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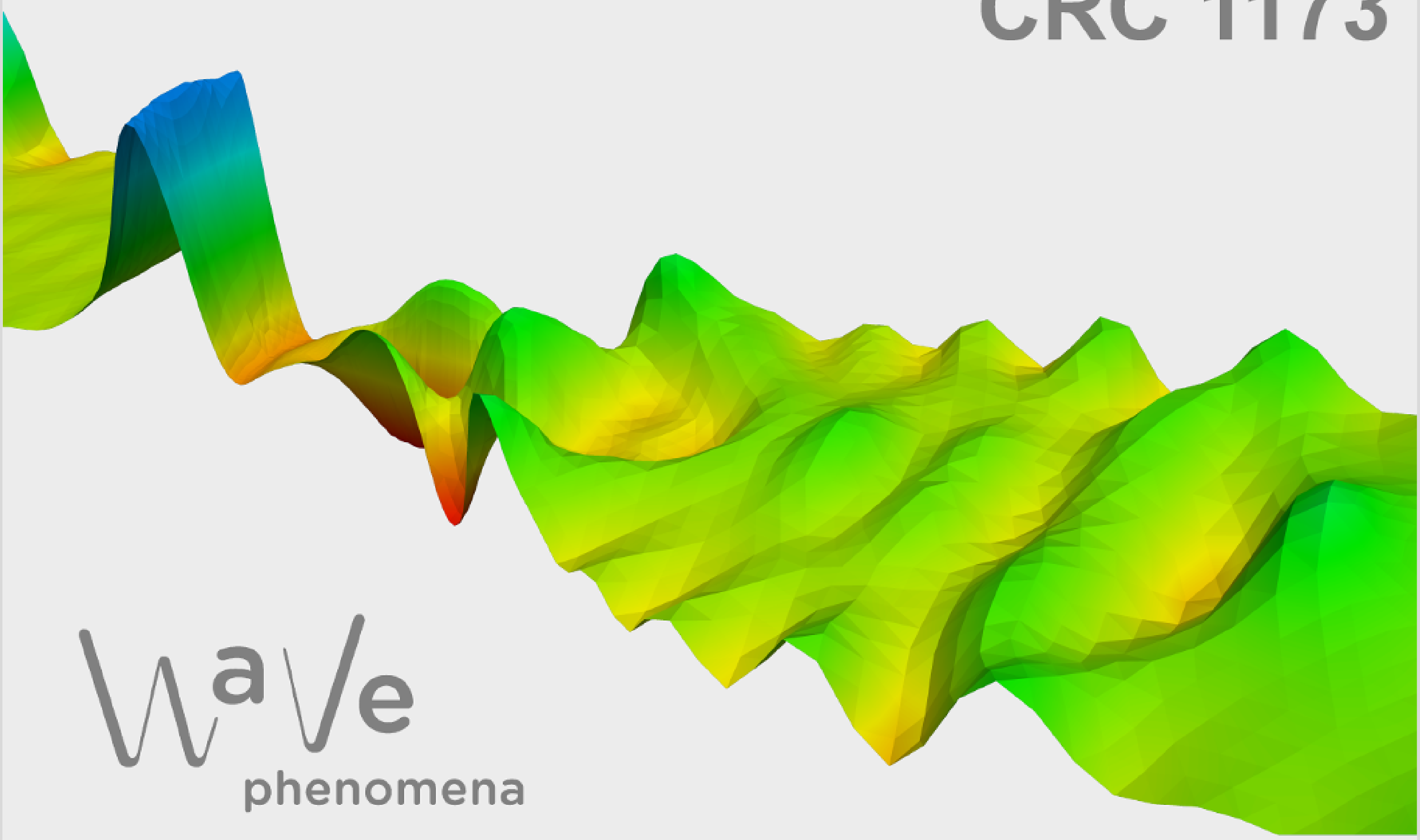

Wave phenomena

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NONLINEAR SCALAR FIELD EQUATION WITH COMPETING NONLOCAL TERMS

PIETRO D'AVENIA, JAROSŁAW MEDERSKI, AND ALESSIO POMPONIO

Abstract. We find radial and nonradial solutions to the following nonlocal problem

$$-\Delta u + \omega u = \big(I_\alpha * F(u)\big) f(u) - \big(I_\beta * G(u)\big) g(u) \text{ in } \mathbb{R}^N$$

under general assumptions, in the spirit of Berestycki and Lions, imposed on f and g, where $N \geq 3$, $0 \leq \beta \leq \alpha < N$, $\omega \geq 0$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions with corresponding primitives F, G, and I_α, I_β are the Riesz potentials. If $\beta > 0$, then we deal with two competing nonlocal terms modelling attractive and repulsive interaction potentials.

1. Introduction

This paper mainly deals with the following problem

$$- \Delta u = \big(I_\alpha * F(u)\big) f(u) - \big(I_\beta * G(u)\big) g(u) \qquad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $0 \leq \beta \leq \alpha < N$, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions with corresponding primitives

$$F(s) = \int_0^s f(t)dt, \quad G(s) = \int_0^s g(t)dt.$$

Moreover $I_\gamma : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential

$$I_\gamma(x) := \frac{\Gamma(\frac{N-\gamma}{2})}{\Gamma(\frac{\gamma}{2})\pi^{N/2} 2^\gamma |x|^{N-\gamma}} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\} \text{ and } \gamma \in (0, N),$$

while we set $I_0 = \delta_0$, namely the identity for the convolution.

If $N = 3$, $\alpha = 2$, $\beta = 0$, $F(s) = \frac{1}{\sqrt{2}}|s|^2$ and $G(s) = s$, then (1.1) is the well-known Choquard, or Choquard-Pekar equation

$$-\Delta u + u = \big(I_2 * |u|^2\big) u \qquad \text{in } \mathbb{R}^N.$$

This equation comes, for instance, from an approximation to the Hartree-Fock theory of a plasma [14, 24]. A variational approach for this case was presented by Lieb [14] and Lions [16].

More generally, if $N \geq 3$, $F(s) = \frac{1}{\sqrt{p}}|s|^p$, for suitable p, $\alpha > 0$ and $G(s) = s$, then weak solutions to (1.1) can be obtained by means of critical points of the associated functional. If, for instance, $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $\beta = 0$, then, according to the work of Moroz and Van Schaftingen [21], the Hardy-Littlewood-Sobolev inequality implies that $\big(I_\alpha * F(u)\big) F(u) \in L^1(\mathbb{R}^N)$ for $u \in H^1(\mathbb{R}^N)$.

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Moreover the associated functional is well-defined and of class $\mathcal{C}^1$ on $H^1(\mathbb{R}^N)$, and its critical points correspond to solutions to

$$-\Delta u + u = \big(I_\alpha * F(u)\big) f(u) \qquad \text{in } \mathbb{R}^N. \tag{1.2}$$

A ground state solution and its properties was obtained in [21]. The same authors in [22] also studied the existence of solutions with a general nonlinearity F in the spirit of the classical result of Berestycki and Lions [6], namely

$$|sf(s)| \le C\big(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}}\big), \quad \lim_{s\to 0} F(s)/|s|^{\frac{N+\alpha}{N}} = \lim_{|s|\to+\infty} F(s)/|s|^{\frac{N+\alpha}{N-2}} = 0, \quad F(s_0)\neq 0, \tag{1.3}$$

for some $s_0 \neq 0$ and $C > 0$, see also a survey [23] and the references therein. Note that, if $\alpha = 0$ in (1.2), since $I_0 * F(u) = F(u)$, (1.3) covers the Berestycki-Lions growth assumptions only for the nonnegative (attractive) nonlinearity $F^2(s) \ge 0$ of the corresponding energy functional (see (3.3) of [6]).

On the other hand, as for instance in the Hartree-Fock theory, the interaction potential could be also repulsive [5, 17], i.e. with $\beta > 0$ and a non-trivial $G(s) \ge 0$. Moreover problems similar to (1.1) may admit some local terms as well, see also [23] and the references therein.

Our aim is to investigate both nonlinear phenomena with both nonlocal terms ($0 < \beta \le \alpha$) in (1.1), since, in the limiting case $\alpha = \beta = 0$, we can fully cover the Berestycki and Lions assumptions [6].

We impose the following assumptions on f and g:

(H_1) there is a constant $C > 0$ and $p \in \left(\frac{2\beta}{N-2}, \frac{N+\beta}{N-2}\right]$ such that $|sf(s)| \le C|s|^{\frac{N+\alpha}{N-2}}$ and $0 \le g(s)s \le C\big(|s|^p + |s|^{\frac{N+\beta}{N-2}}\big)$ for $s \in \mathbb{R}$;

(H_2) $\displaystyle\lim_{s\to 0} \frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}} = \lim_{|s|\to+\infty} \frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}} = 0$;

(H_3) there is $s_0 > 0$ such that $F(s_0) \neq 0$; if $\alpha = \beta$, then we assume also $F(s_0) > G(s_0)$.

Observe that, if $0 \le \beta < \frac{N-2}{2}$, then, due to the continuity of g, we can take $p = 1 \in \left(\frac{2\beta}{N-2}, \frac{N+\beta}{N-2}\right]$.

We remark that these kinds of assumptions follow naturally from the local case, namely when $\alpha = \beta = 0$, and equation (1.1) becomes simply

$$-\Delta u = h(u) \qquad \text{in } \mathbb{R}^N. \tag{1.4}$$

This problem has been studied in [6] and [25, 26], under general assumptions. In particular, in [25, 26] Struwe considered a continuous and odd function $h : \mathbb{R} \to \mathbb{R}$ with primitive $H(s) = \int_0^s h(t)\,dt$ such that

(i) $-\infty \le \limsup_{s\to 0} h(s)s/|s|^{\frac{2N}{N-2}} \le 0$;
(ii) $-\infty \le \limsup_{|s|\to+\infty} h(s)s/|s|^{\frac{2N}{N-2}} \le 0$;
(iii) there is $s_0 > 0$ such that $H(s_0) > 0$.

Observe that the above assumptions contain those in [6]. As usual, by the maximum principle, it is enough to solve (1.4) when $\limsup_{|s|\to+\infty} h(s)s/|s|^{\frac{2N}{N-2}} = 0$. Now, taking F and G even functions such that

$$F^2(s) = \int_0^s \max\{h(t), 0\}\,dt \quad \text{and} \quad G^2(s) = \int_0^s \max\{-h(t), 0\}\,dt, \qquad \text{for } s \ge 0,$$

we get $H(s) = F^2(s) - G^2(s)$ and, in the local case $\alpha = \beta = 0$, assumptions (H_2) and (H_3) are clearly satisfied. Moreover F and G satisfy the following condition

(H_1^*) there is a constant $C > 0$ such that $|(F^2)'(s)s| \le C|s|^{\frac{2N}{N-2}}$ and $0 \le (G^2)'(s)s \le C\big(|s|^2 + |s|^{\frac{2N}{N-2}}\big)$ for $s \in \mathbb{R}$.

This is a slightly weaker variant of (H_1), which is essentially designed for the nonlocal problem. In fact, with our argument, one can provide a different proof of the existence of a radial solution under assumptions (i)–(iii) from [25, 26].
Further progress on the Berestycki-Lions problem (1.4) has been made in [12, 18, 19]; see also the references therein.

We look for a *weak solution* $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ to (1.1), i.e.

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla \psi \, dx = \int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big) f(u)\psi \, dx - \int_{\mathbb{R}^N} \big(I_\beta * G(u)\big) g(u)\psi \, dx$$

for any $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ stands for the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\nabla \cdot\|_2$.

At least formally solutions of (1.1) are critical points of the functional $\mathcal{I} : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\mathcal{I}(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big) F(u) \, dx + \int_{\mathbb{R}^N} \big(I_\beta * G(u)\big) G(u) \, dx,$$

where $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Since $|F(s)| \le C|s|^{\frac{N+\alpha}{N-2}}$ for some constant $C > 0$, we have that $\big(I_\alpha * F(u)\big)F(u) \in L^1(\mathbb{R}^N)$. On the other hand $\big(I_\beta * G(u)\big)G(u) \in L^1_{\text{loc}}(\mathbb{R}^N)$ and need not be integrable in $\mathbb{R}^N$. Therefore $\mathcal{I}$ may be infinite on a dense subset of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and, thus, cannot be Fréchet-differentiable.

We remark also that scaling properties of the problem play a crucial role, but, in our case, seem to be difficult to apply. Indeed, if $\alpha \neq \beta$, then the nonlinear terms

$$\begin{aligned}
\int_{\mathbb{R}^N} \big(I_\alpha * F(u(\lambda\cdot))\big) F(u(\lambda\cdot)) \, dx &= \lambda^{-(N+\alpha)} \int_{\mathbb{R}^N} \big(I_\beta * F(u)\big) F(u) \, dx \\
\int_{\mathbb{R}^N} \big(I_\beta * G(u(\lambda\cdot))\big) G(u(\lambda\cdot)) \, dx &= \lambda^{-(N+\beta)} \int_{\mathbb{R}^N} \big(I_\beta * G(u)\big) G(u) \, dx
\end{aligned}$$

have different scaling coefficients and, in particular, one cannot employ Lagrange multipliers as in [6], rescaling as in [25], or Pohozaev constraint approach as in [18, 19].

Moreover, to recover the lack of compactness due to the fact that we are working in the whole $\mathbb{R}^N$, we start using the invariance of the functional $\mathcal{I}$ with respect to the orthogonal group action $\mathcal{O}(N)$. Hence we may restrict our considerations to the subspace of radial function $\mathcal{D}^{1,2}_{\mathcal{O}(N)}(\mathbb{R}^N)$, however $\mathcal{I}|_{\mathcal{D}^{1,2}_{\mathcal{O}(N)}(\mathbb{R}^N)}$ still preserves the above difficulties and may be infinite.

In this setting, our main result reads as follows.

Theorem 1.1. *Assume that (H_1)–(H_3) hold. Then, there is a nontrivial and radial solution $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ to (1.1) such that $\big(I_\beta * G(u)\big)G(u) \in L^1(\mathbb{R}^N)$.*

Let us describe our variational approach.
We observe that

$$\mathcal{F}(u) := \int_{\mathbb{R}^N} \big(I_\alpha * F(u)\big) F(u) \, dx \tag{1.5}$$

is well-defined on $\mathcal{D}^{1,2}(\mathbb{R}^N)$, however $\mathcal{I}$ may be infinite. Therefore we replace

$$\mathcal{G}(u) := \int_{\mathbb{R}^N} \big(I_\beta * G(u)\big) G(u) \, dx \tag{1.6}$$

with

$$\mathcal{G}_n(u) := \int_{\mathbb{R}^N} \varphi_n(x) \big(I_\beta * G(u)\big) G(u) \, dx, \tag{1.7}$$

where $\{\varphi_n\}_{n\ge1}$ is a sequence of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ radial functions, decreasing with respect to the radius, such that, for every $n \ge 1$, $\varphi_n(x) = 1$ for $x \in B_n$, $\varphi_n(x) = 0$ for $x \in \mathbb{R}^N \setminus B_{2n}$, $0 \le \varphi_n(x) \le$

$1, |x||\nabla\varphi_n(x)| \le c$, and $\varphi_n(x) \le \varphi_k(x)$ for $n \le k$ and $x \in \mathbb{R}^N$ (B_n stands for the ball of radius n centred at 0). Then $\mathcal{G}_n$ is well-defined on $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and

$$\mathcal{I}_n(u) := \int_{\mathbb{R}^N} |\nabla u|^2\,dx - \mathcal{F}(u) + \mathcal{G}_n(u) \tag{1.8}$$

is of class $\mathcal{C}^1$.

The functional $\mathcal{I}_n$ does not satisfy any variant of Ambrosetti-Rabinowitz condition [1], hence it is difficult to find a bounded Palais-Smale sequence on a positive level. Inspired by [10, 11] we apply the variational method in [27, Theorem 2.8] to the functional

$$\mathcal{J}_n := (\sigma, u) \in \mathbb{R} \times \mathcal{D}^{1,2}_{\mathcal{O}(N)}(\mathbb{R}^N) \mapsto \mathcal{I}_n(u(e^\sigma\cdot)) \in \mathbb{R}.$$

We require a new nonlocal variant of the Brezis-Lieb Lemma for a general nonlinearity, see Lemma 2.1, and further compactness properties of $\mathcal{F}(u)$ on $\mathcal{D}^{1,2}_{\mathcal{O}(N)}(\mathbb{R}^N)$ demonstrated in Section 2. Then, letting $n \to +\infty$, the careful analysis of the Mountain Pass levels provides a nontrivial radial solution to (1.1). This approach provides also an alternative proof of the existence of a radial solution in the local case considered in [25,26]. We would like to point out that, contrary to [6,25,26], we no longer use the uniform decay at infinity of radial functions from $\mathcal{D}^{1,2}_{\mathcal{O}(N)}(\mathbb{R}^N)$ (see [6, Radial Lemma A.III]) and the compactness lemma due to Strauss [6, Lemma A.I].

Therefore more can be said in higher dimensions. Let $N \ge 4$, $N \ne 5$ and similarly as Bartsch and Willem in [3] (cf. [12, 18, 19]), let us fix $\tau \in \mathcal{O}(N)$ such that $\tau(x_1, x_2, x_3) = (x_2, x_1, x_3)$ for $x_1, x_2 \in \mathbb{R}^M$ and $x_3 \in \mathbb{R}^{N-2M}$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^N = \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^{N-2M}$ and $2 \le M \le N/2$, with $N - 2M \ne 1$. We define

$$X_\tau := \left\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : u(x) = -u(\tau x) \text{ for all } x \in \mathbb{R}^N\right\}.$$

Clearly, if $u \in X_\tau$ is radial, then $u = 0$. Hence X_τ does not contain nontrivial radial functions. Then let us consider $\mathcal{O} := \mathcal{O}(M) \times \mathcal{O}(M) \times \mathcal{O}(N-2M) \subset \mathcal{O}(N)$ acting isometrically on $\mathcal{D}^{1,2}(\mathbb{R}^N)$ with the subspace of invariant function denoted by $\mathcal{D}^{1,2}_{\mathcal{O}}(\mathbb{R}^N)$. Moreover our functionals are invariant under this action whenever f and g are odd or even.

Our result, in this setting, is

Theorem 1.2. *Assume that (H_1)–(H_3) hold, f and g are odd or even, $N \ge 4$ and $N \ne 5$. Then, there is a nontrivial and nonradial solution $u \in \mathcal{D}^{1,2}_{\mathcal{O}}(\mathbb{R}^N) \cap X_\tau$ to (1.1) such that $\big(I_\beta * G(u)\big)G(u) \in L^1(\mathbb{R}^N)$.*

Observe that in Theorem 1.1 and Theorem 1.2 we can take $G(s) = s$ and $\beta = 0$ and we obtain solutions in $H^1(\mathbb{R}^N)$ solving the Choquard problem (1.2). In fact, dealing with the operator $-\Delta u + u$, more general assumptions imposed on F can be considered, which fully cover situation in [22].

Actually, our argument can be, quite easily, adapted to the following problem

$$-\Delta u + \omega u = \big(I_\alpha * F(u)\big)f(u) - \big(I_\beta * G(u)\big)g(u) \qquad \text{in } \mathbb{R}^N, \tag{1.9}$$

where $\omega > 0$, assuming that

- (H_1') there is a constant $C > 0$ and $p \in \left(\frac{2\beta}{N-2}, \frac{N+\beta}{N-2}\right]$ such that $|sf(s)| \le C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}})$ and $0 \le g(s)s \le C\big(|s|^p + |s|^{\frac{N+\beta}{N-2}}\big)$ for $s \in \mathbb{R}$;
- (H_2') $\displaystyle\lim_{s\to 0} \frac{F(s)}{|s|^{\frac{N+\alpha}{N}}} = \lim_{|s|\to+\infty} \frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}} = 0$;
- (H_3') there is $s_0 > 0$ such that $F(s_0) \ne 0$; we assume also $F(s_0) > G(s_0)$, if $\alpha = \beta > 0$, and $F^2(s_0) > G^2(s_0) + \omega s_0^2$, if $\alpha = \beta = 0$.

Observe that the energy functional associated with (1.9) is given by

$$\mathcal{K}_\omega(u) := \mathcal{I}(u) + \omega \int_{\mathbb{R}^N} |u|^2 \, dx, \quad u \in H^1(\mathbb{R}^N),$$

and may be also infinite due to the possible nonintegrable term $\big(I_\beta * G(u)\big)G(u)$.

Our results for equation (1.9) read as follows.

Theorem 1.3. *Assume that (H_1')–(H_3') hold. Then, there is a nontrivial and radial solution u to (1.9) in $H^1(\mathbb{R}^N)$ such that $\big(I_\beta * G(u)\big)G(u) \in L^1(\mathbb{R}^N)$. Moreover, if f and g are odd or even, $N \geq 4$ and $N \neq 5$, there is also a nontrivial and nonradial solution v to (1.9) in $H^1(\mathbb{R}^N) \cap X_\tau$ such that $\big(I_\beta * G(v)\big)G(v) \in L^1(\mathbb{R}^N)$.*

In particular, if

$$F(s) := \frac{1}{\sqrt{q}}|s|^q \text{ with } 1 < q < \frac{N+\alpha}{N-2}, \text{ and } G(s) := \sqrt{\frac{N-2}{N+\beta}}|s|^{\frac{N+\beta}{N-2}}, \tag{1.10}$$

then F and G satisfy (H_1')–(H_3') if and only if $\omega \in (0, \omega_0)$, where

$$\omega_0 := \begin{cases} \frac{2^*-2q}{2^*(q-1)}\left(\frac{N(q-1)}{2q}\right)^{\frac{2^*-2}{2^*-2q}} & \text{if } \alpha = \beta = 0, \\ +\infty & \text{if } \alpha > 0. \end{cases}$$

Then, finally, we obtain the following corollary.

Corollary 1.4. *Suppose that F and G are given by (1.10).*

(a) For any $\omega \in (0, \omega_0)$ there is a radially symmetric symmetric solution in $H^1(\mathbb{R}^N)$ and a nonradial solution in $H^1(\mathbb{R}^N) \cap X_\tau$ to (1.9).

(b) If $\omega \notin (0, \omega_0)$, then (1.9) has only trivial finite energy solution.

Corollary 1.4 has been known only in the local case $\alpha = \beta = 0$ and the problem appears in nonlinear optics as well as in the the study of Bose–Einstein condensates [9, 20]. Note that solutions exist only for $0 < \omega < \omega_0 < +\infty$, see e.g. [6, 13, 19]. In the nonlocal case, for instance if $N = 3$, $q = 2$ and $\alpha > \beta = 0$, we solve the nonlocal cubic-quintic problem of the nonlinear optics for all $\omega > 0$, where I_α is a nonlocal response function determined by the details of the physical process responsible for the nonlocality [8].

Through the paper we use the following notation.
We denote by $\|\cdot\|_k$ the usual norm in $L^k(\mathbb{R}^N)$, for $k \geq 1$, and by B_R the ball centered in 0 with radius $R > 0$ in $\mathbb{R}^N$. Recall that $2^* = \frac{2N}{N-2}$. Finally C is a generic positive constant which may vary from line to line.

2. Functional setting and compactness properties

We prove our results for $\beta > 0$, the most difficult and fully nonlocal situation. Thus, from now on, we assume that $0 < \beta \leq \alpha < N$ and (H_1)–(H_3) hold, with $p = 1$ when $0 < \beta < \frac{N-2}{2}$. The proofs of the paper are simplified when $\beta = 0$ or $\alpha = \beta = 0$ and we skip these cases.

It is standard to see that the functional $\mathcal{F} : L^{2^*}(\mathbb{R}^N) \to \mathbb{R}$, defined in (1.5) is of class $\mathcal{C}^1$, cf. [22].

In order to control the convergence of $\mathcal{F}$, we need the following nonlocal variant of the Brezis-Lieb Lemma [7] for the general nonlinearity. Note that nonlocal variants for particular nonlinearities have already appeared in [4, Lemma 2.2], [21, Lemma 2.4]. The proofs of [4,21] seem to be difficult to adapt to the general nonlinear term. We provide an independent proof for any continuous f satisfying (H_1) and (H_2).

Lemma 2.1. *Let $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then*

$$\lim_n \left(\int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big) f(u_n)u_n \, dx - \int_{\mathbb{R}^N} \big(I_\alpha * F(u_n - u_0)\big) f(u_n)u_n \, dx \right) = \int_{\mathbb{R}^N} \big(I_\alpha * F(u_0)\big) f(u_0)u_0 \, dx.$$

Proof. We claim that, passing to a subsequence, for any $s \in [0,1]$,

$$\lim_n \int_{\mathbb{R}^N} \big(I_\alpha * (f(u_n)u_n)\big) f(u_n - su_0)u_0 \, dx = \int_{\mathbb{R}^N} \big(I_\alpha * (f(u_0)u_0)\big) f(u_0 - su_0)u_0 \, dx. \tag{2.1}$$

Let $\varepsilon > 0$ and $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $\|u_0 - \psi\|_{2^*} < \varepsilon$. We have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} \big(I_\alpha * (f(u_n)u_n)\big) f(u_n - su_0)u_0 \, dx - \int_{\mathbb{R}^N} \big(I_\alpha * (f(u_0)u_0)\big) f(u_0 - su_0)u_0 \, dx \right| \\
&\quad \le \underbrace{\left| \int_{\mathbb{R}^N} \big(I_\alpha * (f(u_n)u_n)\big) f(u_n - su_0)(u_0 - \psi) \, dx \right|}_{(A)} \\
&\qquad + \underbrace{\left| \int_{\mathbb{R}^N} \big(I_\alpha * (f(u_n)u_n)\big) \big(f(u_n - su_0) - f(u_0 - su_0)\big)\psi \, dx \right|}_{(B)} \\
&\qquad + \underbrace{\left| \int_{\mathbb{R}^N} \Big(\big(I_\alpha * (f(u_n)u_n)\big) - \big(I_\alpha * (f(u_0)u_0)\big)\Big) f(u_0 - su_0)\psi \, dx \right|}_{(C)} \\
&\qquad + \underbrace{\left| \int_{\mathbb{R}^N} \big(I_\alpha * (f(u_0)u_0)\big) f(u_0 - su_0)(\psi - u_0) \, dx \right|}_{(D)}.
\end{aligned}$$

Since $\{u_n\}$ is a bounded sequence in $L^{2^*}(\mathbb{R}^N)$, we deduce by (H_1) that $\{f(u_n)u_n\}$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Moreover, by the continuity, we deduce that $f(u_n)u_n$ converges to $f(u_0)u_0$ a.e. on $\mathbb{R}^N$ along a subsequence. Therefore $f(u_n)u_n$ tends weakly to $f(u_0)u_0$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. As the Riesz potential defines a linear and continuous map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, we obtain that $I_\alpha * (f(u_n)u_n)$ tends weakly to $I_\alpha * (f(u_0)u_0)$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$. Moreover, since $u_n - su_0$ converges to $u_0 - su_0$ in $L^q_{\rm loc}(\mathbb{R}^N)$, for $1 \le q < 2^*$, by (H_1) we infer that $f(u_n - su_0)$ converges to $f(u_0 - su_0)$ in $L^q_{\rm loc}(\mathbb{R}^N)$, for $1 \le q < 2N/(\alpha+2)$.
Then, by the Hardy–Littlewood–Sobolev inequality and since $\{f(u_n - su_0)\}$ is bounded in $L^{\frac{2N}{\alpha+2}}(\mathbb{R}^N)$ we obtain

$$(A) \le C\|f(u_n)u_n\|_{\frac{2N}{N+\alpha}} \|f(u_n - su_0)\|_{\frac{2N}{\alpha+2}} \|u_0 - \psi\|_{2^*} \le C\varepsilon$$

and analogously, $(D) \le C\varepsilon$.
Moreover, denoting by $K := \operatorname{supp}(\psi)$, we have

$$(B) \le C\|f(u_n)u_n\|_{\frac{2N}{N+\alpha}} \|f(u_n - su_0) - f(u_0 - su_0)\|_{L^{\frac{N(N+2\alpha+2)}{(N+\alpha)(\alpha+2)}}(K)} \|\psi\|_{\frac{2N(N+2\alpha+2)}{(N+\alpha)(N-2)}} = o_n(1).$$

Finally, also $(C) = o_n(1)$, since $f(u_0 - su_0)\psi$ belongs to $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, namely the dual space of $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$.

Therefore (2.1) is proved.
Now, for any $n \in \mathbb{N}$, we set $\phi_n(s) = \big(I_\alpha * F(u_n - su_0)\big) f(u_n)u_n$, for $s \in [0,1]$, and we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big) f(u_n)u_n\,dx - \int_{\mathbb{R}^N} \big(I_\alpha * F(u_n - u_0)\big) f(u_n)u_n\,dx \\
&\quad= \int_{\mathbb{R}^N} \big(\phi_n(0) - \phi_n(1)\big) dx = -\int_0^1 \left(\int_{\mathbb{R}^N} \phi_n'(s)\,dx\right) ds \\
&\quad= \int_0^1 \left(\int_{\mathbb{R}^N} \big(I_\alpha * (f(u_n - su_0)u_0)\big) f(u_n)u_n\,dx\right) ds \\
&\quad= \int_0^1 \left(\int_{\mathbb{R}^N} \big(I_\alpha * (f(u_n)u_n)\big) f(u_n - su_0)u_0\,dx\right) ds.
\end{aligned}$$

Hence, by (2.1), taking into account the Lebesgue Dominated Convergence Theorem

$$\begin{aligned}
&\lim_n \left(\int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big) f(u_n)u_n\,dx - \int_{\mathbb{R}^N} \big(I_\alpha * F(u_n - u_0)\big) f(u_n)u_n\,dx\right) \\
&\quad= \lim_n \int_0^1 \left(\int_{\mathbb{R}^N} \big(I_\alpha * (f(u_n)u_n)\big) f(u_n - su_0)u_0\,dx\right) ds \\
&\quad= \int_0^1 \left(\lim_n \int_{\mathbb{R}^N} \big(I_\alpha * (f(u_n)u_n)\big) f(u_n - su_0)u_0\,dx\right) ds \\
&\quad= \int_0^1 \left(\int_{\mathbb{R}^N} \big(I_\alpha * (f(u_0)u_0)\big) f(u_0 - su_0)u_0\,dx\right) ds \\
&\quad= -\int_0^1 \left(\int_{\mathbb{R}^N} \phi_0'(s)\,dx\right) ds = -\int_{\mathbb{R}^N} \left(\int_0^1 \phi_0'(s)\,ds\right) dx \\
&\quad= \int_{\mathbb{R}^N} \big(\phi_0(0) - \phi_0(1)\big) dx = \int_{\mathbb{R}^N} \big(I_\alpha * F(u_0)\big) f(u_0)u_0\,dx.
\end{aligned}$$

□

Now, let $\mathcal{O}' = \mathcal{O}(N)$, or $\mathcal{O}' = \mathcal{O} = \mathcal{O}(M) \times \mathcal{O}(M) \times \mathcal{O}(N-2M) \subset \mathcal{O}(N)$ provided that $N \geq 4$ and $N \neq 5$ with $2 \leq M \leq N/2$ and $N - 2M \neq 1$. Let $\mathcal{D}^{1,2}_{\mathcal{O}'}(\mathbb{R}^N)$ be the subspace of $\mathcal{O}'$-invariant functions. Below we demonstrate the compactness properties in the following lemmas.

Lemma 2.2. *Let $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,2}_{\mathcal{O}'}(\mathbb{R}^N)$. Then*

$$\lim_n \int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big) f(u_n)u_n\,dx = \int_{\mathbb{R}^N} \big(I_\alpha * F(u_0)\big) f(u_0)u_0\,dx.$$

Proof. By Lemma 2.1, we conclude if we prove that

$$\lim_n \int_{\mathbb{R}^N} \big(I_\alpha * F(u_n - u_0)\big) f(u_n)u_n\,dx = 0.$$

Indeed, by the Hardy-Littlewood-Sobolev inequality and (H_1), we have

$$\int_{\mathbb{R}^N} (I_\alpha * F(u_n - u_0)) f(u_n)u_n\,dx \leq C\|F(u_n - u_0)\|_{\frac{2N}{N+\alpha}} \|f(u_n)u_n\|_{\frac{2N}{N+\alpha}} \leq C\|F(u_n - u_0)\|_{\frac{2N}{N+\alpha}}.$$

The fact that $\|F(u_n - u_0)\|_{2N/(N+\alpha)} \to 0$ is a consequence of (H_2) and [19, Lemma A.1]. □

Lemma 2.3. *Let $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then, for any $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$,*

$$\lim_n \int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big) f(u_n)\psi\,dx = \int_{\mathbb{R}^N} \big(I_\alpha * F(u_0)\big) f(u_0)\psi\,dx. \tag{2.2}$$

Proof. Arguing as in the proof of Lemma 2.1 and passing to a subsequence, we have that $f(u_n) \to f(u_0)$ in $L^q_{\rm loc}(\mathbb{R}^N)$, for $1 \le q < 2N/(\alpha+2)$, and $\{I_\alpha * F(u_n)\}$ is bounded in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ and tends weakly to $I_\alpha * F(u_0)$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$. Thus, since

$$\left|\int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big) f(u_n)\psi\, dx - \int_{\mathbb{R}^N} \big(I_\alpha * F(u_0)\big) f(u_0)\psi\, dx\right| \le \int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big)|f(u_n)-f(u_0)||\psi|\, dx + \left|\int_{\mathbb{R}^N} \big(I_\alpha * F(u_n) - I_\alpha * F(u_0)\big) f(u_0)\psi\, dx\right|,$$

using the same arguments as in (B) and (C) in the proof of Lemma 2.1, we get (2.2). $\square$

For what concerns the term with G, at least formally, we define the functional $\mathcal{G}$ as in (1.6). However, if in (H_1), $p < \frac{N+\beta}{N-2}$, the situation is quite different from $\mathcal{F}$ and $\mathcal{G}$ need not be finite. Indeed, in such a case, let us consider the Banach spaces

$$L^\mu(\Omega) + L^\nu(\Omega) := \{u \in \mathcal{M}(\Omega) : u = u_1 + u_2, u_1 \in L^\mu(\Omega), u_2 \in L^\nu(\Omega)\},$$

where $1 \le \mu \le \nu < +\infty$, Ω is an arbitrary subset of $\mathbb{R}^N$, and $\mathcal{M}(\Omega)$ is the set of the real measurable functions defined on Ω, equipped with the norm

$$\|u\|_{\mu,\nu} := \inf_{u=u_1+u_2}(\|u_1\|_{L^\mu(\Omega)} + \|u_2\|_{L^\nu(\Omega)})$$

(see e.g. [2, Section 2] for more details about these spaces).

Observe that if $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$, since $|u|^p \in L^{\frac{2^*}{p}}(\mathbb{R}^N)$ and $|u|^{\frac{N+\beta}{N-2}} \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N)$, by [2, Proposition 2.3] and (H_1), we get

$$G(u) \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N). \tag{2.3}$$

Moreover, since

$$I_\beta * G(u) \le C\big(I_\beta * (|u|^p + |u|^{\frac{N+\beta}{N-2}})\big),$$

by [15, Inequality (9), page 107] and [2, Proposition 2.3] we have

$$I_\beta * G(u) \in L^{\frac{2N}{N-\beta}}(\mathbb{R}^N) + L^{\frac{2N}{(N-2)p-2\beta}}(\mathbb{R}^N). \tag{2.4}$$

However, this does not seem enough to assure that $\mathcal{G}(u) < +\infty$ for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, and so we need a different approach. We replace $\mathcal{G}(u)$ with $\mathcal{G}_n(u)$ given by (1.7) together with the sequence $\{\varphi_n\}$ defined there.

We prove the following lemma.

Lemma 2.4. *For every $n \in \mathbb{N}$, $\mathcal{G}_n \in \mathcal{C}^1(\mathcal{D}^{1,2}(\mathbb{R}^N), \mathbb{R})$.*

Proof. We divide the proof in five steps.

Step 1: *$\mathcal{G}_n$ is well defined.*

Observe that

$$0 \le \mathcal{G}_n(u) \le \int_{B_{2n}} \big(I_\beta * G(u)\big) G(u)\, dx$$

and, by (2.3) and (2.4), $I_\beta * G(u) \in L^{\frac{2N}{N-\beta}}(B_{2n}) + L^{\frac{2N}{(N-2)p-2\beta}}(B_{2n}) \subset L^{\frac{2N}{N-\beta}}(B_{2n})$ and $G(u) \in L^{\frac{2N}{N+\beta}}(B_{2n}) + L^{\frac{2^*}{p}}(B_{2n}) \subset L^{\frac{2N}{N+\beta}}(B_{2n})$. Thus, the Hölder inequality allows us to conclude.

Step 2: *if $\{u_m\} \subset L^{2^*}(\mathbb{R}^N)$ and $u_m \to u$ in $L^{2^*}(\mathbb{R}^N)$, then, up to a subsequence, $I_\beta * G(u_m) \to I_\beta * G(u)$ a.e. in $\mathbb{R}^N$, as $m \to +\infty$.*

Since $u_m \to u$ in $L^{2^*}(\mathbb{R}^N)$, then, up to a subsequence, there exist $\Omega_1 \subset \mathbb{R}^N$ with $|\Omega_1| = 0$ and $w \in L^{2^*}(\mathbb{R}^N)$ such that $|u_m| \le w$ and $u_m \to u$ in $\mathbb{R}^N \setminus \Omega_1$.

Since $w^p + w^{\frac{N+\beta}{N-2}} \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N)$, by [15, Inequality (9), page 107], we have that $I_\beta *$

$\left(w^p + w^{\frac{N+\beta}{N-2}}\right) \in L^{\frac{2N}{N-\beta}}(\mathbb{R}^N) + L^{\frac{2N}{(N-2)p-2\beta}}(\mathbb{R}^N)$ and so, there exists $\Omega_2 \subset \mathbb{R}^N$, with $|\Omega_2| = 0$, such that

$$\frac{w^p(y) + w^{\frac{N+\beta}{N-2}}(y)}{|x-y|^{N-\beta}} \in L^1(\mathbb{R}^N), \qquad \text{for all } x \in \mathbb{R}^N \setminus \Omega_2.$$

Thus, if we fix $x \in \mathbb{R}^N \setminus \Omega_2$, we have that

$$\frac{G(u_m(y))}{|x-y|^{N-\beta}} \to \frac{G(u(y))}{|x-y|^{N-\beta}}, \qquad \text{for all } y \in \mathbb{R}^N \setminus \Omega_1$$

and

$$\frac{G(u_m(y))}{|x-y|^{N-\beta}} \leq C\frac{|u_m(y)|^p + |u_m(y)|^{\frac{N+\beta}{N-2}}}{|x-y|^{N-\beta}} \leq C\frac{w^p(y) + w^{\frac{N+\beta}{N-2}}(y)}{|x-y|^{N-\beta}} \in L^1(\mathbb{R}^N).$$

Hence, by the Lebesgue Dominated Convergence Theorem we can conclude.

Step 3: *$\mathcal{G}_n$ is continuous.*

Let $\{u_m\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be such that $u_m \to u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $m \to +\infty$. Up to a subsequence we have that $u_m \to u$ in $L^{2^*}(\mathbb{R}^N)$, $u_m \to u$ a.e. in $\mathbb{R}^N$, and there exists $w \in L^{2^*}(\mathbb{R}^N)$ such that $|u_m| \leq w$ a.e. in $\mathbb{R}^N$. Thus, since G is continuous, $G(u_m) \to G(u)$ a.e. in $\mathbb{R}^N$ and, by Step 2, $I_\beta * G(u_m) \to I_\beta * G(u)$ a.e. in $\mathbb{R}^N$. Hence

$$\varphi_n(x)\big(I_\beta * G(u_m)\big)G(u_m) \to \varphi_n(x)\big(I_\beta * G(u)\big)G(u) \text{ a.e. in } \mathbb{R}^N, \text{ as } m \to +\infty.$$

Moreover,

$$0 \leq \varphi_n(x)\big(I_\beta * G(u_m)\big)G(u_m) \leq C\varphi_n(x)\big(I_\beta * (w^p + w^{\frac{N+\beta}{N-2}})\big)(w^p + w^{\frac{N+\beta}{N-2}}) \in L^1(\mathbb{R}^N)$$

since, arguing as before, $I_\beta * (w^p + w^{\frac{N+\beta}{N-2}}) \in L^{\frac{2N}{N-\beta}}(B_{2n})$ and $w^p + w^{\frac{N+\beta}{N-2}} \in L^{\frac{2N}{N+\beta}}(B_{2n})$. Thus, the Lebesgue Dominated Convergence Theorem allows us to conclude.

Step 4: *$\mathcal{G}_n$ is differentiable and, for any $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$,*

$$\mathcal{G}_n'(u)[v] = 2\int_{\mathbb{R}^N} \varphi_n(x)\big(I_\beta * G(u)\big)g(u)v\,dx.$$

First we prove that

$$\left|\int_{\mathbb{R}^N} \varphi_n(x)\big(I_\beta * G(u)\big)g(u)v\,dx\right| < +\infty.$$

Observe that

$$\left|\int_{\mathbb{R}^N} \varphi_n(x)\big(I_\beta * G(u)\big)g(u)v\,dx\right| \leq \int_{B_{2n}} \big(I_\beta * G(u)\big)|g(u)||v|\,dx,$$

and, by assumptions on g,

$$|g(u)| \leq C(|u|^{p-1} + |u|^{\frac{\beta+2}{N-2}}) \in \begin{cases} L^{\frac{2N}{\beta+2}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N), & \text{if } 0 < \beta < \frac{N-2}{2}, \\ L^{\frac{2N}{\beta+2}}(\mathbb{R}^N) + L^{\frac{2^*}{p-1}}(\mathbb{R}^N), & \text{if } \frac{N-2}{2} \leq \beta < N. \end{cases} \tag{2.5}$$

In any case we have that $I_\beta * G(u) \in L^{\frac{2N}{N-\beta}}(B_{2n})$ and, by (2.5), $g(u) \in L^{\frac{2N}{\beta+2}}(B_{2n})$. Thus, the Hölder inequality allows us to conclude.

Finally, arguing as before, we prove that the map

$$v \in \mathcal{D}^{1,2}(\mathbb{R}^N) \longmapsto \int_{\mathbb{R}^N} \varphi_n(x)\big(I_\beta * G(u)\big)g(u)v\,dx$$

is continuous and this implies also the claim.

Step 5: *$\mathcal{G}_n'$ is continuous.*

Let $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, with $\|\nabla v\|_2 \leq 1$ and $\{u_m\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be such that $u_m \to u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as

$m \to +\infty$. Up to a subsequence we have that $u_m \to u$ in $L^{2^*}(\mathbb{R}^N)$, $u_m \to u$ a.e. in $\mathbb{R}^N$, and there exists $w \in L^{2^*}(\mathbb{R}^N)$ such that $|u_m| \le w$ a.e. in $\mathbb{R}^N$. Moreover

$$\begin{aligned}\left|\mathcal{G}_n'(u_m)[v] - \mathcal{G}_n'(u)[v]\right| &\le \int_{B_{2n}} \left|\big(I_\beta * G(u_m)\big)g(u_m) - \big(I_\beta * G(u)\big)g(u)\right||v|\,dx \\ &\le C\left(\int_{B_{2n}} \left|\big(I_\beta * G(u_m)\big)g(u_m) - \big(I_\beta * G(u)\big)g(u)\right|^{\frac{2N}{N+2}} dx\right)^{\frac{N+2}{2N}}.\end{aligned}$$

Using also Step 2, we have that $\big(I_\beta * G(u_m)\big)g(u_m) \to [I_\beta * G(u)]g(u)$ a.e. in $\mathbb{R}^N$, and so, observing that, by (H_1),

$$\begin{aligned}0 \le I_\beta * G(u_m) &\le C I_\beta * (w^p + w^{\frac{N+\beta}{N-2}}) \in L^{\frac{2N}{N-\beta}}(B_{2n}),\\ 0 \le I_\beta * G(u) &\le C I_\beta * (|u|^p + |u|^{\frac{N+\beta}{N-2}}) \in L^{\frac{2N}{N-\beta}}(B_{2n}),\end{aligned}$$

and

$$\begin{aligned}|g(u_m)| &\le C(w^{p-1} + w^{\frac{\beta+2}{N-2}}) \in L^{\frac{2N}{\beta+2}}(B_{2n}),\\ |g(u)| &\le C(|u|^{p-1} + |u|^{\frac{\beta+2}{N-2}}) \in L^{\frac{2N}{\beta+2}}(B_{2n}),\end{aligned}$$

we can conclude by the Lebesgue Dominated Convergence Theorem. □

Now we prove this further compactness result.

Lemma 2.5. *Let $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then, for any $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$,*

$$\lim_n \int_{\mathbb{R}^N} \varphi_n(x)(I_\beta * G(u_n))g(u_n)\psi\,dx = \int_{\mathbb{R}^N} (I_\beta * G(u_0))g(u_0)\psi\,dx.$$

Proof. Of course it is enough to show that

$$\lim_n \int_{\mathrm{Spt}(\psi)} \big(I_\beta * G(u_n)\big)g(u_n)\psi\,dx = \int_{\mathrm{Spt}(\psi)} \big(I_\beta * G(u_0)\big)g(u_0)\psi\,dx,$$

recalling that $\mathrm{Spt}(\psi)$ is compact and then, for n large enough, $\mathrm{Spt}(\psi) \subset B_n$.
Since $u_n \rightharpoonup u_0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, up to a subsequence, $u_n \to u_0$ a.e. in $\mathbb{R}^N$ and so $G(u_n) \to G(u_0)$ a.e. in $\mathbb{R}^N$, as $n \to +\infty$.
Moreover $\{G(u_n)\}$ is bounded in $L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N)$. Indeed, by the assumptions on g, the definition of the norm in $L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N)$, and [2, Corollary 2.12], we have

$$\|G(u_n)\|_{\frac{2N}{N+\beta},\frac{2^*}{p}} \le C(\|u_n\|_{2^*}^p + \|u_n\|_{2^*}^{\frac{N+\beta}{N-2}}) \le C.$$

Thus, the reflexivity of $L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N)$ (see [2, Corollary 2.11]) implies that there exists $\tilde{u} \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N)$ such that, up to a subsequence, $G(u_n) \rightharpoonup \tilde{u}$ in $L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N)$. We claim that $\tilde{u} = G(u_0)$.
Indeed, using a classical argument, the weak convergence $G(u_n) \rightharpoonup \tilde{u}$ in $L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N)$ implies that there exists a sequence $\{z_n\} \subset L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N)$ such that, for all $n \in \mathbb{N}$,

$$z_n \in \mathrm{conv}\left(\bigcup_{i=1}^n \{G(u_i)\}\right)$$

and $z_n \to \tilde{u}$ in $L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N)$. Thus, by [2, Proposition 2.8], up to a subsequence, we get that $z_n \to \tilde{u}$ a.e. in $\mathbb{R}^N$, that allows us to conclude.
About the sequence $\{I_\beta * G(u_n)\}$, since by (H_1)

$$0 \le I_\beta * G(u_n) \le C(I_\beta * |u_n|^p + I_\beta * |u_n|^{\frac{N+\beta}{N-2}}) \in L^{\frac{2N}{N-\beta}}(\mathbb{R}^N) + L^{\frac{2N}{(N-2)p-2\beta}}(\mathbb{R}^N),$$

using [2, Corollary 2.12], we have

$$\begin{aligned}\|I_\beta * G(u_n)\|_{\frac{2N}{N-\beta},\frac{2N}{(N-2)p-2\beta}} &\le C(\|I_\beta * |u_n|^{\frac{N+\beta}{N-2}}\|_{\frac{2N}{N-\beta}} + \|I_\beta * |u_n|^p\|_{\frac{2N}{(N-2)p-2\beta}}) \\ &\le C(\|u_n\|_{2^*}^{\frac{N+\beta}{N-2}} + \|u_n\|_{2^*}^p) \le C.\end{aligned}$$

Moreover, observe that the linear functional

$$w \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N) \mapsto I_\beta * w \in L^{\frac{2N}{N-\beta}}(\mathbb{R}^N) + L^{\frac{2N}{(N-2)p-2\beta}}(\mathbb{R}^N)$$

is continuous. Indeed, if $w \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N)$, $w = w_1 + w_2$ with $w_1 \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N)$ and $w_2 \in L^{\frac{2^*}{p}}(\mathbb{R}^N)$, by [15, Inequality (9), page 107] we get

$$\|I_\beta * w\|_{\frac{2N}{N-\beta},\frac{2N}{(N-2)p-2\beta}} \le \|I_\beta * w_1\|_{\frac{2N}{N-\beta}} + \|I_\beta * w_2\|_{\frac{2N}{(N-2)p-2\beta}} \le C(\|w_1\|_{\frac{2N}{N+\beta}} + \|w_2\|_{\frac{2^*}{p}})$$

and, passing to the infimum on $w_1 \in L^{\frac{2N}{N+\beta}}(\mathbb{R}^N)$ and $w_2 \in L^{\frac{2^*}{p}}(\mathbb{R}^N)$, we conclude.
This, combined with the weak convergence $G(u_n) \rightharpoonup G(u_0)$ in $L^{\frac{2N}{N+\beta}}(\mathbb{R}^N) + L^{\frac{2^*}{p}}(\mathbb{R}^N)$, implies that $I_\beta * G(u_n) \rightharpoonup I_\beta * G(u_0)$ in $L^{\frac{2N}{N-\beta}}(\mathbb{R}^N) + L^{\frac{2N}{(N-2)p-2\beta}}(\mathbb{R}^N)$.
Hence, as done for f in Lemma 2.3, we have that

$$\begin{aligned}&\left|\int_{\mathrm{Spt}(\psi)} \big(I_\beta * G(u_n)\big)g(u_n)\psi\,dx - \int_{\mathrm{Spt}(\psi)} \big(I_\beta * G(u_0)\big)g(u_0)\psi\,dx\right| \\ &\quad\le \int_{\mathrm{Spt}(\psi)} \big(I_\beta * G(u_n)\big)|g(u_n) - g(u_0)||\psi|\,dx + \left|\int_{\mathbb{R}^N} \big(I_\beta * G(u_n) - I_\alpha * G(u_0)\big)g(u_0)\psi\,dx\right|.\end{aligned}$$

About the first integral, observe that, the boundedness of $\{u_n\}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ implies also that $u_n \to u_0$ in $L^\tau_{\mathrm{loc}}(\mathbb{R}^N)$, for all $1 \le \tau < 2^*$ and so, for any fixed $1 \le \tau < 2^*$ and $K \subset\subset \mathbb{R}^N$, up to a subsequence, there exists $w_K \in L^\tau(K)$ such that $|u_n| \le w_K$ a.e. in K. Thus, denoting for simplicity $w := w_{\mathrm{Spt}(\psi)}$ and taking for instance

$$\tau = \frac{N(N+2\beta+2)}{(N-2)(N+\beta)},$$

by the assumptions on g we have

$$g(u_n) \to g(u_0) \text{ a.e. in } \mathrm{Spt}(\psi),$$

$$|g(u_n)| \le C(|u_n|^{p-1} + |u_n|^{\frac{\beta+2}{N-2}}) \le C(w^{p-1} + w^{\frac{\beta+2}{N-2}}) \in L^{\frac{N(N+2\beta+2)}{(N+\beta)(\beta+2)}}(\mathrm{Spt}(\psi)),$$

$$|g(u_0)| \le C(|u_0|^{p-1} + |u_0|^{\frac{\beta+2}{N-2}}) \in L^{\frac{N(N+2\beta+2)}{(N+\beta)(\beta+2)}}(\mathrm{Spt}(\psi)).$$

Moreover, the boundedness of $\{I_\beta * G(u_n)\}$ in $L^{\frac{2N}{N-\beta}}(\mathbb{R}^N) + L^{\frac{2N}{(N-2)p-2\beta}}(\mathbb{R}^N)$ implies its boundedness in $L^{\frac{2N}{N-\beta}}(\mathrm{Spt}(\psi)) + L^{\frac{2N}{(N-2)p-2\beta}}(\mathrm{Spt}(\psi)) = L^{\frac{2N}{N-\beta}}(\mathrm{Spt}(\psi))$.
Thus, by the Hölder inequality and the Lebesgue Dominated Convergence Theorem, we have

$$\int_{\mathrm{Spt}(\psi)} \big(I_\beta * G(u_n)\big)|g(u_n) - g(u_0)||\psi|\,dx \le C\Big(\int_{\mathrm{Spt}(\psi)} |g(u_n) - g(u_0)|^{\frac{2N}{N+\beta}}|\psi|^{\frac{2N}{N+\beta}}\,dx\Big)^{\frac{N+\beta}{2N}}$$

$$\leq C\Big(\int_{\mathrm{Spt}(\psi)} |g(u_n)-g(u_0)|^{\frac{N(N+2\beta+2)}{(N+\beta)(\beta+2)}}\,dx\Big)^{\frac{(N+\beta)(\beta+2)}{N(N+2\beta+2)}} = o_n(1).$$

Finally the second integral goes to 0 due to the weak convergence $I_\beta * G(u_n) \rightharpoonup I_\beta * G(u_0)$ in $L^{\frac{2N}{N-\beta}}(\mathbb{R}^N)+L^{\frac{2N}{(N-2)p-2\beta}}(\mathbb{R}^N)$, since $g(u_0)\psi \in L^{\frac{2N}{N+\beta}}(\mathrm{Spt}(\psi)) \subset [L^{\frac{2N}{N-\beta}}(\mathbb{R}^N)+L^{\frac{2N}{(N-2)p-2\beta}}(\mathbb{R}^N)]'$, being

$$\int_{\mathrm{Spt}(\psi)} |g(u_0)\psi|^{\frac{2N}{N+\beta}}\,dx \leq C\int_{\mathrm{Spt}(\psi)} (|u_0|^{p-1}+|u_0|^{\frac{\beta+2}{N-2}})^{\frac{2N}{N+\beta}}|\psi|^{\frac{2N}{N+\beta}}\,dx$$
$$\leq C\Big(\int_{\mathrm{Spt}(\psi)} (|u_0|^{p-1}+|u_0|^{\frac{\beta+2}{N-2}})^{\frac{N(N+2\beta+2)}{(N+\beta)(\beta+2)}}\,dx\Big)^{\frac{2(\beta+2)}{N+2\beta+2}} < +\infty.$$

□

3. Proofs of our main results

Let $X := \mathcal{D}^{1,2}_{\mathcal{O}(N)}(\mathbb{R}^N)$, or $X := \mathcal{D}^{1,2}_{\mathcal{O}}(\mathbb{R}^N)\cap X_\tau$ provided that $N \geq 4$ and $N \neq 5$. As observed before, the functional $\mathcal{I}$ could be also $+\infty$ on X. To avoid this problem, for every $n \geq 1$, we introduce the truncated $\mathcal{C}^1$-functionals $\mathcal{I}_n : X \to \mathbb{R}$ defined by (1.8).

The functionals $\mathcal{I}$ and $\mathcal{I}_n$, $n \geq 1$, satisfy the geometrical assumptions of the Mountain Pass Theorem. Indeed, we prove the following lemma.

Lemma 3.1. *We have:*

(i) there exist $\rho, c > 0$ such that $\mathcal{I}(u) \geq c$ and, for every $n \geq 1$, $\mathcal{I}_n(u) \geq c$ for all $u \in X$ such that $\|\nabla u\|_2 = \rho$;

(ii) there exists $v_0 \in X$ with $\|\nabla v_0\|_2 > \rho$ such that $\mathcal{I}(v_0) < 0$ and, for every $n \geq 1$, $\mathcal{I}_n(v_0) < 0$.

Proof. We prove this lemma only for $\mathcal{I}_n$ since similar and easier arguments hold also for $\mathcal{I}$.
The positivity of G and φ_n, (H_1) and (H_2), the Hardy-Littlewood-Sobolev and Sobolev inequalities imply

$$\mathcal{I}_n(u) \geq \|\nabla u\|_2^2 - C\int_{\mathbb{R}^N}\Big(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\Big)|u|^{\frac{N+\alpha}{N-2}}\,dx \geq \|\nabla u\|_2^2 - C\|u\|_{2^*}^{\frac{2(N+\alpha)}{N-2}} \geq \|\nabla u\|_2^2 - C\|\nabla u\|_2^{\frac{2(N+\alpha)}{N-2}}.$$

Since $2 < \frac{2(N+\alpha)}{N-2}$, we get (i).
Now let us prove (ii).
Case $X = \mathcal{D}^{1,2}_{\mathcal{O}(N)}(\mathbb{R}^N)$. Let $w = s_0\chi_{B_1}$, where s_0 is defined in (H_3), then

$$\mathcal{F}(w) = F^2(s_0)\iint_{B_1\times B_1} I_\alpha(x-y)\,dxdy > 0.$$

We take now $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ radial, non-negative, non-increasing with respect to $|x|$, and such that $\psi(x) = s_0$, for $|x| \leq 1$, and $\psi(x) = 0$, for $|x| \geq \bar{r}$, with $\bar{r} > 1$. If $\bar{r}$ is sufficiently close to 1, using the continuity of $\mathcal{F}$ in $L^{2^*}(\mathbb{R}^N)$, we get also

$$\mathcal{F}(\psi) > 0. \tag{3.1}$$

We consider first the case $\alpha > \beta$.
If we set $\psi_\lambda(x) := \psi(x/\lambda)$, $\lambda > 0$ and since $0 \leq \varphi_n \leq 1$ we have

$$\int_{\mathbb{R}^N} \varphi_n(x)\big(I_\beta * G(\psi_\lambda)\big)G(\psi_\lambda)\,dx \leq \lambda^{N+\beta}\int_{\mathbb{R}^N}\big(I_\beta * G(\psi)\big)G(\psi)\,dx < +\infty.$$

So we infer that

$$\mathcal{I}_n(\psi_\lambda) \leq \lambda^{N-2}\|\nabla\psi\|_2^2 - \lambda^{N+\alpha}\mathcal{F}(\psi) + \lambda^{N+\beta}\mathcal{G}(\psi)$$

and we can conclude considering $v_0 := \psi_\lambda$ with λ large enough, by (3.1).
We now study the case $\alpha = \beta$.
If $G(s_0) = 0$, being, by (H_1), G non-decreasing on $\mathbb{R}_+$, then $G(\psi(x)) = 0$ in $\mathbb{R}^N$ and so we can conclude easily as before.
If, instead, $G(s_0) \neq 0$, by (H_3) we can find $\varepsilon > 0$ sufficiently small such that $(1-\varepsilon)F^2(s_0) > G^2(s_0) > 0$. Moreover there exists $\bar{r} > 1$ sufficiently close to 1 such that

$$1 < \frac{\displaystyle\iint_{B_{\bar{r}}\times B_{\bar{r}}} I_\alpha(x-y)\,dxdy}{\displaystyle\iint_{B_1\times B_1} I_\alpha(x-y)\,dxdy} < \frac{(1-\varepsilon)F^2(s_0)}{G^2(s_0)}$$

and, again by the continuity of $\mathcal{F}$ in $L^{2^*}(\mathbb{R}^N)$,

$$\mathcal{F}(\psi) \geq (1-\varepsilon)F^2(s_0) \iint_{B_1\times B_1} I_\alpha(x-y)\,dxdy > 0.$$

Therefore, by the positivity of G, we deduce that

$$\mathcal{F}(\psi) - \mathcal{G}(\psi) \geq (1-\varepsilon)F^2(s_0) \iint_{B_1\times B_1} I_\alpha(x-y)\,dxdy - G^2(s_0) \iint_{B_{\bar{r}}\times B_{\bar{r}}} I_\alpha(x-y)\,dxdy > 0.$$

Thus we get

$$\mathcal{I}_n(\psi_\lambda) \leq \lambda^{N-2}\|\nabla\psi\|_2^2 - \lambda^{N+\alpha}[\mathcal{F}(\psi) - \mathcal{G}(\psi)],$$

we can conclude again considering $v_0 := \psi_\lambda$ with λ large enough.
Case $X = \mathcal{D}^{1,2}_{\mathcal{O}}(\mathbb{R}^N) \cap X_\tau$.
We take any odd and smooth function $\eta : \mathbb{R} \to [-1,1]$ such that $\eta(s) = 1$ for $s \geq 1$. Then we define $\widetilde{\psi}(x) = \eta(|x_1| - |x_2|)\psi(x)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^{N-2M}$, with the same ψ as before. Observe that $\widetilde{\psi} \in X$. Moreover, arguing as in the previous case, we can find $\bar{r} > 1$, sufficiently close to 1, such that, using the continuity of $\mathcal{F}$ in $L^{2^*}(\mathbb{R}^N)$,

$$\mathcal{F}(\widetilde{\psi}) \geq \frac{1}{2}F^2(s_0) \iint_{B_1\times B_1 \cap \{|x_1|\geq|x_2|+1, |y_1|\geq|y_2|+1\}} I_\alpha(x-y)\,dxdy > 0.$$

Then we argue similarly as in case $X = \mathcal{D}^{1,2}_{\mathcal{O}(N)}(\mathbb{R}^N)$. □

Let

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = v_0\}$$

and

$$c_{\mathcal{I}_n} := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathcal{I}_n(\gamma(t)), \quad c_{\mathcal{I}} := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathcal{I}(\gamma(t)).$$

Our aim is to find a sequence $\{u_n\} \subset X$ such that $\mathcal{I}_n(u_n) = c_{\mathcal{I}_n}$ and $\mathcal{I}_n'(u_n) \to 0$, as $n \to +\infty$. However, due to the general assumptions on F and G, it is not easy to prove the boundedness of such sequence. Therefore, inspired by [10,11], we introduce the functional $\mathcal{J} : \mathbb{R} \times X \to \mathbb{R} \cup \{+\infty\}$

$$\mathcal{J}(\sigma, u) := e^{(N-2)\sigma}\|\nabla u\|_2^2 - e^{(N+\alpha)\sigma}\mathcal{F}(u) + e^{(N+\beta)\sigma}\mathcal{G}(u),$$

and, for every $n \geq 1$, the $\mathcal{C}^1$-functionals $\mathcal{J}_n : \mathbb{R} \times X \to \mathbb{R}$

$$\mathcal{J}_n(\sigma, u) := e^{(N-2)\sigma}\|\nabla u\|_2^2 - e^{(N+\alpha)\sigma}\mathcal{F}(u) + e^{(N+\beta)\sigma} \int_{\mathbb{R}^N} \varphi_n(e^\sigma x)\big(I_\beta * G(u)\big)G(u)\,dx.$$

Observe that, for any $\sigma \in \mathbb{R}$ and $u \in X$, we have that $\mathcal{J}(\sigma, u) = \mathcal{I}(u(e^{-\sigma}\cdot))$ and $\mathcal{J}_n(\sigma, u) = \mathcal{I}_n(u(e^{-\sigma}\cdot))$.

Let

$$\Sigma := \big\{(\sigma,\gamma)\in\mathcal{C}\left([0,1],\mathbb{R}\times X\right) : \big(\sigma(0),\gamma(0)\big)=(0,0) \text{ and } \big(\sigma(1),\gamma(1)\big)=(0,v_0)\big\}$$

and

$$c_{\mathcal{J}_n} := \inf_{(\sigma,\gamma)\in\Sigma}\sup_{t\in[0,1]}\mathcal{J}_n\big(\sigma(t),\gamma(t)\big), \quad c_{\mathcal{J}} := \inf_{(\sigma,\gamma)\in\Sigma}\sup_{t\in[0,1]}\mathcal{J}\big(\sigma(t),\gamma(t)\big).$$

As observed in [10, Lemma 4.1], using the relation, respectively, between $\mathcal{I}$ and $\mathcal{J}$ and $\mathcal{I}_n$ and $\mathcal{J}_n$, we have that

$$c_{\mathcal{I}} = c_{\mathcal{J}}, \qquad c_{\mathcal{I}_n} = c_{\mathcal{J}_n}. \tag{3.2}$$

Since, for any $n\in\mathbb{N}$, $\mathcal{J}_n \le \mathcal{J}_{n+1} \le \mathcal{J}$, we have that the sequence $\{c_{\mathcal{J}_n}\}$ is increasing and bounded from above by $c_{\mathcal{J}}$, and so there exists $\bar{c}>0$ such that $c_{\mathcal{J}_n}\to\bar{c}$, as $n\to+\infty$.

Proposition 3.2. *There is a sequence $\{(\sigma_n,u_n)\}$ in $\mathbb{R}\times X$ such that*

(i) $|\mathcal{J}_n(\sigma_n,u_n)-\bar{c}| = o_n(1)$*;*
(ii) $|\sigma_n| = o_n(1)$*;*
(iii) $\|\mathcal{J}_n'(\sigma_n,u_n)\| = o_n(1)$*;*
(iv) $\{u_n\}$ *is bounded in* X*.*

Proof. In view of (3.2), for any $n\ge 1$ we find $\gamma_{k,n}\in\Gamma$ such that

$$\sup_{t\in[0,1]}\mathcal{I}_k(\gamma_{k,n}(t)) \le c_{\mathcal{J}_k}+\frac{1}{n}$$

and, for sufficiently large k,

$$|c_{\mathcal{J}_k}-\bar{c}| \le \frac{1}{n}$$

also holds. Therefore, passing to a subsequence with a diagonalization argument, we may assume that there exists $\gamma_n\in\Gamma$ such that

$$\sup_{t\in[0,1]}\mathcal{J}_n(0,\gamma_n(t)) \le c_{\mathcal{J}_n}+o_n(1) \quad\text{and}\quad |c_{\mathcal{J}_n}-\bar{c}|\le o_n(1).$$

Thus, by [27, Theorem 2.8], for any $n\ge1$ there is $(\sigma_n,u_n)\in\mathbb{R}\times X$ such that (i)–(iii) hold. Since $\mathcal{J}_n(\sigma_n,u_n)=\bar{c}+o_n(1)$ and $\partial_\sigma\mathcal{J}_n(\sigma_n,u_n)=o_n(1)$, we have

$$\begin{aligned}\left(1-\frac{N-2}{N+\alpha}\right)e^{(N-2)\sigma_n}\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx+\left(1-\frac{N+\beta}{N+\alpha}\right)e^{(N+\beta)\sigma_n}\int_{\mathbb{R}^N}\varphi_n(e^{\sigma_n}x)\big(I_\beta * G(u_n)\big)G(u_n)\,dx\\ -\frac{1}{N+\alpha}e^{(N+\beta)\sigma_n}\int_{\mathbb{R}^N}\big(\nabla\varphi_n(e^{\sigma_n}x)\cdot e^{\sigma_n}x\big)\big(I_\beta * G(u_n)\big)G(u_n)\,dx=\bar{c}+o_n(1).\end{aligned}$$

Since the cut-off functions φ_n are decreasing with respect to the radius, we have that $\nabla\varphi_n(x)\cdot x\le 0$, for any $x\in\mathbb{R}^N$ and so, being $\alpha\ge\beta$, we infer that $\{u_n\}$ is a bounded sequence in X. □

We can now conclude the proof of our main theorems.

Proof of Theorems 1.1 and 1.2. Let $\{(\sigma_n,u_n)\}$ in $\mathbb{R}\times X$ be the sequence found in Proposition 3.2. Then there exists $u_0\in X$ such that $u_n\rightharpoonup u_0$ weakly in X and a.e. on $\mathbb{R}^N$. By Lemma 2.3 and Lemma 2.5, for any $\psi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$, we have that

$$\int_{\mathbb{R}^N}\nabla u_0\cdot\nabla\psi\,dx=\int_{\mathbb{R}^N}\big(I_\alpha * F(u_0)\big)f(u_0)\psi\,dx-\int_{\mathbb{R}^N}\big(I_\beta * G(u_0)\big)g(u_0)\psi\,dx.$$

So we have that u_0 is a weak solution of (1.1). We will prove that $u_0 \neq 0$.
Observe that, by Proposition 3.2, since $\{u_n\}$ is bounded in X and $\partial_u \mathcal{J}_n(\sigma_n, u_n)[u_n] = o_n(1)$, we deduce that there exists $C > 0$ such that, for any $n \geq 1$,

$$\int_{\mathbb{R}^N} \varphi_n(x)\big(I_\beta * G(u_n)\big)g(u_n)u_n\,dx \leq C.$$

Therefore, by Fatou's Lemma

$$\int_{\mathbb{R}^N} \big(I_\beta * G(u_0)\big)g(u_0)u_0\,dx \leq \liminf_n \int_{\mathbb{R}^N} \varphi_n(x)\big(I_\beta * G(u_n)\big)g(u_n)u_n\,dx \leq C. \tag{3.3}$$

For any $m \geq 1$, let

$$\psi_m(x) = \begin{cases} 1 & \text{if } |x| \leq m, \\ \dfrac{2m - |x|}{m} & \text{if } m \leq |x| \leq 2m, \\ 0 & \text{if } |x| \geq 2m. \end{cases}$$

Observe that, for any $m \geq 1$, we have that $\psi_m u_0$ belongs to X. Note that $\psi_m u_0$ has a compact support and $\partial_u \mathcal{J}_n(\sigma_n, u_n)[\psi_m u_0] = o_n(1)$. Therefore, arguing as in Lemma 2.3 and in Lemma 2.5, passing to the limit as $n \to +\infty$, we have that for any $m \geq 1$

$$\int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla(\psi_m u_0)\,dx = \int_{\mathbb{R}^N} \big(I_\alpha * F(u_0)\big)f(u_0)\psi_m u_0\,dx - \int_{\mathbb{R}^N} \big(I_\beta * G(u_0)\big)g(u_0)\psi_m u_0\,dx. \tag{3.4}$$

Being $u_0 \in X$, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla(\psi_m u_0)\,dx - \int_{\mathbb{R}^N} |\nabla u_0|^2\,dx\right| \\ &\quad \leq \int_{\mathbb{R}^N} |\nabla u_0|^2 |\psi_m - 1|\,dx + \int_{\mathbb{R}^N} |\nabla u_0||u_0||\nabla\psi_m|\,dx \\ &\quad \leq \int_{B_m^c} |\nabla u_0|^2\,dx + \Big(\int_{A_m} |\nabla u_0|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_{A_m} |u_0|^{2^*}\,dx\Big)^{\frac{1}{2^*}}\Big(\int_{A_m} |\nabla\psi_m|^N\,dx\Big)^{\frac{1}{N}} \\ &\quad \leq \int_{B_m^c} |\nabla u_0|^2\,dx + C\Big(\int_{B_m^c} |\nabla u_0|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_{B_m^c} |u_0|^{2^*}\,dx\Big)^{\frac{1}{2^*}} \\ &\quad = o_m(1), \end{aligned} \tag{3.5}$$

where $A_m := B_{2m} \setminus B_m$.
Moreover, observe that

$$\big(I_\alpha * F(u_0)\big)f(u_0)\psi_m u_0 \to \big(I_\alpha * F(u_0)\big)f(u_0)u_0, \quad \text{a.e. in } \mathbb{R}^N, \text{ as } m \to +\infty,$$

and

$$\big|\big(I_\alpha * F(u_0)\big)f(u_0)\psi_m u_0\big| \leq \big|\big(I_\alpha * F(u_0)\big)f(u_0)u_0\big| \in L^1(\mathbb{R}^N).$$

Thus, by the Dominated Convergence Theorem, we have that

$$\lim_m \int_{\mathbb{R}^N} \big(I_\alpha * F(u_0)\big)f(u_0)\psi_m u_0\,dx = \int_{\mathbb{R}^N} \big(I_\alpha * F(u_0)\big)f(u_0)u_0\,dx. \tag{3.6}$$

Analogously, we have also that

$$\big(I_\alpha * G(u_0)\big)g(u_0)\psi_m u_0 \to \big(I_\alpha * G(u_0)\big)g(u_0)u_0, \quad \text{a.e. in } \mathbb{R}^N, \text{ as } m \to +\infty,$$

and, using (3.3),

$$0 \leq \big(I_\alpha * G(u_0)\big)g(u_0)\psi_m u_0 \leq \big(I_\alpha * G(u_0)\big)g(u_0)u_0 \in L^1(\mathbb{R}^N).$$

Again the Dominated Convergence Theorem implies

$$\lim_m \int_{\mathbb{R}^N} \big(I_\alpha * G(u_0)\big) g(u_0)\psi_m u_0\,dx = \int_{\mathbb{R}^N} \big(I_\alpha * G(u_0)\big) g(u_0) u_0\,dx. \tag{3.7}$$

Therefore, by (3.4), (3.5), (3.6) and (3.7), we have

$$\int_{\mathbb{R}^N} |\nabla u_0|^2\,dx = \int_{\mathbb{R}^N} \big(I_\alpha * F(u_0)\big) f(u_0)u_0\,dx - \int_{\mathbb{R}^N} \big(I_\beta * G(u_0)\big) g(u_0)u_0\,dx.$$

By Lemma 2.2 and (3.3), since $\partial_u \mathcal{J}_n(\sigma_n,u_n)[u_n] = o_n(1)$, we infer that

$$\begin{aligned}\limsup_n \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx &= \limsup_n \Big[\int_{\mathbb{R}^N} \big(I_\alpha * F(u_n)\big) f(u_n)u_n\,dx - \int_{\mathbb{R}^N} \varphi_n(x)\big(I_\beta * G(u_n)\big) g(u_n)u_n\,dx\Big]\\ &\le \int_{\mathbb{R}^N} \big(I_\alpha * F(u_0)\big) f(u_0)u_0\,dx - \int_{\mathbb{R}^N} \big(I_\beta * G(u_0)\big) g(u_0)u_0\,dx = \int_{\mathbb{R}^N} |\nabla u_0|^2\,dx.\end{aligned}$$

This implies that $u_n \to u_0$ strongly in X. Thus, since $\mathcal{J}_n(\sigma_n,u_n) \to \mathcal{I}(u_0)$, we have that $\mathcal{I}(u_0) = \bar{c} > 0$ and so u_0 is a nontrivial weak solution of (1.1). □

Proof of Theorem 1.3. Proof is a slight modification of our previous arguments and we leave details for the reader. Here we just want to comment (H_3'). The change of assumption in the different cases is due to the scaling properties of the functional $\mathcal{K}_\omega$. Indeed, setting $u_\lambda(x) := u(x/\lambda)$, for $\lambda > 0$, when $\alpha = \beta$ we have

$$\mathcal{K}_\omega(u_\lambda) = \lambda^{N-2}\|\nabla u\|_2^2 + \omega\lambda^N\|u\|_2^2 - \lambda^{N+\alpha}\big(\mathcal{F}(u) - \mathcal{G}(u)\big).$$

Thus, to show the Mountain Pass geometry, if $\alpha = \beta > 0$, we can proceed as in Lemma 3.1, but if $\alpha = \beta = 0$ (the local case), we need a stronger condition, namely we need to take into account the term ωs_0^2 in order to show that $\mathcal{K}_\omega(u_\lambda) < 0$ for large λ (see also [6]). □

Proof of Corollary 1.4. Item (a) follows from Theorem 1.3.
To prove (b), observe that, only in the local case $\alpha = \beta = 0$, ω_0 is finite. Thus, in such a case, if $\omega \ge \omega_0$, then $F^2(s) - G^2(s) - \omega_0 s^2 \le 0$ for $s \in \mathbb{R}$ and there are no nontrivial solutions (see e.g. [6]). If, instead, $\omega \le 0$, similarly as in [22, Theorem 3], if $u \in H^1(\mathbb{R}^N)$ solves (1.9) with (1.10), then we obtain the following Pohozaev identity

$$\|\nabla u\|_2^2 = -\omega\frac{N}{N-2}\|u\|_2^2 + \frac{N+\alpha}{q(N-2)}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^q\big)|u|^q\,dx - \int_{\mathbb{R}^N} \big(I_\beta * |u|^{\frac{N+\beta}{N-2}}\big)|u|^{\frac{N+\beta}{N-2}}\,dx$$

and, taking into account $\mathcal{K}_\omega'(u)[u] = 0$, i.e.

$$\|\nabla u\|_2^2 = -\omega\|u\|_2^2 + \int_{\mathbb{R}^N} \big(I_\alpha * |u|^q\big)|u|^q\,dx - \int_{\mathbb{R}^N} \big(I_\beta * |u|^{\frac{N+\beta}{N-2}}\big)|u|^{\frac{N+\beta}{N-2}}\,dx,$$

we infer that $u = 0$. □

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