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Benjamin Dörich, Marlis Hochbruck

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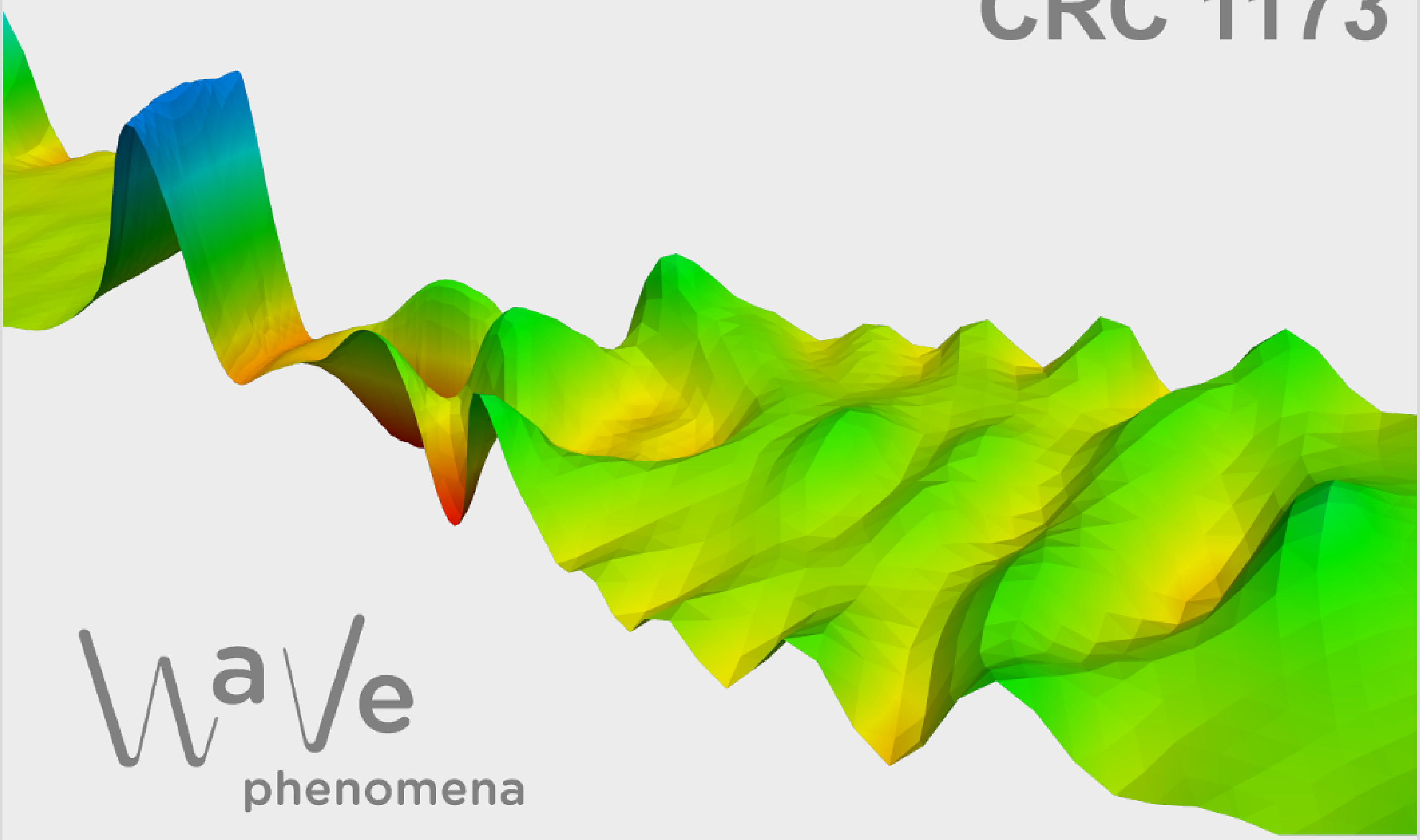

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EXPONENTIAL INTEGRATORS FOR QUASILINEAR WAVE-TYPE EQUATIONS*

BENJAMIN DÖRICH† AND MARLIS HOCHBRUCK†

Abstract. In this paper we propose two exponential integrators of first and second order applied to a class of quasilinear wave-type equations. The analytical framework is an extension of the classical Kato framework and covers quasilinear Maxwell's equations in full space and on a smooth domain as well as a class of quasilinear wave equations. In contrast to earlier works, we do not assume regularity of the solution but only on the data. From this we deduce a well-posedness result upon which we base our error analysis. We include numerical examples to confirm our theoretical findings.

Key words. error analysis, time integration, quasilinear evolution equations, a-priori error bounds, wave equation, Maxwell's equations

AMS subject classifications. Primary: 65M12, 65J15, 65M15. Secondary: 35L05, 35L90, 35Q61.

1. Introduction.

In the present paper we study the time integration of the quasi-linear evolution equation

$$\Lambda(u(t))u'(t) = Au(t) + g(t, u(t)), \quad t \in [0, T], \quad u(0) = u_0, \tag{1.1}$$

posed in some Hilbert space X with a skew-adjoint operator $A : \mathcal{D}(A) \to X$ using exponential integrators. In [18, 19], Kato casted a large class of equations into this form and established suitable well-posedness results. This framework (for $g(t, u) = Q(u)u$) was refined in the doctoral thesis [26] in order to treat certain quasilinear wave and Maxwell's equations. Since these quasilinear equations are important models for nonlinear optics and acoustics, in the recent years a lot of effort has been put in the numerical treatment. We give a brief review in the following paragraph.

In the pioneering works [6, 17, 20, 29] an abstract approximation to nonlinear and quasilinear evolution equation was constructed with the goal to prove existence of solutions. However, one can actually also find approximation rates of the implicit and semi-implicit Euler method in there. By completely new techniques, these results could be improved to optimal order in [15] for equations of the type (1.1) and higher order Runge-Kutta methods were discussed in [14, 21]. In the case of the one-dimensional wave equation equipped with periodic boundary condition, error bounds for semi-discretization in time and full discretization were proved in [9]. These bound were shown for a trigonometric integrator by a sophisticated stability analysis. In [2, 27] continuous and discontinuous Galerkin (dG) methods were used for the space discretization of the Westervelt equation in two and three dimensions and absorbing boundary conditions for this equation were treated in [25]. Concerning full discretization, we further mention the work [5], where error bounds for linear finite elements in space combined with a dG method of order 0 in time for parabolic problems were derived under low regularity assumptions. In the hyperbolic case the thesis [23] treats finite element methods and Runge-Kutta schemes for Maxwell's equations and wave equations. The potential of exponential integrators for Maxwell's equations has been shown by several numerical experiments in [28].

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†Institute for Applied and Numerical Mathematics, Karlsruhe Institute of Technology, 76149 Karlsruhe, Germany (benjamin.doerich@kit.edu, marlis.hochbruck@kit.edu).

Our aim is to provide reliable error bounds for exponential integrators which only depend on the regularity given from the data but not on the additional assumptions on the smoothness of the solution. As a consequence, we only consider first- and second-order methods and analyze the convergence in the appropriate spaces. This approach was already used by the authors in [3].

The theory is inspired by the two following series of papers. In [10, 11, 12] González and Thalhammer constructed and analyzed several exponential integration schemes for parabolic equations. Their analysis heavily relies on the parabolic smoothing properties induced by the analytic semigroup which are not available for skew-adjoint operators. For hyperbolic systems the original Kato framework in [18, 19] was used in [21] to prove error bounds for algebraically stable and coercive Runge-Kutta methods and the same result could be shown in [14, 15] in the refined Kato framework [26].

In contrast to previous works we focus on the time integration of (1.1) by exponential integrators. Recently, in [3] we observed that for semilinear wave equations it is thereby possible to lower the regularity requirements in comparison to standard Runge-Kutta or BDF methods. Our main results show that also for quasilinear wave-type problem less regularity is needed to obtain first- and second-order error bounds compared to the results in [14, 15, 21]. A more detailed version of the results presented in this paper can be found in the doctoral thesis [8].

The rest of the paper is structured as follows. For our error analysis we first introduce the analytical framework in Section 2. In Section 3, we state the main results for the first-order method and add some assumptions required for the second-order scheme. The proofs are given in Section 4 and 5, respectively. We conclude with numerical experiments for both methods combined with a finite element method in space in Section 6.

Notation. For Hilbert spaces X, Y, $\langle \cdot, \cdot \rangle_X$ denotes the scalar product on X and $\mathcal{L}(X, Y)$ the set of all bounded operators $T \colon X \to Y$ equipped with the standard operator norm $\|T\|_{Y \leftarrow X}$ and we set $\mathcal{L}(X) := \mathcal{L}(X, X)$. By $\mathcal{B}_X(r)$ we denote all elements in X with norm less or equal r. Further, we write $W^{k,p}(\Omega)$, $k \in \mathbb{N}_0$, $1 \leq p \leq \infty$, for the Sobolev space of order k with all (weak) derivatives in $L^p(\Omega)$ and abbreviate $H^k(\Omega) := W^{k,2}(\Omega)$.

2. Analytical framework and problem statement.

We introduce the three nested Hilbert spaces

$$Z \hookrightarrow Y \hookrightarrow X$$

continuously and densely embedded, and Y is an exact interpolation space between Z and X, see [22]. The linear operator A is skew-adjoint on $\mathcal{D}(A)$ and the domain satisfies $Y \hookrightarrow \mathcal{D}(A) \hookrightarrow X$ with

$$\|A\|_{X \leftarrow Y} \leq \alpha_{XY}, \qquad \|A\|_{Y \leftarrow Z} \leq \alpha_{YZ}\,.$$

In order to formulate the scheme and state all the assumptions, we rewrite (1.1) as

$$u'(t) = \mathbf{A}(u(t)) + f(t, u(t)), \qquad u(0) = u_0, \tag{2.1}$$

where we use the notation

$$\mathbf{A}(u) = \mathbf{A}_u = \Lambda^{-1}(u)A, \qquad f(t, u) = \Lambda^{-1}(u)g(t, u)\,.$$

Given a numerical approximation $u_n \approx u(t_n)$, $t_n = n\tau$ with stepsize τ, we define the operators

$$\mathbf{A}_n = \mathbf{A}(u_n), \qquad \mathbf{f}_n = f(t_n, u_n)\,, \tag{2.2}$$

and construct the exponential integrators in the following way. We freeze the argument of the differential operator and the semilinear term in (2.1) and solve the resulting linear equation exactly. If we use the last approximation u_n, we obtain the exponential Euler scheme

$$\begin{aligned} u_{n+1} &= e^{\tau \mathbf{A}_n} u_n + \tau\varphi_1(\tau\mathbf{A}_n)\mathbf{f}_n \\ &= u_n + \tau\varphi_1(\tau\mathbf{A}_n)\big(\mathbf{A}_n u_n + \mathbf{f}_n\big)\,, \end{aligned} \tag{2.3}$$

where the function φ_1 is given by

$$\varphi_1(z) = \int_0^1 e^{sz}\, ds\,.$$

We also study a second-order method, inspired by the semi-implicit midpoint rule in [21]. To obtain a method of order 2, one would like to use the average of u_n and u_{n+1}, but this would make the method implicit and thus computationally more expensive. Hence, the idea is to use the last two approximations in order to extrapolate to the average which gives the exponential midpoint rule

$$\begin{aligned} u_{1/2} &= u_0, \\ u_{n+1/2} &= \tfrac{1}{2}\big(3u_n - u_{n-1}\big), && n \geq 1, \\ u_{n+1} &= u_n + \tau\varphi_1(\tau\mathbf{A}_{n+1/2})\big(\mathbf{A}_{n+1/2}u_n + \mathbf{f}_{n+1/2}\big), && n \geq 0. \end{aligned} \tag{2.4}$$

The main computational cost for this two-step method is the same as for the exponential Euler method (2.3).

2.1. A prototypical example. Before we precisely state the assumptions, we focus on an example for equation (1.1). As a prototype consider the quasilinear wave equation from [7] on a bounded domain $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$, with a C^3-boundary $\partial\Omega$ of the form

$$\begin{aligned} \partial_{tt}q(t) + \partial_{tt}K\big(q(t)\big) &= \Delta q(t) + r(t, q(t), \partial_t q(t)), && \text{in } \Omega, \quad t \geq 0\,, \\ q(t) &= 0, && \text{on } \partial\Omega, \quad t \geq 0\,, \end{aligned}$$

with

$$K \in C^4(\mathbb{R}), \qquad 1 + K'(0) > 0, \qquad r \in C^3(\mathbb{R} \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R})\,,$$

and $r(t, \cdot, 0, 0) = 0$ on $\partial\Omega$ for $t \geq 0$. Note that in [7] the term r was not present. We rewrite the equation in a first-order formulation with $u = \big(\, q, \partial_t q\,\big)^T$ and operators

$$\Lambda(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1 + K'(q) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad g(t, u) = \begin{pmatrix} 0 \\ r(t, q, \partial_t q) - K''(q)\big(\partial_t q\big)^2 \end{pmatrix}\,,$$

such that it fits into the framework of (1.1). The Hilbert spaces in this example are

$$\begin{aligned} &X \coloneqq H_0^1(\Omega) \times L^2(\Omega), \qquad Y \coloneqq \big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H_0^1(\Omega), \\ &Z \coloneqq \{q \in H^3(\Omega) \cap H_0^1(\Omega) : \Delta q \in H_0^1(\Omega)\} \times \big(H^2(\Omega) \cap H_0^1(\Omega)\big)\,. \end{aligned}$$

In order to ensure non-degeneracy of (1.1) and to derive the formulation (2.1), we need invertibility of the operator Λ. In the model above, a typical choice is the Kerr-type nonlinearity

$$K(z) = \chi z^3, \quad \chi \in \mathbb{R},$$

see for example [4, 24, 28]. In this model, for the solution q one has to guarantee

$$1 + K'(q) = 1 + 3\chi q^2 > 0, \tag{2.5}$$

which holds for $\chi \geq 0$ independent of q. Since we only consider $d \leq 3$, we employ the continuous embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ with constant C_{emb}, and obtain

$$\|q\|_{L^\infty} \leq C_{\mathrm{emb}} \|q\|_{H^2} \leq C_{\mathrm{emb}} \|u\|_Y .$$

Hence, (2.5) also holds true for $\chi < 0$ if the norm $\|u\|_Y$ is controlled by some radius R satisfying

$$R^2 < \frac{1}{C_{\mathrm{emb}}^2 3 |\chi|} .$$

This radius R plays an important role in the well-posedness and the error analysis in the present paper. From the above considerations it is clear that R is a given quantity of the problem which might have an a priori bound as in the case $\chi < 0$. Further, we need another radius r, which can be chosen arbitrarily, with

$$\|u\|_Z \leq r,$$

in order to establish uniforms bounds in the following.

The scheme can also be applied to quasilinear Maxwell's equations for which an appropriate framework was provided in [26], for instance. Most of the assumptions made in this section are verified therein. For time integration schemes, this framework was first amended in [15].

2.2. General assumptions. We recall that the radius $R < \infty$ is a quantity which is in general given from the problem, but might have no a-priori bound. However, the radius $r < \infty$ can always be chosen arbitrarily large. We drop the dependency of the constants on R and r for the sake of readability, i.e., we always abbreviate $C = C(R, r)$ where C is any constant appearing in the following.

Assumption 2.1 (properties of Λ). *The set $\{\Lambda(y) : y \in \bar{\mathcal{B}}_Y(R)\}$ forms a family of invertible self-adjoint operators in $\mathcal{L}(X)$ such that the ranges $Ran(I \mp \Lambda^{-1}(y)A)$ are dense in X and the inverses $\Lambda^{-1}(y)$ also belong to $\mathcal{L}(Y)$. Moreover, for all $x \in X$ and $y, \widetilde{y} \in \mathcal{B}_Y(R)$, we have for some constants $\lambda_X, \nu_X, \ell > 0$ the bounds*

$$\|\Lambda(y)\|_{X \leftarrow X} \leq \lambda_X, \tag{2.6a}$$

$$\langle x, \Lambda(y)x \rangle_X \geq \nu_X^{-1} \|x\|_X^2, \tag{2.6b}$$

$$\|\Lambda(y) - \Lambda(\widetilde{y})\|_{X \leftarrow X} \leq \ell \|y - \widetilde{y}\|_Y . \tag{2.6c}$$

Further, there are constants ℓ_X, ℓ_Y, ℓ_Z such that for $\phi, \widetilde{\phi} \in \mathcal{B}$:

$$\left\|\Lambda^{-1}(\phi) - \Lambda^{-1}(\widetilde{\phi})\right\|_{V \leftarrow W} \leq \ell_V \left\|\phi - \widetilde{\phi}\right\|_V , \tag{2.6d}$$

with the triples

$$\big(V,W,\mathcal{B}\big) \in \Big\{\big(X,Y,\mathcal{B}_Y(R)\big),\big(Y,Y,\mathcal{B}_Y(R)\big),\big(Z,Z,\mathcal{B}_Z(r)\big)\Big\}.$$

From the previous assumption we immediately infer, with ν_X from (2.6b) and constants ν_Y,ν_Z, that for $\phi\in\mathcal{B}$ it holds

$$\big\|\Lambda^{-1}(\phi)\big\|_{V\leftarrow V} \le \nu_V\,, \tag{2.7}$$

with the tupels

$$\big(V,\mathcal{B}\big) \in \Big\{\big(X,\mathcal{B}_Y(R)\big),\big(Y,\mathcal{B}_Y(R)\big),\big(Z,\mathcal{B}_Z(r)\big)\Big\}.$$

For $\phi\in\mathcal{B}_Y(R)$, we consider the state dependent inner product

$$\langle x,y\rangle_\phi = \langle \Lambda(\phi)x,y\rangle_X, \tag{2.8}$$

which is defined by (2.6a) and (2.6b). The two following properties connect the state dependent norm with the X-norm and also the norms for different states. The assertions can be found in the Appendix of [15].

LEMMA 2.2 (relation between norms). *Let Assumption 2.1 hold.*
(a) For $\phi\in\mathcal{B}_Y(R)$

$$\lambda_X^{-1}\,\|u\|_\phi^2 \le \|u\|_X^2 \le \nu_X\,\|u\|_\phi^2\,.$$

(b) For $\phi,\psi\in\mathcal{B}_Y(R)$

$$\|u\|_\phi \le e^{k_1\tau}\,\|u\|_\psi\,,\qquad \textit{for}\quad \|\phi-\psi\|_Y\le\gamma\tau,$$

where $k_1 = k_1(\gamma) = \frac{1}{2}\nu_X\,\ell\,\gamma$.

The assumptions on Λ also yield bounds and Lipschitz properties of the composed differential operator A_ϕ.

LEMMA 2.3 (properties of $\mathbf{A}_\phi$). *Let Assumption 2.1 hold. Then for $\phi\in\mathcal{B}_Y(R)$*

$$\|\mathbf{A}_\phi\|_{X\leftarrow Y} \le \nu_X\alpha_{XY}, \tag{2.9a}$$

and for $\phi,\psi\in\mathcal{B}_Y(R)\cap\mathcal{B}_Z(r)$

$$\|\mathbf{A}_\phi\|_{Y\leftarrow Z} \le \nu_Y\alpha_{YZ}, \tag{2.9b}$$

$$\|\mathbf{A}_\phi-\mathbf{A}_\psi\|_{X\leftarrow Z} \le L_X\,\|\phi-\psi\|_X\,, \tag{2.9c}$$

$$\|\mathbf{A}_\phi-\mathbf{A}_\psi\|_{Y\leftarrow Z} \le L_Y\,\|\phi-\psi\|_Y\,. \tag{2.9d}$$

Proof. The bound (2.9a) directly follows from Assumption 2.1 and the other statements are proved in [15, Lemma 3.6]. ∎

A key element not only in the proofs of well-posedness is the following assumption. It guarantees that the quasilinear operator behaves well not only with respect to the ground space X but also on the stronger space Z.

ASSUMPTION 2.4 (commutator condition). *We assume that there is a continuous isomorphism $S\colon Z\to X$ such that for $z\in\mathcal{B}_Y(R)\cap\mathcal{B}_Z(r)$*

$$\mathbf{A}_z^S = S\mathbf{A}_zS^{-1} = \mathbf{A}_z + B(z),$$

and there is a constant $\beta \geq 0$ such that

$$\|B(z)\|_{X \leftarrow X} \leq \beta \,.$$

We finally conclude properties of the semilinear term in (2.1). The assumptions are made for the term original term g in (1.1).

Assumption 2.5 (properties of g). *For $V \in \{X, Y, Z\}$ there are constants $L_{g,V}$ such that for $\phi_1, \phi_2 \in \mathcal{B}_Z(r)$ and $t, s \in [0, T]$ it holds*

$$\|g(t, \phi_1) - g(s, \phi_2)\|_V \leq L_{g,V}\big(|t - s| + \|\phi_1 - \phi_2\|_V\big) \,. \tag{2.10}$$

We recall that the above assumptions are all satisfied in the case of the wave and Maxwell's equations considered in Section 2.1, cf. [8, Appendix B]. Employing properties (2.6d) and (2.7) we derive the following result.

Lemma 2.6 (properties of f). *Let Assumptions 2.1 and 2.5 hold.*

(a) The Lipschitz bound (2.10) also holds for f with constants $L_{f,V}$.

(b) For $V \in \{Y, Z\}$ there are constants $C_{f,V,\infty}$ such that for $\phi \in \mathcal{B}_Z(r)$ and $t \in [0, T]$

$$\|f(t, \phi)\|_V \leq C_{f,V,\infty} \,.$$

2.3. Notation and well-posedness. We briefly collect some relevant constant used in the error analysis later. In the following, $\gamma > 0$ denotes a given parameter, which will be determined later.

$$\begin{aligned} k_0 &= (\nu_X \lambda_X)^{1/2} \geq 1, & k_1 &= k_1(\gamma) = \tfrac{1}{2}\,\nu_X \ell \gamma, \\ c_0 &= \|S\|_{X \leftarrow Z} \left\|S^{-1}\right\|_{Z \leftarrow X} k_0 \geq 1, & c_1 &= c_0 \nu_Y \alpha_{YZ}. \end{aligned}$$

We are now in the position to state the necessary extension of the well-posedness result to our problem which is proven in the case of $f(t, u) = Q(u)u$ in [26] and extended in [8, Thm. 5.14] to the general case.

Theorem 2.7. *Let Assumptions 2.1, 2.4, and 2.5 be satisfied. For an initial value*

$$\|u_0\|_Y \leq R_0 \coloneqq \tfrac{1}{4c_0} R, \qquad \|u_0\|_Z \leq r_0 \coloneqq \tfrac{1}{4c_0} r,$$

define the time

$$T \coloneqq \min\Big\{ \frac{\ln 2}{\omega_2}, \frac{R}{4c_0 C_{f,Y,\infty}}, \frac{r}{4c_0 C_{f,Z,\infty}}, \frac{1}{4c_0\big(L_Y r + L_{f,Y}\big)} \Big\}, \tag{2.12}$$

where

$$\omega_2 = \omega_2(\gamma) = k_1(\gamma) + k_0 \beta, \qquad \gamma = \gamma(r) \coloneqq \tfrac{c_1}{c_0} r + 2c_0 C_{f,Y,\infty} \,. \tag{2.13}$$

Then there is a unique solution u of (2.1) with

$$u \in C([0, T], Z) \cap C^1([0, T], Y)$$

satisfying

$$\|u(t)\|_Y \leq R, \qquad \|u(t)\|_Z \leq r$$

on the interval $[0, T]$.

Analogously to the definitions in (2.2) for the numerical approximation u_n, let $u(t)$ be the solution of Theorem 2.7 and define $\widehat{u}_{n+\sigma} = u(t_n + \tau\sigma)$. With this we introduce the notation

$$\widehat{\mathbf{f}}(t) = f(t, u(t)), \qquad \widehat{\mathbf{f}}_{n+\sigma} = f(t_n + \tau\sigma, \widehat{u}_{n+\sigma}),$$
$$\widehat{\mathbf{A}}(t) = \mathbf{A}(u(t)), \qquad \widehat{\mathbf{A}}_{n+\sigma} = \mathbf{A}(\widehat{u}_{n+\sigma}).$$

We further use for a Hilbert space V and a function $v \in C([0,T], V)$

$$\|v\|_{V,\infty} \coloneqq \max_{t\in[0,T]} \|v(t)\|_V .$$

3. Numerical methods and main results. In this section we present error bounds for the schemes (2.3) and (2.4) which are the main results of the paper. In the following we precisely state the necessary regularity of the solution u. Note that in addition all constants depend on quantities introduced in Section 2.

3.1. Exponential Euler. We first consider the first-order scheme proposed in (2.3). For this method we establish uniform error bounds of order one in the X- as well as in the Y-norm.

THEOREM 3.1. *Let u be the solution of* (1.1) *and u_n the approximation obtained from* (2.3)*. If Assumptions* 2.1*,* 2.4*, and* 2.5 *are satisfied, we obtain for $V \in \{X, Y\}$ the error bounds*

$$\|u(t_n) - u_n\|_V \le t_n e^{c_V\, t_n} C_V\, \tau, \quad 0 \le n\tau = t_n \le T,$$

with constants $C_V, c_V > 0$ that only depend on $\|u'\|_{V,\infty}$ and $\|u\|_{Z,\infty}$, but are independent of τ, n and t_n.

By Theorem 2.7, we can control the X-, Y- and Z-norm of u on $[0,T]$. More regularity of the exact solution u and additional assumptions on the data lead to first-order convergence also in the Z-norm, c.f. [15, Thm. 4.5] and [8, Thm. 7.20]. Note however, that the bound in the Y-norm is shown only using regularity given in Theorem 2.7.

3.2. Exponential midpoint rule. In order to derive error bounds of order higher than one for the scheme (2.4), it is not sufficient to use Lipschitz bounds as stated in previous section. Instead, we need to apply Taylor expansion to the terms on the right-hand-side of (2.1). We formulate the necessary differentiability conditions in assumptions. The straightforward but rather lengthy computations to verify them for the quasilinear wave equation and Maxwell's equations can be found in full detail in [8, Appendix B].

ASSUMPTION 3.2 (additional properties of g). *Let $u \in C^1([0,T], Y) \cap C([0,T], Z)$ and consider the map*

$$t \mapsto \widehat{\mathbf{g}}(t) = g(t, u(t)).$$

Then there is a constant $C_{g',Y,\infty}$ with
(a) $t \mapsto \widehat{\mathbf{g}}(t) \in C^1([0,T], Y)$, $\quad \|\widehat{\mathbf{g}}'(t)\|_Y \le C_{g',Y,\infty}$,
and, if in addition, $u \in C^2([0,T], X)$ holds, then there is $C_{g'',X,\infty}$ such that
(b) $t \mapsto \widehat{\mathbf{g}}(t) \in C^2([0,T], X)$, $\quad \|\widehat{\mathbf{g}}''(t)\|_X \le C_{g'',X,\infty}$
with constants only depending on $\|u''\|_{X,\infty}$, $\|u'\|_{Y,\infty}$, and $\|u\|_{Z,\infty}$.

Whereas similar conditions to those in Assumption 3.2 are known from the analysis of semilinear evolution equations, the following is needed for the expansion of the term with the differential operator.

Assumption 3.3 (additional properties of Λ). *Let $u \in C^1([0,T],Y) \cap C([0,T],Z)$ and consider the map*

$$t \mapsto \mathbf{\Lambda^{-1}}(t) \coloneqq \Lambda^{-1}(u(t))\,.$$

For $V \in \{X, Y\}$ and $v \in V$ it holds

(a) $t \mapsto \mathbf{\Lambda^{-1}}(t)v \in C^1([0,T],V), \qquad \left\| \left(\mathbf{\Lambda^{-1}}\right)'(t) \right\|_{V \leftarrow V} \le C_{VV},$

and, if in addition, $u \in C^2([0,T],X)$, it further holds for $y \in Y$

(b) $t \mapsto \mathbf{\Lambda^{-1}}(t)y \in C^2([0,T],X), \qquad \left\| \left(\mathbf{\Lambda^{-1}}\right)''(t) \right\|_{X \leftarrow Y} \le C_{XY},$

with constants C_{XX}, C_{XY}, C_{YY} only depending on $\|u''\|_{X,\infty}$, $\|u'\|_{Y,\infty}$, and $\|u\|_{Z,\infty}$.

A combination of the two preceding assumptions gives us the following differentiability.

Lemma 3.4. *Let $u \in C^1([0,T],Y) \cap C([0,T],Z)$ and consider the map*

$$t \mapsto \widehat{\mathbf{f}}(t) = f(t, u(t))\,.$$

If Assumptions 2.1, 2.5, 3.2, and 3.3 hold, then $\widehat{\mathbf{f}}$ satisfies Assumption 3.2(a). If in addition $u \in C^2([0,T],X)$, then it also satisfies Assumption 3.2(b). The constants $C_{f',Y,\infty}, C_{f'',X,\infty}$ only depend on $\|u''\|_{X,\infty}$, $\|u'\|_{Y,\infty}$, and $\|u\|_{Z,\infty}$.

Further, we easily obtain together with Assumption 2.1 (a) the following differentiability of the differential operator evaluated at a smooth function.

Lemma 3.5. *Let $u \in C^1([0,T],Y) \cap C([0,T],Z)$ and consider the map*

$$t \mapsto \widehat{\mathbf{A}}(t) = \mathbf{\Lambda^{-1}}(t)A\,.$$

If Assumptions 2.1 and 3.3 hold, then for $y \in Y$ and $z \in Z$ it holds

(a) $t \mapsto \widehat{\mathbf{A}}(t)y$ *is* $C^1([0,T],X), \qquad \left\| \widehat{\mathbf{A}}'(t) \right\|_{X \leftarrow Y} \le C^A_{XY},$

(b) $t \mapsto \widehat{\mathbf{A}}(t)z$ *is* $C^1([0,T],Y), \qquad \left\| \widehat{\mathbf{A}}'(t) \right\|_{Y \leftarrow Z} \le C^A_{YZ},$

and, if in addition, $u \in C^2([0,T],X)$, it further holds

(c) $t \mapsto \widehat{\mathbf{A}}(t)z$ *is* $C^2([0,T],X), \qquad \left\| \widehat{\mathbf{A}}''(t) \right\|_{X \leftarrow Z} \le C^A_{XZ},$

with constants $C^A_{XY}, C^A_{YZ}, C^A_{XZ}$ only depending on $\|u''\|_{X,\infty}$, $\|u'\|_{Y,\infty}$, and $\|u\|_{Z,\infty}$.

The first important application of the assumptions in this section is the following regularity result which is intensively used in the error analysis of the second-order scheme.

Lemma 3.6. *If the assumptions of Theorem 2.7 are satisfied and in addition Assumptions 3.2 (a) and 3.3 (a) hold, then the solution u of (2.1) satisfies*

$$u \in C^2([0,T],X) \cap C^1([0,T],Y) \cap C([0,T],Z)\,.$$

Proof. We only need to prove that u' is differentiable in X. To do so, we differentiate the right-hand-side of (2.1), which is possible by the regularity of u in Theorem 2.7, Assumption 3.2 (a), and Lemma 3.5. □

The last step towards the statement of the error bound of the exponential midpoint rule is an adaption of the constants caused by the extrapolated approximation

$u_{n+1/2}$. Since this is not a convex combination of previous approximations, bounds on those are not directly applicable. We hence choose some radius $\widehat{R} > R$ such that Assumption 2.1 on $\Lambda(y)$ for $y \in \mathcal{B}_Y(\widehat{R})$ is still valid. This is necessary for the stability of the numerical schemes and enters later as a mild stepsize restriction $\tau \le \tau_0$ with

$$\tfrac{\widehat{\gamma}\tau_0}{2} < \widehat{R} - R\,, \tag{3.1}$$

where $\widehat{\gamma}$ is chosen below in (3.3). Similarly, we also have to replace the radius r by $\widehat{r} = 2r$. For all computations concerning the exponential midpoint rule we adapt the assumptions of the previous section to the new radii $\widehat{R}$ and $\widehat{r}$ and denoted them by the same name but with an additional hat, e.g., we replace

$$C_{f,X,\infty} = C_{f,X,\infty}(R,r) \quad \text{by} \quad \widehat{C}_{f,X,\infty} = \widehat{C}_{f,X,\infty}(\widehat{R},\widehat{r})\,.$$

Without loss of generality in the following we assume that all constants grow monotonically in the radii such that, e.g., $C_{f,X,\infty} \le \widehat{C}_{f,X,\infty}$ holds. This allows to only simulate up to the time

$$\widehat{T}_{\mathrm{mid}} \coloneqq \min\Big\{\frac{\ln 2}{\widehat{\omega}_2}, \frac{R}{4\widehat{c}_0\widehat{C}_{f,Y,\infty}}, \frac{r}{4\widehat{c}_0\widehat{C}_{f,Z,\infty}}\Big\}, \tag{3.2}$$

where

$$\widehat{\omega}_2 = 2\widehat{k}_1(\widehat{\gamma}) + \widehat{k}_0\widehat{\beta}, \qquad \widehat{\gamma} \coloneqq \frac{\widehat{c}_1}{\widehat{c}_0}r + 2\widehat{c}_0\widehat{C}_{f,Y,\infty}\,. \tag{3.3}$$

If we compare (3.2) to the end time T given in (2.12), then in general the three terms appearing here are smaller than the corresponding ones in (2.12). However, in (3.2) there is one term less, such that one can not decide which time is larger. Hence, we prove the following error bound in X- and the Y-norm on the intersection of both time intervals.

Theorem 3.7. *Let u be the solution of (1.1) and u_n the approximation obtained from (2.4). If Assumptions 2.1, 2.4, and 2.5 are satisfied, and in addition Assumptions 3.2 and 3.3 hold true, and τ_0 is given by (3.1), then for all $\tau \le \tau_0$ the error is bounded by*

$$\|u(t_n) - u_n\|_X + \tau\,\|u(t_n) - u_n\|_Y \le t_n e^{c\,t_n} C\tau^2, \quad 0 \le n\tau = t_n \le \min\{T, \widehat{T}_{mid}\}\,,$$

with constants $C, c > 0$ that only depend on $\|u''\|_{X,\infty}$, $\|u'\|_{Y,\infty}$, and $\|u\|_{Z,\infty}$, but are independent of τ, n and t_n.

Remark 3.8. More regularity of the exact solution u and stronger versions of Assumptions 3.2 and 3.3 also lead to second-order convergence in the Y-norm, see [8, Theorem 7.27].

4. Proof for the exponential Euler method. This section is devoted to the proof of Theorem 3.1, and it is divided into three steps. We first show bounds of the numerical approximations uniformly for all time-steps in the stronger norms and obtain from this well-posedness of the numerical scheme. The analysis closely follows [15]. Next, we derive a suitable representation for the exact solution and bound the defects. In the last step, we solve the error recursion and conclude the main theorem.

4.1. Stability. The first result is a variant of [15, Lemma 3.7] where we use a space that contains all numerical approximations. For $N \in \mathbb{N}$ and $\xi > 0$ we define the space

$$\begin{aligned} E\big(N, R, r, \xi\big) \coloneqq \{\phi = &\big(\phi_0, \dots, \phi_N\big) \in Z^{N+1} \mid \\ &\|\phi_k\|_Y \le R, \|\phi_k\|_Z \le r, \quad k = 0, \dots, N\,, \\ &\|\phi_k - \phi_{k-1}\|_Y \le \xi, \quad k = 1, \dots, N\}\,. \end{aligned} \tag{4.1}$$

It is constructed in such a way that inserting an element in the space $E\big(N, R, r, \xi\big)$ in the numerical scheme yields the following approximation together with the preceding ones to be in $E\big(N+1, R, r, \xi\big)$. Based on the following auxiliary result, this is shown by induction in Lemma 4.2.

Lemma 4.1. *Let $\phi = \big(\phi_0, \dots, \phi_N\big) \in E\big(N, R, r, \tau\gamma\big)$ and $0 \le j \le k \le N$ for $j, k \in \mathbb{N}$. If Assumptions 2.1 and 2.4 hold, then:*

$$\begin{aligned} \Big\| e^{\tau \mathbf{A}_{\phi_k}} e^{\tau \mathbf{A}_{\phi_{k-1}}} \dots e^{\tau \mathbf{A}_{\phi_j}} \Big\|_{X \leftarrow X} &\le k_0 e^{\omega_1 (k-j+1)\tau}, \\ \Big\| e^{\tau \mathbf{A}_{\phi_k}} e^{\tau \mathbf{A}_{\phi_{k-1}}} \dots e^{\tau \mathbf{A}_{\phi_j}} \Big\|_{Y \leftarrow Y} &\le c_0 e^{\omega_2 (k-j+1)\tau}, \\ \Big\| e^{\tau \mathbf{A}_{\phi_k}} e^{\tau \mathbf{A}_{\phi_{k-1}}} \dots e^{\tau \mathbf{A}_{\phi_j}} \Big\|_{Z \leftarrow Z} &\le c_0 e^{\omega_2 (k-j+1)\tau}, \end{aligned}$$

with $\omega_1 = \omega_1(\gamma) = k_1(\gamma)$ and ω_2 given in (2.13).

Proof. The proof can be found in the Appendix of [15]. The modification in the choice of ω_1 and ω_2 is due to $\big\| e^{t \mathbf{A}_\phi} x \big\|_\phi = \|x\|_\phi$ for $x \in X$ with $\|\cdot\|_\phi$ defined in (2.8). ◻

The main result of the section is an extension of [15, Theorem 4.1] and states that the lower bound on the possible simulation time, for which uniform boundedness of the numerical approximations can be guaranteed, is identical to the one of the exact solution.

Lemma 4.2. *Let Assumptions 2.1, 2.4, and 2.5 hold. For T defined in (2.12) and initial values*

$$\|u_0\|_Y \le R_0 \coloneqq \tfrac{1}{4c_0} R, \qquad \|u_0\|_Z \le r_0 \coloneqq \tfrac{1}{4c_0} r,$$

the numerical approximations given by (2.3) satisfy for $N\tau \le T$

$$\big(u_0, \dots, u_N\big) \in E\big(N, R, r, \tau\gamma\big)\,, \tag{4.2}$$

for E defined in (4.1) and γ in (2.13).

Proof. We first introduce an abbreviation for the product of several semigroups

$$\mathbf{S}_i^k \coloneqq \begin{cases} e^{\tau \mathbf{A}_k} \dots e^{\tau \mathbf{A}_i}, & i \le k, \\ I, & i > k, \end{cases} \tag{4.3}$$

and derive the representation

$$\begin{aligned} u_{n+1} &= e^{\tau \mathbf{A}_n} u_n + \tau \varphi_1(\tau \mathbf{A}_n) \mathbf{f}_n \\ &= e^{\tau \mathbf{A}_n} \Big(e^{\tau \mathbf{A}_{n-1}} u_{n-1} + \tau \varphi_1(\tau \mathbf{A}_{n-1}) \mathbf{f}_{n-1} \Big) + \tau \varphi_1(\tau \mathbf{A}_n) \mathbf{f}_n \\ &= \mathbf{S}_0^n u_0 + \tau \sum_{j=0}^{n} \mathbf{S}_{j+1}^n \varphi_1(\tau \mathbf{A}_j) \mathbf{f}_j\,. \end{aligned} \tag{4.4}$$

We prove (4.2) by induction on n. Let (4.2) be true for some $n \le N-1$, i.e. we have $(u_0,\dots,u_n) \in E\big(n,R,r,\gamma\tau\big)$. Then by Lemma 4.1 and the definition of φ_1 we estimate for $j \le n$

$$\big\|\mathbf{S}_{j+1}^n \varphi_1(\tau\mathbf{A}_j)\big\|_{Y\leftarrow Y}\,, \quad \big\|\mathbf{S}_{j+1}^n \varphi_1(\tau\mathbf{A}_j)\big\|_{Z\leftarrow Z} \le c_0 e^{\omega_2(n-j+1)\tau}\,. \tag{4.5}$$

We estimate the Y-norm of u_{n+1} in (4.4) using Lemma 2.6

$$\begin{aligned}\|u_{n+1}\|_Y &\le c_0 e^{\omega_2 t_{n+1}}\|u_0\|_Y + c_0\tau\sum_{j=0}^{n} e^{\omega_2(n-j+1)\tau}\|\mathbf{f}_j\|_Y\\ &\le c_0 e^{\omega_2 t_{n+1}}\big(\|u_0\|_Y + T\,C_{f,Y,\infty}\big)\\ &\le 2c_0\big(R_0 + \tfrac{1}{4c_0}R\big) = R\,,\end{aligned} \tag{4.6}$$

since $t_{n+1} \le T$, where we used the induction hypothesis to bound $\mathbf{f}_j$. Analogously, Lemma 2.6 yields

$$\|u_{n+1}\|_Z \le c_0 e^{\omega_2 T}\big(\|u_0\|_Z + TC_{f,Z,\infty}\big) \le 2\big(\tfrac{r}{4}+\tfrac{r}{4}\big) \le r\,. \tag{4.7}$$

It remains to bound the difference of two successive approximation. We employ (2.9b) and obtain

$$\begin{aligned}\|u_{n+1}-u_n\|_Y &\le \big\|\big(e^{\tau\mathbf{A}_n}-I\big)u_n\big\|_Y + \tau\|\varphi_1(\tau\mathbf{A}_n)\mathbf{f}_n\|_Y\\ &\le \tau\nu_Y\alpha_{YZ}\|\varphi_1(\tau\mathbf{A}_n)u_n\|_Z + \tau\|\varphi_1(\tau\mathbf{A}_n)\mathbf{f}_n\|_Y\,.\end{aligned} \tag{4.8}$$

If we use the representation in (4.4) for u_n, proceeding as in (4.7) we derive

$$\|\varphi_1(\tau\mathbf{A}_n)u_n\|_Z \le c_0 e^{\omega_2 T}\big(\|u_0\|_Z + TC_{f,Z,\infty}\big) \le r\,. \tag{4.9}$$

Applying Lemma 2.6 and (4.5), the second term is bounded by

$$\|\varphi_1(\tau\mathbf{A}_n)\mathbf{f}_n\|_Y \le c_0 e^{\omega_2\tau}\,C_{f,Y,\infty} \le 2c_0\,C_{f,Y,\infty}\,, \tag{4.10}$$

where we used $\tau\omega_2 \le \ln 2$. In total, we arrive at

$$\|u_{n+1}-u_n\|_Y \le \tau\big(\tfrac{c_1}{c_0}r + 2c_0\,C_{f,Y,\infty}\big) = \gamma\tau\,, \tag{4.11}$$

which yields $(u_0,\dots,u_{n+1}) \in E\big(n+1,R,r,\gamma\tau\big)$ and hence closes the proof. ∎

4.2. Defect. In this step we present a recursion for the global error given by

$$e_n := u(t_n) - u_n\,.$$

It is based on a representation of the exact solution, where we replace $\mathbf{A}(u(t))$ by $\mathbf{A}(u_n)$ such that the recurrence relation is driven by the semigroups studied in Lemma 4.1. Hence, one can directly compare the exact solution with the numerical scheme (2.3). In the following proposition we further bound the defects.

Proposition 4.3. *Let Assumption 2.1, 2.4, and 2.5 hold and consider the solution u given by Theorem 2.7 and numerical approximations $(u_n)_n$ given by (2.3). Then the global error satisfies the error recursion*

$$e_{n+1} = e^{\tau\mathbf{A}_n}e_n + \delta_n\,, \tag{4.12}$$

where the defects are bounded by

$$\|\delta_n\|_X \le \left(C_{\sigma,X}\,\tau\,\|e_n\|_X + C_{\delta,X}\,\tau^2\right) e^{\tau\omega_1}\,,$$

with constants $C_{\sigma,X}, C_{\delta,X} > 0$ *that only depend on* $\|u'\|_{X,\infty}$ *and* $\|u\|_{Z,\infty}$*, but are independent of* τ*,* n *and* t_n*.*

Proof. We obtain from equation (2.1) inserting $\mathbf{A}_n$ and $\mathbf{f}_n$ the differential equation

$$\begin{aligned} u'(t) &= \widehat{\mathbf{A}}(t)u(t) + \widehat{\mathbf{f}}(t) \\ &= \mathbf{A}_n u(t) + \mathbf{f}_n \\ &\qquad + \big(\widehat{\mathbf{A}}_n - \mathbf{A}_n\big)u(t) + \big(\widehat{\mathbf{f}}_n - \mathbf{f}_n\big) + \big(\widehat{\mathbf{A}}(t) - \widehat{\mathbf{A}}_n\big)u(t) + \big(\widehat{\mathbf{f}}(t) - \widehat{\mathbf{f}}_n\big) \\ &=: \mathbf{A}_n u(t) + \mathbf{f}_n + \sum_{i=1}^{4} \widetilde{\delta}_{n,i}(t)\,. \end{aligned}$$

By the variation-of-constants formula, the exact solution is given by

$$u(t_{n+1}) = e^{\tau\mathbf{A}_n}u(t_n) + \tau\varphi_1(\tau\mathbf{A}_n)\mathbf{f}_n + \delta_n, \qquad \delta_n := \sum_{i=1}^{4}\delta_{n,i}, \tag{4.13}$$

with the defects

$$\delta_{n,i} = \int_0^\tau e^{(\tau-s)\mathbf{A}_n}\widetilde{\delta}_{n,i}(t_n+s)\,ds\,,$$

which are estimated in the following. Using the Lipschitz bound (2.9c) and the estimates of Lemma 4.1 we infer

$$\begin{aligned} \|\delta_{n,1}\|_X &= \tau\Big\|\int_0^1 e^{(1-s)\tau\mathbf{A}_n}\big(\widehat{\mathbf{A}}_n - \mathbf{A}_n\big)\widehat{u}_{n+s}\,ds\Big\|_X \\ &\le \tau k_0 \int_0^1 e^{(1-s)\tau\omega_1}\,\Big\|\big(\widehat{\mathbf{A}}_n - \mathbf{A}_n\big)\widehat{u}_{n+s}\Big\|_X\,ds \\ &\le \tau k_0 L_X\,\|e_n\|_X \int_0^1 e^{(1-s)\tau\omega_1}\,\|\widehat{u}_{n+s}\|_Z\,ds \\ &\le \tau k_0 L_X e^{\tau\omega_1}\,\|e_n\|_X\,\|u\|_{Z,\infty}\,, \end{aligned} \tag{4.14}$$

and similarly by Lemma 2.6

$$\|\delta_{n,2}\|_X \le \tau k_0 e^{\tau\omega_1} L_{f,X}\,\|e_n\|_X\ .$$

To obtain a local error of order two, we use (2.9c) and estimate

$$\begin{aligned} \|\delta_{n,3}\|_X &\le \tau k_0 \int_0^1 e^{(1-s)\tau\omega_1}\,\Big\|\big(\widehat{\mathbf{A}}_{n+s} - \widehat{\mathbf{A}}_n\big)\widehat{u}_{n+s}\Big\|_X\,ds \\ &\le \tau k_0 L_X \int_0^1 e^{(1-s)\tau\omega_1}\,\|\widehat{u}_{n+s} - \widehat{u}_n\|_X\,\|\widehat{u}_{n+s}\|_Z\,ds \\ &\le \tau^2 k_0 L_X e^{\tau\omega_1}\,\|u'\|_{X,\infty}\,\|u\|_{Z,\infty}\ , \end{aligned} \tag{4.15}$$

as well as by Lemma 2.6

$$\|\delta_{n,4}\|_X \le \tau^2 k_0 e^{\tau\omega_1} L_{f,X}\big(1 + \|u'\|_{X,\infty}\big)\,.$$

We conclude the proof by subtracting (2.3) from (4.13). ∎

Along the same lines we can deduce bounds in the stronger Y-norm, where the additional regularity $u \in C^1([0,T],Y)$ comes into play.

Corollary 4.4. *The defect in* (4.12) *can also be bounded by*

$$\|\delta_n\|_Y \le \big(C_{\sigma,Y}\,\tau\,\|e_n\|_Y + C_{\delta,Y}\,\tau^2\big)\,e^{\tau\omega_2}\,,$$

with constants $C_{\sigma,Y}, C_{\delta,Y} > 0$ *that only depend on* $\|u'\|_{Y,\infty}$ *and* $\|u\|_{Z,\infty}$*, but are independent of* τ*,* n *and* t_n*.*

4.3. Proof of Theorem 3.1. A combination of the stability bounds and the defects yields the global error result.

Proof of Theorem 3.1*.* We only prove the bound in the X-norm here. The error bound in the Y-norm is easily derived replacing Proposition 4.3 by Corollary 4.4 and ω_1 by ω_2.

Using the error recursion in (4.12) and recalling $\mathbf{S}_i^k$ from (4.3), we obtain by a discrete version of the variation-of-constants formula

$$e_{n+1} = e^{\tau\mathbf{A}_n} e_n + \delta_n = \mathbf{S}_0^n e_0 + \sum_{j=0}^{n} \mathbf{S}_{j+1}^n \delta_j\,. \tag{4.16}$$

Similar to (4.5), we conclude by Lemma 4.1, Proposition 4.3, and $e_0 = 0$,

$$\begin{aligned}\|e_{n+1}\|_X &\le \sum_{j=0}^{n} \big\|\mathbf{S}_{j+1}^n\big\|_{X\leftarrow X} \|\delta_j\|_X \\ &\le k_0\tau \sum_{j=0}^{n} e^{\omega_1(n+1-j)\tau} C_\sigma \|e_j\|_X + k_0\tau \sum_{j=0}^{n} e^{\omega_1(n+1-j)\tau} C_\delta \tau\,.\end{aligned}$$

Multiplying with $e^{-\omega_1(n+1)\tau}$ gives

$$e^{-\omega_1(n+1)\tau}\|e_{n+1}\|_X \le C_\sigma k_0 \tau \sum_{j=0}^{n} e^{-\omega_1 j\tau}\|e_j\|_X + k_0\tau \sum_{j=0}^{n} e^{-\omega_1 j\tau} C_\delta \tau\,,$$

and a Gronwall argument yields with $t_{n+1} = (n+1)\tau$

$$e^{-\omega_1 t_{n+1}}\|e_{n+1}\|_X \le t_{n+1} e^{C_\sigma k_0 t_{n+1}} k_0 C_\delta\,\tau\,.$$

Finally, we arrive at

$$\|e_{n+1}\|_X \le t_{n+1} e^{(\omega_1 + C_\sigma k_0) t_{n+1}} k_0 C_\delta\,\tau\,,$$

which completes the proof. ∎

5. Proof for the exponential midpoint rule. The proof of Theorem 3.7 has a very similar structure to the one of Theorem 3.1. We first derive bounds for the numerical approximations. But since we plug in extrapolations of previous approximations, this becomes slightly more technical. In a second step we derive the defects and bound them with the right order. Here, some terms can be treated as for the exponential Euler method and are bounded first. The remaining terms take the additionally required differentiability into account and are more involved. As the structure of the two methods is so similar, there is nothing left to do in the error accumulation, and we may immediately conclude the main result.

5.1. Stability. We again need an auxiliary result on the behavior of the composition of linear flows, but this time evaluated at the extrapolated midpoints. The choice of the larger constants in the space E become clearer in the proceeding lemma.

LEMMA 5.1. *Let* $\phi = \big(\phi_{1/2}, \phi_{3/2} \dots, \phi_{N+1/2}\big) \in E\left(N, \widehat{R}, \widehat{r}, 2\tau\gamma\right)$*. If Assumptions 2.1 and 2.4 hold, we obtain the stability bounds as in (4.1) for* $j \le k$ *and* $j,k \in \{\frac{1}{2}, \frac{3}{2}, \dots, N+\frac{1}{2}\}$ *with* k_0*,* c_0*,* ω_1 *and* ω_2 *replaced by* $\widehat{k}_0$*,* $\widehat{c}_0$*,* $\widehat{\omega}_1$ *and* $\widehat{\omega}_2$*, respectively, where* $\widehat{\omega}_1 \coloneqq 2\widehat{k}_1(\widehat{\gamma})$ *and* $\widehat{\omega}_2$ *is given in (3.3).*

Proof. As for Lemma 4.1, the proof can be found in the Appendix of [15]. ◻

With this we can mimic the bounds in Lemma 4.2. However, we need to take the minimal simulation time $\widehat{T}_{\mathrm{mid}}$ defined in (3.2) into account which is necessary to obtain uniform bounds in the numerical approximations.

LEMMA 5.2. *Let Assumptions 2.1, 2.4, and 2.5 hold. For* $\widehat{T}_{mid}$ *defined in (3.2),* $\tau \le \tau_0$ *with* τ_0 *given in (3.1) and initial values*

$$\|u_0\|_Y \le R_0 \coloneqq \tfrac{1}{4\widehat{c}_0} R, \qquad \|u_0\|_Z \le r_0 \coloneqq \tfrac{1}{4\widehat{c}_0} r,$$

the numerical approximations satisfy for $N\tau \le \widehat{T}_{mid}$

$$(u_0, \dots, u_N) \in E\big(N, R, r, \tau\widehat{\gamma}\big), \quad (u_{1/2}, \dots, u_{N-1/2}) \in E\left(N-1, \widehat{R}, \widehat{r}, 2\tau\widehat{\gamma}\right). \tag{5.1}$$

Proof. We prove the assertion by induction on n and assume (5.1) is true for some $1 \le n \le N-1$. By the choice $u_{1/2} \coloneqq u_0$, the case $n = 1$ is treated as for the exponential Euler. In part (a), we first prove the bounds for the midpoints in order to apply Lemma 5.1.

(a) Using the induction hypothesis, we obtain for $\tau \le \tau_0$ given in (3.1)

$$\big\|u_{n+1/2}\big\|_Y \le \|u_n\|_Y + \tfrac{1}{2}\|u_n - u_{n-1}\|_Y \le R + \tfrac{\widehat{\gamma}\tau}{2} \le \widehat{R}\,,$$

as well as

$$\big\|u_{n+1/2}\big\|_Z \le \tfrac{3}{2}\|u_n\|_Y + \tfrac{1}{2}\|u_{n-1}\|_Z \le 2r = \widehat{r}\,.$$

Let $u_{-1} \coloneqq u_0$, then it holds $u_{1/2} = \frac{3}{2}u_0 - \frac{1}{2}u_{-1}$ and we estimate for $n \ge 1$

$$\big\|u_{n+1/2} - u_{n-1/2}\big\|_Y \le \tfrac{3}{2}\|u_n - u_{n-1}\|_Y + \tfrac{1}{2}\|u_{n-1} - u_{n-2}\|_Y \le 2\widehat{\gamma}\tau\,,$$

which implies $(u_{1/2}, \dots, u_{n+1/2}) \in E\left(n, \widehat{R}, \widehat{r}, 2\tau\widehat{\gamma}\right)$.

(b) With this we may proceed as in Lemma 4.2. The induction hypothesis and Lemma 2.6 yield as in (4.6)

$$\|u_{n+1}\|_Y \le \widehat{c}_0 e^{\widehat{\omega}_2 t_{n+1}}\big(\|u_0\|_Y + \widehat{T}_{\mathrm{mid}}\widehat{C}_{f,Y,\infty}\big) \le R,$$

and analogously

$$\|u_{n+1}\|_Z \le \widehat{c}_0 e^{\widehat{\omega}_2 t_{n+1}} \big(\|u_0\|_Z + \widehat{T}_{\mathrm{mid}} \widehat{C}_{f,Z,\infty}\big) \le r\,.$$

We close as in (4.8), (4.9), (4.10), and (4.11) to obtain

$$\begin{aligned}\|u_{n+1} - u_n\|_Y &= \big\|\big(e^{\tau \mathbf{A}_{n+1/2}} - I\big)\, u_n + \tau\varphi_1(\tau\mathbf{A}_{n+1/2})\mathbf{f}_{n+1/2}\big\|_Y \\ &\le \tau \big\|\mathbf{A}_{n+1/2}\varphi_1(\tau\mathbf{A}_{n+1/2})u_n\big\|_Y + \tau\widehat{c}_0 e^{\widehat{\omega}_2\tau}\widehat{C}_{f,Y,\infty} \\ &\le \widehat{\gamma}\tau\,,\end{aligned}$$

such that $(u_0,\dots,u_{n+1}) \in E\big(n+1, R, r, \gamma\tau\big)$ holds. ∎

5.2. Defects and global error. To shorten the notation, we define analogously to $u_{n+1/2}$ the exact extrapolation and the corresponding operator by

$$\underline{\widehat{u}}_{n+1/2} = \tfrac{1}{2}\,(3\widehat{u}_n - \widehat{u}_{n-1})\,, \quad \underline{\widehat{\mathbf{A}}}_{n+1/2} = \mathbf{A}(\underline{\widehat{u}}_{n+1/2}), \quad \underline{\widehat{\mathbf{f}}}_{n+1/2} = f(t_{n+1/2}, \underline{\widehat{u}}_{n+1/2}),$$

with $\underline{\widehat{u}}_{1/2} = u_0$. We further use the extrapolated error

$$e_{n+1/2} = \underline{\widehat{u}}_{n+1/2} - u_{n+1/2}\,,$$

and resolve this term at the very end. As mentioned before we start with a representation of the exact solution similar to Proposition 4.3 and derive a recursion for the global error. We further provide bounds on the defects.

PROPOSITION 5.3. *Let Assumptions 2.1, 2.4, 2.5, 3.2, and 3.3 be satisfied and consider the solution u given by Lemma 3.6 and numerical approximations $(u_n)_n$ given by (2.4). Then the global error satisfies the error recursion*

$$e_{n+1} = e^{\tau\mathbf{A}_{n+1/2}} e_n + \delta_n\,, \tag{5.2}$$

where the defects are bounded by

$$\begin{aligned}\|\delta_0\|_X &\le C_{\delta,X}\,\tau^2\, e^{\tau\widehat{\omega}_1}\,,\\ \|\delta_n\|_X &\le \big(C_{\sigma,X}\,\tau\,\|e_{n+1/2}\|_X + C_{\delta,X}\,\tau^3\big)\, e^{\tau\widehat{\omega}_1}\,, \qquad n\ge 1\,,\end{aligned}$$

with constants $C_{\sigma,X}, C_{\delta,X}$ only depending on $\|u''\|_{X,\infty}$, $\|u'\|_{Y,\infty}$, and $\|u\|_{Z,\infty}$, but are independent of τ, n and t_n.

Proof. Similar to Proposition 4.3, inserting $\mathbf{A}_{n+1/2}$ and $\mathbf{f}_{n+1/2}$ yields the differential equation

$$\begin{aligned}u'(t) &= \widehat{\mathbf{A}}(t)u(t) + \widehat{\mathbf{f}}(t)\\ &= \mathbf{A}_{n+1/2}u(t) + \mathbf{f}_{n+1/2}\\ &\quad + \Big(\underline{\widehat{\mathbf{A}}}_{n+1/2} - \mathbf{A}_{n+1/2}\Big)\, u(t) + \big(\underline{\widehat{\mathbf{f}}}_{n+1/2} - \mathbf{f}_{n+1/2}\big)\\ &\quad + \Big(\widehat{\mathbf{A}}_{n+1/2} - \underline{\widehat{\mathbf{A}}}_{n+1/2}\Big)\, u(t) + \big(\widehat{\mathbf{f}}_{n+1/2} - \underline{\widehat{\mathbf{f}}}_{n+1/2}\big)\\ &\quad + \Big(\widehat{\mathbf{A}}(t) - \widehat{\mathbf{A}}_{n+1/2}\Big)\, u(t) \quad + \big(\widehat{\mathbf{f}}(t) - \widehat{\mathbf{f}}_{n+1/2}\big)\\ &=: \mathbf{A}_{n+1/2}u(t) + \mathbf{f}_{n+1/2} + \sum_{i=1}^{6} \widetilde{\delta}_{n,i}(t)\,.\end{aligned}$$

As in (4.13), the variation-of-constants formula yields the representation

$$u(t_{n+1}) = e^{\tau \mathbf{A}_{n+1/2}} u(t_n) + \tau \varphi_1(\tau \mathbf{A}_{n+1/2}) \mathbf{f}_{n+1/2} + \delta_n \,, \tag{5.3}$$

where $\delta_n \coloneqq \delta_{n,1} + \ldots + \delta_{n,6}$. We split the proof into two parts. We first bound the four terms that require the Lipschitz bounds only. In the second part, we have to use the additional assumptions on the differentiability. The assertion on the defect from the first step is handled as in Proposition 4.3, and we omit the details.

(a) Along the lines of (4.14) we obtain

$$\|\delta_{n,1}\|_X \le \tau \widehat{k}_0 \widehat{L}_X e^{\tau \widehat{\omega}_1} \left\| e_{n+1/2} \right\|_X \|u\|_{Z,\infty}$$

as well as

$$\|\delta_{n,2}\|_X \le \tau \widehat{k}_0 e^{\tau \widehat{\omega}_1} \widehat{L}_{f,X} \left\| e_{n+1/2} \right\|_X \,.$$

We further derive as in (4.15)

$$\begin{aligned} \|\delta_{n,3}\|_X &\le \tau \widehat{k}_0 \widehat{L}_X \int_0^1 e^{(1-s)\tau \widehat{\omega}_1} \left\| u(t_{n+1/2}) - \underline{\widehat{u}}_{n+1/2} \right\|_X \|\widehat{u}_{n+s}\|_Z \, ds \\ &\le \tau^3 \widehat{k}_0 \widehat{L}_X e^{\tau \widehat{\omega}_1} \tfrac{3}{8} \|u''\|_{X,\infty} \|u\|_{Z,\infty} \,, \end{aligned}$$

as well as $\delta_{n,4}$ by

$$\|\delta_{n,4}\|_X \le \tau^3 \widehat{k}_0 e^{\tau \widehat{\omega}_1} \widehat{L}_{f,X} \tfrac{3}{8} \|u''\|_{X,\infty} \,,$$

where we used Taylor expansion of $u(t_{n+1/2})$ for both defects.

(b) Since the structure of the defects $\delta_{n,5}$ and $\delta_{n,6}$ is very similar, we will only prove the statement for $\delta_{n,5}$. We have

$$\delta_{n,5} = \int_0^\tau e^{(\tau - s)\mathbf{A}_{n+1/2}} d_n(t_n + s) \, ds, \tag{5.4}$$

with the function

$$d_n(t) \coloneqq \Big(\widehat{\mathbf{A}}(t) - \widehat{\mathbf{A}}_{n+1/2} \Big) u(t), \qquad d_n(t_{n+1/2}) = 0 \,. \tag{5.5}$$

Since we need to expand this term in the following, we compute

$$d_n'(t) = \widehat{\mathbf{A}}'(t) u(t) + \big(\widehat{\mathbf{A}}(t) - \widehat{\mathbf{A}}_{n+1/2} \big) u'(t) =: \dot{d}_{n,1}(t) + \dot{d}_{n,2}(t)$$

and observe

$$d_n'(t_{n+1/2}) = \dot{d}_{n,1}(t_{n+1/2}) = \widehat{\mathbf{A}}'(t_{n+1/2}) \widehat{u}_{n+1/2} \,.$$

We also need the derivative of $\dot{d}_{n,1}$ given by

$$\ddot{d}_{n,1}(t) \coloneqq \tfrac{d}{dt} \dot{d}_{n,1}(t) = \widehat{\mathbf{A}}''(t) u(t) + \widehat{\mathbf{A}}'(t) u'(t) \,.$$

Using (5.5) and integration by parts, we expand

$$
\begin{aligned}
d_n(t_n+s) &= \int_0^{s-\tau/2} \dot{d}_{n,1}(t_{n+1/2}+\sigma)\,d\sigma + \int_0^{s-\tau/2} \dot{d}_{n,2}(t_{n+1/2}+\sigma)\,d\sigma \\
&= \left(s-\tfrac{\tau}{2}\right)\dot{d}_{n,1}(t_{n+1/2}) + \int_0^{s-\tau/2} \left(s-\tfrac{\tau}{2}-\sigma\right)\ddot{d}_{n,1}(t_{n+1/2}+\sigma)\,d\sigma \\
&\quad + \int_0^{s-\tau/2} \dot{d}_{n,2}(t_{n+1/2}+\sigma)\,d\sigma\,.
\end{aligned}
$$

Plugging this in (5.4) gives

$$
\begin{aligned}
\delta_{n,5} &= \left(\int_0^{\tau} e^{(\tau-s)\mathbf{A}_{n+1/2}}\left(s-\tfrac{\tau}{2}\right)\,ds\right)\;\dot{d}_{n,1}(t_{n+1/2}) \\
&\quad + \int_0^{\tau} e^{(\tau-s)\mathbf{A}_{n+1/2}} \int_0^{s-\tau/2} \left(s-\tfrac{\tau}{2}-\sigma\right)\ddot{d}_{n,1}(t_{n+1/2}+\sigma)\,d\sigma\,ds \\
&\quad + \int_0^{\tau} e^{(\tau-s)\mathbf{A}_{n+1/2}} \int_0^{s-\tau/2} \dot{d}_{n,2}(t_{n+1/2}+\sigma)\,d\sigma\,ds \\
&= \delta_{n,5}^1+\delta_{n,5}^2+\delta_{n,5}^3.
\end{aligned}
$$

We estimate these terms separately. By integration by parts we obtain

$$
\delta_{n,5}^1 = \left(\frac{1}{2}\int_0^{\tau} e^{(\tau-s)\mathbf{A}_{n+1/2}}\left(s^2-\tau s\right)ds\right)\;\mathbf{A}_{n+1/2}\dot{d}_{n,1}(t_{n+1/2}).
$$

Since Lemma 3.5 implies

$$
\left\|\dot{d}_{n,1}(t_{n+1/2})\right\|_Y = \left\|\widehat{\mathbf{A}}'(t_{n+1/2})\widehat{u}_{n+1/2}\right\|_Y \le C_{YZ}^A\,\|u\|_{Z,\infty}\,, \tag{5.6}
$$

we estimate by Lemma 5.1, (2.9a), and (5.6)

$$
\begin{aligned}
\left\|\delta_{n,5}^1\right\|_X &\le \tfrac{1}{12}\widehat{k}_0\tau^3 e^{\tau\widehat{\omega}_1}\left\|\mathbf{A}_{n+1/2}\,\dot{d}_{n,1}(t_{n+1/2})\right\|_X \\
&\le \tfrac{1}{12}\widehat{k}_0\widehat{\nu}_X\widehat{\alpha}_{XY}\tau^3 e^{\tau\widehat{\omega}_1}\left\|\dot{d}_{n,1}(t_{n+1/2})\right\|_Y \\
&\le \left(\tfrac{1}{12}\widehat{k}_0\widehat{\nu}_X\widehat{\alpha}_{XY}C_{YZ}^A\,\|u\|_{Z,\infty}\right)\tau^3 e^{\tau\widehat{\omega}_1}\,.
\end{aligned}
$$

We further obtain by Lemma 3.5

$$
\begin{aligned}
\left\|\delta_{n,5}^2\right\|_X &\le \tfrac{1}{24}\widehat{k}_0\tau^3 e^{\tau\widehat{\omega}_1}\sup_{t\in[t_n,t_{n+1}]}\left\|\ddot{d}_{n,1}(t)\right\|_X \\
&\le \tfrac{1}{24}\widehat{k}_0\sup_{t\in[t_n,t_{n+1}]}\left(\left\|\widehat{\mathbf{A}}''(t)u(t)\right\|_X + \left\|\widehat{\mathbf{A}}'(t)u'(t)\right\|_X\right)\tau^3 e^{\tau\widehat{\omega}_1}\,, \\
&\le \tfrac{1}{24}\widehat{k}_0\left(C_{XZ}^A\,\|u\|_{Z,\infty} + C_{XY}^A\,\|u'\|_{Y,\infty}\right)\tau^3 e^{\tau\widehat{\omega}_1}\,,
\end{aligned}
$$

as well as

$$\begin{aligned}\left\|\delta_{n,5}^3\right\|_X &\le \tfrac{1}{4}\widehat{k}_0\tau^2 e^{\tau\widehat{\omega}_1} \sup_{t\in[t_n,t_{n+1}]} \left\|\dot{d}_{n,2}(t)\right\|_X \\ &\le \tfrac{1}{4}\widehat{k}_0 \sup_{t\in[t_n,t_{n+1}]} \left\|\big(\widehat{\mathbf{A}}(t)-\widehat{\mathbf{A}}_{n+1/2}\big)u'(t)\right\|_X \tau^2 e^{\tau\widehat{\omega}_1} \\ &\le \Big(\tfrac{1}{8}\widehat{k}_0 C_{XY}^A \left\|u'\right\|_{Y,\infty}\Big)\tau^3 e^{\tau\widehat{\omega}_1}\,.\end{aligned}$$

This gives the assertion for $\delta_{n,5}$. For $\delta_{n,6}$, Taylor expansion, integration by parts as for $\delta_{n,5}^1$, and the bounds provided in Assumption 3.2 yield to the desired estimate.

Subtracting (2.4) from (5.3) closes the proof. ∎

We can finally give a proof the error bound of the exponential midpoint rule.

Proof of Theorem 3.7. By (5.2) we resolve the error recursion as in (4.16) and use the bounds provided in Lemma 5.1 and Proposition 5.3. With the observation

$$\sum_{j=1}^{n}\left\|e_{j+1/2}\right\|_X \le 2\sum_{j=1}^{n}\left\|e_j\right\|_X\ ,$$

the bound in the X-norm is derived analogously to Theorem 3.1. For the convergence in the Y-norm, note that it is sufficient that an adaption of Corollary 4.4 remains valid, which is easily verified. ∎

6. Numerical experiments. We close this paper by illustrating our theoretical findings. We consider the model problem of Section 2.1, given by the quasilinear wave equation

$$\lambda(q)q'' = \Delta q + g(t,q,q'), \tag{6.1}$$

with $\lambda(q) = 1-\frac{1}{10}q^2$ and $g(t,q,q') = \frac{1}{5}q\cdot(q')^2 + r(t)$ where

$$r(t,x) = \sin\big((1+t)(1-|x|^2)^3\big)\,,$$

on the unit disc $\Omega\subseteq\mathbb{R}^2$ subject to homogeneous Dirichlet boundary conditions. Since we focused on the regularity of the solution in the error analysis, we chose the special initial position

$$q_0(x) = -\frac{1}{4}|x|^2\ln(-\ln(\rho(|x|^2)) + C_1(|x|^2-1) + C_2,$$

with $\rho=\frac{2}{5}$ and constants C_1 and C_2 such that $q_0=\Delta q_0 = 0$ holds on $\partial\Omega$. For this position, it holds $q_0\in H^3(\Omega)\setminus H^{3+\varepsilon}(\Omega)$ for any $\varepsilon>0$ due to the term

$$x\mapsto \ln(-\ln(\rho|x|^2))\in H^1(\Omega)\setminus L^\infty(\Omega).$$

For the second component we take $q_0'(x) = -(1-|x|^2)^2$, which is smooth but satisfies $\Delta q_0\ne 0$ on $\partial\Omega$.

6.1. Space discretization. We performed the space discretization by linear Lagrange finite elements. To this end we used the open source Python tool FEniCS [1, version 2018.1.0]. Denoting this ansatz space as $V_h\subseteq H_0^1(\Omega_h)$, with $\Omega_h\subseteq\Omega$, we then seek for $q_h(t) \in V_h$ which solves for all $\phi \in V_h$

$$\begin{aligned}\langle \lambda(q_h(t)) q_h''(t), \phi \rangle_{L^2(\Omega_h)} = &- \langle q_h(t), \phi \rangle_{H_0^1(\Omega_h)} \\ &+ \langle \lambda(q_h(t)) \mathcal{I}_h\big(\lambda(q_h(t))^{-1} g(t, q_h(t), q_h'(t))\big), \phi \rangle_{L^2(\Omega_h)},\end{aligned}$$

where $\mathcal{I}_h$ denotes the interpolation onto V_h. Hence, we solve the system of ordinary differential equations

$$M_h(q_h(t)) q_h''(t) = -L_h q_h(t) + M_h(q_h(t)) g_h(t, q_h(t), q_h'(t)) \tag{6.2}$$

with the mass and stiffness matrix

$$\big(M_h(q_h(t))\big)_{i,j} = \langle \lambda(q_h(t)) \phi_i, \phi_j \rangle_{L^2(\Omega_h)}, \qquad \big(L_h\big)_{i,j} = \langle \nabla \phi_i, \nabla \phi_j \rangle_{L^2(\Omega_h)},$$

and discretized nonlinearity

$$g_h(t, q_h(t)) = \mathcal{I}_h\big(\Lambda(q_h(t))^{-1} g(t, q_h(t), q_h'(t))\big).$$

6.2. Time discretization. We compute the approximations $q_h^{n+1} \approx q(t_{n+1})$ and $v_h^{n+1} \approx q'(t_{n+1})$ by solving a linear version of (6.2) exactly. For the Euler method, given previous approximations q_h^n, v_h^n, we replace $\lambda(q_h(t))$ by $\lambda(q_h^n)$. We obtain the equation

$$M_h(q_h^n) q_h''(t) = -L_h q_h(t) + M_h(q_h^n) g_h(t_n, q_h^n, v_h^n), \qquad t \in [t_n, t_n + \tau], \tag{6.3}$$

where the exact solution yields the next approximations q_h^{n+1}, v_h^{n+1}. We note that this is equivalent to first formulate (6.2) as a first-order system and then apply (2.3).

For the exponential midpoint rule we define the extrapolation terms $q_h^{n+1/2} = \frac{3}{2} q_h^n - \frac{1}{2} q_h^{n-1}$ and $v_h^{n+1/2} = \frac{3}{2} v_h^n - \frac{1}{2} v_h^{n-1}$ and replace $\lambda(q_h(t))$ by $\lambda(q_h^{n+1/2})$ to obtain

$$M_h(q_h^{n+1/2}) q_h''(t) = -L_h q_h(t) + M_h(q_h^{n+1/2}) g_h(t_{n+1/2}, q_h^{n+1/2}, v_h^{n+1/2}). \tag{6.4}$$

The exact solution of of (6.3) and (6.4) is then computed using rational Krylov methods to evaluate the trigonometric matrix functions as it was suggested in [13] and [16]. Under https://doi.org/10.5445/IR/1000131109, the code to reproduce the plots is available.

6.3. Results for the exponential Euler method and the exponential midpoint rule. We computed a reference solution with the midpoint rule on a fine grid with maximal diameter $h_{\mathrm{ref}} = 6 \cdot 10^{-3}$ and step-size $\tau_{\mathrm{ref}} = \frac{1}{360}$. The solution of (6.3) and (6.4) were computed on a coarser mesh with maximal diameter $h_{\max} = 10^{-2}$. and step-sizes τ were chosen such that the quotient $\frac{\tau}{\tau_{\mathrm{ref}}}$ is an integer.

In Figure 1, we depicted the error between the projection of the reference solution and the numerical approximations. The discrete $H_0^1(\Omega) \times L^2(\Omega)$ norm was computed with the mass and stiffness matrix to approximate the error in the X-norm. Since linear Lagrange finite elements are not H^2-conforming, we only included the error of the position in the $H_0^1(\Omega)$ to estimate the Y-norm. One can clearly observe the first- and second-order convergence in time derived in Theorems 3.1 and 3.7. We further mention that the error induced by the space discretization is only relevant in the regime below 10^{-3}.

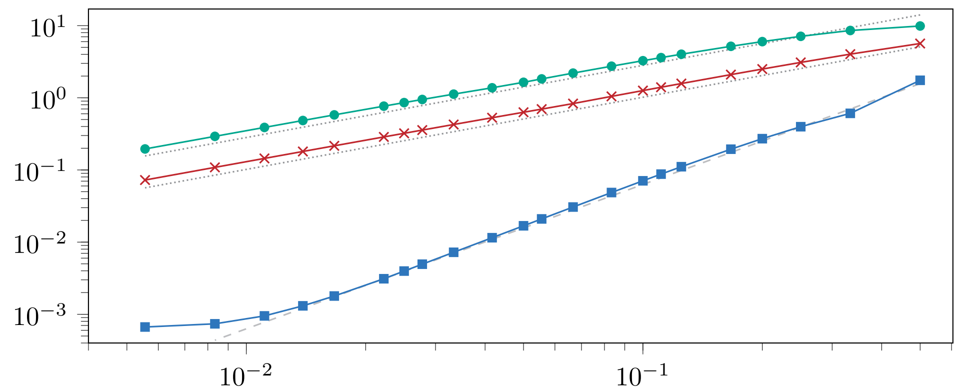

FIG. 1. *Discrete* $L^\infty\Big([0,1], H_0^1(\Omega)\times L^2(\Omega)\Big)$ *error (on the y-axis) of the numerical solution of* (6.1) *computed with* (6.3) *(middle line, red, crosses) and* (6.4) *(lower line, blue, squares) plotted against the step size* τ *(on the x-axis). Further, the discrete* $L^\infty\Big([0,1], H_0^1(\Omega)\Big)$ *error in the velocity* q_0' *computed with* (6.3) *is shown (upper line, green, dots). The gray lines indicate order one (dotted) and two (dashed).*

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