound $O(h^{k+3/2} + \delta^{1/2} h^{k+1})$. Figure 7.2 reports on the spatial convergence rates with $k = 0$. On the left-hand side we can observe that since δ is small enough the first spatial term dominates in the above error estimate, matching the spatial order $O(h^{3/2})$. On the right-hand side, with a large enough δ the second term is dominating, matching the spatial order $O(h)$.

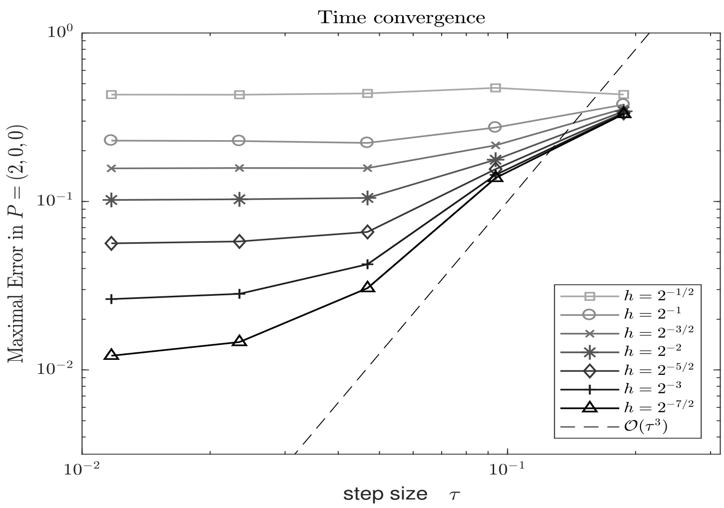

Fig. 7.1 Convergence plot in time for the fully discrete problem, with $\delta = 0.1$

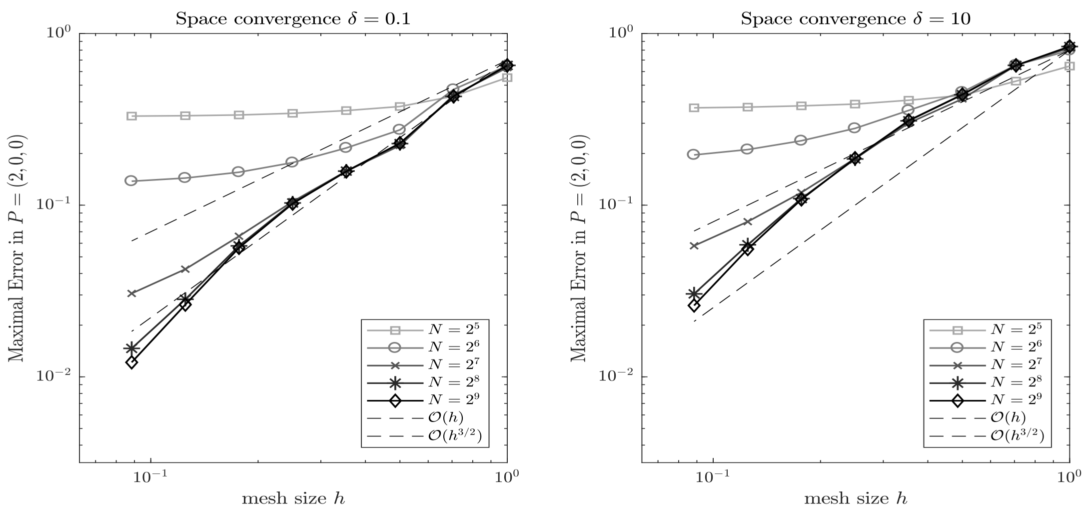

Fig. 7.2 Space convergence plot of the fully discrete system, with varying layer thickness $\delta = 10^{-2}$ (left), and $\delta = 10$ (right).

We conclude our investigations with a visual representation of the scattering arising from a torus with a revolving circle of radius $r = 0.2$, where the outer centres lie on a circle of radius $R = 0.8$. The incidental wave (7.2) with $t_0 = -1$ is scattered by absorbing boundary conditions corresponding to the impedance operator $Z(\partial_t) = \delta\partial_t^{1/2}$ with $\delta = 0.1$ on the torus.

We discretize the described problem in space with 0-th order Raviart–Thomas boundary elements with 2688 degrees of freedom and apply convolution quadrature based on the 3-stage Radau IIA method with $N = 100$ time steps. The left-hand side plot of Figure 7.3 visualizes the frequencies s_k for $k = 1, \dots, mL$, at which the Laplace domain operator $U_h(s_k)$ has to be evaluated. The plot on the right-hand side shows condition numbers and norms of the matrix arising from $A_h(s)$ and its inverse, as one follows the contour depicted before. We observe that the condition number remains relatively mild, which makes iterative solvers accessible to the problem at hand.

Figure 7.4 then shows the total wave E^{tot} on the $x_2 = 0$ plane at different times.

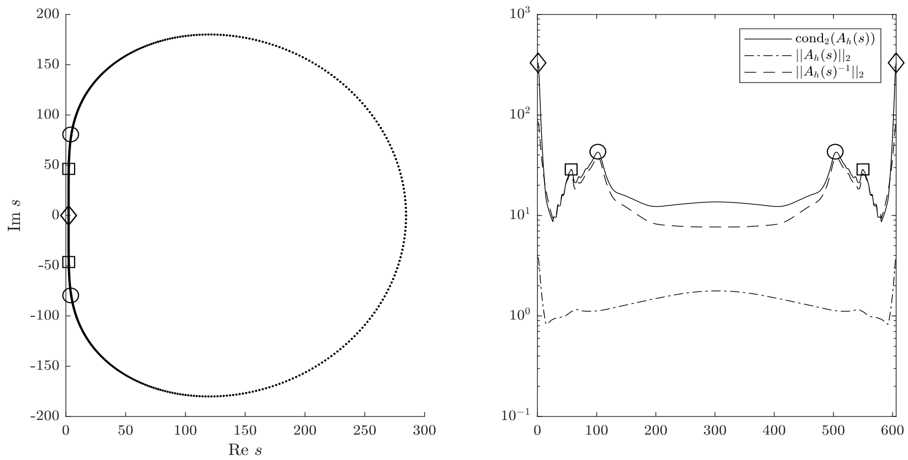

Fig. 7.3 The left-hand side plot shows a plot of the occuring frequencies for the 3-stage Radau IIA method for $N = 100$ and $T = 4$. On the right-hand side, the condition numbers and the euclidean norms of the occuring matrices are shown, as they appear when following the integral contour on the left-hand side. The markers on both plots localize the corresponding spikes of the condition numbers on the integral contour.

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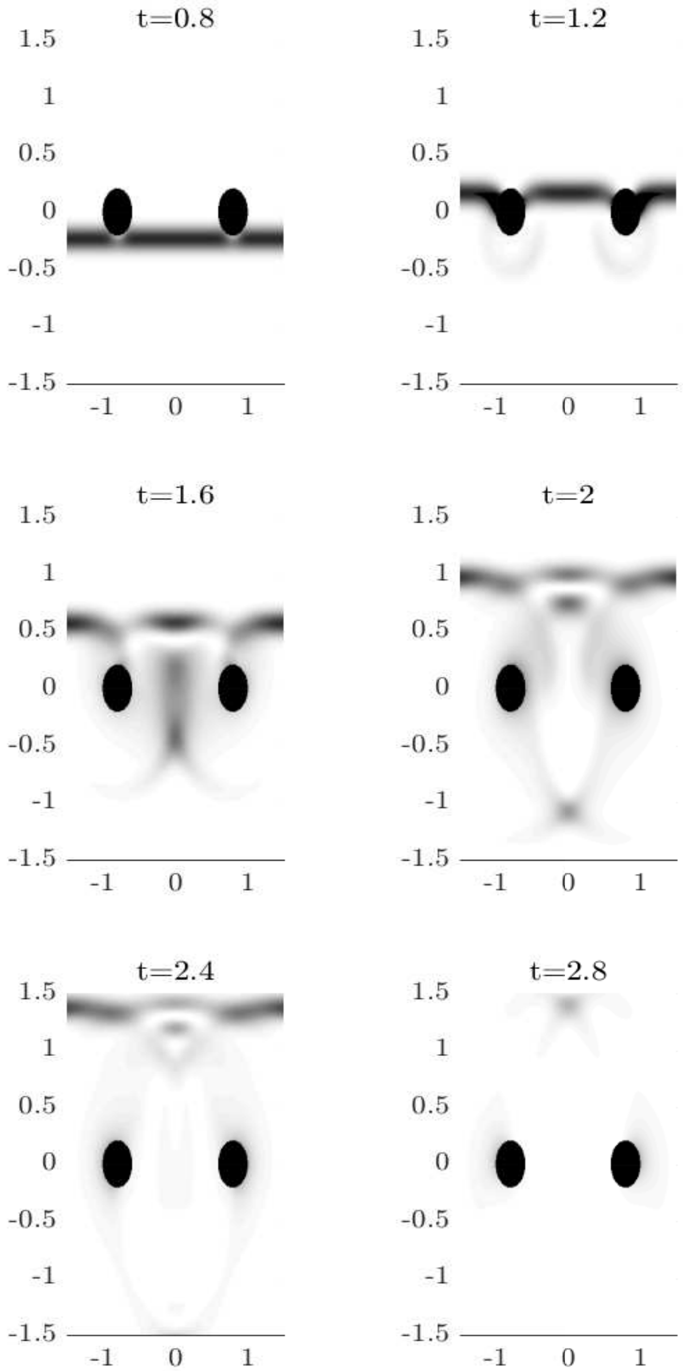

Fig. 7.4 3D-scattering arising from a torus, visualized at different times. Shown is the $y = 0$ plane, through the middle of the scatterer and the boundary condition employed is (1.5) with $\delta = 0.1$

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