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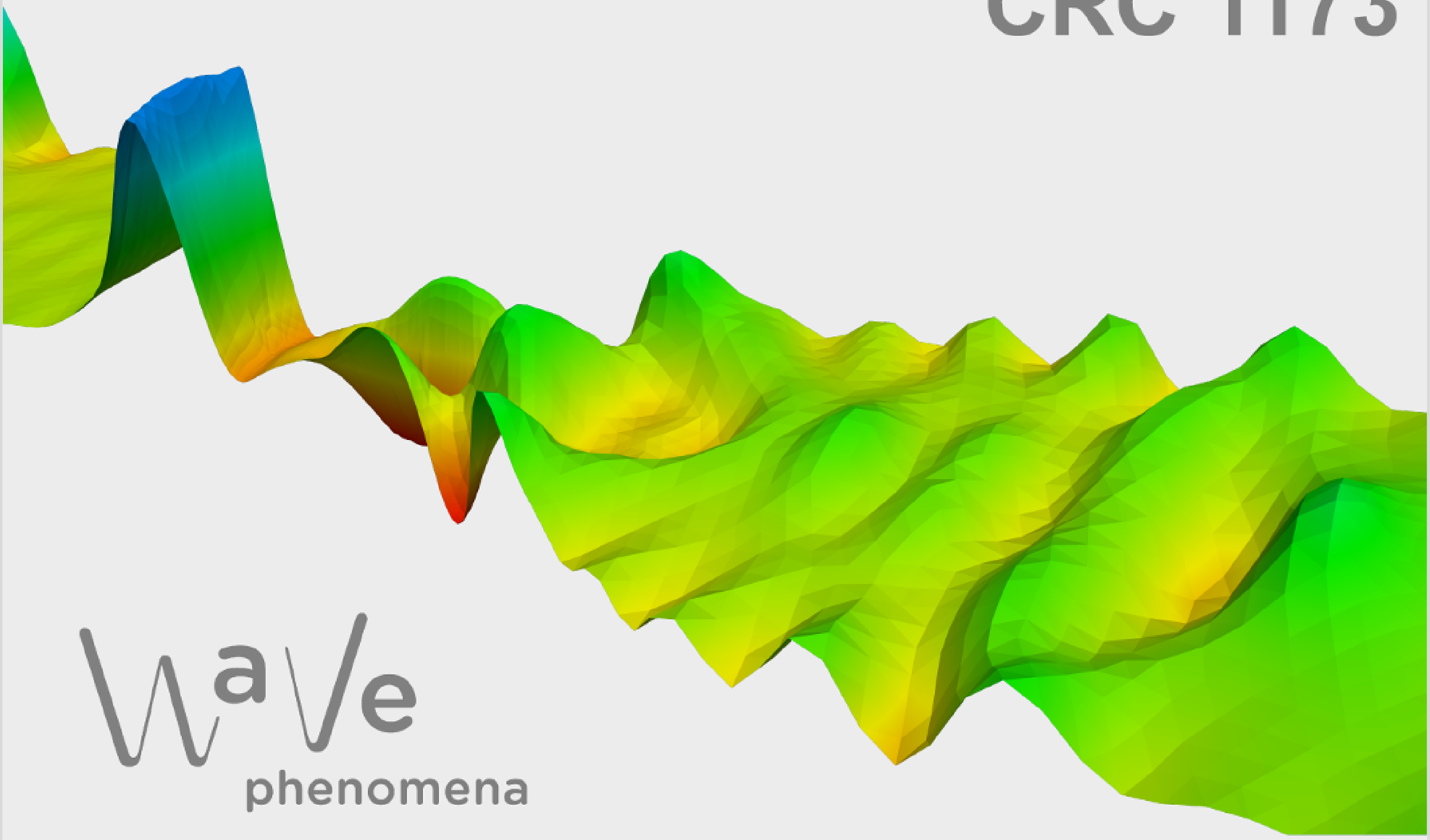

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AN ANNULUS MULTIPLIER AND APPLICATIONS TO THE LIMITING ABSORPTION PRINCIPLE FOR HELMHOLTZ EQUATIONS WITH A STEP POTENTIAL

RAINER MANDEL AND DOMINIC SCHEIDER

Abstract. We consider the Helmholtz equation $-\Delta u + V\,u - \lambda\,u = f$ on $\mathbb{R}^n$ where the potential $V:\mathbb{R}^n \to \mathbb{R}$ is constant on each of the half-spaces $\mathbb{R}^{n-1}\times(-\infty,0)$ and $\mathbb{R}^{n-1}\times(0,\infty)$. We prove an $L^p - L^q$-Limiting Absorption Principle for frequencies $\lambda > \max V$ with the aid of Fourier Restriction Theory and derive the existence of nontrivial solutions of linear and nonlinear Helmholtz equations. As a main analytical tool we develop new $L^p - L^q$ estimates for a singular Fourier multiplier supported in an annulus.

1. Introduction

In this paper we are interested in the Limiting Absorption Principle (LAP) for the Helmholtz equation on $\mathbb{R}^n$ involving a step potential of the form

$$V(x,y) = \begin{cases} V_1 & \text{if } x \in \mathbb{R}^{n-1}, y > 0, \\ V_2 & \text{if } x \in \mathbb{R}^{n-1}, y < 0 \end{cases} \tag{1}$$

where $V_1 \neq V_2$ are two fixed real numbers. We will without loss of generality assume $V_1 > V_2$ in the following. Examples for elliptic problems involving interfaces modelled by potentials of this kind can be found in [14, Theorem 1], [15, Theorem 2] or [28]. To explain the motivation behind our study, we recall the interesting phenomenon called "double scattering". In the context of the Schrödinger equation it means that for sufficiently regular and fast decaying right hand sides f the unique solution of the initial value problem

$$\mathrm{i}\partial_t \psi - \Delta\psi + V\psi = f \quad \text{in } \mathbb{R}^n, \quad \psi(0) = \psi_0,$$

with V as in (1) splits up into two pieces as $t \to \pm\infty$ that correspond to the two different values of V at infinity. This phenomenon is mathematically understood in the one-dimensional case $n = 1$ [24, Theorem 1.2], see also [12, 13]. One byproduct of our results is that such a splitting into two pieces may as well be observed for the solutions of the corresponding Helmholtz equations in $\mathbb{R}^n$ which are obtained through the Limiting Absorption Principle, see for instance the formula (16) where the two parts $f(x,y)1_{(0,\infty)}(\pm y)$ of the right hand side contribute differently to the LAP-solution of the Helmholtz equation. Notice that solutions u of such Helmholtz equations provide monochromatic solutions $\psi(x,t) = \mathrm{e}^{\mathrm{i}\lambda t}u(x)$ of the Schrödinger equation where λ belongs to the L^2-spectrum of the selfadjoint operator $-\Delta + V$

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with domain $H^2(\mathbb{R}^n)$. We prove our LAP in the topology of Lebesgue spaces in order to treat both linear and nonlinear Helmholtz equations. As far as we can see, the more classical results in weighted L^2 spaces resp. $B(\mathbb{R}^n), B^*(\mathbb{R}^n)$ (for the definition, cf. [4, page 4]) by Agmon [1–3] and Agmon-Hörmander [4] do not apply in the nonlinear setting.

Being interested in the LAP for the Helmholtz operator $-\Delta + V$ we fix the notation

$$\mathcal{R}(\mu) := (-\Delta + V - \mu)^{-1} \qquad \text{for } \mu \in \mathbb{C} \setminus \sigma(-\Delta + V).$$

A computation reveals $\sigma(-\Delta + V) = [\min\{V_1, V_2\}, \infty) = [V_2, \infty)$. We aim to prove a LAP that is as strong as the corresponding known result for the constant potential where $V_1 = V_2$. In that case, $\mathcal{R}(\mu)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $(p,q) \in \mathcal{D}$ and

$$\mathcal{D} = \left\{(p,q) \in [1,\infty] \times [1,\infty] : \ \frac{1}{p} > \frac{n+1}{2n}, \ \frac{1}{q} < \frac{n-1}{2n}, \ \frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}\right\} \text{ if } n \geq 3,$$
$$\mathcal{D} = \left\{(p,q) \in [1,\infty] \times [1,\infty] : \ \frac{1}{p} > \frac{n+1}{2n}, \ \frac{1}{q} < \frac{n-1}{2n}, \ \frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} < \frac{2}{n}\right\} \text{ if } n = 2.$$

This is a consequence of results by Kenig-Ruiz-Sogge, Gutiérrez ($n \geq 3$) and Evéquoz ($n = 2$) that we recall in Theorem 5 along with the bibliographical references. For step potentials of the kind (1) we manage to prove the same result provided that the Restriction Conjecture is true. We refer to Section 3 for more information on that topic. Given that this conjecture is open for $n \geq 3$, our LAP relies on the best approximation to the Restriction Conjecture, which is due to Tao (Theorem 8). Accordingly, we deal with exponents coming from the set

$$\tilde{\mathcal{D}} = \left\{(p,q) \in [1,\infty] \times [1,\infty] : \ \frac{1}{p} > \frac{1}{p_*(n)}, \ \frac{1}{q} < \frac{1}{q_*(n)}, \ \frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}\right\} \text{ if } n \geq 3,$$
$$\tilde{\mathcal{D}} = \left\{(p,q) \in [1,\infty] \times [1,\infty] : \ \frac{1}{p} > \frac{1}{p_*(n)}, \ \frac{1}{q} < \frac{1}{q_*(n)}, \ \frac{2}{n+1} \leq \frac{1}{p} - \frac{1}{q} < \frac{2}{n}\right\} \text{ if } n = 2,$$
$$\text{where} \qquad p_*(n) = \frac{2(n+2)}{n+4}, \quad q_*(n) = \frac{2(n+2)}{n}.$$

In particular, $\tilde{\mathcal{D}} = \mathcal{D}$ in the case $n = 2$ because of $p_*(n) = \frac{2n}{n+1} = \frac{4}{3}, q_*(n) = \frac{2n}{n-1} = 4$ and $\tilde{\mathcal{D}} \subsetneq \mathcal{D}$ in the case $n \geq 3$ because of $p_*(n) < \frac{2n}{n+1}, q_*(n) > \frac{2n}{n-1}$. Our main result is the following.

Theorem 1. *Let $n \in \mathbb{N}, n \geq 2$, let V be given by (1) and $\lambda > V_1 > V_2$. Then for all $(p,q) \in \tilde{\mathcal{D}}$ the resolvent estimate*

$$\sup_{0<|\varepsilon|\leq 1} \|\mathcal{R}(\lambda + \mathrm{i}\varepsilon)\|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} < \infty$$

holds. Moreover, the resolvent operators $\mathcal{R}(\lambda + \mathrm{i}\varepsilon)$ converge to nontrivial operators $\mathcal{R}(\lambda \pm \mathrm{i}0)$ as $\pm\varepsilon \searrow 0$ in the weak topology of bounded linear operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. If the Restriction Conjecture is true, then the same holds for exponents $(p,q) \in \mathcal{D}$.

In particular, our resolvent estimates for $n = 2$ coincide with the corresponding estimates for the constant potential whereas the ones for $n \geq 3$ cover a smaller range of parameters. For the important class of selfdual exponents $p = q'$, however, our Theorem 1 is optimal since it gives $6 \leq q < \infty$ for $n = 2$ and $\frac{2(n+1)}{n-1} \leq q \leq \frac{2n}{n-2}$ for $n \geq 3$. Let us mention that our result only covers frequencies in the range $\lambda > V_1 > V_2$ and thus not all frequencies in the (essential) spectrum. We believe that the same estimates can be proved for the remaining frequencies $\lambda \in (V_2, V_1]$ in the spectrum with some technical work. Especially regarding the treatment of Schrödinger or wave equations, uniform estimates with respect to all $\lambda \in \mathbb{C}$ would be very helpful and remain a challenging task for the future.

As an application of Theorem 1 we consider Helmholtz Equations on $\mathbb{R}^n$ involving the potential V from (1). We start with linear problems of the form

$$-\Delta u + Vu - \lambda u = f \quad \text{in } \mathbb{R}^n \tag{2}$$

where $f \in L^p(\mathbb{R}^n)$. Theorem 1 allows, for $(p,q) \in \tilde{\mathcal{D}}$, to define the outgoing solution $u_+ := \mathcal{R}(\lambda + \mathrm{i}0)f \in L^q(\mathbb{R}^n)$ of this equation. Notice that in the context of Helmholtz equations the word "outgoing" is used to distinguish $u_+ = \mathcal{R}(\lambda + \mathrm{i}0)f$ from the corresponding "incoming" solution $u_- := \mathcal{R}(\lambda - \mathrm{i}0)f = \overline{u_+}$, see [4, Definition 6.5]. Combining this with local elliptic regularity theory we obtain the following result.

Corollary 1. *Let $n \in \mathbb{N}, n \geq 2$, $(p,q) \in \tilde{\mathcal{D}}$, let V be given by (1) and $\lambda > V_1 > V_2$. Then for any $f \in L^p(\mathbb{R}^n)$ the Helmholtz equation (2) has a nontrivial "outgoing" resp. "incoming" strong solution u_+ (resp. u_-) $\in L^q(\mathbb{R}^n) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ obtained by the Limiting Absorption Principle, and there holds an estimate of the form*

$$\|u_-\|_{L^q(\mathbb{R}^n)} + \|u_+\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

If the Restriction Conjecture is true, then the same holds for $(p,q) \in \mathcal{D}$.

Here the symbol $\lesssim$ is used in the sense that there exists some constant $C > 0$ depending only on the parameters $V_1, V_2, n, p, q, \lambda$ such that $\|u_-\|_{L^q(\mathbb{R}^n)} + \|u_+\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$. The description of the appropriate radiation conditions for "outgoing" resp. "incoming" solutions remains open and we believe that, here, the results for the ranges $\lambda \in (V_2, V_1), \lambda = V_1$ and $\lambda \in (V_1, \infty)$ will be different. Moreover, it would be nice to provide a reasonable definition of a Herglotz wave. Recall that in the case $V_1 = V_2 = 1$, Herglotz waves are given by $x \mapsto \int_{|\xi|=1} g(\xi) \mathrm{e}^{-\mathrm{i}x\cdot\xi}\, \mathrm{d}\sigma(\xi)$ for square integrable densities g on the sphere. These solutions to the Helmholtz equation (2) for $f = 0$ are of central interest in scattering theory.

In our final result we use the Limiting Absorption Principle from Theorem 1 to prove the existence of solutions to nonlinear Helmholtz equations following the dual variational approach developed by Evéquoz and Weth [20, Theorem 1.2]. We refer to [19, 21, 33, 34, 36] for related results and other approaches to nonlinear Helmholtz equations with constant or periodic potentials.

Corollary 2. *Let $n \in \mathbb{N}, n \geq 2$, let V be given by (1) and assume $\lambda > V_1 > V_2$. Let $\Gamma \in L^\infty(\mathbb{R}^n)$ satisfy $\Gamma > 0$ on $\mathbb{R}^n$ and $\Gamma(x,y) \to 0$ as $|(x,y)| \to \infty$. Then the nonlinear Helmholtz equation*

$$-\Delta u + Vu - \lambda u = \Gamma |u|^{q-2} u \quad \text{in } \mathbb{R}^n \tag{3}$$

has a nontrivial solution in $L^q(\mathbb{R}^n) \cap W^{2,r}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $r < \infty$ provided that $\frac{2(n+1)}{n-1} \leq q < \frac{2n}{n-2}$.

We stress that this result covers the physically relevant special cases of the cubic and quintic nonlinearities for $n = 3$. More refined dual variational techniques as in [16, 18, 20] might be applicable as well to get one or even infinitely many solutions for larger classes of nonlinearities. For the proof of Corollary 2 we concentrate on an adaptation of [20, Theorem 1.2] in order to keep the technicalities at a moderate level. Let us mention that the integrability properties of the solution at infinity are actually slightly better, which can be proved along the lines of [20, Theorem 4.4] with the aid of a bootstrap procedure.

In the proof of Theorem 1 we will use Fourier restriction theory for estimates related to small frequencies, whereas our estimates for intermediate frequency ranges require different tools from Harmonic Analysis that we believe to be interesting in themselves. For instance, we encounter linear operators of the form

$$T_{\lambda,\alpha}h := \mathcal{F}_d^{-1}\left(1_A(\cdot)\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-a^2}}(|\cdot|^2-a^2)^{-\alpha}m(|\cdot|)\mathcal{F}_d h(\cdot)\right) \tag{4}$$

where $\alpha \in \{0, \frac{1}{2}\}$, $m \in C([a,b])$, $\lambda \geq 0$ and $A = \{\xi \in \mathbb{R}^d : a \leq |\xi| \leq b\}$ is an annulus with radii $b > a > 0$. The dimensional parameter will be $d = n-1$. If m is sufficiently smooth and $\lambda = 0$, we expect such operators to behave like so-called Bochner-Riesz operators of negative order – a connection that we will highlight below. Their mapping properties are quite well but, as far as we know, not completely understood, especially for $0 \leq \alpha < \frac{1}{2}$. In our context, however, m is only $\frac{1}{2}$-Hölder continuous and thus we cannot build upon on existing literature about these operators. We first present our result dealing with the one-dimensional case, which will be used in the proof of our LAP in the case $n = d + 1 = 2$. For completeness, we provide an optimal result under the stonger assumption $m \in C^1([a,b])$.

Theorem 2. *Let $\alpha \in [0,1)$, $0 < a < b < \infty, \lambda \geq 0$ and $m \in C^1([a,b])$. Then $T_{\lambda,\alpha} : L^p(\mathbb{R}) \to L^q(\mathbb{R})$ is bounded whenever $\frac{1}{p} - \frac{1}{q} \geq \alpha$, $\frac{1}{p} > \alpha$, $\frac{1}{q} < 1 - \alpha$. This range of exponents is optimal under the given conditions and*

$$\|T_{\lambda,\alpha}h\|_{L^q(\mathbb{R})} \lesssim (1+\lambda)^{2\alpha-\frac{2}{p}+\frac{2}{q}}\|h\|_{L^p(\mathbb{R})}.$$

For $m \in C([a,b])$ this estimate holds whenever $1 \leq p \leq 2 \leq q \leq \infty$, $\frac{1}{p} - \frac{1}{q} \geq \alpha$.

In the higher-dimensional case $d \geq 2$ the matter is more complicated. For our purposes it will be sufficient to prove such estimates for exponents (p,q) belonging to the set

$$D_\alpha := \left\{(p,q) \in [1,\infty]^2 : \frac{1}{p} > \frac{1}{2} + \frac{\alpha}{2d}, \ \frac{1}{q} < \frac{1}{2} - \frac{\alpha}{2d}, \ \frac{1}{p} - \frac{1}{q} \geq \frac{2\alpha}{d+1}\right\} \quad (0 < \alpha < 1) \tag{5}$$

assuming that the symbol m is continuous. In the case $\alpha = 0$ we set $D_0 := [1,2] \times [2,\infty]$.

Theorem 3. *Let $d \in \mathbb{N}, d \geq 2$, $\alpha \in [0,1)$, $0 < a < b < \infty, \lambda \geq 0$ and $m \in C([a,b])$. Then $T_{\lambda,\alpha} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is bounded for all $(p,q) \in D_\alpha$ and we have*

$$\|T_{\lambda,\alpha}h\|_{L^q(\mathbb{R}^d)} \lesssim (1+\lambda)^\gamma \|h\|_{L^p(\mathbb{R}^d)} \qquad \text{for all } \lambda \geq 0$$

for some $\gamma = \gamma_{\alpha,p,q,d} \leq 0$. If additionally $\frac{1}{p} - \frac{1}{q} \geq \frac{2}{d+2}$ is assumed then

$$\gamma \leq 2\alpha - 2 + \frac{1}{p} - \frac{1}{q} \qquad \text{and} \qquad \gamma < 2\alpha - 2 + \frac{1}{p} - \frac{1}{q} \quad \text{if } (p = 1 \text{ or } q = \infty). \tag{6}$$

Remark 1.

(a) *We do not know whether* D_α *is the optimal range for* $\alpha \in (0,1)$ *under the given assumptions on the symbol* m. *In case* $m \in C^1([a,b])$ *it is certainly not because boundedness also holds whenever* $\frac{1}{p} - \frac{1}{q} \geq \frac{d-1+2\alpha}{2d}$, $\frac{1}{p} > \frac{d-1+2\alpha}{2d}$, $\frac{1}{q} < \frac{d+1-2\alpha}{2d}$. *In particular, this is true for the exponents* $q = \infty$, $\frac{d+\alpha}{2d} \geq \frac{1}{p} > \frac{d-1+2\alpha}{2d}$, *none of which is covered by Theorem 3. The proof of this result is a straightforward generalization of the proof of Theorem 2 to the higher-dimensional case. The pointwise bound for the kernel* $|K_\lambda(z)| \lesssim |z|^{\alpha-1}$ *for* $|z| \geq 1+\lambda^2$ *from* (26) *then generalizes to* $|K_\lambda(z)| \lesssim |z|^{\alpha-\frac{d+1}{2}}$.

(b) *The proof actually yields an explicit expression for the decay rate* γ, *which, however, might not be optimal. For that reason we only highlight the important aspect for us, which is* (6). *The assumption* $\frac{1}{p} - \frac{1}{q} \geq \frac{2}{d+2} = \frac{2}{n+1}$ *is designed for our application in the context of the LAP from Theorem 1.*

(c) *The condition* $m \in C([a,b])$ *can be relaxed to* $m \in L^\infty([a,b])$ *in the non-endpoint case* $\frac{1}{p} - \frac{1}{q} > \frac{2\alpha}{d+1}$. *Technically, this is due to the fact that the operators* $\mathcal{T}_{\lambda,s}$ *from* (28) *are still well-defined for* $0 \leq \mathrm{Re}(s) < 1$ *(but not for* $\mathrm{Re}(s) = 1$*). Adapting the interpolation procedure from the Proof of Theorem 3 accordingly, one obtains the result. Moreover, the assumption* $m \in C([a,b])$ *can be replaced by a continuity assumption near* $|\xi| = a$ *(keeping the boundedness assumption) without changing the result.*

As indicated above, Theorem 3 can be extended to certain exponent pairs $(p,q) \in [1,\infty] \times [1,\infty]$ not belonging to D_α provided that m is sufficiently smooth. One example for such an improvement was given in part (a) of the previous remark. The question of optimal ranges of exponents is challenging even in special cases. For instance, let us assume $\lambda = 0$, $m \equiv 1$ and $A = \{\xi \in \mathbb{R}^d : 1 \leq |\xi| \leq 2\}$ where $d \geq 2$. Then the operators $T_\alpha := T_{0,\alpha}$ are given by

$$T_\alpha h = \mathcal{F}_d^{-1}\left(1_A(\cdot)(|\cdot|^2 - 1)^{-\alpha}\mathcal{F}_d h\right).$$

Their mapping properties are identical to those of so-called Bochner-Riesz operators with negative index. The only difference is that the annulus A is replaced by a ball and the singularity of the Fourier multiplier at the inner radius of the annulus now occurs at the boundary of the unit ball $B \subset \mathbb{R}^d$. More precisely, these operators are given by

$$\tilde{T}_\alpha h := \mathcal{F}_d^{-1}\left(1_B(\cdot)(1 - |\cdot|^2)^{-\alpha}\mathcal{F}_d h\right).$$

An alternative description as a convolution operator can be found in [8, p.225]. In the case $\alpha = 0$ this is the ball multiplier which is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ whenever $1 \leq p \leq 2 \leq q \leq \infty$. It is known that the operator $\tilde{T}_0$ is bounded on $L^p(\mathbb{R}^d)$ only for $p = 2$. This is a famous result from 1971 due to Fefferman [23]. Up to our knowledge, it is not known what the optimal range of exponents for the ball multiplier is. In the case $\alpha \in (0,1)$ the optimal region is contained in the set

$$\mathcal{D}_\alpha := \left\{(p,q) \in [1,\infty]^2 : \frac{1}{p} - \frac{1}{q} \geq \frac{2\alpha}{d+1},\ \frac{1}{p} > \frac{d-1+2\alpha}{2d},\ \frac{1}{q} < \frac{d+1-2\alpha}{2d}\right\},$$

see [8, Theorem (iv)]. A standing conjecture is that $\mathcal{D}_\alpha$ in fact coincides with the optimal region, see [32, Conjecture 2]. In the two-dimensional case $d = 2$ the conjecture is true:

Börjeson [8, Theorem (i)] proved the corresponding estimates except for the critical line $\frac{1}{p}-\frac{1}{q}=\frac{2\alpha}{d+1}$ and the missing piece was proved by Bak [6, Theorem 1]. In the higher-dimensional case $d\geq 3$ it is true for $\frac{(d+1)(d-1)}{2(d^2+4d-1)}<\alpha<1$ if d is odd and $\frac{(d+1)(d-2)}{2(d^2+3d-2)}<\alpha<1$ if d is even [32, Theorem 2.13]. We refer to [10, Theorem 1.1], [7, Theorem 4], [8, Theorem (iii)] and [26, Theorem 1] for earlier results in this direction. When restricted to radially symmetric functions $\mathcal{D}_\alpha$ is optimal in all space dimensions and for all $\alpha\in[0,1)$. In the case $\alpha=0$ this follows from [29, Theorem 2] and [31], while for $\alpha\in(0,1)$ this result can be found in [9, Theorem 1]. For estimates in the case $\alpha=0$ and $p=q$ with respect to mixed norms we refer to [11].

In addition to Theorem 3 we will need estimates for the operators $S_\lambda: L^p(\mathbb{R}^d)\to L^s(A)$ and their adjoints $S_\lambda^*: L^{s'}(A)\to L^{p'}(\mathbb{R}^d)$ given by

$$\begin{aligned} S_\lambda h &:= 1_A(\cdot)\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-a^2}}m(|\cdot|)\mathcal{F}_d h(\cdot), \\ S_\lambda^* g &:= \mathcal{F}_d^{-1}\left(1_A(\cdot)\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-a^2}}\overline{m(|\cdot|)}g(\cdot)\right). \end{aligned} \tag{7}$$

Theorem 4. *Let $d\in\mathbb{N}$, $0<a<b<\infty,\lambda\geq 0$ and $m\in L^\infty([a,b])$. Then the operators $S_\lambda: L^p(\mathbb{R}^d)\to L^s(A)$, $S_\lambda^*: L^{s'}(A)\to L^{p'}(\mathbb{R}^d)$ are bounded provided that $1\geq\frac{1}{p}\geq\frac{1}{2}, 1\geq\frac{1}{s}\geq\frac{1}{p'}$ and we have*

$$\|S_\lambda h\|_{L^s(A)}\lesssim\|h\|_{L^p(\mathbb{R}^d)}(1+\lambda)^{\frac{2}{s'}-\frac{2}{p}-\beta}.$$

where $\beta=0$ if $d=1$ and, for sufficiently small $\varepsilon>0$,

$$\beta=\min\left\{\frac{d-1}{p}-\frac{d-1}{s'},\frac{2(\frac{d+1}{p}-\frac{d-1}{s'}-1)}{p_*(d)'}-\varepsilon,\frac{\frac{2}{p_*(d)'}(\frac{1}{p}-\frac{1}{2})}{\frac{1}{p_*(d)}-\frac{1}{2}}-\varepsilon,\frac{2}{p'}\right\}\qquad \text{if } d\geq 2. \tag{8}$$

If the Restriction Conjecture is true, then the same estimate holds for $p_*(d)$ replaced by $\frac{2d}{d+1}$.

The outline of this paper is the following. In Section 2 we first derive a representation formula for the functions $\mathcal{R}(\lambda+\mathrm{i}\varepsilon)f,\mathcal{R}(\lambda\pm\mathrm{i}0)f$ that are of interest in Theorem 1. As a main tool we use one-sided Fourier transforms. In Section 3 we complete the list of required tools from Harmonic Analysis, state all the essential estimates (Propositions 3, 4, 5, 6) and combine them in order to prove Theorem 1. The application to nonlinear PDEs from Corollary 2 is demonstrated as well. In the Sections 4, 5, 6 we subsequently prove Theorem 2, Theorem 3 and Theorem 4. The Propositions are proved in the following four sections.

Before starting our analysis let us fix some notation and conventions. For $p\in[1,\infty]$, we write $L^p(\mathbb{R}^d)$ for the (classical) Lebesgue space of complex-valued p-integrable functions. The corresponding standard norms are denoted by $\|\cdot\|_{L^p(\mathbb{R}^d)}$. Moreover, we write $p'=\frac{p}{p-1}\in[1,\infty]$ for the conjugate exponent. The inner product in $L^2(\mathbb{R}^d)$ is $\langle f,g\rangle_{L^2(\mathbb{R}^d)}=\int_{\mathbb{R}^d}f(x)\overline{g(x)}\,\mathrm{d}x$. The d-dimensional Fourier transform is given by $\mathcal{F}_d g(\xi)=(2\pi)^{-d/2}\int_{\mathbb{R}^d}g(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x$ with inverse $\mathcal{F}_d^{-1}h(\xi)=(\mathcal{F}_d h)(-\xi)$ where $g,h:\mathbb{R}^d\to\mathbb{R}$ are sufficiently regular. At some points it will be convenient to slightly abuse the notation by writing $\mathcal{F}_d^{-1}(g(\xi))(x)$ in place of $\mathcal{F}_d^{-1}(g)(x)$. The space of complex-valued Schwartz functions is $\mathcal{S}(\mathbb{R}^n)$. The sphere of radius μ in $\mathbb{R}^d$ is given by $\mathbb{S}_\mu^{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = \mu\}$ along with its canonical surface measure σ_μ. The corresponding Lebesgue spaces is denoted by $L^s(\mathbb{S}_\mu^{d-1}), s \in [1,\infty]$.

2. The representation formula

In this section we derive a representation formula for the outgoing solution of the Helmholtz equation (2) where V is the step potential from (1), i.e.,

$$V(x,y) = \begin{cases} V_1 & \text{if } x \in \mathbb{R}^{n-1}, y > 0, \\ V_2 & \text{if } x \in \mathbb{R}^{n-1}, y < 0 \end{cases} \qquad \text{with } V_1 > V_2.$$

To this end we solve the perturbed Helmholtz equation

$$-\Delta u_\varepsilon + V(x,y)u_\varepsilon - (\lambda + \mathrm{i}\varepsilon)u_\varepsilon = f \quad \text{in } \mathbb{R}^n \tag{9}$$

where $\lambda > V_1 > V_2$ and $\varepsilon > 0$. We define the one-sided Fourier transforms of $f \in \mathcal{S}(\mathbb{R}^n)$ via

$$(\mathcal{F}_n^\pm f)(\xi,\eta) := (\mathcal{F}_n f_\pm)(\xi,\eta) \qquad \text{where } f_\pm(x,y) := f(x,y)\cdot 1_{(0,\infty)}(\pm y).$$

Proposition 1. *For all $v \in \mathcal{S}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^{n-1}, \eta \in \mathbb{R}$ we have*

$$\mathcal{F}_n^+(-\Delta v)(\xi,\eta) = (|\xi|^2+\eta^2)(\mathcal{F}_n^+ v)(\xi,\eta) + (2\pi)^{-\frac12}\left(\mathrm{i}\eta\mathcal{F}_{n-1}[v(\cdot,0)](\xi) + \mathcal{F}_{n-1}[v'(\cdot,0)](\xi)\right),$$
$$\mathcal{F}_n^-(-\Delta v)(\xi,\eta) = (|\xi|^2+\eta^2)(\mathcal{F}_n^- v)(\xi,\eta) - (2\pi)^{-\frac12}\left(\mathrm{i}\eta\mathcal{F}_{n-1}[v(\cdot,0)](\xi) + \mathcal{F}_{n-1}[v'(\cdot,0)](\xi)\right).$$

Moreover, $\mathrm{ran}(\mathcal{F}_n^+), \mathrm{ran}(\mathcal{F}_n^-)$ *are $L^2(\mathbb{R}^n)$-orthogonal to each other and $\mathcal{F}_n = \mathcal{F}_n^+ + \mathcal{F}_n^-$. Here we denote $\partial_y v(x,y) =: v'(x,y)$.*

We only comment on the orthogonality property. For $f, g \in \mathcal{S}(\mathbb{R}^n)$ Plancherel's identity implies

$$\langle \mathcal{F}_n^+ f, \mathcal{F}_n^- g\rangle_{L^2(\mathbb{R}^n)} = \langle \mathcal{F}_n f_+, \mathcal{F}_n g_-\rangle_{L^2(\mathbb{R}^n)} = \langle f_+, g_-\rangle_{L^2(\mathbb{R}^n)} = 0$$

since the supports of f_+, g_- intersect only in a null set. We introduce $\mu_j := \sqrt{\lambda - V_j} > 0$ and the complex-valued functions $\nu_{j,\varepsilon} : \mathbb{R}^{n-1} \to \mathbb{C}$ via

$$\nu_{j,\varepsilon}(\xi)^2 = \mu_j^2 - |\xi|^2 + \mathrm{i}\varepsilon = \lambda - V_j - |\xi|^2 + \mathrm{i}\varepsilon \quad \text{and} \quad \mathrm{Im}(\nu_{j,\varepsilon}(\xi)) > 0.$$

Notice that $\nu_{j,\varepsilon}(\xi) \to \nu_j(\xi)$ as $\varepsilon \searrow 0$ where

$$\nu_j(\xi) := \begin{cases} (\mu_j^2 - |\xi|^2)^{\frac12} & \text{if } |\xi| \le \mu_j, \\ \mathrm{i}(|\xi|^2 - \mu_j^2)^{\frac12} & \text{if } |\xi| \ge \mu_j. \end{cases} \tag{10}$$

Later we will need the following elementary estimate:

$$1 + |\xi| \lesssim |\nu_j(\xi)|\sqrt{1 + |\nabla\nu_j(\xi)|^2} \lesssim 1 + |\xi| \qquad (\xi \in \mathbb{R}^{n-1}). \tag{11}$$

Proposition 2. *Let $\lambda > V_1 > V_2$ and $f \in S(\mathbb{R}^n)$. Then, for any given $\varepsilon > 0$, the unique solution $u_\varepsilon \in S(\mathbb{R}^n)$ of (9) is given by*

$$
\begin{aligned}
u_\varepsilon(x,y) &= \mathcal{F}_n^{-1}\left(\frac{\mathcal{F}_n^+ f}{|\cdot|^2-\mu_1^2-\mathrm{i}\varepsilon}+\frac{\mathcal{F}_n^- f}{|\cdot|^2-\mu_2^2-\mathrm{i}\varepsilon}\right)(x,y)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}}(m_{1,\varepsilon}g_{+,\varepsilon}+m_{2,\varepsilon}g_{-,\varepsilon})\right)(x)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}}(m_{3,\varepsilon}g_{+,\varepsilon}+m_{4,\varepsilon}g_{-,\varepsilon})\right)(x)
\end{aligned}
\tag{12}
$$

where $g_{+,\varepsilon}(\xi) = \mathcal{F}_n^+ f(\xi, -\nu_{1,\varepsilon}(\xi))$, $g_{-,\varepsilon}(\xi) = \mathcal{F}_n^- f(\xi,\nu_{2,\varepsilon}(\xi))$ and

$$
\begin{aligned}
m_{1,\varepsilon}(\xi) &:= \frac{\mathrm{i}\sqrt{\pi/2}}{\nu_{1,\varepsilon}(\xi)+\nu_{2,\varepsilon}(\xi)}\cdot\left(\operatorname{sign}(y)-\frac{\nu_{2,\varepsilon}(\xi)}{\nu_{1,\varepsilon}(\xi)}\right),\\
m_{2,\varepsilon}(\xi) &:= \frac{\mathrm{i}\sqrt{\pi/2}}{\nu_{1,\varepsilon}(\xi)+\nu_{2,\varepsilon}(\xi)}\cdot(1+\operatorname{sign}(y)),\\
m_{3,\varepsilon}(\xi) &:= \frac{\mathrm{i}\sqrt{\pi/2}}{\nu_{1,\varepsilon}(\xi)+\nu_{2,\varepsilon}(\xi)}\cdot(1-\operatorname{sign}(y)),\\
m_{4,\varepsilon}(\xi) &:= \frac{\mathrm{i}\sqrt{\pi/2}}{\nu_{1,\varepsilon}(\xi)+\nu_{2,\varepsilon}(\xi)}\cdot\left(-\operatorname{sign}(y)-\frac{\nu_{1,\varepsilon}(\xi)}{\nu_{2,\varepsilon}(\xi)}\right).
\end{aligned}
\tag{13}
$$

Proof. From Proposition 1 we obtain

$$(|\xi|^2+\eta^2-(\mu_1^2+\mathrm{i}\varepsilon))\mathcal{F}_n^+u_\varepsilon(\xi,\eta)+(2\pi)^{-\frac12}\left(\mathrm{i}\eta\mathcal{F}_{n-1}[u_\varepsilon(\cdot,0)](\xi)+\mathcal{F}_{n-1}[u_\varepsilon'(\cdot,0)](\xi)\right)=\mathcal{F}_n^+f(\xi,\eta),$$
$$(|\xi|^2+\eta^2-(\mu_2^2+\mathrm{i}\varepsilon))\mathcal{F}_n^-u_\varepsilon(\xi,\eta)-(2\pi)^{-\frac12}\left(\mathrm{i}\eta\mathcal{F}_{n-1}[u_\varepsilon(\cdot,0)](\xi)+\mathcal{F}_{n-1}[u_\varepsilon'(\cdot,0)](\xi)\right)=\mathcal{F}_n^-f(\xi,\eta).$$

So $\nu_{j,\varepsilon}(\xi)^2 = \mu_j^2 + \mathrm{i}\varepsilon - |\xi|^2$ yields the formula

$$
\begin{aligned}
\mathcal{F}_n^+u_\varepsilon(\xi,\eta) &= \frac{\mathcal{F}_n^+f(\xi,\eta)-(2\pi)^{-\frac12}\left(\mathrm{i}\eta\mathcal{F}_{n-1}[u_\varepsilon(\cdot,0)](\xi)+\mathcal{F}_{n-1}[u_\varepsilon'(\cdot,0)](\xi)\right)}{\eta^2-\nu_{1,\varepsilon}(\xi)^2},\\
\mathcal{F}_n^-u_\varepsilon(\xi,\eta) &= \frac{\mathcal{F}_n^-f(\xi,\eta)+(2\pi)^{-\frac12}\left(\mathrm{i}\eta\mathcal{F}_{n-1}[u_\varepsilon(\cdot,0)](\xi)+\mathcal{F}_{n-1}[u_\varepsilon'(\cdot,0)](\xi)\right)}{\eta^2-\nu_{2,\varepsilon}(\xi)^2}.
\end{aligned}
\tag{14}
$$

We now exploit $\operatorname{ran}(\mathcal{F}_n^+) \perp \operatorname{ran}(\mathcal{F}_n^-)$ in order to compute $\mathcal{F}_{n-1}[u_\varepsilon(\cdot,0)](\xi), \mathcal{F}_{n-1}[u_\varepsilon'(\cdot,0)](\xi)$. The Residue Theorem gives for all $\phi \in \mathcal{S}(\mathbb{R}^{n-1})$ and $\zeta(z) := \sqrt{2\pi}\mathrm{e}^{-|z|}$ for $z \in \mathbb{R}$

$$
\begin{aligned}
0 &= \langle \mathcal{F}_n^-u_\varepsilon, \mathcal{F}_n^+(\phi\otimes\zeta)\rangle_{L^2(\mathbb{R}^n)}\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\left(\frac{\mathcal{F}_n^-f(\xi,\eta)+(2\pi)^{-\frac12}(\mathrm{i}\eta\mathcal{F}_{n-1}[u_\varepsilon(\cdot,0)](\xi)+\mathcal{F}_{n-1}[u_\varepsilon'(\cdot,0)](\xi))}{\eta^2-\nu_{2,\varepsilon}(\xi)^2}\cdot\frac{\overline{\hat{\phi}(\xi)}}{-\mathrm{i}\eta+1}\right)\mathrm{d}\xi\,\mathrm{d}\eta\\
&= \int_{\mathbb{R}^{n-1}}\overline{\hat{\phi}(\xi)}\cdot\Bigg[\left(\int_{\mathbb{R}}\frac{\mathcal{F}_n^-f(\xi,\eta)}{(\eta^2-\nu_{2,\varepsilon}(\xi)^2)(-\mathrm{i}\eta+1)}\,\mathrm{d}\eta\right)\\
&\qquad+\frac{\mathcal{F}_{n-1}[u_\varepsilon'(\cdot,0)](\xi)}{\sqrt{2\pi}}\left(\int_{\mathbb{R}}\frac{1}{(\eta^2-\nu_{2,\varepsilon}(\xi)^2)(-\mathrm{i}\eta+1)}\,\mathrm{d}\eta\right)
\end{aligned}
$$

$$+\frac{\mathcal{F}_{n-1}[u_\varepsilon(\cdot,0)](\xi)}{\sqrt{2\pi}}\left(\int_{\mathbb{R}}\frac{\mathrm{i}\eta}{(\eta^2-\nu_{2,\varepsilon}(\xi)^2)(-\mathrm{i}\eta+1)}\,\mathrm{d}\eta\right)\Bigg]\,\mathrm{d}\xi$$

$$=\int_{\mathbb{R}^{n-1}}\frac{\mathrm{i}\pi\overline{\hat{\phi}(\xi)}}{\nu_{2,\varepsilon}(\xi)(1-\mathrm{i}\nu_{2,\varepsilon}(\xi))}\cdot\left(\mathcal{F}_n^- f(\xi,\nu_{2,\varepsilon}(\xi))+\frac{\mathrm{i}\nu_{2,\varepsilon}(\xi)\mathcal{F}_{n-1}[u_\varepsilon(\cdot,0)](\xi)+\mathcal{F}_{n-1}[u_\varepsilon'(\cdot,0)](\xi)}{\sqrt{2\pi}}\right)\mathrm{d}\xi$$

and similarly

$$\begin{aligned}0&=\langle\mathcal{F}_n^+u_\varepsilon,\mathcal{F}_n^-(\phi\otimes\zeta)\rangle_{L^2(\mathbb{R}^n)}\\&=\int_{\mathbb{R}^{n-1}}\frac{\mathrm{i}\pi\overline{\hat{\phi}(\xi)}}{\nu_{1,\varepsilon}(\xi)(1-\mathrm{i}\nu_{1,\varepsilon}(\xi))}\cdot\left(\mathcal{F}_n^+ f(\xi,-\nu_{1,\varepsilon}(\xi))-\frac{\mathrm{i}\nu_{2,\varepsilon}(\xi)\mathcal{F}_{n-1}[u_\varepsilon(\cdot,0)](\xi)+\mathcal{F}_{n-1}[u_\varepsilon'(\cdot,0)](\xi)}{\sqrt{2\pi}}\right)\mathrm{d}\xi.\end{aligned}$$

Since $\phi\in\mathcal{S}(\mathbb{R}^{n-1})$ was arbitrary, we get for almost all $\xi\in\mathbb{R}^{n-1}$

$$\begin{pmatrix}-\mathrm{i}\nu_{1,\varepsilon}(\xi) & 1\\ -\mathrm{i}\nu_{2,\varepsilon}(\xi) & -1\end{pmatrix}\begin{pmatrix}\mathcal{F}_{n-1}[u_\varepsilon(\cdot,0)](\xi)\\ \mathcal{F}_{n-1}[u_\varepsilon'(\cdot,0)](\xi)\end{pmatrix}=\sqrt{2\pi}\begin{pmatrix}\mathcal{F}_n^+ f(\xi,-\nu_{1,\varepsilon}(\xi))\\ \mathcal{F}_n^- f(\xi,\nu_{2,\varepsilon}(\xi))\end{pmatrix}=\sqrt{2\pi}\begin{pmatrix}g_{+,\varepsilon}(\xi)\\ g_{-,\varepsilon}(\xi)\end{pmatrix}.$$

Inverting this linear system we get

$$\begin{pmatrix}\mathcal{F}_{n-1}[u_\varepsilon(\cdot,0)](\xi)\\ \mathcal{F}_{n-1}[u_\varepsilon'(\cdot,0)](\xi)\end{pmatrix}=\frac{\sqrt{2\pi}}{\nu_{1,\varepsilon}(\xi)+\nu_{2,\varepsilon}(\xi)}\begin{pmatrix}\mathrm{i} & \mathrm{i}\\ \nu_{2,\varepsilon}(\xi) & -\nu_{1,\varepsilon}(\xi)\end{pmatrix}\begin{pmatrix}g_{+,\varepsilon}(\xi)\\ g_{-,\varepsilon}(\xi)\end{pmatrix}. \tag{15}$$

From this and $\mathcal{F}_n^++\mathcal{F}_n^-=\mathcal{F}_n$ we get for $x\in\mathbb{R}^{n-1},y\in\mathbb{R}$

$$\begin{aligned}u_\varepsilon(x,y)&=\mathcal{F}_n^{-1}(\mathcal{F}_n^+u_\varepsilon+\mathcal{F}_n^-u_\varepsilon)(x,y)\\
&\overset{(14)}{=}\mathcal{F}_n^{-1}\left(\frac{\mathcal{F}_n^+f(\xi,\eta)}{\eta^2-\nu_{1,\varepsilon}(\xi)^2}+\frac{\mathcal{F}_n^-f(\xi,\eta)}{\eta^2-\nu_{2,\varepsilon}(\xi)^2}\right)(x,y)\\
&\quad-\frac{1}{\sqrt{2\pi}}\mathcal{F}_n^{-1}\left[\left(\frac{\mathrm{i}\eta}{\eta^2-\nu_{1,\varepsilon}(\xi)^2}-\frac{\mathrm{i}\eta}{\eta^2-\nu_{2,\varepsilon}(\xi)^2}\right)\cdot(\mathcal{F}_nu_\varepsilon)(\xi,0)\right](x,y)\\
&\quad-\frac{1}{\sqrt{2\pi}}\mathcal{F}_n^{-1}\left[\left(\frac{1}{\eta^2-\nu_{1,\varepsilon}(\xi)^2}-\frac{1}{\eta^2-\nu_{2,\varepsilon}(\xi)^2}\right)\cdot(\mathcal{F}_nu_\varepsilon)'(\xi,0)\right](x,y)\\
&=\mathcal{F}_n^{-1}\left(\frac{\mathcal{F}_n^+f}{|\cdot|^2-\mu_1^2-\mathrm{i}\varepsilon}+\frac{\mathcal{F}_n^-f}{|\cdot|^2-\mu_2^2-\mathrm{i}\varepsilon}\right)(x,y)\\
&\quad-\frac{1}{2\pi}\mathcal{F}_{n-1}^{-1}\left(\int_{\mathbb{R}}\left(\frac{\mathrm{i}\eta}{\eta^2-\nu_{1,\varepsilon}(\xi)^2}-\frac{\mathrm{i}\eta}{\eta^2-\nu_{2,\varepsilon}(\xi)^2}\right)\mathrm{e}^{\mathrm{i}y\eta}\,\mathrm{d}\eta\cdot(\mathcal{F}_nu_\varepsilon)(\xi,0)\right)(x)\\
&\quad-\frac{1}{2\pi}\mathcal{F}_{n-1}^{-1}\left(\int_{\mathbb{R}}\left(\frac{1}{\eta^2-\nu_{1,\varepsilon}(\xi)^2}-\frac{1}{\eta^2-\nu_{2,\varepsilon}(\xi)^2}\right)\mathrm{e}^{\mathrm{i}y\eta}\,\mathrm{d}\eta\cdot(\mathcal{F}_nu_\varepsilon)'(\xi,0)\right)(x)\\
&=\mathcal{F}_n^{-1}\left(\frac{\mathcal{F}_n^+f}{|\cdot|^2-\mu_{1,\varepsilon}^2}+\frac{\mathcal{F}_n^-f}{|\cdot|^2-\mu_{2,\varepsilon}^2}\right)(x,y)\\
&\quad+\frac{\mathrm{sign}(y)}{2}\mathcal{F}_{n-1}^{-1}\left(\left(\mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}}-\mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}}\right)(\mathcal{F}_nu_\varepsilon)(\cdot,0)\right)(x)\\
&\quad-\frac{\mathrm{i}}{2}\mathcal{F}_{n-1}^{-1}\left(\left(\frac{\mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}}}{\nu_{1,\varepsilon}}-\frac{\mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}}}{\nu_{2,\varepsilon}}\right)(\mathcal{F}_nu_\varepsilon)'(\cdot,0)\right)(x).\end{aligned}$$

Combining this identity with (15) we find (12),(13). □

It will turn out useful to decompose the last lines of (12) according to

$$w_\varepsilon(x,y)+\mathfrak{w}_\varepsilon(x,y)+\mathfrak{W}_\varepsilon(x,y)+W_\varepsilon(x,y)$$

where the small frequencies are collected in w_ε, the large ones in W_ε and the remaining intermediate ranges of frequencies are covered by the terms $\mathfrak{w}_\varepsilon,\mathfrak{W}_\varepsilon$. Formally,

$$\begin{aligned}
w_\varepsilon(x,y) &:= \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}}(1_{|\cdot|\le\mu_1}m_{1,\varepsilon}g_{+,\varepsilon}+1_{|\cdot|\le\mu_1}m_{2,\varepsilon}g_{-,\varepsilon})\right)(x)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}}(1_{|\cdot|\le\mu_1}m_{3,\varepsilon}g_{+,\varepsilon}+1_{|\cdot|\le\mu_2}m_{4,\varepsilon}g_{-,\varepsilon})\right)(x),\\
\mathfrak{w}_\varepsilon(x,y) &:= \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}}1_{\mu_1<|\cdot|\le\mu_2}m_{2,\varepsilon}g_{-,\varepsilon}+\mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}}1_{\mu_1<|\cdot|\le\mu_2}m_{3,\varepsilon}g_{+,\varepsilon}\right)(x),\\
\mathfrak{W}_\varepsilon(x,y) &:= \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}}1_{\mu_1<|\cdot|\le\mu_1+\mu_2}m_{1,\varepsilon}g_{+,\varepsilon}+\mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}}1_{\mu_2<|\cdot|\le\mu_1+\mu_2}m_{2,\varepsilon}g_{-,\varepsilon}\right)(x)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}}1_{\mu_2<|\cdot|\le\mu_1+\mu_2}m_{3,\varepsilon}g_{+,\varepsilon}+\mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}}1_{\mu_2<|\cdot|\le\mu_1+\mu_2}m_{4,\varepsilon}g_{-,\varepsilon}\right)(x),\\
W_\varepsilon(x,y) &:= \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_{1,\varepsilon}}(1_{|\cdot|>\mu_1+\mu_2}m_{1,\varepsilon}g_{+,\varepsilon}+1_{|\cdot|>\mu_1+\mu_2}m_{2,\varepsilon}g_{-,\varepsilon})\right)(x)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_{2,\varepsilon}}(1_{|\cdot|>\mu_1+\mu_2}m_{3,\varepsilon}g_{+,\varepsilon}+1_{|\cdot|>\mu_1+\mu_2}m_{4,\varepsilon}g_{-,\varepsilon})\right)(x).
\end{aligned}$$

In the following section we will state estimates for $w_\varepsilon,\mathfrak{w}_\varepsilon,\mathfrak{W}_\varepsilon,W_\varepsilon$ (see the Propositions 3, 4, 5, 6) that lead to the proof of Theorem 1. Before going on with this we compute the limit of u_ε as $\varepsilon\searrow 0$. The above representation formula for u_ε leads to the definition

$$\begin{aligned}
\big(\mathcal{R}(\lambda+\mathrm{i}0)f\big)(x,y) &:= u_+(x,y)\\
&:= \lim_{\varepsilon\searrow 0}\mathcal{F}_n^{-1}\left(\frac{\mathcal{F}_n^+f}{|\cdot|^2-\mu_1^2-\mathrm{i}\varepsilon}+\frac{\mathcal{F}_n^-f}{|\cdot|^2-\mu_2^2-\mathrm{i}\varepsilon}\right)(x,y)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_1}(m_1g_++m_2g_-)\right)(x)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_2}(m_3g_++m_4g_-)\right)(x)
\end{aligned}\tag{16}$$

where the limit in the first line will be a weak limit in $L^q(\mathbb{R}^n)$. As above, the last two lines of (16) can be rewritten as

$$w(x,y)+\mathfrak{w}(x,y)+\mathfrak{W}(x,y)+W(x,y)$$

where $g_+(\xi)=\mathcal{F}_n^+f(\xi,-\nu_1(\xi))$, $g_-(\xi)=\mathcal{F}_n^-f(\xi,\nu_2(\xi))$ and

$$\begin{aligned}
w(x,y) &:= \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_1}(1_{|\cdot|\le\mu_1}m_1g_++1_{|\cdot|\le\mu_1}m_2g_-)\right)(x)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_2}(1_{|\cdot|\le\mu_1}m_3g_++1_{|\cdot|\le\mu_2}m_4g_-)\right)(x),\\
\mathfrak{w}(x,y) &:= \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_1}1_{\mu_1<|\cdot|\le\mu_2}m_2g_-+\mathrm{e}^{\mathrm{i}|y|\nu_2}1_{\mu_1<|\cdot|\le\mu_2}m_3g_+\right)(x),\\
\mathfrak{W}(x,y) &:= \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_1}1_{\mu_1<|\cdot|\le\mu_1+\mu_2}m_1g_++\mathrm{e}^{\mathrm{i}|y|\nu_1}1_{\mu_2<|\cdot|\le\mu_1+\mu_2}m_2g_-\right)(x)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_2}1_{\mu_2<|\cdot|\le\mu_1+\mu_2}m_3g_++\mathrm{e}^{\mathrm{i}|y|\nu_2}1_{\mu_2<|\cdot|\le\mu_1+\mu_2}m_4g_-\right)(x),\\
W(x,y) &:= \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_1}(1_{|\cdot|>\mu_1+\mu_2}m_1g_++1_{|\cdot|>\mu_1+\mu_2}m_2g_-)\right)(x)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_2}(1_{|\cdot|>\mu_1+\mu_2}m_3g_++1_{|\cdot|>\mu_1+\mu_2}m_4g_-)\right)(x).
\end{aligned}\tag{17}$$

Here,

$$
\begin{aligned}
m_1(\xi) &:= \frac{\mathrm{i}\sqrt{\pi/2}}{\nu_1(\xi)+\nu_2(\xi)}\cdot\left(\operatorname{sign}(y)-\frac{\nu_2(\xi)}{\nu_1(\xi)}\right),\\
m_2(\xi) &:= \frac{\mathrm{i}\sqrt{\pi/2}}{\nu_1(\xi)+\nu_2(\xi)}\cdot\left(1+\operatorname{sign}(y)\right),\\
m_3(\xi) &:= \frac{\mathrm{i}\sqrt{\pi/2}}{\nu_1(\xi)+\nu_2(\xi)}\cdot\left(1-\operatorname{sign}(y)\right),\\
m_4(\xi) &:= \frac{\mathrm{i}\sqrt{\pi/2}}{\nu_1(\xi)+\nu_2(\xi)}\cdot\left(-\operatorname{sign}(y)-\frac{\nu_1(\xi)}{\nu_2(\xi)}\right).
\end{aligned}
\tag{18}
$$

Notice that in the case without a jump in the potential we have $\mu_1=\mu_2=:\mu,\nu_1\equiv\nu_2$ and the formula (16) simplifies to

$$
\big(\mathcal{R}(\lambda+\mathrm{i}0)f\big)(x,y)=\lim_{\varepsilon\searrow 0}\mathcal{F}_n^{-1}\left(\frac{\mathcal{F}_nf}{|\cdot|^2-\mu^2-\mathrm{i}\varepsilon}\right)(x,y)\qquad\text{if } V_1=V_2
$$

because of $m_1+m_3\equiv m_2+m_4\equiv 0$. At several places we shall need the estimates

$$
\begin{aligned}
|m_1(\xi)| &\lesssim (1+|\xi|)^{-1}|\nu_1(\xi)|^{-1} && (\xi\in\mathbb{R}^{n-1}),\\
|m_2(\xi)|+|m_3(\xi)| &\lesssim (1+|\xi|)^{-1} && (\xi\in\mathbb{R}^{n-1}),\\
|m_4(\xi)| &\lesssim (1+|\xi|)^{-1}|\nu_2(\xi)|^{-1} && (\xi\in\mathbb{R}^{n-1}).
\end{aligned}
\tag{19}
$$

3. Proof of Theorem 1 and Corollary 2

We first collect a few tools from Harmonic Analysis that we will need in our estimates. For $n\geq 3$, the first line in (12) and (16) may be analyzed with the aid of Gutiérrez' Limiting Absorption Principle [27, Theorem 6] (see also [30, Theorem 2.3]) for the Helmholtz equation with constant coefficients in $\mathbb{R}^n$. The corresponding result for the case $n=2$ was provided by Evéquoz [17, Theorem 2.1].

Theorem 5 (Gutiérrez, Evéquoz)**.** *Assume $n\in\mathbb{N},n\geq 2,(p,q)\in\mathcal{D}$ and $V\equiv V_1=V_2$. If $\lambda>V_1=V_2$ then the solutions u_ε of* (9) *and $u_+,u_-:=\overline{u_+}$ from* (16) *satisfy*

$$
\|u_+\|_{L^q(\mathbb{R}^n)}+\|u_-\|_{L^q(\mathbb{R}^n)}+\sup_{0<|\varepsilon|\leq 1}\|u_\varepsilon\|_{L^q(\mathbb{R}^n)}\lesssim\|f\|_{L^p(\mathbb{R}^n)}.
$$

More can be said about the qualitative properties of u_+,u_-, especially concerning their behaviour at infinity which is governed by an outgoing respectively incoming Sommerfeld radiation condition that even characterize these solutions of the Helmholtz equation, see for instance [27, Corollary 1] in the case $n\geq 3$. As solutions of the Helmholtz equation (2), the imaginary parts of $u_\pm$ are solutions of the homogeneous Helmholtz equation. Computations reveal (see for instance (5.6) in [37]) that $\operatorname{Im}(u_+)=-\operatorname{Im}(u_-)$ is a multiple of the function $\mathcal{F}_d^{-1}(\mathcal{F}_df\,\mathrm{d}\sigma_\mu)$ where $\mu=\sqrt{\lambda-V_1}=\sqrt{\lambda-V_2}>0$. This corresponds to a Herglotz wave given by the density $\mathcal{F}_df$ on $\mathbb{S}^{d-1}_\mu$. So Theorem 5 implies the following.

Corollary 3. *For $n \in \mathbb{N}, n \geq 2$ and $(p,q) \in \mathcal{D}$ the linear operator $f \mapsto \mathcal{F}_n^{-1}(\mathcal{F}_n f \, \mathrm{d}\sigma_\mu)$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for all $\mu > 0$.*

Another reference for this result and for the optimality of the asserted range can be found in [32, Theorem 2.14]. We will also need several Fourier restriction theorems for the Fourier transforms $\mathcal{F}_n, \mathcal{F}_{n-1}$ restricted to spheres in $\mathbb{R}^{n-1}$ respectively $\mathbb{R}^n$. We use d as the dimensional parameter.

Theorem 6 (Stein-Tomas). *Let $d \in \mathbb{N}, d \geq 2$ and $1 \leq p \leq \frac{2(d+1)}{d+3}$, $\mu > 0$. Then*

$$\|\mathcal{F}_d f\|_{L^2(\mathbb{S}^{d-1}_\mu)} \lesssim \mu^{\frac{d-1}{2} - \frac{d}{p'}} \|f\|_{L^p(\mathbb{R}^d)}, \qquad \|\mathcal{F}_d(g \, \mathrm{d}\sigma_\mu)\|_{L^{p'}(\mathbb{R}^d)} \lesssim \mu^{\frac{d-1}{2} - \frac{d}{p'}} \|g\|_{L^2(\mathbb{S}^{d-1}_\mu)}.$$

The Stein-Tomas Theorem (see [42] or [39, p.386]) is one particularly important estimate that embeds into a whole family of estimates. The Restriction Conjecture says that the estimates

$$\|\mathcal{F}_d h\|_{L^{q'}(\mathbb{S}^{d-1}_\mu)} \lesssim \mu^{\frac{d-1}{q'} - \frac{d}{p'}} \|h\|_{L^p(\mathbb{R}^d)}, \qquad \|\mathcal{F}_d(g \, \mathrm{d}\sigma_\mu)\|_{L^{p'}(\mathbb{R}^d)} \lesssim \mu^{\frac{d-1}{q'} - \frac{d}{p'}} \|g\|_{L^q(\mathbb{S}^{d-1}_\mu)}. \tag{20}$$

hold whenever $p' > \frac{2d}{d-1}, q \geq \left(\frac{d-1}{d+1} p'\right)'$. In the two-dimensional case $d = 2$ the validity of (20) is known since the 1970s [43, Theorem 3], [22, p.33-34]. In the higher-dimensional case, however, the conjecture is still unsolved. Up to our knowledge, the strongest known result in this direction is due to Tao, see [40, Figure 3] and [41, p.1382].

Theorem 7 (Fefferman, Zygmund). *Let $d = 2$ and $p' > \frac{2d}{d-1}, q \geq \left(\frac{d-1}{d+1} p'\right)', \mu > 0$. Then (20) holds.*

Theorem 8 (Tao). *Let $d \in \mathbb{N}, d \geq 3$ and $p' > \frac{2(d+2)}{d}, q \geq \left(\frac{d-1}{d+1} p'\right)', \mu > 0$. Then (20) holds.*

As a consequence, in our analysis we can use (20) for $p' > p_*(d), q \geq \left(\frac{d-1}{d+1} p'\right)'$. We recall $p_*(d) = q_*(d)' = \frac{2d}{d+1}$ in the case $d = 2$ and $p_*(d) = q_*(d)' = \frac{2(d+2)}{d+4}$ in the case $d \geq 3$. All other major technical results are contained in Theorem 2 and Theorem 3 that we presented in the Introduction. In view of the representation formula (16) and the following remarks we will demonstrate Theorem 1 by a separate discussion for four different frequency regimes. Our result for the small frequency parts are the following.

Proposition 3. *Let $n \in \mathbb{N}, n \geq 2$ and $\frac{1}{p} > \frac{1}{p_*(n)}, \frac{1}{q} < \frac{1}{q_*(n)}, \frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1}$. Then we have*

$$\|w\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\leq 1} \|w_\varepsilon\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

In particular, this holds for all $(p,q) \in \tilde{\mathcal{D}}$. If the Restriction Conjecture is true, then these estimates even hold whenever $\frac{1}{p} > \frac{n+1}{2n}, \frac{1}{q} < \frac{n-1}{2n}, \frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1}$ and hence for all $(p,q) \in \mathcal{D}$.

The proof of Proposition 3 will be given in Section 7. In Section 8, we analyze the first of the two terms containing intermediate frequencies. We will prove the following result.

Proposition 4. *Let $n \in \mathbb{N}, n \geq 2$ and $\frac{1}{p} > \frac{1}{p_*(n)}, \frac{1}{q} < \frac{1}{q_*(n)}, \frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1}$. Then we have*

$$\|\mathfrak{w}\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\leq 1} \|\mathfrak{w}_\varepsilon\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

In particular, this holds for all $(p,q)\in\tilde{\mathcal{D}}$. If the Restriction Conjecture is true, then this estimate even holds for $\frac{1}{p}>\frac{n+1}{2n},\frac{1}{q}<\frac{n-1}{2n},\frac{1}{p}-\frac{1}{q}\geq\frac{2}{n+1}$ and thus for all $(p,q)\in\mathcal{D}$.

The estimates related to the second range of intermediate frequencies rely on Theorem 2 ($n=2$) and Theorem 3 ($n\geq 3$). The proof is provided in Section 9.

Proposition 5. *Let $n\in\mathbb{N},n\geq 2$ and $\frac{1}{p}>\frac{n+1}{2n}$, $\frac{1}{q}<\frac{n-1}{2n}$, $\frac{1}{p}-\frac{1}{q}\geq\frac{2}{n+1}$. Then we have*

$$\|\mathfrak{W}\|_{L^q(\mathbb{R}^n)}+\sup_{0<|\varepsilon|\leq 1}\|\mathfrak{W}_\varepsilon\|_{L^q(\mathbb{R}^n)}\lesssim\|f\|_{L^p(\mathbb{R}^n)}.$$

In particular, this holds for all $(p,q)\in\mathcal{D}$.

The estimates for the large frequency parts W,W_ε are the easiest ones. The proof will be presented in Section 10.

Proposition 6. *Let $n\in\mathbb{N},n\geq 2$ and $1\leq p\leq 2\leq q\leq\infty$ with $0\leq\frac{1}{p}-\frac{1}{q}\leq\frac{2}{n}$ and $\frac{1}{p}-\frac{1}{q}<\frac{2}{n}$ if $p=1$ or $q=\infty$. Then we have*

$$\|W\|_{L^q(\mathbb{R}^n)}+\sup_{0<|\varepsilon|\leq 1}\|W_\varepsilon\|_{L^q(\mathbb{R}^n)}\lesssim\|f\|_{L^p(\mathbb{R}^n)}.$$

In particular, this holds for all $(p,q)\in\mathcal{D}$.

Proof of Theorem 1. From Proposition 2 and the representation formulas (12), (16) we get

$$\begin{aligned}
&\|u_+\|_{L^q(\mathbb{R}^n)}+\|u_-\|_{L^q(\mathbb{R}^n)}+\sup_{0<|\varepsilon|\leq 1}\|u_\varepsilon\|_{L^q(\mathbb{R}^n)}\\
&\lesssim\sup_{0<|\varepsilon|\leq 1}\left\|\mathcal{F}_n^{-1}\left(\frac{\mathcal{F}_n^+f}{|\cdot|^2-\mu_1^2-\mathrm{i}\varepsilon}+\frac{\mathcal{F}_n^-f}{|\cdot|^2-\mu_2^2-\mathrm{i}\varepsilon}\right)\right\|_{L^q(\mathbb{R}^n)}\\
&\quad+\|w\|_{L^q(\mathbb{R}^n)}+\|\mathfrak{w}\|_{L^q(\mathbb{R}^n)}+\|\mathfrak{W}\|_{L^q(\mathbb{R}^n)}+\|W\|_{L^q(\mathbb{R}^n)}\\
&\quad+\sup_{0<|\varepsilon|\leq 1}\left(\|w_\varepsilon\|_{L^q(\mathbb{R}^n)}+\|\mathfrak{w}_\varepsilon\|_{L^q(\mathbb{R}^n)}+\|\mathfrak{W}_\varepsilon\|_{L^q(\mathbb{R}^n)}+\|W_\varepsilon\|_{L^q(\mathbb{R}^n)}\right).
\end{aligned}$$

For all exponents $(p,q)\in\tilde{\mathcal{D}}$ the control of each of these terms by the L^p-norm of the right hand side is a consequence of Theorem 5, Proposition 3, Proposition 4, Proposition 5 and Proposition 6 because of $\tilde{\mathcal{D}}\subset\mathcal{D}$. If the Restriction Conjecture is true, then the same statement holds even all $(p,q)\in\mathcal{D}$. □

Proof of Corollary 2. We briefly recall the dual variational technique for nonlinear Helmholtz equations from [20]. We aim at proving the existence of a real-valued function $u\in L^q(\mathbb{R}^n)$ satisfying

$$-\Delta u+Vu-\lambda u=\Gamma|u|^{q-2}u\qquad\text{in }\mathbb{R}^n\tag{21}$$

in the distributional sense. In view of elliptic regularity theory any distributional solution of such an equation will actually belong to $W^{2,r}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $r\in[1,\infty)$. Such solutions of the nonlinear PDE (21) will be obtained by solving the integral equation $u=K(\Gamma|u|^{q-2}u)$ where

$K\phi := \mathrm{Re}(\mathcal{R}(\lambda+\mathrm{i}0)\phi)$ and $\mathcal{R}(\lambda+\mathrm{i}0)$ has the mapping properties stated in Theorem 1. We set $v := \Gamma^{\frac{1}{q'}}|u|^{q-2}u$ and thus look for $v \in L^{q'}(\mathbb{R}^n)$ satisfying

$$|v|^{q'-2}v = \Gamma^{\frac{1}{q}} K(\Gamma^{\frac{1}{q}} v).$$

Since K is symmetric, this equation has a variational structure. So we have to prove the existence of a nontrivial critical point of the functional

$$I(v) := \frac{1}{q'}\int_{\mathbb{R}^n} |v|^{q'} - \frac{1}{2}\int_{\mathbb{R}^n} \left(\Gamma^{\frac{1}{q}} v\right)\left[K\left(\Gamma^{\frac{1}{q}} v\right)\right].$$

This functional has the Mountain Pass geometry, as we will explain and verify below. Moreover, exploiting $\Gamma \to 0$ at infinity, it satisfies the Palais-Smale condition. This can be shown exactly as in [20, Lemma 5.2] where the corresponding statement is proved in the special case $V_1 = V_2$. With these two ingredients we may apply the Mountain Pass Theorem [5, Theorem 2.1] and obtain a nontrivial critical point v of I. Transforming this function back according to $v = \Gamma^{\frac{1}{q'}}|u|^{q-2}u$, we get a nontrivial solution $u = \Gamma^{-\frac{1}{q}}|v|^{q'-2}v = K(\Gamma^{\frac{1}{q}}v) \in L^q(\mathbb{R}^n)$ of the nonlinear Helmholtz equation (3).

We now check that I has the Mountain Pass geometry. First, by choice of q in Corollary 2, the operator $\mathcal{R}(\lambda+\mathrm{i}0) : L^{q'}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded and thus $K : L^{q'}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded as well. Moreover,

$$I(v) \geq \frac{1}{q}\|v\|_{L^{q'}(\mathbb{R}^n)}^{q'} - \frac{1}{2}\|K\|_{L^{q'}(\mathbb{R}^n)\to L^q(\mathbb{R}^n)}\|\Gamma\|_{L^\infty(\mathbb{R}^n)}^{\frac{2}{q}}\|v\|_{L^{q'}(\mathbb{R}^n)}^2$$

and $q' < 2$ imply $I(0) = 0 < \inf_{S_\varrho} I$ for some sufficiently small $\varrho > 0$ where S_ϱ denotes the sphere in $L^{q'}(\mathbb{R}^n)$ with radius ϱ. Finally, $I(tv) \to -\infty$ as $t \to \infty$ for some $v \in L^{q'}(\mathbb{R}^n)$, the proof of which will take the remainder of this section. We adapt an idea from [35, Section 3] and choose the ansatz $v = v_\delta$ where

$$v_\delta(x,y) := \Gamma(x,y)^{-\frac{1}{q}} w(x) \mathrm{e}^{-y} 1_{(\delta,\infty)}(\Gamma(x,y)) 1_{(0,\infty)}(y) \qquad (x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, \delta > 0) \tag{22}$$

with sufficiently small $\delta > 0$ and with a nontrivial Schwartz function w satisfying $\mathrm{supp}(\hat{w}) \subset \mathbb{R}^n \setminus \overline{B_{\mu_2}(0)} = \{\xi \in \mathbb{R}^{n-1} : |\xi| > \mu_2\}$. Notice that $v_\delta \in L^{q'}(\mathbb{R}^n)$ because of $\delta > 0$ and

$$\Gamma^{\frac{1}{q}} v_\delta \to f \quad \text{in } L^{q'}(\mathbb{R}^n) \quad \text{as } \delta \searrow 0 \qquad \text{where } f(x,y) = w(x)\mathrm{e}^{-y}1_{(0,\infty)}(y).$$

Here we used $\Gamma > 0$ on $\mathbb{R}^n$. So we find with the aid of Plancherel's theorem

$$\begin{aligned}
&\lim_{\delta\searrow 0}\int_{\mathbb{R}^n} \left(\Gamma^{\frac{1}{q}} v_\delta\right)\left[K\left(\Gamma^{\frac{1}{q}} v_\delta\right)\right] \,\mathrm{d}(x,y) \\
&= \int_{\mathbb{R}^n} f(Kf)\,\mathrm{d}(x,y) \\
&= \mathrm{Re}\left(\int_{\mathbb{R}^n} (\mathcal{R}(\lambda+\mathrm{i}0)f)\cdot f\,\mathrm{d}(x,y)\right) \\
&\overset{(16)}{=} \mathrm{Re}\left(\int_{\mathbb{R}^n} \mathcal{F}_n^{-1}\left(\frac{\mathcal{F}_n^+ f}{|\cdot|^2-\mu_1^2-\mathrm{i}0} + \frac{\mathcal{F}_n^+ f}{|\cdot|^2-\mu_2^2-\mathrm{i}0}\right)(x,y)\cdot f(x,y)\,\mathrm{d}(x,y)\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \operatorname{Re}\left(\int_{\mathbb{R}^n} \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_1}(m_1 g_+ + m_2 g_-)\right)(x)\cdot f(x,y)\,\mathrm{d}(x,y)\right)\\
&\quad + \operatorname{Re}\left(\int_{\mathbb{R}^n} \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_1}(m_3 g_+ + m_4 g_-)\right)(x)\cdot f(x,y)\,\mathrm{d}(x,y)\right)\\
&= \operatorname{Re}\left(\int_{\mathbb{R}^n} \frac{\mathcal{F}_n^+ f(\xi,\eta)\cdot\overline{\mathcal{F}_n f(\xi,\eta)}}{|\xi|^2+\eta^2-\mu_1^2-\mathrm{i}0} + \frac{\mathcal{F}_n^- f(\xi,\eta)\cdot\overline{\mathcal{F}_n f(\xi,\eta)}}{|\xi|^2+\eta^2-\mu_2^2-\mathrm{i}0}\,\mathrm{d}(\xi,\eta)\right)\\
&\quad + \operatorname{Re}\left(\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}|y|\nu_1(\xi)}(m_1(\xi)g_+(\xi)+m_2(\xi)g_-(\xi))\overline{\mathcal{F}_{n-1}[f(\cdot,y)](\xi)}\,\mathrm{d}(\xi,y)\right)\\
&\quad + \operatorname{Re}\left(\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}|y|\nu_2(\xi)}(m_3(\xi)g_+(\xi)+m_4(\xi)g_-(\xi))\overline{\mathcal{F}_{n-1}[f(\cdot,y)](\xi)}\,\mathrm{d}(\xi,y)\right).
\end{aligned}
$$

Inserting (22) we get

$$
\mathcal{F}_{n-1}[f(\cdot,y)](\xi) = \hat{w}(\xi)\mathrm{e}^{-y}1_{(0,\infty)}(y), \quad \mathcal{F}_n^+ f(\xi,\eta) = \mathcal{F}_n f(\xi,\eta) = \frac{\hat{w}(\xi)}{1+\mathrm{i}\eta}, \quad \mathcal{F}_n^- f \equiv 0.
$$

So our choice of w implies $|\xi|^2+\eta^2 \geq |\xi|^2 > \mu_2^2 > \mu_1^2$ for all $(\xi,\eta)\in \operatorname{supp}(\hat{w})\times\mathbb{R} = \operatorname{supp}(\mathcal{F}_n^+ f) = \operatorname{supp}(\mathcal{F}_n f)$. This has the following consequences:

(i) The principal value symbol $-\mathrm{i}0$ can be omitted in the first two integrals.
(ii) $\nu_j(\xi) = \mathrm{i}|\nu_j(\xi)|$ for $j=1,2$, see (10).
(iii) $g_-(\xi) = \mathcal{F}_n^- f(\xi,\nu_2(\xi)) = 0$ and $g_+(\xi) = \mathcal{F}_n^+ f(\xi,-\nu_1(\xi)) = \frac{\hat{w}(\xi)}{1-\mathrm{i}\nu_1(\xi)} = \frac{\hat{w}(\xi)}{1+|\nu_1(\xi)|}$.

Given that $m_1(\xi)$ is real-valued and positive and $m_3\equiv 0$ for $|\xi|\geq\mu_2>\mu_1, y>0$, see (18) and (ii), this implies

$$
\begin{aligned}
&\int_{\mathbb{R}^n} f(Kf)\,\mathrm{d}(x,y)\\
&= \int_{\mathbb{R}^n} \frac{|\mathcal{F}_n f(\xi,\eta)|^2}{|\xi|^2+\eta^2-\mu_1^2}\,\mathrm{d}(\xi,\eta) + \operatorname{Re}\left(\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}|y|\nu_1(\xi)} m_1(\xi)g_+(\xi)\overline{\mathcal{F}_{n-1}[f(\cdot,y)](\xi)}\,\mathrm{d}(\xi,y)\right)\\
&= \int_{\mathbb{R}^n} \frac{|\mathcal{F}_n f(\xi,\eta)|^2}{|\xi|^2+\eta^2-\mu_1^2}\,\mathrm{d}(\xi,\eta) + \int_{\mathbb{R}^{n-1}} \frac{m_1(\xi)|\hat{w}(\xi)|^2}{1+|\nu_1(\xi)|}\left(\int_0^\infty \mathrm{e}^{-(|\nu_1(\xi)|+1)y}\,\mathrm{d}y\right)\mathrm{d}\xi\\
&= \int_{\mathbb{R}^n} \frac{|\mathcal{F}_n f(\xi,\eta)|^2}{|\xi|^2+\eta^2-\mu_1^2}\,\mathrm{d}(\xi,\eta) + \int_{\mathbb{R}^{n-1}} \frac{m_1(\xi)|\hat{w}(\xi)|^2}{(1+|\nu_1(\xi)|)^2}\,\mathrm{d}\xi > 0.
\end{aligned}
$$

As a consequence, we obtain

$$
\int_{\mathbb{R}^n} \left(\Gamma^{\frac{1}{q}} v_\delta\right)\left[K\left(\Gamma^{\frac{1}{q}} v_\delta\right)\right]\,\mathrm{d}(x,y) > 0
$$

provided that $\delta>0$ is small enough. This finishes the proof of the Mountain Pass Geometry and the claim is proved. □

4. Proof of Theorem 2

In this section we discuss the mapping properties of the operator $T_{\lambda,\alpha}$ from (4) in the one-dimensional case $d = 1$. Before proving the claims from Theorem 2 on that matter, we provide two auxiliary results dealing with singular one-dimensional oscillatory integrals. We use the following well-known estimate.

Proposition 7 (VIII.§1.1.2 Corollary [39] on p.334)**.** *Let $I \subset \mathbb{R}$ be an interval. Then we have for all $b \in W^{1,1}(I)$ the estimate*

$$\left|\int_I \mathrm{e}^{\mathrm{i}c\rho} b(\rho)\,\mathrm{d}\rho\right| \lesssim c^{-1}(|b(0)| + \|b'\|_{L^1(I)})$$

with a constant independent of I and b.

Proposition 8. *Let $\delta \in (0,1)$ and $a \in C^1([0,1])$. Then the following holds for $c > 0$:*

(i) $$\left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho} a(\rho)\rho^{-\delta}\,\mathrm{d}\rho\right| \lesssim (1+c)^{\delta-1}(|a(0)| + \|a'\|_\infty).$$

(ii) $$\left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho} a(\rho)\rho^{-\delta}\,\mathrm{d}\rho - a(0)c^{\delta-1}\int_0^\infty \mathrm{e}^{\mathrm{i}\rho}\rho^{-\delta}\,\mathrm{d}\rho\right| \lesssim c^{-1}\left(|a(0)| + \|a'\|_\infty\right).$$

In particular, there is $M > 0$ independent of a such that

$$\left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho} a(\rho)\rho^{-\delta}\,\mathrm{d}\rho\right| \gtrsim c^{\delta-1}|a(0)| \qquad \text{for } c \geq M(1 + \|a'\|_\infty |a(0)|^{-1})^{1/\delta}.$$

Proof. For $0 < c \leq 1$ the estimate (i) is trivial. For $c > 0$ we get from Proposition 7

$$\begin{aligned}\left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho} a(\rho)\rho^{-\delta}\,\mathrm{d}\rho\right| &\lesssim \left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho}(a(\rho)-a(0))\rho^{-\delta}\,\mathrm{d}\rho\right| + |a(0)|\left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho}\rho^{-\delta}\,\mathrm{d}\rho\right| \\ &\lesssim c^{-1}\int_0^1 |((a(\rho)-a(0))\rho^{-\delta})'|\,\mathrm{d}\rho + |a(0)|c^{\delta-1}\left|\int_0^c \mathrm{e}^{\mathrm{i}\rho}\rho^{-\delta}\,\mathrm{d}\rho\right| \\ &\leq c^{\delta-1}(|a(0)| + \|a'\|_\infty).\end{aligned}$$

The estimate (ii) is similar. For $0 < c \leq 1$ the estimate is trivial, while for $c > 0$ we may exploit Proposition 7 once more to get

$$\begin{aligned}&\left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho} a(\rho)\rho^{-\delta}\,\mathrm{d}\rho - a(0)c^{\delta-1}\int_0^\infty \mathrm{e}^{\mathrm{i}\rho}\rho^{-\delta}\,\mathrm{d}\rho\right| \\ &\leq \left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho}(a(\rho)-a(0))\rho^{-\delta}\,\mathrm{d}\rho\right| + |a(0)|\left|\int_1^\infty \mathrm{e}^{\mathrm{i}c\rho}\rho^{-\delta}\,\mathrm{d}\rho\right| \\ &\lesssim c^{-1}\int_0^1 |((a(\rho)-a(0))\rho^{-\delta})'|\,\mathrm{d}\rho + c^{-1}|a(0)|\left(1 + \int_1^\infty \rho^{-1-\delta}\,\mathrm{d}\rho\right) \\ &\lesssim c^{-1}\left(|a(0)| + \|a'\|_\infty\right).\end{aligned}$$

The second part of (ii) is a direct consequence of the first part since all constants incorporated in $\lesssim$ are independent of a and c. $\square$

The above result allows to determine the exact asymptotics in the singular case $\delta \in (0,1)$ and in particular some lower bound for large c that we will need in the construction of counterexamples. Similar but slightly different results can be obtained for $\delta = 0$.

Proposition 9. *Let $a \in C^1([0,1])$. Then the following holds for all $c > 0$*

(i) $$\left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho} a(\rho)\,\mathrm{d}\rho\right| \lesssim (1+c)^{-1}(|a(0)| + \|a'\|_\infty).$$

(ii) $$\left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho} a(\rho)\,\mathrm{d}\rho - \frac{a(1)\mathrm{e}^{\mathrm{i}c} - a(0)}{\mathrm{i}c}\right| \lesssim (1+c)^{-2}\left(|a(0)| + |a'(0)| + \|a''\|_\infty\right) \quad \textit{if } a \in C^2([0,1]).$$

Proof. Part (i) is proved just as in the singular case, see Proposition 8. For $0 < c \leq 1$ the estimate (ii) is trivial and for $c > 1$ we get via integration by parts and the estimate (i)

$$\left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho} a(\rho)\,\mathrm{d}\rho - \frac{a(1)\mathrm{e}^{\mathrm{i}c} - a(0)}{\mathrm{i}c}\right| = c^{-1}\left|\int_0^1 \mathrm{e}^{\mathrm{i}c\rho} a'(\rho)\,\mathrm{d}\rho\right| \lesssim c^{-2}(\|a''\|_\infty + |a'(0)|).$$

□

Proof of Theorem 2: We have to show that the estimate

$$\|T_{\lambda,\alpha} h\|_{L^q(\mathbb{R})} \lesssim (1+\lambda)^{2\alpha - \frac{2}{p} + \frac{2}{q}} \|h\|_{L^p(\mathbb{R})} \tag{23}$$

holds where $\alpha \in [0,1), \lambda \geq 0$ and $(p,q) \in \mathcal{D}_\alpha$, i.e., $\frac{1}{p} - \frac{1}{q} \geq \alpha$, $\frac{1}{p} > \alpha$, $\frac{1}{q} < 1 - \alpha$. Moreover, we will show that the range of exponents p, q is optimal under these assumptions.

For notational convenience we only consider the special case in Theorem 2 where the annulus is $A = \{\xi \in \mathbb{R} : 1 \leq |\xi| \leq 2\}$, so we consider (see (4)) the operator

$$T_{\lambda,\alpha} h = \mathcal{F}_1^{-1}\left(1_A(\cdot)\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-1}}(|\cdot|^2 - 1)^{-\alpha} m(|\cdot|)\mathcal{F}_1 h(\cdot)\right).$$

We first present the comparatively easy proof for the case $1 \leq p \leq 2 \leq q \leq \infty, \frac{1}{p} - \frac{1}{q} \geq \alpha$ that only requires $m \in C([1,2])$. From the Hausdorff-Young inequality we get in the case $\frac{1}{p} - \frac{1}{q} > \alpha$

$$\begin{aligned}
\|T_{\lambda,\alpha} h\|_{L^q(\mathbb{R})} &\lesssim \left\|\mathcal{F}_1^{-1}\left(1_A(\cdot)\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-1}}(|\cdot|^2 - 1)^{-\alpha} m(|\cdot|)\mathcal{F}_1 h(\cdot)\right)\right\|_{L^q(\mathbb{R})} \\
&\lesssim \left\|\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-1}}(|\cdot|^2 - 1)^{-\alpha} m(|\cdot|)\mathcal{F}_1 h(\cdot)\right\|_{L^{q'}(A)} \\
&\lesssim \left\|\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-1}}(|\cdot|^2 - 1)^{-\alpha} m(|\cdot|)\right\|_{L^{\frac{pq}{q-p}}(A)} \|\mathcal{F}_1 h\|_{L^{p'}(A)} \\
&\lesssim \|m\|_\infty \|\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-1}}(|\cdot|^2 - 1)^{-\alpha}\|_{L^{\frac{pq}{q-p}}(A)} \|\mathcal{F}_1 h\|_{L^{p'}(\mathbb{R})} \\
&\lesssim \|m\|_\infty \left(\int_1^2 \mathrm{e}^{-\frac{pq}{q-p}\lambda\sqrt{r^2-1}}(r^2-1)^{-\frac{\alpha pq}{q-p}}\,\mathrm{d}r\right)^{\frac{q-p}{pq}} \|h\|_{L^p(\mathbb{R})}
\end{aligned}$$

$$\lesssim \|m\|_\infty \left(\int_0^1 \mathrm{e}^{-\lambda\rho}\rho^{1-\frac{2\alpha pq}{q-p}}\,\mathrm{d}\rho\right)^{\frac{q-p}{pq}} \|h\|_{L^p(\mathbb{R})}$$
$$\lesssim \|m\|_\infty (1+\lambda)^{2\alpha-\frac{2}{p}+\frac{2}{q}}\|h\|_{L^p(\mathbb{R})}.$$

The endpoint case $\frac{1}{p}-\frac{1}{q}=\alpha$ is achieved through complex interpolation. Since the procedure is almost the same as in the proof of Theorem 3 below, we omit the details here and remark only that this strategy requires continuity of m.

We continue with the proof of the full result under the assumption $m\in C^1([1,2])$. We use

$$\begin{aligned}
|T_{\lambda,\alpha}h(x)| &= \left|\mathcal{F}_1^{-1}\left(1_A(\cdot)\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-1}}(|\cdot|^2-1)^{-\alpha}m(|\cdot|)\mathcal{F}_1 h\right)(x)\right| \\
&= \frac{1}{\sqrt{2\pi}}\left|\int_A \mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-\lambda\sqrt{|\xi|^2-1}}(|\xi|^2-1)^{-\alpha}m(|\xi|)\mathcal{F}_1 h(\xi)\,\mathrm{d}\xi\right| \\
&= \left|\int_{\mathbb{R}} K_\lambda(x-y)h(y)\,\mathrm{d}y\right| \\
\text{where}\qquad K_\lambda(z) &:= \frac{1}{2\pi}\int_A \mathrm{e}^{\mathrm{i}\xi z}(|\xi|^2-1)^{-\alpha}m(|\xi|)\mathrm{e}^{-\lambda\sqrt{|\xi|^2-1}}\,\mathrm{d}\xi.
\end{aligned}$$

From this identity we get in the case $p\neq 1, q\neq\infty$

$$\|T_{\lambda,\alpha}h\|_{L^q(\mathbb{R})} \lesssim \|K_\lambda * h\|_{L^q(\mathbb{R})} \lesssim \|K_\lambda\|_{L^{\frac{pq}{pq+p-q},\infty}(\mathbb{R})}\|h\|_{L^p(\mathbb{R})}$$

so that we have to show

$$\|K_\lambda\|_{L^{\frac{pq}{pq+p-q},\infty}(\mathbb{R})} \lesssim (1+\lambda)^{2\alpha-\frac{2}{p}+\frac{2}{q}} \qquad \text{if } p\neq 1,\ q\neq\infty. \tag{24}$$

Similarly, in order to cover the cases $p=1$ or $q=\infty$ as well, we need to prove

$$\|K_\lambda\|_{L^{\frac{pq}{pq+p-q}}(\mathbb{R})} \lesssim (1+\lambda)^{2\alpha-\frac{2}{p}+\frac{2}{q}} \qquad \text{if } p=1 \text{ or } q=\infty. \tag{25}$$

The proof of (24),(25) is based on pointwise estimates for the kernel function K_λ. For $|z|\leq 1+\lambda^2$ we will use

$$|K_\lambda(z)| \lesssim \int_1^2 \mathrm{e}^{-\lambda\sqrt{r^2-1}}(r^2-1)^{-\alpha}\,\mathrm{d}r \lesssim \int_0^{\sqrt{3}} \mathrm{e}^{-\lambda\rho}\rho^{1-2\alpha}\,\mathrm{d}\rho \lesssim (1+\lambda)^{2\alpha-2}.$$

For $|z| \geq 1+\lambda^2$ we estimate the kernel with the aid of oscillatory integral theory that uses $m \in C^1([1,2])$. Proposition 8 (i) and Proposition 9 (i) imply

$$\begin{aligned}
|K_\lambda(z)| &\lesssim \left| \int_0^1 \mathrm{e}^{\mathrm{i}\rho z} \rho^{-\alpha}(2+\rho)^{-\alpha} m(1+\rho) \mathrm{e}^{-\lambda\sqrt{\rho^2+2\rho}} \,\mathrm{d}\rho \right| \\
&\overset{\text{Prop. 7}}{\lesssim} \int_0^{|z|^{-1}} \rho^{-\alpha}(2+\rho)^{-\alpha} |m(1+\rho)| \mathrm{e}^{-\lambda\sqrt{\rho^2+2\rho}} \,\mathrm{d}\rho \\
&\quad + \frac{1}{|z|}\left(|z|^\alpha + \int_{|z|^{-1}}^1 \left| \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^{-\alpha}(2+\rho)^{-\alpha} m(1+\rho) \mathrm{e}^{-\lambda\sqrt{\rho^2+2\rho}} \right) \right| \,\mathrm{d}\rho \right) \\
&\lesssim \int_0^{|z|^{-1}} \rho^{-\alpha} \mathrm{e}^{-\lambda\sqrt{\rho}} \,\mathrm{d}\rho + |z|^{\alpha-1} \\
&\quad + \frac{1}{|z|} \int_{|z|^{-1}}^1 \rho^{-\alpha-1} \mathrm{e}^{-\lambda\sqrt{\rho}} \,\mathrm{d}\rho + \frac{\lambda}{|z|} \int_{|z|^{-1}}^1 \rho^{-\alpha-\frac{1}{2}} \mathrm{e}^{-\lambda\sqrt{\rho}} \,\mathrm{d}\rho \\
&\lesssim \lambda^{2\alpha-2} \int_0^{\lambda|z|^{-1/2}} t^{1-2\alpha} \mathrm{e}^{-t} \,\mathrm{d}t + |z|^{\alpha-1} \\
&\quad + \frac{\lambda^{2\alpha}}{|z|} \left(\int_{\lambda|z|^{-1/2}}^{\lambda} t^{-2\alpha}\mathrm{e}^{-t} + t^{-1-2\alpha}\mathrm{e}^{-t} \,\mathrm{d}t \right) \\
&\lesssim \lambda^{2\alpha-2} \int_0^{\lambda|z|^{-1/2}} t^{1-2\alpha} \mathrm{e}^{-t} \,\mathrm{d}t + |z|^{\alpha-1} + \frac{\lambda^{2\alpha}}{|z|} \int_{\lambda|z|^{-1/2}}^{\infty} t^{-1-2\alpha}\mathrm{e}^{-t} \,\mathrm{d}t \\
&\lesssim \lambda^{2\alpha-2} \left(\lambda|z|^{-1/2}\right)^{2-2\alpha} + |z|^{\alpha-1} + \frac{\lambda^{2\alpha}}{|z|} \left(\left(\lambda|z|^{-1/2}\right)^{-2\alpha} + 1 \right) \\
&\lesssim |z|^{\alpha-1}.
\end{aligned} \tag{26}$$

Making use of $\frac{pq}{pq+p-q} \geq \frac{1}{1-\alpha}$ (due to $\frac{1}{p}-\frac{1}{q} \geq \alpha$) we get

$$\begin{aligned}
\|K_\lambda\|_{L^{\frac{pq}{pq+p-q},\infty}(\mathbb{R})} &\lesssim (1+\lambda)^{2\alpha-2} \|1\|_{L^{\frac{pq}{pq+p-q},\infty}([0,1+\lambda^2])} + \||\cdot|^{\alpha-1}\|_{L^{\frac{pq}{pq+p-q},\infty}([1+\lambda^2,\infty))} \\
&\lesssim (1+\lambda)^{2\alpha-\frac{2}{p}+\frac{2}{q}}
\end{aligned}$$

We conclude that (24) holds. Along the same lines we find

$$\|K_\lambda\|_{L^{\frac{pq}{pq+p-q}}(\mathbb{R})} \lesssim (1+\lambda)^{2\alpha-\frac{2}{p}+\frac{2}{q}} \qquad \text{if } p=1 \text{ or } q=\infty$$

because then $\frac{pq}{pq+p-q} > \frac{1}{1-\alpha}$ by assumption. So the sufficiency part of Theorem 2 is proved.

For the construction of a counterexample we assume $m \equiv 1, \lambda = 0$ as well as $\frac{1}{p}-\frac{1}{q} < \alpha$ and $p \neq 1, q \neq \infty$. We want to show that (23) does not hold in this case. To this end we choose β according to

$$\max\left\{1-\alpha-\frac{1}{q}, 0\right\} < \beta < 1-\frac{1}{p}. \tag{27}$$

Then the function $f := \sqrt{2\pi}\mathcal{F}_1^{-1}(1_{[1,2]}(\cdot)(|\cdot|^2-1)^{-\beta})$ belongs to $L^p(\mathbb{R})$ because of

$$|f(x)| = \left|\int_1^2 \mathrm{e}^{\mathrm{i}x\xi}(|\xi|^2-1)^{-\beta}\,\mathrm{d}\xi\right| = \left|\int_0^1 \mathrm{e}^{\mathrm{i}x\rho}\rho^{-\beta}(2+\rho)^{-\beta}\,\mathrm{d}\rho\right| \lesssim (1+|x|)^{\beta-1},$$

see Proposition 8 (i). On the other hand, Proposition 8 (ii) gives in the case $\alpha+\beta<1$

$$|T_{0,\alpha}f(x)| = \left|\int_0^1 \mathrm{e}^{\mathrm{i}x\rho}\rho^{-\alpha-\beta}(2+\rho)^{-\alpha-\beta}\,\mathrm{d}\rho\right| \gtrsim |x|^{\alpha+\beta-1} \quad \text{as } |x|\to\infty.$$

Since our choice for β from (27) implies $q(\alpha+\beta-1)\geq -1$, this estimate gives $T_{0,\alpha}f\notin L^q(\mathbb{R})$. In the case $\alpha+\beta\geq 1$ we slightly modify the counterexample and define $f_\varepsilon := \mathcal{F}_1^{-1}(1_{[1+\varepsilon,2]}(\cdot)(|\cdot|^2-1)^{-\beta})$. Then the sequence (f_ε) is bounded in $L^p(\mathbb{R})$ by the Hausdorff-Young inequality while $|T_{0,\alpha}f_\varepsilon(x)|\to+\infty$ uniformly on a small neighbourhood of $x=0$. Indeed,

$$\begin{aligned}\inf_{|x|\leq\pi/8}|T_{0,\alpha}f_\varepsilon(x)| &= \inf_{|x|\leq\pi/8}\left|\int_{1+\varepsilon}^2 \mathrm{e}^{\mathrm{i}x\xi}(|\xi|^2-1)^{-\beta-\alpha}\,\mathrm{d}\xi\right| \\ &\geq \inf_{|x|\leq\pi/8}\left|\int_{1+\varepsilon}^2 \cos(x\xi)(|\xi|^2-1)^{-\beta-\alpha}\,\mathrm{d}\xi\right| \\ &\geq \cos(\pi/4)\int_{1+\varepsilon}^2 (|\xi|^2-1)^{-\beta-\alpha}\,\mathrm{d}\xi \nearrow \infty \quad \text{as } \varepsilon\to 0.\end{aligned}$$

This shows that $T_{0,\alpha}$ is unbounded for $\frac{1}{p}-\frac{1}{q}<\alpha$ and $p\neq 1, q\neq\infty$.

It remains to show that (23) does not hold either for $p=\frac{1}{\alpha}, q=\infty$ or $p=1, q=\frac{1}{1-\alpha}$ and $\alpha\in[0,1)$. By duality, it suffices to disprove (23) in the former case. This example is constructed as follows. We set for $k\in\mathbb{N}$

$$f_k(y) := \ln(k+1)^{-\alpha}1_{[1,k+1]}(y)y^{-\alpha}\mathrm{e}^{\mathrm{i}y}.$$

Then $\|f_k\|_{L^p(\mathbb{R})}=1$ and

$$\|T_{0,\alpha}f_k\|_{L^\infty(\mathbb{R})} \geq |T_{0,\alpha}f_k(0)| = \frac{1}{2\pi}\ln(k+1)^{-\alpha}\left|\int_1^{k+1} y^{-\alpha}\mathrm{e}^{\mathrm{i}y}\left(\int_1^2 \mathrm{e}^{-\mathrm{i}\xi y}(|\xi|^2-1)^{-\alpha}\,\mathrm{d}\xi\right)\,\mathrm{d}y\right|$$

In the case $\alpha=0$ this implies

$$\begin{aligned}\|T_{0,0}f_k\|_{L^\infty(\mathbb{R})} &\geq \frac{1}{2\pi}\left|\int_1^{k+1}\mathrm{e}^{\mathrm{i}y}\left(\int_1^2 \mathrm{e}^{-\mathrm{i}\xi y}\,\mathrm{d}\xi\right)\,\mathrm{d}y\right| \\ &= \frac{1}{2\pi}\left|\int_1^{k+1}\mathrm{e}^{\mathrm{i}y}\cdot\frac{\mathrm{e}^{-2\mathrm{i}y}-\mathrm{e}^{-\mathrm{i}y}}{-\mathrm{i}y}\,\mathrm{d}y\right| \\ &= \frac{1}{2\pi}\left|\int_1^{k+1}\frac{\mathrm{e}^{-\mathrm{i}y}}{y}-\frac{1}{y}\,\mathrm{d}y\right| \\ &\geq \frac{\ln(k+1)}{2\pi}-\frac{1}{2\pi}\left|\int_1^{k+1}\frac{\mathrm{e}^{-\mathrm{i}y}}{y}\,\mathrm{d}y\right|,\end{aligned}$$

which tends to $+\infty$ as $k\to\infty$. This proves the unboundedness in the case $\alpha=0$, i.e., for $p=q=\infty$. In the case $\alpha\in(0,1)$ we get from Proposition 8 (ii)

$$\begin{aligned}\lim_{y\to\infty} y^{1-\alpha}\mathrm{e}^{\mathrm{i}y}\int_1^2 \mathrm{e}^{-\mathrm{i}\xi y}(|\xi|^2-1)^{-\alpha}\,\mathrm{d}\xi &= \lim_{y\to\infty} y^{1-\alpha}\int_0^1 \mathrm{e}^{-\mathrm{i}\rho y}\rho^{-\alpha}(2+\rho)^{-\alpha}\,\mathrm{d}\rho \\ &= 2^{-\alpha}\int_0^\infty \mathrm{e}^{-\mathrm{i}\rho}\rho^{-\alpha}\,\mathrm{d}\rho \\ &=: \mu\in\mathbb{C}\setminus\{0\}.\end{aligned}$$

Hence, for $k_0\in\mathbb{N}$ sufficiently large and all $k\ge k_0$ we have

$$\begin{aligned}\|T_{0,\alpha}f_k\|_{L^\infty(\mathbb{R})} &\ge |T_{0,\alpha}f_k(0)| \\ &= \frac{1}{2\pi}\ln(k+1)^{-\alpha}\left|\int_1^{k+1} y^{-\alpha}\mathrm{e}^{\mathrm{i}y}\left(\int_1^2 \mathrm{e}^{-\mathrm{i}\xi y}(|\xi|^2-1)^{-\alpha}\,\mathrm{d}\xi\right)\,\mathrm{d}y\right| \\ &= \frac{1}{2\pi}\ln(k+1)^{-\alpha}\left|\int_{k_0}^{k+1} y^{-1}\cdot y^{1-\alpha}\mathrm{e}^{\mathrm{i}y}\left(\int_1^2 \mathrm{e}^{-\mathrm{i}\xi y}(|\xi|^2-1)^{-\alpha}\,\mathrm{d}\xi\right)\,\mathrm{d}y\right| + o(1) \\ &\ge \frac{|\mu|}{4\pi}\ln(k+1)^{-\alpha}\int_{k_0}^{k+1} y^{-1}\,\mathrm{d}y + o(1) \\ &= \frac{|\mu|}{4\pi}\ln(k+1)^{1-\alpha} + o(1),\end{aligned}$$

which tends to $+\infty$ as $k\to\infty$. Hence, the operator $T_{0,\alpha}: L^p(\mathbb{R})\to L^q(\mathbb{R})$ is unbounded for $p=\frac{1}{\alpha}, q=\infty$ and $\alpha\in(0,1)$, which is all we had to show. □

5. Proof of Theorem 3

Theorem 3 is proved with the aid of Stein's Interpolation Theorem [38, Theorem 1] for holomorphic families of operators. So we have to estimate the operators $T_{\lambda,\alpha}$ defined in (4). We first recall our estimates for the operators S_λ from (7) that will provide the desired bounds in the endpoint case $\alpha=0$. Choosing $s=2$ in Theorem 4 we get the following.

Proposition 10. *Let $d\in\mathbb{N}, d\ge 2$, $0<a<b<\infty$ and $m\in L^\infty([a,b])$. Then we have for all $\lambda\ge 0$ and all $p\in[1,2]$*

$$\begin{aligned}\|S_\lambda h\|_{L^2(A)} &\lesssim (1+\lambda)^{-1+(\frac{d+3}{2}-\frac{d+1}{p})_+}\|h\|_{L^p(\mathbb{R}^d)}, \\ \|S_\lambda^* g\|_{L^{p'}(\mathbb{R}^d)} &\lesssim (1+\lambda)^{-1+(\frac{d+3}{2}-\frac{d+1}{p})_+}\|g\|_{L^2(A)}.\end{aligned}$$

As a consequence we obtain the following result.

Proposition 11. *Let $d\in\mathbb{N}, d\ge 2$, $0<a<b<\infty$ and $m\in L^\infty([a,b])$. Then we have for all $\lambda\ge 0$ and all $p\in[1,2], q\in[2,\infty]$*

$$\|T_{\lambda,0}h\|_{L^q(\mathbb{R}^d)} \lesssim (1+\lambda)^{-2+(\frac{d+3}{2}-\frac{d+1}{p})_+ + (\frac{d+3}{2}-\frac{d+1}{q'})_+}\|h\|_{L^p(\mathbb{R}^d)}.$$

Proof. We may assume that m is real-valued nonnegative, otherwise we split the operator into the sum of four such operators according to $m=\mathfrak{m}_1^2-\mathfrak{m}_2^2+\mathrm{i}(\mathfrak{m}_3^2-\mathfrak{m}_4^2)$. But then we have $T_{\lambda,0} = S_\lambda^* S_\lambda$ where m in the definition of S_λ, S_λ is replaced by $\sqrt{m}$ and thus the claim follows from Proposition 10. □

Next we use these estimates in the endpoint case $\alpha = 0$ for the analysis of $T_{\lambda,\alpha}$ from (4) with $\alpha \in (0,1)$. Up to an α-dependent prefactor, these operators may be embedded into the family of operators

$$\mathcal{T}_{\lambda,s} h := \frac{\mathrm{e}^{(1-s)^2}}{\Gamma(1-s)} \mathcal{F}_d^{-1}\left(1_A(\cdot)\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-a^2}}(|\cdot|^2-a^2)^{-s} m(|\cdot|)\mathcal{F}_d h(\cdot)\right). \tag{28}$$

A priori, these operators are well-defined for Schwartz functions $h : \mathbb{R}^d \to \mathbb{C}$ and $s \in \mathbb{C}$ with $0 \le \mathrm{Re}(s) < 1$. We assume $\lambda \ge 0$ and $m \in C([a,b])$. Since we are going to apply Stein's Interpolation Theorem (Theorem 1 in [38]) to the family $(\mathcal{T}_{\lambda,\sigma s})_{s\in S}$ where $S := \{s \in \mathbb{C} : 0 \le \mathrm{Re}(s) \le 1\}$ and $\sigma \in [0,1]$ (including the endpoint case $\sigma = 1$), we need to extend the operators from (28) to the line $\mathrm{Re}(s) = 1$ in a continuous way. Only for this reason we will temporarily assume $m \in C^1([a,b])$, but we will see that this extra assumption is actually not necessary. The extension is based on the representation

$$\begin{aligned}
(\mathcal{T}_{\lambda,s} h)(x) &= \frac{\mathrm{e}^{(1-s)^2}}{\Gamma(1-s)} \int_a^b \mathrm{e}^{-\lambda\sqrt{r^2-a^2}}(r^2-a^2)^{-s} m(r) \mathcal{F}_d^{-1}(\mathcal{F}_d h \,\mathrm{d}\sigma_r)(x)\,\mathrm{d}r \\
&= (1-s)\int_a^b (r-a)^{-s}(\mathcal{A}_{\lambda,s}(r)h)(x)\,\mathrm{d}r \quad \text{where} \\
(\mathcal{A}_{\lambda,s}(r)h)(x) &:= \frac{\mathrm{e}^{(1-s)^2}}{\Gamma(2-s)} \mathrm{e}^{-\lambda\sqrt{r^2-a^2}}(r+a)^{-s} m(r)\mathcal{F}_d^{-1}(\mathcal{F}_d h\,\mathrm{d}\sigma_r)(x).
\end{aligned}$$

Integration by parts motivates the definition

$$(\mathcal{T}_{\lambda,s} h)(x) = (b-a)^{1-s}(\mathcal{A}_{\lambda,s}(b)h)(x) - \int_a^b (r-a)^{1-s}(\mathcal{A}'_{\lambda,s}(r)h)(x)\,\mathrm{d}r \qquad \text{if } \mathrm{Re}(s) = 1.$$

Notice that this expression is well-defined for Schwartz functions $h : \mathbb{R}^d \to \mathbb{C}$ (due to $m \in C^1([a,b])$) and we have

$$\mathcal{T}_{\lambda,1} h = \mathcal{A}_{\lambda,1}(a) h = (2a)^{-1} m(a) \mathcal{F}_d^{-1}(\mathcal{F}_d h\,\mathrm{d}\sigma_a).$$

In order to apply the Interpolation Theorem, we need to check that $(\mathcal{T}_{\lambda,s})_{s\in S}$ is an analytic family of operators in the sense of [38, p.483].

Proposition 12. *For all Schwartz functions $h_1, h_2 : \mathbb{R}^d \to \mathbb{C}$ the map $s \mapsto \int_{\mathbb{R}^d} h_1(\mathcal{T}_{\lambda,s} h_2)\,\mathrm{d}x$ is holomorphic in $\mathring{S}$ and continuous on S.*

Proposition 12 implies that for all $\sigma \in [0,1]$ the family $(\mathcal{T}_{\lambda,\sigma s})_{s\in S}$ is admissible for Stein's Interpolation Theorem. Notice that the original version requires Proposition 12 to hold for step functions, but actually any dense family of functions can be chosen. We use this fact in order to show that $T_{\lambda,\alpha}$, which is an α-dependent multiple of $\mathcal{T}_{\lambda,\alpha}$, is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ whenever $\alpha \in (0,1)$ and $(p,q) \in D_\alpha$ where

$$D_\alpha = \left\{(p,q) \in [1,\infty]^2 : \frac{1}{p} > \frac{1}{2} + \frac{\alpha}{2d},\ \frac{1}{q} < \frac{1}{2} - \frac{\alpha}{2d},\ \frac{1}{p} - \frac{1}{q} \ge \frac{2\alpha}{d+1}\right\},$$

cf. (5). An estimate of the corresponding mapping constant will then provide the result.

Proof of Theorem 3: For notational convenience we only discuss $a = 1, b = 2$. As explained earlier, our proof is based on complex interpolation. We temporarily assume $m \in C^1([1,2])$ in order to make use of Proposition 12 that is needed for Stein's Interpolation Theorem. On the other hand, our estimates will only depend on the L^∞-norm of m so that all results will persist for m belonging to the completion of $C^1([1,2])$ with respect to this norm, namely for $m \in C([1,2])$.

We start with recalling the estimates for the endpoint $\alpha = 0$. From Proposition 11 we deduce

$$\|\mathcal{T}_{\lambda,0} f\|_{L^{q_1}(\mathbb{R}^d)} = \|T_{\lambda,0} f\|_{L^{q_1}(\mathbb{R}^d)} \lesssim (1+\lambda)^{-2+(\frac{d+3}{2}-\frac{d+1}{q_1'})_+ +(\frac{d+3}{2}-\frac{d+1}{p_1})_+} \|f\|_{L^{p_1}(\mathbb{R}^d)} \tag{29}$$

whenever $1 \leq p_1 \leq 2 \leq q_1 \leq \infty$. Those already yield the claim for $\alpha = 0$ so that we may assume $\alpha \in (0,1)$ in the following. On the other hand, for exponents p_2, q_2 satisfying $\frac{1}{p_2} > \frac{d+1}{2d}, \frac{1}{q_2} < \frac{d-1}{2d}, \frac{1}{p_2} - \frac{1}{q_2} \geq \frac{2}{d+1}$ we get for any $s \in S$ with $0 \leq \mathrm{Re}(s) < 1$ from Minkowski's inequality in integral form and Corollary 3

$$\begin{aligned}
\|\mathcal{T}_{\lambda,s} f\|_{L^{q_2}(\mathbb{R}^d)} &= \left\| \frac{\mathrm{e}^{(1-s)^2}}{\Gamma(1-s)} \int_1^2 \mathrm{e}^{-\lambda\sqrt{r^2-1}} (r^2-1)^{-s} m(r) \mathcal{F}_d^{-1}\left((\mathcal{F}_d f \,\mathrm{d}\sigma_r)(\cdot)\,\mathrm{d}r\right) \right\|_{L^{q_2}(\mathbb{R}^d)} \\
&\leq \left| \frac{\mathrm{e}^{(1-s)^2}}{\Gamma(1-s)} \right| \int_1^2 \mathrm{e}^{-\lambda\sqrt{r^2-1}} (r^2-1)^{-\mathrm{Re}(s)} m(r) \left\| \mathcal{F}_d^{-1}\left(\mathcal{F}_d f \,\mathrm{d}\sigma_r\right) \right\|_{L^{q_2}(\mathbb{R}^d)} \,\mathrm{d}r \\
&\leq \left| \frac{\mathrm{e}^{(1-s)^2}}{\Gamma(1-s)} \right| \int_1^2 \mathrm{e}^{-\lambda\sqrt{r^2-1}} (r^2-1)^{-\mathrm{Re}(s)} m(r) r^{-1+\frac{d}{p_2}-\frac{d}{q_2}} \|f\|_{L^{p_2}(\mathbb{R}^d)} \,\mathrm{d}r \\
&\lesssim \left| \frac{\mathrm{e}^{(1-s)^2}}{\Gamma(1-s)} \right| \|m\|_\infty \left(\int_0^{\sqrt{3}} \mathrm{e}^{-\lambda\rho} \rho^{1-2\,\mathrm{Re}(s)} \,\mathrm{d}\rho \right) \|f\|_{L^{p_2}(\mathbb{R}^d)} \\
&\lesssim \left| \frac{\mathrm{e}^{(1-s)^2}}{\Gamma(1-s)} \right| \|m\|_\infty |1-\mathrm{Re}(s)| (1+\lambda)^{2\,\mathrm{Re}(s)-2} \|f\|_{L^{p_2}(\mathbb{R}^d)} \\
&\lesssim \|m\|_\infty (1+\lambda)^{2\,\mathrm{Re}(s)-2} \|f\|_{L^{p_2}(\mathbb{R}^d)}.
\end{aligned} \tag{30}$$

By our choice of the prefactor, which is adapted from [39, p.381], the above estimate is uniform with respect to $s \in S$ such that $0 \leq \mathrm{Re}(s) < 1$. Moreover, as announced earlier, it only depends on the L^∞-norm of m. Hence, the continuity property from Proposition 12 implies that the estimate persists on the closure of this set, namely on the whole strip S. This is a consequence of the Uniform Boundedness Principle. From Proposition 12, (29), (30) we infer that, for any given $\sigma \in [0,1]$, $(\mathcal{T}_{\lambda,\sigma s})_{s \in \mathbb{S}}$ is a holomorphic family of operators of admissible growth in the sense of [38] so that Stein's Interpolation Theorem applies.

We consider three different regimes of exponents $(p,q) \in D_\alpha$, namely

(a) $(p,q) \in \mathcal{D}$, i.e., $\frac{1}{p} > \frac{d+1}{2d}$, $\frac{1}{q} < \frac{d-1}{2d}, \frac{1}{p} - \frac{1}{q} \geq \frac{2}{d+1}$,

(b) $\frac{1}{p} - \frac{1}{q} < \min\left\{\frac{2}{d+1}, \frac{2d}{d+1}(2m_{p,q}-1)\right\}$,

(c) $m_{p,q} \leq \frac{d+1}{2d}$ and $\frac{1}{p} - \frac{1}{q} \geq \frac{2d}{d+1}(2m_{p,q} - 1)$.

Here, $m_{p,q} := \min\{\frac{1}{p}, 1 - \frac{1}{q}\}$. First, for (p, q) as in (a) we do not need interpolation to conclude. Indeed, the above estimate implies

$$\|\mathcal{T}_{\lambda,\alpha} f\|_{L^q(\mathbb{R}^d)} \lesssim (1+\lambda)^\gamma \|f\|_{L^p(\mathbb{R}^d)} \qquad \text{where } \gamma = 2\alpha - 2 < 2\alpha - 2 + \frac{1}{p} - \frac{1}{q}$$

Hence, (6) holds and the claim is proved for such exponents. For exponents (p, q) as in (b) or (c) we use interpolation. Having the above conditions on p_1, q_1, p_2, q_2 in mind, Stein's Interpolation Theorem gives

$$\begin{aligned}
&\|\mathcal{T}_{\lambda,\theta\sigma} f\|_{L^q(\mathbb{R}^d)} \lesssim (1+\lambda)^{2\theta\sigma - 2 + (1-\theta)(\frac{d+3}{2} - \frac{d+1}{p_1})_+ + (1-\theta)(\frac{d+3}{2} - \frac{d+1}{q_1'})_+} \|f\|_{L^p(\mathbb{R}^d)}, \\
&\text{where } \ \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad \theta \in [0,1), \ \sigma \in [0,1].
\end{aligned}$$

Being interested in $\theta\sigma = \alpha$ we thus obtain ($\sigma := \alpha/\theta$)

$$\begin{aligned}
&\|\mathcal{T}_{\lambda,\alpha} f\|_{L^q(\mathbb{R}^d)} \lesssim (1+\lambda)^{2\alpha - 2 + (1-\theta)(\frac{d+3}{2} - \frac{d+1}{p_1})_+ + (1-\theta)(\frac{d+3}{2} - \frac{d+1}{q_1'})_+} \|f\|_{L^p(\mathbb{R}^d)}, \\
&\text{where } \ \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \qquad \theta \in [\alpha, 1), \\
&1 \leq p_1 \leq 2 \leq q_1 \leq \infty, \qquad \frac{1}{p_2} - \frac{1}{q_2} \geq \frac{2}{d+1}, \qquad 1 \geq \frac{1}{p_2}, \frac{1}{q_2'} > \frac{d+1}{2d}.
\end{aligned} \tag{31}$$

In the case (b) one can check that that the choice

$$\theta = \frac{d+1}{2}\left(\frac{1}{p} - \frac{1}{q}\right), \quad p_1 = q_1 = 2, \quad \frac{1}{p_2} = \frac{1}{2} + \frac{2}{d+1}\frac{\frac{1}{p} - \frac{1}{2}}{\frac{1}{p} - \frac{1}{q}}, \quad \frac{1}{q_2} = \frac{1}{2} - \frac{2}{d+1}\frac{\frac{1}{2} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}}.$$

is admissible for (31) and leads to the uuper bound for the operator norm

$$(1+\lambda)^\gamma \quad \text{where } \gamma = 2\alpha - 2 + 2(1-\theta) = 2\alpha - \frac{d+1}{p} + \frac{d+1}{q}.$$

In particular, assuming additionally $\frac{1}{p} - \frac{1}{q} \geq \frac{2}{d+2}$ as in the Theorem, one finds $\gamma \leq 2\alpha - 2 + \frac{1}{p} - \frac{1}{q}$. Given that $p \neq 1$ and $q \neq \infty$ we conclude that (6) holds under this assumption and the claim is proved for such exponents.

It remains to consider exponents (p, q) as in (c). In that case we define $\theta_\varepsilon := 2dm_{p,q} - d - \varepsilon$ for small $\varepsilon > 0$. In the case $m_{p,q} = \frac{1}{p}$ one chooses

$$\theta = \theta_\varepsilon, \quad p_1 = 2, \quad q_1 = \frac{1 - \theta_\varepsilon}{(\frac{1}{2} - \frac{1}{p} + \frac{1}{q} - \frac{\theta_\varepsilon(d-3)}{2(d+1)})_+}, \quad p_2 = \frac{\theta_\varepsilon}{\frac{1}{p} - \frac{1-\theta_\varepsilon}{2}}, \quad q_2 = \frac{\theta_\varepsilon}{\frac{1}{q} - \frac{1-\theta_\varepsilon}{q_1}}$$

and in the case $m_{p,q} = 1 - \frac{1}{q}$ one takes

$$\theta = \theta_\varepsilon, \quad q_1 = 2, \quad p_1 = \left(\frac{1 - \theta_\varepsilon}{(\frac{1}{2} - \frac{1}{p} + \frac{1}{q} - \frac{\theta_\varepsilon(d-3)}{2(d+1)})_+}\right)', \quad q_2 = \frac{\theta_\varepsilon}{\frac{1}{q} - \frac{1-\theta_\varepsilon}{2}}, \quad p_2 = \frac{\theta_\varepsilon}{\frac{1}{p} - \frac{1-\theta_\varepsilon}{p_1}}.$$

A lengthy computation reveals that these choices are admissible for (31) and the bound for the operator norm is $(1+\lambda)^\gamma$ where

$$\gamma = \begin{cases} 2\alpha + \frac{d+1}{p} - \frac{d+1}{q} & \text{, if } 2dm_{p,q} - \frac{d+1}{p} + \frac{d+1}{q} \geq d-1 \\ 2\alpha + 1 - d + 2dm_{p,q} + \frac{d+1}{p} - \frac{d+1}{q} + \varepsilon & \text{, if } 2dm_{p,q} - \frac{d+1}{p} + \frac{d+1}{q} < d-1. \end{cases}$$

Under the additional assumption $\frac{1}{p} - \frac{1}{q} \geq \frac{2}{d+2}$ from the Theorem we get again $\gamma \leq 2\alpha - 2 + \frac{1}{p} - \frac{1}{q}$ and in the case $p = 1$ or $q = \infty$

$$\begin{aligned} \gamma - (2\alpha - 2 + \frac{1}{p} - \frac{1}{q}) &= d + 1 - 2dm_{p,q} - \left(\frac{1}{p} - \frac{1}{q}\right) + \varepsilon \\ &\leq d + 1 - 2dm_{p,q} - \frac{2d}{d+1}(2m_{p,q} + 1) + \varepsilon \\ &= \frac{d^2+1}{d+1} - \frac{2d(d+3)}{d+1} m_{p,q} + \varepsilon \\ &\leq \frac{d^2+1}{d+1} - \frac{2d(d+3)}{d+1}\frac{d+\alpha}{2d} + \varepsilon \\ &\leq \frac{-(3+\alpha)d + 1 - 3\alpha}{d+1} + \varepsilon < 0, \end{aligned}$$

which is all we had to show. □

6. Proof of Theorem 4

We have to prove the estimate

$$\|\mathrm{e}^{-\lambda\sqrt{|\cdot|^2 - a^2}} m(|\cdot|)\mathcal{F}_d h(\cdot)\|_{L^s(A)} \lesssim \|h\|_{L^p(\mathbb{R}^d)}(1+\lambda)^{\frac{2}{s'} - \frac{2}{p} - \beta}.$$

for $m \in L^\infty([a,b]), \lambda \geq 0$ and β as in (8). For simplicity we assume $\mu_1 = 1, \mu_2 = 2$, i.e., $A = \{\xi \in \mathbb{R}^d : 1 < |\xi| \leq 2\}$. We first present the bound given by the Hausdorff-Young inequality, so we assume $d \in \mathbb{N}, 1 \geq \frac{1}{p_1} \geq \frac{1}{2}, 1 \geq \frac{1}{s_1} \geq \frac{1}{p_1'}$. Hölder's inequality implies

$$\begin{aligned} \|S_\lambda h\|_{L^{s_1}(A)} &\lesssim \|\mathcal{F}_d h\|_{L^{p_1'}(A)} \|\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-1}}\|_{L^{\frac{s_1 p_1'}{p_1' - s_1}}(A)} \\ &\lesssim \|h\|_{L^p(\mathbb{R}^d)} \left(\int_1^2 \mathrm{e}^{-\lambda \frac{s_1 p_1'}{p_1'-s_1}\sqrt{r^2-1}} r^{d-1}\,\mathrm{d}r\right)^{\frac{p_1'-s_1}{s_1 p_1'}} \\ &\lesssim \|h\|_{L^p(\mathbb{R}^d)} \left(\int_0^{\sqrt{3}} \mathrm{e}^{-\lambda\frac{s_1 p_1'}{p_1'-s_1}\rho} \rho\,\mathrm{d}\rho\right)^{\frac{1}{s_1} - \frac{1}{p_1'}} \\ &\lesssim \|h\|_{L^p(\mathbb{R}^d)}(1+\lambda)^{-\frac{2}{s_1} + \frac{2}{p_1'}}. \end{aligned}$$

This already gives the claim for $d = 1$. So let us assume $d \geq 2$ from now on. We interpolate the previous estimate with the following one for $1 \geq \frac{1}{p_2} > \frac{1}{p_*(d)}, 1 \geq \frac{1}{s_2} \geq \frac{d+1}{(d-1)p_2'}$. From

Theorem 7 and Theorem 8 we deduce the bound

$$\begin{aligned}\|S_\lambda h\|_{L^{s_2}(A)} &\lesssim \|\mathcal{F}_d h\,\mathrm{e}^{-\lambda\sqrt{|\cdot|^2-1}}\|_{L^{s_2}(A)} \\ &\lesssim \left(\int_1^2 \mathrm{e}^{-\lambda s_2\sqrt{r^2-1}}\left(\int_{\mathbb{S}_r^{d-1}} |\mathcal{F}_d h|^{s_2}\,\mathrm{d}\sigma_r\right)\,\mathrm{d}r\right)^{\frac{1}{s_2}} \\ &\lesssim \left(\int_1^2 \mathrm{e}^{-\lambda s_2\sqrt{r^2-1}} r^{d-1-\frac{ds}{p'}}\|h\|_{L^p(\mathbb{R}^d)}^{s_2}\,\mathrm{d}r\right)^{\frac{1}{s_2}} \\ &\lesssim \|h\|_{L^p(\mathbb{R}^d)}\left(\int_1^2 \mathrm{e}^{-\lambda s_2\sqrt{r^2-1}}\,\mathrm{d}r\right)^{\frac{1}{s_2}} \\ &\lesssim \|h\|_{L^p(\mathbb{R}^d)}(1+\lambda)^{-\frac{2}{s_2}}.\end{aligned}$$

We infer from the Riesz-Thorin Theorem

$$\|S_\lambda h\|_{L^s(A)} \lesssim \|h\|_{L^p(\mathbb{R}^d)}(1+\lambda)^{\frac{2}{s'}-\frac{2}{p_1}-\frac{2\theta}{p_1'}} \tag{32}$$

whenever

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{s} = \frac{1-\theta}{s_1} + \frac{\theta}{s_2}, \qquad 0 \le \theta \le 1,$$
$$1 \ge \frac{1}{p_1} \ge \frac{1}{2}, \quad 1 \ge \frac{1}{s_1} \ge \frac{1}{p_1'}, \quad 1 \ge \frac{1}{p_2} > \frac{1}{p_*(d)}, \quad 1 \ge \frac{1}{s_2} \ge \frac{d+1}{(d-1)p_2'}.$$

In order to get the asserted result we (subsequently) choose for sufficiently small $\tilde\varepsilon > 0$

$$\begin{aligned}&\theta = \min\left\{-1+\frac{d+1}{p}-\frac{d-1}{s'}, 1, \left(\frac{1}{p}-\frac{1}{2}\right)\left(\frac{1}{p_*(d)}-\frac{1-\tilde\varepsilon}{2}\right)^{-1}\right\}, \\ &\frac{\theta}{p_2} = \max\left\{\theta - \frac{d-1}{2}\left(\frac{1}{p}-\frac{1}{s'}\right), \frac{\theta}{p_*(d)}+\frac{\tilde\varepsilon}{2}, \frac{1}{p}-1+\theta\right\}, \qquad \frac{1-\theta}{p_1} = \frac{1}{p}-\frac{\theta}{p_2}, \\ &\max\left\{\frac{1}{s}-1+\theta, \frac{\theta(d+1)}{(d-1)p_2'}\right\} \le \frac{\theta}{s_2} \le \min\left\{\theta, \frac{1}{s}-\frac{1-\theta}{p_1'}\right\}, \qquad \frac{1-\theta}{s_1} = \frac{1}{s}-\frac{\theta}{s_2}.\end{aligned}$$

We briefly explain why this choice is admissible. The inequalities $1 \ge \frac{1}{p_2} > \frac{1}{p_*(d)}$ and $1 \ge \frac{1}{s_2} \ge \frac{d+1}{(d-1)p_2'}$ are immediate consequences of the definition of p_2, s_2. Moreover, $p_2 \ge \frac{\theta}{\frac{1}{p}-1+\theta}$ implies $1 \ge \frac{1}{p_1}$ and after some computations one finds that $\theta \le \min\{-1+\frac{d+1}{p}-\frac{d-1}{s'}, (\frac{1}{p}-\frac{1}{2})(\frac{1}{p_*(d)}-\frac{1}{2}+\frac{\tilde\varepsilon}{2})^{-1}\}$ implies $\frac{1}{p_1} \ge \frac{1}{2}$. Finally, $s_2 \le \frac{\theta}{\frac{1}{s}-1+\theta}$ yields $1 \ge \frac{1}{s_1}$ and $s_2 \ge \frac{\theta}{\frac{1}{s}-\frac{1-\theta}{p_1'}}$ gives $\frac{1}{s_1} \ge \frac{1}{p_1'}$. With this choice we obtain for $\tilde\varepsilon$ sufficiently small (in particular $\tilde\varepsilon \le \varepsilon$)

$$\begin{aligned}&-\frac{2}{p_1}-\frac{2\theta}{p_1'} \\ &= -2\theta - \frac{2(1-\theta)}{p_1}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{p} - 2\theta + \frac{2\theta}{p_2} \\
&= -\frac{2}{p} - 2\theta + 2\max\left\{\theta - \frac{d-1}{2}\left(\frac{1}{p} - \frac{1}{s'}\right), \frac{\theta}{p_*(d)} + \frac{\tilde{\varepsilon}}{2}, \frac{1}{p} - 1 + \theta\right\} \\
&= -\frac{2}{p} - \min\left\{\frac{d-1}{p} - \frac{d-1}{s'}, \frac{2\theta}{p_*(d)'} - \tilde{\varepsilon}, \frac{2}{p'}\right\} \\
&\leq -\frac{2}{p} - \min\left\{\frac{d-1}{p} - \frac{d-1}{s'}, \frac{2(\frac{d+1}{p} - \frac{d-1}{s'} - 1)}{p_*(d)'} - \varepsilon, \frac{2}{p_*(d)'} - \varepsilon, \frac{\frac{2}{p_*(d)'}(\frac{1}{p} - \frac{1}{2})}{\frac{1}{p_*(d)} - \frac{1}{2}} - \varepsilon, \frac{2}{p'}\right\} \\
&= -\frac{2}{p} - \min\left\{\frac{d-1}{p} - \frac{d-1}{s'}, \frac{2(\frac{d+1}{p} - \frac{d-1}{s'} - 1)}{p_*(d)'} - \varepsilon, \frac{\frac{2}{p_*(d)'}(\frac{1}{p} - \frac{1}{2})}{\frac{1}{p_*(d)} - \frac{1}{2}} - \varepsilon, \frac{2}{p'}\right\}.
\end{aligned}
$$

Here, the last equality comes from the fact that the third number inside the bracket of the second last line lies between the fourth and the fifth number. Combining this with (32) gives the desired bound.

7. Proof of Proposition 3

In this section we prove Proposition 3 dealing with the small frequency part w_ε of the solution of the perturbed Helmholtz equation. In order to avoid heavy notation we carry out the estimates for $w = \lim_{\varepsilon \searrow 0} w_\varepsilon$ in detail and briefly discuss the necessary modifications afterwards. We recall from (17) the formula

$$
\begin{aligned}
w(x,y) := {} & \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_1}(1_{|\cdot|\leq\mu_1} m_1 g_+ + 1_{|\cdot|\leq\mu_1} m_2 g_-)\right)(x) \\
& + \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_2}(1_{|\cdot|\leq\mu_1} m_3 g_+ + 1_{|\cdot|\leq\mu_2} m_4 g_-)\right)(x)
\end{aligned} \tag{33}
$$

where $m_1, \ldots, m_4$ were introduced in (18). We recall $\nu_j(\xi) = \sqrt{\mu_j^2 - |\xi|^2}$ for $|\xi| \leq \mu_j$ and $j = 1, 2$. We have to prove the estimate

$$
\|w\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\leq 1} \|w_\varepsilon\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.
$$

under the assumptions $\frac{1}{p} > \frac{1}{p_*(n)}$, $\frac{1}{q} < \frac{1}{q_*(n)}$ and $\frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1}$.

Proof of Proposition 3: For every fixed $x \in \mathbb{R}^{n-1}, y \in \mathbb{R}$ we have

$$
\begin{aligned}
w(x,y) &= (2\pi)^{\frac{1-n}{2}} \int_{|\xi|\leq\mu_1} \mathrm{e}^{\mathrm{i}(x\cdot\xi + |y|\nu_1(\xi))}(m_1(\xi)g_+(\xi) + m_2(\xi)g_-(\xi))\,\mathrm{d}\xi \\
&\quad + (2\pi)^{\frac{1-n}{2}} \int_{|\xi|\leq\mu_2} \mathrm{e}^{\mathrm{i}(x\cdot\xi + |y|\nu_2(\xi))}(1_{|\xi|\leq\mu_1} m_3(\xi)g_+(\xi) + m_4(\xi)g_-(\xi))\,\mathrm{d}\xi \\
&= (2\pi)^{\frac{1-n}{2}} \int_{\mathbb{S}^{n-1}_{\mu_1}} \mathrm{e}^{\mathrm{i}(x\cdot\xi + |y|\eta)} \frac{m_1(\xi)g_+(\xi) + m_2(\xi)g_-(\xi)}{(1 + |\nabla\nu_1(\xi)|^2)^{\frac{1}{2}}} 1_{(0,\infty)}(\eta)\,\mathrm{d}\sigma_{\mu_1}(\xi,\eta) \\
&\quad + (2\pi)^{\frac{1-n}{2}} \int_{\mathbb{S}^{n-1}_{\mu_2}} \mathrm{e}^{\mathrm{i}(x\cdot\xi + |y|\eta)} \frac{1_{|\xi|\leq\mu_1} m_3(\xi)g_+(\xi) + m_4(\xi)g_-(\xi)}{(1 + |\nabla\nu_2(\xi)|^2)^{\frac{1}{2}}} 1_{(0,\infty)}(\eta)\,\mathrm{d}\sigma_{\mu_2}(\xi,\eta).
\end{aligned}
$$

Next we use Theorem 7 in the case $n=2$ and Tao's Fourier Restriction Theorem (Theorem 8) in the case $n\geq 3$. In both cases, $s:=\left(\frac{n-1}{n+1}q\right)'$. Using the estimates $|m_1|\lesssim|\nu_1|^{-1}, |m_2|+|m_3|\lesssim 1, |m_4|\lesssim|\nu_2|^{-1}$, which follow from (19), we obtain

$$\begin{aligned}\|w\|_{L^q(\mathbb{R}^n)} &\lesssim \|(m_1g_+ + m_2g_-)(1+|\nabla\nu_1|^2)^{-\frac12}\|_{L^s(\mathbb{S}^{n-1}_{\mu_1})}\\ &\quad + \|(1_{|\xi|\leq\mu_1}m_3g_+ + m_4g_-)(1+|\nabla\nu_2|^2)^{-\frac12}\|_{L^s(\mathbb{S}^{n-1}_{\mu_2})}\\ &\lesssim \|(|\nu_1|^{-1}|g_+|+|g_-|)(1+|\nabla\nu_1|^2)^{-\frac12}\|_{L^s(\mathbb{S}^{n-1}_{\mu_1})}\\ &\quad + \|(1_{|\xi|\leq\mu_1}|g_+|+|\nu_2|^{-1}|g_-|)(1+|\nabla\nu_2|^2)^{-\frac12}\|_{L^s(\mathbb{S}^{n-1}_{\mu_2})}\\ &\overset{(11)}{\lesssim} \||g_+|+|\nu_1||g_-|\|_{L^s(\mathbb{S}^{n-1}_{\mu_1})} + \|1_{|\xi|\leq\mu_1}|\nu_2||g_+|+|g_-|\|_{L^s(\mathbb{S}^{n-1}_{\mu_2})}\\ &\lesssim \||g_+|+|g_-|\|_{L^s(\mathbb{S}^{n-1}_{\mu_1})} + \|1_{|\xi|\leq\mu_1}|g_+|+|g_-|\|_{L^s(\mathbb{S}^{n-1}_{\mu_2})}\\ &\lesssim \|g_+\|_{L^s(\mathbb{S}^{n-1}_{\mu_1})} + \|g_-\|_{L^s(\mathbb{S}^{n-1}_{\mu_2})}\\ &\lesssim \|\mathcal{F}_n^+f\|_{L^s(\mathbb{S}^{n-1}_{\mu_1})} + \|\mathcal{F}_n^-f\|_{L^s(\mathbb{S}^{n-1}_{\mu_2})}.\end{aligned}$$

Since $\frac1p-\frac1q\geq\frac{2}{n+1}$ implies $s'\geq\left(\frac{n-1}{n+1}p'\right)'$ and $p'>\frac{2(n+2)}{n}$, Theorem 8 applies and we get

$$\|w\|_{L^q(\mathbb{R}^n)}\lesssim\|\mathcal{F}_n^+f\|_{L^s(\mathbb{S}^{n-1}_{\mu_1})}+\|\mathcal{F}_n^-f\|_{L^s(\mathbb{S}^{n-1}_{\mu_2})}\lesssim\|f\|_{L^p(\mathbb{R}^n)},$$

which is all we had to show. Here we used $\mathcal{F}_n^\pm f=\mathcal{F}_n(f_\pm)$ where $f_\pm(x,y)=f(x,y)1_{(0,\infty)}(\pm y)$.

Now we indicate the necessary modifications to get the corresponding uniform estimates for w_ε with respect to $\varepsilon\in(0,1]$. Here, each ν_j in (33) is replaced by $\nu_{j,\varepsilon}$ where $\nu_{j,\varepsilon}(\xi)^2=\mu_j^2-|\xi|^2+\mathrm{i}\varepsilon$ and $\mathrm{Im}(\nu_{j,\varepsilon}(\xi))>0$. In this case we obtain the same estimates as above because the sets $\{(\xi,\mathrm{Re}(\nu_{j,\varepsilon}(\xi))):|\xi|\leq\mu_j\}$ are regular hypersurfaces with the property that the Gaussian curvature has a positive lower bound independent of ε. For such surfaces Tao's result remains true and we may thus argue as above. Notice that the positive imaginary part of $\nu_{1,\varepsilon},\nu_{2,\varepsilon}$ lead to damping factors $\mathrm{e}^{-|y|\,\mathrm{Im}(\nu_{j,\varepsilon}(\xi))}$ in the integrals that may be estimated from above by one. □

8. Proof of Proposition 4

In this section we bound (the first part of) the intermediate frequency terms given by

$$\begin{aligned}\mathfrak{w}(x,y)&=\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}y\nu_1}1_Am_2g_-+\mathrm{e}^{\mathrm{i}y\nu_2}1_Am_3g_+\right)(x)\\ &\overset{(18)}{=}1_{y>0}\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}y\nu_1}1_Am^*\mathcal{F}_n^-f(\cdot,-\nu_2(\cdot))\right)(x)\\ &\qquad+1_{y<0}\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}y\nu_2}1_Am^*\mathcal{F}_n^+f(\cdot,-\nu_1(\cdot))\right)(x)\end{aligned}$$

where $m^*(\xi):=\mathrm{i}\sqrt{2\pi}(\nu_1(\xi)+\nu_2(\xi))^{-1}$. Here, $A=\{\xi\in\mathbb{R}^{n-1}:\mu_1<|\xi|\leq\mu_2\}$ so that $\xi\in A$ implies $\nu_1(\xi)=\mathrm{i}(|\xi|^2-\mu_1)^{1/2}$ and $\nu_2(\xi)=(\mu_2^2-|\xi|^2)^{-1/2}$, see (10). For a bounded complex-valued function $\mathfrak{m}\in L^\infty(A)$ we define the linear operators

$$Q_{\mathfrak{m}}h(x,y):=1_{y<0}\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}y\nu_2}1_A\mathfrak{m}\mathcal{F}_n^+h(\cdot,-\nu_1(\cdot))\right)(x)$$

that we will prove to be bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ under the given assumptions on p, q. Its adjoint is then bounded from $L^{q'}(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ and hence as well for all p, q according to the assumptions. It is given by the formula

$$Q_{\mathfrak{m}}^* h(x,y) := 1_{y>0}\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}y\nu_1}1_A\overline{\mathfrak{m}}\mathcal{F}_n^- h(\cdot,\nu_2(\cdot))\right)(x)$$

because we have for all $h_1, h_2 \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} Q_{\mathfrak{m}}h_1(x,y)\overline{h_2(x,y)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^0\int_{\mathbb{R}^{n-1}} \mathrm{e}^{\mathrm{i}y\nu_2(\xi)}1_A(\xi)\mathfrak{m}(\xi)\mathcal{F}_n^+h_1(\xi,-\nu_1(\xi))\overline{\mathcal{F}_{n-1}[h_2(\cdot,y)](\xi)}\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \int_{\mathbb{R}^{n-1}} 1_A(\xi)\mathfrak{m}(\xi)\mathcal{F}_n^+h_1(\xi,-\nu_1(\xi))\overline{\int_{-\infty}^0 \mathrm{e}^{-\mathrm{i}y\nu_2(\xi)}\mathcal{F}_{n-1}[h_2(\cdot,y)](\xi)\,\mathrm{d}y}\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^{n-1}} 1_A(\xi)\mathfrak{m}(\xi)\mathcal{F}_n^+h_1(\xi,-\nu_1(\xi))\cdot(2\pi)^{\frac12}\overline{\mathcal{F}_n^-h_2(\xi,\nu_2(\xi))}\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^{n-1}}\left(\int_0^\infty 1_A(\xi)\mathcal{F}_{n-1}[h_1(\cdot,y)](\xi)\mathrm{e}^{\mathrm{i}\nu_1(\xi)}\,\mathrm{d}y\right)\mathfrak{m}(\xi)\overline{\mathcal{F}_n^-h_2(\xi,\nu_2(\xi))}\,\mathrm{d}\xi \\
&= \int_0^\infty\int_{\mathbb{R}^{n-1}} \mathcal{F}_{n-1}[h_1(\cdot,y)](\xi)\,\overline{\mathrm{e}^{\mathrm{i}\nu_1(\xi)}1_A(\xi)\overline{\mathfrak{m}(\xi)}\mathcal{F}_n^-h_2(\xi,\nu_2(\xi))}\,\mathrm{d}\xi\,\mathrm{d}y \\
&= \int_0^\infty\int_{\mathbb{R}^{n-1}} h_1(x,y)\,\overline{\mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}\nu_1}1_A\overline{\mathfrak{m}}\mathcal{F}_n^-h_2(\cdot,\nu_2(\cdot))\right)(x)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} h_1(x,y)\overline{Q_{\overline{\mathfrak{m}}}^*h_2(x,y)}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

The following result tells us that it is sufficient to find $L^p - L^q$-bounds for $Q_{\mathfrak{m}}$.

Proposition 13. *For $x \in \mathbb{R}^{n-1}, y > 0$ we have*

$$\mathfrak{w}(x,y) = (Q_{\overline{m^*}}^* f)(x,y) + (Q_{m^*}f)(x,y).$$

Our bounds for Q_m rely on Theorem 4.

Lemma 1. *Let $n \in \mathbb{N}, n \geq 2$ and $\mathfrak{m} \in L^\infty(A)$. Then the linear operator $Q_{\mathfrak{m}} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded whenever $\frac1p > \frac{1}{p_*(n)}, \frac1q < \frac{1}{q_*(n)}, \frac1p - \frac1q \geq \frac{2}{n+1}$. In particular, this holds for all $(p,q) \in \tilde{\mathcal{D}}$. If the Restriction Conjecture is true then it is bounded whenever $\frac1p > \frac{n+1}{2n}, \frac1q < \frac{n-1}{2n}, \frac1p - \frac1q \geq \frac{2}{n+1}$ and hence for all $(p,q) \in \mathcal{D}$.*

Proof. We have

$$\begin{aligned}
Q_{\mathfrak{m}}h(x,y) &= \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}y\nu_2}1_A\mathfrak{m}\mathcal{F}_n^+h(\cdot,-\nu_1(\cdot))\right)(x)\cdot 1_{y<0} \\
&= (2\pi)^{\frac{1-n}{2}}\int_{|\xi|\leq\mu_2} \mathrm{e}^{\mathrm{i}(x\cdot\xi-y\nu_2(\xi))}1_A(\xi)\mathfrak{m}(\xi)\mathcal{F}_n^+h(\xi,-\nu_1(\xi))\,\mathrm{d}\xi\cdot 1_{y<0} \\
&= (2\pi)^{\frac{1-n}{2}}\int_{\mathbb{S}_{\mu_2}^{n-1}} \mathrm{e}^{\mathrm{i}(x\cdot\xi-y\eta)}1_A(\xi)1_{\eta>0}\mathfrak{m}(\xi)\frac{\mathcal{F}_n^+h(\xi,-\nu_1(\xi))}{(1+|\nabla\nu_2(\xi)|^2)^{-\frac12}}\,\mathrm{d}\sigma_{\mu_2}(\xi,\eta)\cdot 1_{y<0}
\end{aligned}$$

In the case $n=2, n\geq 3$ we use Theorem 7, Theorem 8, respectively. Due to $\frac{1}{q}<\frac{1}{q_*(n)}$ and $s:=\left(\frac{n-1}{n+1}q\right)'$ we get the bound

$$\begin{aligned}\|Q_{\mathfrak{m}}h\|_{L^q(\mathbb{R}^n)} &\lesssim \left\|1_A\mathfrak{m}\mathcal{F}_n^+h(\cdot,-\nu_1(\cdot))(1+|\nabla\nu_2(\cdot)|^2)^{-\frac{1}{2}}\right\|_{L^s(\mathbb{S}^{n-1}_{\mu_2})}\\ &\lesssim \|\mathfrak{m}\|_\infty \left\|1_A\mathcal{F}_n^+h(\cdot,-\nu_1(\cdot))\right\|_{L^s(\mathbb{S}^{n-1}_{\mu_2})}\\ &\lesssim \|\mathfrak{m}\|_\infty\left(\int_A\left|\int_0^\infty \mathcal{F}_{n-1}[f(\cdot,z)](\xi)\mathrm{e}^{-z\sqrt{|\xi|^2-\mu_1^2}}\,\mathrm{d}z\right|^s\,\mathrm{d}\xi\right)^{\frac{1}{s}}.\end{aligned}$$

Minkowski's inequality and Theorem 4 imply for β and $d=n-1$ as in (8)

$$\begin{aligned}\|Q_{\mathfrak{m}}h\|_{L^q(\mathbb{R}^n)} &\lesssim \|\mathfrak{m}\|_\infty\int_0^\infty\left(\int_A\left|\mathcal{F}_{n-1}[h(\cdot,z)](\xi)\mathrm{e}^{-z\sqrt{|\xi|^2-\mu_1^2}}\right|^s\,\mathrm{d}\xi\right)^{\frac{1}{s}}\,\mathrm{d}z\\ &\lesssim \|\mathfrak{m}\|_\infty\int_0^\infty \|h(\cdot,z)\|_{L^p(\mathbb{R}^{n-1})}(1+z)^{\frac{2}{s'}-\frac{2}{p}-\beta}\,\mathrm{d}z\\ &\lesssim \|\mathfrak{m}\|_\infty\|h\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

provided that $(\frac{2}{s'}-\frac{2}{p}-\beta)p'<-1$ or equivalently $\beta+\frac{3}{p}-\frac{2}{s'}-1>0$. To prove the main statement of the Lemma, it remains to check this condition for all $(p,q)\in\tilde{\mathcal{D}}$. Indeed, in the case $n=2$, where $d=n-1=1$, this follows from $\beta=0$ and

$$\frac{3}{p}-\frac{2}{s'}-1=3\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{3}{q}-1>2-\frac{3}{4}-1>0.$$

In the case $d=n-1\geq 2$ this is a consequence of the definition of β from (8). We have

$$\begin{aligned}\left(\frac{n-2}{p}-\frac{n-2}{s'}\right)+\frac{3}{p}-\frac{2}{s'}-1 &= \frac{n+1}{p}-\frac{n}{s'}-1 \;=\; \frac{n+1}{p}-\frac{n(n+1)}{(n-1)q}-1\\ &= (n+1)\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{n+1}{(n-1)q}-1\\ &> 2-\frac{n+1}{n-1}\cdot\frac{n}{2(n+2)}-1\\ &= \frac{n^2+n-4}{2(n-1)(n+2)}\\ &> 0,\\ \frac{2}{p'}+\frac{3}{p}-\frac{2}{s'}-1 &= \frac{1}{p}-\frac{2(n+1)}{(n-1)q}+1\\ &= \left(\frac{1}{p}-\frac{1}{q}\right)-\frac{n+3}{(n-1)q}+1\\ &> \frac{2}{n+1}-\frac{n+3}{n-1}\cdot\frac{n}{2(n+2)}+1\end{aligned}$$

$$
\begin{aligned}
&= \frac{(n+3)(n^2+n-4)}{2(n+1)(n-1)(n+2)} \\
&> 0,
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\frac{2}{(p_*(n-1))'}(\frac{1}{p}-\frac{1}{2})}{\frac{1}{p_*(n-1)}-\frac{1}{2}}\right)+\frac{3}{p}-\frac{2}{s'}-1 &= (n-1)\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{3}{p}-\frac{2}{s'}-1 \\
&= \frac{n+2}{p}-\frac{2(n+1)}{(n-1)q}-\frac{n+1}{2} \\
&= \frac{n^2-n-4}{(n-1)p}+\frac{2(n+1)}{n-1}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{n+1}{2} \\
&> \frac{(n^2-n-4)(n+4)}{2(n-1)(n+2)}+\frac{4}{n-1}-\frac{n+1}{2} \\
&= \frac{n^2+n+2}{2(n+2)(n-1)} \\
&> 0,
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{2(\frac{n}{p}-\frac{n-2}{s'}-1)}{p_*(n-1)'}\right)+\frac{3}{p}-\frac{2}{s'}-1 &= \left(\frac{(n-1)n}{(n+1)p}-\frac{(n-1)(n-2)}{(n+1)s'}-\frac{n-1}{n+1}\right)+\frac{3}{p}-\frac{2}{s'}-1 \\
&= \frac{n^2+2n+3}{(n+1)p}-\frac{n^2-n+4}{(n+1)s'}-\frac{2n}{n+1} \\
&= \frac{n^2+2n+3}{n+1}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{n^2-2n-7}{(n+1)(n-1)q}-\frac{2n}{n+1} \\
&> \frac{2(n^2+2n+3)}{(n+1)^2}+1_{n=3}\frac{(n^2-2n-7)n}{2(n+1)(n-1)(n+2)}-\frac{2n}{n+1} \\
&= \frac{2(n+3)}{(n+1)^2}+1_{n=3}\frac{(n^2-2n-7)n}{2(n+1)(n-1)(n+2)} \\
&> 0.
\end{aligned}
$$

Hence, the condition $(\frac{2}{s'}-\frac{2}{p}-\beta)p' < -1$ holds. For the extra claim regarding the Restriction Conjecture, is suffices to prove the above estimates (for $n \geq 3$) where $p_*(n-1)$ is replaced by $\frac{2(n-1)}{n}$ and the estimates $\frac{1}{p} > \frac{n+4}{2(n+2)}, \frac{1}{q} < \frac{n}{2(n+2)}$ are replaced by $\frac{1}{p} > \frac{n-1}{2n}, \frac{1}{q} < \frac{n-1}{2n}$. This can be done as above and one obtains again $\beta+\frac{3}{p}-\frac{2}{s'}-1 > 0$ and the proof is finished. □

Proof of Proposition 4: This is a consequence of Proposition 13 and Lemma 1 because

$$\|\mathfrak{w}\|_{L^q(\mathbb{R}^n)} \lesssim \|Q^*_{m^*}f\|_{L^q(\mathbb{R}^n)} + \|Q_{m^*}f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

provided that $\frac{1}{p} > \frac{1}{p_*(n)}, \frac{1}{q} < \frac{1}{q_*(n)}, \frac{1}{p}-\frac{1}{q} \geq \frac{2}{n+1}$ holds. If the Restriction Conjecture holds, $p_*(n)$ and $q_*(n)$ can be replaced by $\frac{2n}{n+1}$ and $\frac{2n}{n-1}$, respectively. □

9. Proof of Proposition 5

We now prove our estimates for those intermediate frequencies collected in $\mathfrak{W}_\varepsilon$. As above, we avoid the technicalities for $\varepsilon > 0$ by considering only the most singular limit term

$$\mathfrak{W}(x,y) = \lim_{\varepsilon\searrow 0}\mathfrak{W}_\varepsilon(x,y) = \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_1}1_{\mu_1<|\cdot|\le\mu_1+\mu_2}m_1g_+ + \mathrm{e}^{\mathrm{i}|y|\nu_1}1_{\mu_2<|\cdot|\le\mu_1+\mu_2}m_2g_-\right)(x)$$
$$+ \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_2}1_{\mu_2<|\cdot|\le\mu_1+\mu_2}m_3g_+ + \mathrm{e}^{\mathrm{i}|y|\nu_2}1_{\mu_2<|\cdot|\le\mu_1+\mu_2}m_4g_-\right)(x).$$

We use $\nu_j(\xi) = \mathrm{i}\sqrt{|\xi|^2-\mu_j^2}$ for $|\xi| > \mu_j$. To prove Proposition 5 wehave to show that for all $n\in\mathbb{N}, n\ge 2$ and all p,q such that $\frac{1}{p} > \frac{n+1}{2n}$, $\frac{1}{q} < \frac{n-1}{2n}$ and $\frac{1}{p}-\frac{1}{q} \ge \frac{2}{n+1}$ the following estimate holds

$$\|\mathfrak{W}\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\le 1}\|\mathfrak{W}_\varepsilon\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof of Proposition 5: We introduce the annuli $A_j := \{\xi\in\mathbb{R}^{n-1} : \mu_j \le |\xi| \le \mu_1+\mu_2\}$ for $j = 1,2$. Then we have for fixed $x\in\mathbb{R}^n, y\in\mathbb{R}$

$$\begin{aligned}
\mathfrak{W}(x,y) &= \mathcal{F}_{n-1}^{-1}\left(1_{A_1}\mathrm{e}^{\mathrm{i}|y|\nu_1}m_1\mathcal{F}_n^+f(\cdot,-\nu_1(\cdot))\right)(x)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(1_{A_2}\mathrm{e}^{\mathrm{i}|y|\nu_1}m_2\mathcal{F}_n^-f(\cdot,\nu_2(\cdot))\right)(x)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(1_{A_2}\mathrm{e}^{\mathrm{i}|y|\nu_2}m_3\mathcal{F}_n^+f(\cdot,-\nu_1(\cdot))\right)(x)\\
&\quad+\mathcal{F}_{n-1}^{-1}\left(1_{A_2}\mathrm{e}^{\mathrm{i}|y|\nu_2}m_4\mathcal{F}_n^-f(\cdot,\nu_2(\cdot))\right)(x)\\
&= \int_0^\infty \mathcal{F}_{n-1}^{-1}\left(1_{A_1}(\xi)\mathrm{e}^{-(|y|+z)\sqrt{|\xi|^2-\mu_1^2}}m_1(\xi)\mathcal{F}_{n-1}[f(\cdot,z)](\xi)\right)(x)\,\mathrm{d}z\\
&\quad+\int_{-\infty}^0 \mathcal{F}_{n-1}^{-1}\left(1_{A_2}(\xi)\mathrm{e}^{-|y|\sqrt{|\xi|^2-\mu_1^2}+z\sqrt{|\xi|^2-\mu_2^2}}m_2(\xi)\mathcal{F}_{n-1}[f(\cdot,z)](\xi)\right)(x)\,\mathrm{d}z\\
&\quad+\int_0^\infty \mathcal{F}_{n-1}^{-1}\left(1_{A_2}(\xi)\mathrm{e}^{-|y|\sqrt{|\xi|^2-\mu_2^2}-z\sqrt{|\xi|^2-\mu_1^2}}m_3(\xi)\mathcal{F}_{n-1}[f(\cdot,z)](\xi)\right)(x)\,\mathrm{d}z\\
&\quad+\int_{-\infty}^0 \mathcal{F}_{n-1}^{-1}\left(1_{A_2}(\xi)\mathrm{e}^{-|y|\sqrt{|\xi|^2-\mu_2^2}+z\sqrt{|\xi|^2-\mu_2^2}}m_4(\xi)\mathcal{F}_{n-1}[f(\cdot,z)](\xi)\right)(x)\,\mathrm{d}z\\
&= \int_0^\infty \mathcal{F}_{n-1}^{-1}\left(1_{A_1}(\xi)\mathrm{e}^{-(|y|+|z|)\sqrt{|\xi|^2-\mu_1^2}}(|\xi|^2-\mu_1^2)^{-\frac{1}{2}}\tilde{m}_1(|\xi|)\mathcal{F}_{n-1}[f(\cdot,z)](\xi)\right)(x)\,\mathrm{d}z\\
&\quad+\int_{-\infty}^0 \mathcal{F}_{n-1}^{-1}\left(1_{A_2}(\xi)\mathrm{e}^{-(|y|+|z|)\sqrt{|\xi|^2-\mu_2^2}}\tilde{m}_2(|\xi|)\mathcal{F}_{n-1}[f(\cdot,z)](\xi)\right)(x)\,\mathrm{d}z\\
&\quad+\int_0^\infty \mathcal{F}_{n-1}^{-1}\left(1_{A_2}(\xi)\mathrm{e}^{-(|y|+|z|)\sqrt{|\xi|^2-\mu_2^2}}\tilde{m}_3(|\xi|)\mathcal{F}_{n-1}[f(\cdot,z)](\xi)\right)(x)\,\mathrm{d}z\\
&\quad+\int_{-\infty}^0 \mathcal{F}_{n-1}^{-1}\left(1_{A_2}(\xi)\mathrm{e}^{-(|y|+|z|)\sqrt{|\xi|^2-\mu_2^2}}(|\xi|^2-\mu_2^2)^{-\frac{1}{2}}\tilde{m}_4(|\xi|)\mathcal{F}_{n-1}[f(\cdot,z)](\xi)\right)(x)\,\mathrm{d}z.
\end{aligned}$$

Here, the functions $\tilde{m}_1,\dots,\tilde{m}_4$ are defined by

$$\tilde{m}_1(|\xi|) = m_1(\xi)(|\xi|^2-\mu_1^2)^{\frac{1}{2}} = \frac{\sqrt{\pi/2}}{\nu_1(\xi)+\nu_2(\xi)}\cdot\left(\mathrm{sign}(y)\nu_1(\xi)-\nu_2(\xi)\right),$$

$$\tilde{m}_4(|\xi|) = m_4(\xi)(|\xi|^2 - \mu_2^2)^{\frac{1}{2}} = \frac{\sqrt{\pi/2}}{\nu_1(\xi) + \nu_2(\xi)} \cdot (-\operatorname{sign}(y)\nu_2(\xi) - \nu_1(\xi))$$

and

$$\tilde{m}_2(|\xi|) = m_2(\xi)\mathrm{e}^{|y|(\sqrt{|\xi|^2-\mu_2^2}-\sqrt{|\xi|^2-\mu_1^2})} = \frac{\mathrm{i}\sqrt{\pi/2}(1+\operatorname{sign}(y))}{\nu_1(\xi)+\nu_2(\xi)} \cdot \mathrm{e}^{|y|(\sqrt{|\xi|^2-\mu_2^2}-\sqrt{|\xi|^2-\mu_1^2})},$$

$$\tilde{m}_3(|\xi|) = m_3(\xi)\mathrm{e}^{|z|(\sqrt{|\xi|^2-\mu_2^2}-\sqrt{|\xi|^2-\mu_1^2})} = \frac{\mathrm{i}\sqrt{\pi/2}(1-\operatorname{sign}(y))}{\nu_1(\xi)+\nu_2(\xi)} \cdot \mathrm{e}^{|z|(\sqrt{|\xi|^2-\mu_2^2}-\sqrt{|\xi|^2-\mu_1^2})}.$$

Notice that $\tilde{m}_1, \ldots, \tilde{m}_4$ are indeed radially symmetric because so are ν_1, ν_2. From $|\nu_1(\xi) + \nu_2(\xi)| \gtrsim 1$ and $\mu_1 < \mu_2$ we infer that all four terms are bounded independently of y, z. So we apply Theorem 2 ($n = 2$) resp. Theorem 3 ($n \geq 3$) to bound these integrals for any fixed $y, z \in \mathbb{R}$. The assumptions of these theorems are satisfied because our assumption in Proposition 5 implies for $d = n - 1$ and $\alpha \in \{0, \frac{1}{2}\}$

$$\frac{1}{p} > \frac{n+1}{2n} \geq \frac{1}{2} + \frac{\alpha}{2d}, \qquad \frac{1}{q} < \frac{n-1}{2n} \leq \frac{1}{2} - \frac{\alpha}{2d}, \qquad \frac{1}{p} - \frac{1}{q} \geq \frac{2}{n+1} \geq \frac{2\alpha}{d+1}.$$

From the above-mentioned Theorems we get

$$\|\mathfrak{W}(\cdot, y)\|_{L^q(\mathbb{R}^{n-1})} \lesssim \int_{\mathbb{R}} (1 + |y| + |z|)^\gamma \|f(\cdot, z)\|_{L^p(\mathbb{R}^{n-1})} \,\mathrm{d}z$$

where $\gamma \leq -1 + \frac{1}{p} - \frac{1}{q}$ and $\gamma < -1 + \frac{1}{p} - \frac{1}{q}$ for $p = 1$ or $q = \infty$. In the latter case the classical version of Young's convolution inequality applies and gives

$$\begin{aligned}
\|\mathfrak{W}\|_{L^q(\mathbb{R}^n)} &\lesssim \|(1 + |\cdot|)^\gamma * \|f(\cdot, z)\|_{L^p(\mathbb{R}^{n-1})}\|_{L^q(\mathbb{R})} \\
&\lesssim \|(1 + |\cdot|)^\gamma\|_{L^{\frac{pq}{pq-q+p}}(\mathbb{R})} \left(\int_{\mathbb{R}} \|f(\cdot, z)\|_{L^p(\mathbb{R}^{n-1})}^p \,\mathrm{d}z \right)^{\frac{1}{p}} \\
&\lesssim \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

In the former case Young's convolution inequality in weak Lebesgue spaces [25, Theorem 1.4.25] is applicable and yields

$$\begin{aligned}
\|\mathfrak{W}\|_{L^q(\mathbb{R}^n)} &\lesssim \|(1 + |\cdot|)^{-1+\frac{1}{p}-\frac{1}{q}} * \|f(\cdot, z)\|_{L^p(\mathbb{R}^{n-1})}\|_{L^q(\mathbb{R})} \\
&\lesssim \|(1 + |\cdot|)^{-1+\frac{1}{p}-\frac{1}{q}}\|_{L^{\frac{pq}{pq-q+p},w}(\mathbb{R})} \left(\int_{\mathbb{R}} \|f(\cdot, z)\|_{L^p(\mathbb{R}^{n-1})}^p \,\mathrm{d}z \right)^{\frac{1}{p}} \\
&\lesssim \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

□

10. Proof of Proposition 6

We recall that have to prove the following: For all $n \in \mathbb{N}, n \geq 2$ and $p', q \in [2, \infty]$ we have

$$\|W\|_{L^q(\mathbb{R}^n)} + \sup_{0<|\varepsilon|\leq 1} \|W_\varepsilon\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

provided that $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{n}$ and $\frac{1}{p} - \frac{1}{q} < \frac{2}{n}$ if $p = 1$ or $q = \infty$. Here,

$$\begin{aligned} W(x,y) &= \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_1} 1_{|\cdot|>\mu_1+\mu_2}(m_1 g_+ + m_2 g_-)\right)(x) \\ &\quad + \mathcal{F}_{n-1}^{-1}\left(\mathrm{e}^{\mathrm{i}|y|\nu_2} 1_{|\cdot|>\mu_1+\mu_2}(m_3 g_+ + m_4 g_-)\right)(x). \end{aligned}$$

and W_ε is given by the same formula with ν_1, ν_2 replaced by $\nu_{1,\varepsilon}, \nu_{2,\varepsilon}$, respectively. We recall $g_+(\xi) = \mathcal{F}_n^+ f(\xi, -\nu_1(\xi))$ and $g_-(\xi) = \mathcal{F}_n^-(\xi, \nu_2(\xi))$.

Proof of Proposition 6: Again we concentrate on the estimates for W since the corresponding modifications for W_ε are purely technical. We recall that for $|\xi| \geq R := \mu_1 + \mu_2$ we have $|m_1(\xi)| + \ldots + |m_4(\xi)| \lesssim (1 + |\xi|)^{-1}$ as well as $\mathrm{i}\nu_j(\xi) = -\sqrt{|\xi|^2 - \mu_j^2} \leq -c(|\xi| + 1)$ for some $c > 0$, see (19) and (10). So the Hausdorff-Young inequality implies

$$\begin{aligned} \|W(\cdot,y)\|_{L^q(\mathbb{R}^{n-1})} &\lesssim \left\|\mathrm{e}^{\mathrm{i}|y|\nu_1}(m_1 g_+ + m_2 g_-) + \mathrm{e}^{\mathrm{i}|y|\nu_2}(m_3 g_+ + m_4 g_-)\right\|_{L^{q'}(\mathbb{R}^{n-1}\setminus B_R(0))} \\ &\lesssim \left\|\mathrm{e}^{-c|y|(|\cdot|+1)}(|\cdot|+1)^{-1}(|g_+| + |g_-|)\right\|_{L^{q'}(\mathbb{R}^{n-1}\setminus B_R(0))} \\ &\lesssim \int_{\mathbb{R}} \left\|\mathrm{e}^{-c(|y|+|z|)(|\cdot|+1)}(|\cdot|+1)^{-1}\mathcal{F}_{n-1}[f(\cdot,z)]\right\|_{L^{q'}(\mathbb{R}^{n-1})}\,\mathrm{d}z. \end{aligned}$$

In the last step we applied Minkowski's inequality in integral form. Using now Hölder's and then the Hausdorff-Young inequality we get

$$\begin{aligned} \|W\|_{L^q(\mathbb{R}^n)} &\leq \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left\|\frac{\mathrm{e}^{-c(|z|+|y|)(|\cdot|+1)}}{|\cdot|+1}\right\|_{L^{\frac{pq}{q-p}}(\mathbb{R}^{n-1})} \|\mathcal{F}_{n-1}[f(\,\cdot\,,z)]\|_{L^{p'}(\mathbb{R}^{n-1})}\,\mathrm{d}z\right)^q \mathrm{d}y\right)^{\frac{1}{q}} \\ &\lesssim \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left\|\frac{\mathrm{e}^{-c(|z|+|y|)(|\cdot|+1)}}{|\cdot|+1}\right\|_{L^{\frac{pq}{q-p}}(\mathbb{R}^{n-1})} \|f(\,\cdot\,,z)\|_{L^{p}(\mathbb{R}^{n-1})}\,\mathrm{d}z\right)^q \mathrm{d}y\right)^{\frac{1}{q}} \\ &\lesssim \|K * F\|_{L^q(\mathbb{R})} \end{aligned}$$

where

$$K(w) = \left\|(|\cdot|+1)^{-1}\,\mathrm{e}^{-c|w|(|\cdot|+1)}\right\|_{L^{\frac{pq}{q-p}}(\mathbb{R}^{n-1})} \quad \text{and} \quad F(w) = \|f(\,\cdot\,,w)\|_{L^p(\mathbb{R}^{n-1})}\,.$$

For $w > 0$ we have

$$\begin{aligned} K(w) &\eqsim \mathrm{e}^{-cw}\left(\int_0^1 r^{n-2}\,\mathrm{d}r + \int_1^\infty r^{n-2-\frac{pq}{q-p}}\;\mathrm{e}^{-c\frac{pq}{q-p}wr}\,\mathrm{d}r\right)^{\frac{q-p}{pq}} \\ &= \mathrm{e}^{-cw}\left(\int_0^1 r^{n-2}\,\mathrm{d}r + w^{-(n-1)+\frac{pq}{q-p}}\int_w^\infty t^{n-2-\frac{pq}{q-p}}\;\mathrm{e}^{-c\frac{pq}{q-p}t}\,\mathrm{d}t\right)^{\frac{q-p}{pq}} \\ &\eqsim 1_{[1,\infty)}(w)\mathrm{e}^{-cw} + 1_{(0,1)}(w) w^{-(n-1)\frac{q-p}{pq}+1}. \end{aligned}$$

Thus $K \in L^{pq/(pq+p-q),\infty}(\mathbb{R})$ if and only if

$$\left(-(n-1)\frac{q-p}{pq}+1\right)\cdot\frac{pq}{pq+p-q} \geq -1, \quad \text{equivalently} \quad \frac{1}{p}-\frac{1}{q} \leq \frac{2}{n},$$

which is precisely the assumption in the proposition. Similarly, $K \in L^{pq/(pq+p-q)}(\mathbb{R})$ if and only if $\frac{1}{p} - \frac{1}{q} < \frac{2}{n}$. So, as in the proof of Proposition 5, the classical and weak-space versions of Young's convolution inequality imply $\|W\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$, which is all we had to show. $\square$

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Conflict of interest

The authors declare that they have no conflict of interest.

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