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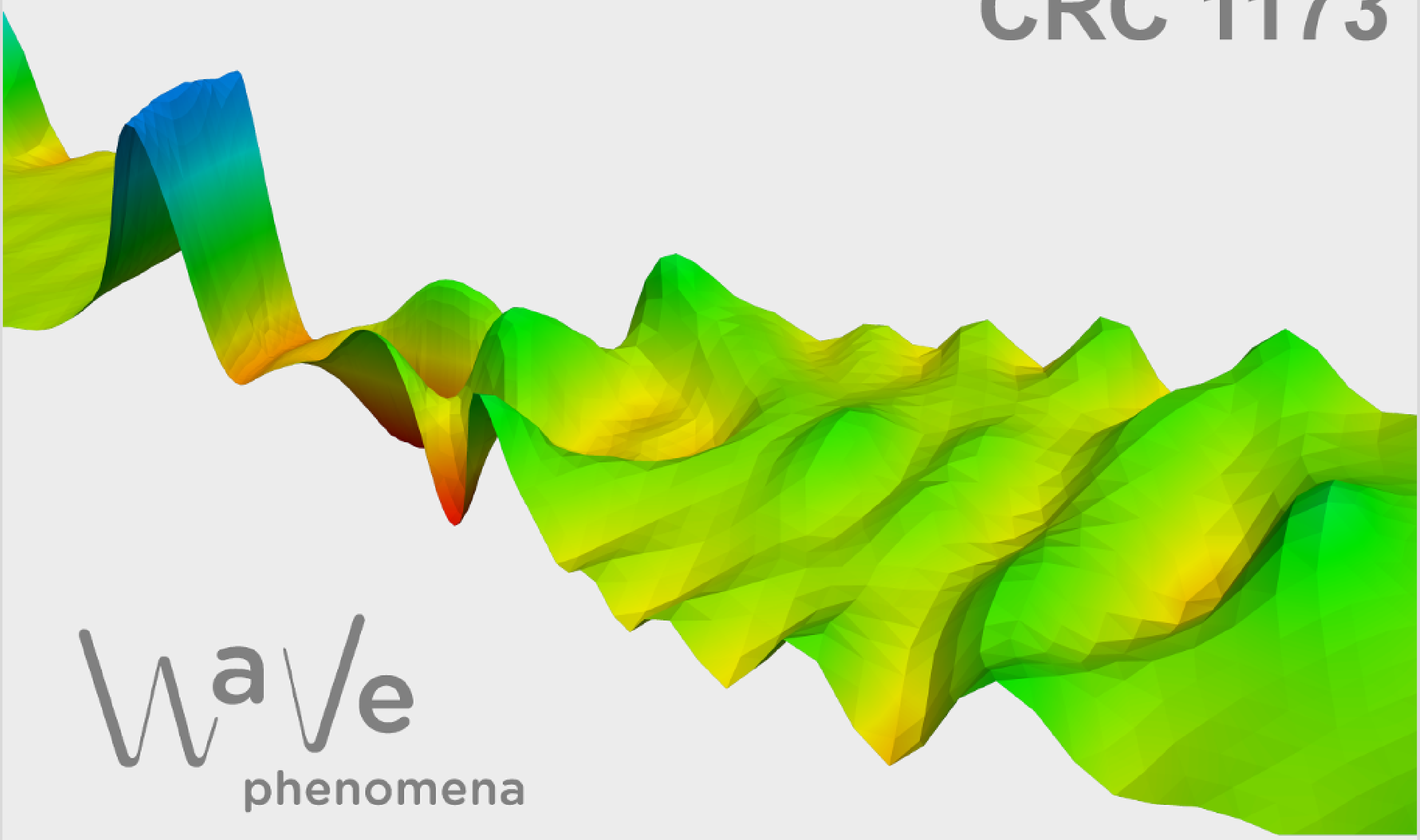

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RESOLVENT ESTIMATES FOR TIME-HARMONIC MAXWELL'S EQUATIONS IN THE PARTIALLY ANISOTROPIC CASE

ROBERT SCHIPPA

ABSTRACT. We prove resolvent estimates in L^p-spaces for time-harmonic Maxwell's equations in two spatial dimensions and in three dimensions in the partially anisotropic case. In the two-dimensional case the estimates are sharp. We consider anisotropic permittivity and permeability, which are both taken to be time-independent and spatially homogeneous. For the proof we diagonalize time-harmonic Maxwell's equations to equations involving Half-Laplacians. We apply these estimates to localize eigenvalues for perturbations by potentials and to derive a limiting absorption principle in intersections of L^p-spaces.

1. INTRODUCTION

Maxwell's equations describe electro-magnetic waves and consequently the propagation of light. We refer to the physics' literature for further query (cf. [17, 6]). Time-dependent Maxwell's equations in media, in three spatial dimensions, and in the absence of charges relate *electric and magnetic field* $(\mathcal{E}, \mathcal{B}) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ with *displacement and magnetizing fields* $(\mathcal{D}, \mathcal{H}) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ and the *electric and magnetic current* $(\mathcal{J}_e, \mathcal{J}_m) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$:

$$(1) \qquad \begin{cases} \partial_t \mathcal{D} &= \nabla \times \mathcal{H} + \mathcal{J}_e, \\ \partial_t \mathcal{B} &= -\nabla \times \mathcal{E} + \mathcal{J}_m. \end{cases} \qquad \nabla \cdot \mathcal{D} = \nabla \cdot \mathcal{B} = \nabla \cdot \mathcal{J}_e = \nabla \cdot \mathcal{J}_m = 0,$$

In physical applications, the magnetic current vanishes. Here we consider the more general case to highlight symmetry between the electric and magnetic field.

In the following we consider the time-harmonic, monochromatic ansatz

$$(2) \qquad \begin{aligned} \mathcal{D}(t,x) &= e^{i\omega t} D(x), \quad \mathcal{H}(t,x) = e^{i\omega t} H(x), \\ \mathcal{J}_e(t,x) &= e^{i\omega t} J_e(x), \quad \mathcal{J}_m(t,x) = e^{i\omega t} J_m(x) \end{aligned}$$

with $\omega \in \mathbb{R}$.

Furthermore, (1) is supplemented with the material laws

$$(3) \qquad \mathcal{D}(t,x) = \varepsilon \mathcal{E}(t,x), \quad \mathcal{B}(t,x) = \mu \mathcal{H}(t,x),$$

where $\varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\varepsilon_i, \mu > 0$. Requiring ε and μ to be symmetric and positive definite is a physically natural assumption. Liess stated already on [18, p. 63] that the material laws (3) in the general case are physically equivalent to ε diagonal and μ scalar. We plan to analyze the fully anisotropic case

$$\varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3), \quad \mu = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$$

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in forthcoming work [19], where we shall see in detail that we can reduce the analysis in the general case to scalar μ. Material laws with scalar μ are frequently used in optics (cf. [20, Section 2]). Then (1) becomes under (2) and (3)

$$P(\omega, D)\begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} J_e \\ J_m \end{pmatrix}, \quad P(\omega, D) = \begin{pmatrix} i\omega & -\mu^{-1}\nabla\times \\ \nabla\times(\varepsilon^{-1}\cdot) & i\omega \end{pmatrix}. \tag{4}$$

In this work we consider Maxwell's equations in two spatial dimensions and the partially anisotropic case in three dimensions. The time-dependent form of Maxwell's equations in two dimensions corresponds to electric and magnetic fields and currents of the form

$$\begin{aligned} \mathcal{E}_i(t,x) &= \mathcal{E}_i(t,x_1,x_2), \quad i=1,2; \quad \mathcal{E}_3 = 0; \\ \mathcal{B}_i &= 0, \quad i=1,2; \quad \mathcal{B}_3(t,x) = \mathcal{B}_3(t,x_1,x_2); \\ \mathcal{J}_{ei}(t,x) &= \mathcal{J}_{ei}(t,x_1,x_2), \quad i=1,2; \quad \mathcal{J}_{e3} = 0; \\ \mathcal{J}_{mi}(t,x) &= 0, \quad i=1,2; \quad \mathcal{J}_{m3}(t,x) = \mathcal{J}_{m3}(t,x_1,x_2). \end{aligned}$$

(1) simplifies to (cf. [1]):

$$\left\{ \begin{array}{ll} \partial_t \mathcal{D} & = \nabla_\perp \mathcal{H} + \mathcal{J}_e, \quad \nabla\cdot D = \nabla\cdot J_e = 0, \\ \partial_t \mathcal{B} & = -\nabla\times\mathcal{E} + \mathcal{J}_m, \end{array} \right. \tag{5}$$

where $\mathcal{D}, \mathcal{E}, \mathcal{J}_e : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{B}, \mathcal{H}, \mathcal{J}_m : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$, $\nabla_\perp = (\partial_2, -\partial_1)^t$, and we suppose (3) with $\mu > 0$, and $(\varepsilon^{ij})_{i,j} \in \mathbb{R}^{2\times 2}$ denoting a symmetric, positive definite matrix. Note that in the two-dimensional case $\mathcal{B}$, $\mathcal{H}$, and $\mathcal{J}_m$ are regarded as scalar quantities, which are not required to be divergence-free.
We can rewrite (5) under (2) and (3) as
(6)

$$P(\omega, D)\begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} J_e \\ J_m \end{pmatrix}, \quad P(\omega, D) = \begin{pmatrix} i\omega & 0 & -\mu^{-1}\partial_2 \\ 0 & i\omega & \mu^{-1}\partial_1 \\ \partial_2\varepsilon_{11} - \partial_1\varepsilon_{21} & \partial_2\varepsilon_{12} - \partial_1\varepsilon_{22} & i\omega \end{pmatrix},$$

denoting with ε_{ij} the components of the inverse of ε.

In the following let $d \in \{2,3\}$, $m(2) = 3$, $m(3) = 6$, and

$$\begin{aligned} L^p_0(\mathbb{R}^2) &= \{(f_1,f_2,f_3) \in L^p(\mathbb{R}^2)^3 \,:\, \partial_1 f_1 + \partial_2 f_2 = 0 \text{ in } \mathcal{S}'(\mathbb{R}^2)\}, \\ L^p_0(\mathbb{R}^3) &= \{(f_1,\dots,f_6) \in L^p(\mathbb{R}^3)^6 \,:\, \nabla\cdot(f_1,f_2,f_3) = \nabla\cdot(f_4,f_5,f_6) = 0 \text{ in } \mathcal{S}'(\mathbb{R}^3)\}. \end{aligned}$$

In this paper we are concerned with the resolvent estimates

$$\|(D,H)\|_{L^q_0(\mathbb{R}^d)} = \|P(\omega,D)^{-1}(J_{e0},J_{m0})\|_{L^q_0(\mathbb{R}^d)} \lesssim \kappa_{p,q}(\omega)\|(J_{e0},J_{m0})\|_{L^p_0(\mathbb{R}^d)}. \tag{7}$$

However, as will be clear from perceiving $P(\omega, D)$ as a Fourier multiplier, $P(\omega,D)^{-1}$ cannot even be explained in the distributional sense. The remedy will be to consider $\omega \in \mathbb{C}\backslash\mathbb{R}$ and derive estimates independent of the distance to the real axis. Then limits can be explained. This is referred to as limiting absorption principle. We shall further derive explicit formulae for the resulting limits. To find the estimates, we diagonalize $P(\omega,D)^{-1}$, which leads us to consider resolvent estimates for the Half-Laplacian.

We digress for a moment to elaborate on L^p-L^q-estimates for the fractional Laplacian and applications. Let $s \in (0,d)$. For $\omega \in \mathbb{C}\backslash[0,\infty)$ we consider the resolvents as Fourier multiplier:

$$((-\Delta)^{s/2}-\omega)^{-1}f = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{\hat{f}(\xi)}{(|\xi|^s-\omega)^{-1}} e^{ix.\xi} d\xi \tag{8}$$

for $f:\mathbb{R}^d \to \mathbb{C}$ in some suitable a priori class, e.g., $f \in \mathcal{S}(\mathbb{R}^d)$. In the present context, resolvent estimates for the Half-Laplacian $\|((-\Delta)^{\frac{1}{2}}-\omega)^{-1}\|_{p\to q}$ are most important. There is a huge body of literature on resolvent estimates for the Laplacian $(-\Delta-\omega)^{-1}: L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$. This is due to versatile applications to uniform Sobolev estimates and unique continuation (cf. [14]), the localization of eigenvalues for Schrödinger operators with complex potential (cf. [5, 7, 8]), or limiting absorption principles (cf. [11]). Kenig–Ruiz–Sogge [14] showed that uniform resolvent estimates in $\omega \in \mathbb{C}\backslash[0,\infty)$ for $d \geq 3$ hold if and only if

$$\frac{1}{p}-\frac{1}{q}=\frac{2}{d} \text{ and } \frac{2d}{d+3} < p < \frac{2d}{d+1}. \tag{9}$$

By homogeneity and scaling, we find

$$\|(-\Delta-\omega)^{-1}\|_{p\to q} = |\omega|^{-1+\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\big(-\Delta-\frac{\omega}{|\omega|}\big)^{-1}\|_{p\to q} \quad \forall \omega \in \mathbb{C}\backslash[0,\infty). \tag{10}$$

Thus, it suffices to consider $|\omega|=1$ to discuss boundedness. To the best of the author's knowledge, Kwon–Lee [15] showed the currently widest range of resolvent estimates for the fractional Laplacian outside the uniform boundedness range. To state the range of admissible L^p-L^q-estimates, we shall use notations from [15]. Let $I^2 = \{(x,y)\in\mathbb{R}^2 \,|\, 0\le x,y\le 1\}$, and let $(x,y)'=(1-x,1-y)$ for $(x,y)\in I^2$. For $\mathcal{R}\subseteq I$ we set $\mathcal{R}'=\{(x,y)'\,|\,(x,y)\in\mathcal{R}\}$.
For $X_1,\dots,X_n \in I^2$, $[X_1,\dots,X_n]$ denotes the convex hull. We set $(X,Y) = [X,Y]\backslash\{X,Y\}$ and $[X,Y) = [X,Y]\backslash\{Y\}$.
Kwon–Lee [15, Proposition 6.1] showed boundedness of the resolvent of the fractional Laplacian $((-\Delta)^{\frac{s}{2}}-z)^{-1}$ for $(1/p,1/q)\in\mathcal{R}_0^{\frac{s}{2}}$ with

$$\mathcal{R}_0^{\frac{s}{2}} = \mathcal{R}_0^{\frac{s}{2}}(d) = \{(x,y)\in I^2 \,|\, 0\le x-y\le \frac{s}{d}\}\backslash\{(1,\frac{d-s}{d}),(\frac{s}{d},0)\}.$$

Uniform estimates for the Laplacian are due to Gutiérrez [11]. She showed that uniform estimates for $\omega\in\{z\in\mathbb{C}: |z|=1,\, z\neq 1\}$ hold if and only if $(1/p,1/q)$ lies in the set

$$\mathcal{R}_1 = \mathcal{R}_1(d) = \{(x,y)\in\mathcal{R}_0^1(d) : \frac{2}{d+1}\le x-y\le\frac{2}{d},\, x>\frac{d+1}{d},\, y<\frac{d-1}{d}\}. \tag{11}$$

Failure outside this range was known before (cf. [14, 2]) due to the connection to Bochner-Riesz operators with negative index. Clearly, there are more estimates available outside of (11) if one allows for dependence on ω. E.g.,

$$\|(-\Delta-\omega)^{-1}\|_{L^2\to L^2} \sim \operatorname{dist}(\omega,[0,\infty))^{-1}. \tag{12}$$

Kwon–Lee [15] analyzed estimates outside the uniform boundedness range in detail and covered a broad range. Still, estimates with dependence on z can be used to localize eigenvalues for Schrödinger operators with complex potentials (cf. [5]).

Returning to (1), Cossetti–Mandel analyzed the isotropic case $\varepsilon,\mu>0$, also in the spatially inhomogeneous case in [4]. In the isotropic case, iterating (1) and using the divergence conditions yields Helmholtz equations for D and H. This approach was carried out in [4]. In the anisotropic case this strategy becomes less straight-forward. Instead we choose to diagonalize the Fourier multiplier to get into the position to use resolvent estimates for the fractional Laplacian. A similar diagonalization was used for the derivation of Strichartz estimates in [21]. Kwon–Lee–Seo [16] previously used a diagonalization to prove resolvent estimates for the Lamé operator. In three spatial dimensions we consider $\mu > 0$ and $\varepsilon = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Diagonalizing the symbol to operators involving the Half-Laplacian works in the *partially anisotropic case*, i.e.,

$$\#\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \leq 2. \tag{13}$$

It turns out that in the fully anisotropic case

$$\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3 \neq \varepsilon_1,$$

diagonalizing the multiplier introduces singularities, and this case has to be treated differently. This will be subject of forthcoming work (cf. [19]).

After finding the resolvent estimates stated below in Theorem 1.1, we shall see that a limiting absorption principle $L^p \to L^q$ fails. We will see that a suitable local result still holds and estimates $L^{p_1} \cap L^{p_2} \to L^q$. Furthermore, localization of eigenvalues for perturbartions is deduced.

For $d \in \{2, 3\}$ and $(1/p, 1/q) \in I^2$, define

$$\gamma_{p,q} = \gamma_{p,q}(d) = \max\{0, 1 - \frac{d+1}{2}(\frac{1}{p} - \frac{1}{q}), \frac{d+1}{2} - \frac{d}{p}, \frac{d}{q} - \frac{d-1}{2}\}.$$

To state resolvent estimates more precisely, we introduce the regions from [15, p. 1421] specifying the value of $\gamma_{p,q}(d)$:

$$\begin{aligned}
\mathcal{P} &= \mathcal{P}(d) = \{(x,y) \in I^2 \,:\, x - y \geq \frac{2}{d+1}, \quad x > \frac{d+1}{2d}, \quad y < \frac{d-1}{2d}\},\\
\mathcal{T} &= \mathcal{T}(d) = \{(x,y) \in I^2 \,:\, 0 \leq x - y < \frac{2}{d+1}, \quad \frac{d-1}{d+1}(1-x) \leq y \leq \frac{d+1}{d-1}(1-x)\},\\
\mathcal{Q} &= \mathcal{Q}(d) = \{(x,y) \in I^2 \,:\, y < \frac{d-1}{d+1}(1-x), \quad y \leq x < \frac{d+1}{2d}\}.
\end{aligned}$$

We set additionally

$$\begin{aligned}
&H = (\frac{1}{2}, \frac{1}{2}), \quad D = (\frac{d-1}{2d}, \frac{d-1}{2d}), \quad D' = (\frac{d+1}{2d}, \frac{d+1}{2d}), \quad P_* = (\frac{3}{10}, \frac{3}{10}),\\
&B = (\frac{d+1}{2d}, \frac{(d-1)^2}{2d(d+1)}), \quad B' = (\frac{d^2+4d-1}{2d(d+1)}, \frac{d-1}{2d}), \quad P_o = (\frac{2}{5}, \frac{3}{10})
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}_1^{\frac{s}{2}} &= \mathcal{R}_1^{\frac{s}{2}}(d) = \mathcal{P}(d) \cap \mathcal{R}_0^{\frac{s}{2}}(d),\\
\mathcal{R}_2^{\frac{s}{2}} &= \mathcal{R}_2^{\frac{s}{2}}(d) = (\mathcal{T}(d) \cap \mathcal{R}_0^{\frac{s}{2}}(d)) \backslash ([D, H) \cup [D', H)),\\
\mathcal{R}_3^{\frac{s}{2}} &= \mathcal{R}_3^{\frac{s}{2}}(d) = \mathcal{Q}(d) \cap \mathcal{R}_0^{\frac{s}{2}}(d).
\end{aligned}$$

Set

$$\begin{aligned}
\kappa_{p,q}^{(\frac{1}{2})}(z) &= |z|^{-1+d\left(\frac{1}{p}-\frac{1}{q}\right)+\gamma_{p,q}} \text{dist}(z, [0,\infty))^{-\gamma_{p,q}},\\
\kappa_{p,q}(z) &= |z|^{-1+d\left(\frac{1}{p}-\frac{1}{q}\right)+\gamma_{p,q}} \text{dist}(z, \mathbb{R}))^{-\gamma_{p,q}}.
\end{aligned}$$

Kwon–Lee [15, Conjecture 3, p. 1462] conjectured that

$$\kappa_{p,q}^{(\frac{1}{2})}(z) \sim_{p,q,d} \|((-\Delta)^{1/2} - z)^{-1}\|_{p\to q} \tag{14}$$

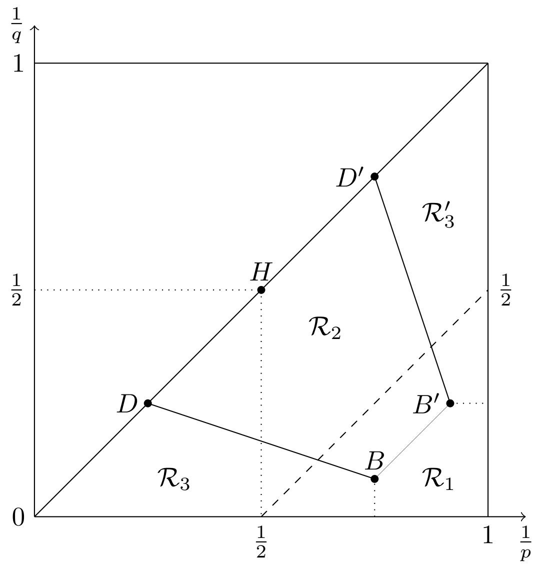
in $\big(\bigcup_{i=1}^3 \mathcal{R}_i^{\frac{1}{2}}\big) \cup \big(\mathcal{R}_3^{\frac{1}{2}}\big)'$. For $d=3$ set

$$\tilde{\mathcal{R}}_2 = [B, B', P_o', H, P_o] \backslash ([P_o, H) \cup [P_o', H) \cup [B, B']), \quad \tilde{\mathcal{R}}_3 = \mathcal{R}_3 \backslash [D, P_o, P_*],$$

$$\tilde{\mathcal{R}}_2^{\frac{1}{2}} = \mathcal{R}_0^{\frac{1}{2}}(d) \cap \tilde{\mathcal{R}}_2(d), \quad \tilde{\mathcal{R}}_3^{\frac{1}{2}} = \mathcal{R}_3^{\frac{1}{2}}(d) \backslash [D, P_0, P_*].$$

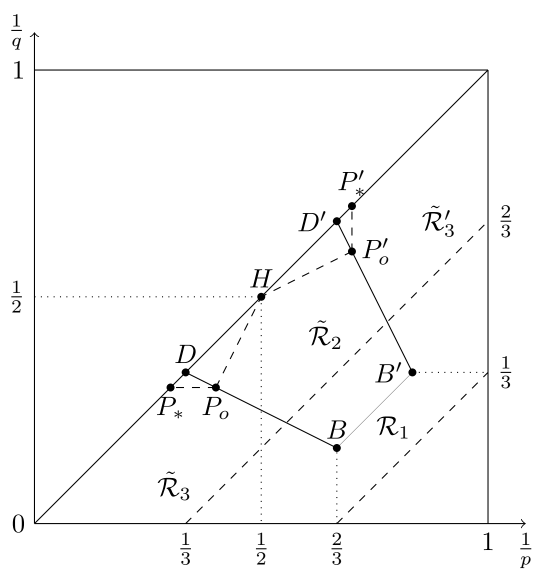
Kwon–Lee [15, Theorem 6.2, p. 1462] verified the conjecture for $d=2$ and for $d=3$ in the restricted range in $\mathcal{R}_1^{\frac{1}{2}} \cup \big(\bigcup_{i=2}^3 \tilde{\mathcal{R}}_i^{\frac{1}{2}}\big) \cup (\tilde{\mathcal{R}}_3^{\frac{1}{2}})'$.

For $d=2$ the regions for the Laplacian look as follows:



The dashed line indicates the boundary of $\mathcal{R}_0^{\frac{1}{2}}$, by which the regions for resolvent estimates of the Laplacian have to be intersected to find the corresponding regions for the Half-Laplacian.

For $d=3$ we have the following regions:



Here, the dashed lines further exclude regions close to the diagonal as the conjecture stated above is unproved in three dimensions.

We prove the following for time-harmonic Maxwell's equations:

Theorem 1.1. *Let $1 < p, q < \infty$, $d \in \{2,3\}$, and $\omega \in \mathbb{C}\backslash\mathbb{R}$. Let $P(\omega, D)$ as in (6) for $d = 2$, and as in (4) for $d = 3$. Then, $P(\omega, D)^{-1} : L^p_0(\mathbb{R}^d) \to L^q_0(\mathbb{R}^d)$ is bounded if and only if $(1/p, 1/q) \in \mathcal{R}_0^{\frac{1}{2}}(d)$.*
1. If $d = 2$, then

$$\|P(\omega, D)^{-1}\|_{L^p_0(\mathbb{R}^d)\to L^q_0(\mathbb{R}^d)} \sim \kappa_{p,q}(\omega) \tag{15}$$

is true for $(1/p, 1/q) \in \mathcal{R}_0^{\frac{1}{2}}(2)$.

2. If $d = 3$ with ε satisfying (13), then (15) is true for $(1/p, 1/q) \in \mathcal{I}^3 = \big(\bigcup_{i=1}^3 \tilde{\mathcal{R}}_i^{\frac{1}{2}}(d)\big) \cup \big(\tilde{\mathcal{R}}_3^{\frac{1}{2}}(d)\big)'$.

Contrary to the Laplacian, we cannot include the boundary of $\mathcal{I}$ as Riesz transforms are involved in the proof of the boundedness. It is well-known that the Riesz transforms are bounded on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, but neither on L^1 nor on L^∞. If the bound of the resolvent is independent of $\text{dist}(z, \mathbb{R})$, we can consider the limit $z \to 0$. This leads to a (global) limiting absorption principle. However, since $\gamma_{p,q} > 0$ for p and q as in Theorem 1.1, we have the following:

Corollary 1.2. *There is no global Limiting Absorption Principle for (5) or (1).*

We shall introduce the notion of a local LAP for functions with compact frequency support. Roughly speaking, in the local part the resolvent estimates are equivalent to resolvent estimates for the Laplacian, and uniform estimates $L^{p_1} \to L^q$ are possible. In the global part the multiplier is smooth, but provides merely the smoothing of the Half-Laplacian. We use different $L^{p_2} \to L^q$-estimates for this region. This gives uniform $L^{p_1} \cap L^{p_2} \to L^q$-estimates for z in a compact set away from the origin and a limiting absorption principle in the same spaces.

Theorem 1.3 (Local LAP for Time-Harmonic Maxwell's equations)**.** *Let $1 < p_1, p_2, q < \infty$, and let $d \in \{2,3\}$. If $(1/p_1, 1/q) \in \mathcal{P}(d)$, $(1/p_2, 1/q) \in \mathcal{R}_0^{\frac{1}{2}}(d)$, then $P(\omega, D)^{-1} : L^{p_1}_0(\mathbb{R}^d) \cap L^{p_2}_0(\mathbb{R}^d) \to L^q_0(\mathbb{R}^d)$ is bounded uniformly for $\omega \in \mathbb{C}\backslash\mathbb{R}$ in a compact set away from the origin. Furthermore, for $\omega \in \mathbb{R}\backslash 0$ there are limiting operators $P_\pm(\omega) : L^{p_1}_0(\mathbb{R}^d) \cap L^{p_2}_0(\mathbb{R}^d) \to L^q_0(\mathbb{R}^d)$ such that $(D, B) = P_\pm(\omega)(J_e, J_m)$ satisfy*

$$P(\omega, D)(D, B) = (J_e, J_m) \tag{16}$$

in $\mathcal{S}'(\mathbb{R}^d)^{m(d)}$.

At last, we use the ω-dependent resolvent estimates to localize eigenvalues for operators $P(\omega, D) + V$ acting in L^q. For this purpose, we consider for $\ell > 0$ and $(1/p, 1/q) \in \bigcup_{i=2}^3 \tilde{\mathcal{R}}_i^{\frac{1}{2}} \cup \big(\tilde{\mathcal{R}}_3^{\frac{1}{2}}\big)'$ the region, where uniform resolvent estimates are possible:

$$\begin{aligned} \mathcal{Z}_{p,q}(\ell) &= \{\omega \in \mathbb{C}\backslash\mathbb{R} \,:\, \kappa_{p,q}(\omega) \le \ell\} \\ &= \{\omega \in \mathbb{C}\backslash\mathbb{R} \,:\, |\omega|^{-\alpha_{p,q}} |\omega|^{\gamma_{p,q}} |\Im\omega|^{-\gamma_{p,q}} \le \ell\}, \quad \alpha_{p,q} = 1 - d\big(\frac{1}{p} - \frac{1}{q}\big). \end{aligned} \tag{17}$$

Describing the regions, we start with observing the symmetry in the real and imaginary part. For $\alpha_{p,q} = 0$, $\ell < 1$, we find $\mathcal{Z}_{p,q}(\ell) = \emptyset$. For $\ell \ge 1$, $\mathcal{Z}_{p,q}(\ell)$ describes a cone around the y-axis with aperture getting larger. For $\alpha_{p,q} > 0$ the boundaries become slightly curved. Pictorial representations for $\Re\omega > 0$ were provided in [15, Figures 9 (a)-(c)]. For the left half plane, the region is obtained by reflection along the imaginary axis.

Let C be the constant such that

$$\|P(\omega, D)^{-1}\|_{L^p_0(\mathbb{R}^d) \to L^q_0(\mathbb{R}^d)} \leq C\kappa_{p,q}(z). \tag{18}$$

Corollary 1.4. *Let $d \in \{2, 3\}$, $\ell > 0$. For $d = 2$ let $(1/p, 1/q) \in \bigcup_{i=2}^3 \mathcal{R}_i^{\frac{1}{2}} \cup (\mathcal{R}_3^{\frac{1}{2}})'$ and for $d = 3$, $(1/p, 1/q) \in \bigcup_{i=2}^3 \tilde{\mathcal{R}}_i^{\frac{1}{2}} \cup \tilde{\mathcal{R}}_3'$. Let $C > 0$ be as in (18). Suppose that there is $t \in (0, 1)$ such that*

$$\|V\|_{\frac{pq}{q-p}} \leq t(C\ell)^{-1}.$$

If $E \in \mathbb{C}\backslash\mathbb{R}$ is an eigenvalue of $P + V$ acting in L^q_0, then E must lie in $\mathbb{C}\backslash\mathcal{Z}_{p,q}(\ell)$.

Proof. The short argument is standard by now (cf. [15, 16]), but contained for the sake of completeness. Let $u \in L^q_0(\mathbb{R}^d)$ be an eigenfunction of $P + V$ with eigenvalue $E \in \mathbb{C}\backslash\mathbb{R}$ and suppose that $E \in \mathcal{Z}_{p,q}(\ell)$. By Hölder's inequality, we find $-(P - E)u = (V - (P - E + V))u = Vu \in L^p$. By definition of $\mathcal{Z}_{p,q}(\ell)$, we find

$$\|(P - E)^{-1}\|_{p\to q} \leq C\kappa_{p,q}(E) \leq C\ell.$$

By the triangle and Hölder's inequality, we find

$$\|(P-E)^{-1}(P-E)u\|_q \leq C\ell(\|(P-E+V)u\|_p + \|Vu\|_p) \leq C\ell\|V\|_{\frac{pq}{q-p}}\|u\|_q \leq t\|u\|_q,$$

which implies $u = 0$ as $t < 1$. Hence, $E \notin \mathcal{Z}_{p,q}(\ell)$. □

We end the introduction with remarks on further applications of the diagonalization: It appears as if the diagonalization can also be applied to time-harmonic Maxwell's equations with periodic boundary conditions. We refer to the works [3, 12] for context. Finally, it would be interesting to investigate L^p-L^q-Carleman estimates, which were shown to fail for the Lamé operator in [16].

Outline of the paper. In Section 2 we diagonalize time-harmonic Maxwell's equations in Fourier space to reduce the resolvent estimates to estimates for the Half-Laplacian. We also give examples for lower resolvent bounds in terms of the Half-Laplacian. In Section 3 we argue how an LAP fails in L^p-spaces, but can be salvaged in intersections of L^p-spaces. We also give solution formulae derived from the local LAP.

2. Reduction to resolvent estimates for the Half-Laplacian

Let $\omega \in \mathbb{C}\backslash\mathbb{R}$. We diagonalize $P(\omega, D)$ as in (6) and as in (4) in the partially anisotropic case. We shall see that the transformation matrices are essentially Riesz transforms. This allows to bound the resolvents with estimates for the Half-Laplacian. We will make repeated use of the following multiplier theorem:

Theorem 2.1 ([9, Theorem 6.2.7, p. 446])**.** *Let $m : \mathbb{R}^n\backslash 0 \to \mathbb{C}$ be a bounded function that satisfies*

$$|\partial^\alpha m(\xi)| \leq D_\alpha |\xi|^{-|\alpha|} \qquad (\xi \in \mathbb{R}^n\backslash 0) \tag{19}$$

for $|\alpha| \leq \lfloor \frac{n}{2} \rfloor + 1$. Then, $\mathfrak{m}_p : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ given by $f \mapsto (m\hat{f})\check{}$ defines a bounded mapping with

$$\|\mathfrak{m}_p\|_{L^p\to L^p} \leq C_n \max(p, (p-1)^{-1})(A + \|m\|_{L^\infty}), \tag{20}$$

where

$$A = \max(D_\alpha,\ |\alpha| \leq \lfloor \frac{n}{2} \rfloor + 1).$$

As pointed out in [9], for a zero-homogeneous function

$$m(\lambda \xi) = \lambda^{i\tau} m(\xi), \quad \tau \in \mathbb{R}, \tag{21}$$

m is an L^p-multiplier for $1 < p < \infty$. Differentiating the above display with respect to ∂_ξ^α, we obtain

$$\lambda^{|\alpha|} \partial_\xi^\alpha m(\lambda \xi) = \lambda^{i\tau} \partial_\xi^\alpha m(\xi)$$

and (19) is satisfied with $C_\alpha = \sup_{|\theta|=1} |\partial^\alpha m(\theta)|$.

2.1. **Maxwell's equations in 2d.** Let $u = (D_1, D_2, B)$. We denote $(\varepsilon^{-1})_{ij} = (\varepsilon_{ij})_{i,j}$. To reduce to estimates for the Half-Laplacian, we diagonalize the symbol:

$$(P(\omega, D)u)\widehat{(\xi)} = p(\omega, \xi)\hat{u}(\xi) = i \begin{pmatrix} \omega & 0 & -\xi_2 \mu^{-1} \\ 0 & \omega & \xi_1 \mu^{-1} \\ \xi_2 \varepsilon_{11} - \xi_1 \varepsilon_{12} & \xi_2 \varepsilon_{12} - \xi_1 \varepsilon_{22} & \omega \end{pmatrix} \hat{u}(\xi). \tag{22}$$

Let

$$\tilde{\varepsilon} = \begin{pmatrix} \varepsilon_{22} & -\varepsilon_{12} \\ -\varepsilon_{12} & \varepsilon_{11} \end{pmatrix}$$

denote the adjugate matrix of ε^{-1}. Let $\|\xi\|_{\varepsilon'}^2 = \langle \xi, \mu^{-1} \tilde{\varepsilon} \xi \rangle$, and $\xi' = \xi / \|\xi\|_{\varepsilon'}$. p can be diagonalized to

$$d(\omega, \xi) = i \mathrm{diag}(\omega, \omega - \|\xi\|_{\varepsilon'}, \omega + \|\xi\|_{\varepsilon'}). \tag{23}$$

We align the corresponding eigenvectors as columns to

$$m(\xi) = \begin{pmatrix} \varepsilon_{22}\xi_1' - \varepsilon_{12}\xi_2' & -\xi_2' \mu^{-1} & \xi_2' \mu^{-1} \\ \varepsilon_{11}\xi_2' - \varepsilon_{12}\xi_1' & \xi_1' \mu^{-1} & -\xi_1' \mu^{-1} \\ 0 & -1 & -1 \end{pmatrix}. \tag{24}$$

For the inverse matrix we compute

$$m^{-1}(\xi) = \begin{pmatrix} \mu^{-1}\xi_1' & \mu^{-1}\xi_2' & 0 \\ \frac{\xi_1'\varepsilon_{21} - \xi_2'\epsilon_{11}}{2} & \frac{\varepsilon_{22}\xi_1' - \varepsilon_{21}\xi_2'}{2} & -\frac{1}{2} \\ \frac{\xi_2'\varepsilon_{11} - \xi_1'\varepsilon_{12}}{2} & \frac{\xi_2'\varepsilon_{12} - \xi_1'\varepsilon_{22}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{25}$$

With the above, we have

$$p(\omega, \xi) = m(\xi) d(\omega, \xi) m^{-1}(\xi). \tag{26}$$

We observe that $(m^{-1}(D)(u))_1 = 0$ by the divergence condition. Moreover, $m^{-1}(D)$ and $m(D)$ are uniformly bounded from $L^p \to L^p$ for $1 < p < \infty$ with a constant only depending on p, ε, μ as the entries are linear combinations of Riesz transformations after change of variables.
Therefore, we find

$$\|P(\omega, D)^{-1}\|_{L^p \to L^q} \lesssim \|d_+(\omega, D)\|_{L^p \to L^q} + \|d_-(\omega, D)\|_{L^p \to L^q} \tag{27}$$

with

$$d_\pm(\omega, D) : L^p(\mathbb{R}^2) \to L^q(\mathbb{R}^2), \quad (d_\pm f)\widehat{(\xi)} = \frac{1}{\omega \pm \|\xi\|_{\varepsilon'}} \hat{f}(\xi). \tag{28}$$

These are in the scope of the resolvent estimates from [15] yielding

$$\begin{aligned}\|P(\omega,D)^{-1}\|_{L^p\to L^q} &\lesssim |\omega|^{-1+2\left(\frac{1}{p}-\frac{1}{q}\right)+\gamma_{p,q}} \\ &\qquad \times (\operatorname{dist}(\omega,[0,\infty))^{-\gamma_{p,q}} + \operatorname{dist}(\omega,(-\infty,0])^{-\gamma_{p,q}}) \\ &\lesssim |\omega|^{-1+2\left(\frac{1}{p}-\frac{1}{q}\right)+\gamma_{p,q}} \operatorname{dist}(\omega,\mathbb{R})^{-\gamma_{p,q}}\end{aligned}$$

for $1<p,q<\infty$, $(1/p,1/q)\in\mathcal{R}_0^{\frac{1}{2}}$.

To show the necessary part, we shall see that

$$\|P(\omega,D)^{-1}\|_{L^p\to L^q} \sim \|((-\Delta)^{\frac{1}{2}}+\omega)^{-1}\|_{L^p\to L^q} + \|((-\Delta)^{\frac{1}{2}}-\omega)^{-1}\|_{L^p\to L^q}. \tag{29}$$

For this we consider generalized Riesz transforms

$$(\mathcal{R}_j^{\varepsilon'} f)\widehat{\ }(\xi) = \frac{\xi_j}{\|\xi\|_{\varepsilon'}}\hat{f}(\xi). \tag{30}$$

These satisfy for $1<p<\infty$

$$\|f\|_{L^p(\mathbb{R}^2)} \sim_{p,\varepsilon,\mu} \|\mathcal{R}_1^{\varepsilon'} f\|_{L^p(\mathbb{R}^2)} + \|\mathcal{R}_2^{\varepsilon'} f\|_{L^p(\mathbb{R}^2)}. \tag{31}$$

In fact, as already used above, $\|\mathcal{R}_j^{\varepsilon'} f\|_{L^p} \lesssim_{p,\varepsilon,\mu} \|f\|_{L^p}$ for $1<p<\infty$ as a consequence of Theorem 2.1. For the reverse bound, we decompose $f=f_1+f_2$ via a smooth partition of unity such that $|\xi_i|\gtrsim\|\xi\|_{\varepsilon'}$ for $(\xi_1,\xi_2)\in\operatorname{supp}(\hat{f}_i)$. Let χ_i be zero homogeneous Fourier multipliers such that $f_i=\chi_i f$. Set $((\mathcal{R}_i^{\varepsilon'})^{-1}f)\widehat{\ }(\xi) = \frac{\|\xi\|_{\varepsilon'}}{\xi_i}\hat{f}(\xi)$. By Theorem 2.1,

$$\|\big(\mathcal{R}_i^{\varepsilon'}\big)^{-1}\chi_i f\|_{L^p} \lesssim_{p,\varepsilon,\mu} \|\chi_i f\|_{L^p}.$$

Consequently,

$$\|f\|_{L^p} \le \|f_1\|_{L^p} + \|f_2\|_{L^p} \le \sum_{i=1}^2 \|\big(\mathcal{R}_i^{\varepsilon'}\big)^{-1}\mathcal{R}_i^{\varepsilon'} f_i\|_{L^p} \lesssim_{p,\varepsilon,\mu} \sum_{i=1}^2 \|\mathcal{R}_i^{\varepsilon'} f_i\|_p.$$

With (31) in mind, we show (29) by considering the data

$$v = \begin{pmatrix} -2\mathcal{R}_2^{\varepsilon'} f & 2\mathcal{R}_1^{\varepsilon'} f & 0\end{pmatrix}^t. \tag{32}$$

Clearly, $\partial_1 v_1 + \partial_2 v_2 = 0$. We compute

$$m^{-1}(D)v = \mu\begin{pmatrix}0 & 1 & -1\end{pmatrix}^t \hat{f}(\xi).$$

We further compute

$$P(\omega,D)^{-1}v = -\begin{pmatrix}\mathcal{R}_2^{\varepsilon'}(d_-+d_+)f & \mathcal{R}_1^{\varepsilon'}(d_-+d_+)f & \mu(d_--d_+)f\end{pmatrix}^t,$$

and it follows by (31)

$$\|P(\omega,D)^{-1}v\|_{L^q} \sim \|(d_-+d_+)f\|_{L^q} + \mu\|(d_--d_+)f\|_{L^q} \sim \|d_-f\|_{L^q} + \|d_+f\|_{L^q}$$

as claimed. Since $\|v\|_{L^p}\sim\|f\|_{L^p}$, by choosing f suitably, we find

$$\|P(\omega,D)^{-1}\|_{L^p\to L^q} \gtrsim \max(\|d_-\|_{L^p\to L^q}, \|d_+\|_{L^p\to L^q}) \sim \|d_-\|_{L^p\to L^q} + \|d_+\|_{L^p\to L^q}.$$

The proof of Theorem 1.1 for $d=2$ is complete.

2.2. **Maxwell's equations in 3d in the partially anisotropic case.** We consider $P(\omega, D)$ as in (1) with $\varepsilon = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $\mu > 0$. Here we consider the partially anisotropic case $a^{-1} = \varepsilon_1 \neq \varepsilon_2 = \varepsilon_3 = b^{-1}$ and suppose that $\mu = 1$ without loss of generality, to which we can reduce by linear substitution. The isotropic case under more general assumptions was considered in [4]. For $\xi \in \mathbb{R}^3$ we denote

$$\begin{aligned} \|\xi\|^2 &= \xi_1^2 + \xi_2^2 + \xi_3^2, \quad & \|\xi\|_\varepsilon^2 &= b\xi_1^2 + a\xi_2^2 + a\xi_3^2, \\ \xi' &= \xi/\|\xi\|, & \tilde{\xi} &= \xi/\|\xi\|_\varepsilon. \end{aligned}$$

We write further

$$(\nabla \times u)\widehat{\ }(\xi) = -ib(\xi)\hat{u}(\xi), \quad b(\xi) = \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}.$$

Here, the symbol of $P(\omega, D)$ is given by

$$\begin{pmatrix} i\omega 1_{3\times 3} & ib(\xi) \\ -ib(\xi)\varepsilon^{-1} & i\omega 1_{3\times 3} \end{pmatrix} = m(\xi) d(\xi) m^{-1}(\xi)$$

with

$$d(\xi) = i\,\mathrm{diag}(\omega, \omega, \omega - \sqrt{b}\|\xi\|, \omega + \sqrt{b}\|\xi\|, \omega - \|\xi\|_\varepsilon, \omega + \|\xi\|_\varepsilon).$$

We find the following corresponding eigenvectors, which are normalized to zero-homogeneous entries. Eigenvectors to $i\omega$ are

$$\begin{aligned} v_1^t &= \big(0, 0, 0, \xi_1', \xi_2', \xi_3'\big), \\ v_2^t &= \big(\frac{\tilde{\xi}_1}{a}, \frac{\tilde{\xi}_2}{b}, \frac{\tilde{\xi}_3}{b}, 0, 0, 0\big). \end{aligned}$$

Eigenvectors to $i\omega \mp i\sqrt{b}\|\xi\|$ are given by

$$\begin{aligned} v_3^t &= \big(0, -\frac{\xi_3'}{\sqrt{b}}, \frac{\xi_2'}{\sqrt{b}}, -((\xi_2')^2 + (\xi_3')^2), \xi_1'\xi_2', \xi_1'\xi_3'\big), \\ v_4^t &= \big(0, \frac{\xi_3'}{\sqrt{b}}, -\frac{\xi_2'}{\sqrt{b}}, -((\xi_2')^2 + (\xi_3')^2), \xi_1'\xi_2', \xi_1'\xi_3'\big). \end{aligned}$$

Eigenvectors to $i\omega \mp i\|\xi\|_\varepsilon$ are given by

$$\begin{aligned} v_5^t &= \big(\tilde{\xi}_2^2 + \tilde{\xi}_3^2, -\tilde{\xi}_1\tilde{\xi}_2, -\tilde{\xi}_1\tilde{\xi}_3, 0, -\tilde{\xi}_3, \tilde{\xi}_2\big), \\ v_6^t &= \big(-(\tilde{\xi}_2^2 + \tilde{\xi}_3^2), \tilde{\xi}_1\tilde{\xi}_2, \tilde{\xi}_1\tilde{\xi}_3, 0, -\tilde{\xi}_3, \tilde{\xi}_2\big). \end{aligned}$$

Set

$$m(\xi) = (v_1, \dots, v_6).$$

We find

$$m^{-1}(\xi) =$$

$$\begin{pmatrix} 0 & 0 & 0 & \xi_1' & \xi_2' & \xi_3' \\ ab\tilde{\xi}_1 & ab\tilde{\xi}_2 & ab\tilde{\xi}_3 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon} \frac{\tilde{\xi}_3}{\tilde{\xi}_2^2 + \tilde{\xi}_3^2} & \frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon} \frac{\tilde{\xi}_2}{\tilde{\xi}_2^2 + \tilde{\xi}_3^2} & -1/2 & \frac{\xi_1'\xi_2'}{2(\xi_2'^2 + \xi_3'^2)} & \frac{\xi_1'\xi_3'}{2(\xi_2'^2 + \xi_3'^2)} \\ 0 & \frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon} \frac{\tilde{\xi}_3}{\tilde{\xi}_2^2 + \tilde{\xi}_3^2} & -\frac{\sqrt{b}\|\xi\|}{2\|\xi\|_\varepsilon} \frac{\tilde{\xi}_2}{\tilde{\xi}_2^2 + \tilde{\xi}_3^2} & -1/2 & \frac{\xi_1'\xi_2'}{2(\xi_2'^2 + \xi_3'^2)} & \frac{\xi_1'\xi_3'}{2(\xi_2'^2 + \xi_3'^2)} \\ a/2 & -\frac{b\tilde{\xi}_1\tilde{\xi}_2}{2(\tilde{\xi}_2^2 + \tilde{\xi}_3^2)} & -\frac{b\tilde{\xi}_1\tilde{\xi}_3}{2(\tilde{\xi}_2^2 + \tilde{\xi}_3^2)} & 0 & -\frac{\xi_3'\|\xi\|_\varepsilon}{2\|\xi\|(\xi_2'^2 + \xi_3'^2)} & \frac{\|\xi\|_\varepsilon \xi_2'}{2\|\xi\|(\xi_2'^2 + \xi_3'^2)} \\ -a/2 & \frac{b\tilde{\xi}_1\tilde{\xi}_2}{2(\tilde{\xi}_2^2 + \tilde{\xi}_3^2)} & \frac{b\tilde{\xi}_1\tilde{\xi}_3}{2(\tilde{\xi}_2^2 + \tilde{\xi}_3^2)} & 0 & -\frac{|\xi|_\varepsilon}{2\|\xi\|} \frac{\xi_3'}{(\xi_2'^2 + \xi_3'^2)} & \frac{\|\xi\|_\varepsilon \xi_2'}{2\|\xi\|(\xi_2'^2 + \xi_3'^2)} \end{pmatrix}.$$

We observe that the matrix becomes singular for $|\xi_2|+|\xi_3| \to 0$. Therefore, we renormalize $v_3,\dots,v_6$ with $a(\xi) = \frac{(\xi_2^2+\xi_3^2)^{\frac{1}{2}}}{(\|\xi\| \|\xi\|_\varepsilon)^{\frac{1}{2}}}$. We compute by elementary matrix operations (adding and subtracting the third and fourth, and fifth and sixth eigenvector, respectively) that

$$|\det m(\xi)| \sim \begin{vmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_2' & \xi_3' \\ 0 & 0 & 0 & 0 & -\tilde{\xi}_3 & \tilde{\xi}_2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_3' & -\xi_2' & 0 & 0 & 0 \\ 0 & \tilde{\xi}_2 & \tilde{\xi}_3 & 0 & 0 & 0 \end{vmatrix} \sim (\xi_2' \tilde{\xi}_2 + \xi_3' \tilde{\xi}_3)^2.$$

Let $\tilde{m}$ denote the renormalization, where we replace $v_3,\dots,v_6$ in m with $v_3/a(\xi),\dots,v_6/a(\xi)$. $\tilde{m}$ has determinant comparable to 1. By Cramer's rule, the entries of $\tilde{m}^{-1}$ are polynomials in the entries of $\tilde{m}$. Hence, we do not give the expressions for $\tilde{m}^{-1}$. It is enough to check that the Fourier multipliers associated with the entries in $\tilde{m}$ are all uniformly L^p-bounded, for which we use Theorem 2.1.

We turn to the proof that the entries of $v_i/a(\xi)$, $i=3,\dots,6$, are multipliers bounded in L^p for $1<p<\infty$.

Entries of v_3:

- $(v_3)_2/a(\xi)$: We have to show that

$$\frac{\xi_3 (\|\xi\| \|\xi\|_{\tilde{\varepsilon}})^{1/2}}{\|\xi\| (\xi_2^2+\xi_3^2)^{1/2}} = \frac{\xi_3}{(\xi_2^2+\xi_3^2)^{1/2}} \big(\frac{\|\xi\|_\varepsilon}{\|\xi\|} \big)^{1/2}$$

  is a multiplier. This is the case because $\frac{\xi_3}{(\xi_2^2+\xi_3^2)^{1/2}}$ is a Riesz transform in (x_2,x_3) and the second factor $\big(\frac{\|\xi\|_\varepsilon}{\|\xi\|}\big)^{1/2}$ is zero-homogeneous and smooth away from the origin, hence, in the scope of Theorem 2.1.
- $(v_3)_3/a(\xi)$ is a multiplier by symmetry in ξ_2 and ξ_3 and the previous considerations.
- $(v_3)_4/a(\xi)$: We find

$$\frac{(\xi_2^2+\xi_3^2)}{\|\xi\|^2 (\xi_2^2+\xi_3^2)^{1/2}} \cdot (\|\xi\| \|\xi\|_\varepsilon)^{1/2} = \frac{(\xi_2^2+\xi_3^2)^{1/2}}{\|\xi\|} \cdot \big(\frac{\|\xi\|_\varepsilon}{\|\xi\|} \big)^{1/2}$$

  to be a Fourier multiplier as it is zero-homogeneous and smooth away from the origin.
- $(v_3)_5/a(\xi)$: Consider

$$\frac{\xi_1 \xi_2}{\|\xi\|^2 (\xi_2^2+\xi_3^2)^{1/2}} (\|\xi\| \|\xi\|_\varepsilon)^{1/2} = \frac{\xi_1}{\|\xi\|} \cdot \frac{\xi_2}{(\xi_2^2+\xi_3^2)^{1/2}} \cdot \big(\frac{\|\xi\|_\varepsilon}{\|\xi\|} \big)^{1/2},$$

  which is again a Fourier multiplier because the first and third expression are zero-homogeneous and smooth in $\mathbb{R}^n \backslash 0$, the second is again a Riesz transform in two variables.
- $(v_3)_6/a(\xi)$ can be handled like the previous case.

Entries of $v_4/a(\xi)$: The entries are Fourier multipliers because they coincide up to a factor with entries from $v_3/a(\xi)$.

Entries of $v_5/a(\xi)$:

- $(v_5/a(\xi))_1$: We find
$$\frac{\xi_2^2+\xi_3^2}{\|\xi\|_\varepsilon^2}\cdot\frac{(\|\xi\|\|\xi\|_\varepsilon)^{1/2}}{(\xi_2^2+\xi_3^2)^{1/2}}=\frac{(\xi_2^2+\xi_3^2)^{1/2}}{\|\xi\|_\varepsilon}\cdot\big(\frac{\|\xi\|}{\|\xi\|_\varepsilon}\big)^{1/2}.$$
This is a multiplier because it is a product of two smooth away from the origin, zero-homogeneous functions.
- $(v_5/a(\xi))_2$: We find
$$\frac{\tilde{\xi}_1\tilde{\xi}_2}{(\xi_2^2+\xi_3^2)^{1/2}}(\|\xi\|\|\xi\|_\varepsilon)^{1/2}=\frac{\xi_1}{\|\xi\|_\varepsilon}\cdot\frac{\xi_2}{(\xi_2^2+\xi_3^2)^{1/2}}\cdot\big(\frac{\|\xi\|}{\|\xi\|_\varepsilon}\big)^{1/2}.$$
The first and third factor are zero-homogeneous and smooth away from the origin; the second is up to a constant a Riesz transform in two variables.
- $(v_5)_3/a(\xi)$: This can be handled like $(v_5)_2$ because of symmetry in ξ_2 and ξ_3.
- $(v_5)_5/a(\xi)$: We find
$$\frac{\tilde{\xi}_3}{(\xi_2^2+\xi_3^2)^{1/2}}(\|\xi\|\|\xi\|_\varepsilon)^{1/2}=\frac{\xi_3}{(\xi_2^2+\xi_3^2)^{1/2}}\cdot\big(\frac{\|\xi\|}{\|\xi\|_\varepsilon}\big)^{1/2};$$
the first factor corresponds to a Riesz transform in two variables; the second factor is in the scope of Theorem 2.1.
- $(v_5)_6/a(\xi)$: The same arguments as for $(v_5)_5/a(\xi)$ apply by symmetry.

The entries of $v_6/a(\xi)$ can be handled like the entries of $v_5/a(\xi)$. The proof of L^p-boundedness of $\tilde{m}$ is complete. This proves the upper bound for the resolvent.

Below let $\widehat{(\mathcal{R}_i f)}(\xi)=\frac{\xi_i}{\|\xi\|}\hat{f}(\xi)$. To show the lower bound, we consider the following initial data:
$$J_{0e}=\begin{pmatrix}0\\-\mathcal{R}_3 f\\\mathcal{R}_2 f\end{pmatrix},\quad J_{0m}=\underline{0}.$$
Note that $\nabla\cdot J_{0e}=0$ and again, the initial data is also physically meaningful as the magnetic current vanishes.

Let $\widehat{d_\pm f}(\xi)=(\omega\pm\sqrt{b}|\xi|)^{-1}\hat{f}(\xi)$. We compute

(33)
$$(dm^{-1})(\xi)\begin{pmatrix}\hat{J}_{0e}\\\hat{J}_{0b}\end{pmatrix}=\frac{\sqrt{b}}{2}\begin{pmatrix}0\\0\\\widehat{d_-f}\\-\widehat{d_+f}\\0\\0\end{pmatrix},\quad\begin{pmatrix}D\\B\end{pmatrix}=i\begin{pmatrix}0\\-\mathcal{R}_3(d_-f+d_+f)\\\mathcal{R}_2(d_-f+d_+f)\\-((\mathcal{R}_2^2+\mathcal{R}_3^2)(d_-f-d_+f)\\\mathcal{R}_1\mathcal{R}_2(d_-f-d_+f)\\\mathcal{R}_1\mathcal{R}_3(d_-f-d_+f).\end{pmatrix}$$
We shall see that

(34)
$$\|(D,B)\|_{L_0^q}\gtrsim\|d_-f+d_+f\|_{L^q}+\|d_-f-d_+f\|_{L^q}\gtrsim\|d_-f\|_{L^q}+\|d_+f\|_{L^q}.$$
This will be the case, if f has frequency support in a conic neighbourhood of the ξ_3-axis or is spherically symmetric. Indeed, if f has frequency support in a conic neighbourhood of the ξ_3-axis, then $\mathcal{R}_3$ is invertible and $\|\mathcal{R}_2 g\|_{L^p}\ll\|g\|_{L^p}$ by Theorem 2.1, such that (34) follows from considering the second and fourth component in (33).

In case of spherical symmetry, we still find
$$\|\mathcal{R}_3 f\|_{L^p}\sim\|f\|_{L^p},\quad\|\mathcal{R}_3 d_\pm f\|_{L^p}\sim\|d_\pm f\|_{L^p}.$$

If we can choose g such that the operator norms of $d_\pm$ are approximated, we find

$$\|(D,H)\|_{L^q_0} \gtrsim (\|d_-\|_{L^p \to L^q} + \|d_+\|_{L^p \to L^q}) \|f\|_{L^p}.$$

Lastly, if f is supported in a conic neighbourhood of the ξ_3-axis, or is spherically symmetric, we find $\|(J_{e0}, J_{m0})\|_{L^p_0} \sim \|f\|_{L^p}$. To see that it suffices to consider the frequency support of f as such, we recall the examples from [15, Section 5.2], giving the claimed lower bound for the operator norm of the resolvent of the fractional Laplacian: a Knapp type example, which can be realized with frequency support in a conic neighbourhood of the ξ_3-axis [15, p. 1458], and a spherically symmetric example related with the surface measure on the sphere [15, p. 1459]. This finishes the proof of Theorem 1.1. □

3. Local and global LAP

Let $P(\omega, D)$ be as in the previous section. In the following we want to investigate the limit of

$$P(\omega \pm i\delta, D)^{-1} f \text{ as } \delta \to 0, \quad \omega \in \mathbb{R}\backslash 0,$$

by which we construct solutions to time-harmonic Maxwell's equations. By scaling we see that the following estimates are uniform in ω, provided it varies in a compact set away from the origin. We further suppose that $\omega > 0$, the case $\omega < 0$ can be treated with the obvious modifications.

We work with the following notions:

Definition 3.1. Let $d \in \{2,3\}$, $1 < p, q < \infty$, $\omega \in \mathbb{R}\backslash 0$, and $0 < \delta < 1/2$. We say that a global LAP holds if $P(\omega \pm i\delta, D)^{-1} : L^p_0 \to L^q_0$ are bounded uniformly in $\delta > 0$, and there are operators $P_\pm(\omega) : L^p_0 \to L^q_0$ such that

$$P(\omega \pm i\delta, D)^{-1} f \to P_\pm(\omega) f \text{ as } \delta \to 0 \tag{35}$$

in $\mathcal{S}'(\mathbb{R}^d)^{m(d)}$.

We say that a local LAP holds if for any $\beta \in C^\infty_c(\mathbb{R}^d)$, $P(\omega \pm i\delta, D)^{-1}\beta(D) : L^p_0 \to L^q_0$ are bounded uniformly in $\delta > 0$, and there are operators $P^{loc}_\pm(\omega) : L^p_0 \to L^q_0$ such that

$$P(\omega \pm i\delta, D)^{-1}\beta(D) f \to P^{loc}_\pm(\omega) f \tag{36}$$

in $\mathcal{S}'(\mathbb{R}^d)$.

In the following let $0 < |\delta| < 1/2$. By the above diagonalization, it is equivalent to consider uniform boundedness of

$$d^{\varepsilon'}_\pm(\omega + i\delta) : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d), \quad (d^{\varepsilon'}_\pm(\omega + i\delta) f)\widehat{\ }(\xi) = \frac{\hat{f}(\xi)}{\|\xi\|_{\varepsilon'} \pm (\omega + i\delta)}.$$

Hence, by the results of the previous section, the global LAP fails due to the lack of uniform resolvent estimates for the Half-Laplacian in L^p-spaces. This is recorded in Corollary 1.2.

Regarding the local LAP, we observe that the operator

$$(d^{\varepsilon'}_+(\omega \pm i\delta) f)\widehat{\ }(\xi) = \frac{\beta(\xi)\hat{f}(\xi)}{\|\xi\|_{\varepsilon'} + (\omega \pm i\delta)}$$

for $\beta \in C_c^\infty$, $0 < \delta < 1/2$ is bounded from $L^p \to L^q$ for $1 \leq p \leq q \leq \infty$ by Young's inequality, with the obvious limit as $\delta \to 0$. Thus, we focus on

$$(d_\delta f)\widehat{(\xi)} := (d_-(\omega \pm i\delta) f)\widehat{(\xi)} = \frac{\beta(\xi)\hat{f}(\xi)}{\|\xi\|_{\varepsilon'} - (\omega \pm i\delta)} \tag{37}$$

with $0 < \delta < \delta_0 \ll 1$, where $\beta \in C_c^\infty(\mathbb{R}^n)$.

We can be more precise about the limiting operators: For $t \in \mathbb{R}$ recall Sokhotsky's formula, which hold in the sense of distributions:

$$\lim_{\varepsilon \downarrow 0} \frac{1}{t \pm i\varepsilon} = v.p. \frac{1}{t} \mp i\pi \delta_0(t),$$

where δ_0 denotes the delta-distribution at the origin.
Let

$$\mathcal{R}_\pm^{loc} f = \lim_{\delta \to \pm 0} d_\delta f.$$

We find

$$\mathcal{R}_\pm^{loc} f = v.p. \int \frac{\beta(\xi) e^{ix\xi}}{\|\xi\|_{\varepsilon'} - 1} \hat{f}(\xi) d\xi \mp i\pi \int e^{ix\xi} \beta(\xi) \delta(\|\xi\|_{\varepsilon'} - 1) \hat{f}(\xi) d\xi,$$

and by the diagonalization formulae, we find that the limiting operators can be expressed as linear combinations involving possibly generalized Riesz transforms, $\mathcal{R}_\pm^{loc}$, and d_+. We recall the L^p-L^q-mapping properties of $\mathcal{R}_\pm^{loc}$.

We observe that

$$(\mathcal{R}_-^{loc} - \mathcal{R}_+^{loc}) f = 2\pi i \int_{\{\|\xi\|_\varepsilon = 1\}} \beta(\xi) e^{ix\xi} \hat{f}(\xi) d\sigma(\xi).$$

This operator, modulo the bounded operator given by convolution with $\mathcal{F}^{-1}\beta$ and linear change of variables $\xi \to \zeta$ such that $\|\xi\|_{\varepsilon'} = \|\zeta\|$, is known as *restriction-extension operator* (cf. [13, 15]) and is a special case of the Bochner-Riesz operator of negative index:

$$(\mathcal{B}^\alpha f)\widehat{(\xi)} = \frac{1}{\Gamma(1-\alpha)} \frac{\hat{f}(\xi)}{(1 - |\xi|^2)_+^\alpha}, \quad 0 < \alpha \leq \frac{d+2}{2},$$

which, for $\alpha \geq 1$, is defined by analytic continuation. Hence, for $\alpha = 1$, it matches the restriction extension operator. The restriction–extension operator is well-understood due to the works of Börjeson [2], Sogge [22], and Gutiérrez [10, 11]. The most recent results for Bochner–Riesz operators of negative index are due to Kwon–Lee [15]. Gutiérrez showed that $\mathcal{B}^1 : L^p \to L^q$ is bounded if and only if $(1/p, 1/q) \in \mathcal{P}(d)$ with

$$\mathcal{P}(d) = \{(x,y) \in [0,1]^2 \, : \, x - y \geq \frac{2}{d+1}, \; x > \frac{d+1}{2d}, \; y < \frac{d-1}{2d}\}.$$

She used this to show uniform resolvent estimates for

$$(-\Delta - z)^{-1} : L^p \to L^q, \quad z \in \mathbb{S}^1 \backslash \{1\} \text{ for } (1/p, 1/q) \in \mathcal{R}_1(d).$$

By the same argument for d_δ (cf. [15, Proposition 4.1]) and the diagonalization, the uniform bounds for $P(\omega \pm i\delta, D)^{-1}\beta(D) : L_0^p \to L_0^q$ with $1 < p, q < \infty$, $(1/p, 1/q) \in \mathcal{P}(d)$ follow.

We finish the proof of Theorem 1.3: Let $1 < p_1, p_2, q < \infty$, $\omega \in \mathbb{R}\backslash 0$, $\beta \in C_c^\infty$ with $\beta \equiv 1$ on $\{\max(\|\xi\|, \|\xi\|_\varepsilon) \leq 2\omega\}$ and decompose

$$\begin{pmatrix} \hat{J}_e \\ \hat{J}_m \end{pmatrix} = \beta(\xi) \begin{pmatrix} \hat{J}_e \\ \hat{J}_m \end{pmatrix} + (1-\beta)(\xi) \begin{pmatrix} \hat{J}_e \\ \hat{J}_m \end{pmatrix} =: \hat{J}_1 + \hat{J}_2.$$

By the local LAP, we find uniform bounds for $0 < \delta < 1/2$

$$\|P(\omega \pm i\delta, D)^{-1} J_1\|_{L_0^q} \lesssim \|J_1\|_{L_0^{p_1}}$$

provided that $(\frac{1}{p_1}, \frac{1}{q}) \in \mathcal{P}(d)$. The estimate

$$\|P(\omega \pm i\delta, D)^{-1} J_2\|_{L_0^q} \lesssim \|J_2\|_{L_0^{p_2}}$$

follows for $0 \leq \frac{1}{p_2} - \frac{1}{q} \leq \frac{1}{d}$ by Theorem 2.1 and properties of the Bessel kernel. The limiting operators $P_\pm(\omega)$ were described above: We have

$$P(\omega \pm i\delta, D)^{-1}(J_e, J_m) \to P_\pm(\omega)(J_e, J_m)$$

in $\mathcal{S}'(\mathbb{R}^d)^{m(d)}$.
Let $(D, B)^\pm_\delta = P(\omega \pm i\delta, D)^{-1}(J_e, J_m)$ and $(D, B)^\pm = P_\pm(\omega)(J_e, J_m)$. At last, we show that

$$P(\omega, D)(D, B)^\pm = (J_e, J_m). \tag{38}$$

For this purpose, we show that for $\delta \to 0$ we have

$$P(\omega, D)(D, B)^\pm_\delta \to (J_e, J_m) \tag{39}$$

in $\mathcal{S}'(\mathbb{R}^d)^{m(d)}$. As $(D, B)^\pm_\delta \to (D, B)^\pm$ in $\mathcal{S}'(\mathbb{R}^d)^{m(d)}$, this would conclude the proof. To show (39), we return to the diagonalization (26)

$$p(\tilde{\omega}, \xi) = i m(\xi) d(\tilde{\omega}, \xi) m^{-1}(\xi) \text{ for } \tilde{\omega} \in \mathbb{C}.$$

We find for $\omega \in \mathbb{R}$:

$$\begin{aligned} p(\omega, \xi) p^{-1}(\omega \pm i\delta, \xi) &= m(\xi) d(\omega, \xi) d(\omega \pm i\delta, \xi)^{-1} m^{-1}(\xi) \\ &= m(\xi)(1_{m(d) \times m(d)} \mp i\delta d(\omega \pm i\delta, \xi)^{-1}) m^{-1}(\xi) \\ &= 1_{m(d) \times m(d)} \pm \delta p(\omega \pm i\delta, \xi)^{-1}. \end{aligned}$$

Hence,

$$P(\omega, D)(D, B)^\pm_\delta = (J_e, J_m) \pm \delta P(\omega \pm i\delta, D)^{-1}(J_e, J_m)$$

and

$$\|P(\omega, D)(D, B)^\pm_\delta - (J_e, J_m)\|_{L_0^q(\mathbb{R}^d)} \lesssim \delta \|(J_e, J_m)\|_{L_0^{p_1} \cap L_0^{p_2}} \to 0.$$

In particular, (39) holds true in $\mathcal{S}'(\mathbb{R}^d)^{m(d)}$. The proof of Theorem 1.3 is complete. □

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References

[1] D. Boito, L. N. S. de Andrade, G. de Sousa, R. Gama, and C. Y. M. London. On Maxwell's electrodynamics in two spatial dimensions. *arXiv e-prints*, page arXiv:1809.07368, September 2018.

[2] Lennart Börjeson. Estimates for the Bochner-Riesz operator with negative index. *Indiana Univ. Math. J.*, 35(2):225–233, 1986.

[3] Jean Bourgain, Peng Shao, Christopher D. Sogge, and Xiaohua Yao. On L^p-resolvent estimates and the density of eigenvalues for compact Riemannian manifolds. *Comm. Math. Phys.*, 333(3):1483–1527, 2015.

[4] Lucrezia Cossetti and Rainer Mandel. A limiting absorption principle for Helmholtz systems and time-harmonic isotropic Maxwell's equations. *arXiv e-prints*, page arXiv:2009.05087, September 2020.

[5] Jean-Claude Cuenin. Eigenvalue bounds for Dirac and fractional Schrödinger operators with complex potentials. *J. Funct. Anal.*, 272(7):2987–3018, 2017.

[6] Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman lectures on physics. Vol. 2: Mainly electromagnetism and matter.* Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1964.

[7] Rupert L. Frank. Eigenvalue bounds for Schrödinger operators with complex potentials. *Bull. Lond. Math. Soc.*, 43(4):745–750, 2011.

[8] Rupert L. Frank. Eigenvalue bounds for Schrödinger operators with complex potentials. III. *Trans. Amer. Math. Soc.*, 370(1):219–240, 2018.

[9] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[10] Susana Gutiérrez. A note on restricted weak-type estimates for Bochner-Riesz operators with negative index in $\mathbf{R}^n$, $n \geq 2$. *Proc. Amer. Math. Soc.*, 128(2):495–501, 2000.

[11] Susana Gutiérrez. Non trivial L^q solutions to the Ginzburg-Landau equation. *Math. Ann.*, 328(1-2):1–25, 2004.

[12] Jonathan Hickman. Uniform L^p Resolvent Estimates on the Torus. *arXiv e-prints*, page arXiv:1907.08131, July 2019.

[13] Eunhee Jeong, Yehyun Kwon, and Sanghyuk Lee. Uniform Sobolev inequalities for second order non-elliptic differential operators. *Adv. Math.*, 302:323–350, 2016.

[14] C. E. Kenig, A. Ruiz, and C. D. Sogge. Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. *Duke Math. J.*, 55(2):329–347, 1987.

[15] Yehyun Kwon and Sanghyuk Lee. Sharp resolvent estimates outside of the uniform boundedness range. *Comm. Math. Phys.*, 374(3):1417–1467, 2020.

[16] Yehyun Kwon, Sanghyuk Lee, and Ihyeok Seo. Resolvent estimates for the Lamé operator and failure of Carleman estimates. *arXiv e-prints*, page arXiv:1912.12620, December 2019.

[17] L. D. Landau and E. M. Lifschitz. *Lehrbuch der theoretischen Physik ("Landau-Lifschitz"). Band VIII.* Akademie-Verlag, Berlin, fifth edition, 1990. Elektrodynamik der Kontinua. [Electrodynamics of continua], Translated from the second Russian edition by Georg Dautcourt, Helmut Günther, Horst Heino v. Borzeskowski [Horst-Heino von Borzeszkowski] and Stefan-Ludwig Drechsler, Translation edited by Gerd Lehmann, With the collaboration of Lifschitz and L. P. Pitajewski [L. P. Pitaevskiĭ], With a foreword by P. Ziesche and Lehmann.

[18] Otto Liess. Decay estimates for the solutions of the system of crystal optics. *Asymptotic Anal.*, 4(1):61–95, 1991.

[19] Rainer Mandel and Robert Schippa. Solutions to anisotropic Maxwell's equations.

[20] Jerome V. Moloney and Alan C. Newell. Nonlinear optics. *Phys. D*, 44(1-2):1–37, 1990.

[21] Robert Schippa and Roland Schnaubelt. On quasilinear Maxwell's equations in two dimensions.

[22] Christopher D. Sogge. Oscillatory integrals and spherical harmonics. *Duke Math. J.*, 53(1):43–65, 1986.

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